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MAREK KOLK

Piecewise Polynomial Collocation for Volterra Integral Equations with Singularities

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Supervisor:

PhD Prof. Arvet Pedas
University of Tartu
Tartu, Estonia

Opponents:

PhD Prof. Ezio Venturino
University of Torino
Torino, Italy

PhD Prof. Jaan Janno
Tallinn University of Technology
Tallinn, Estonia

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Chapter 1

Introduction

An integral equation is a functional equation in which the unknown function appears under an integral sign. The origins of quantitative theory of integral equations with variable (upper) limits of integration go back to the early 19th century work of Abel. Systematic study of integral equations started from the works of Volterra [71], where he derived an integral equation of the form

$$u(x) = \int_0^x K(x,y)u(y)\,dy + f(x), \quad x \in [0,b], \quad b > 0 \tag{1.1.1}$$

(later called Volterra integral equations of the second kind). Here f and K (the kernel of the integral equation) are given real-valued functions, u is the function which we have to find.

Volterra integral equations arise naturally in many mathematical models of various physical and biological phenomena. In particular, they occur in areas such as viscoelasticity [16, 21, 30], damped vibrations [43], study of epidemics [72], population dynamics [18], identification of memory kernels in heat conduction [28, 29] and financial mathematics [45]; further examples can be found in monographs by Brunner and Houwen [12] and Linz [44].

Fredholm continued a comprehensive study on integral equations in [24], where he gave the necessary and sufficient conditions for solvability of integral equations of the form

$$u(x) = \int_0^b K(x,y)u(y)\,dy + f(x), \quad x \in [0,b] \tag{1.1.2}$$

(later called Fredholm integral equations of the second kind). Also equation (1.1.1) can be written as a Fredholm equation (1.1.2) if we set $K(x,y) = 0$ for $y > x$. The classical Fredholm theory therefore also applies to Volterra equations, but loses much of its power. On the other hand, a direct study of Volterra equations yields many results which cannot be obtained with the Fredholm theory.

The main objects of study in the present thesis are numerical methods for solving Volterra integral equations of the form (1.1.1) where the forcing function f is at least continuous on the interval $[0, b]$ and the kernel K is continuous on the triangle $\{(x, y) : 0 < y < x \leq b\}$.

The study of numerical methods for solving integral equations has received considerable attention in the past. The survey papers by Baker [5, 6, 7], Brunner [9, 11], Venturino [68] and the monographs by Atkinson [2], Baker [4], Brunner [12], Brunner and Houwen [13], Hackbusch [27], Kress [41], Linz [44], Vainikko [64], Vainikko, Pedas and Uba [65] give a good picture of these developments and contain an extensive bibliography. We also refer to the papers [33, 50, 51, 52] for convergence and stability analysis of collocation methods with smooth splines for Volterra integral equations. The present thesis is most closely related to the works [14, 53, 56] where a discussion about the convergence of piecewise polynomial collocation methods for solving Volterra integral equations with weakly singular kernels is given. The basis for our treatment is the paper by Pedas and Vainikko [57] where an analysis about the regularity of a solution of the equation (1.1.1) with different singularities is presented.

In particular, we consider a class of kernels $K(x, y)$ which are m times ($m \geq 0$) continuously differentiable on $\{(x, y) : 0 \leq y < x \leq b\}$ and there exists a real number $\nu \in (-\infty, 1)$ such that the estimate

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j K(x, y) \right| \leq c \left\{ \begin{array}{ll} 1 & , \nu + i < 0 \\ 1 + |\log(x - y)| & , \nu + i = 0 \\ (x - y)^{-\nu - i} & , \nu + i > 0 \end{array} \right\} \tag{1.1.3}$$

holds with a positive constant $c = c(K)$ for all $x, y \in \{(x, y) : 0 \leq y < x \leq b\}$ and for all non-negative integers i and j such that $i + j \leq m$. It follows from (1.1.3) by $i = j = 0$ that

$$|K(x, y)| \leq c \left\{ \begin{array}{ll} 1 & , \nu < 0 \\ 1 + |\log(x - y)| & , \nu = 0 \\ (x - y)^{-\nu} & , \nu > 0 \end{array} \right\}, \quad 0 \leq y < x \leq b.$$

Thus, if $0 \leq \nu < 1$, then the kernel $K(x, y)$ may have a weak singularity as $y \to x$:

$$|K(x, y)| \leq c\,(1 + |\log(x - y)|), \quad \nu = 0,$$

$$|K(x, y)| \leq c\,(x - y)^{-\nu}, \quad 0 < \nu < 1,$$

where $0 \leq y < x \leq b$. If $\nu < 0$, then the kernel $K(x, y)$ itself is bounded on $\{(x, y) : 0 \leq y < x \leq b\}$ but its derivatives may be singular as $y \to x$. Note that the class of kernels allowed in the present thesis is an adaption of the class of functions that were introduced by Vainikko in [64] for multidimensional Fredholm integral equations with weakly singular kernels.

We also consider a more complicated situation for equation (1.1.1) where the kernel $K(x, y)$, in addition to a diagonal singularity (a singularity as $y \to x$), may

have a boundary singularity (a singularity as $y \to 0$). More precisely, in addition to the kernels K satisfying (1.1.3), we consider a set of kernels $K(x,y)$ which are m times ($m \geq 0$) continuously differentiable on $\{(x,y) : 0 < y < x \leq b\}$ and there exist two real numbers ν and λ such that the estimate

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j K(x,y) \right| \leq c \left\{ \begin{array}{ll} 1 & , \nu + i < 0 \\ 1 + |\log(x-y)| & , \nu + i = 0 \\ (x-y)^{-\nu-i} & , \nu + i > 0 \end{array} \right\} y^{-\lambda-j} \tag{1.1.4}$$

holds with a positive constant $c = c(K)$ for all $x, y \in \{(x,y) : 0 < y < x \leq b\}$ and for all non-negative integers i and j such that $i + j \leq m$. It follows from [57] that under the conditions

$$\nu < 1, \quad \lambda < \min\{1, 1-\nu\}$$

a kernel $K(x,y)$ satisfying (1.1.4) is still at most weakly singular in the sense that

$$\sup_{0<x<b} \int_0^b |K(x,y)|\, dy \leq c < \infty.$$

For example, $K(x,y)$ may have the form

$$K(x,y) = K_1(x,y)(x-y)^{-\nu}\, y^{-\lambda} + K_2(x,y), \quad 0 < y < x \leq b, \tag{1.1.5}$$

where $0 < \nu < 1$, $\lambda < 1 - \nu$ and $K_1(x,y)$ and $K_2(x,y)$ are some smooth functions for $0 \leq y \leq x \leq b$.

For Volterra equations (1.1.1) with bounded kernels, the smoothness of the kernel K and the forcing function f determines the smoothness of the solution u on the entire interval of integration $[0,b]$. If we allow weakly singular kernels $K(x,y)$ satisfying (1.1.3) or (1.1.4), then the derivatives of the resulting solutions $u(x)$ may become unbounded at the initial point $x = 0$ of the interval of integration $[0,b]$, see Theorems 4.2.1-4.2.3 below (see also [13, 14, 47]). This complicates constructing high order numerical methods for solving such equations.

In collocation methods, the singular behavior of the exact solution can be taken into account by using special graded grids with the nodes

$$x_j = b\left(\frac{j}{N}\right)^r, \quad j = 0, \ldots, N, \; N \in \mathbb{N}, \; r \in \mathbb{R}, \; r \geq 1. \tag{1.1.6}$$

Here r is a parameter describing the non-uniformity of the grid: if $r > 1$, then the grid points (1.1.6) are more densely clustered near the left endpoint of the interval $[0,b]$ where the solution of the equation (1.1.1) may be singular. By using a collocation method based on piecewise polynomials of degree $m-1$ ($m \geq 1$) and (1.1.6) one can reach a convergence order $O(N^{-m})$ for sufficiently large values of r. For instance, in the case of kernels (1.1.5) with $0 < \nu < 1$, $0 \leq \lambda < 1$, $\nu + \lambda < 1$ the convergence behavior of order $O(N^{-m})$ is available for

$$r \geq \frac{m}{1 - \nu - \lambda}, \tag{1.1.7}$$

see [36, 37, 38] (for $\lambda = 0$ this result follows also from the corresponding results of [10, 11, 12, 13, 14, 53, 56]). Thus, by taking the local polynomial degree m sufficiently large, we can obtain an order of convergence as high as we like, provided that r is so large that the condition (1.1.7) is fulfilled. However, although the piecewise polynomial collocation method on graded grids turns out to be stable for solving weakly singular integral equations (see [31]), the realization of this method in the case of strongly graded grids by large values of r may lead to unstable behavior of numerical results.

To avoid problems associated with the use of strongly graded grids or just improving the convergence order of piecewise polynomial collocation methods for some special case, we first perform a change of variables in the equation (1.1.1) so that the singularities of the derivatives of the exact solution u will be milder or disappear and after that we solve the transformed equation by the piecewise polynomial collocation method on a mildly graded or uniform grid. Our approach is based on the ideas and results of [55, 56, 57], see also [8, 19, 20, 26, 48, 58, 66, 67, 69, 73].

In this thesis, we also take a deeper look at the local superconvergence of the collocation method. The word "superconvergence" was introduced in the early 1970s to describe phenomena arising in the solution of two point boundary-value problems and related partial differential equations (see more in [61]). For certain piecewise-polynomial approximation methods, the order of the convergence at the knots, or other points of interval, is higher than one might have expected from the order of the piecewise polynomials employed. Such a phenomenon was observed for Galerkin [22] and collocation [17] methods, under a condition that the collocation points are very carefully chosen. Subsequently, the usage has been extended to cover similar phenomena for Fredholm integral equations (1.1.2), under appropriate assumptions on K and f (see, for example [2, 54, 58, 61, 64]). Superconvergence phenomena of piecewise polynomial collocation methods for Volterra integral equations (1.1.1) with kernels $K(x, y)$ satisfying (1.1.3) is studied in [14, 53, 56]. In the present thesis, we extend the corresponding investigations to equation (1.1.1) with a wider class of kernels $K(x, y)$ satisfying (1.1.4). Moreover, the attainable order of global and local convergence of proposed algorithms based on a suitable smoothing transformation will be studied.

This thesis is organized as follows.

Chapters 1 and 2 have an introductory character. In Chapter 2, we introduce some definitions and basic results which we use in this work. We also prove some auxiliary estimates for later use in the proofs of main lemmas and theorems of Chapters 4 and 5.

In Chapter 3, we introduce a wide class of smoothing transformations and prove some new results about their smoothing properties. These results refine and complement the corresponding results of [56]. We also derive some estimates for

kernels and interpolation operators which will be used in Chapters 4 and 5.

In Chapter 4, we introduce the basic assumptions about the kernel K and forcing function f of equation (1.1.1) and formulate the underlying results about the regularity and possible singularities of a solution of equation (1.1.1). After that we introduce a class of numerical methods for solving (1.1.1) using suitable smoothing transformations described in Chapter 3 and piecewise polynomial collocation methods on graded grids. We prove the convergence of proposed algorithms and derive global convergence estimates for different new classes of kernels K and forcing functions f. These results generalize and complement the corresponding results of [8, 10, 12, 13, 14, 20, 53, 55, 56].

Chapter 5 is devoted to the study of local superconvergence properties of proposed numerical schemes introduced in Chapter 4. This chapter is an extended expansion of the corresponding results of [39]. Using special collocation points, error estimates at the specific points on $[0, b]$ are given, showing a more rapid convergence than the global uniform convergence on $[0, b]$ available by piecewise polynomials. These results generalize, refine and complete the corresponding results of [14, 53, 56]. The basic idea for the proofs of the superconvergence theorems in this chapter is common for all of them. As there exist some important technical nuances for different types of kernels, we have divided this chapter into subsections and have studied superconvergence phenomena for different type of kernels separately. The convergence estimates for equation (1.1.1) with kernels satisfying (1.1.3) are in good agreement with the results in [53] where the smoothing of the solution has not been considered.

In Chapter 6, we introduce various test problems and compare the computational experiments with the theoretical results which we obtained in Chapters 4 and 5. The numerical results totally support the theoretical analysis.

Most of the results given in Chapters 3-6 are published in [35, 36, 37, 38, 39], although the thesis contains also several new results which have not published yet. In some cases the results in this thesis are stated and proved in a more general form than in our published papers.

Chapter 2

Notations and Basic Results

In this chapter we introduce some basic notations and formulate some well-known results but we also prove some additional results which we need later.

2.1 Notations

Throughout this work c denotes a positive constant which may have different values in different occurrences. By $\mathbb{N} = \{1, 2, \dots\}$ we denote the set of all positive integers, by $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$ the set of integers and by $\mathbb{R} = (-\infty, \infty)$ the set of real numbers. Additionally we use the sets $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{R}_+ = [0, \infty)$.

By $C^m(D)$ $(D \subset \mathbb{R}^n, m \in \mathbb{N}_0, n \in \mathbb{N})$ we denote the set of continuous and m times continuously differentiable functions $v : D \to \mathbb{R}$.

By $C[a, b]$ we denote the Banach space of continuous functions $v : [a, b] \to \mathbb{R}$ with the norm

$$\|v\|_{C[a,b]} = \|v\|_\infty = \max_{a \le x \le b} |v(x)|, \quad v \in C[a, b].$$

By $C^m[a, b]$, $m \in \mathbb{N}_0$, we denote the Banach space of m times continuously differentiable functions $v : [a, b] \to \mathbb{R}$ with the norm

$$\|v\|_{C^m[a,b]} = \sum_{i=0}^{m} \|v^{(i)}\|_\infty, \quad v \in C^m[a, b], \quad C^0[a, b] = C[a, b].$$

By $L^\infty(a, b)$ we denote the set of measurable functions $v : [a, b] \to \mathbb{R}$, such that

$$\inf_{D \subset [a,b] : \mu(D) = 0} \sup_{x \in [a,b] \setminus D} |v(x)| < \infty,$$

where $\mu(D)$ is the Lebesgue measure of the set D. The set $L^\infty(a,b)$ is a Banach space with the norm

$$\|v\|_{L^\infty(a,b)} = \|v\|_\infty = \inf_{D\subset[a,b]:\mu(D)=0} \sup_{x\in[a,b]\setminus D} |v(x)|, \quad v \in L^\infty(a,b).$$

Let E and F be Banach spaces. By $\mathcal{L}(E,F)$ we denote the Banach space of all linear continuous operators $A : E \to F$ with the norm

$$\|A\|_{\mathcal{L}(E,F)} = \sup_{x\in E, \|x\|_E\le 1} \|Ax\|_F, \quad A \in \mathcal{L}(E,F).$$

2.2 Linear Operators

In this work, we use the following well-known results from the theory of linear operators (see, for example [2, 27, 41]).

Theorem 2.2.1. (*Banach-Steinhaus theorem*). *Let $A : E \to F$ be a bounded linear operator and let $A_n : E \to F$ be a sequence of bounded linear operators from a Banach space E into a normed space F. For pointwise convergence $A_n x \to Ax, n \to \infty$, for all $x \in E$ it is necessary and sufficient that $\|A_n\| \le c$ for all $n \in \mathbb{N}$ with some constant c and that $A_n x \to Ax$, $n \to \infty$, for all $x \in G$ where G is some dense subset of E.*

Theorem 2.2.2. *Let E be a Banach space, and let $A \in \mathcal{L}(E,E)$ be a bounded linear operator from E into E with $\|A\|_{\mathcal{L}(E,E)} < 1$. Then there exists $(I-A)^{-1} \in \mathcal{L}(E,E)$, and*

$$\|(I-A)^{-1}\|_{\mathcal{L}(E,E)} \le \frac{1}{1-\|A\|_{\mathcal{L}(E,E)}},$$

where I is the identity mapping in E.

Theorem 2.2.3. *Let E and F be Banach spaces. If the operators $A, B \in \mathcal{L}(E,F)$ are such that A is invertible ($A^{-1} \in \mathcal{L}(F,E)$) and $\|B\|_{\mathcal{L}(E,F)}\|A^{-1}\|_{\mathcal{L}(F,E)} < 1$ then $A+B$ is invertible and the estimate*

$$\|(A+B)^{-1}\|_{\mathcal{L}(F,E)} \le \frac{\|A^{-1}\|_{\mathcal{L}(F,E)}}{1-\|B\|_{\mathcal{L}(E,F)}\|A^{-1}\|_{\mathcal{L}(F,E)}}$$

holds.

Theorem 2.2.4. *Let E, G be normed spaces and let F be a Banach space. Let A be a compact operator mapping E into F and let $B_n : F \to G$ be a pointwise convergent sequence of bounded linear operators with limit operator $B : F \to G$. Then*

$$\|(B_n - B)A\| \to 0, \quad n \to \infty.$$

Theorem 2.2.5. (*Fredholm alternative*). *Let E be a Banach space, and let $A \in \mathcal{L}(E,E)$ be a compact operator. Then the equation $x = Ax + f$, $f \in E$ has a unique solution $x \in E$ if and only if the homogeneous equation $z = Az$ has only the trivial solution $z = 0$. In such a case, the operator $I - A$ has a bounded inverse $(I-A)^{-1} \in \mathcal{L}(E,E)$.*

A linear operator $A : E \to F$ is called compact if A transforms every bounded set of E into a relatively compact set of F. A linear compact operator $A : E \to F$ is continuous. A subset $M \subset E$ is called relatively compact, if all sequences $(x_n) \subset M$ contain a subsequence converging in E.

2.3 Some Basic Results

For differentiating compositions we need the following result (see, for example, [40, p. 111]).

Theorem 2.3.1. *Let $m \in \mathbb{N}_0$, $f \in C^m[a_1,b_1]$ and $g \in C^m[a_2,b_2]$ be real-valued functions such that f maps $[a_1,b_1]$ into $[a_2,b_2]$. Then the derivatives of the composition function $h(t) = g(f(t))$ can be expressed by Faà di Bruno formula*

$$h^{(j)}(t) = \sum \frac{j!}{k_1! \cdots k_j!} g^{(k)}(f(t)) \left(\frac{f'(t)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(j)}(t)}{j!}\right)^{k_j}, \quad 0 \le j \le m, \tag{2.3.1}$$

where $t \in [a_1,b_1]$, $k = k_1 + \cdots + k_j$ and the sum is taken over all $k_1, \ldots, k_j \in \mathbb{N}_0$ for which $k_1 + 2k_2 + \cdots + jk_j = j$.

In this thesis, we use a Gamma function $\Gamma(x)$ defined as Euler's integral (see [1, p. 255]):

$$\Gamma(x) = \int_0^\infty e^{-s}\, s^{x-1}\, ds, \quad x > 0. \tag{2.3.2}$$

Gamma function is related with Beta function $B(x,y)$ by the following equalities (see [1, p. 258]):

$$\int_0^1 s^\lambda (1-s)^\nu\, ds = B(1+\nu, 1+\lambda) = \frac{\Gamma(1+\nu)\Gamma(1+\lambda)}{\Gamma(2+\nu+\lambda)}, \quad \nu, \lambda > -1. \tag{2.3.3}$$

We also use the following result: for every $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, $n \ge 2$ the inequality

$$\sum_{i=1}^n i^\alpha \le c \left\{ \begin{array}{ll} n^{1+\alpha} & \text{for } \alpha > -1 \\ \log n & \text{for } \alpha = -1 \\ 1 & \text{for } \alpha < -1 \end{array} \right\} \tag{2.3.4}$$

holds with a positive constant c not depending on n.

Next, we prove some results for estimating special integrals and functions. For given real numbers $\gamma, b > 0$ and for any $x \in [0, b]$ consider the functions

$$g(x) = x^\gamma (1 + |\log x|), \quad f(x) = x^\gamma \left(\frac{1}{\gamma} + |\log x| \right). \tag{2.3.5}$$

Remark 2.3.1. *Functions g and f in formula (2.3.5) are continuous and bounded on $[0, b]$ and the function f is monotonically increasing on $[0, b]$ for any $\gamma > 0$. Moreover, we can estimate*

$$x^\gamma (1 + |\log x|) \le c, \quad x \in [0, b], \quad \gamma > 0, \tag{2.3.6}$$

and

$$c_1 f(x) \le g(x) \le c_2 f(x), \quad x \in [0, b], \quad \gamma > 0, \tag{2.3.7}$$

where $c = c(\gamma)$; $c_2 \ge c_1$ are some positive constants.

Indeed, for $\gamma \ge 1$ we have

$$g(x) = \gamma x^\gamma \left(\frac{1}{\gamma} + \frac{1}{\gamma} |\log x| \right) \le \gamma x^\gamma \left(\frac{1}{\gamma} + |\log x| \right) = \gamma f(x).$$

For $0 < \gamma < 1$ we write directly

$$g(x) = x^\gamma (1 + |\log x|) \le x^\gamma \left(\frac{1}{\gamma} + |\log x| \right) = f(x).$$

We have got that $g(x) \le c_2 f(x)$. The lower bound can be shown in similar way. So, the inequality (2.3.7) is true. Next, we comment, why f is monotonically increasing on $[0, b]$. For $x \ge 1$ it is trivial. For $0 < x < 1$ we obtain

$$f'(x) = \gamma x^{\gamma - 1} \left(\frac{1}{\gamma} + |\log x| \right) - x^\gamma \frac{1}{x} = \gamma x^{\gamma - 1} |\log x| \ge 0, \quad 0 < x < 1.$$

Thus the function f is monotonically increasing on $[0, b]$.

Lemma 2.3.1. *Assume that $b, \alpha, \beta \in \mathbb{R}$ are such that $b > 0$, $\alpha > -1$ and $\alpha + \beta > -1$. Then for every $t \in [0, b]$ the following estimates*

$$\int_t^b (s - t)^\alpha s^\beta \, ds \le c (b - t)^{\alpha + 1} b^\beta \tag{2.3.8}$$

and

$$\int_t^b (s - t)^\alpha s^\beta (1 + |\log s|) \, ds \le c (b - t)^{\alpha + 1} b^\beta (1 + |\log b|) \tag{2.3.9}$$

hold with a positive constant c not depending on t and b.

Proof. First we show (2.3.8). Let $\beta \geq 0$. Then

$$\begin{aligned} I_{\alpha,\beta} :&= \int_t^b (s-t)^\alpha \, s^\beta \, ds \\ &\leq b^\beta \int_t^b (s-t)^\alpha \, ds = \frac{1}{\alpha+1} \, b^\beta \, (b-t)^{\alpha+1} \leq c \, (b-t)^{\alpha+1} b^\beta, \end{aligned} \tag{2.3.10}$$

with the constant $c = \frac{1}{\alpha+1}$. Now, let $\beta < 0$. Then

$$I_{\alpha,\beta} = \int_t^b \left(\frac{s-t}{s} \right)^{-\beta} (s-t)^{\alpha+\beta} \, ds. \tag{2.3.11}$$

As the function $u(s) = \frac{s-t}{s}$ is increasing on the interval $[t, b]$ and $\alpha + \beta > -1$ then

$$\begin{aligned} I_{\alpha,\beta} &\leq \left(\frac{b-t}{b} \right)^{-\beta} \int_t^b (s-t)^{\alpha+\beta} \, ds \\ &= \frac{1}{\alpha+\beta+1} \left(\frac{b-t}{b} \right)^{-\beta} (b-t)^{\alpha+\beta+1} \leq c \, (b-t)^{\alpha+1} \, b^\beta. \end{aligned} \tag{2.3.12}$$

We have proved (2.3.8). Next we show (2.3.9). First, for $\beta > 0$ we have

$$\hat{I}_{\alpha,\beta} := \int_t^b (s-t)^\alpha \, s^\beta \, (1 + |\log s|) \, ds \leq c \, b^\beta \, (1 + |\log b|) \int_t^b (s-t)^\alpha \, ds, \tag{2.3.13}$$

as we can use the inequality (2.3.7) and by Remark 2.3.1 the function $u(s) = s^\beta \left(\frac{1}{\beta} + |\log s| \right)$ is monotonically increasing on $[t, b]$. Now (2.3.9) follows from (2.3.8) and (2.3.13). Next we assume $\beta \leq 0$. Then

$$\begin{aligned} \hat{I}_{\alpha,\beta} &= \int_t^b (s-t)^\alpha \, s^{\frac{\alpha+\beta+1}{2}} \, s^{-\frac{\alpha-\beta+1}{2}} \, (1 + |\log s|) \, ds \\ &\leq c \, b^{\frac{\alpha+\beta+1}{2}} (1 + |\log b|) \int_t^b (s-t)^\alpha \, s^{-\frac{\alpha-\beta+1}{2}} \, ds. \end{aligned} \tag{2.3.14}$$

As $\alpha - \frac{\alpha-\beta+1}{2} = \frac{\alpha+\beta-1}{2} > -1$ then we can use (2.3.8),

$$\hat{I}_{\alpha,\beta} \leq c \, b^{\frac{\alpha+\beta+1}{2}} (1 + |\log b|) \, (b-t)^{\alpha+1} \, b^{-\frac{\alpha-\beta+1}{2}} = c \, (b-t)^{\alpha+1} \, b^\beta \, (1 + |\log b|). \tag{2.3.15}$$

□

Lemma 2.3.2. *Assume that $a, b, \alpha, \beta \in \mathbb{R}$ are such that $b > a \geq 0$, $\alpha > -1$, $\beta > -1$ and $\alpha + \beta > -1$. Then for every $t \geq b$ the following estimate*

$$\int_a^b (t-s)^\alpha s^\beta \, ds \leq c\,(b-a)\,(t-a)^\alpha b^\beta \tag{2.3.16}$$

holds with a positive constant c not depending on a, b and t.

Proof. **a)** First, we consider the case $\alpha \geq 0$. Then for $\beta \geq 0$ we obtain

$$I_{\alpha,\beta} := \int_a^b (t-s)^\alpha s^\beta \, ds \leq (t-a)^\alpha b^\beta \int_a^b ds = (b-a)\,(t-a)^\alpha b^\beta \tag{2.3.17}$$

and for $-1 < \beta < 0$ we obtain

$$\begin{aligned} I_{\alpha,\beta} &\leq (t-a)^\alpha \int_a^b (s-a)^{\frac{1+\beta}{2}} (s-a)^{-\frac{1+\beta}{2}} s^\beta \, ds \\ &\leq (t-a)^\alpha (b-a)^{\frac{1+\beta}{2}} \int_a^b (s-a)^{-\frac{1+\beta}{2}} s^\beta \, ds. \end{aligned} \tag{2.3.18}$$

As $-\frac{1+\beta}{2} > -1$ and also $-\frac{1+\beta}{2} + \beta > -1$ we may use (2.3.8):

$$I_{\alpha,\beta} \leq c\,(b-a)\,(t-a)^\alpha b^\beta. \tag{2.3.19}$$

b) Now we consider the case $-1 < \alpha < 0$. If $b = t$ then we get for $\beta \geq 0$,

$$I_{\alpha,\beta} \leq b^\beta \int_a^b (b-s)^\alpha \, ds = \frac{1}{\alpha+1} b^\beta (b-a)^{\alpha+1} \leq c\,(b-a)\,(t-a)^\alpha b^\beta, \tag{2.3.20}$$

and for $-1 < \beta < 0$ we obtain

$$\begin{aligned} I_{\alpha,\beta} &= \int_a^b [(b-a)-(s-a)]^\alpha s^\beta \, ds = \int_0^{b-a} [(b-a)-z]^\alpha (z+a)^\beta \, dz \\ &= (b-a)^{\alpha+1} \int_0^1 (1-s)^\alpha [(b-a)s+a]^\beta s^{\beta-\beta} \, ds \\ &= (b-a)^{\alpha+1} \int_0^1 (1-s)^\alpha s^\beta \left(\frac{s}{(b-a)s+a}\right)^{-\beta} ds. \end{aligned} \tag{2.3.21}$$

Note that the function $u(s) = \frac{s}{(b-a)s+a}$ is increasing on $[0,1]$. Thus

$$\begin{aligned} I_{\alpha,\beta} &\leq c\,(b-a)^{\alpha+1}\,b^{\beta}\int\limits_0^1 (1-s)^{\alpha}\,s^{\beta}\,ds = c\,(b-a)^{\alpha+1}\,b^{\beta}\,B(1+\alpha,1+\beta) \\ &\leq c^{'}\,(b-a)\,(b-a)^{\alpha}\,b^{\beta}, \end{aligned} \tag{2.3.22}$$

where $B(1+\alpha,1+\beta)$ is the Beta function (2.3.3) which is bounded with some constant c as $\alpha,\beta > -1$. We summarize our result as follows:

$$\int\limits_a^b (b-s)^{\alpha}\,s^{\beta}\,ds \leq c\,(b-a)\,(t-a)^{\alpha}\,b^{\beta}, \quad b = t, \quad -1 < \alpha < 0. \tag{2.3.23}$$

Finally we may study the case $t > b$ for $-1 < \alpha < 0$. Write

$$I_{\alpha,\beta} = \int\limits_a^b \left(\frac{b-s}{t-s}\right)^{-\alpha} (b-s)^{\alpha}\,s^{\beta}\,ds. \tag{2.3.24}$$

Function $u(s) = \frac{b-s}{t-s}$ is decreasing on interval $[a,b]$. So we can bring $u(s)$ out from the integral (2.3.24) with $s = a$ and for the remaining part of the integral we can use the estimate (2.3.23). Thus

$$I_{\alpha,\beta} \leq c\left(\frac{b-a}{t-a}\right)^{-\alpha} (b-a)^{\alpha+1}\,b^{\beta} = c\,(b-a)\,(t-a)^{\alpha}\,b^{\beta}. \tag{2.3.25}$$

We have proved the inequality (2.3.16).

□

2.4 Grids and Spline Spaces

For given $b \in \mathbb{R}$, $b > 0$, $N \in \mathbb{N}$, let

$$\Delta_N = \{x_0, x_1, \ldots, x_N \,:\, 0 = x_0 < x_1 < \cdots < x_N = b\} \tag{2.4.1}$$

be a partition (a grid) of the interval [1)] $[0, b]$.

Definition 2.4.1. *A grid Δ_N is called regular if*

$$\max_{i=1,\ldots,N} (x_i - x_{i-1}) \to 0 \quad \textit{as} \quad N \to \infty\,. \tag{2.4.2}$$

Definition 2.4.2. *A grid $\Delta_N = \Delta_N^r = \Delta_N^{(r,b)}$ $(r \in [1, \infty))$ is called graded, if the grid points are defined by the formula*

$$x_i = b\left(\frac{i}{N}\right)^r, \quad i = 0, \ldots, N. \tag{2.4.3}$$

Here the parameter r characterizes the degree of non-uniformity of the grid Δ_N^r. In case $r = 1$ we get a uniform grid. If $r > 1$ then the nodes $x_0, \ldots, x_N$ of the grid Δ_N^r are more densely clustered near the left endpoint of the interval $[0, b]$.

For a given integer $m \in \mathbb{N}$ and $0 \le k \le m-2$ let

$$\begin{aligned} S_{m-1}^{(-1)}(\Delta_N) &= \{z_N \,:\, z_N|_{[x_{j-1},x_j]} \in \pi_{m-1},\, j = 1, \ldots, N\}, \\ S_{m-1}^{(k)}(\Delta_N) &= \{z_N \in C^k[0,b] \,:\, z_N|_{[x_{j-1},x_j]} \in \pi_{m-1},\, j = 1, \ldots, N\} \end{aligned} \tag{2.4.4}$$

be the spline spaces of piecewise polynomial functions on the grid Δ_N. Here $z_N|_{[x_{j-1},x_j]}$ $(j = 1, \ldots, N)$ is the restriction of $u(t)$, $t \in [0, b]$, to the subinterval $[x_{j-1}, x_j] \subset [0, b]$ and π_{m-1} denotes the set of polynomials of degree not exceeding $m-1$. Note that the elements of $S_{m-1}^{(-1)}(\Delta_N)$ may have jump discontinuities at the interior knots $x_1, \ldots, x_{N-1}$ of the grid Δ_N.

Next, we formulate some results about graded grids (2.4.3) which will be used in this thesis. We denote the lengths of the subintervals $[x_{i-1}, x_i] \subset [0, b]$ by

$$h_i = x_i - x_{i-1} \quad i = 1, \ldots, N. \tag{2.4.5}$$

For graded grids we can use the estimates

$$h_i \le c\, N^{-r}\, i^{r-1} = c\, N^{-1} \left(\frac{i}{N}\right)^{r-1} \le c'\, N^{-1}, \quad i = 1, \ldots, N, \tag{2.4.6}$$

[1)] For ease of notation we suppress the index N in $x_i = x_i^{(N)}$, $i = 0, \ldots, N$, indicating the dependence of the grid points on N.

where c is a positive constant not depending on i and N. Indeed,

$$\begin{aligned} x_i - x_{i-1} &= b\left(\frac{i}{N}\right)^r - b\left(\frac{i-1}{N}\right)^r = b\,N^{-r}(i^r - (i-1)^r) \\ &= b\,r\,N^{-r}\xi^{r-1} \le b\,r\,N^{-r}\,i^{r-1}\ , \quad \xi \in (i-1,i), \quad i = 1,\dots,N. \end{aligned} \tag{2.4.7}$$

Thus the inequalities (2.4.6) hold with $c = br$. From (2.4.6) it follows that graded grids are regular.

Further, let $\beta \in \mathbb{R}$. Then

$$2^{-r\,|\beta|}\,x_i^\beta \le x_{i-1}^\beta \le 2^{r\,|\beta|}\,x_i^\beta, \quad i = 2,\dots,N. \tag{2.4.8}$$

Indeed, for $\beta \ge 0$ the upper bound is obvious and for the lower bound we can write

$$x_i^\beta = \left(\frac{i^r}{N^r}\frac{(i-1)^r}{(i-1)^r}\right)^\beta = x_{i-1}^\beta\left(\frac{i}{i-1}\right)^{r\beta} \le 2^{r|\beta|}x_{i-1}^\beta.$$

For $\beta < 0$ the lower bound is obvious and for the upper bound we can write

$$x_{i-1}^\beta = \left(\frac{(i-1)^r}{N^r}\frac{i^r}{i^r}\right)^\beta = x_i^\beta\left(\frac{i-1}{i}\right)^{r\beta} \le 2^{r|\beta|}x_i^\beta.$$

Thus the inequalities (2.4.8) hold.

Now we show the estimate

$$x_i - x_{i-1} \le 2^r\,(x_{i-1} - x_{i-2}), \quad i = 2,\dots,N. \tag{2.4.9}$$

Indeed,

$$\begin{aligned} x_i - x_{i-1} &= \frac{i^r - (i-1)^r}{N^r} = \frac{i^r\left(1 - \left(\frac{i-1}{i}\right)^r\right)}{N^r} \le \frac{i^r\left(1 - \left(\frac{i-2}{i-1}\right)^r\right)}{N^r} \\ &= \left(\frac{i}{i-1}\right)^r(x_{i-1} - x_{i-2}) \le 2^r(x_{i-1} - x_{i-2}). \end{aligned}$$

Next, we prove the inequality

$$\left(\frac{i}{N}\right)^\beta \frac{1 + |\log h_i|}{1 + |\log x_i|} \le c\left\{\begin{array}{ll} N^{-\beta} & \text{for } \beta < 0 \\ \log N & \text{for } \beta \ge 0 \end{array}\right\}, \quad i = 1,\dots,N, N > 1, \tag{2.4.10}$$

where the positive constant c is not depending on i and N. If $i = 1$ then $\frac{1+|\log h_i|}{1+|\log x_i|} = 1$ and (2.4.10) holds. Now, let $i \ge 2$. Then

$$\begin{aligned} \frac{1 + |\log h_i|}{1 + |\log x_i|} &= \frac{1 + \left|\log\left(x_i\left(1 - \frac{x_{i-1}}{x_i}\right)\right)\right|}{1 + |\log x_i|} \le c\frac{1 + |\log x_i| + \left|\log\left(1 - \frac{(i-1)^r}{i^r}\right)\right|}{1 + |\log x_i|} \\ &\le c'\left(1 + \left|\log\left(\left(\frac{i-1}{i}\right)^{r-1}\frac{1}{i}\right)\right|\right) \le c''(1 + |\log i^{-1}|). \end{aligned}$$

We formulate the following result:

$$\frac{1+|\log h_i|}{1+|\log x_i|} \le c\,(1+|\log i^{-1}|), \quad i=1,\dots,N. \tag{2.4.11}$$

Estimate (2.4.10) follows from (2.4.11) because of $i^\beta(1+|\log i^{-1}|) \le c$ for $\beta<0$. If $\beta \ge 0$ then $\left(\frac{i}{N}\right)^\beta \le c$ and (2.4.10) follows from (2.4.11). A corollary from (2.4.10) is that

$$\left(\frac{i}{N}\right)^\beta (1+|\log h_i|) \le c \left\{ \begin{array}{ll} N^{-\beta}\log N & \text{for } \beta<0 \\ \log N & \text{for } \beta \ge 0 \end{array} \right\}, \quad i=1,\dots,N, N>1. \tag{2.4.12}$$

Lemma 2.4.1. *Let $N \in \mathbb{N}$ and let $x_0,\dots,x_N$ be the node points (2.4.3) of the graded grid. Assume $\alpha,\beta \in (-1,\infty)$ such that $\alpha+\beta>-1$. Then for every $x \in [x_{j-1},x_j]$, $j=1,\dots,N$, and for $k=1,\dots,j$ the following estimates*

$$\int_{x_{k-1}}^{\min\{x_k,x\}} (x-s)^\alpha s^\beta\,ds \le c\,(x_k-x_{k-1})\,(x_j-x_{k-1})^\alpha x_k^\beta, \tag{2.4.13}$$

$$\begin{aligned} &\int_{x_{k-1}}^{\min\{x_k,x\}} (x-s)^\alpha s^\beta (1+|\log(x-s)|)\,ds \\ &\qquad \le c\,(x_k-x_{k-1})\,(x_j-x_{k-1})^\alpha x_k^\beta (1+|\log(x_j-x_{k-1})|), \end{aligned} \tag{2.4.14}$$

$$\begin{aligned} &\int_{x_{k-1}}^{\min\{x_k,x\}} (x-s)^\alpha s^\beta \frac{1+|\log(x-s)|}{1+|\log s|}\,ds \\ &\qquad \le c\,(x_k-x_{k-1})\,(x_j-x_{k-1})^\alpha x_k^\beta \frac{1+|\log(x_j-x_{k-1})|}{1+|\log x_k|}, \end{aligned} \tag{2.4.15}$$

hold with a positive constant c not depending on k,j and N.

Proof. First we show (2.4.13). If $j=1$, then $k=1$, $x \in [x_0,x_1]$, $x_0=0$ and $min\{x_1,x\}=x$. By Lemma 2.3.2,

$$\begin{aligned} \int_{x_0}^{x} (x-s)^\alpha s^\beta\,ds &\le c\,(x-x_0)\,(x-x_0)^\alpha x^\beta = c\,x^{\alpha+\beta+1} \le c\,x_1^{\alpha+\beta+1} \\ &= c\,(x_1-x_0)\,(x_1-x_0)^\alpha x_1^\beta = c\,(x_k-x_{k-1})\,(x_j-x_{k-1})^\alpha x_k^\beta. \end{aligned} \tag{2.4.16}$$

Now we consider the case $j=2,\dots,N$. Then for $k=1,\dots,j-1$ we have $\min\{x_k,x\}=x_k$ and by Lemma 2.3.2,

$$\int_{x_{k-1}}^{x_k} (x-s)^\alpha s^\beta\,ds \le c\,(x_k-x_{k-1})\,(x-x_{k-1})^\alpha x_k^\beta. \tag{2.4.17}$$

If $\alpha \geq 0$ then $(x - x_{k-1})^\alpha \leq (x_j - x_{k-1})^\alpha$ and (2.4.13) holds. If $\alpha < 0$ then we may write

$$\begin{aligned}(x - x_{k-1})^\alpha &= (x_j - x_{k-1})^\alpha \left(\frac{x - x_{k-1}}{x_j - x_{k-1}}\right)^\alpha \leq (x_j - x_{k-1})^\alpha \left(\frac{x_{j-1} - x_{k-1}}{x_j - x_{k-1}}\right)^\alpha \\ &= (x_j - x_{k-1})^\alpha \left(1 - \frac{x_j - x_{j-1}}{x_j - x_{k-1}}\right)^\alpha \leq (x_j - x_{k-1})^\alpha \left(1 - \frac{x_j - x_{j-1}}{x_j - x_{j-2}}\right)^\alpha.\end{aligned}$$

From (2.4.9) it follows

$$\begin{aligned}(x - x_{k-1})^\alpha &\leq (x_j - x_{k-1})^\alpha \left(\frac{x_{j-1} - x_{j-2}}{x_j - x_{j-2}}\right)^\alpha \leq (x_j - x_{k-1})^\alpha \left(\frac{2(x_j - x_{j-1})}{x_{j-1} - x_{j-2}}\right)^{|\alpha|} \\ &\leq (x_j - x_{k-1})^\alpha \left(\frac{2^{r+1}(x_{j-1} - x_{j-2})}{x_{j-1} - x_{j-2}}\right)^{|\alpha|} = 2^{(r+1)|\alpha|}(x_j - x_{k-1})^\alpha.\end{aligned} \tag{2.4.18}$$

Estimate (2.4.13) yields from (2.4.17) and (2.4.18). It remains to consider the case $k = j$. Here we have $\min\{x_k, x\} = x$ and by Lemma 2.3.2,

$$\begin{aligned}\int_{x_{k-1}}^{x} (x - s)^\alpha s^\beta \, ds &\leq c\,(x - x_{k-1})\,(x - x_{k-1})^\alpha x^\beta = c\,(x - x_{k-1})^{\alpha+1} x^\beta \\ &\leq c\,(x_j - x_{k-1})^{\alpha+1} x^\beta = c\,(x_k - x_{k-1})\,(x_j - x_{k-1})^\alpha x^\beta.\end{aligned} \tag{2.4.19}$$

If $\beta \geq 0$ then $x^\beta \leq x_j^\beta = x_k^\beta$. If $\beta < 0$ then $x^\beta \leq x_{j-1}^\beta$ and we may use (2.4.8). We have proved (2.4.13).

Now we consider (2.4.14). First, note that the function

$$u(x - s) = (x - s)^\gamma \left(\frac{1}{\gamma} + |\log(x - s)|\right)$$

is increasing on interval $[0, x_j - x_{k-1}]$ for any positive γ (see Remark 2.3.1). Using this for $\alpha > 0$ we can write

$$\begin{aligned}I_{k,j} :&= \int_{x_{k-1}}^{\min\{x_k,x\}} (x - s)^\alpha s^\beta \,(1 + |\log(x - s)|)\, ds \\ &\leq c\,(x_j - x_{k-1})^\alpha \,(1 + |\log(x_j - x_{k-1})|) \int_{x_{k-1}}^{\min\{x_k,x\}} s^\beta \, ds.\end{aligned} \tag{2.4.20}$$

Estimate (2.4.14) follows now from (2.4.13). Next, let $-1 < \alpha \leq 0$. Then for $\beta \geq 0$ we have

$$\begin{aligned} I_{k,j} &\leq x_k^{\beta} \int\limits_{x_{k-1}}^{\min\{x_k,x\}} (x-s)^{\frac{\alpha+1}{2}-\frac{1-\alpha}{2}} \left(1+|\log(x-s)|\right) ds \\ &\leq c\, x_k^{\beta} \left(x_j - x_{k-1}\right)^{\frac{\alpha+1}{2}} \left(1+|\log(x_j-x_{k-1})|\right) \int\limits_{x_{k-1}}^{\min\{x_k,x\}} (x-s)^{-\frac{1-\alpha}{2}} ds. \end{aligned} \tag{2.4.21}$$

Estimate (2.4.14) yields now from (2.4.13). Finally we assume $-1 < \beta < 0$. Thus

$$\begin{aligned} I_{k,j} &= \int\limits_{x_{k-1}}^{\min\{x_k,x\}} (x-s)^{\frac{\alpha+\beta+1}{2}-\frac{\beta+1-\alpha}{2}} \left(1+|\log(x-s)|\right) s^{\beta} ds \\ &\leq c\, \left(x_j - x_{k-1}\right)^{\frac{\alpha+\beta+1}{2}} \left(1+|\log(x_j-x_{k-1})|\right) \int\limits_{x_{k-1}}^{\min\{x_k,x\}} (x-s)^{\frac{\alpha-\beta-1}{2}} s^{\beta} ds. \end{aligned} \tag{2.4.22}$$

Since $\alpha, \beta \in (-1, 0]$, $\frac{\alpha-\beta-1}{2} > -1$ and $\frac{\alpha-\beta-1}{2} + \beta > -1$, we can use (2.4.13). We have proved (2.4.14).

The last estimate (2.4.15) follows from (2.4.14), since $\frac{1}{1+|\log s|} \leq c\, \frac{1}{1+|\log x_k|}$ for $s \in [x_{k-1}, x_k]$, $k = 1, \ldots, j$.

□

Lemma 2.4.2. *Let $\beta \in \mathbb{R}$, $N \in \mathbb{N}$, $N \geq 2$ and $x_0, \ldots, x_N$ be the node points of graded grid* (2.4.3)*. Then for every $k = 1, \ldots, N$ the following estimate*

$$\sum_{i=1}^{k} \left(\frac{i}{N}\right)^{\beta} \frac{1+|\log(x_k - x_{i-1})|}{1+|\log x_i|} \leq c \left\{ \begin{array}{lll} N \log N & \textit{for} & \beta \geq -1 \\ N^{-\beta} & \textit{for} & \beta < -1 \end{array} \right\} \tag{2.4.23}$$

holds with a positive constant c not depending on k, i and N.

Proof. Let $M = \frac{k}{2}$ if k is even and $M = \frac{k+1}{2}$ if k is odd. For $i = 1, \ldots, M$,

$$\begin{aligned} \frac{1+|\log(x_k - x_{i-1})|}{1+|\log x_i|} &\leq c\, \frac{1+|\log(x_k - x_{M-1})|}{1+|\log x_i|} = c\, \frac{1+|\log(x_k(1-\left(\frac{M-1}{k}\right)^r))|}{1+|\log x_i|} \\ &\leq c'\, \frac{1+|\log x_k| + |\log(1-\left(\frac{1}{2}\right)^r)|}{1+|\log x_k|} \leq c''. \end{aligned} \tag{2.4.24}$$

On the other hand, from (2.4.11) it follows that

$$\frac{1+|\log(x_k-x_{i-1})|}{1+|\log x_i|} \le c\frac{1+|\log(x_i-x_{i-1})|}{1+|\log x_i|} \le c^{'}\,(1+|\log i^{-1}|), \quad i=1,\dots,k. \tag{2.4.25}$$

We denote

$$I_k := \sum_{i=1}^{k}\left(\frac{i}{N}\right)^{\beta}\frac{1+|\log(x_k-x_{i-1})|}{1+|\log x_i|}. \tag{2.4.26}$$

Thus

$$\begin{aligned} I_k &= \left(\sum_{i=1}^{M}+\sum_{i=M+1}^{k}\right)\left(\left(\frac{i}{N}\right)^{\beta}\frac{1+|\log(x_k-x_{i-1})|}{1+|\log x_i|}\right) \\ &\le cN^{-\beta}\sum_{i=1}^{M} i^{\beta}+\sum_{i=M+1}^{k}\left(\frac{i}{N}\right)^{\beta}(1+|\log i^{-1}|). \end{aligned} \tag{2.4.27}$$

If $\beta \ge 0$ then

$$I_k \le c\,N^{-\beta}\,k^{\beta+1}+c\,k\left(\frac{k}{N}\right)^{\beta}(1+|\log k^{-1}|) \le c^{'}\,N\log N. \tag{2.4.28}$$

If $-1<\beta<0$ then

$$\begin{aligned} I_k &\le c\,N+c\,N^{-\beta}(1+|\log k^{-1}|)\sum_{i=M+1}^{k} i^{\beta} \le c^{'}\,N+c^{'}\,N^{-\beta}\,k^{1+\beta}\,\log N \\ &\le c^{''}\,N\log N. \end{aligned} \tag{2.4.29}$$

If $\beta=-1$ then

$$\begin{aligned} I_k &\le c\,N\,(1+\log k)+c\,N\,k^{-1}\,(1+|\log k^{-1}|)\sum_{i=M+1}^{k}\frac{k}{i} \\ &\le c^{'}\,N\,\log N+c^{'}\,N\,k^{-1}\,\log N\sum_{i=M+1}^{k} 2 \le c^{''}\,N\log N. \end{aligned} \tag{2.4.30}$$

Finally for $\beta<-1$ we get from (2.4.27),

$$I_k \le c\,N^{-\beta}+N^{-\beta}\sum_{i=M+1}^{k} i^{\beta}(1+|\log i^{-1}|) \le c^{'}\,N^{-\beta}\sum_{i=M+1}^{k} i^{\frac{\beta-1}{2}}\,i^{\frac{\beta+1}{2}}(1+|\log i^{-1}|). \tag{2.4.31}$$

As $\frac{\beta-1}{2}<-1$ and $\frac{\beta+1}{2}<0$ then $i^{\frac{\beta+1}{2}}(1+|\log i^{-1}|)\le c$ and

$$I_k \le c\,N^{-\beta}\sum_{i=M+1}^{k} i^{\frac{\beta-1}{2}} \le c^{'}\,N^{-\beta}. \tag{2.4.32}$$

We have proved the inequality (2.4.23).

□

2.5 Basic Function Sets

Let $b \in \mathbb{R}$, $b > 0$, $m \in \mathbb{N}$. First we introduce the following two sets of functions $C^{m,\theta}(0,b]$ and $C_*^{m,\theta}(0,b]$:

The set $\mathbf{C^{m,\theta}(0,b]}$, $\theta < 1$, consists of continuous functions $u : [0,b] \to \mathbb{R}$ which are m times continuously differentiable on $(0,b]$ and the estimates

$$|u^{(j)}(x)| \le c \left\{ \begin{array}{lll} 1 & \text{for} & j < 1-\theta \\ 1 + |\log x| & \text{for} & j = 1-\theta \\ x^{1-\theta-j} & \text{for} & j > 1-\theta \end{array} \right\} \tag{2.5.1}$$

hold with a constant $c = c(u)$ for all $x \in (0,b]$ and $j = 0, \dots, m$.

The set $\mathbf{C_*^{m,\theta}(0,b]}$, $\theta < 1$, consists of continuous functions $u_* : [0,b] \to \mathbb{R}$ which are m times continuously differentiable on $(0,b]$ and the estimates

$$|u_*^{(j)}(x)| \le c \left\{ \begin{array}{lll} 1 & \text{for} & j < 1-\theta \\ x^{1-\theta-j}(1 + |\log x|) & \text{for} & j \ge 1-\theta \end{array} \right\} \tag{2.5.2}$$

hold with a constant $c = c(u_*)$ for all $x \in (0,b]$ and $j = 0, \dots, m$.

For example, the function $u(x) = x^k \log x$ belongs to $C^{m,1-k}(0,b]$ for $k \in \mathbb{N}$, $k \le m$. The function $u_*(x) = x^\alpha \log x$ belongs to $C_*^{m,1-\alpha}(0,b]$ for $\alpha \in (0,1)$. Note that

$$C^m[0,b] \subset C^{m,\theta}(0,b] \subset C_*^{m,\theta}(0,b] \subset C[0,b], \quad m \in \mathbb{N}, \quad \theta < 1.$$

Further, let $m \in \mathbb{N}_0$, $\nu, \lambda \in (-\infty, 1)$ and

$$D_b = \{(x,y) : 0 < y < x \le b\}, \quad D_b' = \{(x,y) : 0 \le y < x \le b\}. \tag{2.5.3}$$

We denote by $\bar{D}_b$ the closure of D_b, $\bar{D}_b = \{(x,y) : 0 \le y \le x \le b\}$. We introduce the following three classes of kernels.

$\mathbf{W^{m,\nu}(D_b')}$ consists of m times continuously differentiable functions $K : D_b' \to \mathbb{R}$ satisfying the inequalities

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j K(x,y) \right| \le c \left\{ \begin{array}{ll} 1 & ,\nu + i < 0 \\ 1 + |\log(x-y)| & ,\nu + i = 0 \\ (x-y)^{-\nu-i} & ,\nu + i > 0 \end{array} \right\} \tag{2.5.4}$$

with a constant $c = c(K)$ for all $(x,y) \in D_b'$ and for all nonnegative integers $i, j \in \mathbb{N}_0$, $i + j \le m$.

$\mathbf{W^{m,\nu,\lambda}(D_b)}$ consists of m times continuously differentiable functions $K : D_b \to \mathbb{R}$ satisfying the inequalities

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j K(x,y) \right| \le c \left\{ \begin{array}{ll} 1 & ,\nu + i < 0 \\ 1 + |\log(x-y)| & ,\nu + i = 0 \\ (x-y)^{-\nu-i} & ,\nu + i > 0 \end{array} \right\} y^{-\lambda-j} \tag{2.5.5}$$

with a constant $c = c(K)$ for all $(x, y) \in D_b$ and for all nonnegative integers $i, j \in \mathbb{N}_0$, $i + j \leq m$ and additionally for $\nu + i < 0$ the equality $\lim_{y \to x} \left(\frac{\partial}{\partial x}\right)^i K(x, y) = 0$ holds.

$\mathbf{W}_*^{\mathbf{m},\nu,\lambda}(\mathbf{D_b})$ consists of functions $K_* \in W^{m,\nu,\lambda}(D_b)$ that in addition to (2.5.5) satisfy for $\lambda + j > 0$ the inequalities

$$\left|\left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j K_*(x, y)\right| \leq c \left\{ \begin{array}{ll} 1 & , \nu + i < 0 \\ 1 + |\log(x - y)| & , \nu + i = 0 \\ (x - y)^{-\nu - i} & , \nu + i > 0 \end{array} \right\} \frac{y^{-\lambda - j}}{1 + |\log y|} \tag{2.5.6}$$

with a constant $c = c(K_*)$ for all $(x, y) \in D_b$ and for all nonnegative integers $i, j \in \mathbb{N}_0$, $i + j \leq m$.

For example, the kernel

$$K(x, y) = g(x, y)(x - y)^{-\nu}, \quad g \in C^m(\bar{D}_b),$$

belongs to the set $W^{m,\nu}(D_b')$ for any $\nu < 1$. The kernel

$$K(x, y) = g(x, y)(x - y)^k \log(x - y), \quad g \in C^m(\bar{D}_b),$$

belongs to the set $W^{m,-k}(D_b')$ for any $k \in \mathbb{N}_0$, $k \leq m$. Moreover, the kernel

$$K(x, y) = g(x, y)(x - y)^{-\nu} y^{-\lambda}, \quad g \in C^m(\bar{D}_b),$$

belongs to the set $W^{m,\nu,\lambda}(D_b)$ for any $\nu, \lambda < 1$, $\nu + \lambda < 1$. Note that

$$W^{m,\nu}(D_b') \subset W_*^{m,\nu,0}(D_b) \subset W^{m,\nu,0}(D_b) \quad \text{for } 0 \leq \nu < 1,$$

$$W_*^{m,\nu,\lambda}(D_b) \subset W^{m,\nu,\lambda}(D_b) \subset W^{m,\nu,\lambda'}(D_b) \quad \text{for } \lambda < \lambda' < 1, \nu < 1.$$

2.6 Interpolation Operator

Let $m, N \in \mathbb{N}$ and let Δ_N be a grid (2.4.1) for the interval $[0, b]$, $b > 0$. In every subinterval $[x_{j-1}, x_j]$ $(j = 1, \ldots, N)$ we introduce m interpolation points

$$x_{ij} = x_{i-1} + \eta_j (x_i - x_{i-1}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, m, \tag{2.6.1}$$

where $\eta_1, \ldots, \eta_m$ are some fixed (collocation) parameters not depending on i and N and such that

$$0 \leq \eta_1 < \cdots < \eta_m \leq 1. \tag{2.6.2}$$

Next we introduce an interpolation operator $P_N = P_N^{(m-1)}$ which assigns to every continuous function $u \in C[0, b]$ the piecewise polynomial function $P_N u \in S_{m-1}^{(-1)}(\Delta_N)$ such that $P_N u$ interpolates u at the nodes (2.6.1):

$$(P_N u)(x_{ij}) = u(x_{ij}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, m. \tag{2.6.3}$$

Thus, $(P_N u)(x)$ is independently defined on every subinterval $[x_{i-1}, x_i]$, $i = 1, \ldots, N$, and may be discontinuous at the interior points $x = x_i$, $i = 1, \ldots, N-1$, of Δ_N. In this case we may treat $P_N u$ as a two-valued function at these interior points. If $\eta_1 = 0$ and $\eta_m = 1$ then the function $P_N u$ is continuous on [0,b].

Lemma 2.6.1. *[64] Let $P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$ be determined by the conditions (2.6.3). Then $P_N \in \mathcal{L}(C[x_{i-1}, x_i], C[x_{i-1}, x_i])$ $(i = 1, \ldots, N)$ and $P_N \in \mathcal{L}(C[0,b], L^\infty(0,b))$. Moreover, the norms of these operators are uniformly bounded*

$$\max_{i=1,\ldots,N} \|P_N\|_{\mathcal{L}(C[x_{i-1},x_i],C[x_{i-1},x_i])} \le c, \quad N \in \mathbb{N}, \tag{2.6.4}$$

$$\|P_N\|_{\mathcal{L}(C[0,b],L^\infty(0,b))} \le c, \quad N \in \mathbb{N}. \tag{2.6.5}$$

Here c is a positive constant which is independent of i and N.

Lemma 2.6.2. *[15] Let $u \in C^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta < 1$ and assume that a graded grid (2.4.3) and the interpolation points (2.6.1) are used. Then for the operator P_N defined by (2.6.3) the following estimate*

$$\|u - P_N u\|_\infty \le c \left\{ \begin{array}{lll} N^{-m} & \text{for } m < 1-\theta & , r \ge 1 \\ N^{-m} & \text{for } m = 1-\theta & , r > 1 \\ N^{-m}(1+\log N) & \text{for } m = 1-\theta & , r = 1 \\ N^{-m} & \text{for } m > 1-\theta & , r \ge \frac{m}{1-\theta} \\ N^{-r(1-\theta)} & \text{for } m > 1-\theta & , 1 \le r < \frac{m}{1-\theta} \end{array} \right\} \tag{2.6.6}$$

holds with a positive constant c independent of N.

Lemma 2.6.3. *[15, 53] Let $T : L^\infty(0,b) \to C[0,b]$ be a linear compact operator. Suppose that the grid (2.4.1) is regular i.e. satisfies (2.4.2). Then for the operator P_N defined by (2.6.3) we have*

$$\|T - P_N T\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \to 0 \text{ as } N \to \infty. \tag{2.6.7}$$

For the next lemma we use the idea of the proof in [53] where the interpolation error for arbitrary, regular, quasi-uniform and graded grids is studied.

Lemma 2.6.4. *Let $u \in C[0,b] \cap C^m(0,b]$, $m \in \mathbb{N}$, $b \in \mathbb{R}$, $b > 0$ and assume that a grid (2.4.1) and the interpolation points (2.6.1) are used. Then for the operator $P_N = P_N^{(m-1)}$ defined by (2.6.3) the following estimate*

$$|u(x) - (P_N u)(x)| \le c \int_x^{x_i} (s-x)^{m-1} \, |u^{(m)}(s)| \, ds, \quad x \in [x_{i-1}, x_i] \tag{2.6.8}$$

holds for $i = 1, \ldots, N$ with a positive constant c not depending on i and N.

Proof. Let $u \in C[0,b] \cap C^m(0,b]$. For every $i = 1, \dots, N$ we choose a polynomial $\omega_{N,i} \in \pi_{m-1}$ of degree $m-1$. Then for $x \in [x_{i-1}, x_i]$ $(i = 1, \dots, N)$ we have

$$\begin{aligned} u(x) - (P_N u)(x) &= u(x) - \omega_{N,i}(x) + \omega_{N,i}(x) - (P_N u)(x) \\ &= u(x) - \omega_{N,i}(x) + (P_N \omega_{N,i})(x) - (P_N u)(x) \\ &= (I - P_N)(u(x) - \omega_{N,i}(x)). \end{aligned} \tag{2.6.9}$$

By Lemma 2.6.1 $\|P_N\|_{\mathcal{L}(C[x_{i-1},x_i],C[x_{i-1},x_i])} \le c$ and therefore

$$\begin{aligned} |u(x) - (P_N u)(x)| &= |(I - P_N)(u(x) - \omega_{N,i}(x))| \\ &\le (1 + \|P_N\|_{\mathcal{L}(C[x_{i-1},x_i],C[x_{i-1},x_i])})\, |u(x) - \omega_{N,i}(x)| \\ &\le c\, |u(x) - \omega_{N,i}(x)|. \end{aligned} \tag{2.6.10}$$

Next, we fix $\omega_{N,i}(x)$ as a Taylor polynomial for the function $u(x)$ at $x = x_i$:

$$\omega_{N,i}(x) = \sum_{n=0}^{m-1} \frac{u^{(n)}(x_i)(x - x_i)^n}{n!}, \quad x \in [x_{i-1}, x_i]. \tag{2.6.11}$$

The integral form of the reminder term of the $(m-1)$th order Taylor approximation of u at $x = x_i$ gives us for $x \in [x_{i-1}, x_i]$ the inequality

$$\begin{aligned} |u(x) - \omega_{N,i}(x)| &= \frac{1}{(m-1)!} \left| \int_{x_i}^{x} (x - s)^{m-1}\, u^{(m)}(s)\, ds \right| \\ &\le c \int_{x}^{x_i} (s - x)^{m-1}\, |u^{(m)}(s)|\, ds. \end{aligned} \tag{2.6.12}$$

The estimate (2.6.8) now follows from (2.6.10) and (2.6.12).

□

Chapter 3

Smoothing Transformations

In this chapter, we first (Sections 3.1-3.2) introduce a wide class of smoothing transformations for functions $u \in C^{m,\theta}(0,b]$ and $u \in C_*^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta < 1$. If $u \in C^{m,\theta}(0,b]$ then the function u is continuous on the closed interval $[0,b]$ but its derivatives may be unbounded at the left endpoint of the interval $[0,b]$. Let $\varphi : [0,a] \to [0,b]$ be a continuous function which maps $[0,a]$ onto $[0,b]$, $a > 0$. We are interested in such changes $x = \varphi(t)$ of the variable x that the singularities of $u(\varphi(t))$ as a function of t will be milder. Here we use the ideas of [48, 56].

Section 3.3 includes the estimates for kernels $K_\varphi(t,s)$ and the derivatives $\frac{\partial}{\partial s}K_\varphi(t,s)$. We use these results in the superconvergence theorems in Chapter 5.

In Section 3.4, we prove some estimates for $u - P_N u$ on subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots, N$, as well as the estimates on full interval $[0,b]$. The main results of this section are given in Lemmas 3.2.3 and 3.4.4. These results are partly published in [36, 37, 38, 39].

3.1 Definitions

For given $m \in \mathbb{N}$, $\rho \in \mathbb{R}$, $\rho \geq 1$ and for positive real numbers $a, b > 0$, let

$$\varphi : [0,a] \to [0,b] \tag{3.1.1}$$

be a transformation which maps $[0,a]$ onto $[0,b]$ such that

$$\varphi \in C^m(0,a] \cap C^1[0,a], \tag{3.1.2}$$

$$c_1\, t^{\rho-1} \leq \varphi^{'}(t) \leq c_2\, t^{\rho-1}, \quad 0 \leq t \leq a, \tag{3.1.3}$$

$$|\varphi^{(j)}(t)| \leq c\, t^{\rho-j}, \quad 0 < t \leq a, \quad j = 0, \ldots, m, \tag{3.1.4}$$

where $c > 0, c_2 \geq c_1 > 0$ are some constants. The set of the transformations φ satisfying (3.1.1)- (3.1.4) is denoted by $\Phi^{m,\rho}$.

A consequence of (3.1.1)-(3.1.3) is that

$$|\varphi(t) - \varphi(s)| \geq c\,|t-s|\;(t^{\rho-1} + s^{\rho-1}), \quad s,t \in [0,a], \tag{3.1.5}$$

with a constant $c > 0$ not depending on $t, s \in [0,a]$. Indeed, due to symmetry we may assume that $s < t$. Then, following [56], we have

$$\varphi(t) - \varphi(s) = \int_s^t \varphi^{'}(\tau)\,d\tau \geq \frac{c_1}{\rho}(t^\rho - s^\rho), \quad s < t.$$

Therefore,

$$\frac{\varphi(t) - \varphi(s)}{(t-s)(t^{\rho-1} + s^{\rho-1})} \geq \frac{c_1(t^\rho - s^\rho)}{\rho(t-s)(t^{\rho-1} + s^{\rho-1})} = \frac{c_1(1-\xi^\rho)}{\rho(1-\xi)(1+\xi^{\rho-1})},$$

where $\xi = \frac{s}{t} \in [0,1)$. Denote

$$h(\xi) = \frac{c_1(1-\xi^\rho)}{\rho(1-\xi)(1+\xi^{\rho-1})}, \quad 0 \leq \xi < 1.$$

Clearly, $h(\xi) > 0$ for $0 \leq \xi < 1$. Using l'Hospital's rule we obtain that

$$\lim_{\xi \to 1} h(\xi) = \frac{c_1}{2} > 0.$$

Thus, after extension to the point $\xi = 1$, function h is positive and continuous on $[0,1]$, hence its infimum is positive. This completes the proof of (3.1.5).

Remark 3.1.1. *An important subset of $\Phi^{m,\rho}$ is given by the functions*

$$\varphi \in \Phi^{m,\rho} \cap C^m[0,a]. \tag{3.1.6}$$

Actually, this is the class of transformations which was introduced in [56].

The simplest example of $\varphi \in \Phi^{m,\rho}$ is given by

$$\varphi(t) = \frac{b}{a^\rho}\,t^\rho, \quad 0 \leq t \leq a, \quad a,b > 0, \quad \rho \geq 1. \tag{3.1.7}$$

Clearly $\varphi \in C^m(0,a] \cap C^1[0,a]$, $m \in \mathbb{N}$. Note that for $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$ we must take $\rho \in \mathbb{N}$ in case $1 \leq \rho \leq m$ and may take $\rho \in \mathbb{R}$ in case $\rho > m$.

Another example of $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$ is

$$\varphi(t) = 2^{\rho/2}\,b\,\left(\sin\left(\frac{\pi}{4a}\,t\right)\right)^\rho, \quad 0 \leq t \leq a, \quad \mathbb{N} \ni \rho \leq m. \tag{3.1.8}$$

A class of examples (see [56]) is given by the functions $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$ satisfying the conditions

$$\varphi^{'}(t) > 0, \quad 0 < t \leq a, \tag{3.1.9}$$

$$\varphi^{(j)}(0) = 0, \quad j = 0,1,\ldots,\rho-1; \quad \varphi^{(\rho)}(0) \neq 0, \quad \mathbb{N} \ni \rho \leq m. \tag{3.1.10}$$

Note that the idea of smoothing integrands by introducing a suitable change of variables has been used also for evaluating various types of integrals, see, for example [3, 23, 42, 46, 49, 60, 70]. We will use this idea in Chapter 4 for constructing high order methods for solving the equation (1.1.1).

3.2 Smoothing Properties

In the case $\rho = 1$ transformations $\varphi \in \Phi^{m,\rho}$ are isomorphic and have a continuous inverse. We are interested in transformations φ with $\rho > 1$ as they possess a smoothing property for a function $u_\varphi(t) = u(\varphi(t))$ with singularities of $u(x)$ at $x = 0$.

Lemma 3.2.1. *Assume that:*

1. $u \in C^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta, b \in \mathbb{R}$, $\theta < 1$, $b > 0$;
2. $\varphi \in \Phi^{m,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, \rho \in \mathbb{R}$, $\rho \geq 1, a > 0$;
3. $u_\varphi(t) = u(\varphi(t))$, $t \in [0,a]$.

Then

$$u_\varphi \in \left\{ \begin{array}{ll} C^{m,1-\rho}(0,a] & \text{for } \theta < 0 \\ C_*^{m,1-\rho}(0,a] & \text{for } \theta = 0 \\ C^{m,1-\rho(1-\theta)}(0,a] & \text{for } 0 < \theta < 1 \end{array} \right\} \tag{3.2.1}$$

and for $0 < t \leq a$, $j = 1, \ldots, m$,

$$|u_\varphi^{(j)}(t)| \leq c \left\{ \begin{array}{ll} t^{\rho-j} & \text{for } \theta < 0 \\ t^{\rho-j}(1 + |\log t|) & \text{for } \theta = 0 \\ t^{\rho(1-\theta)-j} & \text{for } 0 < \theta < 1 \end{array} \right\}, \tag{3.2.2}$$

where c is a positive constant not depending on t.

Proof. Since $u \in C^{m,\theta}(0,b]$ and $\varphi \in \Phi^{m,\rho}$, $u_\varphi(t) = u(\varphi(t))$ is a continuous function on $[0,a]$ which is m times continuously differentiable on $(0,a]$. By Theorem 2.3.1,

$$u_\varphi^{(j)}(t) = \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N}_0, \\ k_1+2k_2+\cdots+jk_j=j}} \frac{j!}{k_1! \cdots k_j!} u^{(k_1+\cdots+k_j)}(\varphi(t)) \left(\frac{\varphi'(t)}{1!} \right)^{k_1} \cdots \left(\frac{\varphi^{(j)}(t)}{j!} \right)^{k_j}, \tag{3.2.3}$$

where $0 < t \leq a$ and $j = 1, \ldots, m$. We shall estimate all the terms in the sum

(3.2.3) for $0 < t \le a$. From (2.5.1), (3.1.2)-(3.1.4) it follows that

$$
\begin{aligned}
S_j(t) &:= \left| u^{(k_1+\cdots+k_j)}(\varphi(t)) \left(\frac{\varphi'(t)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(j)}(t)}{j!}\right)^{k_j} \right| \\
&\le c \left\{ \begin{array}{ll} 1 & , k_1+\cdots+k_j < 1-\theta \\ 1+|\log t^{\rho}| & , k_1+\cdots+k_j = 1-\theta \\ t^{\rho(1-\theta-(k_1+\cdots+k_j))} & , k_1+\cdots+k_j > 1-\theta \end{array} \right\} t^{(\rho-1)k_1+(\rho-2)k_2+\cdots+(\rho-j)k_j} \\
&\le c' \left\{ \begin{array}{ll} t^{\rho(k_1+\cdots+k_j)-j} & , k_1+\cdots+k_j < 1-\theta \\ t^{\rho(k_1+\cdots+k_j)-j}(1+|\log t|) & , k_1+\cdots+k_j = 1-\theta \\ t^{\rho(1-\theta)-j} & , k_1+\cdots+k_j > 1-\theta \end{array} \right\} \\
&= c'\, t^{\rho-j} \left\{ \begin{array}{ll} t^{\rho(k_1+\cdots+k_j)-\rho} & , k_1+\cdots+k_j < 1-\theta \\ t^{\rho(1-\theta)-\rho}(1+|\log t|) & , k_1+\cdots+k_j = 1-\theta \\ t^{\rho(1-\theta)-\rho} & , k_1+\cdots+k_j > 1-\theta \end{array} \right\}.
\end{aligned}
\tag{3.2.4}
$$

In the case $0 < \theta < 1$ we have $0 < 1-\theta < 1$. As $k_1 + 2k_2 + \cdots + jk_j = j$ then the sum $k_1 + \cdots + k_j \ge 1 > 1-\theta$ is true for all $j = 1, \ldots, m$ and we can estimate

$$S_j(t) \le c\, t^{\rho(1-\theta)-j}, \quad 0 < t \le a, \quad j = 1, \ldots, m.$$

Thus (3.2.2) holds for $0 < \theta < 1$. We can also write

$$|u_{\varphi}^{(j)}(t)| \le c \left\{ \begin{array}{ll} 1 & \text{if } j \le \rho(1-\theta) \\ t^{\rho(1-\theta)-j} & \text{if } j > \rho(1-\theta) \end{array} \right\}, \quad 0 < t \le a, \quad j = 1, \ldots, m. \tag{3.2.5}$$

From this and (2.5.1) it follows that the function $u_{\varphi} \in C^{m,1-\rho(1-\theta)}(0, a]$.

In the case $\theta = 0$ we have $\rho(1-\theta) - \rho = 0$. As for every $j = 1, \ldots, m$ the sum $k_1 + \cdots + k_j \ge 1 = 1-\theta$, then we can estimate (3.2.4) as follows:

$$S_j(t) \le c\, t^{\rho-j}(1+|\log t|), \quad 0 < t \le a, \quad j = 1, \ldots, m.$$

Thus (3.2.2) holds for $\theta = 0$. We can also write

$$|u_{\varphi}^{(j)}(t)| \le c \left\{ \begin{array}{ll} 1 & \text{if } j < \rho \\ t^{\rho-j}(1+|\log t|) & \text{if } j \ge \rho \end{array} \right\}, \quad 0 < t \le a, \quad j = 1, \ldots, m. \tag{3.2.6}$$

By (2.5.2) the function $u_{\varphi} \in C_*^{m,1-\rho}(0, a]$.

In the case $\theta < 0$ we have $\rho(1-\theta) - \rho > 0$ and

$$S_j(t) \le c\, t^{\rho-j} \left\{ \begin{array}{ll} t^{\rho(k_1+\cdots+k_j)-\rho} & , k_1+\cdots+k_j < 1-\theta \\ 1 & , k_1+\cdots+k_j \ge 1-\theta \end{array} \right\}. \tag{3.2.7}$$

Since also $\rho(k_1 + \cdots + k_j) - \rho \ge 0$ then we can estimate

$$S_j(t) \le c t^{\rho-j}, \quad 0 < t \le a, \quad j = 1, \ldots, m.$$

Due to (2.5.1) the function $u_\varphi \in C^{m,1-\rho}(0,a]$. This proves our lemma.

□

Lemma 3.2.2. *Assume that:*

1. $u_* \in C_*^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta, b \in \mathbb{R}$, $\theta < 1$, $b > 0$;
2. $\varphi \in \Phi^{m,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, \rho \in \mathbb{R}$, $\rho \geq 1, a > 0$;
3. $u_{*_\varphi}(t) = u_*(\varphi(t))$, $t \in [0,a]$.

Then

$$u_{*_\varphi} \in \left\{ \begin{array}{ll} C^{m,1-\rho}(0,a] & \text{for } \theta < 0 \\ C_*^{m,1-\rho(1-\theta)}(0,a] & \text{for } 0 \leq \theta < 1 \end{array} \right\} \tag{3.2.8}$$

and for $0 < t \leq a$, $j = 1, \ldots, m$,

$$|u_{*_\varphi}^{(j)}(t)| \leq c \left\{ \begin{array}{ll} t^{\rho-j} & \text{for } \theta < 0 \\ t^{\rho(1-\theta)-j}(1 + |\log t|) & \text{for } 0 \leq \theta < 1 \end{array} \right\}, \tag{3.2.9}$$

where c is a positive constant not depending on t.

Proof. Since $u_* \in C_*^{m,\theta}(0,b]$ and $\varphi \in \Phi^{m,\rho}$, $u_{*_\varphi}(t) = u_*(\varphi(t))$ is a continuous function on $[0,a]$ which is m times continuously differentiable on $(0,a]$. It is easy to see that the same analysis of the proof of Lemma 3.2.1, works also here. For example, from (2.5.2), (3.1.2)-(3.1.4) we can write (3.2.4) in the form

$$\begin{aligned} S_j(t) &\leq c \left\{ \begin{array}{ll} t^{\rho(k_1+\cdots+k_j)-j} & , k_1 + \cdots + k_j < 1 - \theta \\ t^{\rho(1-\theta)-j}(1 + |\log t^\rho|) & , k_1 + \cdots + k_j \geq 1 - \theta \end{array} \right\} \\ &\leq c' t^{\rho-j} \left\{ \begin{array}{ll} t^{\rho(k_1+\cdots+k_j)-\rho} & , k_1 + \cdots + k_j < 1 - \theta \\ t^{\rho(1-\theta)-\rho}(1 + |\log t|) & , k_1 + \cdots + k_j \geq 1 - \theta \end{array} \right\}. \end{aligned} \tag{3.2.10}$$

In the case $\theta < 0$ we can estimate

$$S_j(t) \leq c t^{\rho-j}, \quad 0 < t \leq a, \quad j = 1, \ldots, m.$$

In the case $0 \leq \theta < 1$ we have $k_1 + \cdots + k_j \geq 1 - \theta$ for all $j = 1, \ldots, m$ and we can write

$$S_j(t) \leq c t^{\rho(1-\theta)-j}(1 + |\log t|), \quad 0 < t \leq a, \quad j = 1, \ldots, m.$$

Now we use (2.5.1) and (2.5.2) to conclude the proof.

□

Lemma 3.2.3. *Assume that:*

1. $u \in C^{m,\theta}(0,b]$ *and* $u_* \in C_*^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta, b \in \mathbb{R}$, $b > 0$, $\theta < 1$;

2. $\psi \in \Phi^{m,\rho} \cap C^m[0,a]$, $\psi : [0,a] \to [0,b]$, $a, \rho \in \mathbb{R}$, $\rho \geq 1, a > 0$;
3. $u_\psi(t) = u(\psi(t))$ *and* $u_{*_\psi}(t) = u_*(\psi(t))$, $t \in [0,a]$.

Then $u_\psi \in C^{m,\theta_\rho}(0,a]$ *and* $u_{*_\psi} \in C_*^{m,\theta_\rho}(0,a]$ *with*

$$\theta_\rho = 1 - \rho\,(1-\theta). \tag{3.2.11}$$

Proof. The proof is based on the ideas used in the proof of Lemma 3.2.1 and in [56]. First we consider $u \in C^{m,\theta}(0,b]$. Since $\psi \in \Phi^{m,\rho} \cap C^m[0,a]$, $u_\psi(t) = u(\psi(t))$ is a continuous function on $[0,a]$ which is m times continuously differentiable on $(0,a]$. Due to (3.1.4) and $\psi \in C^m[0,a]$ we can estimate

$$|\psi^{(j)}(t)| \leq c \left\{ \begin{array}{ll} 1 & \text{if } j > \rho \\ t^{\rho - j} & \text{if } j \leq \rho \end{array} \right\}, \quad 0 < t \leq a, \quad j = 1, \ldots, m. \tag{3.2.12}$$

By Theorem 2.3.1,

$$u_\psi^{(j)}(t) = \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N}_0, \\ k_1 + 2k_2 + \cdots + jk_j = j}} \frac{j!}{k_1! \cdots k_j!} u^{(k_1+\cdots+k_j)}(\psi(t)) \left(\frac{\psi'(t)}{1!} \right)^{k_1} \cdots \left(\frac{\psi^{(j)}(t)}{j!} \right)^{k_j}, \tag{3.2.13}$$

where $0 < t \leq a$ and $j = 1, \ldots, m$. We shall estimate all the terms in the sum (3.2.13) for $0 < t \leq a$. Let $i \in \mathbb{N}$ be a fixed index defined as

$$\left\{ \begin{array}{ll} i = j & \text{if } j \leq \rho \\ \rho - 1 < i \leq \rho & \text{if } j > \rho \end{array} \right\}.$$

Then due to (2.5.1), (3.1.2)-(3.1.4) and (3.2.12) we can write

$$\begin{aligned} S_j(t) &:= \left| u^{(k_1+\cdots+k_j)}(\psi(t)) \left(\frac{\psi'(t)}{1!} \right)^{k_1} \cdots \left(\frac{\psi^{(j)}(t)}{j!} \right)^{k_j} \right| \\ &\leq c \left\{ \begin{array}{ll} 1 & , k_1 + \cdots + k_j < 1 - \theta \\ 1 + |\log t^\rho| & , k_1 + \cdots + k_j = 1 - \theta \\ t^{\rho(1-\theta-(k_1+\cdots+k_j))} & , k_1 + \cdots + k_j > 1 - \theta \end{array} \right\} t^{(\rho-1)k_1 + (\rho-2)k_2 + \cdots + (\rho-i)k_i} \\ &\leq c' \left\{ \begin{array}{ll} t^{(\rho-1)k_1 + (\rho-2)k_2 + \cdots + (\rho-i)k_i} & , k_1 + \cdots + k_j < 1 - \theta \\ t^{(\rho-1)k_1 + (\rho-2)k_2 + \cdots + (\rho-i)k_i}(1 + |\log t|) & , k_1 + \cdots + k_j = 1 - \theta \\ t^{\rho(1-\theta) - \rho(k_{i+1} + \cdots + k_j) - (k_1 + 2k_2 + \cdots + ik_i)} & , k_1 + \cdots + k_j > 1 - \theta \end{array} \right\}. \end{aligned} \tag{3.2.14}$$

Note that

$$(\rho - 1)k_1 + (\rho - 2)k_2 + \cdots + (\rho - i)k_i \geq 0. \tag{3.2.15}$$

Since $k_1 + 2k_2 + \cdots + jk_j = j$ then

$$\rho(k_{i+1} + \cdots + k_j) + k_1 + 2k_2 + \cdots + ik_i = j + [(\rho - i - 1)k_{i+1} + \cdots + (\rho - j)k_j].$$

For $k_1 + \cdots + k_j = 1 - \theta$ we can write

$$\begin{aligned}(\rho-1)k_1 + (\rho-2)k_2 + \cdots + (\rho-i)k_i =& \rho(1-\theta) - j \\ & - [(\rho-i-1)k_{i+1} + \cdots + (\rho-j)k_j],\end{aligned} \tag{3.2.16}$$

and we obtain (3.2.14) in the form

$$S_j(t) \le c \left\{ \begin{array}{ll} 1 & , k_1 + \cdots + k_j < 1-\theta \\ t^{\rho(1-\theta)-j}\, t^{-[(\rho-i-1)k_{i+1}+\cdots+(\rho-j)k_j]}(1+|\log t|) & , k_1 + \cdots + k_j = 1-\theta \\ t^{\rho(1-\theta)-j}\, t^{-[(\rho-i-1)k_{i+1}+\cdots+(\rho-j)k_j]} & , k_1 + \cdots + k_j > 1-\theta \end{array} \right\}. \tag{3.2.17}$$

By the definition of i,

$$-[(\rho-i-1)k_{i+1} + \cdots + (\rho-j)k_j] \ge 0. \tag{3.2.18}$$

Note that in case $k_1 + \cdots + k_j = 1-\theta$ and $j > \rho(1-\theta)$, due to (3.2.16) and (3.2.15) we have

$$-[(\rho-i-1)k_{i+1} + \cdots + (\rho-j)k_j] > 0$$

and

$$t^{-[(\rho-i-1)k_{i+1}+\cdots+(\rho-j)k_j]}(1+|\log t|) \le c.$$

Thus

$$S_j(t) \le c \left\{ \begin{array}{ll} 1 & , j < \rho(1-\theta) \\ 1+|\log t| & , j = \rho(1-\theta) \\ t^{\rho(1-\theta)-j} & , j > \rho(1-\theta) \end{array} \right\}, \quad 0 < t \le a, \quad j = 1, \ldots, m, \tag{3.2.19}$$

and due to (2.5.1) the function $u_\psi \in C^{m,1-\rho(1-\theta)}(0,a]$.

Finally, we observe the function $u_* \in C_*^{m,\theta}(0,b]$. It is easy to see that the same analysis done for $u \in C^{m,\theta}(0,b]$, works also here. For example, from (2.5.2) and (3.1.2)-(3.1.4) we get (3.2.17) in the form

$$\begin{aligned} & S_j(t) \\ & \le c \left\{ \begin{array}{ll} 1 & , k_1 + \cdots + k_j < 1-\theta \\ t^{\rho(1-\theta)-j}\, t^{-[(\rho-i-1)k_{i+1}+\cdots+(\rho-j)k_j]}(1+|\log t|) & , k_1 + \cdots + k_j \ge 1-\theta \end{array} \right\} \\ & \le c' \left\{ \begin{array}{ll} 1 & , j < \rho(1-\theta) \\ t^{\rho(1-\theta)-j}(1+|\log t|) & , j \ge \rho(1-\theta) \end{array} \right\}. \end{aligned} \tag{3.2.20}$$

From (2.5.2) it follows that $u_{*_\psi} \in C_*^{m,1-\rho(1-\theta)}(0,a]$.

□

3.3 Estimates for Kernels

Lemma 3.3.1. *Let* $K \in W^{0,\nu,\lambda}(D_b)$, $\nu, \lambda \in (-\infty, 1)$, $\varphi \in \Phi^{1,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, b \in (0,\infty)$, $\rho \in [1,\infty)$. *Then*

$$|K_\varphi(t,s)| \le c \left\{ \begin{array}{ll} s^{(\rho-1)(1-\lambda)} & ,\nu < 0 \\ s^{(\rho-1)(1-\lambda)}\,(1 + |\log(t-s)|) & ,\nu = 0 \\ s^{(\rho-1)(1-\nu-\lambda)}\,(t-s)^{-\nu} & ,\nu > 0 \end{array} \right\} s^{-\lambda}, \tag{3.3.1}$$

where $(t,s) \in D_a$ *and* $K_\varphi(t,s) = \varphi^{'}(s)\,K(\varphi(t), \varphi(s))$.

Proof. Due to (2.5.5) with $i, j = 0$,

$$\begin{aligned} |K_\varphi(t,s)| &\le |\varphi^{'}(s)|\,|K(\varphi(t), \varphi(s))| \\ &\le c\,|\varphi^{'}(s)| \left\{ \begin{array}{ll} 1 & ,\nu < 0 \\ 1 + |\log(\varphi(t) - \varphi(s))| & ,\nu = 0 \\ (\varphi(t) - \varphi(s))^{-\nu} & ,\nu > 0 \end{array} \right\} (\varphi(s))^{-\lambda}. \end{aligned} \tag{3.3.2}$$

Now we can use estimates (3.1.4) and (3.1.5) for φ:

$$\begin{aligned} |K_\varphi(t,s)| &\le c\,s^{\rho-1} \left\{ \begin{array}{ll} 1 & ,\nu < 0 \\ 1 + |\log((t-s)(t^{\rho-1} + s^{\rho-1}))| & ,\nu = 0 \\ (t-s)^{-\nu}(t^{\rho-1} + s^{\rho-1})^{-\nu} & ,\nu > 0 \end{array} \right\} s^{-\rho\lambda} \\ &\le c^{'} \left\{ \begin{array}{ll} s^{(\rho-1)(1-\lambda)} & ,\nu < 0 \\ s^{(\rho-1)(1-\lambda)}\,(1 + |\log(t-s)|) & ,\nu = 0 \\ s^{(\rho-1)(1-\nu-\lambda)}(t-s)^{-\nu} & ,\nu > 0 \end{array} \right\} s^{-\lambda}. \end{aligned} \tag{3.3.3}$$

□

Lemma 3.3.2. *Let* $K \in W_*^{0,\nu,\lambda}(D_b)$, $\nu \in (-\infty, 1)$, $\lambda \in (0,1)$, $\varphi \in \Phi^{1,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, b \in (0,\infty)$, $\rho \in [1,\infty)$. *Then*

$$|K_\varphi(t,s)| \le c \left\{ \begin{array}{ll} s^{(\rho-1)(1-\lambda)} & ,\nu < 0 \\ s^{(\rho-1)(1-\lambda)}\,(1 + |\log(t-s)|) & ,\nu = 0 \\ s^{(\rho-1)(1-\nu-\lambda)}\,(t-s)^{-\nu} & ,\nu > 0 \end{array} \right\} \frac{s^{-\lambda}}{1 + |\log s|}, \tag{3.3.4}$$

where $(t,s) \in D_a$ *and* $K_\varphi(t,s) = \varphi^{'}(s)\,K(\varphi(t), \varphi(s))$.

Proof. We can use the same idea as in the proof of Lemma 3.3.1 using (2.5.6).

□

Lemma 3.3.3. *Let* $K \in W^{0,\nu}(D_b^{'})$, $\nu \in (-\infty, 1)$, $\varphi \in \Phi^{1,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, b \in (0,\infty)$, $\rho \in [1,\infty)$. *Then*

$$|K_\varphi(t,s)| \le c \left\{ \begin{array}{ll} s^{\rho-1} & ,\nu < 0 \\ s^{\rho-1}\,(1 + |\log(t-s)|) & ,\nu = 0 \\ s^{(\rho-1)(1-\nu)}\,(t-s)^{-\nu} & ,\nu > 0 \end{array} \right\}, \tag{3.3.5}$$

where $(t,s) \in D_a^{'}$ *and* $K_\varphi(t,s) = \varphi^{'}(s)\, K(\varphi(t), \varphi(s))$.

Proof. We can use the same idea as in the proof of Lemma 3.3.1 using (2.5.4).

□

Lemma 3.3.4. *Let* $K \in W^{1,\nu,\lambda}(D_b)$, $\nu, \lambda \in (-\infty, 1)$, $\varphi \in \Phi^{2,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, b \in (0,\infty)$, $\rho \in [1,\infty)$. *Then*

$$\left|\frac{\partial}{\partial s} K_\varphi(t,s)\right| \le c \left\{ \begin{array}{ll} s^{(\rho-1)(1-\lambda)} & ,\nu < -1 \\ s^{(\rho-1)(1-\lambda)}[1 + s^{\rho}\,(1 + |\log(t-s)|)] & ,\nu = -1 \\ s^{(\rho-1)(1-\lambda)}[1 + s^{1-\nu(\rho-1)}\,(t-s)^{-\nu-1}] & ,-1 < \nu < 0 \\ s^{(\rho-1)(1-\lambda)+1}[s^{-1}(1 + |\log(t-s)|) + (t-s)^{-1}] & ,\nu = 0 \\ s^{(\rho-1)(1-\nu-\lambda)+1}\,(t-s)^{-\nu}\,[s^{-1} + (t-s)^{-1}] & ,\nu > 0 \end{array} \right\} s^{-\lambda-1}, \tag{3.3.6}$$

where c *is a positive constant not depending on* t *and* s, $(t,s) \in D_a$ *and* $K_\varphi(t,s) = \varphi^{'}(s)\, K(\varphi(t), \varphi(s))$.

Proof. First we find

$$\frac{\partial}{\partial s} K_\varphi(t,s) = [\varphi^{'}(s)]^2 \frac{\partial}{\partial y} K(\varphi(t), y)\bigg|_{y=\varphi(s)} + \varphi^{''}(s) K(\varphi(t), \varphi(s)). \tag{3.3.7}$$

Here

$$\frac{\partial}{\partial y} K(\varphi(t), y)\bigg|_{y=\varphi(s)} = \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) K(x,y) - \frac{\partial}{\partial x} K(x,y)\right]_{\substack{x=\varphi(t)\\ y=\varphi(s)}}. \tag{3.3.8}$$

We use the estimates (2.5.5) for $i = 0, j = 1$ and for $i = 1, j = 0$,

$$\begin{aligned} \left|\frac{\partial}{\partial y} K(\varphi(t), y)\right|_{y=\varphi(s)} &\le \left|\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) K(x,y)\right|_{\substack{x=\varphi(t)\\ y=\varphi(s)}} + \left|\frac{\partial}{\partial x} K(x,y)\right|_{\substack{x=\varphi(t)\\ y=\varphi(s)}} \\ &\le c \left\{ \begin{array}{ll} 1 & ,\nu < 0 \\ 1 + |\log(\varphi(t) - \varphi(s))| & ,\nu = 0 \\ (\varphi(t) - \varphi(s))^{-\nu} & ,\nu > 0 \end{array} \right\} (\varphi(s))^{-\lambda-1} \\ &\quad + c \left\{ \begin{array}{ll} 1 & ,\nu + 1 < 0 \\ 1 + |\log(\varphi(t) - \varphi(s))| & ,\nu + 1 = 0 \\ (\varphi(t) - \varphi(s))^{-\nu-1} & ,\nu + 1 > 0 \end{array} \right\} (\varphi(s))^{-\lambda}. \end{aligned} \tag{3.3.9}$$

Now we can use the estimates (3.1.4) and (3.1.5) for φ,

$$\left|\frac{\partial}{\partial y}K(\varphi(t),y)\right|_{y=\varphi(s)} \leq c\left\{\begin{array}{ll} s^{-\rho} & ,\nu<0 \\ s^{-\rho}\left(1+|\log((t-s)(t^{\rho-1}+s^{\rho-1}))|\right) & ,\nu=0 \\ s^{-\rho}(t-s)^{-\nu}(t^{\rho-1}+s^{\rho-1})^{-\nu} & ,\nu>0 \end{array}\right\} s^{-\rho\lambda}$$

$$+\,c\left\{\begin{array}{ll} 1 & ,\nu+1<0 \\ 1+|\log((t-s)(t^{\rho-1}+s^{\rho-1}))| & ,\nu+1=0 \\ (t-s)^{-\nu-1}(t^{\rho-1}+s^{\rho-1})^{-\nu-1} & ,\nu+1>0 \end{array}\right\} s^{-\rho\lambda}$$

$$\leq c'\left\{\begin{array}{ll} s^{-\rho} & ,\nu<-1 \\ s^{-\rho}+1+|\log(t-s)| & ,\nu=-1 \\ s^{-\rho}\,+s^{(\rho-1)(-\nu-1)}(t-s)^{-\nu-1} & ,-1<\nu<0 \\ s^{1-\rho}\left[s^{-1}(1+|\log(t-s)|)+\,(t-s)^{-1}\right] & ,\nu=0 \\ s^{(\rho-1)(-\nu-1)}\,(t-s)^{-\nu}\left[s^{-1}+(t-s)^{-1}\right] & ,\nu>0 \end{array}\right\} s^{-\rho\lambda}. \tag{3.3.10}$$

From (3.1.4) we get $|\varphi'(s)|^2 \leq cs^{2\rho-2}$. Write

$$2\rho-2-\rho\lambda=(\rho-1)(1-\lambda)+\rho-\lambda-1.$$

Thus from (3.3.10) it follows that

$$\left|[\varphi'(s)]^2\frac{\partial}{\partial y}K(\varphi(t),y)\right|_{y=\varphi(s)} \leq$$

$$\leq c\left\{\begin{array}{ll} s^{(\rho-1)(1-\lambda)} & ,\nu<-1 \\ s^{(\rho-1)(1-\lambda)}[1+s^{\rho}\,(1+|\log(t-s)|)] & ,\nu=-1 \\ s^{(\rho-1)(1-\lambda)}[1+s^{1-\nu(\rho-1)}\,(t-s)^{-\nu-1}] & ,-1<\nu<0 \\ s^{(\rho-1)(1-\lambda)+1}[s^{-1}(1+|\log(t-s)|)+(t-s)^{-1}] & ,\nu=0 \\ s^{(\rho-1)(1-\nu-\lambda)+1}\,(t-s)^{-\nu}\,[s^{-1}+(t-s)^{-1}] & ,\nu>0 \end{array}\right\} s^{-\lambda-1}. \tag{3.3.11}$$

Using (2.5.5) with $i=0$ and $j=0$ we get

$$\left|\varphi''(s)K(\varphi(t),\varphi(s))\right| \leq c\,|\varphi''(s)|\left\{\begin{array}{ll} 1 & ,\nu<0 \\ 1+|\log(t-s)| & ,\nu=0 \\ s^{-\nu(\rho-1)}\,(t-s)^{-\nu} & ,\nu>0 \end{array}\right\} s^{-\rho\lambda}. \tag{3.3.12}$$

Further, from (3.1.4) we have $|\varphi''(s)| \leq c\,s^{\rho-2}$. As

$$\rho-2-\rho\lambda=(\rho-1)(1-\lambda)-\lambda-1$$

then

$$\left|\varphi''(s)K(\varphi(t),\varphi(s))\right| \leq c\left\{\begin{array}{ll} s^{(\rho-1)(1-\lambda)} & ,\nu<0 \\ s^{(\rho-1)(1-\lambda)}(1+|\log(t-s)|) & ,\nu=0 \\ s^{(\rho-1)(1-\nu-\lambda)}\,(t-s)^{-\nu} & ,\nu>0 \end{array}\right\} s^{-\lambda-1}. \tag{3.3.13}$$

Estimate (3.3.6) follows from (3.3.7), (3.3.11) and (3.3.13).

□

Lemma 3.3.5. *Let $K \in W_*^{1,\nu,\lambda}(D_b)$, $\nu \in (-\infty, 1)$, $\lambda \in (0,1)$, $\varphi \in \Phi^{2,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, b \in (0,\infty)$, $\rho \in [1,\infty)$. Then*

$$\left|\frac{\partial}{\partial s} K_\varphi(t,s)\right| \le c \left\{ \begin{array}{ll} s^{(\rho-1)(1-\lambda)} & ,\nu < -1 \\ s^{(\rho-1)(1-\lambda)}[1 + s^\rho\,(1 + |\log(t-s)|)] & ,\nu = -1 \\ s^{(\rho-1)(1-\lambda)}[1 + s^{1-\nu(\rho-1)}\,(t-s)^{-\nu-1}] & ,-1 < \nu < 0 \\ s^{(\rho-1)(1-\lambda)+1}[s^{-1}(1 + |\log(t-s)|) + (t-s)^{-1}] & ,\nu = 0 \\ s^{(\rho-1)(1-\nu-\lambda)+1}\,(t-s)^{-\nu}\,[s^{-1} + (t-s)^{-1}] & ,\nu > 0 \end{array} \right\} \frac{s^{-\lambda-1}}{1 + |\log s|}, \tag{3.3.14}$$

where c is a positive constant not depending on t and s, $(t,s) \in D_a$ and $K_\varphi(t,s) = \varphi'(s)\,K(\varphi(t), \varphi(s))$.

Proof. We have $\lambda + j > 0$ for $j = 0$ and $j = 1$. Using (2.5.6), we can get the estimate (3.3.14) similarly as in the proof of Lemma 3.3.4.

□

Lemma 3.3.6. *Let $K \in W^{1,\nu}(D_b')$, $\nu \in (-\infty, 1)$, $\varphi \in \Phi^{2,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, b \in (0,\infty)$, $\rho \in [1,\infty)$. If $\varphi \in \Phi^{2,\rho} \cap C^2[0,a]$ then*

$$\left|\frac{\partial}{\partial s} K_\varphi(t,s)\right| \le c \left\{ \begin{array}{ll} s^{\max\{\rho-2,0\}} & ,\nu < -1 \\ s^{\max\{\rho-2,0\}} + s^{2\rho-2}(1 + |\log(t-s)|) & ,\nu = -1 \\ s^{\max\{\rho-2,0\}} + s^{(\rho-1)(1-\nu)}\,(t-s)^{-\nu-1} & ,-1 < \nu < 0 \\ s^{\rho-1}\,[s^{\max\{1-\rho,-1\}}\,(1 + |\log(t-s)|) + (t-s)^{-1}] & ,\nu = 0 \\ s^{(\rho-1)(1-\nu)}\,(t-s)^{-\nu}\,[s^{\max\{1-\rho,-1\}} + (t-s)^{-1}] & ,\nu > 0 \end{array} \right\} \tag{3.3.15}$$

and if $\varphi \in \Phi^{2,\rho}$ then

$$\left|\frac{\partial}{\partial s} K_\varphi(t,s)\right| \le c \left\{ \begin{array}{ll} s^{\rho-2} & ,\nu < -1 \\ s^{\rho-2}\,[1 + s^\rho(1 + |\log(t-s)|)] & ,\nu = -1 \\ s^{\rho-2}\,[1 + s^{1-\nu(\rho-1)}\,(t-s)^{-\nu-1}] & ,-1 < \nu < 0 \\ s^{\rho-1}\,[s^{-1}\,(1 + |\log(t-s)|) + (t-s)^{-1}] & ,\nu = 0 \\ s^{(\rho-1)(1-\nu)}\,(t-s)^{-\nu}\,[s^{-1} + (t-s)^{-1}] & ,\nu > 0 \end{array} \right\}. \tag{3.3.16}$$

Here c is a positive constant not depending on t and s, $(t,s) \in D_a$ and $K_\varphi(t,s) = \varphi'(s)\,K(\varphi(t), \varphi(s))$.

Proof. We follow the proof of Lemma 3.3.4. Let $\varphi \in \Phi^{2,\rho}$. Then

$$\frac{\partial}{\partial s} K_\varphi(t,s) = [\varphi'(s)]^2 \frac{\partial}{\partial y} K(\varphi(t), y)\Big|_{y=\varphi(s)} + \varphi''(s) K(\varphi(t), \varphi(s)). \tag{3.3.17}$$

We use estimates (2.5.4) for $i = 0, j = 1$ and for $i = 1, j = 0$. Similarly to the proof of Lemma 3.3.4, we use the estimates (3.1.4) and (3.1.5) for φ:

$$\begin{aligned}\left|\frac{\partial}{\partial y}K(\varphi(t),y)\right|_{y=\varphi(s)} &\le \left|\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)K(x,y)\right|_{\substack{x=\varphi(t)\\ y=\varphi(s)}} + \left|\frac{\partial}{\partial x}K(x,y)\right|_{\substack{x=\varphi(t)\\ y=\varphi(s)}} \\ &\le c\left\{\begin{array}{ll} 1 & ,\nu<0 \\ 1+|\log(t-s)| & ,\nu=0 \\ (t-s)^{-\nu}(t^{\rho-1}+s^{\rho-1})^{-\nu} & ,\nu>0 \end{array}\right\} \\ &+ c\left\{\begin{array}{ll} 1 & ,\nu+1<0 \\ 1+|\log(t-s)| & ,\nu+1=0 \\ (t-s)^{-\nu-1}(t^{\rho-1}+s^{\rho-1})^{-\nu-1} & ,\nu+1>0 \end{array}\right\}.\end{aligned} \tag{3.3.18}$$

Thus

$$\left|\frac{\partial}{\partial y}K(\varphi(t),y)\right|_{y=\varphi(s)} \le c\left\{\begin{array}{ll} 1 & ,\nu<-1 \\ 1+|\log(t-s)| & ,\nu=-1 \\ s^{-\nu(\rho-1)}\,s^{-(\rho-1)}\,(t-s)^{-\nu-1} & ,\nu>-1 \end{array}\right\}. \tag{3.3.19}$$

From (3.1.4) we get $[\varphi'(s)]^2 \le c\,s^{2\rho-2}$ and

$$\left|[\varphi'(s)]^2\frac{\partial}{\partial y}K(\varphi(t),y)\right|_{y=\varphi(s)} \le c\left\{\begin{array}{ll} s^{2\rho-2} & ,\nu<-1 \\ s^{2\rho-2}\,(1+|\log(t-s)|) & ,\nu=-1 \\ s^{(\rho-1)(1-\nu)}\,(t-s)^{-\nu-1} & ,\nu>-1 \end{array}\right\}. \tag{3.3.20}$$

Next, we use (2.5.4) with $i = 0$ and $j = 0$ for the formula

$$\left|\varphi''(s)K(\varphi(t),\varphi(s))\right| \le c\,|\varphi''(s)|\left\{\begin{array}{ll} 1 & ,\nu<0 \\ 1+|\log(t-s)| & ,\nu=0 \\ s^{-\nu(\rho-1)}\,(t-s)^{-\nu} & ,\nu>0 \end{array}\right\}. \tag{3.3.21}$$

Due to (3.1.4) we have $|\varphi''(s)| \le c\,s^{\rho-2}$. Now the estimate (3.3.16) follows directly from (3.3.17), (3.3.20) and (3.3.21).

Further, let $\varphi \in \Phi^{2,\rho} \cap C^2[0,a]$. Then we have $\varphi \in C^2[0,a]$ and using this with the estimate (3.1.4) we can write

$$|\varphi''(s)| \le c\,s^{\max\{\rho-2,0\}}, \quad 0 < s \le a, \quad \rho \ge 1.$$

Now (3.3.15) follows from (3.3.17), (3.3.20) and (3.3.21).

□

3.4 Interpolation Error Estimates

Lemma 3.4.1. *Assume that:*

1. $u \in C^{m,\theta}(0,b]$, $u_* \in C_*^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta, b \in \mathbb{R}$, $\theta < 1$, $b > 0$;
2. $\varphi \in \Phi^{m,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, \rho \in \mathbb{R}$, $\rho \geq 1, a > 0$;
3. $u_\varphi(t) = u(\varphi(t))$, $u_{*_\varphi}(t) = u_*(\varphi(t))$, $t \in [0,a]$;
4. *a grid* Δ_N *on* $[0,a]$ *defined by* (2.4.1) *and the interpolation points* (2.6.1) *are used.*

Then for the operator $P_N = P_N^{(m-1)}$ *defined by* (2.6.3) *and for* $i = 1, \ldots, N$ *the following estimates*

$$\sup_{t\in[t_{i-1},t_i]} |u_\varphi(t) - (P_N u_\varphi)(t)| \leq c\, h_i^m \left\{ \begin{array}{ll} t_i^{\rho-m} & \text{for } \theta < 0 \\ t_i^{\rho-m}(1 + |\log t_i|) & \text{for } \theta = 0 \\ t_i^{\rho(1-\theta)-m} & \text{for } 0 < \theta < 1 \end{array} \right\} \tag{3.4.1}$$

and

$$\sup_{t\in[t_{i-1},t_i]} |u_{*_\varphi}(t) - (P_N u_{*_\varphi})(t)| \leq c\, h_i^m \left\{ \begin{array}{ll} t_i^{\rho-m} & \text{for } \theta < 0 \\ t_i^{\rho(1-\theta)-m}(1 + |\log t_i|) & \text{for } 0 \leq \theta < 1 \end{array} \right\} \tag{3.4.2}$$

hold with a positive constant c *not depending on* i *and* N. *Here* $h_i = t_i - t_{i-1}$, $i = 1, \ldots, N$.

Proof. Let $u \in C^{m,\theta}(0,b]$. Then $u_\varphi(t) = u(\varphi(t))$ is a continuous function on $[0,a]$ and $u_\varphi(t)$ is m times continuously differentiable on $(0,a]$. By Lemma 2.6.4 and Lemma 3.2.1 (inequalities (3.2.2)) for every $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, N$, we have the estimate

$$\begin{aligned} |u_\varphi(t) - (P_N u_\varphi)(t)| &\leq c \int_t^{t_i} (s-t)^{m-1} |u_\varphi^{(m)}(s)|\, ds \\ &\leq c' \int_t^{t_i} (s-t)^{m-1} \left\{ \begin{array}{ll} s^{\rho-m} & , \theta < 0 \\ s^{\rho-m}(1 + |\log s|) & , \theta = 0 \\ s^{\rho(1-\theta)-m} & , 0 < \theta < 1 \end{array} \right\} ds. \end{aligned} \tag{3.4.3}$$

From Lemma 2.3.1 we obtain

$$|u_\varphi(t) - (P_N u_\varphi)(t)| \le c\,(t_i - t)^m \left\{ \begin{array}{ll} t_i^{\rho-m} & , \theta < 0 \\ t_i^{\rho-m}(1 + |\log t_i|) & , \theta = 0 \\ t_i^{\rho(1-\theta)-m} & , 0 < \theta < 1 \end{array} \right\}, \tag{3.4.4}$$
$$t \in [t_{i-1}, t_i], \quad i = 1, \dots, N.$$

Taking $t = t_{i-1}$, (3.4.1) yields.

Finally, let $u_* \in C_*^{m,\theta}(0,b]$. Then $u_{*_\varphi}(t) = u_*(\varphi(t))$ is a continuous function on $[0,a]$ and $u_{*_\varphi}(t)$ is m times continuously differentiable on $(0,a]$. Similarly by Lemma 2.6.4 and Lemma 3.2.2 for every $t \in [t_{i-1}, t_i]$, $i = 1, \dots, N$, we have the estimate

$$\begin{aligned} |u_{*_\varphi}(t) - (P_N u_{*_\varphi})(t)| &\le c \int_t^{t_i} (s-t)^{m-1}\, |u_{*_\varphi}^{(m)}(s)|\, ds \\ &\le c' \int_t^{t_i} (s-t)^{m-1} \left\{ \begin{array}{ll} s^{\rho-m} & , \theta < 0 \\ s^{\rho(1-\theta)-m}(1 + |\log s|) & , 0 \le \theta < 1 \end{array} \right\} ds. \end{aligned} \tag{3.4.5}$$

We obtain our claim (3.4.2) by Lemma 2.3.1.

□

Lemma 3.4.2. *Assume that:*

1. $u \in C^{m,\theta}(0,b]$, $u_* \in C_*^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta, b \in \mathbb{R}$, $\theta < 1$, $b > 0$;
2. $\psi \in \Phi^{m,\rho} \cap C^m[0,a]$, $\psi : [0,a] \to [0,b]$, $a, \rho \in \mathbb{R}$, $\rho \ge 1, a > 0$;
3. $u_\psi(t) = u(\psi(t))$, $u_{*_\psi}(t) = u_*(\psi(t))$, $t \in [0,a]$;
4. *a grid* Δ_N *on* $[0,a]$ *defined by* (2.4.1) *and the interpolation points* (2.6.1) *are used.*

Then for the operator $P_N = P_N^{(m-1)}$ *defined by* (2.6.3) *and for* $i = 1, \dots, N$ *the following estimates*

$$\sup_{t \in [t_{i-1}, t_i]} |u_\psi(t) - (P_N u_\psi)(t)| \le c\, h_i^m \left\{ \begin{array}{ll} 1 & , m < \rho(1-\theta) \\ 1 + |\log t_i| & , m = \rho(1-\theta) \\ t_i^{\rho(1-\theta)-m} & , m > \rho(1-\theta) \end{array} \right\} \tag{3.4.6}$$

and

$$\sup_{t \in [t_{i-1}, t_i]} |u_{*_\psi}(t) - (P_N u_{*_\psi})(t)| \le c\, h_i^m \left\{ \begin{array}{ll} 1 & , m < \rho(1-\theta) \\ t_i^{\rho(1-\theta)-m}(1 + |\log t_i|) & , m \ge \rho(1-\theta) \end{array} \right\} \tag{3.4.7}$$

hold with a positive constant c *not depending on* i *and* N. *Here* $h_i = t_i - t_{i-1}$, $i = 1, \dots, N$.

Proof. Let $u \in C^{m,\theta}(0,b]$. Then $u_\psi(t) = u(\psi(t))$ is a continuous function on $[0,a]$ and $u_\psi(t)$ is m times continuously differentiable on $(0,a]$. By Lemma 3.2.3 we have $u_\psi \in C^{m,1-\rho(1-\theta)}(0,a]$ and by (2.5.1) and Lemma 2.6.4 for every $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, N$, we have the estimate

$$\begin{aligned} |u_\psi(t) - (P_N u_\psi)(t)| &\le c \int_t^{t_i} (s-t)^{m-1} \, |u_\psi^{(m)}(s)| \, ds \\ &\le c' \int_t^{t_i} (s-t)^{m-1} \left\{ \begin{array}{ll} 1 & , m < \rho(1-\theta) \\ 1 + |\log s| & , m = \rho(1-\theta) \\ s^{\rho(1-\theta)-m} & , m > \rho(1-\theta) \end{array} \right\} ds. \end{aligned} \tag{3.4.8}$$

By Lemma 2.3.1 we obtain

$$\begin{aligned} |u_\psi(t) - (P_N u_\psi)(t)| &\le c\,(t_i - t)^m \left\{ \begin{array}{ll} 1 & , m < \rho(1-\theta) \\ 1 + |\log t_i| & , m = \rho(1-\theta) \\ t_i^{\rho(1-\theta)-m} & , m > \rho(1-\theta) \end{array} \right\}, \\ t &\in [t_{i-1}, t_i], \quad i = 1, \ldots, N. \end{aligned} \tag{3.4.9}$$

Taking $t = t_{i-1}$, (3.4.6) yields.

Finally, let $u_* \in C_*^{m,\theta}(0,b]$. Then $u_{*_\psi}(t) = u_*(\psi(t))$ is a continuous function on $[0,a]$ and $u_{*_\psi}(t)$ is m times continuously differentiable on $(0,a]$. By Lemma 3.2.3 we have $u_{*_\psi} \in C_*^{m,1-\rho(1-\theta)}(0,a]$ and by (2.5.2) and Lemma 2.6.4 for every $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, N$, we have the estimate

$$\begin{aligned} |u_{*_\psi}(t) - (P_N u_{*_\psi})(t)| &= c \int_t^{t_i} (t-s)^{m-1} \, |u_{*_\psi}^{(m)}(s)| \, ds \\ &\le c' \int_t^{t_i} (s-t)^{m-1} \left\{ \begin{array}{ll} 1 & , m < \rho(1-\theta) \\ s^{\rho(1-\theta)-m}(1 + |\log s|) & , m \ge \rho(1-\theta) \end{array} \right\} ds. \end{aligned} \tag{3.4.10}$$

We obtain our claim (3.4.7) using Lemma 2.3.1.

□

Remark 3.4.1. *Let the conditions of Lemma 3.4.2 be fulfilled and let $u \in C^{m,\theta}(0,b]$, $0 < \theta < 1$. Then from Lemma 3.4.1 it follows that we can use inequality*

$$\sup_{t \in [t_{i-1}, t_i]} |u_\psi(t) - (P_N u_\psi)(t)| \le c\, h_i^m \, t_i^{\rho(1-\theta)-m}, \quad i = 1, \ldots, N, \tag{3.4.11}$$

instead of (3.4.6).

Remark 3.4.2. *Let the conditions of Lemma 3.4.2 be fulfilled and let* $u_* \in C_*^{m,\theta}(0,b]$, $0 \le \theta < 1$. *Then from Lemma 3.4.1 it follows that we can use the inequality*

$$\sup_{t\in[t_{i-1},t_i]} |u_{*_\psi}(t) - (P_N u_{*_\psi})(t)| \le c\, h_i^m\, t_i^{\rho(1-\theta)-m}(1+|\log t_i|), \quad i=1,\dots,N, \tag{3.4.12}$$

instead of (3.4.7).

Lemma 3.4.3. *Assume that:*

1. $u \in C^{m,\theta}(0,b]$, $u_* \in C_*^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta, b \in \mathbb{R}$, $\theta < 1$, $b > 0$;
2. $\varphi \in \Phi^{m,\rho}$, $\varphi : [0,a] \to [0,b]$, $a, \rho \in \mathbb{R}$, $\rho \ge 1, a > 0$;
3. $u_\varphi(t) = u(\varphi(t))$, $u_{*_\varphi}(t) = u_*(\varphi(t))$, $t \in [0,a]$;
4. *a graded grid* Δ_N^r *on* $[0,a]$ *defined by* (2.4.3), $r \ge 1$, *and the interpolation points* (2.6.1) *are used.*

Then for the operator $P_N = P_N^{(m-1)}$ *defined by* (2.6.3) *the following estimates hold:*

if $\theta < 0$ *then*

$$\|u_\varphi - P_N u_\varphi\|_\infty \le c\, E_{N,1}^{(m,\theta,\rho,r)}, \quad \|u_{*_\varphi} - P_N u_{*_\varphi}\|_\infty \le c\, E_{N,1}^{(m,\theta,\rho,r)}, \tag{3.4.13}$$

where

$$E_{N,1}^{(m,\theta,\rho,r)} = \left\{ \begin{array}{ll} N^{-m} & \text{for } r \ge \frac{m}{\rho} \\ N^{-r\rho} & \text{for } 1 \le r < \frac{m}{\rho} \end{array} \right\}; \tag{3.4.14}$$

if $\theta = 0$ *then*

$$\|u_\varphi - P_N u_\varphi\|_\infty \le c\, E_{N,2}^{(m,\theta,\rho,r)}, \quad \|u_{*_\varphi} - P_N u_{*_\varphi}\|_\infty \le c\, E_{N,2}^{(m,\theta,\rho,r)}, \tag{3.4.15}$$

where

$$E_{N,2}^{(m,\theta,\rho,r)} = \left\{ \begin{array}{ll} N^{-m} & \text{for } r > \frac{m}{\rho} \\ N^{-r\rho}(1+\log N) & \text{for } 1 \le r \le \frac{m}{\rho} \end{array} \right\}; \tag{3.4.16}$$

if $0 < \theta < 1$ *then*

$$\|u_\varphi - P_N u_\varphi\|_\infty \le c \left\{ \begin{array}{ll} N^{-m} & \text{for } r \ge \frac{m}{\rho(1-\theta)} \\ N^{-r\rho\,(1-\theta)} & \text{for } 1 \le r < \frac{m}{\rho(1-\theta)} \end{array} \right\}, \tag{3.4.17}$$

$$\|u_{*_\varphi} - P_N u_{*_\varphi}\|_\infty \le c \left\{ \begin{array}{ll} N^{-m} & \text{for } r > \frac{m}{\rho(1-\theta)} \\ N^{-r\rho\,(1-\theta)}(1+\log N) & \text{for } 1 \le r \le \frac{m}{\rho(1-\theta)} \end{array} \right\}, \tag{3.4.18}$$

where c *is a positive constant not depending on* N.

Proof. First we consider $u \in C^{m,\theta}(0,b]$. By Lemma 3.4.1 we have

$$\begin{aligned}\|u_\varphi - P_N u_\varphi\|_\infty &\le c \max_{i=1,\dots,N} \sup_{t\in[t_{i-1},t_i]} |u_\varphi(t) - (P_N u_\varphi)(t)| \\ &\le c' \max_{i=1,\dots,N} h_i^m \left\{ \begin{array}{ll} t_i^{\rho-m} & \text{for } \theta < 0 \\ t_i^{\rho-m}(1+|\log t_i|) & \text{for } \theta = 0 \\ t_i^{\rho(1-\theta)-m} & \text{for } 0<\theta<1 \end{array} \right\}.\end{aligned} \tag{3.4.19}$$

Now we can use (2.4.3) and (2.4.6). Thus

$$\|u_\varphi - P_N u_\varphi\|_\infty \le c\, N^{-m} \max_{i=1,\dots,N} \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r\rho-m} & \text{for } \theta < 0 \\ \left(\frac{i}{N}\right)^{r\rho-m}(1+|\log \frac{i}{N}|) & \text{for } \theta = 0 \\ \left(\frac{i}{N}\right)^{r\rho(1-\theta)-m} & \text{for } 0<\theta<1 \end{array} \right\}. \tag{3.4.20}$$

Inequalities (3.4.13), (3.4.15) and (3.4.17) for u_φ follow directly after we choose a suitable r. Note that

$$\left(\frac{i}{N}\right)^{r\rho-m}(1+|\log\frac{i}{N}|) \le c \quad \text{if } r > \frac{m}{\rho}.$$

In a similar way, by Lemma 3.4.1 we can estimate

$$\|u_{*_\varphi} - P_N u_{*_\varphi}\|_\infty \le c\, N^{-m} \max_{i=1,\dots,N} \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r\rho-m} & ,\theta < 0 \\ \left(\frac{i}{N}\right)^{r\rho(1-\theta)-m}(1+|\log \frac{i}{N}|) & ,0 \le \theta < 1 \end{array} \right\}. \tag{3.4.21}$$

After we choose a suitable r, this completes the proof of (3.4.13), (3.4.15) and (3.4.18) for the function u_{*_φ}.

□

Lemma 3.4.4. *Assume that:*

1. $u \in C^{m,\theta}(0,b]$, $u_* \in C_*^{m,\theta}(0,b]$, $m \in \mathbb{N}$, $\theta, b \in \mathbb{R}$, $\theta < 1$, $b > 0$;
2. $\psi \in \Phi^{m,\rho} \cap C^m[0,a]$, $\psi : [0,a] \to [0,b]$, $a, \rho \in \mathbb{R}$, $\rho \ge 1, a > 0$;
3. $u_\psi(t) = u(\psi(t))$, $u_{*_\psi}(t) = u_*(\psi(t))$, $t \in [0,a]$;
4. *a graded grid* Δ_N^r *on* $[0,a]$ *defined by* (2.4.3), $r \ge 1$, *and the interpolation points* (2.6.1) *are used.*

Then for the operator $P_N = P_N^{(m-1)}$ *defined by* (2.6.3) *the following estimates hold:*

$$\|u_\psi - P_N u_\psi\|_\infty \le c\, E_{N,1}^{(m,\theta,\rho,r)}, \qquad \theta \le 0, \tag{3.4.22}$$

$$\|u_\psi - P_N u_\psi\|_\infty \le c\, E_{N,2}^{(m,\theta,\rho,r)}, \qquad 0 < \theta < 1, \tag{3.4.23}$$

$$\|u_{*_\psi} - P_N u_{*_\psi}\|_\infty \le c\, E_{N,3}^{(m,\theta,\rho,r)}, \qquad \theta < 1, \tag{3.4.24}$$

where c is a positive constant not depending on N and

$$E_{N,1}^{(m,\theta,\rho,r)} = \left\{ \begin{array}{lll} N^{-m} & , m < \rho(1-\theta) & , r \geq 1 \\ N^{-m} & , m = \rho(1-\theta) & , r > 1 \\ N^{-m}(1+\log N) & , m = \rho(1-\theta) & , r = 1 \\ N^{-m} & , m > \rho(1-\theta) & , r \geq \frac{m}{\rho(1-\theta)} \\ N^{-r\rho\,(1-\theta)} & , m > \rho(1-\theta) & , 1 \leq r < \frac{m}{\rho(1-\theta)} \end{array} \right\}, \quad (3.4.25)$$

$$E_{N,2}^{(m,\theta,\rho,r)} = \left\{ \begin{array}{ll} N^{-m} & , r \geq \frac{m}{\rho(1-\theta)} \\ N^{-r\rho\,(1-\theta)} & , 1 \leq r < \frac{m}{\rho(1-\theta)} \end{array} \right\}, \quad (3.4.26)$$

$$E_{N,3}^{(m,\theta,\rho,r)} = \left\{ \begin{array}{ll} N^{-m} & , r > \frac{m}{\rho(1-\theta)} \\ N^{-r\rho\,(1-\theta)}(1+\log N) & , 1 \leq r \leq \frac{m}{\rho(1-\theta)} \end{array} \right\}. \quad (3.4.27)$$

Proof. First we show (3.4.22). Let $\theta \leq 0$. Then by Lemma 3.4.2 and due to (2.4.3) and (2.4.6) we have

$$\begin{aligned} \|u_\psi - P_N u_\psi\|_\infty &\leq c \max_{i=1,\dots,N} \sup_{t\in[t_{i-1},t_i]} |u_\psi(t) - (P_N u_\psi)(t)| \\ &\leq c' \max_{i=1,\dots,N} h_i^m \left\{ \begin{array}{ll} 1 & , m < \rho(1-\theta) \\ 1 + |\log t_i| & , m = \rho(1-\theta) \\ t_i^{\rho(1-\theta)-m} & , m > \rho(1-\theta) \end{array} \right\} \\ &\leq c'' N^{-m} \max_{i=1,\dots,N} \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{rm-m} & , m < \rho(1-\theta) \\ \left(\frac{i}{N}\right)^{rm-m}(1+|\log \frac{i}{N}|) & , m = \rho(1-\theta) \\ \left(\frac{i}{N}\right)^{r\rho(1-\theta)-m} & , m > \rho(1-\theta) \end{array} \right\}. \end{aligned} \quad (3.4.28)$$

Thus (3.4.22) follows after we choose a suitable r. Similarly, we can show (3.4.24) for $\theta < 1$. By Lemma 3.4.2,

$$\begin{aligned} &\|u_{*_\psi} - P_N u_{*_\psi}\|_\infty \\ &\quad \leq c\, N^{-m} \max_{i=1,\dots,N} \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{rm-m} & , m < \rho(1-\theta) \\ \left(\frac{i}{N}\right)^{r\rho(1-\theta)-m}(1+|\log \frac{i}{N}|) & , m \geq \rho(1-\theta) \end{array} \right\}. \end{aligned} \quad (3.4.29)$$

Now (3.4.24) follows easily for $m \geq \rho(1-\theta)$. If $m < \rho(1-\theta)$ then $r \geq 1 > \frac{m}{\rho(1-\theta)}$ and (3.4.24) holds.

Finally, estimate (3.4.23) follows from Remark 3.4.1 and Lemma 3.4.3, inequality (3.4.17).

□

Chapter 4

Smoothing and Collocation Method

In this chapter we describe the integral equation which is the subject of this thesis and construct a class of numerical methods for solving it. On the basis of results obtained in Chapter 3 we first regularize the solution of equation (4.1.1) by introducing a suitable new independent variable so that the singularities of the derivatives of the solution will be milder or even disappear. After that we solve the transformed equation by a piecewise polynomial collocation method on mildly graded or uniform grid and discuss the attainable rate of convergence of the obtained approximations. The main results of this chapter are formulated by Theorems 4.4.1-4.4.5. They are partly published in [35, 36, 37, 38, 39].

The idea of smoothing the solution by introducing a suitable change of variables has been considered widely in [56, 58], see also [8, 19, 20, 26, 48, 66, 67, 69, 73].

4.1 Integral Equation

We consider a linear Volterra integral equations of the form

$$u(x) = \int_0^x K(x,y)u(y)dy + f(x), \quad 0 \le x \le b, \quad b > 0. \tag{4.1.1}$$

We assume that the forcing function $f : [0,b] \to \mathbb{R}$ is continuous. The kernel $K(x,y)$ may have a diagonal singularity as $y \to x$. For example, the kernel $K(x,y)$ may have the form

$$K(x,y) = g_1(x,y)(x-y)^{-\nu} + g_2(x,y)\log(x-y) + g_3(x,y), \quad (x,y) \in D_b^{'},$$

where $g_1, g_2, g_3 \in C^m(\bar{D}_b)$, $m \in \mathbb{N}_0$, $\nu < 1$. We also examine a more complicated situation for equation (4.1.1) where the kernel $K(x,y)$, in addition to a diagonal

singularity, may have a boundary singularity (a singularity as $y \to 0$). For example, the the kernel $K(x,y)$ may have the form

$$K(x,y) = g_1(x,y)(x-y)^{-\nu}y^{-\lambda} + g_2(x,y)\log(x-y)y^{-\lambda} + g_3(x,y)y^{-\lambda}, \quad (x,y) \in D_b,$$

where $g_1, g_2, g_3 \in C^m(\bar{D}_b)$, $m \in \mathbb{N}_0$, $\nu, \lambda < 1$, $\nu + \lambda < 1$. More precisely, we assume that $K \in W^{m,\nu}(D_b')$ or $K \in W^{m,\nu,\lambda}(D_b)$ or $K \in W_*^{m,\nu,\lambda}(D_b)$. Here $m \in \mathbb{N}_0$, $\nu, \lambda < 1$, $\nu + \lambda < 1$.

We define an integral operator T by the following formula:

$$(T\,z)(x) = \int_0^x K(x,y)z(y)dy, \quad x \in [0,b]. \tag{4.1.2}$$

Thus equation (4.1.1) has the operator equation form

$$u = Tu + f. \tag{4.1.3}$$

The proof of the following theorem can be found in [57].

Theorem 4.1.1. *Let $K \in W^{0,\nu}(D_b')$ or $K \in W^{0,\nu,\lambda}(D_b)$ with $\nu, \lambda \in (-\infty, 1)$ such that $\nu + \lambda < 1$. Then T, defined by (4.1.2), is compact as an operator from $L^\infty(0,b)$ into $C[0,b]$ (and hence also from $L^\infty(0,b)$ to $L^\infty(0,b)$ and from $C[0,b]$ to $C[0,b]$).*

4.2 Regularity of the Solution

The regularity of the solution of equation (4.1.1) can be described by the following Theorems 4.2.1-4.2.3, see [57].

Theorem 4.2.1. *Let $K \in W^{m,\nu}(D_b')$ and $f \in C^{m,\nu}(0,b]$ where $m \geq 1$, $\nu < 1$. Then equation (4.1.1) has a unique solution and it belongs to $C^{m,\nu}(0,b]$.*

Theorem 4.2.2. *Let $K \in W^{m,\nu,\lambda}(D_b)$ where $m \geq 1$, $\nu < 1$, $\lambda < \min\{1, 1-\nu\}$. Then equation (4.1.1) has a unique solution u and the following is true:*

1. *if $\nu \notin \mathbb{Z}$ and $f \in C^{m,\nu+\lambda}(0,b]$, then $u \in C^{m,\nu+\lambda}(0,b]$;*
2. *if $f \in C_*^{m,\nu+\lambda}(0,b]$, then $u \in C_*^{m,\nu+\lambda}(0,b]$ (for $\nu \in \mathbb{Z}$ as well as for $\nu \notin \mathbb{Z}$).*

Theorem 4.2.3. *Let $K \in W_*^{m,\nu,\lambda}(D_b)$, $f \in C^{m,\nu+\lambda}(0,b]$ where $m \geq 1$, $\nu \in \mathbb{Z}$, $\nu \leq 0$, $\lambda < \min\{1, 1-\nu\}$. Then equation (4.1.1) has a unique solution $u \in C^{m,\nu+\lambda}(0,b]$.*

4.3 Description of the Numerical Method

The approach proposed in this work for the numerical solution of equation (4.1.1) can be described by the following three steps.

Step 1. Let $\varphi \in \Phi^{m,\rho}$ be the transformation (3.1.1)-(3.1.4) from $[0,a]$ onto $[0,b]$, where $a,b,\rho \in \mathbb{R}$, $\rho \geq 1$, $m \in \mathbb{N}$ and $a,b > 0$. Introducing in (4.1.1) the change of variables $x = \varphi(t)$, $y = \varphi(s)$, $s,t \in [0,a]$ we obtain an integral equation of the form

$$u_\varphi(t) = \int_0^t K_\varphi(t,s)u_\varphi(s)ds + f_\varphi(t)\,, \quad 0 \leq t \leq a\,, \tag{4.3.1}$$

where

$$u_\varphi(t) = u(\varphi(t)), \tag{4.3.2}$$

$$f_\varphi(t) = f(\varphi(t)), \tag{4.3.3}$$

$$K_\varphi(t,s) = \varphi^{'}(s)\, K(\varphi(t),\varphi(s)). \tag{4.3.4}$$

Step 2. We find an approximation v_N to u_φ, the solution of equation (4.3.1), determining v_N by the standard collocation method as follows.

For the collocation method, we choose $N \in \mathbb{N}$. Now, let $\Delta_N^{(r,a)}$ be a graded grid on $[0,a]$ with grid points

$$t_i = a\left(\frac{i}{N}\right)^r, \quad r \geq 1, \quad i = 0,\ldots,N. \tag{4.3.5}$$

Next, we choose the collocation parameters (2.6.2):

$$0 \leq \eta_1 < \cdots < \eta_m \leq 1, \quad m \in \mathbb{N}. \tag{4.3.6}$$

We construct m collocation points in every subinterval $[t_{i-1},t_i]$ as follows:

$$t_{ij} = t_{i-1} + \eta_j(t_i - t_{i-1})\,, \quad i = 1,\ldots,N, \quad j = 1,\ldots,m. \tag{4.3.7}$$

We find an approximation $v_N = v_{N,m,r,\varphi}$ to u_φ, the solution of equation (4.3.1) by the following collocation conditions:

$$v_N \in S_{m-1}^{(-1)}(\Delta_N^{(r,a)}), \quad N,m \in \mathbb{N},\ r \geq 1, \tag{4.3.8}$$

$$v_N(t_{ij}) = \int_0^{t_{ij}} K_\varphi(t_{ij},s)v_N(s)\,ds + f_\varphi(t_{ij}), \quad i = 1,\ldots,N, \quad j = 1,\ldots,m, \tag{4.3.9}$$

with t_{ij} given by formula (4.3.7).

Step 3. We determine an approximation $u_N = u_{N,m,r,\varphi}$ to u, the solution of equation (4.1.1), setting

$$u_N(x) = v_N(\varphi^{-1}(x)), \quad 0 \leq x \leq b. \tag{4.3.10}$$

Remark 4.3.1. *For the simplest transformation* (3.1.7),

$$\varphi(t) = \frac{b}{a^\rho} t^\rho, \quad 0 \le t \le a, \quad a, b > 0, \quad \rho \ge 1, \tag{4.3.11}$$

we have

$$\varphi^{-1}(x) = \frac{a}{b^{\frac{1}{\rho}}} x^{\frac{1}{\rho}}, \quad 0 \le x \le b.$$

Remark 4.3.2. *If $\varphi(t) = t$, $t \in [0, b]$, (i.e. $\rho = 1$ and $a = b$ in* (4.3.11)*), then the method described above coincides with the standard collocation method for solving equation* (4.1.1) *on the graded grids.*

Remark 4.3.3. *If we use the parameters $\eta_1 = 0$, $\eta_m = 1$ in* (4.3.6) *then the resulting collocation approximation v_N belongs to the smoother polynomial spline space $S_{m-1}^{(0)}(\Delta_N^{(r,a)})$.*

Remark 4.3.4. *The conditions* (4.3.8)-(4.3.9) *form a linear system of algebraic equations whose exact form is determined by the choice of a basis in the space $S_{m-1}^{(-1)}(\Delta_N^{(r,a)})$ (or in the space $S_{m-1}^{(0)}(\Delta_N^{(r,a)})$ if $\eta_1 = 0$, $\eta_m = 1$).*

For example, in each interval $[t_{i-1}, t_i] \subset [0, a]$ $(i = 1, \dots, N)$ we may use the representation

$$v_N(t) = \sum_{j=1}^{m} \beta_{ij} L_{ij}(t), \quad t \in [t_{i-1}, t_i], \quad i = 1, \dots, N, \tag{4.3.12}$$

where $L_{ij}(t)$ is the Lagrange fundamental polynomial of degree $m - 1$,

$$L_{ij}(t) = \prod_{\substack{k=1 \\ k \ne j}}^{m} \frac{t - t_{ik}}{t_{ij} - t_{ik}}, \quad t \in [t_{i-1}, t_i], \quad i = 1, \dots, N. \tag{4.3.13}$$

It follows from (4.3.13) that

$$L_{ij}(t_{ik}) = \left\{ \begin{array}{ll} 1 & \text{if } k = j \\ 0 & \text{if } k \ne j \end{array} \right\}, \quad k = 1, \dots, m. \tag{4.3.14}$$

The collocation conditions (4.3.9) then lead to a linear system of equations for the coefficients $\{\beta_{ij}\}$:

$$\begin{aligned} \beta_{ij} = f_\varphi(t_{ij}) &+ \sum_{k=1}^{i-1} \sum_{p=1}^{m} \left(\int_{t_{k-1}}^{t_k} K_\varphi(t_{ij}, s) L_{kp}(s)\, ds \right) \beta_{kp} \\ &+ \sum_{p=1}^{m} \left(\int_{t_{i-1}}^{t_{ij}} K_\varphi(t_{ij}, s) L_{ip}(s)\, ds \right) \beta_{ip}, \quad i = 1, \dots, N, \quad j = 1, \dots, m. \end{aligned} \tag{4.3.15}$$

The linear system (4.3.15) can be solved step by step. First, we solve the system (4.3.15) for $i = 1$ to find the quantities $\beta_{11}, \ldots, \beta_{1m}$. Using the known values $\beta_{11}, \ldots, \beta_{1m}$, we solve the system (4.3.15) for $i = 2$ and find the quantities $\beta_{21}, \ldots, \beta_{2m}$. We follow these steps until we find $\beta_{N1}, \ldots, \beta_{Nm}$ using the known values of $\beta_{i1}, \ldots, \beta_{im}$, $i = 1, \ldots, N-1$.

Using (4.3.4) we define an integral operator T_φ by

$$(T_\varphi z)(t) = \int_0^t K_\varphi(t,s)\, z(s)\, ds, \quad t \in [0,a]. \tag{4.3.16}$$

Thus (4.3.1) has operator equation form

$$u_\varphi = T_\varphi u_\varphi + f_\varphi. \tag{4.3.17}$$

Similarly to (2.6.3) we define an interpolation operator $P_N = P_N^{(m-1)}$ which assigns to every continuous function $z \in C[0,a]$ its piecewise polynomial function $P_N z \in S_{m-1}^{(-1)}(\Delta_N^{(r,a)})$ such that $P_N z$ interpolates z at the nodes (4.3.7):

$$(P_N z)(t_{ij}) = z(t_{ij}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, m. \tag{4.3.18}$$

Thus the conditions (4.3.8)-(4.3.9) have the operator equation representation

$$v_N = P_N T_\varphi v_N + P_N f_\varphi. \tag{4.3.19}$$

4.4 Convergence Analysis and Error Estimates

Lemma 4.4.1. *Let $K \in W^{0,\nu}(D_b^{'})$, $b \in (0,\infty)$, $\nu \in (-\infty, 1)$. Assume that $\varphi \in \Phi^{1,\rho}$ with $\rho \in [1,\infty)$ and $a \in (0,\infty)$. Then T_φ is compact as an operator from $L^\infty(0,a)$ to $C[0,a]$ (and hence also from $L^\infty(0,a)$ to $L^\infty(0,a)$ and from $C[0,a]$ to $C[0,a]$).*

Proof. On the basis of (3.1.1)-(3.1.4) and (4.3.4) we obtain that $K_\varphi(t,s)$ is continuous on $D_a^{'}$. By Lemma 3.3.3,

$$|K_\varphi(t,s)| \le c \left\{ \begin{array}{ll} 1 & ,\nu < 0 \\ 1 + |\log(t-s)| & ,\nu = 0 \\ (t-s)^{-\nu} & ,\nu > 0 \end{array} \right\}, \quad (t,s) \in D_a^{'}.$$

Thus the kernel K_φ is weakly singular and the assertions of lemma follow.

□

Lemma 4.4.2. *Let $K \in W^{0,\nu,\lambda}(D_b)$, $b \in (0,\infty)$, $\nu, \lambda \in (-\infty, 1)$ such that $\nu + \lambda < 1$. Assume that $\varphi \in \Phi^{1,\rho}$ with $\rho \in [1,\infty)$ and $a \in (0,\infty)$. Then T_φ is compact as an operator from $L^\infty(0,a)$ to $C[0,a]$ (and hence also from $L^\infty(0,a)$ to $L^\infty(0,a)$ and from $C[0,a]$ to $C[0,a]$).*

Proof. On the basis of (3.1.1)-(3.1.4) and (4.3.4) we obtain that $K_\varphi(t,s)$ is continuous on D_a. By Lemma 3.3.1,

$$|K_\varphi(t,s)| \le c \left\{ \begin{array}{ll} 1 & ,\nu < 0 \\ 1 + |\log(t-s)| & ,\nu = 0 \\ (t-s)^{-\nu} & ,\nu > 0 \end{array} \right\} s^{-\lambda}, \quad (t,s) \in D_a.$$

By Theorem 4.1.1 the assertions of lemma follow.

□

Theorem 4.4.1. *Let $K \in W^{0,\nu}(D_b^{'})$ or $K \in W^{0,\nu,\lambda}(D_b)$ and let $f \in C[0,b]$, $b \in (0,\infty)$, $\nu, \lambda \in (-\infty, 1)$ such that $\nu + \lambda < 1$. Then the equation* (4.1.1) *has a unique solution $u \in C[0,b]$.*

Furthermore, assume that $\varphi \in \Phi^{1,\rho}$ with $\rho \in [1,\infty)$ and $a \in (0,\infty)$. Then the equation (4.3.1) *has a unique solution $u_\varphi \in C[0,a]$.*

Proof. Due to Theorem 4.1.1 the operator T defined by (4.1.2) is compact from $C[0,b]$ to $C[0,b]$. By Lemmas 4.4.1 and 4.4.2 also the operator T_φ defined by (4.3.16) is compact from $C[0,a]$ to $C[0,a]$. If $f \in C[0,b]$ (from (4.3.3) it follows that $f_\varphi \in C[0,a]$) then the assertions theorem follow from Theorem 2.2.5 since the homogeneous equations $u = Tu$ and $u_\varphi = T_\varphi u_\varphi$ have respectively in $C[0,b]$ and in $C[0,a]$ only the trivial solutions $u = 0$ and $u_\varphi = 0$.

□

Next, we prove the convergence of the method described in Section 4.3.

Theorem 4.4.2. *Let $K \in W^{0,\nu}(D_b^{'})$ or $K \in W^{0,\nu,\lambda}(D_b)$, $f \in C[0,b]$, $b \in (0,\infty)$, $\nu, \lambda \in (-\infty, 1)$ such that $\nu + \lambda < 1$. Furthermore, assume that $\varphi \in \Phi^{1,\rho}$ with $\rho \in [1,\infty)$, $a \in (0,\infty)$ and the interpolation nodes* (4.3.7) *in which the grid points* (4.3.5) *and the parameters* (4.3.6) *are used.*

Then conditions (4.3.8)-(4.3.9) *determine for sufficiently large values of N, say $N \ge N_0$, a unique approximation v_N to u_φ, the solution of equation* (4.3.1)*, and*

$$\|u_\varphi - v_N\|_\infty \to 0 \quad \text{as } N \to \infty. \tag{4.4.1}$$

Moreover, the setting (4.3.10) *determines for $N \ge N_0$ a unique approximation u_N to u, the solution of equation* (4.1.1)*, and*

$$\|u - u_N\|_\infty \to 0 \quad \text{as } N \to \infty, \tag{4.4.2}$$

$$\|u - u_N\|_\infty \le c\,\|u_\varphi - P_N u_\varphi\|_\infty, \tag{4.4.3}$$

where c is some positive constant and $P_N = P_N^{(m-1)}$ is defined by (4.3.18).

Proof. The proof is a standard discussion based on the compactness of the operator $T_\varphi : L^\infty(0,a) \to L^\infty(0,a)$. Let the operator T_φ be defined by the formula (4.3.16). By Theorem 4.4.1 and (4.3.17) there exists a unique solution $u_\varphi = (I - T_\varphi)^{-1} f_\varphi \in C[0,a] \subset L^\infty(0,a)$. Here I is the identity mapping and $(I - T_\varphi)^{-1}$ is bounded as an operator from $L^\infty(0,a)$ into $C[0,a] \subset L^\infty(0,a)$.

Next, we consider the equation (4.3.19). By Lemma 3.4.3 and Lemma 3.4.4, $\|u_\varphi - P_N u_\varphi\|_\infty \to 0$ as $N \to \infty$ for every $u_\varphi \in C[0,a]$. Then from the compactness of T_φ (see Lemmas 4.4.1 and 4.4.2) by Lemma 2.6.3 it follows that

$$\|T_\varphi - P_N T_\varphi\|_{\mathcal{L}(L^\infty(0,a),L^\infty(0,a))} \to 0 \quad \text{as } N \to \infty. \tag{4.4.4}$$

Now we can pick N_0 such that

$$\|T_\varphi - P_N T_\varphi\|_{\mathcal{L}(L^\infty(0,a),L^\infty(0,a))} \|(I - T_\varphi)^{-1}\|_{\mathcal{L}(L^\infty(0,a),L^\infty(0,a))} < 1, \quad N \geq N_0. \tag{4.4.5}$$

Further we can write $I - P_N T_\varphi$ in the form

$$I - P_N T_\varphi = [I - (P_N T_\varphi - T_\varphi)(I - T_\varphi)^{-1}]\,(I - T_\varphi). \tag{4.4.6}$$

Due to (4.4.5) and by Theorem 2.2.3 it follows that $I - P_N T_\varphi$ is invertible and by Theorem 2.2.3 the estimate

$$\|(I - P_N T_\varphi)^{-1}\| \leq \frac{\|(I - T_\varphi)^{-1}\|}{1 - \|T_\varphi - P_N T_\varphi\|\,\|(I - T_\varphi)^{-1}\|} \leq c, \quad N \geq N_0, \tag{4.4.7}$$

holds with a positive constant c not depending on N, here all the norms are in the space $\mathcal{L}(L^\infty(0,a), L^\infty(0,a))$. We obtain that equation (4.3.19) for sufficiently large $N \geq N_0$ has a unique solution $v_N \in S_{m-1}^{(-1)}(\Delta_N^{(r,a)})$. For this and u_φ, the solution of equation (4.3.17), we have that

$$v_N = (I - P_N T_\varphi)^{-1} P_N f_\varphi, \quad P_N f_\varphi = (P_N - P_N T_\varphi) u_\varphi.$$

Thus

$$u_\varphi - v_N = (I - P_N T_\varphi)^{-1}(u_\varphi - P_N u_\varphi), \quad N \geq N_0. \tag{4.4.8}$$

Therefore, by (4.4.7),

$$\|u_\varphi - v_N\|_\infty \leq c\,\|u_\varphi - P_N u_\varphi\|_\infty, \quad N \geq N_0. \tag{4.4.9}$$

By Lemma 3.4.3 and Lemma 3.4.4 $\|u_\varphi - P_N u_\varphi\|_\infty \to 0$ as $N \to \infty$ for every $u_\varphi \in C[0,a]$ and hence (4.4.1) holds.

Finally, as the equation (4.3.19) has a unique solution v_N for sufficiently large N, then the setting (4.3.10) determines a unique approximation u_N to u, solution of the equation (4.1.1). Further,

$$\sup_{0 \leq x \leq b} |u(x) - u_N(x)| = \sup_{0 \leq t \leq a} |u(\varphi(t)) - u_N(\varphi(t))| = \|u_\varphi - v_N\|_\infty. \tag{4.4.10}$$

Now (4.4.2) and (4.4.3) follow from (4.4.9) and (4.4.1).

□

In the following three theorems, we give the estimates for the norm $\|u-u_N\|_\infty$.

Theorem 4.4.3. *Let $K \in W^{m,\nu}(D_b')$, $f \in C^{m,\nu}(0,b]$, $m \in \mathbb{N}$, $b \in (0,\infty)$, $\nu \in (-\infty,1)$. Let $\varphi \in \Phi^{m,\rho}$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes* (4.3.7) *in which the grid points* (4.3.5) *and the parameters* (4.3.6) *are used.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0 \geq 2$, a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following is true:*

1) if $0 < \nu < 1$, then for $\varphi \in \Phi^{m,\rho}$ and $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$ we have

$$\|u-u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & , r \geq \frac{m}{\rho(1-\nu)} \\ N^{-r\rho(1-\nu)} & , 1 \leq r < \frac{m}{\rho(1-\nu)} \end{array} \right\}; \tag{4.4.11}$$

2) if $\nu \leq 0$, then for $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$ we have

$$\|u-u_N\|_\infty \leq c \left\{ \begin{array}{lll} N^{-m} & , m < \rho(1-\nu) & , r \geq 1 \\ N^{-m} & , m = \rho(1-\nu) & , r > 1 \\ N^{-m}\log N & , m = \rho(1-\nu) & , r = 1 \\ N^{-m} & , m > \rho(1-\nu) & , r \geq \frac{m}{\rho(1-\nu)} \\ N^{-r\rho(1-\nu)} & , m > \rho(1-\nu) & , 1 \leq r < \frac{m}{\rho(1-\nu)} \end{array} \right\}; \tag{4.4.12}$$

3) if $\nu = 0$, then for $\varphi \in \Phi^{m,\rho}$ we have

$$\|u-u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & \textit{for } r > \frac{m}{\rho} \\ N^{-r\rho}\log N & \textit{for } 1 \leq r \leq \frac{m}{\rho} \end{array} \right\}; \tag{4.4.13}$$

4) if $\nu < 0$, then for $\varphi \in \Phi^{m,\rho}$ we have

$$\|u-u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & \textit{for } r \geq \frac{m}{\rho} \\ N^{-r\rho} & \textit{for } 1 \leq r < \frac{m}{\rho} \end{array} \right\}. \tag{4.4.14}$$

Here c is a positive constant not depending on N.

Proof. We choose an arbitrary transformation $\varphi \in \Phi^{m,\rho}$ or $\psi \in \Phi^{m,\rho} \cap C^m[0,a]$ such that $\varphi, \psi : [0,a] \to [0,b]$, $\rho \geq 1$, $a, b > 0$. Let

$$u_\varphi(t) = u(\varphi(t)), \quad u_\psi(t) = u(\psi(t)), \quad t \in [0,a].$$

By Theorem 4.4.1 the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$ or $u_\psi \in C[0,a]$. Moreover, by

Theorem 4.2.1 the solution u belongs to $C^{m,\nu}(0,b]$. By Lemmas 3.2.1 and 3.2.3 we can write

$$\left\{ \begin{array}{lll} u_\varphi \in C^{m,1-\rho}(0,a], & u_\psi \in C^{m,1-\rho(1-\nu)}(0,a] & \text{for } \nu < 0 \\ u_\varphi \in C_*^{m,1-\rho}(0,a], & u_\psi \in C^{m,1-\rho}(0,a] & \text{for } \nu = 0 \\ u_\varphi \in C^{m,1-\rho(1-\nu)}(0,a], & u_\psi \in C^{m,1-\rho(1-\nu)}(0,a] & \text{for } 0 < \nu < 1 \end{array} \right\}. \quad (4.4.15)$$

The estimates (4.4.11)-(4.4.14) follow from Theorem 4.4.2 (see inequality (4.4.3)) and from the Lemmas 3.4.3 and 3.4.4.

□

Theorem 4.4.4. *Let $K \in W^{m,\nu,\lambda}(D_b)$, $f \in C_*^{m,\nu+\lambda}(0,b]$, $m \in \mathbb{N}$, $b \in (0,\infty)$, $\nu, \lambda \in (-\infty, 1)$ such that $\nu + \lambda < 1$. Let $\varphi \in \Phi^{m,\rho}$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes* (4.3.7) *in which the grid points* (4.3.5) *and the parameters* (4.3.6) *are used.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0 \geq 2$, a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following is true:*

1) if $\nu + \lambda < 0$, then for $\varphi \in \Phi^{m,\rho}$ we have

$$\|u - u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & \text{for } r \geq \frac{m}{\rho} \\ N^{-r\rho} & \text{for } 1 \leq r < \frac{m}{\rho} \end{array} \right\}; \quad (4.4.16)$$

2) in all other cases: $0 \leq \nu + \lambda < 1$ for $\varphi \in \Phi^{m,\rho}$ and $\nu + \lambda < 1$ for $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$, we have

$$\|u - u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & \text{for } r > \frac{m}{\rho(1-\nu-\lambda)} \\ N^{-r\rho(1-\nu-\lambda)} \log N & \text{for } 1 \leq r \leq \frac{m}{\rho(1-\nu-\lambda)} \end{array} \right\}. \quad (4.4.17)$$

Here c is positive constant not depending on N.

Proof. We choose an arbitrary transformation $\varphi \in \Phi^{m,\rho}$ or $\psi \in \Phi^{m,\rho} \cap C^m[0,a]$ such that $\varphi, \psi : [0,a] \to [0,b]$, $\rho \geq 1$, $a, b > 0$. Let

$$u_\varphi(t) = u(\varphi(t)), \quad u_\psi(t) = u(\psi(t)), \quad t \in [0,a].$$

By Theorem 4.4.1 the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$ or $u_\psi \in C[0,a]$. Moreover, by Theorem 4.2.2 the solution u belongs to $C_*^{m,\nu+\lambda}(0,b]$. By Lemmas 3.2.2 and 3.2.3 we can write

$$\left\{ \begin{array}{lll} u_\varphi \in C^{m,1-\rho}(0,a], & u_\psi \in C_*^{m,1-\rho(1-\nu-\lambda)}(0,a] & \text{for } \nu + \lambda < 0 \\ u_\varphi \in C_*^{m,1-\rho(1-\nu-\lambda)}(0,a], & u_\psi \in C_*^{m,1-\rho(1-\nu-\lambda)}(0,a] & \text{for } 0 \leq \nu + \lambda < 1 \end{array} \right\}. \quad (4.4.18)$$

The estimates (4.4.16)-(4.4.17) follow from Theorem 4.4.2 and from the Lemmas 3.4.3 and 3.4.4.

□

Theorem 4.4.5. *Assume $f \in C^{m,\nu+\lambda}(0,b]$, $m \in \mathbb{N}$, $b \in (0,\infty)$, $\nu, \lambda \in (-\infty, 1)$ such that $\nu + \lambda < 1$. Let one of the following conditions be true*

1. $K \in W^{m,\nu,\lambda}(D_b)$, $\nu \notin \mathbb{Z}$;
2. $K \in W_*^{m,\nu,\lambda}(D_b)$, $\nu \in \mathbb{Z}$.

Furthermore, let the interpolation nodes (4.3.7) *in which the grid points* (4.3.5) *and the parameters* (4.3.6) *are used.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0 \geq 2$, a unique approximation u_N to u, the solution of the equation* (4.1.1), *and the following is true:*

1) if $0 < \nu + \lambda < 1$, then for $\varphi \in \Phi^{m,\rho}$ and $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$ we have

$$\|u - u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & , r \geq \frac{m}{\rho(1-\nu-\lambda)} \\ N^{-r\rho\,(1-\nu-\lambda)} & , 1 \leq r < \frac{m}{\rho(1-\nu-\lambda)} \end{array} \right\}; \tag{4.4.19}$$

2) if $\nu + \lambda \leq 0$, then for $\varphi \in \Phi^{m,\rho} \cap C^m[0,a]$ we have

$$\|u - u_N\|_\infty \leq c \left\{ \begin{array}{lll} N^{-m} & , m < \rho(1-\nu-\lambda) & , r \geq 1 \\ N^{-m} & , m = \rho(1-\nu-\lambda) & , r > 1 \\ N^{-m}\log N & , m = \rho(1-\nu-\lambda) & , r = 1 \\ N^{-m} & , m > \rho(1-\nu-\lambda) & , r \geq \frac{m}{\rho(1-\nu-\lambda)} \\ N^{-r\rho\,(1-\nu-\lambda)} & , m > \rho(1-\nu-\lambda) & , 1 \leq r < \frac{m}{\rho(1-\nu-\lambda)} \end{array} \right\}; \tag{4.4.20}$$

3) if $\nu + \lambda = 0$, then for $\varphi \in \Phi^{m,\rho}$ we have

$$\|u - u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & \textit{for } r > \frac{m}{\rho} \\ N^{-r\rho}\log N & \textit{for } 1 \leq r \leq \frac{m}{\rho} \end{array} \right\}; \tag{4.4.21}$$

4) if $\nu + \lambda < 0$, then for $\varphi \in \Phi^{m,\rho}$ we have

$$\|u - u_N\|_\infty \leq c \left\{ \begin{array}{ll} N^{-m} & \textit{for } r \geq \frac{m}{\rho} \\ N^{-r\rho} & \textit{for } 1 \leq r < \frac{m}{\rho} \end{array} \right\}. \tag{4.4.22}$$

Here c is positive constant not depending on N.

Proof. We choose an arbitrary transformation $\varphi \in \Phi^{m,\rho}$ or $\psi \in \Phi^{m,\rho} \cap C^m[0,a]$ such that $\varphi, \psi : [0,a] \to [0,b]$, $\rho \geq 1$, $a, b > 0$. Let

$$u_\varphi(t) = u(\varphi(t)), \quad u_\psi(t) = u(\psi(t)), \quad t \in [0,a].$$

By Theorem 4.4.1 the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$ or $u_\psi \in C[0,a]$.

Moreover, by Theorem 4.2.2 for $K \in W^{m,\nu,\lambda}(D_b)$ and $\nu \notin \mathbb{Z}$, by Theorem 4.2.3 for $K \in W_*^{m,\nu,\lambda}(D_b)$ and $\nu \in \mathbb{Z}$, the solution u belongs to $C^{m,\nu+\lambda}(0,b]$. By Lemmas 3.2.1 and 3.2.3 we can write

$$\left\{ \begin{array}{lll} u_\varphi \in C^{m,1-\rho}(0,a], & u_\psi \in C^{m,1-\rho(1-\nu-\lambda)}(0,a] & \text{for } \nu+\lambda < 0 \\ u_\varphi \in C_*^{m,1-\rho}(0,a], & u_\psi \in C^{m,1-\rho}(0,a] & \text{for } \nu+\lambda = 0 \\ u_\varphi \in C^{m,1-\rho(1-\nu-\lambda)}(0,a], & u_\psi \in C^{m,1-\rho(1-\nu-\lambda)}(0,a] & \text{for } 0 < \nu+\lambda < 1 \end{array} \right\}. \tag{4.4.23}$$

The estimates (4.4.19)-(4.4.22) follow from Theorem 4.4.2 and from the Lemmas 3.4.3 and 3.4.4.

□

Remark 4.4.1. *For the simplest transformation* (3.1.7)*,*

$$\varphi(t) = \frac{b}{a^\rho} t^\rho, \quad 0 \leq t \leq a, \quad a, b > 0, \quad \rho \geq 1,$$

we have $\varphi(t) = t$ if $a = b$ and $\rho = 1$. In this case $\varphi \in C^m[0,b]$, $m \in \mathbb{N}$. Theorems 4.4.3-4.4.5 establish the order of global convergence of a piecewise polynomial collocation method applied directly (without any change of variables) to the integral equation (4.1.1)*.*

Remark 4.4.2. *It follows from Theorems 4.4.3-4.4.5 that the accuracy*

$$\|u - u_N\|_\infty = \|u_\varphi - v_N\|_\infty \leq c\, N^{-m}$$

can be achieved on a mildly graded or uniform grid. For that we can choose ρ as large as necessary.

Chapter 5

Superconvergence Results

Theorems 4.4.3-4.4.5 suggest that by using a collocation method based on piecewise polynomials of degree $m-1$ ($m \in \mathbb{N}$) and graded grids of type (4.3.5), one can reach a convergence order

$$\|u - u_N\|_\infty \leq c\, N^{-m}, \quad N \geq N_0, \tag{5.0.1}$$

for sufficiently large values of the smoothing parameter ρ or the grid parameter r.

In (5.0.1) the order m cannot be improved so far as piecewise polynomials of the order $m-1$ are used for the approximation. In this chapter we will show that the convergence order at the points $\varphi(t_{ij})$, $i = 1, \ldots, N$, $j = 1, \ldots, m$, will be higher than $O(N^{-m})$ for a careful choice of collocation parameters (4.3.6) and for assuming somewhat higher regularity of the functions f and K. Here the function φ is a suitable smoothing transformation introduced in Chapter 3.

We use the idea which is employed in [39, 56, 64]. As the kernel K may have different types of singularities, there exist some technical differences in the proofs of the theorems of the current chapter. For ease of reading we have divided the chapter to different sections depending on the type of diagonal singularity of the kernel K. We will study superconvergence phenomena separately for the cases $0 < \nu < 1$, $\nu = 0$ and $\nu < 0$. The results of this chapter are partly published in [36, 37, 39].

5.1 Introduction

In this chapter, we assume that the collocation parameters

$$0 \leq \eta_1 < \cdots < \eta_m \leq 1 \tag{5.1.1}$$

(see (2.6.2) and (4.3.6)) are the node points of a quadrature formula

$$\int_0^1 g(s)\,ds = \sum_{j=1}^{m} w_j g(\eta_j) + E_m(g), \tag{5.1.2}$$

which is exact for all polynomials of degree m (i.e., the rest term $E_m(g)$ vanishes for each polynomial g of degree not greater than m). Actually, the weights $w_j = w_j^{(m)}$, $j = 1, \ldots, m$, of the formula (5.1.2) will not be used in our analysis. The assumption about the existence of a quadrature formula in the form (5.1.2) which is sharp for all polynomials of degree m is used in the proofs of Theorems 5.2.1-5.4.1.

Let $\Delta_N^{(r,a)}$, $N \in \mathbb{N}$, $r \geq 1$, be a graded grid (4.3.5) on $[0, a]$ and let t_{ij} be the collocation points (4.3.7),

$$t_{ij} = t_{i-1} + \eta_j(t_i - t_{i-1}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, m. \tag{5.1.3}$$

Assume that $\varphi \in \Phi^{m+1,\rho}$ satisfies the conditions (3.1.1)-(3.1.4) for $\rho \in [1, \infty)$. Let u be the exact solution of the equation (4.1.1) and u_φ be the exact solution of the equation (4.3.1), v_N is the approximation of u_φ found by collocation method (4.3.8)-(4.3.9), and u_N is the approximation of u defined by (4.3.10). We denote

$$\gamma_N^{(m,\nu,\lambda,\rho,r)} = \max_{\substack{i=1,\ldots,N\\ j=1,\ldots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| = \max_{\substack{i=1,\ldots,N\\ j=1,\ldots,m}} |u_\varphi(t_{ij}) - v_N(t_{ij})|. \tag{5.1.4}$$

Theorem 5.1.1. *Let $K \in W^{0,\nu}(D_b')$ or $K \in W^{0,\nu,\lambda}(D_b)$, $f \in C[0, b]$, $b \in (0, \infty)$, $\nu, \lambda \in (-\infty, 1)$ such that $\nu + \lambda < 1$. Furthermore, assume that $\varphi \in \Phi^{1,\rho}$ with $\rho \in [1, \infty)$, $a \in (0, \infty)$ and the interpolation nodes* (4.3.7) *in which the grid points* (4.3.5) *and the parameters* (4.3.6) *are used.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0$, a unique approximation v_N to u_φ, the solution of equation* (4.3.1)*, and a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following estimate*

$$\gamma_N^{(m,\nu,\lambda,\rho,r)} \leq c \sup_{t\in[0,a]} \left| \int_0^t K_\varphi(t, s)(u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s))\,ds \right| \tag{5.1.5}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,\nu,\lambda,\rho,r)}$ is defined by (5.1.4).

Proof. By Lemmas 4.4.1 and 4.4.2 the operator T_φ, defined by (4.3.16), is compact from $L^\infty(0, a)$ to $C[0, a]$. It follows from Theorem 4.4.2 that equation (4.3.19) has a unique solution v_N for $N \geq N_0$. For it and u_φ, the solution to (4.3.1), we have that

$$v_N = P_N^{(m-1)} T_\varphi v_N + P_N^{(m-1)} f_\varphi, \tag{5.1.6}$$

$$P_N^{(m-1)}u_\varphi = P_N^{(m-1)}T_\varphi u_\varphi + P_N^{(m-1)}f_\varphi. \tag{5.1.7}$$

After subtracting (5.1.7) and (5.1.6) we obtain

$$\begin{aligned} v_N - P_N^{(m-1)}u_\varphi &= P_N^{(m-1)}T_\varphi v_N - P_N^{(m-1)}T_\varphi u_\varphi \\ &= P_N^{(m-1)}T_\varphi(v_N - P_N^{(m-1)}u_\varphi) + P_N^{(m-1)}T_\varphi(P_N^{(m-1)}u_\varphi - u_\varphi). \end{aligned}$$

Hence

$$(I - P_N^{(m-1)}T_\varphi)(v_N - P_N^{(m-1)}u_\varphi) = P_N^{(m-1)}T_\varphi(P_N^{(m-1)}u_\varphi - u_\varphi), \quad N \geq N_0. \tag{5.1.8}$$

From (4.4.7) and (5.1.6) it follows that

$$\|v_N - P_N^{(m-1)}u_\varphi\|_\infty \leq c\,\|T_\varphi(P_N^{(m-1)}u_\varphi - u_\varphi)\|_\infty, \quad N \geq N_0, \tag{5.1.9}$$

with a constant $c > 0$ which is independent of N. Further, let u be the solution to (4.1.1) and u_N be the approximation for u, determined by (4.3.10). Then

$$\begin{aligned} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| &= |u_\varphi(t_{ij}) - v_N(t_{ij})| = |(P_N^{(m-1)}u_\varphi)(t_{ij}) - v_N(t_{ij})| \\ &\leq c\,\|v_N - P_N^{(m-1)}u_\varphi\|_\infty, \quad i = 1, \ldots, N, \quad j = 1, \ldots, m. \end{aligned} \tag{5.1.10}$$

Therefore, due to (5.1.4), (5.1.9) and (5.1.10),

$$\begin{aligned} \gamma_N^{(m,\nu,\lambda,\rho,r)} &\leq c\,\|T_\varphi(P_N^{(m-1)}u_\varphi - u_\varphi)\|_\infty \\ &= c \sup_{t\in[0,a]} \left| \int_0^t K_\varphi(t,s)\left(u_\varphi(s) - (P_N^{(m-1)}u_\varphi)(s)\right) ds \right|, \quad N \geq N_0. \end{aligned} \tag{5.1.11}$$

This completes the proof.

□

Remark 5.1.1. *If $K \in W^{0,\nu}(D_b')$, $\nu \in (-\infty, 1)$ and $\varphi(t) = t$, $t \in [0,b]$, then Theorem 5.1.1 coincides with Lemma 3.3.1. in [53].*

In the following theorems, we have to estimate $\gamma_N^{(m,\nu,\lambda,\rho,r)}$ defined by (5.1.4). For that we use the inequality (5.1.5), writing it in the following form:

$$\gamma_N^{(m,\nu,\lambda,\rho,r)} \leq c \max_{k=1,\ldots,N} \left| \sup_{t\in[t_{k-1},t_k]} |I_1^{(k)}(t)| + \sup_{t\in[t_{k-1},t_k]} |I_2^{(k)}(t)| \right|, \quad [t_{k-1}, t_k] \subset [0,a], \tag{5.1.12}$$

where

$$I_1^{(k)}(t) := \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} K_\varphi(t,s)\left[u_\varphi(s) - (P_N^{(m-1)}u_\varphi)(s)\right] ds, \quad t \in [t_{k-1}, t_k], \tag{5.1.13}$$

and

$$I_2^{(k)}(t) := \left(\int_{t_0}^{t_1} + \int_{t_{k-2}}^{t_{k-1}} + \int_{t_{k-1}}^{t} \right) K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1}, t_k]. \tag{5.1.14}$$

In the formula (5.1.14), we keep in mind that there is only the integral part $\int_{t_0}^{t}$ if $k = 1$ and there is only the integral part $\int_{t_0}^{t_1} + \int_{t_{k-1}}^{t}$ if $k = 2$. We also define

$$I_1^{(k)}(t) = 0, \qquad \text{if } k < 4. \tag{5.1.15}$$

In addition to the parameters $\eta_1, \ldots, \eta_m$ (see (5.1.1)) we introduce a parameter $\eta_{m+1} \in [0,1]$ such that $\eta_{m+1} \neq \eta_j$, $j = 1, \ldots, m$. The choice of η_{m+1} is arbitrary but we assume that it is somehow fixed. Using $\eta_1, \ldots, \eta_{m+1}$, we can define $m+1$ node points on every subinterval $[t_{i-1}, t_i]$:

$$t_{ij} = t_{i-1} + \eta_j (t_i - t_{i-1}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, m+1. \tag{5.1.16}$$

Similarly to $P_N^{(m-1)}$ let $P_N^{(m)}$ be an interpolation operator which assigns to any continuous function $z \in C[0,a]$ a piecewise polynomial function $P_N^{(m)} z$ so that $P_N^{(m)} z$ is on every subinterval $[t_{i-1}, t_i]$ a polynomial of degree not exceeding m and

$$(P_N^{(m)} z)(t_{ij}) = z(t_{ij}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, m+1. \tag{5.1.17}$$

As the quadrature formula (5.1.2) is exact for all polynomials of degree not exceeding m, we get

$$\int_{t_{i-1}}^{t_i} z(s)\, ds = \frac{t_i - t_{i-1}}{2} \sum_{j=1}^{m} w_j z(t_{ij}), \quad i = 1, \ldots, N, \tag{5.1.18}$$

where z is a polynomial of degree not exceeding m. Therefore

$$\int_{t_{i-1}}^{t_i} \left[(P_N^{(m-1)} u_\varphi)(s) - (P_N^{(m)} u_\varphi)(s) \right] ds = 0, \quad i = 1, \ldots, N. \tag{5.1.19}$$

5.2 Superconvergence: the Case $0 < \nu < 1$

Theorem 5.2.1. *For a given $m \in \mathbb{N}$, assume that $K \in W^{m+1,\nu,\lambda}(D_b)$ and $f \in C^{m+1,\nu+\lambda}(0,b]$ with $b \in (0,\infty)$, $\nu \in (0,1)$, $\lambda \in (-\infty,1)$ such that $\nu + \lambda < 1$. Let $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes* (5.1.3) *be generated by the graded grid* (4.3.5) *and by the collocation parameters $\eta_1, \ldots, \eta_m$ (see* (5.1.1)*) of a quadrature formula* (5.1.2) *which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0$, a unique approximation v_N to u_φ, the solution of equation* (4.3.1)*, and a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following estimate*

$$\gamma_N^{(m,\nu,\lambda,\rho,r)} = \max_{\substack{i=1,\ldots,N \\ j=1,\ldots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \leq c\, E_N^{(m,\nu,\lambda,\rho,r)} \tag{5.2.1}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,\nu,\lambda,\rho,r)}$ is defined by (5.1.4) *and*

$$E_N^{(m,\nu,\lambda,\rho,r)} = \left\{ \begin{array}{ll} N^{-(m+1-\nu)} & \text{for } r \geq \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \\ N^{-2r\rho\,(1-\nu-\lambda)} & \text{for } 1 \leq r < \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \end{array} \right\}. \tag{5.2.2}$$

Proof. By Theorem 4.4.1, equation (4.1.1) has a unique solution $u \in C[0,b]$ and equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$. Due to (4.4.23) we know that $u_\varphi \in C^{m+1,1-\rho(1-\nu-\lambda)}(0,a]$. Moreover, by Theorem 4.4.2 we have for $N \geq N_0$ a unique approximation v_N to u_φ. Since the conditions of Theorem 5.1.1 are fulfilled, we may use the inequalities (5.1.5) and (5.1.12):

$$\gamma_N^{(m,\nu,\lambda,\rho,r)} \leq c \max_{k=1,\ldots,N} \left| \sup_{t\in[t_{k-1},t_k]} |I_1^{(k)}(t)| + \sup_{t\in[t_{k-1},t_k]} |I_2^{(k)}(t)| \right|, \quad [t_{k-1},t_k] \subset [0,a], \tag{5.2.3}$$

where $I_1^{(k)}(t)$ is defined by (5.1.13) and $I_2^{(k)}(t)$ is defined by (5.1.14).

We fix the index k such that $t \in [t_{k-1},t_k]$, $k = 1,\ldots,N$. First, we estimate (5.1.14):

$$I_2^{(k)}(t) = \left(\int_{t_0}^{t_1} + \int_{t_{k-2}}^{t_{k-1}} + \int_{t_{k-1}}^{t} \right) K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1},t_k]. \tag{5.2.4}$$

For $i = 1, \ldots, k$ we consider the integral

$$\begin{aligned} I_3^{(i)}(t) &:= \left| \int\limits_{t_{i-1}}^{\min\{t_i,t\}} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \right| \\ &\le c \sup_{s\in[t_{i-1},t_i]} \left| u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right| \int\limits_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds. \end{aligned} \tag{5.2.5}$$

Note that $\min\{t_i, t\} = t_i$ for $i < k$ and $\min\{t_i, t\} = t$ for $i = k$, $t \in [t_{k-1}, t_k]$. Using (3.3.1) and (2.4.13), we obtain

$$\begin{aligned} \int\limits_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds &\le c \int\limits_{t_{i-1}}^{\min\{t_i,t\}} s^{(\rho-1)(1-\nu-\lambda)-\lambda} (t-s)^{-\nu}\, ds \\ &\le c' (t_i - t_{i-1})(t_k - t_{i-1})^{-\nu}\, t_i^{(\rho-1)(1-\nu-\lambda)-\lambda}, \quad i = 1, \ldots, k. \end{aligned} \tag{5.2.6}$$

Let $h_i = t_i - t_{i-1}$, $i = 1, \ldots, N$. Then we can write

$$(t_k - t_{i-1})^{-\nu} \le [(k-i+1)h_i]^{-\nu} = (k-i+1)^{-\nu} h_i^{-\nu}, \quad 0 < \nu < 1. \tag{5.2.7}$$

Due to (5.2.5), (5.2.6) and (5.2.7),

$$I_3^{(i)}(t) \le c\, h_i^{1-\nu}\, t_i^{\rho(1-\nu-\lambda)-(1-\nu)} (k-i+1)^{-\nu} \sup_{s\in[t_{i-1},t_i]} \left| u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right|.$$

Now we can use (3.4.6) for $u_\varphi \in C^{m+1,1-\rho(1-\nu-\lambda)}(0,a]$. This together with $h_i \le c\, N^{-1}(i/N)^{r-1}$ yields

$$\begin{aligned} I_3^{(i)}(t) &\le c\, h_i^{m+1-\nu} (k-i+1)^{-\nu} \left\{ \begin{array}{ll} t_i^{\rho(1-\nu-\lambda)-(1-\nu)} & , m < \rho(1-\nu-\lambda) \\ t_i^{\rho(1-\nu-\lambda)-(1-\nu)} (1 + |\log t_i|) & , m = \rho(1-\nu-\lambda) \\ t_i^{2\rho(1-\nu-\lambda)-(m+1-\nu)} & , m > \rho(1-\nu-\lambda) \end{array} \right\} \\ &\le c' N^{-(m+1-\nu)} \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+\rho(1-\nu-\lambda))-(m+1-\nu)} & , m < \rho(1-\nu-\lambda) \\ \left(\frac{i}{N}\right)^{r(m+\rho(1-\nu-\lambda))-(m+1-\nu)} (1 + |\log t_i|) & , m = \rho(1-\nu-\lambda) \\ \left(\frac{i}{N}\right)^{2r\rho(1-\nu-\lambda)-(m+1-\nu)} & , m > \rho(1-\nu-\lambda) \end{array} \right\}. \end{aligned} \tag{5.2.8}$$

If $m \le \rho(1-\nu-\lambda)$ then due to $0 < 1-\nu < 1$ we have

$$r(m + \rho(1-\nu-\lambda)) - (m+1-\nu) \ge 2rm - m - (1-\nu) > 0.$$

Thus

$$\left(\frac{i}{N}\right)^{r(m+\rho(1-\nu-\lambda))-(m+1-\nu)} (1 + |\log t_i|) \le c$$

and

$$I_3^{(i)}(t) \le c\, N^{-(m+1-\nu)}, \quad \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} < 1 \le r, \quad m \le \rho(1-\nu-\lambda).$$

If $m > \rho(1-\nu-\lambda)$ then

$$N^{-(m+1-\nu)} \left(\frac{i}{N}\right)^{2r\rho(1-\nu-\lambda)-(m+1-\nu)} \le c \left\{ \begin{array}{ll} N^{-(m+1-\nu)} & \text{for } r \ge \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \\ N^{-2r\rho(1-\nu-\lambda)} & \text{for } 1 \le r < \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \end{array} \right\}.$$

We obtained

$$I_3^{(i)}(t) \le c\, E_N^{(m,\nu,\lambda,\rho,r)}, \quad i = 1, \dots, k, \quad t \in [t_{k-1}, t_k]. \tag{5.2.9}$$

Due to (5.2.4) and (5.2.9) we have

$$|I_2^{(k)}(t)| \le I_3^{(1)}(t) + I_3^{(k-1)}(t) + I_3^{(k)}(t) \le c\, E_N^{(m,\nu,\lambda,\rho,r)},\ t \in [t_{k-1}, t_k],\ k = 1, \dots, N. \tag{5.2.10}$$

Next, we consider $I_1^{(k)}(t)$ for $k = 4, \dots, N$,

$$I_1^{(k)}(t) = \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1}, t_k]. \tag{5.2.11}$$

Using (5.1.19), we can write

$$\begin{aligned} I_1^{(k)}(t) = & \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left[K_\varphi(t,s) - K_\varphi(t,t_{i/2}) \right] \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \\ & + \sum_{i=2}^{k-2} K_\varphi(t,t_{i/2}) \int_{t_{i-1}}^{t_i} \left[u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right] ds, \end{aligned} \tag{5.2.12}$$

where $t_{i/2} = \frac{t_{i-1}+t_i}{2}$, $i = 2, \dots, k-2$; $k = 4, \dots, N$. We have for any $s \in [t_{i-1}, t_i]$, $i = 2, \dots, k-2$,

$$K_\varphi(t,s) - K_\varphi(t,t_{i/2}) = (s - t_{i/2}) \frac{\partial}{\partial s} K_\varphi(t,s) \Big|_{s=\xi}, \quad \xi \in (s, t_{i/2}). \tag{5.2.13}$$

By Lemma 3.3.4 we obtain for $\xi \in (s, t_{i/2})$ that

$$|K_\varphi(t,s) - K_\varphi(t,t_{i/2})| \le c\,(t_i - t_{i-1})\, \xi^{(\rho-1)(1-\nu-\lambda)-\lambda} (t-\xi)^{-\nu} [\xi^{-1} + (t-\xi)^{-1}]. \tag{5.2.14}$$

Since $t \in [t_{k-1}, t_k]$, $s \in [t_{i-1}, t_i]$, $\xi \in (s, t_{i/2})$, $i = 2, \dots, k-2$, $k \ge 4$, we have

$$c_1 \le \frac{\xi}{t_i} \le c_2, \quad c_3 \le \frac{t-\xi}{t_k - t_{i-1}} \le c_4,$$

where $c_2 \geq c_1 > 0$ and $c_4 \geq c_3 > 0$ are some constants which do not depend on i and k (see (2.4.8) and (2.4.9)). Thus for $i = 2, \ldots, k-2$ we obtain

$$\begin{aligned} &\int_{t_{i-1}}^{t_i} |K_\varphi(t,s) - K_\varphi(t,t_{i/2})|\, ds \\ &\leq c\,(t_i - t_{i-1})^2\, t_i^{(\rho-1)(1-\nu-\lambda)-\lambda}(t_k - t_{i-1})^{-\nu}[t_i^{-1} + (t_k - t_{i-1})^{-1}]. \end{aligned} \tag{5.2.15}$$

Due to (3.4.6) and (5.2.15) we have

$$\begin{aligned} I_4^{(k)}(t) &:= \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left|K_\varphi(t,s) - K_\varphi(t,t_{i/2})\right| \left|u_\varphi(s) - (P_N^{(m-1)}u_\varphi)(s)\right| ds \\ &\leq c \sum_{i=2}^{k-2} h_i^{m+2} t_i^{\rho(1-\nu-\lambda)-(1-\nu)}(t_k - t_{i-1})^{-\nu}[t_i^{-1} + (t_k - t_{i-1})^{-1}]\times \\ &\qquad \times \left\{ \begin{array}{ll} 1 & , m < \rho(1-\nu-\lambda) \\ 1 + |\log t_i| & , m = \rho(1-\nu-\lambda) \\ t_i^{\rho(1-\nu-\lambda)-m} & , m > \rho(1-\nu-\lambda) \end{array} \right\}. \end{aligned} \tag{5.2.16}$$

Since $h_i t_i^{-1} \leq c\, i^{-1}$ and $h_i(t_k - t_{i-1})^{-1} \leq c\,(k-i+1)^{-1}$, $i = 2, \ldots, k-2$, we have

$$\begin{aligned} I_4^{(k)}(t) &\leq c \sum_{i=2}^{k-2} h_i^{m+1-\nu}(k-i+1)^{-\nu}[i^{-1} + (k-i+1)^{-1}]\times \\ &\qquad \times \left\{ \begin{array}{ll} t_i^{\rho(1-\nu-\lambda)-(1-\nu)} & , m < \rho(1-\nu-\lambda) \\ t_i^{\rho(1-\nu-\lambda)-(1-\nu)}(1 + |\log t_i|) & , m = \rho(1-\nu-\lambda) \\ t_i^{2\rho(1-\nu-\lambda)-(m+1-\nu)} & , m > \rho(1-\nu-\lambda) \end{array} \right\} \\ &\leq c^{'}\, N^{-(m+1-\nu)} \sum_{i=2}^{k-2} [i^{-1}(k-i+1)^{-\nu} + (k-i+1)^{-1-\nu}]\times \\ &\qquad \times \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+\rho(1-\nu-\lambda))-(m+1-\nu)} & , m < \rho(1-\nu-\lambda) \\ \left(\frac{i}{N}\right)^{r(m+\rho(1-\nu-\lambda))-(m+1-\nu)}(1 + |\log t_i|) & , m = \rho(1-\nu-\lambda) \\ \left(\frac{i}{N}\right)^{2r\rho(1-\nu-\lambda)-(m+1-\nu)} & , m > \rho(1-\nu-\lambda) \end{array} \right\}. \end{aligned} \tag{5.2.17}$$

If $m \leq \rho(1-\nu-\lambda)$ then $r(m + \rho(1-\nu-\lambda)) - (m+1-\nu) > 0$ and

$$I_4^{(k)}(t) \leq c\, N^{-(m+1-\nu)} \sum_{i=2}^{k-2} [i^{-1}(k-i+1)^{-\nu} + (k-i+1)^{-1-\nu}] \leq c^{'} N^{-(m+1-\nu)}. \tag{5.2.18}$$

If $m > \rho(1-\nu-\lambda)$ then (5.2.18) holds for $r \geq \frac{m+1-\nu}{2\rho(1-\nu-\lambda)}$. We denote

$$\gamma := 2r\rho(1-\nu-\lambda) - (m+1-\nu).$$

Then for $1 \le r < \frac{m+1-\nu}{2\rho(1-\nu-\lambda)}$ we have $\gamma < 0$ and

$$I_4^{(k)}(t) \le c\, N^{-2\rho(1-\nu-\lambda)} \sum_{i=2}^{k-2} \left[i^{\gamma-1}(k-i+1)^{-\nu} + i^{\gamma}(k-i+1)^{-1-\nu} \right] \le c^{'}\, N^{-2r\rho(1-\lambda-\nu)}. \tag{5.2.19}$$

From (5.2.18) and (5.2.19) we obtain

$$I_4^{(k)}(t) \le c\, E_N^{(m,\nu,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \ldots, N. \tag{5.2.20}$$

Finally, we consider

$$I_5^{(k)}(t) := \sum_{i=2}^{k-2} |K_\varphi(t, t_{i/2})| \int_{t_{i-1}}^{t_i} \left| u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right| ds, \quad t_{i/2} \in [t_{i-1}, t_i], t \in [t_{k-1}, t_k]. \tag{5.2.21}$$

By (3.4.6) and Lemma 3.3.1 we have

$$\begin{aligned}
I_5^{(k)}(t) &\le c \sum_{i=2}^{k-2} h_i^{m+2} t_i^{(\rho-1)(1-\nu-\lambda)-\lambda} (t_k - t_{i-1})^{-\nu} \times \\
&\times \left\{ \begin{array}{ll} 1 & , m+1 < \rho(1-\nu-\lambda) \\ 1 + |\log t_i| & , m+1 = \rho(1-\nu-\lambda) \\ t_i^{\rho(1-\nu-\lambda)-m-1} & , m+1 > \rho(1-\nu-\lambda) \end{array} \right\} \\
&\le c^{'} \sum_{i=2}^{k-2} h_i^{m+2-\nu} (k-i+1)^{-\nu} \times \\
&\times \left\{ \begin{array}{ll} t_i^{\rho(1-\nu-\lambda)-(1-\nu)} & , m+1 < \rho(1-\nu-\lambda) \\ t_i^{\rho(1-\nu-\lambda)-(1-\nu)}(1 + |\log t_i|) & , m+1 = \rho(1-\nu-\lambda) \\ t_i^{2\rho(1-\nu-\lambda)-m-2+\nu} & , m+1 > \rho(1-\nu-\lambda) \end{array} \right\} \\
&\le c^{''}\, N^{-m-2+\nu} \sum_{i=2}^{k-2} (k-i+1)^{-\nu} \times \\
&\times \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+1+\rho(1-\nu-\lambda))-m-2+\nu} & , m+1 < \rho(1-\nu-\lambda) \\ \left(\frac{i}{N}\right)^{r(m+1+\rho(1-\nu-\lambda))-m-2+\nu} (1 + |\log \frac{i}{N}|) & , m+1 = \rho(1-\nu-\lambda) \\ \left(\frac{i}{N}\right)^{2r\rho(1-\nu-\lambda)-m-2+\nu} & , m+1 > \rho(1-\nu-\lambda) \end{array} \right\}.
\end{aligned} \tag{5.2.22}$$

If $m+1 \le \rho(1-\nu-\lambda)$ then $r(m+1+\rho(1-\nu-\lambda)) - m - 2 + \nu > 0$ and

$$I_5^{(k)}(t) \le c\, N^{-m-2+\nu} \sum_{i=2}^{k-2} (k-i+1)^{-\nu} \le c^{'}\, N^{-m-1}. \tag{5.2.23}$$

If $m+1 > \rho(1-\nu-\lambda)$ then we can write

$$\begin{aligned} I_5^{(k)}(t) &\le c\, N^{-2r\rho(1-\nu-\lambda)} \sum_{i=2}^{k-2} i^{2r\rho(1-\nu-\lambda)-m-2+\nu}(k-i+1)^{-\nu} \\ &\le c' \left\{ \begin{array}{ll} N^{-(m+1-\nu)} & \text{for } r \ge \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \\ N^{-2r\rho(1-\nu-\lambda)} & \text{for } 1 \le r < \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \end{array} \right\}. \end{aligned} \tag{5.2.24}$$

From (5.2.22)-(5.2.24) we obtain

$$I_5^{(k)}(t) \le c\, E_N^{(m,\nu,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \dots, N. \tag{5.2.25}$$

Due to (5.2.12), (5.2.16), (5.2.20), (5.2.21) and (5.2.25),

$$|I_1^{(k)}(t)| \le c(I_4^{(k)}(t) + I_5^{(k)}(t)) \le c'\, E_N^{(m,\nu,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \dots, N. \tag{5.2.26}$$

Estimate (5.2.1) follows from (5.2.3), (5.2.10) and (5.2.26).

□

Remark 5.2.1. *If $0 < \nu + \lambda < 1$ then Theorem 5.2.1 holds also for any transformation $\varphi \in \Phi^{m+1,\rho}$. This follows from the proof of Theorem 4.4.5 (see (4.4.23)). In this case the solution $u_\varphi(t) = u(\varphi(t))$ of the equation (4.3.1) belongs to the set $C^{m+1,1-\rho(1-\nu-\lambda)}(0,a]$ for the transformations $\varphi \in \Phi^{m+1,\rho}$ and for $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$ as well. Thus all the estimates, which we use in the proof of Theorem 5.2.1, hold.*

Theorem 5.2.2. *For a given $m \in \mathbb{N}$, assume that $K \in W^{m+1,\nu,\lambda}(D_b)$ and $f \in C_*^{m+1,\nu+\lambda}(0,b]$ with $b \in (0,\infty)$, $\nu \in (0,1)$, $\lambda \in (-\infty,1)$ such that $\nu + \lambda < 1$. Let $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes (5.1.3) be generated by the graded grid (4.3.5) and by the collocation parameters $\eta_1, \dots, \eta_m$ (see (5.1.1)) of a quadrature formula (5.1.2) which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) determine for sufficiently large values of N, say $N \ge N_0 \ge 2$, a unique approximation v_N to u_φ, the solution of equation (4.3.1), and a unique approximation u_N to u, the solution of equation (4.1.1), and the following estimate

$$\gamma_N^{(m,\nu,\lambda,\rho,r)} = \max_{\substack{i=1,\dots,N \\ j=1,\dots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \le c\, E_N^{(m,\nu,\lambda,\rho,r)} \tag{5.2.27}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,\nu,\lambda,\rho,r)}$ is defined by (5.1.4) and

$$E_N^{(m,\nu,\lambda,\rho,r)} = \left\{ \begin{array}{ll} N^{-(m+1-\nu)} & \text{for } r > \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \\ N^{-2r\rho(1-\nu-\lambda)} \log N & \text{for } 1 \le r \le \frac{m+1-\nu}{2\rho(1-\nu-\lambda)} \end{array} \right\}. \tag{5.2.28}$$

Proof. By Theorem 4.4.1, the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$. Due to (4.4.18) we know that $u_\varphi \in C_*^{m+1,1-\rho(1-\nu-\lambda)}(0,a]$. Moreover, by Theorem 4.4.2 we have for $N \geq N_0$ a unique approximation v_N to u_φ. Since the conditions of Theorem 5.1.1 are fulfilled, we may use the inequalities (5.1.5) and (5.1.12):

$$\gamma_N^{(m,\nu,\lambda,\rho,r)} \leq c \max_{k=1,\dots,N} \left| \sup_{t\in[t_{k-1},t_k]} |I_1^{(k)}(t)| + \sup_{t\in[t_{k-1},t_k]} |I_2^{(k)}(t)| \right|, \quad [t_{k-1},t_k] \subset [0,a], \tag{5.2.29}$$

where $I_1^{(k)}(t)$ is defined by (5.1.13) and $I_2^{(k)}(t)$ is defined by (5.1.14).

Next, we can follow the proof of Theorem 5.2.1, since the estimates for the kernels K_φ remain the same and for

$$\sup_{s\in[t_{i-1},t_i]} \left| u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right|, \quad i = 1,\dots,N,$$

instead of (3.4.6) we use the estimates (3.4.7):

$$\begin{aligned} &\sup_{s\in[t_{i-1},t_i]} |u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s)| \\ &\qquad \leq c\, h_i^m \left\{ \begin{array}{ll} 1 & , m < \rho(1-\nu-\lambda) \\ t_i^{\rho(1-\nu-\lambda)-m}(1+|\log t_i|) & , m \geq \rho(1-\nu-\lambda) \end{array} \right\}. \end{aligned} \tag{5.2.30}$$

□

Remark 5.2.2. *If $0 \leq \nu + \lambda < 1$ then Theorem 5.2.2 holds also for any transformation $\varphi \in \Phi^{m+1,\rho}$. This follows from the proof of Theorem 4.4.4 (see (4.4.18)). In this case the solution $u_\varphi(t) = u(\varphi(t))$ of the equation (4.3.1) belongs to the set $C_*^{m+1,1-\rho(1-\nu-\lambda)}(0,a]$ for the transformations $\varphi \in \Phi^{m+1,\rho}$ and for $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$ as well. Thus all the estimates, which we use in the proof of Theorem 5.2.2, hold.*

Theorem 5.2.3. *For a given $m \in \mathbb{N}$, assume that $K \in W^{m+1,\nu}(D_b')$ and $f \in C^{m+1,\nu}(0,b]$ with $b \in (0,\infty)$, $\nu \in (0,1)$. Let $\varphi \in \Phi^{m+1,\rho}$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes (5.1.3) be generated by the graded grid (4.3.5) and by the collocation parameters $\eta_1,\dots,\eta_m$ (see (5.1.1)) of a quadrature formula (5.1.2) which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) determine for sufficiently large values of N, say $N \geq N_0$, a unique approximation v_N to u_φ, the solution of equation (4.3.1), and a unique approximation u_N to u, the solution of equation (4.1.1), and the following estimate

$$\gamma_N^{(m,\nu,0,\rho,r)} = \max_{\substack{i=1,\dots,N \\ j=1,\dots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \leq c\, E_N^{(m,\nu,\rho,r)} \tag{5.2.31}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,\nu,0,\rho,r)}$ is defined by (5.1.4) *and*

$$E_N^{(m,\nu,\rho,r)} = \left\{ \begin{array}{ll} N^{-(m+1-\nu)} & \text{for } r \geq \frac{m+1-\nu}{2\rho(1-\nu)} \\ N^{-2r\rho\,(1-\nu)} & \text{for } 1 \leq r < \frac{m+1-\nu}{2\rho(1-\nu)} \end{array} \right\}. \tag{5.2.32}$$

Proof. Since $W^{m+1,\nu}(D_b') \subset W^{m+1,\nu,0}(D_b)$ for $0 < \nu < 1$ then (5.2.31) follows from Theorem 5.2.1 and Remark 5.2.1 where we take $\lambda = 0$.

□

5.3 Superconvergence: the Case $\nu = 0$

In this section, we often use the following inequality:

$$\left(\frac{i}{N}\right)^{\gamma}(1 + |\log t_i|) \leq c\left(\frac{i}{N}\right)^{\gamma}(1 + |\log \frac{i}{N}|) \leq c', \quad i = 1, \ldots, N, \tag{5.3.1}$$

where $\gamma > 0$, t_i is given by formula (4.3.5), c and c' are some positive constants not depending on i and N.

Theorem 5.3.1. *For a given $m \in \mathbb{N}$, assume that $K \in W_*^{m+1,0,\lambda}(D_b)$ and $f \in C^{m+1,\lambda}(0,b]$ with $b \in (0,\infty)$, $\lambda \in (0,1)$. Let $\varphi \in \Phi^{m+1,\rho}$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes* (5.1.3) *be generated by the graded grid* (4.3.5) *and by the collocation parameters $\eta_1, \ldots, \eta_m$ (see* (5.1.1)*) of a quadrature formula* (5.1.2) *which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0 \geq 2$, a unique approximation v_N to u_φ, the solution of equation* (4.3.1)*, and a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following estimate*

$$\gamma_N^{(m,0,\lambda,\rho,r)} = \max_{\substack{i=1,\ldots,N \\ j=1,\ldots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \leq c\, E_N^{(m,0,\lambda,\rho,r)} \tag{5.3.2}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,0,\lambda,\rho,r)}$ is defined by (5.1.4) *and*

$$E_N^{(m,0,\lambda,\rho,r)} = \left\{ \begin{array}{ll} N^{-m-1}\log N & \text{for } r \geq \frac{m+1}{2\rho(1-\lambda)} \\ N^{-2r\rho\,(1-\lambda)} & \text{for } 1 \leq r < \frac{m+1}{2\rho(1-\lambda)} \end{array} \right\}. \tag{5.3.3}$$

Proof. By Theorem 4.4.1, the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$. Due to (4.4.23) we

know that $u_\varphi \in C^{m+1,1-\rho(1-\lambda)}(0,a]$. Moreover, by Theorem 4.4.2 we have for $N \geq N_0$ a unique approximation v_N to u_φ. Due to Theorem 5.1.1 we may use the inequalities (5.1.5) and (5.1.12):

$$\gamma_N^{(m,0,\lambda,\rho,r)} \leq c \max_{k=1,\dots,N} \left| \sup_{t\in[t_{k-1},t_k]} |I_1^{(k)}(t)| + \sup_{t\in[t_{k-1},t_k]} |I_2^{(k)}(t)| \right|, \quad [t_{k-1},t_k] \subset [0,a], \tag{5.3.4}$$

where $I_1^{(k)}(t)$ is defined by (5.1.13) and $I_2^{(k)}(t)$ is defined by (5.1.14).

We fix the index k such that $t \in [t_{k-1},t_k]$, $k = 1,\dots,N$. First, we estimate (5.1.14):

$$I_2^{(k)}(t) = \left(\int_{t_0}^{t_1} + \int_{t_{k-2}}^{t_{k-1}} + \int_{t_{k-1}}^{t} \right) K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1},t_k]. \tag{5.3.5}$$

In analogy to the proof of Theorem 5.2.1 we consider for $i = 1,\dots,k$ the integral

$$\begin{aligned} I_3^{(i)}(t) &:= \left| \int_{t_{i-1}}^{\min\{t_i,t\}} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \right| \\ &\leq c \sup_{s\in[t_{i-1},t_i]} \left| u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right| \int_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds. \end{aligned} \tag{5.3.6}$$

Note that $\min\{t_i,t\} = t_i$ for $i < k$ and $\min\{t_i,t\} = t$ for $i = k$, $t \in [t_{k-1},t_k]$. Using (3.3.4) and (2.4.15), we obtain

$$\begin{aligned} \int_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds &\leq c \int_{t_{i-1}}^{\min\{t_i,t\}} s^{(\rho-1)(1-\lambda)-\lambda} \frac{1+|\log(t-s)|}{1+|\log s|}\, ds \\ &\leq c' \,(t_i - t_{i-1})(1+|\log(t_k - t_{i-1})|) \frac{t_i^{\rho(1-\lambda)-1}}{1+|\log t_i|}. \end{aligned} \tag{5.3.7}$$

Let $h_i = t_i - t_{i-1}$, $i = 1,\dots,N$. Due to (5.3.6), (3.4.1) and (5.3.7),

$$I_3^{(i)}(t) \leq c\, h_i^{m+1} t_i^{\rho(1-\lambda)-m} t_i^{\rho(1-\lambda)-1} \frac{1+|\log(t_i - t_{i-1})|}{1+|\log t_i|}. \tag{5.3.8}$$

From (2.4.6) we get

$$I_3^{(i)}(t) \leq c\, N^{-m-1} \left(\frac{i}{N} \right)^{2r\rho(1-\lambda)-m-1} \frac{1+|\log h_i|}{1+|\log t_i|}. \tag{5.3.9}$$

With the help of (2.4.10) we obtain

$$I_3^{(i)}(t) \le c \begin{cases} N^{-m-1}\log N & \text{for } r \ge \frac{m+1}{2\rho(1-\lambda)} \\ N^{-2r\rho(1-\lambda)} & \text{for } 1 \le r < \frac{m+1}{2\rho(1-\lambda)}. \end{cases}$$

Therefore

$$I_3^{(i)}(t) \le c\, E_N^{(m,0,\lambda,\rho,r)}, \quad i = 1,\dots,k, \quad t \in [t_{k-1}, t_k]. \tag{5.3.10}$$

Due to (5.3.5) and (5.3.10) we have

$$|I_2^{(k)}(t)| \le I_3^{(1)}(t) + I_3^{(k-1)}(t) + I_3^{(k)}(t) \le c\, E_N^{(m,0,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k],\ k = 1,\dots,N. \tag{5.3.11}$$

Next, we consider $I_1^{(k)}(t)$ for $k = 4,\dots,N$,

$$I_1^{(k)}(t) = \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1}, t_k]. \tag{5.3.12}$$

Using (5.1.19), we can write

$$\begin{aligned} I_1^{(k)}(t) = &\sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left[K_\varphi(t,s) - K_\varphi(t,t_{i/2}) \right] \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \\ &+ \sum_{i=2}^{k-2} K_\varphi(t,t_{i/2}) \int_{t_{i-1}}^{t_i} \left[u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right] ds, \end{aligned} \tag{5.3.13}$$

where $t_{i/2} = \frac{t_{i-1}+t_i}{2}$, $i = 2,\dots,k-2$; $k = 4,\dots,N$. We have for any $s \in [t_{i-1}, t_i]$, $i = 2,\dots,k-2$,

$$K_\varphi(t,s) - K_\varphi(t,t_{i/2}) = (s - t_{i/2}) \frac{\partial}{\partial s} K_\varphi(t,s) \Big|_{s=\xi}, \quad \xi \in (s, t_{i/2}). \tag{5.3.14}$$

By Lemma 3.3.5 we obtain for $\xi \in (s, t_{i/2})$ that

$$|K_\varphi(t,s) - K_\varphi(t,t_{i/2})| \le c(t_i - t_{i-1}) \frac{\xi^{(\rho-1)(1-\lambda)-\lambda}[\xi^{-1}(1 + |\log(t-\xi)|) + (t-\xi)^{-1}]}{1 + |\log \xi|}. \tag{5.3.15}$$

Since $t \in [t_{k-1}, t_k]$, $s \in [t_{i-1}, t_i]$, $\xi \in (s, t_{i/2})$, $i = 2,\dots,k-2$, $k \ge 4$, we have

$$c_1 \le \frac{\xi}{t_i} \le c_2, \quad c_3 \le \frac{t-\xi}{t_k - t_{i-1}} \le c_4,$$

where $c_2 \geq c_1 > 0$ and $c_4 \geq c_3 > 0$ are some constants which do not depend on i and k (see (2.4.8) and (2.4.9)). Hence for $i = 2, \ldots, k-2$ we have

$$\int_{t_{i-1}}^{t_i} |K_\varphi(t,s) - K_\varphi(t,t_{i/2})|\, ds$$
$$\leq c\,(t_i - t_{i-1})^2\, t_i^{(\rho-1)(1-\lambda)-\lambda}\, \frac{t_i^{-1}(1 + |\log(t_k - t_{i-1})|) + (t_k - t_{i-1})^{-1}}{1 + |\log t_i|}. \tag{5.3.16}$$

Due to (3.4.1) and (5.3.16),

$$I_4^{(k)}(t) := \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left|K_\varphi(t,s) - K_\varphi(t,t_{i/2})\right| \left|u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s)\right| ds$$
$$\leq c \sum_{i=2}^{k-2} h_i^{m+2} t_i^{\rho(1-\lambda)-m}\, \frac{t_i^{\rho(1-\lambda)-1}[t_i^{-1}(1 + |\log(t_k - t_{i-1})|) + (t_k - t_{i-1})^{-1}]}{1 + |\log t_i|}. \tag{5.3.17}$$

Since $h_i t_i^{-1} \leq c\, i^{-1}$ and $h_i (t_k - t_{i-1})^{-1} \leq c\,(k-i+1)^{-1}$, $i = 2, \ldots, k-2$, we have

$$I_4^{(k)}(t) \leq c \sum_{i=2}^{k-2} h_i^{m+1} t_i^{2\rho(1-\lambda)-m-1}\, \frac{t_i^{-1} h_i (1 + |\log(t_k - t_{i-1})|) + (k-i+1)^{-1}}{1 + |\log t_i|}$$
$$\leq c^{'}\, N^{-m-1} \sum_{i=2}^{k-2} \left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-1} \frac{i^{-1}(1 + |\log(t_k - t_{i-1})|) + (k-i+1)^{-1}}{1 + |\log t_i|}. \tag{5.3.18}$$

Now, for $r \geq \frac{m+1}{2\rho(1-\lambda)}$ we have $2r\rho(1-\lambda) - m - 1 \geq 0$ and

$$I_4^{(k)}(t) \leq c\, N^{-m-2} \sum_{i=2}^{k-2} \left(\frac{i}{N}\right)^{-1} \frac{1 + |\log(t_k - t_{i-1})|}{1 + |\log t_i|} + c\, N^{-m-1} \sum_{i=2}^{k-2} (k-i+1)^{-1}. \tag{5.3.19}$$

By Lemma 2.4.2 we obtain

$$I_4^{(k)}(t) \leq c\, N^{-m-2}\, N \log N + c\, N^{-m-1} \log N \leq c^{'}\, N^{-m-1} \log N. \tag{5.3.20}$$

If $1 \leq r < \frac{m+1}{2\rho(1-\lambda)}$ then from (5.3.18) we have

$$I_4^{(k)}(t) \leq c\, N^{-m-2} \sum_{i=2}^{k-2} \left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-2} \frac{1 + |\log(t_k - t_{i-1})|}{1 + |\log t_i|}$$
$$+ c\, N^{-2r\rho(1-\lambda)} \sum_{i=2}^{k-2} i^{2r\rho(1-\lambda)-m-1} (k-i+1)^{-1}. \tag{5.3.21}$$

Since $2r\rho(1-\lambda) - m - 2 < -1$, we obtain by Lemma 2.4.2 that

$$I_4^{(k)}(t) \leq c\, N^{-m-2}\, N^{-2r\rho(1-\lambda)+m+2} + c\, N^{-2r\rho(1-\lambda)} \leq c^{'}\, N^{-2r\rho(1-\lambda)}. \tag{5.3.22}$$

Summarizing (5.3.20) and (5.3.22):

$$I_4^{(k)}(t) \leq c\, E_N^{(m,0,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \ldots, N. \tag{5.3.23}$$

Finally, for $t \in [t_{k-1}, t_k]$, $k = 4, \ldots, N$, we consider

$$I_5^{(k)}(t) := \sum_{i=2}^{k-2} |K_\varphi(t, t_{i/2})| \int\limits_{t_{i-1}}^{t_i} \left| u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right| ds, \quad t_{i/2} \in [t_{i-1}, t_i]. \tag{5.3.24}$$

By (3.4.1), (2.4.6) and Lemma 3.3.2 we have

$$\begin{aligned} I_5^{(k)}(t) &\leq c \sum_{i=2}^{k-2} h_i^{m+2} t_i^{\rho(1-\lambda)-m-1} t_i^{\rho(1-\lambda)-1} \frac{1 + |\log(t - t_{i/2})|}{1 + |\log t_{i/2}|} \\ &\leq c^{'}\, N^{-m-2} \sum_{i=2}^{k-2} \left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-2} \frac{1 + |\log(t_k - t_{i-1})|}{1 + |\log t_i|}. \end{aligned} \tag{5.3.25}$$

By Lemma 2.4.2,

$$I_5^{(k)}(t) \leq c \left\{ \begin{array}{ll} N^{-m-1} \log N & \text{for } r \geq \frac{m+1}{2\rho(1-\lambda)} \\ N^{-2r\rho(1-\lambda)} & \text{for } 1 \leq r < \frac{m+1}{2\rho(1-\lambda)} \end{array} \right\} = c\, E_N^{(m,0,\lambda,\rho,r)}. \tag{5.3.26}$$

By (5.3.13), (5.3.17), (5.3.23), (5.3.24) and (5.3.26),

$$|I_1^{(k)}(t)| \leq c(I_4^{(k)}(t) + I_5^{(k)}(t)) \leq c^{'}\, E_N^{(m,0,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \ldots, N. \tag{5.3.27}$$

Estimate (5.3.2) follows from (5.3.4), (5.3.11) and (5.3.27).

□

Theorem 5.3.2. *For a given $m \in \mathbb{N}$, assume that $K \in W_*^{m+1,0,\lambda}(D_b)$ and $f \in C^{m+1,\lambda}(0,b]$ with $b \in (0,\infty)$, $\lambda \in (-\infty, 0]$. Let $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes* (5.1.3) *be generated by the graded grid* (4.3.5) *and by the collocation parameters $\eta_1, \ldots, \eta_m$ (see* (5.1.1)*) of a quadrature formula* (5.1.2) *which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0 \geq 2$, a unique approximation v_N to u_φ, the solution of equation* (4.3.1)*, and a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following estimate*

$$\gamma_N^{(m,0,\lambda,\rho,r)} = \max_{\substack{i=1,\ldots,N \\ j=1,\ldots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \leq c\, E_N^{(m,0,\lambda,\rho,r)} \tag{5.3.28}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,0,\lambda,\rho,r)}$ is defined by (5.1.4) *and*

$$E_N^{(m,0,\lambda,\rho,r)} = \left\{ \begin{array}{ll} N^{-m-1}\log N & \text{for } r > \frac{m+1}{2\rho(1-\lambda)} \\ N^{-m-1}\log^2 N & \text{for } r = \frac{m+1}{2\rho(1-\lambda)} \\ N^{-2r\rho\,(1-\lambda)}\log N & \text{for } 1 \le r < \frac{m+1}{2\rho(1-\lambda)} \end{array} \right\}. \tag{5.3.29}$$

Proof. By Theorem 4.4.1, the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$. Due to (4.4.23) we know that $u_\varphi \in C^{m+1,1-\rho(1-\lambda)}(0,a]$. Moreover, by Theorem 4.4.2 we have for $N \ge N_0$ a unique approximation v_N to u_φ. Since the conditions of Theorem 5.1.1 are fulfilled, we may use the inequalities (5.1.5) and (5.1.12):

$$\gamma_N^{(m,0,\lambda,\rho,r)} \le c \max_{k=1,\dots,N} \left| \sup_{t\in[t_{k-1},t_k]} |I_1^{(k)}(t)| + \sup_{t\in[t_{k-1},t_k]} |I_2^{(k)}(t)| \right|, \quad [t_{k-1},t_k] \subset [0,a], \tag{5.3.30}$$

where $I_1^{(k)}(t)$ is defined by (5.1.13) and $I_2^{(k)}(t)$ is defined by (5.1.14).

We fix the index k such that $t \in [t_{k-1},t_k]$, $k = 1,\dots,N$. First, we estimate (5.1.14):

$$I_2^{(k)}(t) = \left(\int_{t_0}^{t_1} + \int_{t_{k-2}}^{t_{k-1}} + \int_{t_{k-1}}^{t} \right) K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1},t_k]. \tag{5.3.31}$$

In analogy to the proofs of Theorems 5.2.1 and 5.3.1 we consider for $i = 1,\dots,k$ the integral

$$\begin{aligned} I_3^{(i)}(t) &:= \left| \int_{t_{i-1}}^{\min\{t_i,t\}} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \right| \\ &\le c \sup_{s\in[t_{i-1},t_i]} \left| u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right| \int_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds. \end{aligned} \tag{5.3.32}$$

Note that $\min\{t_i,t\} = t_i$ for $i < k$ and $\min\{t_i,t\} = t$ for $i = k$, $t \in [t_{k-1},t_k]$. Using (3.3.1) and (2.4.14), we obtain

$$\begin{aligned} \int_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds &\le c \int_{t_{i-1}}^{\min\{t_i,t\}} s^{(\rho-1)(1-\lambda)-\lambda} \left(1 + |\log(t-s)|\right) ds \\ &\le c' \left(t_i - t_{i-1}\right) t_i^{\rho(1-\lambda)-1} (1 + |\log(t_k - t_{i-1})|). \end{aligned} \tag{5.3.33}$$

Let $h_i = t_i - t_{i-1}$, $i = 1, \dots, N$. Due to (5.3.32), (3.4.6) and (5.3.33),

$$I_3^{(i)}(t) \le c\, h_i^{m+1} \left\{ \begin{array}{ll} t_i^{\rho(1-\lambda)-1} & , m < \rho(1-\lambda) \\ t_i^{\rho(1-\lambda)-1}(1+|\log t_i|) & , m = \rho(1-\lambda) \\ t_i^{2\rho(1-\lambda)-m-1} & , m > \rho(1-\lambda) \end{array} \right\} (1+|\log h_i|)$$
$$\le c^{'}\, N^{-m-1} \log N \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+\rho(1-\lambda))-m-1} & , m < \rho(1-\lambda) \\ \left(\frac{i}{N}\right)^{r(m+\rho(1-\lambda))-m-1}(1+|\log t_i|) & , m = \rho(1-\lambda) \\ \left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-1} & , m > \rho(1-\lambda) \end{array} \right\}. \tag{5.3.34}$$

If $m < \rho(1-\lambda)$ then $r(m+\rho(1-\lambda)) - m - 1 > 2rm - m - 1 \ge 0$ and

$$I_3^{(i)}(t) \le c\, N^{-m-1} \log N, \quad m < \rho(1-\lambda).$$

If $m = \rho(1-\lambda)$ then $r(m+\rho(1-\lambda)) - m - 1 = 2rm - m - 1 \ge 0$ and

$$I_3^{(i)}(t) \le c \left\{ \begin{array}{ll} N^{-m-1} \log N & \text{for } r > \frac{m+1}{2\rho(1-\lambda)} \\ N^{-m-1} \log^2 N & \text{for } r = \frac{m+1}{2\rho(1-\lambda)} \end{array} \right\}.$$

If $m > \rho(1-\lambda)$ then

$$I_3^{(i)}(t) \le c \begin{cases} N^{-m-1} \log N & \text{for } r \ge \frac{m+1}{2\rho(1-\lambda)} \\ N^{-2r\rho(1-\lambda)} \log N & \text{for } 1 \le r < \frac{m+1}{2\rho(1-\lambda)}. \end{cases}$$

We obtained

$$I_3^{(i)}(t) \le c\, E_N^{(m,0,\lambda,\rho,r)}, \quad i = 1, \dots, k, \quad t \in [t_{k-1}, t_k]. \tag{5.3.35}$$

Due to (5.3.31) and (5.3.35) we have

$$|I_2^{(k)}(t)| \le I_3^{(1)}(t) + I_3^{(k-1)}(t) + I_3^{(k)}(t) \le c\, E_N^{(m,0,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k],\ k = 1, \dots, N. \tag{5.3.36}$$

Next, we consider $I_1^{(k)}(t)$ for $k = 4, \dots, N$,

$$I_1^{(k)}(t) = \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1}, t_k]. \tag{5.3.37}$$

Using (5.1.19), we can write

$$\begin{aligned} I_1^{(k)}(t) = & \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left[K_\varphi(t,s) - K_\varphi(t,t_{i/2}) \right] \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \\ & + \sum_{i=2}^{k-2} K_\varphi(t,t_{i/2}) \int_{t_{i-1}}^{t_i} \left[u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right] ds, \end{aligned} \tag{5.3.38}$$

where $t_{i/2} = \frac{t_{i-1}+t_i}{2}$, $i = 2, \dots, k-2$; $k = 4, \dots, N$. We have for any $s \in [t_{i-1}, t_i]$, $i = 2, \dots, k-2$,

$$K_\varphi(t,s) - K_\varphi(t,t_{i/2}) = (s - t_{i/2}) \frac{\partial}{\partial s} K_\varphi(t,s) \Big|_{s=\xi}, \quad \xi \in (s, t_{i/2}). \tag{5.3.39}$$

By Lemma 3.3.4 we obtain for $\xi \in (s, t_{i/2})$ that

$$\begin{aligned} &|K_\varphi(t,s) - K_\varphi(t,t_{i/2})| \\ &\quad \le c(t_i - t_{i-1})\, \xi^{(\rho-1)(1-\lambda)-\lambda} [\xi^{-1}(1 + |\log(t-\xi)|) + (t-\xi)^{-1}]. \end{aligned} \tag{5.3.40}$$

Since $t \in [t_{k-1}, t_k]$, $s \in [t_{i-1}, t_i]$, $\xi \in (s, t_{i/2})$, $i = 2, \dots, k-2$, $k \ge 4$, we have

$$c_1 \le \frac{\xi}{t_i} \le c_2, \quad c_3 \le \frac{t-\xi}{t_k - t_{i-1}} \le c_4,$$

where $c_2 \ge c_1 > 0$ and $c_4 \ge c_3 > 0$ are some constants which do not depend on i and k (see for example (2.4.8) and (2.4.9)). Hence

$$\begin{aligned} &\int_{t_{i-1}}^{t_i} |K_\varphi(t,s) - K_\varphi(t,t_{i/2})|\, ds \le c\,(t_i - t_{i-1})^2\, t_i^{(\rho-1)(1-\lambda)-\lambda} \times \\ &\qquad \times [t_i^{-1}(1 + |\log(t_k - t_{i-1})|) + (t_k - t_{i-1})^{-1}], \quad i = 2, \dots, k-2. \end{aligned} \tag{5.3.41}$$

Due to (3.4.6) and (5.3.41) we have

$$\begin{aligned} I_4^{(k)}(t) &:= \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left|K_\varphi(t,s) - K_\varphi(t,t_{i/2})\right| \left|u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s)\right| ds \\ &\le c \sum_{i=2}^{k-2} h_i^{m+2} t_i^{\rho(1-\lambda)-1} [t_i^{-1}(1 + |\log(t_k - t_{i-1})|) + (t_k - t_{i-1})^{-1}] \times \\ &\qquad \times \left\{ \begin{array}{ll} 1 & , m < \rho(1-\lambda) \\ 1 + |\log t_i| & , m = \rho(1-\lambda) \\ t_i^{\rho(1-\lambda)-m} & , m > \rho(1-\lambda) \end{array} \right\}. \end{aligned} \tag{5.3.42}$$

Since $h_i t_i^{-1} \le c\, i^{-1}$ and $h_i (t_k - t_{i-1})^{-1} \le c\,(k-i+1)^{-1}$, $i = 2, \dots, k-2$, we can

estimate $I_4^{(k)}(t)$ as follows

$$
\begin{aligned}
I_4^{(k)}(t) &\le c\sum_{i=2}^{k-2} h_i^{m+1}[i^{-1}(1+|\log(t_k-t_{i-1})|)+(k-i+1)^{-1}]\times \\
&\qquad \times \left\{ \begin{array}{ll} t_i^{\rho(1-\lambda)-1} & ,m<\rho(1-\lambda) \\ t_i^{\rho(1-\lambda)-1}(1+|\log t_i|) & ,m=\rho(1-\lambda) \\ t_i^{2\rho(1-\lambda)-m-1} & ,m>\rho(1-\lambda) \end{array} \right\} \\
&\le c^{'} N^{-m-1}\sum_{i=2}^{k-2}[i^{-1}(1+|\log(t_k-t_{i-1})|)+(k-i+1)^{-1}]\times \\
&\qquad \times \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+\rho(1-\lambda))-m-1} & ,m<\rho(1-\lambda) \\ \left(\frac{i}{N}\right)^{r(m+\rho(1-\lambda))-m-1}(1+|\log t_i|) & ,m=\rho(1-\lambda) \\ \left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-1} & ,m>\rho(1-\lambda) \end{array} \right\}.
\end{aligned}
\tag{5.3.43}
$$

If $m<\rho(1-\lambda)$ then $r(m+\rho(1-\lambda))-m-1>0$ and

$$
\begin{aligned}
I_4^{(k)}(t) &\le c\,N^{-r(m+\rho(1-\lambda))}\log N\sum_{i=2}^{k-2} i^{r(m+\rho(1-\lambda))-m-2} + c\,N^{-m-1}\sum_{i=2}^{k-2}(k-i+1)^{-1} \\
&\le c^{'} N^{-r(m+\rho(1-\lambda))}\,N^{r(m+\rho(1-\lambda))-m-1}\log N + c^{'} N^{-m-1}\log N \\
&\le c^{''} N^{-m-1}\log N.
\end{aligned}
\tag{5.3.44}
$$

If $m=\rho(1-\lambda)$ then $r(m+\rho(1-\lambda))-m-1=2rm-m-1\ge 0$. We write

$$
I_4^{(k)}(t) \le c\,N^{-m-1}\sum_{i=2}^{k-2}(k-i+1)^{-1}\left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-1}(1+|\log\frac{i}{N}|)+c\,I_5^{(k)},
\tag{5.3.45}
$$

where

$$
I_5^{(k)} = N^{-m-2}\sum_{i=2}^{k-2}\left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-2}(1+|\log(t_k-t_{i-1})|)\,(1+|\log t_i|).
\tag{5.3.46}
$$

Further,

$$
I_4^{(k)}(t) \le c\left\{ \begin{array}{ll} N^{-m-1}\log N & \text{for } r>\frac{m+1}{2\rho(1-\lambda)} \\ N^{-m-1}\log^2 N & \text{for } r=\frac{m+1}{2\rho(1-\lambda)} \end{array} \right\} + c\,I_5^{(k)}.
\tag{5.3.47}
$$

It remains to estimate $I_5^{(k)}$. First we write $I_5^{(k)}$ in the form

$$
I_5^{(k)} = N^{-m-2}\sum_{i=2}^{k-2}\left(\frac{i}{N}\right)^{-1}\frac{1+|\log(t_k-t_{i-1})|}{1+|\log t_i|}\left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-1}(1+|\log t_i|)^2.
\tag{5.3.48}
$$

Since

$$\left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-1}(1+|\log t_i|)^2 \le c \quad \text{for} \quad 2r\rho(1-\lambda)-m-1>0,$$

we obtain by Lemma 2.4.2 that

$$I_5^{(k)} \le c \left\{ \begin{array}{ll} N^{-m-1}\log^3 N & \text{for } m\,r\rho+|\lambda|=1 \\ N^{-m-1}\log N & \text{otherwise} \end{array} \right\}. \tag{5.3.49}$$

Actually, in the case $m=1$, $r=1$, $\rho=1$ and $\lambda=0$ we can improve the estimate (5.3.49). Indeed, in this case $K \in W_*^{2,0,0}(D_b)$. Similarly to the proof of Lemma 3.3.4, we can write

$$\begin{aligned} \left|\frac{\partial}{\partial s}K_\varphi(t,s)\right| &\le \left|[\varphi^{'}(s)]^2\frac{\partial}{\partial y}K(\varphi(t),y)\right|_{y=\varphi(s)} + \left|\varphi^{''}(s)K(\varphi(t),\varphi(s))\right| \\ &\le c\left(s^{-1}\frac{1+|\log(t-s)|}{1+|\log s|}+(t-s)^{-1}+|\varphi^{''}(s)|(1+|\log(t-s)|)\right). \end{aligned} \tag{5.3.50}$$

From $\varphi \in C^{m+1}[0,a]$ it follows that $|\varphi^{''}(s)| \le c$. Thus

$$\left|\frac{\partial}{\partial s}K_\varphi(t,s)\right| \le c\left(s^{-1}\frac{1+|\log(t-s)|}{1+|\log s|}+(t-s)^{-1}\right), \quad (t,s)\in D_a. \tag{5.3.51}$$

Now, instead of (5.3.40), we can use inequality

$$|K_\varphi(t,s)-K_\varphi(t,t_{i/2})| \le c(t_i-t_{i-1})\left[\xi^{-1}\frac{1+|\log(t-\xi)|}{1+|\log\xi|}+(t-\xi)^{-1}\right]. \tag{5.3.52}$$

Using this, we can prove with the same idea we used before (cf. (5.3.39)-(5.3.47)), that

$$I_5^{(k)} \le c \left\{ \begin{array}{ll} N^{-m-1}\log^2 N & \text{for } m\,r\rho+|\lambda|=1 \\ N^{-m-1}\log N & \text{otherwise} \end{array} \right\}. \tag{5.3.53}$$

Now it follows from (5.3.29), (5.3.47) and (5.3.53) that

$$I_4^{(k)}(t) \le c\,E_N^{(m,0,\lambda,\rho,r)}, \quad m=\rho(1-\lambda). \tag{5.3.54}$$

If $m > \rho(1-\lambda)$ then

$$\begin{aligned} I_4^{(k)} &\le c\,N^{-2\rho(1-\lambda)}\left[\log N\sum_{i=2}^{k-2} i^{2r\rho(1-\lambda)-m-2}+\sum_{i=2}^{k-2}(k-i+1)^{-1}i^{2r\rho(1-\lambda)-m-1}\right] \\ &\le c^{'}\left\{ \begin{array}{ll} N^{-m-1}\log N & \text{if } r>\frac{m+1}{2\rho(1-\lambda)} \\ N^{-m-1}\log^2 N & \text{if } r=\frac{m+1}{2\rho(1-\lambda)} \\ N^{-2r\rho(1-\lambda)}\log N & \text{if } 1\le r<\frac{m+1}{2\rho(1-\lambda)} \end{array} \right\}. \end{aligned} \tag{5.3.55}$$

From (5.3.44), (5.3.54) and (5.3.55) we obtain

$$I_4^{(k)}(t) \le c\, E_N^{(m,0,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \dots, N. \tag{5.3.56}$$

Finally, for $t \in [t_{k-1}, t_k]$, $k = 4, \dots, N$, we consider

$$I_6^{(k)}(t) := \sum_{i=2}^{k-2} |K_\varphi(t, t_{i/2})| \int_{t_{i-1}}^{t_i} \left| u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right| ds, \quad t_{i/2} \in [t_{i-1}, t_i]. \tag{5.3.57}$$

By Lemma 3.3.1 and (3.4.6) we have

$$\begin{aligned}
I_6^{(k)}(t) &\le c \sum_{i=2}^{k-2} h_i^{m+2} t_i^{(\rho-1)(1-\lambda)-\lambda} (1 + |\log(t_k - t_{i-1})|) \times \\
&\qquad \times \left\{ \begin{array}{ll} 1 & , m+1 < \rho(1-\lambda) \\ 1 + |\log t_i| & , m+1 = \rho(1-\lambda) \\ t_i^{\rho(1-\lambda)-m-1} & , m+1 > \rho(1-\lambda) \end{array} \right\} \\
&\le c' \sum_{i=2}^{k-2} h_i^{m+2} (1 + |\log(t_k - t_{i-1})|) \times \\
&\qquad \times \left\{ \begin{array}{ll} t_i^{\rho(1-\lambda)-1} & , m+1 < \rho(1-\lambda) \\ t_i^{\rho(1-\lambda)-1}(1 + |\log t_i|) & , m+1 = \rho(1-\lambda) \\ t_i^{2\rho(1-\lambda)-m-2} & , m+1 > \rho(1-\lambda) \end{array} \right\} \\
&\le c'' N^{-m-2} \sum_{i=2}^{k-2} (1 + |\log(t_k - t_{i-1})|) \times \\
&\qquad \times \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+1+\rho(1-\lambda))-m-2} & , m+1 < \rho(1-\lambda) \\ \left(\frac{i}{N}\right)^{r(m+1+\rho(1-\lambda))-m-2} (1 + |\log \frac{i}{N}|) & , m+1 = \rho(1-\lambda) \\ \left(\frac{i}{N}\right)^{2r\rho(1-\lambda)-m-2} & , m+1 > \rho(1-\lambda) \end{array} \right\}.
\end{aligned} \tag{5.3.58}$$

If $m + 1 \le \rho(1-\lambda)$ then $r(m + 1 + \rho(1-\lambda)) - m - 2 \ge m > 0$ and

$$I_6^{(k)}(t) \le c\, N^{-m-1} \log N. \tag{5.3.59}$$

If $m + 1 > \rho(1-\lambda)$ we can then write

$$\begin{aligned}
I_6^{(k)}(t) &\le c\, N^{-2r\rho(1-\lambda)} \log N \sum_{i=2}^{k-2} i^{2r\rho(1-\lambda)-m-2} \\
&\le c' \left\{ \begin{array}{ll} N^{-m-1} \log N & \text{for } r > \frac{m+1}{2\rho(1-\lambda)} \\ N^{-m-1} \log^2 N & \text{for } r = \frac{m+1}{2\rho(1-\lambda)} \\ N^{-2r\rho(1-\lambda)} \log N & \text{for } 1 \le r < \frac{m+1}{2\rho(1-\lambda)} \end{array} \right\}.
\end{aligned} \tag{5.3.60}$$

From (5.3.58)-(5.3.60) we obtain

$$I_6^{(k)}(t) \le c\, E_N^{(m,0,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \dots, N. \tag{5.3.61}$$

Due to (5.3.38), (5.3.42), (5.3.56), (5.3.57) and (5.3.61),

$$|I_1^{(k)}(t)| \le c(I_4^{(k)}(t) + I_6^{(k)}(t)) \le c'\, E_N^{(m,0,\lambda,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \dots, N. \tag{5.3.62}$$

Estimate (5.3.28) follows from (5.3.30), (5.3.36) and (5.3.62).

□

Theorem 5.3.3. *For given $m \in \mathbb{N}$, assume that $K \in W^{m+1,0,\lambda}(D_b)$ and $f \in C_*^{m+1,\lambda}(0,b]$ with $b \in (0,\infty)$, $\lambda \in (-\infty, 1)$. Let $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes* (5.1.3) *be generated by the graded grid* (4.3.5) *and by the collocation parameters $\eta_1, \dots, \eta_m$ (see* (5.1.1)*) of a quadrature formula* (5.1.2) *which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \ge N_0 \ge 2$, a unique approximation v_N to u_φ, the solution of equation* (4.3.1)*, and a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following estimate*

$$\gamma_N^{(m,0,\lambda,\rho,r)} = \max_{\substack{i=1,\dots,N \\ j=1,\dots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \le c\, E_N^{(m,0,\lambda,\rho,r)} \tag{5.3.63}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,0,\lambda,\rho,r)}$ is defined by (5.1.4) *and*

$$E_N^{(m,0,\lambda,\rho,r)} = \left\{ \begin{array}{ll} N^{-m-1} \log N & \text{for } r > \frac{m+1}{2\rho(1-\lambda)} \\ N^{-m-1} \log^3 N & \text{for } r = \frac{m+1}{2\rho(1-\lambda)} \\ N^{-2\rho\, r\, (1-\lambda)} \log^2 N & \text{for } 1 \le r < \frac{m+1}{2\rho(1-\lambda)} \end{array} \right\}. \tag{5.3.64}$$

Proof. By Theorem 4.4.1, the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$. Due to (4.4.18) we know that $u_\varphi \in C_*^{m+1,1-\rho(1-\lambda)}(0,a]$. Next, we follow the proof of Theorem 5.3.2. We use the same estimates for the kernel $K_\varphi(t,s)$ and instead of (3.4.6),

$$\sup_{s \in [t_{i-1}, t_i]} |u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s)| \le c\, h_i^m \left\{ \begin{array}{ll} 1 & , m < \rho(1-\lambda) \\ 1 + |\log t_i| & , m = \rho(1-\lambda) \\ t_i^{\rho(1-\lambda)-m} & , m > \rho(1-\lambda) \end{array} \right\}, \tag{5.3.65}$$

we use the estimate (3.4.7):

$$\sup_{s\in[t_{i-1},t_i]} |u_\varphi(s) - (P_N^{(m-1)}u_\varphi)(s)| \\ \leq c\, h_i^m \left\{ \begin{array}{ll} 1 & , m < \rho(1-\lambda) \\ t_i^{\rho(1-\lambda)-m}(1+|\log t_i|) & , m \geq \rho(1-\lambda) \end{array} \right\}. \tag{5.3.66}$$

□

Remark 5.3.1. *If $0 \leq \nu + \lambda < 1$ then Theorem 5.3.3 holds also for any transformation $\varphi \in \Phi^{m+1,\rho}$. This follows from the proof of Theorem 4.4.4 (see (4.4.18)). In this case the solution $u_\varphi(t) = u(\varphi(t))$ of the equation (4.3.1) belongs to the set $C_*^{m+1,1-\rho(1-\lambda)}(0,a]$ for the transformations $\varphi \in \Phi^{m+1,\rho}$ and for $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$ as well. Thus all the estimates, which we use in the proof of Theorem 5.3.3, hold.*

Theorem 5.3.4. *For a given $m \in \mathbb{N}$, assume $K \in W^{m+1,0}(D_b')$, $f \in C^{m+1,0}(0,b]$, $b \in (0,\infty)$. Let $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes (5.1.3) be generated by the graded grid (4.3.5) and by the collocation parameters $\eta_1, \ldots, \eta_m$ (see (5.1.1)) of a quadrature formula (5.1.2) which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) determine for sufficiently large values of N, say $N \geq N_0 \geq 2$, a unique approximation v_N to u_φ, the solution of equation (4.3.1), and a unique approximation u_N to u, the solution of equation (4.1.1), and the following estimate

$$\gamma_N^{(m,0,0,\rho,r)} = \max_{\substack{i=1,\ldots,N \\ j=1,\ldots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \leq c\, E_N^{(m,\rho,r)} \tag{5.3.67}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,0,0,\rho,r)}$ is defined by (5.1.4) and

$$E_N^{(m,\rho,r)} = \left\{ \begin{array}{ll} N^{-m-1}\log N & \text{for } r > \frac{m+1}{2\rho} \\ N^{-m-1}\log^2 N & \text{for } r = \frac{m+1}{2\rho} \\ N^{-2r\rho}\log N & \text{for } 1 \leq r < \frac{m+1}{2\rho} \end{array} \right\}. \tag{5.3.68}$$

Proof. Since $W^{m+1,0}(D_b') \subset W_*^{m+1,0,0}(D_b)$ then (5.3.67) follows from Theorem 5.3.2 where we take $\lambda = 0$.

□

5.4 Superconvergence: the Case $\nu < 0$

Theorem 5.4.1. *For given $m \in \mathbb{N}$, assume that $K \in W^{m+1,\nu}(D_b')$ and $f \in C^{m+1,\nu}(0,b]$ with $b \in (0,\infty)$, $\nu \in (-\infty,0)$. Let $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,a]$, $\varphi : [0,a] \to [0,b]$, $\rho \in [1,\infty)$, $a \in (0,\infty)$. Furthermore, let the interpolation nodes* (5.1.3) *be generated by the graded grid* (4.3.5) *and by the collocation parameters $\eta_1, \ldots, \eta_m$ (see* (5.1.1)*) of a quadrature formula* (5.1.2) *which is exact for all polynomials of degree not exceeding m.*

Then settings (4.3.8)-(4.3.10) *determine for sufficiently large values of N, say $N \geq N_0 \geq 2$, a unique approximation v_N to u_φ, the solution of equation* (4.3.1)*, and a unique approximation u_N to u, the solution of equation* (4.1.1)*, and the following estimate*

$$\gamma_N^{(m,\nu,0,\rho,r)} = \max_{\substack{i=1,\ldots,N \\ j=1,\ldots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| \leq c\, E_N^{(m,\nu,\rho,r)} \tag{5.4.1}$$

holds with a positive constant c not depending on N. Here $\gamma_N^{(m,\nu,0,\rho,r)}$ is defined by (5.1.4) *and*

$$E_N^{(m,\nu,\rho,r)} = \left\{ \begin{array}{ll} N^{-m-1} & \text{for } r > \frac{m+1}{\rho(2-\nu)} \\ N^{-m-1}\log N & \text{for } r = \frac{m+1}{\rho(2-\nu)} \\ N^{-r\rho\,(2-\nu)} & \text{for } 1 \leq r < \frac{m+1}{\rho(2-\nu)} \end{array} \right\}. \tag{5.4.2}$$

Proof. By Theorem 4.4.1, the equation (4.1.1) has a unique solution $u \in C[0,b]$ and the equation (4.3.1) has a unique solution $u_\varphi \in C[0,a]$. Due to (4.4.15) we know that $u_\varphi \in C^{m+1,1-\rho(1-\nu)}(0,a]$. Moreover, by Theorem 4.4.2 we have for $N \geq N_0$ a unique approximation v_N to u_φ. Due to Theorem 5.1.1 we may use the inequalities (5.1.5) and (5.1.12):

$$\gamma_N^{(m,\nu,0,\rho,r)} \leq c \max_{k=1,\ldots,N} \left| \sup_{t\in[t_{k-1},t_k]} |I_1^{(k)}(t)| + \sup_{t\in[t_{k-1},t_k]} |I_2^{(k)}(t)| \right|, \quad [t_{k-1},t_k] \subset [0,a], \tag{5.4.3}$$

where $I_1^{(k)}(t)$ is defined by (5.1.13) and $I_2^{(k)}(t)$ is defined by (5.1.14).

We fix the index k such that $t \in [t_{k-1},t_k]$, $k = 1,\ldots,N$. First, we estimate (5.1.14):

$$I_2^{(k)}(t) = \left(\int_{t_0}^{t_1} + \int_{t_{k-2}}^{t_{k-1}} + \int_{t_{k-1}}^{t} \right) K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1},t_k]. \tag{5.4.4}$$

In analogy to the proof of Theorem 5.2.1 for $i = 1,\ldots,k$ we consider the

integral

$$\begin{aligned} I_3^{(i)}(t) &:= \left| \int\limits_{t_{i-1}}^{\min\{t_i,t\}} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \right| \\ &\le c \sup_{s\in[t_{i-1},t_i]} \left| u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right| \int\limits_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds. \end{aligned} \tag{5.4.5}$$

Note that $\min\{t_i,t\} = t_i$ for $i < k$ and $\min\{t_i,t\} = t$ for $i = k$, $t \in [t_{k-1}, t_k]$. Using (3.3.5) and (2.4.13), we obtain

$$\int\limits_{t_{i-1}}^{\min\{t_i,t\}} |K_\varphi(t,s)|\, ds \le c \int\limits_{t_{i-1}}^{\min\{t_i,t\}} s^{\rho-1}\, ds \le c\,(t_i - t_{i-1})\, t_i^{\rho-1}. \tag{5.4.6}$$

Let $h_i = t_i - t_{i-1}$, $i = 1, \dots, N$. Due to (5.4.5), (3.4.6) and (5.4.6),

$$\begin{aligned} I_3^{(i)}(t) &\le c\, h_i^{m+1} \left\{ \begin{array}{ll} t_i^{\rho-1} & , m < \rho(1-\nu) \\ t_i^{\rho-1}(1 + |\log t_i|) & , m = \rho(1-\nu) \\ t_i^{\rho(2-\nu)-m-1} & , m > \rho(1-\nu) \end{array} \right\} \\ &\le c^{'} N^{-m-1} \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+\rho)-m-1} & , m < \rho(1-\nu) \\ \left(\frac{i}{N}\right)^{r(m+\rho)-m-1} (1 + |\log t_i|) & , m = \rho(1-\nu) \\ \left(\frac{i}{N}\right)^{r\rho(2-\nu)-m-1} & , m > \rho(1-\nu) \end{array} \right\}. \end{aligned} \tag{5.4.7}$$

If $m < \rho(1-\nu)$ then $r(m+\rho) - m - 1 \ge 0$. Thus

$$I_3^{(i)}(t) \le c\, N^{-m-1}, \quad 1 \le r, \quad m < \rho(1-\nu).$$

If $m = \rho(1-\nu)$ then $r(m+\rho) - m - 1 \ge 0$ and

$$I_3^{(i)}(t) \le c \left\{ \begin{array}{ll} N^{-m-1} \log N & , r = \rho = 1 \Leftrightarrow r = \frac{m+1}{\rho(2-\nu)} \\ N^{-m-1} & , \text{ otherwise} \end{array} \right\}.$$

If $m > \rho(1-\nu)$ then

$$I_3^{(i)}(t) \le c \begin{cases} N^{-m-1} & \text{for } r \ge \frac{m+1}{\rho(2-\nu)} \\ N^{-r\rho(2-\nu)} & \text{for } 1 \le r < \frac{m+1}{\rho(2-\nu)}. \end{cases}$$

We obtained

$$I_3^{(i)}(t) \le c\, E_N^{(m,\nu,\rho,r)}, \quad i = 1, \dots, k, \quad t \in [t_{k-1}, t_k]. \tag{5.4.8}$$

From (5.4.4) and (5.4.8) we get also

$$|I_2^{(k)}(t)| \le I_3^{(1)}(t) + I_3^{(k-1)}(t) + I_3^{(k)}(t) \le c\, E_N^{(m,\nu,\rho,r)}, \quad t \in [t_{k-1}, t_k],\ k = 1, \dots, N. \tag{5.4.9}$$

Next, we consider $I_1^{(k)}(t)$ for $k = 4, \ldots, N$,

$$I_1^{(k)}(t) = \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} K_\varphi(t,s) \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds, \quad t \in [t_{k-1}, t_k]. \tag{5.4.10}$$

Using (5.1.19), we can write

$$\begin{aligned} I_1^{(k)}(t) = & \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left[K_\varphi(t,s) - K_\varphi(t,t_{i/2}) \right] \left[u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right] ds \\ & + \sum_{i=2}^{k-2} K_\varphi(t,t_{i/2}) \int_{t_{i-1}}^{t_i} \left[u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right] ds, \end{aligned} \tag{5.4.11}$$

where $t_{i/2} = \frac{t_{i-1}+t_i}{2}$, $i = 2, \ldots, k-2$; $k = 4, \ldots, N$. We have for any $s \in [t_{i-1}, t_i]$, $i = 2, \ldots, k-2$,

$$K_\varphi(t,s) - K_\varphi(t,t_{i/2}) = (s - t_{i/2}) \frac{\partial}{\partial s} K_\varphi(t,s) \Big|_{s=\xi}, \quad \xi \in (s, t_{i/2}). \tag{5.4.12}$$

By Lemma 3.3.6 we obtain for $\xi \in (s, t_{i/2})$ that

$$\begin{aligned} & |K_\varphi(t,s) - K_\varphi(t,t_{i/2})| \\ & \le c\,(t_i - t_{i-1}) \left\{ \begin{array}{ll} \xi^{\max\{\rho-2,0\}} & \text{for } \nu < -1 \\ \xi^{\max\{\rho-2,0\}} + \xi^{2\rho-2}(1 + |\log(t-\xi)|) & \text{for } \nu = -1 \\ \xi^{\max\{\rho-2,0\}} + \xi^{(\rho-1)(1-\nu)}(t-\xi)^{-\nu-1} & \text{for } -1 < \nu < 0 \end{array} \right\}. \end{aligned} \tag{5.4.13}$$

Since $t \in [t_{k-1}, t_k]$, $s \in [t_{i-1}, t_i]$, $\xi \in (s, t_{i/2})$, $i = 2, \ldots, k-2$, $t_i < t_{k-1}$, we have

$$c_1 \le \frac{t-\xi}{t-s} \le c_2, \tag{5.4.14}$$

where $c_2 \ge c_1 > 0$ are some constants which do not depend on i and k. We denote

$$I_4^{(k)}(t) := \sum_{i=2}^{k-2} \int_{t_{i-1}}^{t_i} \left| K_\varphi(t,s) - K_\varphi(t,t_{i/2}) \right| \left| u_\varphi(s) - (P_N^{(m-1)} u_\varphi)(s) \right| ds. \tag{5.4.15}$$

If $m < \rho(1-\nu)$ then by (3.4.6), (5.4.13) and (5.4.14),

$$\begin{aligned} I_4^{(k)}(t) & \le c \sum_{i=2}^{k-2} h_i^{m+1} \int_{t_{i-1}}^{t_i} \left\{ \begin{array}{ll} 1 & , \nu < -1 \\ 1 + |\log(t-s)| & , \nu = -1 \\ (t-s)^{-\nu-1} & , -1 < \nu < 0 \end{array} \right\} ds \\ & \le c' N^{-m-1} \int_{t_1}^{t_{k-2}} \left\{ \begin{array}{ll} 1 & , \nu < -1 \\ 1 + |\log(t-s)| & , \nu = -1 \\ (t-s)^{-\nu-1} & , -1 < \nu < 0 \end{array} \right\} ds \le c'' N^{-m-1}. \end{aligned} \tag{5.4.16}$$

If $m = \rho(1-\nu)$ then by (3.4.6), (5.4.13) and (5.4.14),

$$I_4^{(k)}(t) \le c \sum_{i=2}^{k-2} h_i^{m+1}(1+|\log t_i|) \int_{t_{i-1}}^{t_i} \left\{ \begin{array}{ll} 1 & ,\nu < -1 \\ t_i^{2\rho-2}(1+|\log(t-s)|) & ,\nu = -1 \\ t_i^{(\rho-1)(1-\nu)}(t-s)^{-\nu-1} & ,-1 < \nu < 0 \end{array} \right\} ds. \tag{5.4.17}$$

Further, for $\nu < -1$ we have

$$\sum_{i=2}^{k-2} h_i^{m+1}(1+|\log t_i|) \int_{t_{i-1}}^{t_i} ds \le c\, N^{-m-2} \sum_{i=2}^{k-2} (1+|\log \frac{i}{N}|) \le c^{'}\, N^{-m-1}. \tag{5.4.18}$$

Next, for $\rho > 1$ we use the inequalities

$$t_i^{2\rho-2}(1+|\log t_i|) \le c, \quad t_i^{(\rho-1)(1-\nu)}(1+|\log t_i|) \le c.$$

Thus for $-1 \le \nu < 0$ we have

$$\begin{aligned} & I_4^{(k)}(t) \\ & \le c \sum_{i=2}^{k-2} h_i^{m+1} \left\{ \begin{array}{ll} 1 & ,\rho > 1 \\ 1+|\log t_i| & ,\rho = 1 \end{array} \right\} \int_{t_{i-1}}^{t_i} \left\{ \begin{array}{ll} 1+|\log(t-s)| & ,\nu = -1 \\ (t-s)^{-\nu-1} & ,-1 < \nu < 0 \end{array} \right\} ds \\ & \le c^{'}\, N^{-m-1} \max_{i=2,\dots,k-2} \left\{ \begin{array}{ll} 1 & ,\rho > 1 \\ \left(\frac{i}{N}\right)^{(r-1)(m+1)} (1+|\log \frac{i}{N}|) & ,\rho = 1 \end{array} \right\} \times \\ & \quad \times \int_{t_1}^{t_{k-2}} \left\{ \begin{array}{l} 1+|\log(t-s)| \\ (t-s)^{-\nu-1} \end{array} \right\} ds \le c^{''} \left\{ \begin{array}{ll} N^{-m-1} & ,r > \frac{m+1}{\rho(2-\nu)} \\ N^{-m-1}\log N & ,r = \frac{m+1}{\rho(2-\nu)} \end{array} \right\}. \end{aligned} \tag{5.4.19}$$

If $m > \rho(1-\nu)$ then by (3.4.6) and (5.4.15),

$$\begin{aligned} I_4^{(k)}(t) \le c \sum_{i=2}^{k-2} h_i^{m+1} t_i^{\rho(1-\nu)-m} \times \\ \times \int_{t_{i-1}}^{t_i} \left\{ \begin{array}{ll} t_i^{\max\{\rho-2,0\}} & \text{for } \nu < -1 \\ t_i^{\max\{\rho-2,0\}} + t_i^{2\rho-2}(1+|\log(t-s)|) & \text{for } \nu = -1 \\ t_i^{\max\{\rho-2,0\}} + t_i^{(\rho-1)(1-\nu)}(t-s)^{-\nu-1} & \text{for } -1 < \nu < 0 \end{array} \right\} ds. \end{aligned} \tag{5.4.20}$$

Further,

$$I_{4,1}^{(k)}(t) := \sum_{i=2}^{k-2} h_i^{m+2} t_i^{\rho(1-\nu)-m+\max\{\rho-2,0\}} \le c \sum_{i=2}^{k-2} h_i^{m+2} t_i^{\rho(2-\nu)-m-2}$$
$$\le c^{'} N^{-r\rho(2-\nu)} \sum_{i=2}^{k-2} i^{r\rho(2-\nu)-m-2} \le c^{''} \left\{ \begin{array}{ll} N^{-m-1} & , r > \frac{m+1}{\rho(2-\nu)} \\ N^{-m-1} \log N & , r = \frac{m+1}{\rho(2-\nu)} \\ N^{-r\rho(2-\nu)} & , 1 \le r < \frac{m+1}{\rho(2-\nu)} \end{array} \right\}. \tag{5.4.21}$$

Next,

$$I_{4,2}^{(k)}(t) : = \sum_{i=2}^{k-2} h_i^{m+1} \left\{ \begin{array}{ll} t_i^{\rho(1-\nu)-m+2\rho-2} & , \nu = -1 \\ t_i^{\rho(1-\nu)-m+\rho(1-\nu)-(1-\nu)} & , -1 < \nu < 0 \end{array} \right\} \times$$
$$\times \int_{t_{i-1}}^{t_i} \left\{ \begin{array}{ll} 1 + |\log(t-s)| & , \nu = -1 \\ (t-s)^{-\nu-1} & , -1 < \nu < 0 \end{array} \right\} ds. \tag{5.4.22}$$

Therefore

$$I_{4,2}^{(k)}(t) \le c \max_{i=2,\dots,k-2} \left\{ \begin{array}{ll} h_i^{m+1} t_i^{\rho(2-\nu)-m-1} t_i^{\rho-1} & , \nu = -1 \\ h_i^{m+1} t_i^{\rho(2-\nu)-m-1} t_i^{-\nu(\rho-1)} & , -1 < \nu < 0 \end{array} \right\} \times$$
$$\times \int_{t_1}^{t_{k-2}} \left\{ \begin{array}{ll} 1 + |\log(t-s)| & , \nu = -1 \\ (t-s)^{-\nu-1} & , -1 < \nu < 0 \end{array} \right\} ds. \tag{5.4.23}$$

We may write

$$I_{4,2}^{(k)}(t) \le c \max_{i=2,\dots,k-2} \{h_i^{m+1} t_i^{\rho(2-\nu)-m-1}\} \le c^{'} N^{-r\rho(2-\nu)} \max_{i=2,\dots,k-2} \{i^{r\rho(2-\nu)-m-1}\}. \tag{5.4.24}$$

It follows that

$$I_{4,2}^{(k)}(t) \le c \left\{ \begin{array}{ll} N^{-m-1} & , r \ge \frac{m+1}{\rho(2-\nu)} \\ N^{-r\rho(2-\nu)} & , 1 \le r < \frac{m+1}{\rho(2-\nu)} \end{array} \right\}. \tag{5.4.25}$$

From (5.4.16), (5.4.19), (5.4.21), (5.4.22) and (5.4.25) we obtain

$$I_4^{(k)}(t) \le c \left(E_N^{(m,\nu,\rho,r)} + I_{4,1}^{(k)}(t) + I_{4,2}^{(k)}(t) \right) \le c^{'} E_N^{(m,\nu,\rho,r)},$$
$$t \in [t_{k-1}, t_k], \quad k = 4, \dots, N. \tag{5.4.26}$$

Finally, for $t \in [t_{k-1}, t_k]$, $k = 4, \dots, N$, we consider

$$I_5^{(k)}(t) := \sum_{i=2}^{k-2} |K_\varphi(t, t_{i/2})| \int_{t_{i-1}}^{t_i} \left| u_\varphi(s) - (P_N^{(m)} u_\varphi)(s) \right| ds, \quad t_{i/2} \in [t_{i-1}, t_i]. \tag{5.4.27}$$

By (3.4.6) and Lemma 3.3.3 we have

$$
\begin{aligned}
I_5^{(k)}(t) &\le c\sum_{i=2}^{k-2} h_i^{m+2} t_i^{\rho-1} \left\{ \begin{array}{ll} 1 & , m+1 < \rho(1-\nu) \\ 1+|\log t_i| & , m+1 = \rho(1-\nu) \\ t_i^{\rho(1-\nu)-m-1} & , m+1 > \rho(1-\nu) \end{array} \right\} \\
&\le c^{'} \sum_{i=2}^{k-2} h_i^{m+2} \left\{ \begin{array}{ll} t_i^{\rho-1} & , m+1 < \rho(1-\nu) \\ t_i^{\rho-1}(1+|\log t_i|) & , m+1 = \rho(1-\nu) \\ t_i^{\rho(2-\nu)-m-2} & , m+1 > \rho(1-\nu) \end{array} \right\} \\
&\le c^{''} N^{-m-2} \sum_{i=2}^{k-2} \left\{ \begin{array}{ll} \left(\frac{i}{N}\right)^{r(m+1+\rho)-m-2} & , m+1 < \rho(1-\nu) \\ \left(\frac{i}{N}\right)^{r(m+1+\rho)-m-2} (1+|\log \frac{i}{N}|) & , m+1 = \rho(1-\nu) \\ \left(\frac{i}{N}\right)^{r\rho(2-\nu)-m-2} & , m+1 > \rho(1-\nu) \end{array} \right\}.
\end{aligned}
\tag{5.4.28}
$$

If $m+1 \le \rho(1-\nu)$ then $r(m+1+\rho)-m-2 \ge 0$ and

$$
I_5^{(k)}(t) \le c\, N^{-m-2} \sum_{i=2}^{k-2} (1+|\log \frac{i}{N}|) \le c^{'} N^{-m-1}, \quad r \ge 1 > \frac{m+1}{\rho(2-\nu)}. \tag{5.4.29}
$$

If $m+1 > \rho(1-\nu)$ then we can write

$$
I_5^{(k)}(t) \le c\, N^{-r\rho(2-\nu)} \sum_{i=2}^{k-2} i^{r\rho(2-\nu)-m-2} \le c^{'} \left\{ \begin{array}{ll} N^{-m-1} & \text{for } r > \frac{m+1}{\rho(2-\nu)} \\ N^{-m-1}\log N & \text{for } r = \frac{m+1}{\rho(2-\nu)} \\ N^{-r\rho(2-\nu)} & \text{for } 1 \le r < \frac{m+1}{\rho(2-\nu)} \end{array} \right\}. \tag{5.4.30}
$$

From (5.4.28)-(5.4.30) we obtain

$$
I_5^{(k)}(t) \le c\, E_N^{(m,\nu,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \ldots, N. \tag{5.4.31}
$$

Due to (5.4.11), (5.4.26), (5.4.27) and (5.4.31),

$$
|I_1^{(k)}(t)| \le c(I_4^{(k)}(t) + I_5^{(k)}(t)) \le c^{'} E_N^{(m,\nu,\rho,r)}, \quad t \in [t_{k-1}, t_k], \quad k = 4, \ldots, N. \tag{5.4.32}
$$

Estimate (5.4.1) follows from (5.4.3), (5.4.9) and (5.4.32).

□

Chapter 6

Numerical Experiments

6.1 Introduction

Let us consider equation (4.1.1) with $b = 1$:

$$u(x) = \int_0^x K(x,y)\, u(y)\, dy + f(x), \quad x \in [0,1]. \tag{6.1.1}$$

In (6.1.1), we make the variable changes $x = \varphi(t), y = \varphi(s)$ where $\varphi : [0,1] \to [0,1]$ is the smoothing transformation (3.1.1)-(3.1.4) with $a = 1$. We use the following three transformations:

$$\varphi_1(t) := t^\rho, \quad \rho \geq 1, \tag{6.1.2}$$

$$\varphi_2(t) := 2^{\rho/2} \left(\sin\left(\frac{\pi}{4}t\right)\right)^\rho, \quad \rho \geq 1, \tag{6.1.3}$$

$$\varphi_3(t) := \binom{2\rho}{\rho} \frac{2\rho+1}{2^{2\rho}} \int_0^t (s\,(2-s))^\rho\, ds, \quad \rho \in \mathbb{N}. \tag{6.1.4}$$

Here $\varphi_1(t) = t^\rho$ is the most natural transformation and is used also in the works [19, 36, 37, 38, 39, 55]. The transformation (6.1.3) is taken from [56]. Clearly $\varphi_1, \varphi_2 \in \Phi^{m,\rho}$ for arbitrary $\rho \in [1,\infty)$ and $m \in \mathbb{N}$. Moreover, $\varphi_1, \varphi_2 \in \Phi^{m,\rho} \cap C^m[0,1]$ for $\rho \in \mathbb{N}$ or for $\rho \in (m,\infty)$. A sample of nontrivial polynomial functions $\varphi_3 \in \Phi^{m+1,\rho+1} \cap C^{m+1}[0,1]$, $\rho, m \in \mathbb{N}$, is constructed similarly as in [26]. For φ_3 we can write

$$\varphi_3(t) = \frac{3}{2}t^2(-\frac{1}{3}t + 1), \quad \text{if } \rho = 1,$$

$$\frac{\partial}{\partial t}\varphi_3(t) = \binom{2\rho}{\rho} \frac{2\rho+1}{2^{2\rho}} (t\,(2-t))^\rho, \quad \rho \in \mathbb{N}.$$

After a variable change we obtain the equation (4.3.1),

$$u_\rho(t) = \int_0^t K_\rho(t,s)\, u_\rho(s)\, ds + f_\rho(t), \quad t \in [0,1]. \tag{6.1.5}$$

Equation (6.1.1) was solved numerically by method (4.3.8)-(4.3.10) for

$$m = 3, \quad \eta_1 = \frac{5-\sqrt{15}}{10}, \quad \eta_2 = \frac{1}{2}, \quad \eta_3 = \frac{5+\sqrt{15}}{10}, \tag{6.1.6}$$

$$v_N(t_{i-1} + (t_i - t_{i-1})z) = \sum_{j=1}^{m} \beta_{ij} \prod_{\substack{k=1\\k\neq j}}^{m} \frac{z-\eta_k}{\eta_j - \eta_k}, \quad z \in [0,1], \tag{6.1.7}$$

where v_N is the approximate solution of (6.1.5) and β_{ij} $(i = 1,\dots,N;\ j = 1,\dots,m)$ are some unknown constants which we find from the linear system (4.3.15). Here η_1, η_2, η_3 are the node points of the three-point Gauss-Legendre quadrature rule (5.1.2) which is exact for all polynomials of degree not exceeding $2m-1$ (for $m = 3$ we have $2m - 1 = 5$). The numerical values of the node points of Gauss-Legendre quadrature rule can be found, for example, in [1]. Most of the numerical tests in this chapter are performed with the parameter $m = 3$. Some special samples are performed with other values of m and with other node points $\eta_1, \dots, \eta_m$. We will explicitly state different values of m in the related examples.

The numerical results of this chapter are calculated with Fortran programming language using 16-digit double precision arithmetic in a 32-bit computer. The analytical part is studied and tested with Maple. Weakly singular integrals in the linear system (4.3.15) were found with the quadrature rule introduced in [62] (see also [32, 63]):

$$\int_0^\sigma g(x)\, dx \approx x_1 g(\xi) + \sum_{i=2}^{N_0} \frac{x_i - x_{i-1}}{2} \sum_{j=1}^{n} w_j g(\xi_{ij}), \quad \sigma \in (0,\infty), \quad n \in \mathbb{N}, \tag{6.1.8}$$

where the integrand $g(x)$ can have singularity only at $x = 0$,

$$\xi = \frac{x_1}{2}, \quad \xi_{ij} = x_{i-1} + \frac{\xi_j + 1}{2}(x_i - x_{i-1}). \tag{6.1.9}$$

Here $-1 \le \xi_1 < \cdots < \xi_n \le 1$ are the nodes of the n-point Gauss-Legendre quadrature rule (we used $n = 10$ in our calculations), $x_0, \dots, x_{N_0}$ are the nodes of the graded grid (2.4.3) on the interval $[0,\sigma]$, $N_0 \in \mathbb{N}$. The number of steps N_0 in the quadrature rule (6.1.8) were doubled until a reasonable error tolerance for the collocation method was achieved. If the integrand in (6.1.8) diverges as $(x-\sigma)^\theta$, $0 < \theta < 1$, near the point $x = \sigma$, we use the identity (see [59, p. 138-139])

$$\int_\alpha^\sigma g(x)\, dx = \frac{1}{1-\theta} \int_0^{(\sigma-\alpha)^{1-\theta}} x^{\frac{\theta}{1-\theta}} g(\sigma - x^{\frac{1}{1-\theta}})\, dx, \quad 0 \le \alpha < \sigma. \tag{6.1.10}$$

For destroying the diagonal singularity of the kernel $K(t,s)$ at $t=s$ we use the singularity subtraction technique introduced by Kantorovich and Krylov in [34] (see also [25, 59, 65]):

$$\int_{a_1}^{b_1} K(t,s)F(s)\,ds = \int_{a_1}^{b_1} K(t,s)(F(s)-F(t))\,ds + F(t)\int_{a_1}^{b_1} K(t,s)\,ds, \quad 0 \le a_1 < b_1, \tag{6.1.11}$$

where the integrand $K(t,s)(F(s)-F(t))$ is smoother than the integrand $K(t,s)F(s)$ and, the integral $\int_{a_1}^{b_1} K(t,s)\,ds$ is computed analytically or numerically.

In the following tables, we use the quantities $\varepsilon_N^{(\rho,r)}$ and $\gamma_N^{(\rho,r)}$. The quantities $\varepsilon_N^{(\rho,r)}$ are approximate values of the norm $\|u-u_N\|_\infty$, calculated as follows:

$$\begin{gathered}\varepsilon_N^{(\rho,r)} = \max_{i,j} |u(\varphi(\tau_{ij})) - u_N(\varphi(\tau_{ij}))| = \max_{i,j} |u_\varphi(\tau_{ij}) - v_N(\tau_{ij})|, \\ \tau_{ij} = t_{i-1} + \frac{j}{20}(t_i - t_{i-1}), \quad i=1,\dots,N, \quad j=0,\dots,20.\end{gathered} \tag{6.1.12}$$

Here u is the exact solution of (6.1.1), u_φ is the exact solution of (6.1.5), v_N is the approximation to u_φ determined by (4.3.8) and (4.3.9), and u_N is the approximation to u determined by (4.3.10):

$$u_N(x) = v_N(\varphi^{-1}(x)), \quad x \in [0,1]. \tag{6.1.13}$$

In the following tables the ratios

$$\delta_N^{(\rho,r)} = \frac{\varepsilon_{N/2}^{(\rho,r)}}{\varepsilon_N^{(\rho,r)}} \tag{6.1.14}$$

characterizing the observed convergence rate are also presented. For the local superconvergence results we use the quantities:

$$\gamma_N^{(\rho,r)} = \max_{\substack{i=1,\dots,N\\ j=1,\dots,m}} |u(\varphi(t_{ij})) - u_N(\varphi(t_{ij}))| = \max_{\substack{i=1,\dots,N\\ j=1,\dots,m}} |u_\varphi(t_{ij}) - v_N(t_{ij})| \tag{6.1.15}$$

(see (5.1.4)) and the ratios

$$\tilde{\delta}_N^{(\rho,r)} = \frac{\gamma_{N/2}^{(\rho,r)}}{\gamma_N^{(\rho,r)}}. \tag{6.1.16}$$

We give the expected values of the ratios $\delta_N^{(\rho,r)}$ and $\tilde{\delta}_N^{(\rho,r)}$ in the last row of each table. If the theoretical estimate includes a logarithm, then in the last row of the following tables for $\frac{\log N/2}{\log N}$ we use the values $\frac{\log 128}{\log 256} = 0.875$, $\frac{\log 512}{\log 1024} = 0.9$,

$\frac{\log 1024}{\log 2048} \approx 0.909$ for the greatest value of $N = 256$, $N = 1024$ and $N = 2048$ respectively.

The errors in the following tables are written in the form

$$p\,E\,-\,d = p\,10^{-d}, \quad p > 0, \quad d \in \mathbb{Z}.$$

We will further comment the following tables in Section 6.2. The behavior of the numerical results are similar in other sections and we point out only the most important differences.

6.2 Case $K \in W^{m+1,0,\lambda}(D_b)$ and $f \in C_*^{m+1,\lambda}(0,b]$

We consider the equation (6.1.1), where

$$K(x,y) = \log(x-y)\,y^{-1/2}, \quad (x,y) \in D_1, \tag{6.2.1}$$

$$f(x) = x^{1/2}\,(1+\log x) - x\left(\log^2 x - \log x + 1 - \frac{\pi^2}{6}\right), \tag{6.2.2}$$

$$u(x) = x^{1/2}(1+\log x), \quad x \in [0,1]. \tag{6.2.3}$$

Clearly $K \in W^{m+1,0,1/2}(D_1)$ and $f, u \in C_*^{m+1,1/2}(0,1]$ for arbitrary $m \in \mathbb{N}$. For numerical results we use $m = 3$ and the node points (6.1.6). In this case numerical results for $\rho = 1$ are published also in [35].

From Theorem 4.4.4 it follows that for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r > \frac{6}{\rho} \\ 2^{r\rho/2}\left(\frac{\log(N/2)}{\log N}\right) & \text{if } 1 \le r \le \frac{6}{\rho} \end{array} \right\}. \tag{6.2.4}$$

From Theorem 5.3.3 it follows that for sufficiently large values of N we have

$$\tilde{\delta_N}^{(\rho,r)} = \left\{ \begin{array}{ll} 2^4\left(\frac{\log(N/2)}{\log N}\right) & \text{if } r > \frac{4}{\rho} \\ 2^4\left(\frac{\log(N/2)}{\log N}\right)^3 & \text{if } r = \frac{4}{\rho} \\ 2^{r\rho}\left(\frac{\log(N/2)}{\log N}\right)^2 & \text{if } 1 \le r < \frac{4}{\rho} \end{array} \right\}. \tag{6.2.5}$$

Tables 6.2.1-6.2.2 describe global convergence errors $\|u - u_N\|_\infty$ for examples (6.2.1)-(6.2.3) and for the sine transformation φ_2. The expected theoretical results are presented in (6.2.4). Since $\nu + \lambda = 1/2 > 0$ the ratios (6.2.4) must hold for all the functions $\varphi_1, \varphi_2, \varphi_3$ with $\rho \in [1,\infty)$, according to Theorem 4.4.4. The observed errors $\varepsilon_N^{(\rho,r)}$ are overall in good agreement with Theorem 4.4.4 and (6.2.4). We will comment some interesting features which we can observe also in other tables in the current chapter.

Table 6.2.1: Examples (6.2.1)-(6.2.3), $\nu = 0$, $\lambda = 1/2$, sine transformation φ_2, global convergence (6.2.4), uniform grid.

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(2,1)}$ $\delta_N^{(2,1)}$	$\varepsilon_N^{(3,1)}$ $\delta_N^{(3,1)}$	$\varepsilon_N^{(3.5,1)}$ $\delta_N^{(3.5,1)}$	$\varepsilon_N^{(5,1)}$ $\delta_N^{(5,1)}$	$\varepsilon_N^{(6,1)}$ $\delta_N^{(6,1)}$	$\varepsilon_N^{(6.5,1)}$ $\delta_N^{(6.5,1)}$
16	2.4 E - 1	2.0 E - 2	4.1 E - 3	2.6 E - 3	1.1 E - 4	1.8 E - 4	1.3 E - 4
	1.23	**1.94**	**2.32**	**3.00**	**2.65**	**5.72**	**7.04**
32	1.9 E - 1	1.0 E - 2	1.8 E - 3	8.7 E - 4	4.1 E - 5	3.1 E - 5	1.8 E - 5
	1.24	**1.96**	**2.40**	**3.04**	**3.70**	**6.23**	**7.56**
64	1.6 E - 1	5.2 E - 3	7.3 E - 4	2.9 E - 4	1.1 E - 5	4.9 E - 6	2.4 E - 6
	1.25	**1.98**	**2.46**	**3.07**	**4.20**	**6.55**	**7.89**
128	1.2 E - 1	2.6 E - 3	3.0 E - 4	9.3 E - 5	2.6 E - 6	7.5 E - 7	3.1 E - 7
	1.27	**1.99**	**2.51**	**3.10**	**4.50**	**6.77**	**7.97**
256	9.9 E - 2	1.3 E - 3	1.2 E - 4	3.0 E - 5	5.8 E - 7	1.1 E - 7	3.9 E - 8
	1.28	**1.99**	**2.54**	**3.12**	**4.70**	**6.94**	**7.97**
512	7.7 E - 2	6.6 E - 4	4.7 E - 5	9.7 E - 6	1.2 E - 7	1.6 E - 8	4.9 E - 9
	1.29	**2.00**	**2.57**	**3.13**	**4.84**	**7.06**	**8.00**
1024	6.0 E - 2	3.3 E - 4	1.8 E - 5	3.1 E - 6	2.6 E - 8	2.3 E - 9	6.1 E - 10
	1.30	**2.00**	**2.59**	**3.15**	**4.94**	**7.16**	**8.00**
2048	4.6 E - 2	1.7 E - 4	7.0 E - 6	9.8 E - 7	5.2 E - 9	3.2 E - 10	7.6 E - 11
	1.29	1.82	2.57	3.06	5.14	7.27	8.00

Table 6.2.2: Examples (6.2.1)-(6.2.3), $\nu = 0$, $\lambda = 1/2$, sine transformation φ_2, global convergence (6.2.4), graded grid.

N	$\varepsilon_N^{(1,2)}$ $\delta_N^{(1,2)}$	$\varepsilon_N^{(1,3)}$ $\delta_N^{(1,3)}$	$\varepsilon_N^{(1,4)}$ $\delta_N^{(1,4)}$	$\varepsilon_N^{(2,2)}$ $\delta_N^{(2,2)}$	$\varepsilon_N^{(3,1.5)}$ $\delta_N^{(3,1.5)}$	$\varepsilon_N^{(1,6)}$ $\delta_N^{(1,6)}$	$\varepsilon_N^{(3,3)}$ $\delta_N^{(3,3)}$
16	9.9 E - 2	3.5 E - 2	1.1 E - 2	1.3 E - 3	7.3 E - 4	2.9 E - 3	6.8 E - 4
	1.65	**2.32**	**3.30**	**3.98**	**3.89**	**6.13**	**7.77**
32	6.0 E - 2	1.5 E - 2	3.4 E - 3	3.3 E - 4	1.9 E - 4	4.8 E - 4	8.8 E - 5
	1.70	**2.41**	**3.42**	**3.99**	**4.03**	**6.48**	**8.00**
64	3.5 E - 2	6.3 E - 3	1.0 E - 3	8.3 E - 5	4.7 E - 5	7.4 E - 5	1.1 E - 5
	1.75	**2.48**	**3.50**	**4.00**	**4.12**	**6.57**	**8.05**
128	2.0 E - 2	2.5 E - 3	2.9 E - 4	2.1 E - 5	1.1 E - 5	1.1 E - 5	1.4 E - 6
	1.78	**2.52**	**3.55**	**4.00**	**4.20**	**6.79**	**8.05**
256	1.1 E - 2	1.0 E - 3	8.1 E - 5	5.2 E - 6	2.7 E - 6	1.7 E - 6	1.7 E - 7
	1.81	**2.55**	**3.60**	**4.00**	**4.26**	**6.95**	**8.03**
512	6.3 E - 3	3.9 E - 4	2.2 E - 5	1.3 E - 6	6.3 E - 7	2.4 E - 7	2.1 E - 8
	1.83	**2.58**	**3.64**	**4.00**	**4.31**	**7.07**	**8.02**
1024	3.4 E - 3	1.5 E - 4	6.2 E - 6	3.2 E - 7	1.5 E - 7	3.4 E - 8	2.6 E - 9
	1.84	**2.60**	**3.67**	**4.00**	**4.34**	**7.17**	**8.01**
2048	1.9 E - 3	5.9 E - 5	1.7 E - 6	8.1 E - 8	3.4 E - 8	4.7 E - 9	3.3 E - 10
	1.82	2.57	3.64	3.64	4.32	7.27	8.00

First, we point out that sometimes the numerical results are noticeably better than the expected theoretical ones for specially chosen ρ values. When we choose the parameters ρ and r such that the product $r\rho$ remains the same (in this case the theoretical estimates have the same order) we can sometimes see much better results for the parameter $\rho > 1$. One reason for this is that, for example, if we have functions of the type $g(x) = \sqrt{x}$, then $g(\varphi(t)) = \sqrt{t^\rho}$ may become sufficiently smooth for even but not for odd values of $\rho \in \mathbb{N}$. If we know more about the behavior of the solution u, then we may try to choose more carefully the transformation φ and the parameter ρ to achieve better convergence results.

In Tables 6.2.1-6.2.2, we see this phenomenon for the parameters $\rho = 2, r = 1$ (compare with $\rho = 1, r = 2$) and $\rho = 2, r = 2$ (compare with $\rho = 1, r = 4$). The same is true for the smoothing function φ_1 instead of φ_2 (these results are not printed out in the current thesis but we have tested them as well). If we take $\rho = 2$ and $\varphi(t) = t^2$ then

$$u_\varphi(t) = \sqrt{t^2}(1 + \log t^2) = t(1 + \log t^2).$$

The new function u_φ not only belongs to $C_*^{m,0}(0,1]$ but actually u_φ belongs to the smoother set $C^{m,0}(0,1] \subset C_*^{m,0}(0,1]$. We can also see, that the kernel

$$K_\varphi(t,s) = 2s\log(t^2 - s^2)(s^2)^{-1/2} = 2[\log(t-s) + \log(t+s)]$$

belongs to $W^{m,0}(D_1')$ and K_φ does not possess any boundary singularity. Since also $f \in C^{m,0}(0,1]$ we can use instead of Theorem 4.4.4 other results, for example, the estimates (4.4.19) (since the solution $u_\varphi \in C^{m,0}(0,1]$). In our case it follows from (4.4.19) that

$$\delta_N^{(\rho,r)} \le c\ 2^{-r\rho/2} \quad \text{for } 1 \le r < \tfrac{6}{\rho}\ .$$

We can see this for the parameters $\rho = 2, r = 1$ in Table 6.2.1 and for $\rho = 2, r = 2$ in Table 6.2.2.

The second phenomenon we see in most of the tables is that, if the parameters ρ and r are close to the values where the optimal theoretical convergence rate must occur, then the ratios $\delta_N^{(\rho,r)}$ are somewhat smaller than the predicted theoretical estimates, but we can still see that $\delta_N^{(\rho,r)}$ converges slowly to the theoretical value. We can observe this in Tables 6.2.1-6.2.2 in the cases $\rho = 5, r = 1$; $\rho = 6, r = 1$ and $\rho = 1, r = 6$. If we take the parameters ρ and r a little bit larger than the optimal values, then the ratios $\delta_N^{(\rho,r)}$ are in very good agreement with the theoretical results (see $\rho = 6.5, r = 1$ in Table 6.2.1).

To illustrate that it is not possible to improve the ratios $\delta_N^{(\rho,r)}$ we usually have added into the tables a set of parameters ρ and r which are much larger than the ones required for optimal theoretical convergence. Usually the only effect we may observe is the small growth of the actual errors $\varepsilon_N^{(\rho,r)}$, see $\rho = 3, r = 3$ in Table 6.2.2 (compare with $\rho = 6.5$, $r = 1$ in Table 6.2.1). During the calculations we used other larger values as well, but we did not see any improvements.

6.2. *Case* $K \in W^{m+1,0,\lambda}(D_b)$ *and* $f \in C_*^{m+1,\lambda}(0,b]$

Another typical behavior of the actual errors $\varepsilon_N^{(\rho,r)}$ is that they are a little smaller already for the starting values of N for uniform grid (compare $\rho = 3$, $r = 1$ with $\rho = 1$, $r = 3$ and $\rho = 6$, $r = 1$ with $\rho = 1$, $r = 6$).

Table 6.2.3: Examples (6.2.1)-(6.2.3), $\nu = 0$, $\lambda = 1/2$, sine transformation φ_2, local superconvergence (6.2.5), uniform grid.

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(1.5,1)}$ $\tilde{\delta}_N^{(1.5,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(3,1)}$ $\tilde{\delta}_N^{(3,1)}$	$\gamma_N^{(3.5,1)}$ $\tilde{\delta}_N^{(3.5,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(6,1)}$ $\tilde{\delta}_N^{(6,1)}$
16	3.8 E - 2	8.6 E - 3	8.9 E - 4	4.4 E - 5	2.2 E - 5	6.4 E - 6	4.3 E - 6
	1.34	**2.00**	**3.23**	**5.08**	**8.13**	**13.36**	**14.29**
32	2.8 E - 2	4.3 E - 3	2.8 E - 4	8.7 E - 6	2.7 E - 6	4.8 E - 7	3.0 E - 7
	1.40	**2.11**	**3.39**	**5.58**	**8.62**	**13.64**	**14.31**
64	2.0 E - 2	2.0 E - 3	8.1 E - 5	1.6 E - 6	3.2 E - 7	3.5 E - 8	2.1 E - 8
	1.45	**2.21**	**3.50**	**5.92**	**8.95**	**13.90**	**14.54**
128	1.4 E - 2	9.2 E - 4	2.3 E - 5	2.6 E - 7	3.6 E - 8	2.5 E - 9	1.5 E - 9
	1.51	**2.29**	**3.58**	**6.18**	**9.19**	**14.13**	**14.69**
256	9.2 E - 3	4.0 E - 4	6.5 E - 6	4.3 E - 8	3.9 E - 9	1.8 E - 10	1.0 E - 10
	1.55	**2.35**	**3.64**	**6.37**	**9.38**	**14.32**	**14.79**
512	6.0 E - 3	1.7 E - 4	1.8 E - 6	6.7 E - 9	4.1 E - 10	1.2 E - 11	6.7 E - 12
	1.60	**2.41**	**3.68**	**6.52**	**9.54**	**14.48**	**14.65**
1024	3.7 E - 3	7.1 E - 5	4.9 E - 7	1.0 E - 9	4.3 E - 11	8.6 E - 13	4.6 E - 13
	1.64	**2.45**	**3.71**	**6.65**	**9.67**	**14.61**	**10.77**
2048	2.3 E - 3	2.9 E - 5	1.3 E - 7	1.5 E - 10	4.5 E - 12	5.9 E - 14	4.3 E - 14
	1.65	2.34	3.31	6.61	9.35	12.02	14.55

Tables 6.2.3-6.2.4 describe the local superconvergence errors for examples (6.2.1)-(6.2.3) and for the sine transformation φ_2. The expected theoretical results are presented in (6.2.5). The observed ratios $\tilde{\delta}_N^{(\rho,r)}$ are overall in good agreement with Theorem 5.3.3 and (6.2.5). We point out that some loss of accuracy is present in our calculations in Tables 6.2.3-6.2.4 for $N = 2048$, mostly where $\gamma_N^{(\rho,r)} \approx 10^{-14}$, see $\rho = 6, r = 1$ and $\rho = 3, r = 1.5$.

The same phenomenon about special values of ρ we described for global convergence (Tables 6.2.1-6.2.2) can be seen in Tables 6.2.3-6.2.4 as well: compare $\rho = 2, r = 1$ with $\rho = 1, r = 2$; $\rho = 4, r = 1$ with $\rho = 1, r = 4$.

When there was some kind of deficit of the ratios $\delta_N^{(\rho,r)}$ near the optimal values of ρ and r then for the local superconvergence we have the opposite effect: the ratios $\tilde{\delta}_N^{(\rho,r)}$ are often a little bit better than the expected theoretical results for the optimal values of the parameters ρ and r. See $\rho = 4, r = 1$; $\rho = 2, r = 2$; $\rho = 1, r = 6$ in Tables 6.2.3-6.2.4.

By Remark 5.3.1 we can use real values of $\rho \in [1, \rho)$ and $\varphi \in \Phi^{m+1,\rho}$ instead of $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,1]$. Our calculations confirm this for the different values of $\rho \in \mathbb{R}$, in Table 6.2.3 see $\rho = 1.5, r = 1$ and $\rho = 3.5, r = 1$.

Table 6.2.4: Examples (6.2.1)-(6.2.3), $\nu = 0$, $\lambda = 1/2$, sine transformation φ_2, local superconvergence (6.2.5), graded grid.

N	$\gamma_N^{(1,2)}$ $\tilde{\delta}_N^{(1,2)}$	$\gamma_N^{(1,3)}$ $\tilde{\delta}_N^{(1,3)}$	$\gamma_N^{(1,4)}$ $\tilde{\delta}_N^{(1,4)}$	$\gamma_N^{(2,2)}$ $\tilde{\delta}_N^{(2,2)}$	$\gamma_N^{(3,1.5)}$ $\tilde{\delta}_N^{(3,1.5)}$	$\gamma_N^{(1,6)}$ $\tilde{\delta}_N^{(1,6)}$	$\gamma_N^{(3,3)}$ $\tilde{\delta}_N^{(3,3)}$
16	9.2 E - 3	1.4 E - 3	2.5 E - 4	8.1 E - 6	4.5 E - 6	4.8 E - 5	5.2 E - 5
	2.48	**5.15**	**9.27**	**12.22**	**13.30**	**21.08**	**12.26**
32	3.7 E - 3	2.6 E - 4	2.7 E - 5	6.6 E - 7	3.3 E - 7	2.3 E - 6	4.2 E - 6
	2.74	**5.68**	**10.68**	**13.15**	**13.61**	**18.03**	**12.96**
64	1.4 E - 3	4.6 E - 5	2.5 E - 6	5.0 E - 8	2.5 E - 8	1.3 E - 7	3.2 E - 7
	2.94	**6.04**	**11.41**	**13.67**	**14.01**	**14.48**	**13.40**
128	4.6 E - 4	7.7 E - 6	2.2 E - 7	3.7 E - 9	1.8 E - 9	8.7 E - 9	2.4 E - 8
	3.10	**6.16**	**11.79**	**14.00**	**14.28**	**14.77**	**13.90**
256	1.5 E - 4	1.2 E - 6	1.9 E - 8	2.6 E - 10	1.2 E - 10	5.9 E - 10	1.7 E - 9
	3.21	**6.30**	**12.31**	**14.23**	**14.48**	**14.91**	**14.23**
512	4.6 E - 5	2.0 E - 7	1.5 E - 9	1.8 E - 11	8.5 E - 12	3.9 E - 11	1.2 E - 10
	3.30	**6.48**	**12.70**	**14.41**	**14.20**	**15.02**	**14.44**
1024	1.4 E - 5	3.1 E - 8	1.2 E - 10	1.3 E - 12	6.0 E - 13	2.6 E - 12	8.5 E - 12
	3.36	**6.62**	**13.02**	**14.56**	**11.84**	**15.45**	**14.50**
2048	4.2 E - 6	4.6 E - 9	9.3 E - 12	8.8 E - 14	5.1 E - 14	1.7 E - 13	5.8 E - 13
	3.31	6.61	12.02	12.02	14.55	14.55	14.55

Table 6.2.5: Examples (6.2.1)-(6.2.3), $\nu = 0$, $\lambda = 1/2$, polynomial transformation φ_3, global convergence (6.2.4).

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(1,1.25)}$ $\delta_N^{(1,1.25)}$	$\varepsilon_N^{(2,1)}$ $\delta_N^{(2,1)}$	$\varepsilon_N^{(3,1)}$ $\delta_N^{(3,1)}$	$\varepsilon_N^{(4,1)}$ $\delta_N^{(4,1)}$	$\varepsilon_N^{(5,1)}$ $\delta_N^{(5,1)}$	$\varepsilon_N^{(6,1)}$ $\delta_N^{(6,1)}$
16	2.2 E - 2	1.1 E - 2	5.1 E - 3	1.9 E - 3	2.4 E - 4	3.5 E - 4	2.3 E - 4
	1.94	**2.32**	**2.28**	**3.99**	**4.00**	**5.03**	**7.67**
32	1.1 E - 2	4.8 E - 3	2.3 E - 3	4.8 E - 4	5.9 E - 5	6.9 E - 5	3.0 E - 5
	1.96	**2.35**	**2.38**	**4.02**	**3.08**	**5.84**	**8.01**
64	5.7 E - 3	2.0 E - 3	9.5 E - 4	1.2 E - 4	1.9 E - 5	1.2 E - 5	3.8 E - 6
	1.98	**2.36**	**2.45**	**4.02**	**3.90**	**6.32**	**7.82**
128	2.9 E - 3	8.6 E - 4	3.9 E - 4	2.9 E - 5	4.9 E - 6	1.9 E - 6	4.8 E - 7
	1.99	**2.37**	**2.50**	**4.02**	**4.33**	**6.63**	**7.98**
256	1.4 E - 3	3.6 E - 4	1.6 E - 4	7.3 E - 6	1.1 E - 6	2.8 E - 7	6.1 E - 8
	1.99	**2.38**	**2.53**	**4.01**	**4.59**	**6.84**	**8.00**
512	7.3 E - 4	1.5 E - 4	6.1 E - 5	1.8 E - 6	2.5 E - 7	4.1 E - 8	7.6 E - 9
	2.00	**2.38**	**2.56**	**4.01**	**4.77**	**6.99**	**8.00**
1024	3.6 E - 4	6.6 E - 5	2.4 E - 5	4.5 E - 7	5.2 E - 8	5.9 E - 9	9.5 E - 10
	2.00	**2.38**	**2.58**	**4.00**	**4.89**	**7.11**	**8.00**
2048	1.8 E - 4	2.7 E - 5	9.3 E - 6	1.1 E - 7	1.1 E - 8	8.3 E - 10	1.2 E - 10
	1.82	2.16	2.57	3.64	5.14	7.27	8.00

Table 6.2.6: Examples (6.2.1)-(6.2.3), $\nu = 0$, $\lambda = 1/2$, polynomial transformation φ_3, local superconvergence (6.2.5).

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(1,1.25)}$ $\tilde{\delta}_N^{(1,1.25)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(1,1.75)}$ $\tilde{\delta}_N^{(1,1.75)}$	$\gamma_N^{(3,1)}$ $\tilde{\delta}_N^{(3,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(2,2)}$ $\tilde{\delta}_N^{(2,2)}$
16	1.0 E - 3	3.3 E - 4	6.7 E - 5	2.8 E - 5	1.6 E - 5	5.1 E - 6	9.4 E - 6
	3.21	**4.60**	**4.80**	**9.38**	**12.75**	**14.74**	**13.13**
32	3.3 E - 4	7.1 E - 5	1.4 E - 5	3.0 E - 6	1.3 E - 6	3.5 E - 7	7.1 E - 7
	3.37	**4.83**	**5.42**	**9.76**	**13.44**	**14.70**	**13.68**
64	9.7 E - 5	1.5 E - 5	2.6 E - 6	3.0 E - 7	9.6 E - 8	2.4 E - 8	5.2 E - 8
	3.49	**4.98**	**5.82**	**9.98**	**13.38**	**14.91**	**14.05**
128	2.8 E - 5	3.0 E - 6	4.4 E - 7	3.0 E - 8	7.0 E - 9	1.6 E - 9	3.7 E - 9
	3.57	**5.07**	**6.11**	**10.14**	**14.09**	**15.00**	**14.30**
256	7.8 E - 6	5.8 E - 7	7.3 E - 8	3.0 E - 9	4.9 E - 10	1.1 E - 10	2.6 E - 10
	3.63	**5.14**	**6.32**	**10.25**	**14.30**	**15.06**	**14.48**
512	2.1 E - 6	1.1 E - 7	1.2 E - 8	2.9 E - 10	3.5 E - 11	7.0 E - 12	1.8 E - 11
	3.67	**5.19**	**6.48**	**10.35**	**14.46**	**15.06**	**14.51**
1024	5.8 E - 7	2.2 E - 8	1.8 E - 9	2.8 E - 11	2.4 E - 12	4.7 E - 13	1.2 E - 12
	3.70	**5.23**	**6.62**	**10.42**	**14.58**	**5.62**	**12.72**
2048	1.6 E - 7	4.2 E - 9	2.7 E - 10	2.7 E - 12	1.6 E - 13	8.3 E - 14	9.7 E - 14
	3.31	4.68	6.61	9.35	12.02	14.55	14.55

Tables 6.2.5-6.2.6 describe examples (6.2.1)-(6.2.3) for the polynomial transformation φ_3. For the given $\rho \in \mathbb{N}$ we have $\varphi_3 \in \Phi^{m+1,\rho+1} \cap C^{m+1}[0,1]$, $m \in \mathbb{N}$. The numerical results in Tables 6.2.5-6.2.6 are similar to the ones we observed in Tables 6.2.1-6.2.4 for φ_2. Special choices of parameters are $\rho = 1, r = 1$ and $\rho = 3, r = 1$.

Table 6.2.6 shows the local superconvergence errors. We have some loss of accuracy in calculations using $N = 2048$ for $\rho = 4, r = 1$ and $\rho = 2, r = 2$. Most of the sets of parameters ρ and r produce a little bit better ratios $\tilde{\delta_N}^{(\rho,r)}$ than the expected theoretical ones in (6.2.5) (this is related with even and odd values of ρ).

Next, we consider equation (6.1.1), where

$$K(x,y) = \log(x-y)\, y^{-8/10}, \quad (x,y) \in D_1, \tag{6.2.6}$$

$$f(x) = 1. \tag{6.2.7}$$

Here the exact solution u is unknown. For the numerical tests we use the approximation u_N obtained with the parameters $m = 3$, $N = 8192$, $\rho = 8$, $r = 2$ and $\varphi(t) = t^8$:

$$u(x) \approx u_{8192}(x) = v_{8192}(x^{1/8}). \tag{6.2.8}$$

By Theorem 4.2.2 the solution u must be an element at least in the set $C_*^{m,8/10}(0,1]$, $m \in \mathbb{N}$. It may happen that because of a very smooth function $f(x) = 1$ the solution u belongs to a smoother set $C_*^{m,\theta}(0,1]$ for some $\theta < 8/10$ but the calculations in Tables 6.2.7-6.2.8 are in good agreement with (6.2.9) and (6.2.10).

Clearly $K \in W^{m+1,0,8/10}(D_1)$ and $f \in C^{m+1}[0,1] \subset C_*^{m+1,8/10}(0,1]$ for arbitrary $m \in \mathbb{N}$. From Theorem 4.4.4 it follows that for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r > \frac{15}{\rho} \\ 2^{r\rho/5}\left(\frac{\log(N/2)}{\log N}\right) & \text{if } 1 \le r \le \frac{15}{\rho} \end{array} \right\}. \tag{6.2.9}$$

From Theorem 5.3.3 it follows that for sufficiently large values of N we have

$$\tilde{\delta_N}^{(\rho,r)} = \left\{ \begin{array}{ll} 2^4\left(\frac{\log(N/2)}{\log N}\right) & \text{if } r > \frac{10}{\rho} \\ 2^4\left(\frac{\log(N/2)}{\log N}\right)^3 & \text{if } r = \frac{10}{\rho} \\ 2^{2r\rho/5}\left(\frac{\log(N/2)}{\log N}\right)^2 & \text{if } 1 \le r < \frac{10}{\rho} \end{array} \right\}. \tag{6.2.10}$$

Table 6.2.7: Examples (6.2.6)-(6.2.8), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, global convergence (6.2.9), uniform grid.

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(4,1)}$ $\delta_N^{(4,1)}$	$\varepsilon_N^{(7,1)}$ $\delta_N^{(7,1)}$	$\varepsilon_N^{(10,1)}$ $\delta_N^{(10,1)}$	$\varepsilon_N^{(14,1)}$ $\delta_N^{(14,1)}$	$\varepsilon_N^{(15,1)}$ $\delta_N^{(15,1)}$	$\varepsilon_N^{(16,1)}$ $\delta_N^{(16,1)}$
16	9.1 E - 1	5.7 E - 1	2.9 E - 2	1.8 E - 2	2.5 E - 3	3.0 E - 3	3.5 E - 3
	1.00	**1.28**	**1.27**	**4.99**	**5.70**	**6.72**	**6.61**
32	9.0 E - 1	4.5 E - 1	2.3 E - 2	3.7 E - 3	4.3 E - 4	4.4 E - 4	5.3 E - 4
	1.00	**1.38**	**1.95**	**4.71**	**5.33**	**7.48**	**7.43**
64	9.0 E - 1	3.3 E - 1	1.2 E - 2	7.8 E - 4	8.1 E - 5	5.9 E - 5	7.2 E - 5
	1.01	**1.49**	**2.31**	**4.32**	**5.69**	**7.83**	**7.81**
128	9.0 E - 1	2.2 E - 1	5.1 E - 3	1.8 E - 4	1.4 E - 5	7.5 E - 6	9.2 E - 6
	1.01	**1.57**	**2.39**	**4.12**	**5.90**	**7.00**	**7.96**
256	8.9 E - 1	1.4 E - 1	2.1 E - 3	4.4 E - 5	2.4 E - 6	1.1 E - 6	1.2 E - 6
	1.01	**1.63**	**2.40**	**4.04**	**6.04**	**6.99**	**8.00**
512	8.9 E - 1	8.5 E - 2	8.9 E - 4	1.1 E - 5	4.0 E - 7	1.5 E - 7	1.4 E - 7
	1.01	**1.66**	**2.41**	**4.01**	**6.15**	**7.11**	**8.03**
1024	8.8 E - 1	5.1 E - 2	3.7 E - 4	2.7 E - 6	6.5 E - 8	2.2 E - 8	1.8 E - 8
	1.01	**1.67**	**2.42**	**4.00**	**6.23**	**7.19**	**8.03**
2048	8.7 E - 1	3.1 E - 2	1.5 E - 4	6.8 E - 7	1.0 E - 8	3.0 E - 9	2.2 E - 9
	1.04	1.58	2.40	3.64	6.33	7.27	8.00

For examples (6.2.6)-(6.2.8), global convergence results are reported in Tables 6.2.7-6.2.8 and the superconvergence results are reported in Tables 6.2.9-6.2.10. By the numerical results, special ρ values seem to be $\rho = 5$ and $\rho = 10$.

We have some loss of accuracy in our calculations with $N = 2048$ in Tables 6.2.9-6.2.10 for $\rho = 10, r = 1$, $\rho = 14, r = 1$ and $\rho = 4, r = 4$ where $\tilde{\delta}_N^{(\rho,r)} \approx 10^{-11}$. The ratios $\tilde{\delta}_N^{(\rho,r)}$ are smaller than one for $\rho = 1, r = 1$ but note that the actual errors $\gamma_N^{(\rho,r)}$ are relatively small. The superconvergence results for graded grids

Table 6.2.8: Examples (6.2.6)-(6.2.8), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, global convergence (6.2.9), graded grid.

N	$\varepsilon_N^{(2,2)}$ $\delta_N^{(2,2)}$	$\varepsilon_N^{(1,7)}$ $\delta_N^{(1,7)}$	$\varepsilon_N^{(5,1.6)}$ $\delta_N^{(5,1.6)}$	$\varepsilon_N^{(3,3)}$ $\delta_N^{(3,3)}$	$\varepsilon_N^{(1,10)}$ $\delta_N^{(1,10)}$	$\varepsilon_N^{(1,14)}$ $\delta_N^{(1,14)}$	$\varepsilon_N^{(1,16)}$ $\delta_N^{(1,16)}$
16	7.5 E - 1	5.7 E - 1	6.4 E - 2	1.6 E - 1	2.5 E - 1	1.4 E - 1	1.2 E - 1
	1.13	**1.51**	**3.59**	**2.77**	**2.18**	**3.13**	**3.76**
32	6.7 E - 1	3.8 E - 1	1.8 E - 2	5.6 E - 2	1.2 E - 1	4.5 E - 2	3.1 E - 2
	1.18	**1.78**	**3.38**	**2.98**	**2.48**	**4.40**	**5.47**
64	5.7 E - 1	2.1 E - 1	5.3 E - 3	1.9 E - 2	4.7 E - 2	1.0 E - 2	5.7 E - 3
	1.25	**2.02**	**3.17**	**3.08**	**3.08**	**5.28**	**6.68**
128	4.5 E - 1	1.0 E - 1	1.7 E - 3	6.1 E - 3	1.5 E - 2	1.9 E - 3	8.6 E - 4
	1.32	**2.20**	**3.08**	**3.13**	**3.35**	**5.74**	**7.34**
256	3.4 E - 1	4.8 E - 2	5.4 E - 4	2.0 E - 3	4.6 E - 3	3.4 E - 4	1.2 E - 4
	1.40	**2.30**	**3.05**	**3.17**	**3.48**	**5.90**	**7.66**
512	2.5 E - 1	2.1 E - 2	1.8 E - 4	6.2 E - 4	1.3 E - 3	5.7 E - 5	1.5 E - 5
	1.46	**2.36**	**3.04**	**3.20**	**3.55**	**6.04**	**7.89**
1024	1.7 E - 1	8.8 E - 3	5.8 E - 5	1.9 E - 4	3.7 E - 4	9.5 E - 6	1.9 E - 6
	1.52	**2.40**	**3.03**	**3.22**	**3.60**	**6.15**	**8.03**
2048	1.1 E - 1	3.7 E - 3	1.9 E - 5	6.0 E - 5	1.0 E - 4	1.5 E - 6	2.4 E - 7
	1.58	2.40	2.76	3.17	3.64	6.33	8.00

Table 6.2.9: Examples (6.2.6)-(6.2.8), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, local superconvergence (6.2.10), uniform grid.

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(5,1)}$ $\tilde{\delta}_N^{(5,1)}$	$\gamma_N^{(8,1)}$ $\tilde{\delta}_N^{(8,1)}$	$\gamma_N^{(9,1)}$ $\tilde{\delta}_N^{(9,1)}$	$\gamma_N^{(10,1)}$ $\tilde{\delta}_N^{(10,1)}$	$\gamma_N^{(14,1)}$ $\tilde{\delta}_N^{(14,1)}$
16	1.2 E - 2	9.7 E - 2	5.4 E - 2	1.7 E - 3	1.4 E - 3	7.2 E - 4	2.6 E - 4
	0.75	**1.23**	**2.38**	**3.70**	**8.85**	**16.75**	**9.67**
32	1.6 E - 2	7.9 E - 2	2.3 E - 2	4.6 E - 4	1.6 E - 4	4.3 E - 5	2.7 E - 5
	0.81	**1.55**	**3.15**	**6.86**	**10.98**	**17.60**	**11.83**
64	2.0 E - 2	5.1 E - 2	7.2 E - 3	6.6 E - 5	1.5 E - 5	2.5 E - 6	2.3 E - 6
	0.84	**1.89**	**3.71**	**7.71**	**10.85**	**14.85**	**13.14**
128	2.3 E - 2	2.7 E - 2	2.0 E - 3	8.6 E - 6	1.3 E - 6	1.6 E - 7	1.7 E - 7
	0.85	**2.19**	**3.93**	**7.74**	**10.50**	**16.37**	**13.03**
256	2.7 E - 2	1.2 E - 2	5.0 E - 4	1.1 E - 6	1.3 E - 7	1.0 E - 8	1.3 E - 8
	0.87	**2.41**	**3.93**	**7.70**	**10.38**	**15.39**	**13.38**
512	3.2 E - 2	5.1 E - 3	1.3 E - 4	1.4 E - 7	1.2 E - 8	6.6 E - 10	9.9 E - 10
	0.88	**2.55**	**3.86**	**7.73**	**10.39**	**14.71**	**14.59**
1024	3.6 E - 2	2.0 E - 3	3.3 E - 5	1.9 E - 8	1.2 E - 9	4.5 E - 11	6.8 E - 11
	0.88	**2.63**	**3.81**	**7.77**	**10.52**	**2.02**	**2.79**
2048	4.1 E - 2	7.7 E - 4	8.6 E - 6	2.4 E - 9	1.1 E - 10	2.2 E - 11	2.4 E - 11
	1.09	2.51	3.31	7.59	10.02	12.02	14.55

Table 6.2.10: Examples (6.2.6)-(6.2.8), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, local superconvergence (6.2.10), graded grid.

N	$\gamma_N^{(2,2)}$ $\tilde{\delta}_N^{(2,2)}$	$\gamma_N^{(1,5)}$ $\tilde{\delta}_N^{(1,5)}$	$\gamma_N^{(1,6)}$ $\tilde{\delta}_N^{(1,6)}$	$\gamma_N^{(1,8)}$ $\tilde{\delta}_N^{(1,8)}$	$\gamma_N^{(3,3)}$ $\tilde{\delta}_N^{(3,3)}$	$\gamma_N^{(1,10)}$ $\tilde{\delta}_N^{(1,10)}$	$\gamma_N^{(4,4)}$ $\tilde{\delta}_N^{(4,4)}$
16	8.5 E - 2	9.9 E - 2	1.2 E - 1	8.5 E - 2	1.3 E - 2	4.3 E - 2	2.8 E - 4
	0.83	**0.84**	**1.17**	**2.78**	**6.96**	**3.41**	**10.94**
32	1.0 E - 1	1.2 E - 1	1.0 E - 1	3.1 E - 2	1.9 E - 3	1.3 E - 2	2.6 E - 5
	1.02	**1.18**	**1.84**	**3.00**	**7.33**	**5.25**	**11.17**
64	1.0 E - 1	1.0 E - 1	5.4 E - 2	1.0 E - 2	2.6 E - 4	2.4 E - 3	2.3 E - 6
	1.27	**1.62**	**2.59**	**4.67**	**8.38**	**7.73**	**12.74**
128	8.0 E - 2	6.2 E - 2	2.1 E - 2	2.2 E - 3	3.1 E - 5	3.1 E - 4	1.8 E - 7
	1.55	**2.08**	**3.20**	**5.49**	**9.17**	**9.62**	**12.89**
256	5.2 E - 2	3.0 E - 2	6.5 E - 3	4.0 E - 4	3.3 E - 6	3.2 E - 5	1.4 E - 8
	1.82	**2.50**	**3.36**	**6.44**	**9.60**	**10.97**	**13.96**
512	2.8 E - 2	1.2 E - 2	1.9 E - 3	6.2 E - 5	3.5 E - 7	2.9 E - 6	1.0 E - 9
	2.05	**2.83**	**3.83**	**6.86**	**9.87**	**11.94**	**14.95**
1024	1.4 E - 2	4.2 E - 3	5.1 E - 4	9.1 E - 6	3.5 E - 8	2.5 E - 7	6.8 E - 11
	2.24	**3.06**	**4.12**	**7.24**	**10.07**	**12.42**	**2.72**
2048	6.1 E - 3	1.4 E - 3	1.2 E - 4	1.3 E - 6	3.5 E - 9	2.0 E - 8	2.5 E - 11
	2.51	3.31	4.36	7.59	10.02	12.02	14.55

show a little bit smaller ratios $\tilde{\delta}_N^{(\rho,r)}$, in this case note that the actual errors $\gamma_N^{(\rho,r)}$ are relatively large for the starting values of N, but they decrease slowly near the theoretical values.

6.3 Case $K \in W_*^{m+1,0,\lambda}(D_b)$ and $f \in C^{m+1,\lambda}(0,b]$

We consider the equation (6.1.1), where

$$K(x,y) = \log(x-y)\frac{y^{-8/10}}{1-\log y}, \quad (x,y) \in D_1, \tag{6.3.1}$$

$$f(x) = 1. \tag{6.3.2}$$

Here the exact solution u is unknown. For the numerical tests we use the approximation u_N obtained with the parameters $m = 3$, $N = 8192$, $\rho = 8$, $r = 2$ and $\varphi(t) = t^8$:

$$u(x) \approx u_{8192}(x) = v_{8192}(x^{1/8}). \tag{6.3.3}$$

By Theorem 4.2.2 the solution u must be an element at least in the set $C^{m,8/10}(0,1]$, $m \in \mathbb{N}$. It may happen that because of a very smooth function $f(x) = 1$ the solution u belongs to a smoother set $C^{m,\theta}(0,1]$ for some $\theta < 8/10$ but the calculations in Tables 6.3.1-6.3.2 are in good agreement with (6.3.4) and (6.3.5). For the numerical results we use $m = 3$ and the node points (6.1.6).

6.3. *Case* $K \in W_*^{m+1,0,\lambda}(D_b)$ *and* $f \in C^{m+1,\lambda}(0,b]$

Clearly $K \in W_*^{m+1,0,8/10}(D_1)$ and $f \in C^{m+1}[0,1] \subset C^{m+1,8/10}(0,1]$ for arbitrary $m \in \mathbb{N}$. Since $\nu + \lambda = 8/10 > 0$ then from Theorem 4.4.5 it follows that for any transformation $\varphi \in \Phi^{m,\rho}$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r \geq \frac{15}{\rho} \\ 2^{r\rho/5} & \text{if } 1 \leq r < \frac{15}{\rho} \end{array} \right\}. \tag{6.3.4}$$

From Theorem 5.3.1 it follows that for any transformation $\varphi \in \Phi^{m+1,\rho}$ and for sufficiently large values of N we have

$$\tilde{\delta_N}^{(\rho,r)} = \left\{ \begin{array}{ll} 2^4 \left(\frac{\log(N/2)}{\log N}\right) & \text{if } r \geq \frac{10}{\rho} \\ 2^{2r\rho/5} & \text{if } 1 \leq r < \frac{10}{\rho} \end{array} \right\}. \tag{6.3.5}$$

Examples (6.3.1)-(6.3.3) are similar to the examples (6.2.6)-(6.2.8), where we do not have the part $1/(1 - \log y)$ in the kernel $K(x,y)$. Still, there exist differences in the theory (compare (6.2.9), (6.2.10) with (6.3.4), (6.3.5)) and in the numerical results as well.

Table 6.3.1: Examples (6.3.1)-(6.3.3), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, global convergence (6.3.4), uniform grid.

N	$\varepsilon_N^{(1,1)}$	$\varepsilon_N^{(4,1)}$	$\varepsilon_N^{(7,1)}$	$\varepsilon_N^{(10,1)}$	$\varepsilon_N^{(14,1)}$	$\varepsilon_N^{(15,1)}$	$\varepsilon_N^{(16,1)}$
	$\delta_N^{(1,1)}$	$\delta_N^{(4,1)}$	$\delta_N^{(7,1)}$	$\delta_N^{(10,1)}$	$\delta_N^{(14,1)}$	$\delta_N^{(15,1)}$	$\delta_N^{(16,1)}$
16	4.9 E - 1	2.5 E - 2	2.3 E - 3	3.0 E - 4	5.2 E - 4	5.7 E - 4	6.1 E - 4
	1.06	**1.80**	**1.71**	**5.01**	**3.92**	**3.67**	**3.42**
32	4.7 E - 1	1.4 E - 2	8.5 E - 4	6.0 E - 5	1.3 E - 4	1.5 E - 4	1.8 E - 4
	1.06	**1.78**	**1.65**	**6.43**	**5.75**	**5.58**	**5.42**
64	4.4 E - 1	7.8 E - 3	3.2 E - 4	9.3 E - 6	2.3 E - 5	2.8 E - 5	3.3 E - 5
	1.07	**1.76**	**2.63**	**7.24**	**6.86**	**6.77**	**6.68**
128	4.1 E - 1	4.4 E - 3	1.2 E - 4	1.3 E - 6	3.4 E - 6	4.1 E - 6	4.9 E - 6
	1.07	**1.75**	**2.62**	**7.64**	**7.46**	**7.41**	**7.37**
256	3.8 E - 1	2.5 E - 3	4.6 E - 5	1.7 E - 7	4.5 E - 7	5.5 E - 7	6.7 E - 7
	1.08	**1.75**	**2.62**	**7.84**	**7.75**	**7.73**	**7.70**
512	3.6 E - 1	1.5 E - 3	1.8 E - 5	2.1 E - 8	5.8 E - 8	7.2 E - 8	8.7 E - 8
	1.08	**1.74**	**2.62**	**7.98**	**7.90**	**7.89**	**7.88**
1024	3.3 E - 1	8.3 E - 4	6.7 E - 6	2.7 E - 9	7.4 E - 9	9.1 E - 9	1.1 E - 8
	1.09	**1.74**	**2.62**	**5.13**	**8.09**	**8.04**	**7.99**
2048	3.0 E - 1	4.8 E - 4	2.6 E - 6	5.2 E - 10	9.1 E - 10	1.1 E - 9	1.4 E - 9
	1.15	1.74	2.64	4.00	6.96	8.00	8.00

Tables 6.3.1-6.3.2 describe global convergence errors for examples (6.3.1)-(6.3.3) with the exponential transformation $\varphi_1(t) = t^\rho$. The results for the uniform grid in Table 6.3.1 are mostly in good agreement with the theoretical one (6.3.4), but the choices of parameters $\rho = 10, r = 1$ and $\rho = 14, r = 1$ are much better than

Table 6.3.2: Examples (6.3.1)-(6.3.3), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, global convergence (6.3.4), graded grid.

N	$\varepsilon_N^{(2,2)}$ $\delta_N^{(2,2)}$	$\varepsilon_N^{(1,7)}$ $\delta_N^{(1,7)}$	$\varepsilon_N^{(5,1.6)}$ $\delta_N^{(5,1.6)}$	$\varepsilon_N^{(3,3)}$ $\delta_N^{(3,3)}$	$\varepsilon_N^{(1,10)}$ $\delta_N^{(1,10)}$	$\varepsilon_N^{(1,14)}$ $\delta_N^{(1,14)}$	$\varepsilon_N^{(1,16)}$ $\delta_N^{(1,16)}$
16	1.1 E - 1	4.5 E - 2	1.9 E - 4	4.0 E - 3	1.9 E - 2	1.1 E - 2	9.5 E - 3
	1.63	**2.49**	**5.27**	**3.43**	**3.72**	**6.01**	**7.29**
32	7.0 E - 2	1.8 E - 2	3.6 E - 5	1.2 E - 3	5.1 E - 3	1.8 E - 3	1.3 E - 3
	1.67	**2.56**	**3.78**	**3.44**	**3.86**	**6.58**	**8.35**
64	4.2 E - 2	7.1 E - 3	9.6 E - 6	3.4 E - 4	1.3 E - 3	2.7 E - 4	1.6 E - 4
	1.69	**2.59**	**3.69**	**3.45**	**3.92**	**6.77**	**8.79**
128	2.5 E - 2	2.6 E - 3	2.6 E - 6	9.9 E - 5	3.4 E - 4	3.9 E - 5	1.8 E - 5
	1.70	**2.60**	**3.62**	**3.46**	**3.94**	**6.84**	**8.95**
256	1.5 E - 2	2.6 E - 3	7.2 E - 7	2.9 E - 5	8.5 E - 5	5.7 E - 6	2.0 E - 6
	1.71	**2.61**	**3.57**	**3.46**	**3.95**	**6.88**	**8.76**
512	8.5 E - 3	2.6 E - 4	2.0 E - 7	8.3 E - 6	2.2 E - 5	8.4 E - 7	2.3 E - 7
	1.72	**2.62**	**3.53**	**3.46**	**3.96**	**6.90**	**8.21**
1024	4.9 E - 3	2.6 E - 4	5.7 E - 8	2.4 E - 6	5.4 E - 6	1.2 E - 7	2.8 E - 8
	1.72	**2.62**	**3.49**	**3.47**	**3.97**	**6.91**	**8.10**
2048	2.9 E - 3	2.6 E - 5	1.6 E - 8	6.9 E - 7	1.4 E - 6	1.8 E - 8	3.4 E - 9
	1.74	2.64	3.03	3.48	4.00	6.96	8.00

the expected theoretical ones. The results for graded grids are reported in Table 6.3.2. We can see, that the related values $\rho = 1, r = 10$ and $\rho = 1, r = 14$ do not produce better results than the theoretical ones.

Tables 6.3.3-6.3.4 describe superconvergence errors for examples (6.3.1)-(6.3.3). We can see some roundoff errors for the values $\gamma_N^{(\rho,r)} \approx 10^{-11}$. Typically the numerical results in Tables 6.3.3-6.3.4 are slightly worse but close to the expected theoretical ones in (6.3.5) for the smaller values of ρ, r and are slightly better for the optimal values of ρ and r.

6.4 Case $K \in W^{m+1,\nu,\lambda}(D_b)$ and $f \in C^{m+1,\nu+\lambda}(0,b]$

We consider the equation (6.1.1), where

$$K(x,y) = (x-y)^{-1/2}\, y^{-3/10}, \quad (x,y) \in D_1, \tag{6.4.1}$$

$$f(x) = x^{1/5}\left(1 - \frac{\Gamma(\frac{1}{2})\Gamma(\frac{9}{10})}{\Gamma(\frac{7}{5})} x^{1/5} - x^{4/5} + \frac{\Gamma(\frac{1}{2})\Gamma(\frac{17}{10})}{\Gamma(\frac{22}{10})} x\right), \tag{6.4.2}$$

$$u(x) = x^{1/5} - x, \quad x \in [0,1]. \tag{6.4.3}$$

Table 6.3.3: Examples (6.3.1)-(6.3.3), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, local superconvergence (6.3.5), uniform grid.

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(5,1)}$ $\tilde{\delta}_N^{(5,1)}$	$\gamma_N^{(8,1)}$ $\tilde{\delta}_N^{(8,1)}$	$\gamma_N^{(9,1)}$ $\tilde{\delta}_N^{(9,1)}$	$\gamma_N^{(10,1)}$ $\tilde{\delta}_N^{(10,1)}$	$\gamma_N^{(14,1)}$ $\tilde{\delta}_N^{(14,1)}$
16	5.4 E - 2	5.7 E - 4	1.3 E - 5	1.3 E - 5	1.7 E - 5	2.1 E - 5	3.0 E - 4
	1.00	**2.99**	**5.36**	**7.87**	**7.18**	**6.54**	**21.35**
32	5.5 E - 2	1.9 E - 4	2.5 E - 6	1.6 E - 6	2.4 E - 6	3.3 E - 6	1.4 E - 5
	1.03	**2.99**	**4.94**	**10.85**	**10.37**	**9.90**	**26.93**
64	5.3 E - 2	6.4 E - 5	5.1 E - 7	1.5 E - 7	2.3 E - 7	3.3 E - 7	5.2 E - 7
	1.05	**2.98**	**4.73**	**12.57**	**12.22**	**12.22**	**15.61**
128	5.1 E - 2	2.1 E - 5	1.1 E - 7	1.2 E - 8	1.9 E - 8	2.7 E - 8	3.3 E - 8
	1.08	**2.98**	**4.62**	**12.55**	**12.72**	**12.49**	**11.71**
256	4.7 E - 2	7.2 E - 6	2.3 E - 8	9.6 E - 10	1.5 E - 9	2.2 E - 9	2.8 E - 9
	1.10	**2.99**	**4.56**	**10.71**	**13.69**	**11.99**	**13.53**
512	4.3 E - 2	2.4 E - 6	5.1 E - 9	8.9 E - 11	1.1 E - 10	1.8 E - 10	2.1 E - 10
	1.12	**2.99**	**4.52**	**3.44**	**4.30**	**14.56**	**10.62**
1024	3.8 E - 2	8.0 E - 7	1.1 E - 9	2.6 E - 11	2.5 E - 11	1.2 E - 11	2.0 E - 11
	1.14	**2.99**	**4.48**	**1.20**	**1.20**	**1.20**	**1.88**
2048	3.4 E - 2	2.7 E - 7	2.5 E - 10	2.2 E - 11	2.1 E - 11	1.0 E - 11	1.1 E - 11
	1.32	3.03	4.00	9.19	12.13	14.55	14.55

Table 6.3.4: Examples (6.3.1)-(6.3.3), $\nu = 0$, $\lambda = 8/10$, exponential transformation φ_1, local superconvergence (6.3.5), graded grid.

N	$\gamma_N^{(2,2)}$ $\tilde{\delta}_N^{(2,2)}$	$\gamma_N^{(1,5)}$ $\tilde{\delta}_N^{(1,5)}$	$\gamma_N^{(1,6)}$ $\tilde{\delta}_N^{(1,6)}$	$\gamma_N^{(1,8)}$ $\tilde{\delta}_N^{(1,8)}$	$\gamma_N^{(3,3)}$ $\tilde{\delta}_N^{(3,3)}$	$\gamma_N^{(1,10)}$ $\tilde{\delta}_N^{(1,10)}$	$\gamma_N^{(4,4)}$ $\tilde{\delta}_N^{(4,4)}$
16	5.9 E - 3	6.3 E - 3	2.8 E - 3	9.6 E - 4	1.8 E - 5	5.0 E - 4	6.3 E - 5
	2.54	**3.22**	**4.00**	**6.95**	**7.88**	**11.16**	**4.95**
32	2.3 E - 3	1.9 E - 3	7.0 E - 4	1.4 E - 4	2.3 E - 6	4.4 E - 5	1.3 E - 5
	2.69	**3.49**	**4.52**	**7.80**	**10.63**	**13.02**	**8.11**
64	8.7 E - 4	5.6 E - 4	1.5 E - 4	1.8 E - 5	2.2 E - 7	3.4 E - 6	1.6 E - 6
	2.79	**3.54**	**4.81**	**8.34**	**12.16**	**14.33**	**10.86**
128	3.1 E - 4	1.6 E - 4	3.2 E - 5	2.1 E - 6	1.8 E - 8	2.4 E - 7	1.5 E - 7
	2.85	**3.67**	**4.96**	**8.64**	**13.13**	**14.85**	**12.48**
256	1.1 E - 4	4.3 E - 5	6.5 E - 6	2.5 E - 7	1.4 E - 9	1.6 E - 8	1.2 E - 8
	2.89	**3.76**	**5.05**	**8.80**	**13.44**	**15.19**	**13.04**
512	3.8 E - 5	1.1 E - 5	1.3 E - 6	2.8 E - 8	1.0 E - 10	1.1 E - 9	8.9 E - 10
	2.92	**3.82**	**5.07**	**8.85**	**4.10**	**15.35**	**14.47**
1024	1.3 E - 5	3.0 E - 6	2.5 E - 7	3.1 E - 9	2.5 E - 11	6.9 E - 11	6.2 E - 11
	2.94	**3.86**	**5.10**	**8.92**	**1.19**	**6.90**	**3.76**
2048	4.4 E - 6	7.7 E - 7	5.0 E - 8	3.5 E - 10	2.1 E - 11	9.9 E - 12	1.6 E - 11
	3.03	4.00	5.28	9.19	12.13	14.55	14.55

Note that $\nu = 1/2$ and $\lambda = 3/10$. Clearly $K \in W^{m+1,1/2,3/10}(D_1)$ and $f, u \in C^{m+1,1/5}(0,1]$ for arbitrary $m \in \mathbb{N}$. For the numerical results we use $m = 3$ and the node points (6.1.6).

Since $\nu + \lambda = 1/2 + 3/10 = 1/5 > 0$ from Theorem 4.4.5 it follows that for any transformation $\varphi \in \Phi^{m,\rho}$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r \geq \frac{15}{\rho} \\ 2^{r\rho/5} & \text{if } 1 \leq r < \frac{15}{\rho} \end{array} \right\}. \tag{6.4.4}$$

From Theorem 5.2.1 and Remark 5.2.1 it follows that for any transformation $\varphi \in \Phi^{m+1,\rho}$ and for sufficiently large values of N we have

$$\tilde{\delta_N}^{(\rho,r)} = \left\{ \begin{array}{ll} 2^{3.5} & \text{if } r \geq \frac{8.75}{\rho} \\ 2^{2r\rho/5} & \text{if } 1 \leq r < \frac{8.75}{\rho} \end{array} \right\}. \tag{6.4.5}$$

Table 6.4.1: Examples (6.4.1)-(6.4.3), $\nu = 1/2$, $\lambda = 3/10$, sine transformation φ_2, global convergence (6.4.4), uniform grid.

N	$\varepsilon_N^{(1,1)}$	$\varepsilon_N^{(4,1)}$	$\varepsilon_N^{(5,1)}$	$\varepsilon_N^{(6,1)}$	$\varepsilon_N^{(9,1)}$	$\varepsilon_N^{(11,1)}$	$\varepsilon_N^{(16,1)}$
	$\delta_N^{(1,1)}$	$\delta_N^{(4,1)}$	$\delta_N^{(5,1)}$	$\delta_N^{(6,1)}$	$\delta_N^{(9,1)}$	$\delta_N^{(11,1)}$	$\delta_N^{(16,1)}$
16	8.8 E + 1	5.5 E - 3	2.4 E - 4	8.1 E - 4	8.3 E - 4	1.4 E - 3	4.5 E - 3
	1.94	**1.75**	**12.78**	**2.31**	**9.14**	**8.56**	**8.31**
32	4.5 E + 1	3.1 E - 3	1.9 E - 5	3.5 E - 4	9.0 E - 5	1.7 E - 4	5.4 E - 4
	1.82	**1.75**	**11.85**	**2.30**	**9.28**	**8.80**	**7.58**
64	2.5 E + 1	1.8 E - 3	1.6 E - 6	1.5 E - 4	9.7 E - 6	1.9 E - 5	7.1 E - 5
	1.78	**1.74**	**11.68**	**2.30**	**4.58**	**8.99**	**9.40**
128	1.4 E + 1	1.0 E - 3	1.4 E - 7	6.6 E - 5	2.1 E - 6	2.1 E - 6	7.6 E - 6
	1.76	**1.74**	**10.94**	**2.30**	**3.48**	**8.93**	**8.81**
256	8.0 E + 0	5.9 E - 4	1.3 E - 8	2.9 E - 5	6.1 E - 7	2.4 E - 7	8.6 E - 7
	1.76	**1.74**	**10.70**	**2.30**	**3.48**	**8.76**	**8.51**
512	4.5 E + 0	3.4 E - 4	1.2 E - 9	1.3 E - 5	1.8 E - 7	2.7 E - 8	1.0 E - 7
	1.77	**1.74**	**8.12**	**2.30**	**3.48**	**7.83**	**8.48**
1024	2.6 E + 0	1.9 E - 4	1.4 E - 10	5.5 E - 6	5.0 E - 8	3.4 E - 9	1.2 E - 8
	1.78	**1.74**	**8.05**	**2.30**	**3.48**	**4.60**	**8.38**
2048	1.4 E + 0	1.1 E - 4	1.8 E - 11	2.4 E - 6	1.4 E - 8	7.5 E - 10	1.4 E - 9
	1.15	1.74	2.00	2.30	3.48	4.59	8.00

Tables 6.4.1-6.4.2 describe global convergence errors $\|u - u_N\|_\infty$ for examples (6.4.1)-(6.4.3) and for the sine transformation φ_2. The numerical results in Tables 6.4.1-6.4.2 are in good agreement with the theoretical estimates (6.4.4). The actual errors $\varepsilon_N^{(\rho,r)}$ are quite large for the parameters $\rho = 1, r = 1$ and thus also the ratios $\delta_N^{(\rho,r)}$ are better than the expected theoretical ones in (6.4.4). We can see the same for the superconvergence in Table 6.4.3.

Table 6.4.2: Examples (6.4.1)-(6.4.3), $\nu = 1/2$, $\lambda = 3/10$, sine transformation φ_2, global convergence (6.4.4), graded grid.

N	$\varepsilon_N^{(1,4)}$ $\delta_N^{(1,4)}$	$\varepsilon_N^{(1,5)}$ $\delta_N^{(1,5)}$	$\varepsilon_N^{(3,2)}$ $\delta_N^{(3,2)}$	$\varepsilon_N^{(3,3)}$ $\delta_N^{(3,3)}$	$\varepsilon_N^{(7.5,1.5)}$ $\delta_N^{(7.5,1.5)}$	$\varepsilon_N^{(1,16)}$ $\delta_N^{(1,16)}$	$\varepsilon_N^{(9,3)}$ $\delta_N^{(9,3)}$
16	1.9 E - 1	7.9 E - 2	5.2 E - 3	9.7 E - 4	1.4 E - 3	2.6 E - 1	4.4 E - 2
	5.26	**4.40**	**2.30**	**3.49**	**10.72**	**33.58**	**24.35**
32	3.6 E - 2	1.8 E - 2	2.2 E - 3	2.8 E - 4	1.3 E - 4	7.8 E - 3	1.8 E - 3
	1.75	**2.01**	**2.30**	**3.48**	**9.18**	**22.31**	**8.33**
64	2.1 E - 2	8.9 E - 3	9.7 E - 4	8.0 E - 5	1.4 E - 5	3.5 E - 4	2.2 E - 4
	1.75	**2.00**	**2.30**	**3.48**	**9.28**	**12.21**	**8.66**
128	1.2 E - 2	4.5 E - 3	4.2 E - 4	2.3 E - 5	1.5 E - 6	2.9 E - 5	2.5 E - 5
	1.75	**2.00**	**2.30**	**3.48**	**9.22**	**8.97**	**8.86**
256	6.8 E - 3	2.2 E - 3	1.8 E - 4	6.6 E - 6	1.7 E - 7	3.2 E - 6	2.8 E - 6
	1.74	**2.00**	**2.30**	**3.48**	**7.78**	**9.31**	**8.81**
512	3.9 E - 3	1.1 E - 3	8.0 E - 5	1.9 E - 6	2.1 E - 8	3.4 E - 7	3.2 E - 7
	1.74	**2.00**	**2.30**	**3.48**	**4.76**	**9.22**	**8.64**
1024	2.2 E - 3	5.6 E - 4	3.5 E - 5	5.4 E - 7	4.5 E - 9	3.7 E - 8	3.7 E - 8
	1.74	**2.00**	**2.30**	**3.48**	**4.76**	**8.91**	**8.54**
2048	1.3 E - 3	2.8 E - 4	1.5 E - 5	1.6 E - 7	9.4 E -10	4.2 E - 9	4.4 E - 9
	1.74	2.00	2.30	3.48	4.76	8.00	8.00

An interesting choice of parameters is $\rho = 5, r = 1$ because then the smoothing function φ_2 eliminates the boundary singularity from the first derivatives of the solution u:

$$u(\varphi_2(t)) = \left(2^{2.5}\left(\sin\left(\frac{\pi}{4}t\right)\right)^5\right)^{1/5} - 2^{2.5}\left(\sin\left(\frac{\pi}{4}t\right)\right)^5 \approx c_1\, t - c_2\, t^5,$$

where c_1, c_2 are some constants and t is small. We can see this effect very clearly in Table 6.4.1, compare it with the parameters $\rho = 1, r = 5$ in Table 6.4.2.

Tables 6.4.3-6.4.4 describe the local superconvergence errors for examples (6.4.1)-(6.4.3). The numerical results in Tables 6.4.3-6.4.4 are in good agreement with the theoretical one (6.4.5). A characteristic feature here is, that near the optimal values of ρ and r, the ratios $\tilde{\delta_N}^{(\rho,r)}$ are large for the small values of N but they will decrease if N will increase. See, for example, $\rho = 5.5, r = 1$; $\rho = 1, r = 4$; $\rho = 3, r = 2$ and $\rho = 1, r = 6$, the same can be seen in Tables 6.4.1-6.4.2 for global convergence and for the parameters $\rho = 11, r = 1$ and $\rho = 7.5, r = 1.5$. The ratios $\tilde{\delta_N}^{(\rho,r)}$ are larger for the parameters $\rho = 1, r = 1$ and $\rho = 1, r = 2$ (compare with $\rho = 2, r = 1$) but they probably also will decrease if N will increase as we can see for other choices of the parameters ρ and r.

Next, we illustrate a very similar experiment to (6.4.1)-(6.4.3) by switching the

Table 6.4.3: Examples (6.4.1)-(6.4.3), $\nu = 1/2$, $\lambda = 3/10$, sine transformation φ_2, local superconvergence (6.4.5), uniform grid.

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(5,1)}$ $\tilde{\delta}_N^{(5,1)}$	$\gamma_N^{(5.5,1)}$ $\tilde{\delta}_N^{(5.5,1)}$	$\gamma_N^{(9,1)}$ $\tilde{\delta}_N^{(9,1)}$	$\gamma_N^{(14,1)}$ $\tilde{\delta}_N^{(14,1)}$
8	2.1 E + 2	1.6 E + 0	9.2 E - 4	2.4 E - 3	3.2 E - 3	1.1 E - 2	5.3 E - 2
	2.45	**4.26**	**9.50**	**10.78**	**11.34**	**12.82**	**18.33**
16	8.6 E + 1	3.7 E - 1	9.7 E - 5	2.2 E - 4	2.8 E - 4	8.2 E - 4	2.9 E - 3
	1.92	**4.55**	**5.38**	**12.43**	**12.49**	**10.08**	**11.63**
32	4.5 E + 1	8.2 E - 2	1.8 E - 5	1.8 E - 5	2.3 E - 5	8.2 E - 5	2.5 E - 4
	1.81	**4.84**	**3.63**	**11.80**	**11.74**	**13.02**	**12.16**
64	2.5 E + 1	1.7 E - 2	5.0 E - 6	1.5 E - 6	1.9 E - 6	6.3 E - 6	2.1 E - 5
	1.77	**5.24**	**3.10**	**11.81**	**11.65**	**12.32**	**11.76**
128	1.4 E + 1	3.2 E - 3	1.6 E - 6	1.3 E - 7	1.7 E - 7	5.1 E - 7	1.8 E - 6
	1.76	**6.02**	**3.07**	**11.16**	**11.19**	**11.50**	**11.82**
256	8.0 E + 0	5.4 E - 4	5.2 E - 7	1.2 E - 8	1.5 E - 8	4.4 E - 8	1.5 E - 7
	1.76	**2.33**	**3.05**	**11.13**	**8.30**	**11.16**	**11.07**
512	4.5 E + 0	2.3 E - 4	1.7 E - 7	1.0 E - 9	1.8 E - 9	4.0 E - 9	1.3 E - 8
	1.77	**1.79**	**3.04**	**10.80**	**4.60**	**11.14**	**11.16**
1024	2.6 E + 0	1.3 E - 4	5.6 E - 8	9.7 E - 11	3.9 E - 10	3.6 E - 10	1.2 E - 9
	1.32	1.74	3.03	4.00	4.59	11.31	11.31

Table 6.4.4: Examples (6.4.1)-(6.4.3), $\nu = 1/2$, $\lambda = 3/10$, sine transformation φ_2, local superconvergence (6.4.5), graded grid.

N	$\gamma_N^{(1,2)}$ $\tilde{\delta}_N^{(1,2)}$	$\gamma_N^{(1,4)}$ $\tilde{\delta}_N^{(1,4)}$	$\gamma_N^{(1,5)}$ $\tilde{\delta}_N^{(1,5)}$	$\gamma_N^{(3,2)}$ $\tilde{\delta}_N^{(3,2)}$	$\gamma_N^{(1,6)}$ $\tilde{\delta}_N^{(1,6)}$	$\gamma_N^{(1,9)}$ $\tilde{\delta}_N^{(1,9)}$	$\gamma_N^{(9,3)}$ $\tilde{\delta}_N^{(9,3)}$
8	2.4 E + 1	1.6 E + 0	9.9 E - 1	1.2 E - 3	8.5 E - 1	1.2 E + 0	2.5 E - 1
	3.03	**8.66**	**13.43**	**7.92**	**18.55**	**27.51**	**6.91**
16	7.8 E + 0	1.8 E - 1	7.3 E - 2	1.5 E - 4	4.6 E - 2	4.4 E - 2	3.6 E - 2
	3.07	**10.08**	**16.44**	**9.53**	**22.29**	**26.48**	**19.64**
32	2.5 E + 0	1.8 E - 2	4.5 E - 3	1.6 E - 5	2.0 E - 3	1.7 E - 3	1.8 E - 3
	3.15	**11.01**	**18.09**	**10.45**	**22.01**	**19.52**	**12.11**
64	8.1 E - 1	1.6 E - 3	2.5 E - 4	1.5 E - 6	9.3 E - 5	8.5 E - 5	1.5 E - 4
	3.23	**11.76**	**22.66**	**11.04**	**17.60**	**14.24**	**12.42**
128	2.5 E - 1	1.4 E - 4	1.1 E - 5	1.4 E - 7	5.3 E - 6	6.0 E - 6	1.2 E - 5
	3.29	**9.48**	**4.93**	**9.11**	**11.98**	**11.77**	**12.02**
256	7.6 E - 2	1.5 E - 5	2.2 E - 6	1.5 E - 8	4.4 E - 7	5.1 E - 7	1.0 E - 6
	3.34	**3.06**	**4.03**	**5.28**	**5.29**	**11.90**	**11.67**
512	2.3 E - 2	4.7 E - 6	5.5 E - 7	2.8 E - 9	8.3 E - 8	4.3 E - 8	8.6 E - 8
	3.37	**3.05**	**4.01**	**5.28**	**5.29**	**11.57**	**11.15**
1024	6.7 E - 3	1.6 E - 6	1.4 E - 7	5.4 E - 10	1.6 E - 8	3.7 E - 9	7.7 E - 9
	1.74	3.03	4.00	5.28	5.28	11.31	11.31

values of ν and λ. Thus we consider the equation (6.1.1), where

$$K(x,y) = (x-y)^{-3/10}\, y^{-1/2}, \quad (x,y) \in D_1, \tag{6.4.6}$$

$$f(x) = x^{1/5}\left(1 - \frac{\Gamma(\frac{7}{10})\Gamma(\frac{7}{10})}{\Gamma(\frac{7}{5})} x^{1/5} - x^{4/5} + \frac{\Gamma(\frac{7}{10})\Gamma(\frac{3}{2})}{\Gamma(\frac{22}{10})} x\right), \tag{6.4.7}$$

$$u(x) = x^{1/5} - x, \quad x \in [0,1]. \tag{6.4.8}$$

Note that the solution u is the same as in examples (6.4.1)-(6.4.3) and the forcing function f contains Gamma functions with arguments that differ from those of the functions (6.4.2). This time $\nu = 3/10$ and $\lambda = 1/2$. Clearly $K \in W^{m+1,3/10,1/2}(D_1)$ and $f, u \in C^{m+1,1/5}(0,1]$ for arbitrary $m \in \mathbb{N}$. For the numerical results we use $m = 3$ and the node points (6.1.6).

Since $\nu + \lambda = 3/10 + 1/2 = 1/5 > 0$ it follows from Theorem 4.4.5 that for any transformation $\varphi \in \Phi^{m,\rho}$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r \geq \frac{15}{\rho} \\ 2^{r\rho/5} & \text{if } 1 \leq r < \frac{15}{\rho} \end{array} \right\}. \tag{6.4.9}$$

The equality (6.4.9) is the same as (6.4.4).

From Theorem 5.2.1 and from Remark 5.2.1 it follows that for any transformation $\varphi \in \Phi^{m+1,\rho}$ and for sufficiently large values of N we have

$$\tilde{\delta_N}^{(\rho,r)} = \left\{ \begin{array}{ll} 2^{3.7} & \text{if } r \geq \frac{9.25}{\rho} \\ 2^{2r\rho/5} & \text{if } 1 \leq r < \frac{9.25}{\rho} \end{array} \right\}. \tag{6.4.10}$$

The equality (6.4.10) differs from (6.4.5) by the optimal convergence rate $2^{3.7}$ instead of $2^{3.5}$ and by the choice of ρ and r.

We can compare global convergence results in Table 6.4.5 with the results in Table 6.4.1. The overall picture is very similar which is in good agreement with the theoretical estimates (6.4.9) and (6.4.4).

We see more differences for the local superconvergence results in Table 6.4.6 and in Tables 6.4.3-6.4.4. For example, compare $\rho = 9, r = 3$ in Table 6.4.6 with $\rho = 9, r = 3$ in Table 6.4.4. The numerical results are in good agreement with the theoretical ones in (6.4.10) but the same effect about the better ratios for the small values of N, that we have described for examples (6.4.1)-(6.4.3), is true also here.

Other numerical results for the equation (6.1.1) can be found with $K \in W^{m+1,\nu,\lambda}(D_b)$ and $f \in C^{m+1,\nu+\lambda}(0,b]$ in [36] (for φ_1, $m = 3$, $\nu = 2/5$ and $\lambda = 1/5$), in [39] (for φ_1, $m = 3$, $\nu = \frac{3}{10}$ and $\lambda = \frac{2}{10}$) and in [37] (for φ_1, $m = 3$, $\nu = \frac{1}{4}$ and $\lambda = \frac{1}{2}$).

Table 6.4.5: Examples (6.4.6)-(6.4.8), $\nu = 3/10$, $\lambda = 1/2$, sine transformation φ_2, global convergence (6.4.9), uniform grid.

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(4,1)}$ $\delta_N^{(4,1)}$	$\varepsilon_N^{(5,1)}$ $\delta_N^{(5,1)}$	$\varepsilon_N^{(6,1)}$ $\delta_N^{(6,1)}$	$\varepsilon_N^{(9,1)}$ $\delta_N^{(9,1)}$	$\varepsilon_N^{(11,1)}$ $\delta_N^{(11,1)}$	$\varepsilon_N^{(16,1)}$ $\delta_N^{(16,1)}$
16	9.5 E + 0	5.5 E - 3	4.1 E - 5	8.1 E - 4	3.3 E - 4	7.2 E - 4	2.5 E - 3
	1.88	**1.76**	**8.36**	**2.31**	**8.49**	**8.21**	**7.66**
32	5.1 E + 0	3.1 E - 3	4.9 E - 6	3.5 E - 4	3.9 E - 5	8.8 E - 5	3.3 E - 4
	1.70	**1.75**	**8.22**	**2.30**	**5.32**	**8.19**	**7.93**
64	3.0 E + 0	1.8 E - 3	5.9 E - 7	1.5 E - 4	7.4 E - 6	1.1 E - 5	4.2 E - 5
	1.62	**1.75**	**8.13**	**2.30**	**3.48**	**8.15**	**8.02**
128	1.8 E + 0	1.0 E - 3	7.3 E - 8	6.6 E - 5	2.1 E - 6	1.3 E - 6	5.2 E - 6
	1.59	**1.74**	**8.08**	**2.30**	**3.48**	**8.10**	**8.04**
256	1.2 E + 0	5.9 E - 4	9.0 E - 9	2.9 E - 5	6.1 E - 7	1.6 E - 7	6.5 E - 7
	1.58	**1.74**	**8.05**	**2.30**	**3.48**	**8.07**	**8.04**
512	7.3 E - 1	3.4 E - 4	1.1 E - 9	1.3 E - 5	1.8 E - 7	2.0 E - 8	8.0 E - 8
	1.57	**1.74**	**8.03**	**2.30**	**3.48**	**5.85**	**8.04**
1024	4.7 E - 1	1.9 E - 4	1.4 E - 10	5.5 E - 6	5.0 E - 8	3.4 E - 9	1.0 E - 8
	1.57	**1.74**	**8.00**	**2.30**	**3.48**	**4.60**	**8.02**
2048	3.0 E - 1	1.1 E - 4	1.8 E - 11	2.4 E - 6	1.4 E - 8	7.5 E - 10	1.2 E - 9
	1.15	1.74	2.00	2.30	3.48	4.59	8.00

Table 6.4.6: Examples (6.4.6)-(6.4.8), $\nu = 3/10$, $\lambda = 1/2$, sine transformation φ_2, local superconvergence (6.4.10).

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(6,1)}$ $\tilde{\delta}_N^{(6,1)}$	$\gamma_N^{(7,1)}$ $\tilde{\delta}_N^{(7,1)}$	$\gamma_N^{(10.5,1)}$ $\tilde{\delta}_N^{(10.5,1)}$	$\gamma_N^{(9,3)}$ $\tilde{\delta}_N^{(9,3)}$
8	2.4 E + 1	4.2 E - 1	1.1 E - 3	1.2 E - 4	1.6 E - 4	6.8 E - 4	1.4 E - 2
	2.54	**2.60**	**7.01**	**15.70**	**14.35**	**12.33**	**10.66**
16	9.4 E + 0	1.6 E - 1	1.6 E - 4	7.4 E - 6	1.1 E - 5	5.5 E - 5	1.3 E - 3
	1.87	**2.63**	**7.50**	**10.93**	**13.37**	**12.76**	**11.65**
32	5.0 E + 0	6.2 E - 2	2.2 E - 5	6.8 E - 7	8.4 E - 7	4.3 E - 6	1.1 E - 4
	1.69	**2.64**	**3.56**	**5.28**	**13.22**	**12.69**	**13.24**
64	3.0 E + 0	2.4 E - 2	6.0 E - 6	1.3 E - 7	6.4 E - 8	3.4 E - 7	8.3 E - 6
	1.62	**2.65**	**3.08**	**5.28**	**10.54**	**12.83**	**13.02**
128	1.8 E + 0	8.9 E - 3	2.0 E - 6	2.4 E - 8	6.1 E - 9	2.7 E - 8	6.3 E - 7
	1.59	**2.66**	**3.06**	**5.28**	**6.97**	**12.97**	**12.78**
256	1.2 E + 0	3.3 E - 3	6.4 E - 7	4.6 E - 9	8.7 E - 10	2.1 E - 9	5.0 E - 8
	1.58	**2.66**	**3.05**	**5.28**	**6.97**	**13.18**	**12.87**
512	7.3 E - 1	1.3 E - 3	2.1 E - 7	8.7 E - 10	1.2 E - 10	1.6 E - 10	3.9 E - 9
	1.57	**2.66**	**3.04**	**5.28**	**6.97**	**15.90**	**13.04**
1024	4.7 E - 1	4.7 E - 4	6.9 E - 8	1.7 E - 10	1.8 E - 11	9.8 E - 12	3.0 E - 10
	1.32	1.74	3.03	5.28	6.96	13.00	13.00

6.5 Case $K \in W^{m+1,0}(D_b^{'})$ and $f \in C^{m+1,0}(0,b]$

We consider the equation (6.1.1), where

$$K(x,y) = \log(x-y), \quad (x,y) \in D_1^{'}, \tag{6.5.1}$$

$$f(x) = x\,(1+\log x) - \frac{1}{2}x^2\left(\log^2 x - \log x + \frac{1}{2} - \frac{\pi^2}{6}\right), \tag{6.5.2}$$

$$u(x) = x(1+\log x), \quad x \in [0,1]. \tag{6.5.3}$$

Clearly $K \in W^{m+1,0}(D_1^{'})$ and $f, u \in C^{m+1,0}(0,1]$ for arbitrary $m \in \mathbb{N}$. For current test problem we use $m = 4$ and the node points $\eta_1, \ldots, \eta_4$ of the four-point Gauss-Legendre quadrature rule (5.1.2):

$$\begin{aligned} m = 4, \quad \eta_1 &= \frac{7-\sqrt{21+14\sqrt{\frac{6}{5}}}}{14}, \quad \eta_2 = \frac{7-\sqrt{21-14\sqrt{\frac{6}{5}}}}{14}, \\ \eta_3 &= \frac{7+\sqrt{21-14\sqrt{\frac{6}{5}}}}{14}, \quad \eta_4 = \frac{7+\sqrt{21+14\sqrt{\frac{6}{5}}}}{14}. \end{aligned} \tag{6.5.4}$$

From Theorem 4.4.3 it follows that for any transformation $\varphi \in \Phi^{m,\rho} \cap C^m[0,1]$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{lll} 2^4 & \text{if } \rho > 4 & ,r \geq 1 \\ 2^4 & \text{if } \rho = 4 & ,r > 1 \\ 2^4\left(\frac{\log(N/2)}{\log N}\right) & \text{if } \rho = 4 & ,r = 1 \\ 2^4 & \text{if } \rho < 4 & ,r \geq \frac{4}{\rho} \\ 2^{r\rho} & \text{if } \rho < 4 & ,1 \leq r < \frac{4}{\rho} \end{array} \right\}, \tag{6.5.5}$$

and for any transformation $\varphi \in \Phi^{m,\rho}$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^4 & \text{if } r > \frac{4}{\rho} \\ 2^{r\rho}\left(\frac{\log(N/2)}{\log N}\right) & \text{if } 1 \leq r \leq \frac{4}{\rho} \end{array} \right\}. \tag{6.5.6}$$

From Theorem 5.3.4 it follows that for any transformation $\varphi \in \Phi^{m+1,\rho} \cap C^{m+1}[0,1]$ and for sufficiently large values of N we have

$$\tilde{\delta_N}^{(\rho,r)} = \left\{ \begin{array}{ll} 2^5\left(\frac{\log(N/2)}{\log N}\right) & \text{if } r > \frac{2.5}{\rho} \\ 2^5\left(\frac{\log(N/2)}{\log N}\right)^2 & \text{if } r = \frac{2.5}{\rho} \\ 2^{2r\rho}\left(\frac{\log(N/2)}{\log N}\right) & \text{if } 1 \leq r < \frac{2.5}{\rho} \end{array} \right\}. \tag{6.5.7}$$

Tables 6.5.1-6.5.2 describe global convergence errors $\|u - u_N\|_\infty$ for examples (6.5.1)-(6.5.3) and for the exponential transformation φ_1. In Table 6.5.1, we only

Table 6.5.1: Examples (6.5.1)-(6.5.3), $\nu = 0$, exponential transformation $\varphi_1 \in \Phi^{m,\rho} \cap C^m[0,1]$, global convergence (6.5.5)

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(2,1)}$ $\delta_N^{(2,1)}$	$\varepsilon_N^{(2,1.25)}$ $\delta_N^{(2,1.25)}$	$\varepsilon_N^{(1,2.5)}$ $\delta_N^{(1,2.5)}$	$\varepsilon_N^{(1,4)}$ $\delta_N^{(1,4)}$	$\varepsilon_N^{(4,1)}$ $\delta_N^{(4,1)}$	$\varepsilon_N^{(5,1)}$ $\delta_N^{(5,1)}$
8	1.0 E - 2	4.3 E - 4	1.5 E - 4	4.7 E - 4	9.7 E - 5	3.8 E - 5	1.4 E - 4
	1.98	**3.99**	**5.65**	**5.65**	**15.87**	**16.79**	**16.29**
16	5.2 E - 3	1.1 E - 4	2.7 E - 5	8.3 E - 5	6.1 E - 6	2.2 E - 6	8.8 E - 6
	1.99	**4.00**	**5.66**	**5.66**	**15.99**	**16.61**	**16.43**
32	2.6 E - 3	2.7 E - 5	4.7 E - 6	1.5 E - 5	3.8 E - 7	1.3 E - 7	5.3 E - 7
	1.99	**4.00**	**5.66**	**5.66**	**16.00**	**14.05**	**16.34**
64	1.3 E - 3	6.7 E - 6	8.3 E - 7	2.6 E - 6	2.4 E - 8	9.6 E - 9	3.3 E - 8
	2.00	**4.00**	**5.66**	**5.66**	**16.00**	**12.84**	**16.22**
128	6.6 E - 4	1.7 E - 6	1.5 E - 7	4.6 E - 7	1.5 E - 9	7.5 E - 10	2.0 E - 9
	2.00	**4.00**	**5.66**	**5.66**	**16.00**	**13.36**	**16.14**
256	3.3 E - 4	4.2 E - 7	2.6 E - 8	8.1 E - 8	9.3 E - 11	5.6 E - 11	1.2 E - 10
	2.00	**4.00**	**5.66**	**5.66**	**16.00**	**13.74**	**16.09**
512	1.7 E - 4	1.0 E - 7	4.6 E - 9	1.4 E - 8	5.8 E - 12	4.1 E - 12	7.8 E - 12
	2.00	**4.00**	**5.66**	**5.66**	**16.00**	**14.02**	**16.14**
1024	8.3 E - 5	2.6 E - 8	8.1 E - 10	2.5 E - 9	3.6 E - 13	2.9 E - 13	4.8 E - 13
	2.00	4.00	5.66	5.66	16.00	14.40	16.00

Table 6.5.2: Examples (6.5.1)-(6.5.3), $\nu = 0$, exponential transformation $\varphi_1 \in \Phi^{m,\rho}$, global convergence (6.5.6)

N	$\varepsilon_N^{(1.5,1)}$ $\delta_N^{(1.5,1)}$	$\varepsilon_N^{(2.5,1)}$ $\delta_N^{(2.5,1)}$	$\varepsilon_N^{(1.25,2)}$ $\delta_N^{(1.25,2)}$	$\varepsilon_N^{(1.5,2)}$ $\delta_N^{(1.5,2)}$	$\varepsilon_N^{(1.9,2)}$ $\delta_N^{(1.9,2)}$	$\varepsilon_N^{(1.5,3)}$ $\delta_N^{(1.5,3)}$	$\varepsilon_N^{(3.5,2)}$ $\delta_N^{(3.5,2)}$
8	1.9 E - 3	8.4 E - 5	1.7 E - 4	1.4 E - 4	1.7 E - 5	2.4 E - 5	1.9 E - 4
	2.32	**4.17**	**3.61**	**6.34**	**12.48**	**15.46**	**15.14**
16	8.0 E - 4	2.0 E - 5	4.8 E - 5	2.2 E - 5	1.4 E - 6	1.5 E - 6	1.3 E - 5
	2.40	**4.48**	**4.16**	**6.63**	**12.64**	**16.52**	**15.95**
32	3.3 E - 4	4.5 E - 6	1.2 E - 5	3.2 E - 6	1.1 E - 7	9.4 E - 8	7.9 E - 7
	2.46	**4.69**	**4.47**	**6.83**	**12.75**	**17.26**	**16.16**
64	1.4 E - 4	9.6 E - 7	2.6 E - 6	4.8 E - 7	8.4 E - 9	5.4 E - 9	4.9 E - 8
	2.50	**4.83**	**4.68**	**6.98**	**12.84**	**17.05**	**16.16**
128	5.5 E - 5	2.0 E - 7	5.5 E - 7	6.8 E - 8	6.6 E - 10	3.2 E - 10	3.0 E - 9
	2.54	**4.93**	**4.82**	**7.09**	**12.92**	**16.51**	**16.12**
256	2.2 E - 5	4.0 E - 8	1.2 E - 7	9.6 E - 9	5.1 E - 11	1.9 E - 11	1.9 E - 10
	2.56	**5.02**	**4.93**	**7.19**	**12.99**	**16.24**	**16.08**
512	8.4 E - 6	8.1 E - 9	2.3 E - 8	1.3 E - 9	3.9 E - 12	1.2 E - 12	1.2 E - 11
	2.59	**5.08**	**5.01**	**7.26**	**13.05**	**16.12**	**16.20**
1024	3.2 E - 6	1.6 E - 9	4.7 E - 9	1.8 E - 10	3.0 E - 13	7.4 E - 14	7.2 E - 13
	2.55	5.09	5.09	7.20	12.54	16.00	16.00

use values $\rho \in \mathbb{N}$ and in Table 6.5.2, we allow also values $\rho \in \mathbb{R}$. The numerical results in Tables 6.5.1-6.5.2 are in very good agreement with the theoretical estimates (6.5.5)-(6.5.6). We mention the special choice of parameters $\rho = 4$, $r = 1$ for which the numerical results are also in good agreement with the theoretical estimate

$$\|u - u_N\|_\infty \leq c\, 2^4 \left(\frac{\log(N/2)}{\log N} \right).$$

We can see form Table 6.5.2 that there exist important differences between $\varphi_1 \in \Phi^{m,\rho} \cap C^m[0,1]$ and $\varphi_1 \in \Phi^{m,\rho}$. Compare $\rho = 1$, $r = 2.5$ and $\rho = 2$, $r = 1.25$ with $\rho = 2.5$, $r = 1$ and $\rho = 1.25$, $r = 2$.

Table 6.5.3: Examples (6.5.1)-(6.5.3), $\nu = 0$, exponential transformation φ_1, local superconvergence (6.5.7)

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(1,1.5)}$ $\tilde{\delta}_N^{(1,1.5)}$	$\gamma_N^{(1.5,1)}$ $\tilde{\delta}_N^{(1.5,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(2,1.25)}$ $\tilde{\delta}_N^{(2,1.25)}$	$\gamma_N^{(1.25,2)}$ $\tilde{\delta}_N^{(1.25,2)}$	$\gamma_N^{(5,1)}$ $\tilde{\delta}_N^{(5,1)}$
4	4.7 E - 4	1.4 E - 4	2.7 E - 5	1.2 E - 5	3.4 E - 6	5.6 E - 6	2.1 E - 4
	3.29	**6.40**	**4.54**	**13.08**	**25.46**	**25.85**	**21.59**
8	1.4 E - 4	2.2 E - 5	6.0 E - 6	9.3 E - 7	1.3 E - 7	2.2 E - 7	9.8 E - 6
	3.42	**6.73**	**5.20**	**13.46**	**26.42**	**26.42**	**25.57**
16	4.1 E - 5	3.3 E - 6	1.2 E - 6	6.9 E - 8	5.0 E - 9	8.2 E - 9	3.8 E - 7
	3.52	**6.94**	**5.65**	**13.76**	**27.19**	**23.40**	**26.87**
32	1.2 E - 5	4.7 E - 7	2.1 E - 7	5.0 E - 9	1.8 E - 10	3.5 E - 10	1.4 E - 8
	3.58	**7.09**	**5.97**	**14.02**	**27.65**	**17.37**	**26.77**
64	3.3 E - 6	6.7 E - 8	3.4 E - 8	3.6 E - 10	6.7 E - 12	2.0 E - 11	5.3 E - 10
	3.63	**7.19**	**6.20**	**14.23**	**27.84**	**20.45**	**27.85**
128	9.1 E - 7	9.3 E - 9	5.5 E - 9	2.5 E - 11	2.4 E - 13	9.8 E - 13	1.9 E - 11
	3.67	**7.27**	**6.39**	**14.38**	**28.32**	**22.38**	**28.10**
256	2.5 E - 7	1.3 E - 9	8.7 E - 10	1.7 E - 12	8.5 E - 15	4.4 E - 14	6.8 E - 13
	3.50	7.00		14.00	24.50		28.00

Table 6.5.3 describes the local superconvergence errors for examples (6.5.1)-(6.5.3). The expected theoretical results are presented in (6.5.7) for the transformations $\varphi_1 \in \Phi^{m+1,\rho} \cap C^m[0,1]$. The numerical results are in good agreement with the theoretical ones. We can compare the parameters $\rho = 1, r = 1.5$ with $\rho = 1.5, r = 1$ and $\rho = 2, r = 1.25$ with $\rho = 1.25, r = 2$ where the condition $\varphi_1 \in \Phi^{m+1,\rho} \cap C^{m+1}[0,1]$ is important.

6.6 Case $K \in W^{m+1,\nu}(D_b^{'})$ and $f \in C^{m+1,\nu}(0,b]$, $\nu > 0$

We consider the equation (6.1.1), where

$$K(x,y) = (x-y)^{-5/9}, \quad (x,y) \in D_1^{'}, \tag{6.6.1}$$

$$f(x) = x^{4/9}\left(1 - \frac{\Gamma(\frac{4}{9})\Gamma(\frac{13}{9})}{\Gamma(\frac{17}{9})}x^{4/9} - x^{5/9} + \frac{81}{52}x\right), \tag{6.6.2}$$

$$u(x) = x^{4/9} - x, \quad x \in [0,1]. \tag{6.6.3}$$

Clearly $K \in W^{m+1,5/9}(D_1')$ and $f, u \in C^{m+1,5/9}(0,1]$ for arbitrary $m \in \mathbb{N}$. For the numerical results we use $m = 3$ and the node points (6.1.6).

Since $\nu = 5/9 > 0$ from Theorem 4.4.3 it follows that for any transformation $\varphi \in \Phi^{m,\rho}$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r \geq \frac{6.75}{\rho} \\ 2^{4r\rho/9} & \text{if } 1 \leq r < \frac{6.75}{\rho} \end{array} \right\}. \tag{6.6.4}$$

From Theorem 5.2.3 it follows that for any transformation $\varphi \in \Phi^{m+1,\rho}$ and for sufficiently large values of N we have

$$\tilde{\delta}_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^{3+4/9} & \text{if } r \geq \frac{3.875}{\rho} \\ 2^{8r\rho/9} & \text{if } 1 \leq r < \frac{3.875}{\rho} \end{array} \right\}. \tag{6.6.5}$$

Table 6.6.1: Examples (6.6.1)-(6.6.3), $\nu = 5/9$, sine transformation φ_2, global convergence (6.6.4), uniform grid.

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(2,1)}$ $\delta_N^{(2,1)}$	$\varepsilon_N^{(3,1)}$ $\delta_N^{(3,1)}$	$\varepsilon_N^{(4.5,1)}$ $\delta_N^{(4.5,1)}$	$\varepsilon_N^{(6,1)}$ $\delta_N^{(6,1)}$	$\varepsilon_N^{(7,1)}$ $\delta_N^{(7,1)}$	$\varepsilon_N^{(10,1)}$ $\delta_N^{(10,1)}$
16	8.4 E - 2	2.0 E - 3	7.0 E - 4	3.9 E - 5	9.5 E - 5	1.9 E - 4	7.9 E - 4
	1.44	**1.86**	**2.57**	**8.42**	**8.77**	**8.72**	**8.54**
32	5.8 E - 2	1.1 E - 3	2.7 E - 4	4.6 E - 6	1.1 E - 5	2.2 E - 5	9.3 E - 5
	1.38	**1.86**	**2.54**	**8.27**	**8.85**	**8.72**	**8.50**
64	4.2 E - 2	5.8 E - 4	1.1 E - 4	5.6 E - 7	1.2 E - 6	2.5 E - 6	1.1 E - 5
	1.37	**1.85**	**2.53**	**8.19**	**8.35**	**8.79**	**8.61**
128	3.1 E - 2	3.1 E - 4	4.3 E - 5	6.8 E - 8	1.5 E - 7	2.9 E - 7	1.3 E - 6
	1.37	**1.85**	**2.52**	**8.13**	**8.11**	**8.80**	**8.61**
256	2.3 E - 2	1.7 E - 4	1.7 E - 5	8.4 E - 9	1.8 E - 8	3.3 E - 8	1.5 E - 7
	1.37	**1.85**	**2.52**	**8.09**	**6.35**	**8.71**	**8.55**
512	1.6 E - 2	9.1 E - 5	6.7 E - 6	1.0 E - 9	2.8 E - 9	3.8 E - 9	1.7 E - 8
	1.37	**1.85**	**2.52**	**8.07**	**6.35**	**8.48**	**8.46**
1024	1.2 E - 2	4.9 E - 5	2.7 E - 6	1.3 E - 10	4.5 E - 10	4.5 E - 10	2.0 E - 9
	1.36	**1.85**	**2.52**	**8.04**	**6.35**	**8.12**	**8.36**
2048	8.8 E - 3	2.7 E - 5	1.1 E - 6	1.6 E - 11	7.0 E - 11	5.5 E - 11	2.4 E - 10
	1.36	1.85	2.52	4.00	6.35	8.00	8.00

Tables 6.6.1-6.6.2 describe global convergence errors $\|u - u_N\|_\infty$ for examples (6.6.1)-(6.6.3) and for the sine transformation φ_2. The results in Tables 6.6.1-6.6.2 are in very good agreement with the theoretical estimates (6.6.4).

Table 6.6.2: Examples (6.6.1)-(6.6.3), $\nu = 5/9$, sine transformation φ_2, global convergence (6.6.4), graded grid.

N	$\varepsilon_N^{(1,2)}$ $\delta_N^{(1,2)}$	$\varepsilon_N^{(1,3)}$ $\delta_N^{(1,3)}$	$\varepsilon_N^{(2,2)}$ $\delta_N^{(2,2)}$	$\varepsilon_N^{(1,4.5)}$ $\delta_N^{(1,4.5)}$	$\varepsilon_N^{(1,6)}$ $\delta_N^{(1,6)}$	$\varepsilon_N^{(1,7)}$ $\delta_N^{(1,7)}$	$\varepsilon_N^{(8,2)}$ $\delta_N^{(8,2)}$
16	2.3 E - 2	6.5 E - 3	1.7 E - 4	1.1 E - 3	6.7 E - 4	1.0 E - 3	2.8 E - 3
	1.87	**2.53**	**3.43**	**4.02**	**8.82**	**8.77**	**8.90**
32	1.2 E - 2	2.6 E - 3	4.9 E - 5	2.8 E - 4	7.6 E - 5	1.2 E - 4	3.1 E - 4
	1.86	**2.52**	**3.43**	**4.00**	**6.40**	**8.75**	**8.63**
64	6.5 E - 3	1.0 E - 3	1.4 E - 5	7.1 E - 5	1.2 E - 5	1.4 E - 5	3.6 E - 5
	1.86	**2.52**	**3.43**	**4.00**	**6.35**	**8.72**	**8.64**
128	3.5 E - 3	4.0 E - 4	4.2 E - 6	1.8 E - 5	1.9 E - 6	1.6 E - 6	4.2 E - 6
	1.85	**2.52**	**3.43**	**4.00**	**6.35**	**8.63**	**8.65**
256	1.9 E - 3	1.6 E - 4	1.2 E - 6	4.4 E - 6	3.0 E - 7	1.8 E - 7	4.8 E - 7
	1.85	**2.52**	**3.43**	**4.00**	**6.35**	**8.52**	**8.62**
512	1.0 E - 3	6.4 E - 5	3.6 E - 7	1.1 E - 6	4.7 E - 8	2.1 E - 8	5.6 E - 8
	1.85	**2.52**	**3.43**	**4.00**	**6.35**	**8.41**	**8.53**
1024	5.5 E - 4	2.5 E - 5	1.0 E - 7	2.8 E - 7	7.3 E - 9	2.5 E - 9	6.6 E - 9
	1.85	**2.52**	**3.43**	**4.00**	**6.35**	**8.32**	**8.43**
2048	3.0 E - 4	1.0 E - 5	3.0 E - 8	6.9 E - 8	1.2 E - 9	3.0 E - 10	7.8 E -10
	1.85	2.52	3.43	4.00	6.35	8.00	8.00

Note that an interesting choice of parameters is $\rho = 4.5, r = 1$ because then the smoothing function φ_2 eliminates the boundary singularity from the first derivatives of the solution u:

$$u(\varphi_2(t)) = \left(2^{4.5/2}\left(\sin\left(\frac{\pi}{4}t\right)\right)^{4.5}\right)^{4/9} - 2^{4.5/2}\left(\sin\left(\frac{\pi}{4}t\right)\right)^{4.5} \approx c_1\, t^2 - c_2\, t^{4.5},$$

where c_1, c_2 are some constants and t is small. We can see this effect in Table 6.6.1, compare it with the parameters $\rho = 1, r = 4.5$ in Table 6.6.2.

Tables 6.6.3-6.6.4 describe the local superconvergence errors for examples (6.6.1)-(6.6.3). The superconvergence results in Tables 6.6.3-6.6.4 are in good agreement with the theoretical one (6.6.5).

The ratios $\tilde{\delta}_N^{(\rho,r)}$ are much better for the choice of parameters $\rho = 1, r = 1$ and we have seen this for many tests. Nevertheless, it is not always true. We can observe for the superconvergence results in Tables 6.6.3-6.6.4 that the ratios $\tilde{\delta}_N^{(\rho,r)}$ are much better than the theoretical ones for smaller values of N but they will decreas if N will increase (see $\rho = 3, r = 1$; $\rho = 1, r = 3$; $\rho = 1, r = 1.5$).

Note that for the local superconvergence phenomenon the choice of the node points $\eta_1, \ldots, \eta_m$ is important. For illustrating we use same examples (6.6.1)-(6.6.3) taking $m = 2$ with the arbitrary node points:

$$\eta_1 = 0.1, \quad \eta_2 = 0.9.$$

Table 6.6.3: Examples (6.6.1)-(6.6.3), $\nu = 5/9$, sine transformation φ_2, local superconvergence (6.6.5), uniform grid.

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(1.5,1)}$ $\tilde{\delta}_N^{(1.5,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(2.5,1)}$ $\tilde{\delta}_N^{(2.5,1)}$	$\gamma_N^{(3,1)}$ $\tilde{\delta}_N^{(3,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(10,1)}$ $\tilde{\delta}_N^{(10,1)}$
8	2.3 E - 1	4.4 E - 3	1.6 E - 4	3.2 E - 5	7.4 E - 5	1.9 E - 4	4.3 E - 3
	2.80	**4.46**	**6.55**	**8.83**	**9.42**	**9.46**	**9.93**
16	8.2 E - 2	9.8 E - 4	2.4 E - 5	3.6 E - 6	7.9 E - 6	2.0 E - 5	4.3 E - 4
	2.78	**4.41**	**3.69**	**5.73**	**9.66**	**10.52**	**10.34**
32	2.9 E - 2	2.2 E - 4	3.6 E - 6	6.2 E - 7	8.1 E - 7	1.9 E - 6	4.2 E - 5
	2.77	**4.35**	**3.62**	**4.88**	**10.14**	**10.42**	**10.64**
64	1.1 E - 2	5.1 E - 5	9.9 E - 7	1.3 E - 7	8.0 E - 8	1.8 E - 7	3.9 E - 6
	2.76	**4.41**	**3.47**	**4.75**	**10.11**	**10.32**	**10.84**
128	3.9 E - 3	1.2 E - 5	2.8 E - 7	2.7 E - 8	7.9 E - 9	1.7 E - 8	3.6 E - 7
	2.75	**2.93**	**3.45**	**4.70**	**6.37**	**10.61**	**10.84**
256	1.4 E - 3	4.0 E - 6	8.2 E - 8	5.7 E - 9	1.2 E - 9	1.6 E - 9	3.3 E - 8
	2.74	**2.54**	**3.44**	**4.68**	**6.36**	**10.75**	**10.87**
512	5.1 E - 4	1.6 E - 6	2.4 E - 8	1.2 E - 9	2.0 E - 10	1.5 E - 10	3.1 E - 9
	2.73	**2.53**	**3.44**	**4.67**	**6.35**	**10.65**	**10.88**
1024	1.9 E - 4	6.1 E - 7	7.0 E - 9	2.6 E - 10	3.1 E - 11	1.4 E - 11	2.8 E - 10
	1.85	2.52	3.43	4.67	6.35	10.89	10.89

Table 6.6.4: Examples (6.6.1)-(6.6.3), $\nu = 5/9$, sine transformation φ_2, local superconvergence (6.6.5), graded grid.

N	$\gamma_N^{(1,1.5)}$ $\tilde{\delta}_N^{(1,1.5)}$	$\gamma_N^{(1,2)}$ $\tilde{\delta}_N^{(1,2)}$	$\gamma_N^{(1,2.5)}$ $\tilde{\delta}_N^{(1,2.5)}$	$\gamma_N^{(1,3)}$ $\tilde{\delta}_N^{(1,3)}$	$\gamma_N^{(2,2)}$ $\tilde{\delta}_N^{(2,2)}$	$\gamma_N^{(1,10)}$ $\tilde{\delta}_N^{(1,10)}$	$\gamma_N^{(8,2)}$ $\tilde{\delta}_N^{(8,2)}$
8	4.8 E - 2	1.2 E - 2	4.3 E - 3	3.0 E - 3	3.5 E - 4	1.6 E - 1	4.9 E - 3
	4.51	**7.40**	**11.70**	**15.26**	**12.53**	**44.54**	**2.89**
16	1.1 E - 2	1.6 E - 3	3.7 E - 4	2.0 E - 4	2.8 E - 5	3.5 E - 3	1.7 E - 3
	4.52	**7.48**	**11.80**	**14.18**	**11.31**	**18.65**	**10.73**
32	2.3 E - 3	2.1 E - 4	3.1 E - 5	1.4 E - 5	2.5 E - 6	1.9 E - 4	1.6 E - 4
	4.52	**7.50**	**11.68**	**13.11**	**10.07**	**13.57**	**10.60**
64	5.2 E - 4	2.8 E - 5	2.7 E - 6	1.1 E - 6	2.2 E - 7	1.4 E - 5	1.5 E - 5
	4.51	**5.64**	**4.69**	**11.75**	**10.99**	**11.80**	**10.89**
128	1.1 E - 4	5.0 E - 6	5.7 E - 7	8.9 E - 8	2.0 E - 8	1.2 E - 6	1.4 E - 6
	4.51	**3.45**	**4.68**	**6.37**	**10.95**	**11.26**	**10.84**
256	2.6 E - 5	1.4 E - 6	1.2 E - 7	1.4 E - 8	1.9 E - 9	1.1 E - 7	1.3 E - 7
	3.73	**3.44**	**4.67**	**6.36**	**10.94**	**11.06**	**10.88**
512	6.8 E - 6	4.2 E - 7	2.6 E - 8	2.2 E - 9	1.7 E - 10	9.5 E - 9	1.2 E - 8
	2.54	**3.44**	**4.67**	**6.35**	**11.05**	**10.98**	**10.89**
1024	2.7 E - 6	1.2 E - 7	5.6 E - 9	3.5 E - 10	1.5 E - 11	8.7 E - 10	1.1 E - 9
	2.52	3.43	4.67	6.35	10.89	10.89	10.89

In Table 6.6.5, we can see that there is no better convergence order than $O(N^{-2})$ even for large values of ρ and r. Compare the results with Table 6.6.6 where the node points

$$\eta_1 = \frac{3-\sqrt{3}}{6}, \quad \eta_2 = \frac{3+\sqrt{3}}{6},$$

of the two-point Gauss-Legendre quadrature rule (5.1.2) are used.

Table 6.6.5: Examples (6.6.1)-(6.6.3), $\nu = 5/9$, sine transformation φ_2, local superconvergence, $\eta_1 = 0.1, \eta_2 = 0.9$.

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(1.5,1)}$ $\tilde{\delta}_N^{(1.5,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(2.5,1)}$ $\tilde{\delta}_N^{(2.5,1)}$	$\gamma_N^{(3,1)}$ $\tilde{\delta}_N^{(3,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(3,3)}$ $\tilde{\delta}_N^{(3,3)}$
16	6.3 E - 1	2.1 E - 1	8.9 E - 2	7.2 E - 2	7.4 E - 2	9.2 E - 2	5.7 E - 1
	2.17	**3.58**	**3.92**	**3.92**	**3.96**	**4.09**	**7.14**
32	2.9 E - 1	5.9 E - 2	2.3 E - 2	1.8 E - 2	1.9 E - 2	2.3 E - 2	8.0 E - 2
	2.33	**3.64**	**3.88**	**3.85**	**3.85**	**3.86**	**4.11**
64	1.3 E - 1	1.6 E - 2	5.9 E - 3	4.8 E - 3	4.8 E - 3	5.9 E - 3	2.0 E - 2
	2.45	**3.70**	**3.90**	**3.88**	**3.87**	**3.85**	**3.86**
128	5.1 E - 2	4.4 E - 3	1.5 E - 3	1.2 E - 3	1.3 E - 3	1.5 E - 3	5.1 E - 3
	2.53	**3.76**	**3.92**	**3.91**	**3.90**	**3.89**	**3.85**
256	2.0 E - 2	1.2 E - 3	3.8 E - 4	3.1 E - 4	3.2 E - 4	3.9 E - 4	1.3 E - 3
	2.59	**3.80**	**3.94**	**3.93**	**3.93**	**3.91**	**3.89**
512	7.8 E - 3	3.1 E - 4	9.7 E - 5	8.0 E - 5	8.2 E - 5	1.0 E - 4	3.4 E - 4
	2.63	**3.84**	**3.96**	**3.96**	**3.95**	**3.95**	**3.91**
1024	3.0 E - 3	8.0 E - 5	2.5 E - 5	2.0 E - 5	2.1 E - 5	2.5 E - 5	8.6 E - 5
	2.66	**3.86**	**3.97**	**3.96**	**3.96**	**3.95**	**3.95**
2048	1.1 E - 3	2.1 E - 5	6.1 E - 6	5.1 E - 6	5.2 E - 6	6.4 E - 6	2.2 E - 5

6.7 Case $K \in W^{m+1,\nu}(D_b')$ and $f \in C^{m+1,\nu}(0,b]$, $\nu < 0$

We consider the equation (6.1.1), where

$$K(x,y) = (x-y)^{1/4} + 1, \quad (x,y) \in D_1', \tag{6.7.1}$$

$$f(x) = \frac{1}{5}x^{5/4} - \frac{4}{45}x^{9/4} - \frac{\Gamma(\frac{5}{4})\Gamma(\frac{9}{4})}{\Gamma(\frac{7}{2})}x^{10/4} + \frac{1}{2}x^2 - 2x + 1, \tag{6.7.2}$$

$$u(x) = x^{5/4} - x + 1, \quad x \in [0,1]. \tag{6.7.3}$$

Clearly $K \in W^{m+1,-1/4}(D_1')$ and $f, u \in C^{m+1,-1/4}(0,1]$ for arbitrary $m \in \mathbb{N}$. For the numerical results we use $m = 3$ and the node points (6.1.6).

Table 6.6.6: Examples (6.6.1)-(6.6.3), $\nu = 5/9$, sine transformation φ_2, local superconvergence, Gauss-Legendre points for $m = 2$.

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(1.5,1)}$ $\tilde{\delta}_N^{(1.5,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(2.5,1)}$ $\tilde{\delta}_N^{(2.5,1)}$	$\gamma_N^{(3,1)}$ $\tilde{\delta}_N^{(3,1)}$	$\gamma_N^{(4,1)}$ $\tilde{\delta}_N^{(4,1)}$	$\gamma_N^{(3,3)}$ $\tilde{\delta}_N^{(3,3)}$
16	2.4 E - 1	4.8 E - 3	9.0 E - 4	1.5 E - 3	1.7 E - 3	2.4 E - 3	1.6 E - 2
	2.82	**4.47**	**4.36**	**5.62**	**5.66**	**5.48**	**6.38**
32	8.5 E - 2	1.1 E - 3	2.1 E - 4	2.7 E - 4	3.0 E - 4	4.3 E - 4	2.4 E - 3
	2.80	**4.44**	**4.96**	**5.51**	**5.26**	**5.18**	**5.90**
64	3.1 E - 2	2.4 E - 4	4.2 E - 5	4.8 E - 5	5.6 E - 5	8.4 E - 5	4.1 E - 4
	2.78	**4.39**	**5.17**	**5.40**	**5.40**	**5.40**	**5.36**
128	1.1 E - 2	5.5 E - 5	8.0 E - 6	8.9 E - 6	1.0 E - 5	1.6 E - 5	7.7 E - 5
	2.76	**4.36**	**5.25**	**5.45**	**5.46**	**5.46**	**5.51**
256	4.0 E - 3	4.4 E - 5	1.5 E - 6	1.6 E - 6	1.9 E - 6	2.8 E - 6	1.4 E - 5
	2.75	**2.56**	**5.34**	**5.50**	**5.51**	**5.52**	**5.51**
512	1.4 E - 3	2.6 E - 6	2.9 E - 7	3.0 E - 7	3.5 E - 7	5.1 E - 7	2.5 E - 6
	2.74	**2.54**	**5.41**	**5.49**	**5.50**	**5.50**	**5.56**
1024	5.3 E - 4	2.5 E - 6	5.3 E - 8	5.4 E - 8	6.3 E - 8	9.4 E - 8	4.6 E - 7
	2.74	**2.53**	**4.85**	**5.49**	**5.49**	**5.50**	**5.52**
2048	1.9 E - 4	2.5 E - 7	1.1 E - 8	9.9 E - 9	1.1 E - 8	1.7 E - 8	8.3 E - 8
	1.85	2.52	3.43	4.67	5.44	5.44	5.44

From Theorem 4.4.3 it follows that for any transformation $\varphi \in \Phi^{m,\rho} \cap C^m[0,1]$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } \rho > 2.4 \quad , r \geq 1 \\ 2^3 & \text{if } \rho = 2.4 \quad , r > 1 \\ 2^3 \left(\frac{\log (N/2)}{\log N} \right) & \text{if } \rho = 2.4 \quad , r = 1 \\ 2^3 & \text{if } \rho < 2.4 \quad , r \geq \frac{2.4}{\rho} \\ 2^{5r\rho/4} & \text{if } \rho < 2.4 \quad , 1 \leq r < \frac{2.4}{\rho} \end{array} \right\}, \tag{6.7.4}$$

and for any transformation $\varphi \in \Phi^{m,\rho}$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r \geq \frac{3}{\rho} \\ 2^{r\rho} & \text{if } 1 \leq r < \frac{3}{\rho} \end{array} \right\}. \tag{6.7.5}$$

From Theorem 5.4.1 it follows that for any transformation $\varphi \in \Phi^{m,\rho} \cap C^m[0,1]$ and for sufficiently large values of N we have

$$\tilde{\delta_N}^{(\rho,r)} = \left\{ \begin{array}{ll} 2^4 & \text{if } r > \frac{16}{9\rho} \\ 2^4 \left(\frac{\log (N/2)}{\log N} \right) & \text{if } r = \frac{16}{9\rho} \\ 2^{9r\rho/4} & \text{if } 1 \leq r < \frac{16}{9\rho} \end{array} \right\}. \tag{6.7.6}$$

Table 6.7.1: Examples (6.7.1)-(6.7.3), $\nu = -1/4$, exponential transformation $\varphi_1 \in \Phi^{m,\rho} \cap C^m[0,1]$, global convergence (6.7.4)

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(1,1.5)}$ $\delta_N^{(1,1.5)}$	$\varepsilon_N^{(1,2)}$ $\delta_N^{(1,2)}$	$\varepsilon_N^{(2,1)}$ $\delta_N^{(2,1)}$	$\varepsilon_N^{(2,1.1)}$ $\delta_N^{(2,1.1)}$	$\varepsilon_N^{(1,2.4)}$ $\delta_N^{(1,2.4)}$	$\varepsilon_N^{(3,1)}$ $\delta_N^{(3,1)}$
8	1.7 E - 3	4.5 E - 4	1.2 E - 4	1.5 E - 4	9.1 E - 5	6.7 E - 5	1.9 E - 4
	2.39	**3.68**	**5.66**	**5.66**	**6.73**	**8.00**	**7.83**
16	6.9 E - 4	1.2 E - 4	2.1 E - 5	2.7 E - 5	1.4 E - 5	8.3 E - 6	2.4 E - 5
	2.39	**3.67**	**5.66**	**5.66**	**6.73**	**8.00**	**7.93**
32	2.9 E - 4	3.3 E - 5	3.8 E - 6	4.8 E - 6	2.0 E - 6	1.0 E - 6	3.0 E - 6
	2.38	**3.67**	**5.66**	**5.66**	**6.73**	**8.00**	**7.97**
64	1.2 E - 4	9.0 E - 6	6.7 E - 7	8.5 E - 7	3.0 E - 7	1.3 E - 7	3.8 E - 7
	2.38	**3.67**	**5.66**	**5.66**	**6.73**	**8.00**	**7.99**
128	5.1 E - 5	2.5 E - 6	1.2 E - 7	1.5 E - 7	4.5 E - 8	1.6 E - 8	4.8 E - 8
	2.38	**3.67**	**5.66**	**5.66**	**6.73**	**8.00**	**7.99**
256	2.1 E - 5	6.7 E - 7	2.1 E - 8	2.6 E - 8	6.6 E - 9	2.0 E - 9	6.0 E - 9
	2.38	**3.67**	**5.66**	**5.66**	**6.73**	**8.00**	**8.00**
512	9.0 E - 6	1.8 E - 7	3.7 E - 9	4.7 E - 9	9.8 E - 10	2.5 E - 10	7.5 E - 10
	2.38	**3.67**	**5.66**	**5.66**	**6.73**	**8.00**	**7.99**
1024	3.8 E - 6	5.0 E - 8	6.5 E - 10	8.3 E - 10	1.5 E - 10	3.2 E - 11	9.4 E - 11
	2.38	3.67	5.66	5.66	6.73	8.00	8.00

Table 6.7.2: Examples (6.7.1)-(6.7.3), $\nu = -1/4$, exponential transformation $\varphi_1 \in \Phi^{m,\rho}$, global convergence (6.7.5)

N	$\varepsilon_N^{(1.2,1)}$ $\delta_N^{(1.2,1)}$	$\varepsilon_N^{(1.5,1)}$ $\delta_N^{(1.5,1)}$	$\varepsilon_N^{(1.78,1)}$ $\delta_N^{(1.78,1)}$	$\varepsilon_N^{(1.1,2)}$ $\delta_N^{(1.1,2)}$	$\varepsilon_N^{(1.2,2)}$ $\delta_N^{(1.2,2)}$	$\varepsilon_N^{(2.4,1)}$ $\delta_N^{(2.4,1)}$	$\varepsilon_N^{(2.5,1)}$ $\delta_N^{(2.5,1)}$
8	6.3 E - 4	8.7 E - 4	4.2 E - 4	4.8 E - 5	9.0 E - 5	7.8 E - 5	9.3 E - 5
	1.77	**2.73**	**3.72**	**3.30**	**4.52**	**4.65**	**5.25**
16	3.5 E - 4	3.2 E - 4	1.1 E - 4	1.6 E - 5	2.0 E - 5	1.7 E - 5	1.8 E - 5
	1.93	**2.75**	**3.66**	**3.72**	**4.82**	**4.23**	**4.77**
32	1.8 E - 4	1.2 E - 4	3.1 E - 5	4.3 E - 6	4.1 E - 6	4.0 E - 6	3.7 E - 6
	2.04	**2.77**	**3.61**	**4.07**	**4.99**	**4.67**	**5.13**
64	9.0 E - 5	4.2 E - 5	8.6 E - 6	1.1 E - 6	8.3 E - 7	8.5 E - 7	7.2 E - 7
	2.10	**2.79**	**3.56**	**4.26**	**5.10**	**4.90**	**5.34**
128	4.3 E - 5	1.5 E - 5	2.4 E - 6	2.5 E - 7	1.6 E - 7	1.7 E - 7	1.4 E - 7
	2.15	**2.80**	**3.53**	**4.38**	**5.16**	**5.04**	**5.46**
256	2.0 E - 5	5.3 E - 6	6.9 E - 7	5.6 E - 8	3.1 E - 8	3.4 E - 8	2.5 E - 8
	2.18	**2.80**	**3.51**	**4.45**	**5.20**	**5.13**	**5.53**
512	9.1 E - 6	1.9 E - 6	2.0 E - 7	1.3 E - 8	6.0 E - 9	6.7 E - 9	4.5 E - 9
	2.21	**2.81**	**3.49**	**4.50**	**5.23**	**5.18**	**5.58**
1024	4.1 E - 6	6.8 E - 7	5.6 E - 8	2.8 E - 9	1.2 E - 9	1.3 E - 9	8.1 E - 10
	2.30	2.83	3.43	4.59	5.28	5.28	5.66

Tables 6.7.1-6.7.2 describe global convergence errors $\|u - u_N\|_\infty$ for examples (6.7.1)-(6.7.3) and for the exponential transformation φ_1. In Table 6.7.1, we only use values $\rho \in \mathbb{N}$ and in Table 6.7.2, we allow also values $\rho \in \mathbb{R}$. The numerical results in Tables 6.7.1-6.7.2 are in very good agreement with the theoretical estimates (6.7.4) and (6.7.5).

As we can see form Table 6.7.2, there exist important differences between $\varphi_1 \in \Phi^{m,\rho} \cap C^m[0,1]$ and $\varphi_1 \in \Phi^{m,\rho}$. Compare $\rho = 1$, $r = 1.5$ and $\rho = 2$, $r = 1.1$ with $\rho = 1.5$, $r = 1$ and $\rho = 1.1$, $r = 2$, also $\rho = 1, r = 2.4$ with $\rho = 2.4, r = 1$.

Table 6.7.3: Examples (6.7.1)-(6.7.3), $\nu = -1/4$, exponential transformation φ_1, local superconvergence (6.7.6)

N	$\gamma_N^{(1,1)}$ $\tilde{\delta}_N^{(1,1)}$	$\gamma_N^{(1,1.2)}$ $\tilde{\delta}_N^{(1,1.2)}$	$\gamma_N^{(1.2,1)}$ $\tilde{\delta}_N^{(1.2,1)}$	$\gamma_N^{(1,1.5)}$ $\tilde{\delta}_N^{(1,1.5)}$	$\gamma_N^{(1.5,1)}$ $\tilde{\delta}_N^{(1.5,1)}$	$\gamma_N^{(2,1)}$ $\tilde{\delta}_N^{(2,1)}$	$\gamma_N^{(3,1)}$ $\tilde{\delta}_N^{(3,1)}$
4	1.0 E - 4	5.4 E - 5	9.6 E - 6	2.3 E - 5	2.1 E - 5	1.1 E - 5	7.4 E - 5
	4.91	**6.68**	**7.53**	**10.80**	**7.48**	**16.20**	**13.89**
8	2.1 E - 5	8.0 E - 6	1.3 E - 6	2.1 E - 6	2.8 E - 6	6.6 E - 7	5.3 E - 6
	4.82	**6.55**	**3.99**	**10.59**	**7.57**	**16.48**	**15.17**
16	4.3 E - 6	1.2 E - 6	3.2 E - 7	2.0 E - 7	3.7 E - 7	4.0 E - 8	3.5 E - 7
	4.79	**6.50**	**4.46**	**10.50**	**7.69**	**16.58**	**15.86**
32	9.0 E - 7	1.9 E - 7	7.1 E - 8	1.9 E - 8	4.9 E - 8	2.4 E - 9	2.2 E - 8
	4.77	**6.50**	**4.73**	**10.45**	**7.77**	**16.60**	**16.18**
64	1.9 E - 7	2.9 E - 8	1.5 E - 8	1.8 E - 9	6.3 E - 9	1.5 E - 10	1.4 E - 9
	4.76	**6.50**	**4.89**	**10.43**	**7.84**	**16.73**	**16.33**
128	3.9 E - 8	4.6 E - 9	3.1 E - 9	1.8 E - 10	8.0 E - 10	8.7 E - 12	8.4 E - 11
	4.76	**6.50**	**4.99**	**10.47**	**7.89**	**16.52**	**16.68**
256	8.3 E - 9	6.9 E - 10	6.2 E - 10	1.7 E - 11	1.0 E - 10	5.3 E - 13	5.0 E - 12
	4.76	6.50		10.38		16.00	16.00

Table 6.7.3 describes the local superconvergence errors for examples (6.7.1)-(6.7.3). The expected theoretical results are presented in (6.7.6) for the transformations $\varphi_1 \in \Phi^{m+1,\rho} \cap C^{m+1}[0,1]$. The superconvergence results in Table 6.7.3 are in good agreement with the theoretical ones. The condition $\varphi_1 \in \Phi^{m+1,\rho} \cap C^{m+1}[0,1]$ is important as we can see for $\rho = 1, r = 1.2$ and $\rho = 1.2, r = 1$, also for $\rho = 1, r = 1.5$ and $\rho = 1.5, r = 1$.

Next, we consider the equation (6.1.1) (see [51]), where

$$K(x,y) = -2, \quad (x,y) \in D_1^{'}, \tag{6.7.7}$$

$$f(x) = \frac{3\sin x - \cos x + 3e^x}{2} \tag{6.7.8}$$

$$u(x) = \frac{\sin x + \cos x + e^x}{2}, \quad x \in [0,1]. \tag{6.7.9}$$

6.7. *Case* $K \in W^{m+1,\nu}(D_b')$ *and* $f \in C^{m+1,\nu}(0,b]$, $\nu < 0$

Here $K \in W^{m,\nu}(D_1')$ and $f, u \in C^m[0,1] \subset C^{m,\nu}(0,1]$ for arbitrary $m \in \mathbb{N}$ and $\nu < 1$. For the numerical results we use $m = 3$ and the node points (6.1.6).

Table 6.7.4: Examples (6.7.7)-(6.7.9), $\nu < 1 - m$, exponential transformation φ_1, global convergence (6.7.10)-(6.7.11).

N	$\varepsilon_N^{(1,1)}$ $\delta_N^{(1,1)}$	$\varepsilon_N^{(1.5,1)}$ $\delta_N^{(1.5,1)}$	$\varepsilon_N^{(1,1.5)}$ $\delta_N^{(1,1.5)}$	$\varepsilon_N^{(2,1)}$ $\delta_N^{(2,1)}$	$\varepsilon_N^{(2.5,1)}$ $\delta_N^{(2.5,1)}$	$\varepsilon_N^{(1,2.5)}$ $\delta_N^{(1,2.5)}$	$\varepsilon_N^{(3.5,1)}$ $\delta_N^{(3.5,1)}$
16	3.0 E - 6	3.6 E - 4	9.6 E - 6	3.7 E - 5	7.9 E - 5	4.0 E - 5	2.4 E - 4
	7.93	**2.83**	**7.71**	**7.60**	**7.48**	**7.30**	**7.24**
32	3.8 E - 7	1.3 E - 4	1.3 E - 6	4.9 E - 6	1.1 E - 5	5.5 E - 6	3.3 E - 5
	7.97	**2.83**	**7.86**	**7.80**	**7.73**	**7.64**	**7.61**
64	4.8 E - 8	4.4 E - 5	1.6 E - 7	6.2 E - 7	1.4 E - 6	7.2 E - 7	4.3 E - 6
	7.98	**2.83**	**7.93**	**7.90**	**7.87**	**7.82**	**7.80**
128	6.0 E - 9	1.6 E - 5	2.0 E - 8	7.9 E - 8	1.7 E - 7	9.2 E - 8	5.5 E - 7
	7.99	**2.83**	**7.96**	**7.95**	**6.59**	**7.91**	**7.90**
256	7.5 E - 10	5.6 E - 6	2.5 E - 9	9.9 E - 9	2.6 E - 8	1.2 E - 8	7.0 E - 8
	7.99	**2.83**	**7.98**	**7.97**	**5.66**	**7.95**	**7.95**
512	9.4 E - 11	2.0 E - 6	3.2 E - 10	1.2 E - 9	4.7 E - 9	1.5 E - 9	8.8 E - 9
	7.95	**2.83**	**7.98**	**7.98**	**5.66**	**7.97**	**7.97**
1024	1.2 E - 11	6.9 E - 7	4.0 E - 11	1.6 E - 10	8.3 E - 10	1.8 E - 10	1.1 E - 9
	7.64	**2.83**	**7.89**	**7.97**	**5.66**	**7.97**	**7.98**
2048	1.5 E - 12	2.5 E - 7	5.0 E - 12	2.0 E - 11	1.5 E - 10	2.3 E - 11	1.4 E - 10
	8.00	2.83	8.00	8.00	5.66	8.00	8.00

From Theorem 4.4.3 it follows by $\nu < 1 - m$ that for any transformation $\varphi \in \Phi^{m,\rho} \cap C^m[0,1]$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = 2^3, \quad \rho \geq 1 \quad , r \geq 1, \tag{6.7.10}$$

and for any transformation $\varphi \in \Phi^{m,\rho}$ and for sufficiently large values of N we have

$$\delta_N^{(\rho,r)} = \left\{ \begin{array}{ll} 2^3 & \text{if } r \geq \frac{3}{\rho} \\ 2^{r\rho} & \text{if } 1 \leq r < \frac{3}{\rho} \end{array} \right\}. \tag{6.7.11}$$

The numerical results in Table 6.7.4 are perfectly in agreement with the theoretical results (6.7.10) and (6.7.11). Note that there is a difference between $\rho \in \mathbb{N}$ and $\rho \in \mathbb{R}$ for $1 \leq \rho < 3$.

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Sisukokkuvõte

Tükiti polünomiaalne kollokatsioonimeetod iseärasustega Volterra integraalvõrrandi jaoks

Käesolevas töös vaadeldakse lineaarset Volterra II liiki integraalvõrrandit

$$u(x) = \int_0^x K(x,y)u(y)\,dy + f(x), \quad x \in [0,b], \quad b > 0, \tag{7.1.1}$$

kus otsitavaks on u. Vabaliikme f kohta tehakse teatud eeldused, mis on täidetud kõigi lõigus $[0,b]$ pidevate ja m korda pidevalt diferentseeruvate funktsioonide korral ning võimaldavad vaadelda ka selliseid funktsioone, mille tuletised mingist järgust alates võivad olla tõkestamata punkti 0 läheduses.

Tuuma $K(x,y)$ kohta eeldatakse, et $K(x,y)$ on m korda pidevalt diferentseeruv kolmnurgas $\{(x,y) : 0 \le y < x \le b\}$, kusjuures leidub reaalarv $\nu < 1$ nii, et kõigi tingimust $i + j \le m$ rahuldavate mittenegatiivsete täisarvude i ja j korral kehtib võrratus

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j K(x,y) \right| \le c \left\{ \begin{array}{ll} 1 & ,\nu + i < 0 \\ 1 + |\log(x-y)| & ,\nu + i = 0 \\ (x-y)^{-\nu-i} & ,\nu + i > 0 \end{array} \right\}, \tag{7.1.2}$$

kus $c = c(K)$ on mingi positiivne konstant ja $0 \le x \le b$, $0 \le y < x$. Tingimusest (7.1.2) järeldub $i = j = 0$ korral, et kehtib hinnang

$$|K(x,y)| \le c \left\{ \begin{array}{ll} 1 & ,\nu < 0 \\ 1 + |\log(x-y)| & ,\nu = 0 \\ (x-y)^{-\nu} & ,\nu > 0 \end{array} \right\}, \quad 0 \le y < x \le b.$$

Seega, kui $0 \le \nu < 1$, siis tuum $K(x,y)$ võib omada logaritmilist või astmelist iseärasust diagonaalil $y = x$:

$$|K(x,y)| \le c\,(1 + |\log(x-y)|), \quad \nu = 0,$$

$$|K(x,y)| \le c\,(x-y)^{-\nu}, \quad 0 < \nu < 1,$$

kus $0 \le y < x \le b$. Kui $\nu < 0$, siis tuum $K(x,y)$ on tõkestatud hulgal $\{(x,y) : 0 \le y < x \le b\}$, kuid tema tuletised võivad $y \to x$ korral tõkestamatult kasvada.

Vaadeldakse ka juhtu, kus $K(x,y)$ võib lisaks diagonaalsele iseärasusele (s.t. iseärasusele kui $y \to x$) omada veel iseärasust $y \to 0$ korral. Täpsemalt, vaadeldakse tuumade $K(x,y)$ klassi, kus $K(x,y)$ on m korda pidevalt diferentseeruv kolmnurgas $\{(x,y) : 0 < y < x \le b\}$, kusjuures leiduvad reaalarvud $\nu < 1$ ja $\lambda < 1$

nii, et $\nu + \lambda < 1$ ning kõigi tingimust $i + j \leq m$ rahuldavate mittenegatiivsete täisarvude i ja j korral kehtib võrratus

$$\left| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j K(x,y) \right| \leq c \left\{ \begin{array}{ll} 1 & ,\nu + i < 0 \\ 1 + |\log(x-y)| & ,\nu + i = 0 \\ (x-y)^{-\nu-i} & ,\nu + i > 0 \end{array} \right\} y^{-\lambda-j}, \tag{7.1.3}$$

kus $c = c(K)$ on mingi positiivne konstant ja $0 \leq x \leq b$, $0 < y < x$. Näiteks võib tuum $K(x,y)$ omada kuju

$$K(x,y) = K_1(x,y)(x-y)^{-\nu} y^{-\lambda} + K_2(x,y), \quad 0 < y < x \leq b,$$

kus $0 < \nu < 1$, $\lambda < 1 - \nu$ ja $K_1(x,y)$ ning $K_2(x,y)$ on mingid m korda pidevalt diferentseeruvad funktsioonid, kui $0 \leq y \leq x \leq b$.

Võrrandi (7.1.1) lahend u on sel juhul pidev funktsioon lõigul $[0,b]$, kuid tema tuletised võivad olla tõkestamata punktis 0.

Võrrandi (7.1.1) ligikaudsel lahendamisel tükiti polünomiaalse kollokatsioonimeetodiga saab lahendi u iseärast käitumist arvesse võtta, kui kasutada selliseid ebaühtlaseid võrke, kus võrgupunktid asuvad tihedamalt lõigu $[0,b]$ vasakpoolse otspunkti ümbruses. Kuid tugevalt ebaühtlaste võrkude kasutamine võib soodustada ümardamisvigade kuhjumist ning praktiliste arvutuste läbiviimisel põhjustada teatavat numbrilist ebastabiilsust, kui võrgupunktide arv on küllalt suur.

Võrrandi (7.1.1) ligikaudseks lahendamiseks on töös vaadeldud meetodit, mis tugineb võrrandi teisendamisel kujule, mille lahendi tuletiste iseärasused on nõrgemad, kui lähtevõrrandi lahendi tuletiste iseärasused. Teisendatud võrrandi lahendamiseks kasutatakse kollokatsioonimeetodit ühtlasel või nõrgalt ebaühtlasel võrgul tükiti polünomiaalsete koordinaatfunktsioonide korral.

Dissertatsioon koosneb kuuest peatükist. Esimeses kahes peatükis antakse ülevaade tööst ja tõestatakse rida abitulemusi, mida läheb vaja lähislahendite vigade hindamisel.

Kolmandas peatükis tuuakse sisse töös olulist rolli mängiv siluvate teisenduste klass ja uuritakse sellesse klassi kuuluvate teisenduste omadusi.

Neljandas peatükis käsitletakse võrrandi (7.1.1) ligikaudset lahendamist meetodiga, mis tugineb siluvale muutujate vahetusele ja kollokatsioonimeetodile tükiti polünomiaalsete koordinaatfunktsioonide korral. On tõestatud vaadeldava meetodi koondumine ning tuletatud rida koonduvuskiiruse hinnanguid mitmesuguste tuumade ja vabaliikmete jaoks erinevate siluvate teisenduste ja võrgu ebaühtlust kirjeldava parameetri väärtuste korral. Saadud tulemustest järeldub muuhulgas, kuidas tuleb valida vastav siluv muutujate vahetus ja kuidas tuleb tihendada võrku integreerimislõigu $[0,b]$ vasakpoolse otspunkti ümbruses nii, et saavutada meetodi kõrgeim võimalik koonduvusjärk $O(N^{-m})$, kui teisendatud võrrandi lahendi ligikaudseks leidmiseks kasutatakse $m-1$ järku ($m \geq 1$) tükiti polünomiaalset aproksimatsiooni.

Viiendas peatükis esitatakse uusi superkoondumise tulemusi lõigu $[0, b]$ spetsiaalsetes punktides tingimusel, et lahendi, vabaliikme ja tuuma iseärasusi siluv teisendus on sobivalt valitud ning kollokatsioonipunktide valik rahuldab teatud täiendavaid eeldusi.

Viimases peatükis on kontrollitud teoreetiliste hinnangute täpsust ulatuslike numbriliste eksperimentide läbiviimise teel. Testülesannete lahendamisel saadud arvulistest tulemustest järeldub, et töös saadud veahinnangud on järgu poolest mitteparandatavad.

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Curriculum Vitae

Marek Kolk

Born : November 27, 1978, Viljandi, Estonia

Citizenship : Estonian

Address : Institute of Mathematics, J. Liivi 2, 50409 Tartu, Estonia

Phone : +372 56 62 64

E-mail : marek.kolk@ut.ee

Education

1985 - 1994	Puiatu Primary School in Viljandimaa
1994 - 1997	Carl Robert Jakobson Gymnasium in Viljandi
1997 - 2002	Faculty of Mathematics, University of Tartu, *Baccalaureus Scientiarum* in mathematics
2002 - 2004	Faculty of Mathematics, University of Tartu, Institute of Applied Mathematics, *Magister Scientiarum* in mathematics
2004 - 2010	Faculty of Mathematics, University of Tartu, PhD student at the Institute of Applied Mathematics

Professional employment

since 2008	Institute of Applied Mathematics, University of Tartu, researcher

Scientific Work

The main field of interest is high order numerical methods for solving weakly singular integral and integro-differential equations. Results have been published in six papers and presented at the following six conferences:

"The 11th Mathematical Modelling and Analysis" in Jurmala, Latvia (2006),

"The 12th Mathematical Modelling and Analysis" in Trakai, Lithuania (2007),

"The 13th Mathematical Modelling and Analysis" in Kääriku, Estonia (2008),

"The 7th Methods of Algebra and Analysis" in Tartu, Estonia (2008),

"The 8th European Conference on Numerical Mathematics and Advanced Applications" in Uppsala, Sweden (2009),

"The 8th International Conference of Numerical Analysis and Applied Mathematics" in Rhodes, Greece (2010).

Elulookirjeldus

Marek Kolk

Sünniaeg ja -koht : 27. november 1978, Viljandi, Eesti

Kodakondsus : Eesti

Telefon : +372 56 62 64

E-kiri : marek.kolk@ut.ee

Haridus

1985 - 1994	Puiatu Põhikool Viljandimaal
1994 - 1997	Viljandi Carl Robert Jakobsoni nimeline Gümnaasium
1997 - 2002	Tartu Ülikooli matemaatika-informaatikateaduskond, *baccalaureus scientiarum* matemaatika erialal
2002 - 2004	Tartu Ülikooli matemaatika-informaatikateaduskond, *magister scientiarum* matemaatika erialal
2004 - 2010	Tartu Ülikooli matemaatika-informaatikateaduskond, doktoriõpe matemaatika instituudis

Erialane teenistuskäik

alates 2008	Tartu Ülikooli matemaatika-informaatikateaduskond, matemaatika instituudi teadur

Teadustegevus

Peamine uurimisvaldkond on efektiivsed numbrilised meetodid integraal- ja integro-diferentsiaalvõrrandite ligikaudseks lahendamiseks. Tulemused on publitseeritud kuues teadusartiklis ja esitatud kuuel teaduskonverentsil:

"The 11th Mathematical Modelling and Analysis" Jurmalas (2006),

"The 12th Mathematical Modelling and Analysis" Trakais (2007),

"The 13th Mathematical Modelling and Analysis" Käärikul (2008),

"Algebra ja analüüsi meetodid VII" Tartus (2008),

"The 8th European Conference on Numerical Mathematics and Advanced Applications" Uppsalas (2009),

"The 8th International Conference of Numerical Analysis and Applied Mathematics" Rhodosel (2010).

List of Publications

1. T. Diogo, M. Kolk, P. Lima, A. Pedas. High-order methods for weakly singular Volterra integro-differential equations. Constanda, C.; Perez, M. E. Integral Methods in Science and Engineering, Volume 2: Computational Methods, Birkhäuser, Boston (2010), 151-160.

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