DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
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Graded q-differential algebras and algebraic models in noncommutative geometry

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Introduction

Given a differential module E over a commutative ring with a differential d, where d is an endomorphism of a module E satisfying $d^2 = 0$, one can measure the non-exactness of the sequence $E \xrightarrow{d} E \xrightarrow{d} E$ by the homologies of this differential module E which can be viewed as the study of the equation $d^2 = 0$. This equation is a basis for several important structures in modern algebra, differential geometry and theoretical physics to point out only three of them which are the homological algebra [46], the theory of de Rham cohomologies on a smooth manifolds and the BRST-quantization in gauge field theories. The theory of de Rham cohomologies on a smooth manifold M originated with the work of de Rham when he proved that $H^*(\Omega(M)) \cong H^*(M, \mathbb{R})$ for the de Rham algebra of differential forms on a manifold M, i.e. the cohomologies of the de Rham complex are isomorphic to the real cohomologies of a manifold, and this immediately provided a link between the analysis on a manifold and its topology. From an algebraic point of view the de Rham algebra of differential forms on a smooth manifold is a graded differential algebra.

An idea to generalize the concept of a differential module and to elaborate the corresponding algebraic structures by giving the mentioned above basic property of differential $d^2 = 0$ a more general form $d^N = 0$, $N \geq 2$ seems to be very natural. Taking the equation $d^N = 0$ as a starting point one should choose a space where a calculus with $d^N = 0$ will be constructed. As a calculus with $d^N = 0$ may be considered as a generalization of $d^2 = 0$ and taking into account that there is an exterior calculus of differential forms with exterior differential $d^2 = 0$ on a smooth manifold one way to construct $d^N = 0$ is to take a smooth manifold and to consider objects on this manifold more general than the differentials forms. This approach was proposed and studied in [22, 23, 24]. The second approach arises within the framework of noncommutative geometry and it is based on q-deformed structures such as graded q-Leibniz rule, graded q-commutator, graded inner q-derivation, where q is a primitive Nth root of unity.

The first approach to $d^N = 0$ is based on the tensors with mixed symmetries. Let us remind that a differential p-form

$$\omega = \omega_{i_1 i_2 \ldots i_p}\, dx^{i_1} dx^{i_2} \ldots dx^{i_p},$$

can be identified with the skew-symmetric tensor field $\{\omega_{i_1 i_2 \ldots i_p}\}$. If we identify a differential p-form with the skew-symmetric tensor field then locally an exterior differential d can be written in the form

$$d\{\omega_{i_1 i_2 \ldots i_p}\} = \{\mathcal{A} \circ \partial\,(\omega_{i_1 i_2 \ldots i_p})\},$$

where $\mathcal{A}$ is the alternation with respect to subscripts $i_1, i_2, \ldots, i_p$ and ∂ : $\{\omega_{i_1 i_2 \ldots i_p}\} \longrightarrow \{\frac{\partial}{\partial x^j}(\omega_{i_1 i_2 \ldots i_p})\}$. In this case $d^2 = 0$ follows from the fact that partial derivatives commute with each other and differentiating and alternating twice we get zero. The basic idea of the first approach is to consider tensor fields with a more general kind of symmetry which is determined by Young diagrams and to replace the alternation $\mathcal{A}$ by Young symmetrizer $\mathcal{Y}$. In order to be more precise let us assume that we have a sequence of Young diagrams $(Y_p)_{p \in \mathbb{N}}$, where Young diagram Y_p consists of p boxes. This sequence of Young diagrams induces the sequence of vector spaces $\{\Omega^p_{Y_p}\}_{p \in \mathbb{N}}$ of smooth covariant tensor fields of degree p with symmetry determined by the Young diagram Y_p. Then the operator

$$d = \mathcal{Y}_p \circ \partial : \Omega^p_{Y_p} \longrightarrow \Omega^{p+1}_{Y_{p+1}},$$

where $\mathcal{Y}_p$ is the Young symmetrizer of a diagram Y_p, is of degree one. It can be proved that if each Young diagram Y_p has strictly less columns than N then $d^N = 0$.

The second approach was proposed and studied in the series of papers [34, 19, 20, 21] and it has led to the structures such as differential N-complex, N-cochain complex, generalized cohomologies of N-cochain complex and graded q-differential algebra, where q is a primitive Nth root of unity. In the paper [34] the author developed the algebraic structures based on $d^N = 0$ such as N-complex, homologies of N-complex as well as constructed an algebra of differential forms with exterior differential d satisfying $d^N = 0$ on a vector space $\mathbb{R}^n$, considered a connection in a vector bundle and studied characteristic classes constructed by means of differential forms with exterior differential $d^N = 0$. It is worth noting that in the case of a primitive cubic root of unity the curvature form of connection resembles the Chern-Simons form which is widely used in gauge theory and topological quantum field theories [44, 49].

A notion of graded q-differential algebra was introduced in [20] and it may be viewed as a generalization of a graded differential algebra. It is well known that a connection and its curvature are basic elements of the theory of fiber bundles and they play an important role not only in a modern differential geometry but also in theoretical physics namely in a gauge field theory. The development of a theory of connections in fibre bundles has been closely related to the development of a theoretical physics. The advent of supersymmetric field theories in the 70's of the previous century aroused interest for $\mathbb{Z}_2$-graded structures which became known in theoretical physics under the name of superstructures for instance supermanifold, super algebra, super Lie algebra, super Lie group, super vector bundle and so on. This direction of development has led to a concept of superconnection which was introduced and studied in [42]. The emergence of noncommutative geometry in the 80's of the previous century was a powerful spur to the development of a theory of connections on modules [16, 21, 22, 25, 26, 27, 28, 44]. A basic algebraic structure used in the theory of connections on modules is a graded differential algebra. A graded differential algebra is an algebraic model for the de Rham algebra of differential forms on a smooth manifold. Consequently considering a concept of graded q-differential algebra which is more general structure than a graded differential algebra we can develop a generalization of the theory of connections on modules. One of the aims of this thesis is to present and study algebraic structures based on the relation $d^N = 0$ and to generalize a concept of connection and its curvature applying a concept of graded q-differential algebra to the theory of connections on modules.

Chapter 1 is devoted to N-complexes and their cohomologies. In Section 1.1. we present the basic notions of homological algebra such as differential module, the homology of differential module, graded module, cochain complex, cohomologies of a cochain complex and cosimplicial module. We give statements which will be useful in what follows such as the exact triangle of homologies for an exact sequence of differential modules, the Künneth formula for cohomologies for tensor product of cochain complexes. There are also several examples of cochain complexes such as the cochain complex Hochshild cochains and Chevalley-Eilenberg complex of a Lie algebra. In Section 1.2. we start with a calculus of q-numbers which will play a very important role in this thesis. Then we present the notions of N-differential module, generalized homology of order m of N-differential module and give several propositions about homology of N-differential modules. In Section 1.3. we consider N-complexes and their cohomologies. We begin with the definition of a cochain N-complex of modules. We also remind a reader the notions such as graded q-commutator, graded q-derivation of degree m, graded q-Leibniz rule and inner graded q-derivation. Then we present the exact sequence in cohomologies for a short exact sequence of N-complexes. We show that N-complex can be constructed with the help of pre-cosimplicial K-module. Next we prove Theorem 1.3.7 which will be very important in the next chapters. This theorem is very useful in the sense that we can construct various cochain N-complexes by means of this theorem. Theorem 1.3.7 asserts if their exist an element v of grading one of a graded associative unital algebra $\mathscr{A}$ which satisfies $v^N \in \mathscr{Z}(\mathscr{A})$, where $\mathscr{Z}(\mathscr{A})$ is the graded center of $\mathscr{A}$, then the inner graded q-derivation ad_v^q is N-differential. We prove (Theorem 1.3.8.) that the generalized cohomologies of cochain N-complex of Theorem 1.3.7 are trivial. We end this section with a generalized Clifford algebra, explaining how this type of algebra can be equipped with the structure of cochain N-complex by means of Theorem 1.3.7.

Chapter 2 is devoted to the concepts of graded differential and graded q-differential algebras. In Section 2.1. we give a brief overview of graded differential algebras which play an important role in the modern differential geometry. We mention two well known examples of graded differential algebras in differential geometry which are the de Rham algebra of differential forms on a smooth manifold and the graded differential algebra of cochains on the Lie algebra $\mathfrak{g}$ of a Lie group G. The next example of a graded differential algebra which is briefly described in this section is the universal graded differential envelope $\Omega(\mathscr{A})$ of a graded associative unital algebra $\mathscr{A}$. We also remind a reader the notion of first order (coordinate) differential calculus over an associative unital algebra, which is widely used in the noncommutative geometry. At the end of Section 2.1. we describe the structure of the reduced Wess-Zumino algebra of differential forms on a reduced quantum plane. In Section 2.2 we give the definition of a graded q-differential algebra. We introduce the algebra of polynomials and endow it with the structure of graded q-differential algebra. We introduce two operators D, ∇ and the polynomials f_k, which are defined with the help of recurrent relation. We prove the Theorem 2.2.4 which give explicit power expansion formulae for the operator D and the polynomials f_k. Section 2.3. is devoted to the algebra of differential forms on a reduced quantum plane where our approach is based on the notion of graded q-differential algebra.

Chapter 3 is devoted to the generalization of the theory of connection by means of the notion of graded q-differential algebra. Section 3.1. has an introductory character and makes reader familiar with the basic notions of differential geometry such as connection and its curvature in the context of vector bundles. We generalize a concept of Ω-connection proposed by M. Dubois-Violette in [20], where Ω is a graded differential algebra with differential d. We use an algebraic approach based on the concept of graded q-differential algebra to define a notion of N-connection and show that in the case of $N = 2$ we get the algebraic analog of a classical connection. To better understand the structure of N-connection we introduce the notions of dual N-connection, N-connection consistent with a Hermitian structure of module. We define the notion of curvature of N-connection and prove that it satisfies the analog of Bianchi identity. At the end of this section we prove that every projective module admits an N-connection. In Section 3.2. we introduce a construction of $\mathbb{Z}_N$-connection, which can be viewed as a generalization of $\mathbb{Z}_2$-graded connection (superconnection). In Section 3.3 we consider the local structure of N-connection and its curvature, introducing the notion of matrix of N-connection. We express the components of the curvature of N-connection in the terms of the matrix of N-connection, this allows us to define a curvature matrix of N-connection. We consider the form of the curvature matrix of N-connection in two special cases, when $N = 2$ and $N = 3$. Making use of the algebra of polynomials introduced in Section 2.2 we consider the N-curvature form of N-connection form and give the explicit power expansion formulae for N-curvature form.

Chapter 1

N-complexes and cohomologies

1.1 Cochain complexes and cohomologies

Let K be a commutative ring with a unit, E be a left K-module and $\mathrm{End}\, E$ be the left K-module of endomorphisms of E. If E, F are left K-modules then the K-module of homomorphism from E to F will be denoted by $\mathrm{Hom}\,(E, F)$.

Definition 1.1.1. A module E endowed with an endomorphism $d \in \mathrm{End}\, E$ is said to be *a differential module* with *differential* or *coboundary operator* d if endomorphism d satisfies $d^2 = 0$. In the case when K is a field a differential module E will be referred to as a *differential vector space.*

It is easy to see that

$$\mathrm{Ker}\, d = \{u \in E : du = 0\}, \quad \mathrm{Im}\, d = \{u \in E : \exists v \in E,\, u = dv\},$$

are the submodules of a module E. From the nilpotency property of a differential $d^2 = 0$ it follows $\mathrm{Im}\, d \subset \mathrm{Ker}\, d \subset E$, and one can measure the non-exactness of the sequence $E \xrightarrow{d} E \xrightarrow{d} E$ by means of the quotient module $H(E) = \mathrm{Ker}\, d/\mathrm{Im}\, d$ which will be referred to as *the homology of a differential module E.*

Let E, F be differential modules respectively with differentials $d : E \longrightarrow E$, $d' : F \longrightarrow F$.

Definition 1.1.2. A homomorphism of modules $\phi \in \mathrm{Hom}\,(E, F)$ is said to be a homomorphism of differential modules E, F if it satisfies $\phi \circ d = d' \circ \phi$.

If $\phi : E \longrightarrow F$ is a homomorphism of differential modules respectively with differentials d, d' and $u \in \operatorname{Ker} d$ then $d'(\phi(u)) = d' \circ \phi(u) = \phi \circ d(u) = \phi(0) = 0'$, where $0, 0'$ are the zeroes of differential modules E, E'. Hence we have $\phi\,(\operatorname{Ker} d) \subset \operatorname{Ker} d'$ and analogously $\phi\,(\operatorname{Im} d) \subset \operatorname{Im} d'$. Consequently the mapping $\phi_* : H(E) \longrightarrow H(F)$ defined by

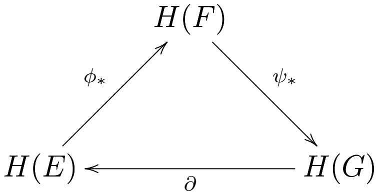
$$\phi_*([u]) = [\phi(u)], \quad u \in \operatorname{Ker} d, \tag{1.1.1}$$

where $[u] \in H(E), [\phi(u)] \in H(F)$ are the homology classes of u and $\phi(u)$, is the homomorphism of homologies of differential modules E, F. Thus a homomorphism $\phi : E \longrightarrow F$ of differential modules induces the homomorphism $\phi_* : H(E) \longrightarrow H(F)$ of their homologies. Let us remind that a sequence of homomorphisms of differential modules

$$\ldots \to E_{i-1} \overset{\phi_i}{\to} E_i \overset{\phi_{i+1}}{\to} E_{i+1} \to \ldots$$

is said to be an *exact sequence* if for any i we have $\operatorname{Im} \phi_i = \operatorname{Ker} \phi_{i+1}$. It can be proved [20]

Proposition 1.1.3. *If* $0 \to E \overset{\phi}{\to} F \overset{\psi}{\to} G \to 0$ *is an exact sequence of homomorphisms of differential modules then there exists a homomorphism of homologies* $\partial : H(G) \to H(E)$ *such that the triangle of homomorphisms*



(1.1.2)

is exact.

Proof. Let us prove $\operatorname{Im} \phi_* = \operatorname{Ker} \psi_*$ which means the exactness of the triangle of homomorphisms (1.1.2) at $H(F)$. Let d_E, d_F, d_G be differentials correspondingly of differential modules E, F, G. If $[y] \in \operatorname{Im} \phi_* \subset H(F)$ then there exists $[x] \in H(E)$ such that $\phi_*([x]) = [y]$. Making use of (1.1.1) we get $[y] = [\phi(x)]$ which means that elements $y, \phi(x)$ belong to the same homology class. Consequently there exists $y' \in F$ such that $y - \phi(x) = d_F y'$. Applying a homomorphism ψ to the both sides of this relation we get

$$\psi(y - \phi(x)) = \psi(d_F y') \Rightarrow \psi(y) - \psi \circ \phi(x) = d_G(\psi(y')).$$

Denoting $z = \psi(y')$ and taking into account the exactness of the sequence $E \xrightarrow{\phi} F \xrightarrow{\psi} G$ which for any x give $\psi \circ \phi(x) = 0$ we finally get $\psi(y) = d_G z$. Hence $[\psi(y)] = 0$ and making use of (1.1.1) we get $[\psi(y)] = \psi_*([y]) = 0$ which gives $[y] \in \operatorname{Ker} \psi_*$. Thus $\operatorname{Im} \phi_* \subset \operatorname{Ker} \psi_*$ and proving similarly $\operatorname{Ker} \psi_* \subset \operatorname{Im} \phi_*$ we finally get $\operatorname{Im} \phi_* = \operatorname{Ker} \psi_*$.

In order to construct the homomorphism $\partial : H(G) \longrightarrow H(E)$ we begin with $[z] \in H(G)$ which means $d_G z = 0$. Since ψ is surjective there exists $y \in F$ such that $\psi(y) = z$. We have $0 = d_G z = d_G(\psi(y)) = \psi(d_F(y))$. Hence $d_F(y) \in \operatorname{Ker} \psi$ but $\operatorname{Ker} \psi = \operatorname{Im} \phi$. Thus there exists $x \in E$ such that $\phi(x) = d_F(y)$. Applying a differential d_F to both sides

$$d_F(\phi(x)) = d_F^2(y) \Rightarrow \phi(d_E(x)) = 0,$$

and taking into account that ϕ is injective we conclude $d_E(x) = 0$. We define the hohomorphism $\partial : H(G) \longrightarrow H(E)$ by $\partial([z]) = [x]$. It can be shown that the triangle of homomorphisms (1.1.2) is exact at $H(E)$ and $H(G)$. $\square$

Let us mention that an exact sequence of homomorphisms of modules

$$0 \rightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} G \rightarrow 0$$

is called a *short exact sequence of homomorphisms* and in the case of differential modules a homomorphism $\partial : H(G) \longrightarrow H(E)$ is called *a connecting homomorphism of a short exact sequence.*

Definition 1.1.4. Let Γ be an additive group. A module E is said to be a Γ-graded module if it is given together with a direct sum decomposition into submodules $E^i \subset E$ labeled by $i \in \Gamma$, i.e.

$$E = \oplus_{i \in \Gamma} E^i.$$

We will call an element $u \in E^i$ a homogeneous element of grading i of Γ-graded module E. The grading of a homogeneous element $u \in E^i$ will be denoted by $|u| \in \Gamma$, i.e. $|u| = i$. If $E' = \oplus_{i \in \Gamma} \acute{E}^i$ is another Γ-graded module then a homomorphism of modules $\phi : E \longrightarrow E'$ is said to be a homomorphism of Γ-graded modules if $\phi(E^i) \subset \acute{E}^i$, i.e. a homomorphism ϕ preserves Γ-graded structures of E, E'.

We will use the following three types of Γ-gradations:

i) Γ-gradations where $\Gamma = \mathbb{Z}$ is the additive group of integers. In this case we will call $E = \oplus_{i\in\mathbb{Z}} E^i$ a $\mathbb{Z}$-graded module.

ii) Γ-gradations where $\Gamma = \mathbb{Z}_N$ is an additive group of residue classes modulo N, i.e. $\mathbb{Z}_N = \{\overline{0}, \overline{1}, \ldots, \overline{N-1}\}$, where $\overline{i}$ is a residue class modulo N. In this case a module $E = \oplus_{i\in\mathbb{Z}_N} E^i$ will be called $\mathbb{Z}_N$-graded or N-graded module.

ii) Γ-gradations where $\Gamma = \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ is the group of residue classes of modulo 2 with two elements $\overline{0}, \overline{1}$ and we will write $E = E_+ \oplus E_-$ instead of $E = E_{\overline{0}} \oplus E_{\overline{1}}$. In this case we will call E a semi-graded module. Obviously a semi-grade module is a particular case of N-graded module and we treat it separately since the $\mathbb{Z}_2$-graded structures are of great importance in supergeometry and supersymmetric field theories.

It is worth mentioning that in what follows the $\mathbb{Z}$-graded and $\mathbb{Z}_N$-graded structures will be most frequently used gradations. Obviously if we have a $\mathbb{Z}$-graded module E then it is always possible to pass from $\mathbb{Z}$-gradation of E to $\mathbb{Z}_N$-gradation. In most structures which will be defined and studied in this thesis can be used either $\mathbb{Z}$-gradation or $\mathbb{Z}_N$-gradation and in order not to mention this every time we adopt a convention that the term "graded module" means either $\mathbb{Z}$-graded module or $\mathbb{Z}_N$-graded module. In order to simplify notations we will denote a graded module by $E = \oplus_{i\in\mathbb{Z}} E^i$ having in mind that in this formula $\mathbb{Z}$ can be replaced by $\mathbb{Z}_N$.

Definition 1.1.5. A $\mathbb{Z}$-graded module $E = \oplus_{i\in\mathbb{Z}} E^i$ is said to be a positive graded module if for every $i < 0$ a corresponding submodule E^i is trivial, i.e. $E^i = 0$ for $i < 0$. In the case of a positive graded module E we will use notation $E = \oplus_{i\in\mathbb{N}} E^i$ which means that for $i < 0$ a corresponding submodule E^i is trivial.

Definition 1.1.6. A differential module E with differential d is said to be a *(positive) cochain complex* with differential or *coboundary operator* d if E is a (positive) graded module $E = \oplus_{i\in\mathbb{Z}} E^i$ and d is a homogeneous mapping of degree 1 with respect to a graded structure of E, i.e. $d : E^i \to E^{i+1}$. A *semi-cochain complex* E is a semi-graded differential module $E = E_+ \oplus E_-$ with differential d satisfying $d : E_\pm \longrightarrow E_\mp$.

If E is a cochain complex with coboundary operator d then in what follows we will call d either a differential of a cochain complex or a coboundary operator usually using the latter when studying cohomology of a cochain complex. Let E, E' be two cochain complexes correspondingly with differentials d, d'. A homomorphism of cochain complexes $\phi : E \longrightarrow E'$ is a homomorphism of graded differential modules, i.e. it satisfies $\phi(E^i) \subset \acute{E}^i, \phi \circ d = d' \circ \phi$. If E, E' are two semi-cochain complexes then $\phi : E \longrightarrow E'$ is a homomorphism of semi-cochain complexes if it is a homomorphism of differential modules satisfying $\phi(E_\pm) \subset E'_\pm$. The homology $H(E)$ of a cochain complex E has the structure of a graded module which is induced by a graded structure of E as follows

$$H^i(E) = \operatorname{Ker} d \cap E^i / \operatorname{Im} d \cap E^i.$$

Hence $H(E) = \oplus_{i\in\mathbb{Z}} H^i(E)$, and the homology $H(E)$ is usually referred to as a *cohomology of a cochain complex* E. Similarly a cohomology $H(E)$ of a semi-cochain complex is a semi-graded module $H(E) = H(E_+) \oplus H(E_-)$.

Let E_1, E_2 be two cochain complexes correspondingly with differentials d_1, d_2. The tensor product of graded modules E_1, E_2 is the graded module whose gradation is defined as follows

$$E_1 \otimes E_2 = \oplus_{k\in\mathbb{Z}} (E_1 \otimes E_2)^k, \tag{1.1.3}$$

where $(E_1 \otimes E_2)^k = \oplus_{i+j=k} E_1^i \otimes E_2^j$. Let us define

$$d(u \otimes v) = d_1(u) \otimes v + (-1)^{|u|} u \otimes d_2(v), \tag{1.1.4}$$

where u is a homogeneous element of E_1 and $v \in E_2$. Clearly

$$d : (E_1 \otimes E_2)^k \longrightarrow (E_1 \otimes E_2)^{k+1}.$$

Since $d_1^2 = d_2^2 = 0$ we have

$$\begin{aligned} d^2(u \otimes v) &= d(d_1(u) \otimes v) + (-1)^{|u|} d(u \otimes d_2(v)) \\ &= d_1^2(u) \otimes v + (-1)^{|u|+1} d_1(u) \otimes d_2(v) \\ &\qquad + (-1)^{|u|} d_1(u) \otimes d_2(v) + (-1)^{2|u|} u \otimes d_2^2(v) = 0 \end{aligned}$$

Consequently the tensor product of graded modules $E_1 \otimes E_2$ with graded structure defined by (1.1.3) and with differential d defined by (1.1.4) is the cochain complex which will be referred to as *the tensor product of cochain complexes* E_1, E_2. It is well known that the cohomology of the tensor product of two cochain complexes is equal to the tensor product of cohomologies of these cochain complexes, i.e.

Proposition 1.1.7. *Let K be a field of characteristic zero and E_1, E_2 be cochain complexes. Then*

$$H(E_1 \otimes E_2) = H(E_1) \otimes H(E_2). \tag{1.1.5}$$

The statement (1.1.5) of Proposition 1.1.7 bears the name of *Künneth formula.*

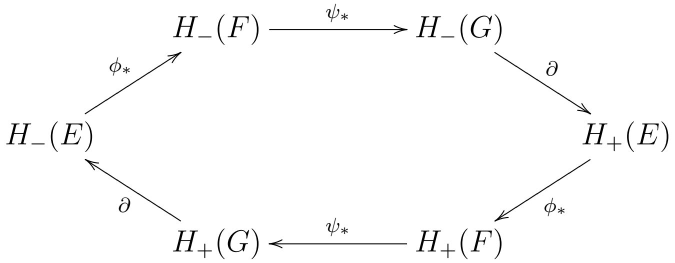
From Proposition 1.1.3 it follows that given an exact sequence of homomorphisms of cochain complexes

$$0 \to E \xrightarrow{\phi} F \xrightarrow{\psi} G \to 0$$

we have the exact sequence of homomorphisms of cohomologies of these cochain complexes

$$\ldots \xrightarrow{\partial} H^i(E) \xrightarrow{\phi_*} H^i(F) \xrightarrow{\psi_*} H^i(G) \xrightarrow{\partial} H^{i+1}(E) \to \ldots$$

In the case of a semi-cochain complex $E = E_+ \oplus E_-$ the corresponding exact sequence of homomorphisms of cohomologies can be represented in the form of the hexagon diagram



A cochain complex can be constructed by means of a pre-cosimplial module.

Definition 1.1.8. A positive graded module $E = \oplus_{n \in \mathbb{N}} E^n$ together with homomorphisms of degree one $f_0, f_1, f_2, \ldots, f_n, \ldots$, where

$$E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_0} E^2 \xrightarrow{f_0} \ldots,$$

and

$$E^{n-1} \xrightarrow{f_n} E^n \xrightarrow{f_n} E^{n+1} \xrightarrow{f_n} \ldots (n \geq 1),$$

is said to be a *pre-cosimplicial module* if homomorphisms $f_0, f_1, \ldots, f_n, \ldots$ satisfy the relations

$$f_j \circ f_i = f_i \circ f_{j-1}, \quad i < j. \tag{1.1.6}$$

Homomorphisms $f_0, f_1, \ldots, f_n, \ldots$ are called the *coface homomorphisms* of a pre-cosimplial module E.

From this definition it follows that in the case of a pre-cosimplicial module we have the following sequence of modules (or submodules of E)

$$E^0 \xrightarrow{\substack{f_1\\ f_0}} E^1 \xrightarrow{\substack{f_2\\ f_1\\ f_0}} E^2 \xrightarrow{\substack{f_3\\ f_2\\ f_1\\ f_0}} E^3 \xrightarrow{\substack{f_4\\ f_3\\ f_2\\ f_1\\ f_0}} \ldots \xrightarrow{\substack{f_n\\ \vdots\\ f_2\\ f_1\\ f_0}} E^n \xrightarrow{\substack{f_{n+1}\\ f_n\\ \vdots\\ f_2\\ f_1\\ f_0}} E^{n+1} \xrightarrow{\substack{f_{n+2}\\ f_{n+1}\\ f_n\\ \vdots\\ f_2\\ f_1\\ f_0}} \ldots$$

together with coface homomorphisms $f_0, f_1, f_2, \ldots, f_n, \ldots$. The above diagram shows that each part $E^n \longrightarrow E^{n+1}$ of this sequence is equipped with the $n+2$ coface homomorphisms $f_0, f_1, \ldots, f_{n+1}$. For example in the case of the part $E^1 \longrightarrow E^2$ we have three coface homomorphisms $f_0, f_1, f_2 : E^1 \longrightarrow E^2$ which satisfy

$$f_1 \circ f_0 = f_0^2, \quad f_2 \circ f_0 = f_0 \circ f_1, \quad f_2 \circ f_1 = f_1^2.$$

For each $n \in \mathbb{N}$ we define $d : E^n \longrightarrow E^{n+1}$ by

$$d = \sum_{i=0}^{n+1} (-1)^i f_i. \tag{1.1.7}$$

Calculating d^2 we get

$$\begin{aligned}
d^2 &= \sum_{j=0}^{n+1} (-1)^j f_j \sum_{i=0}^{n} (-1)^i f_i = \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} f_j \circ f_i \\
&= \sum_{i \geq j \geq 0}^{n} (-1)^{i+j} f_j \circ f_i + \sum_{j=1, j>i}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} f_j \circ f_i \\
&= \sum_{i \geq j \geq 0}^{n} (-1)^{i+j} f_j \circ f_i + \sum_{j=1, j>i}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} f_i \circ f_{j-1}
\end{aligned}$$

$$= \sum_{i \geq j \geq 0}^{n} (-1)^{i+j} f_j \circ f_i - \sum_{i \geq j \geq 0}^{n} (-1)^{i+j} f_j \circ f_i = 0. \qquad (1.1.8)$$

Hence d is the differential of a positive graded module E and E is the positive cochain complex. The differential d defined in (1.1.7) is called *the simplicial differential*, and the cohomology $H(E) = \oplus_{n \in \mathbb{N}} E^n$ of this positive cochain complex is called *the cohomology of pre-cosimplicial module.*

Definition 1.1.9. A pre-cosimplicial module E with coface homomorphisms $f_0, f_1, f_2, \ldots, f_n, \ldots$ is said to be *a cosimplicial module* if a positive graded module E is endowed in addition to coface homomorphisms with *codegeneracy homomorphisms* $s_0, s_1, s_2, \ldots, s_n, \ldots$, where for each $n \geq 0$ and $i \in \{0, 1, 2, \ldots, n\}$ the homomorphisms $s_i : E^{n+1} \longrightarrow E^n$ satisfy

$$s_j s_i = s_i s_{j+1}, \qquad i \leq j \qquad (1.1.9)$$

and

$$s_j f_i = \begin{cases} f_i s_{j-1}, & i < j \\ \mathrm{Id}_{E_k}, & i = j \text{ or } i = j+1, k \geq j \\ f_{i-1} s_j, & i \geq j \end{cases} \qquad (1.1.10)$$

Definition 1.1.10. An element $u \in E^n$ of a cosimplicial module E is said to be *a normalized cochain of degree* n if $s_i(u) = 0$ for any $i \in \{0, 1, 2, \ldots, n-1\}$.

Let us denote by $N^n(E) \subset E^n$ the submodule of normalized cochains of degree n and by $N(E) \subset E$ the graded submodule of normalized cochains. It is easily proved that if $u \in N^n(E)$ then $du \in N^{n+1}(E)$, where d is the simplicial differential. Hence $N(E)$ is the subcomplex of a cochain complex E. It can also be proved [46] that the cohomology of the cochain complex of normalized cochains $N(E)$ is isomorphic to cohomology of a cochain complex E, i.e. $H(N(E) \equiv N(E))$.

Let $\mathscr{A}$ be an associative unital K-algebra and $\mathscr{M}$ be an $\mathscr{A}$-bimodule. An *$\mathscr{M}$-valued Hochschild n-cochain* is a K-linear mapping $\omega : \otimes^n \mathscr{A} \longrightarrow \mathscr{M}$, where $\otimes^n \mathscr{A} = \underbrace{\mathscr{A} \otimes \mathscr{A} \otimes \ldots \otimes \mathscr{A}}_{n}$. Evidently the set of all $\mathscr{M}$-valued Hochschild n-cochains is a K-module which we denote by $C^n(\mathscr{A}, \mathscr{M})$. If we identify $C^0(\mathscr{A}, \mathscr{M}) \equiv \mathscr{M}$ then $C(\mathscr{A}, \mathscr{M}) = \oplus_{n \in \mathbb{N}} C^n(\mathscr{A}, \mathscr{M})$ is the positive graded module of $\mathscr{M}$-valued Hochschild cochains. For each $n \in \mathbb{N}$ we define the homomorphisms of degree one $f_0, f_1, \ldots, f_{n+1} : E^n \longrightarrow E^{n+1}$ as follows

$$
\begin{aligned}
&f_0(\omega)(x_0, x_1, \ldots, x_n) = x_0\, \omega(x_1, x_2, \ldots, x_n),\\
&f_i(\omega)(x_0, x_1, \ldots, x_n) = \omega(x_0, x_1, \ldots, x_{i-2}, x_{i-1}x_i, x_{i+1}, \ldots, x_n),\\
&f_{n+1}(\omega)(x_0, x_1, \ldots, x_n) = \omega(x_0, x_1, \ldots, x_{n-1})\, x_n,
\end{aligned}
$$

where $1 \leq i \leq n, x_0, x_1, \ldots, x_n \in \mathscr{A}$. For instance if $n = 0$ then an element $m \in C^0(\mathscr{A}, \mathscr{M}) \equiv \mathscr{M}$ is an element of an $\mathscr{A}$-bimodule $\mathscr{M}$, and the above formulae take on the form

$$
f_0(m)(x_0) = x_0\, m, \quad f_1(m)(x_0) = m\, x_0.
$$

Hence in this simple case the homomorphisms $f_0, f_1 : E^0 \longrightarrow E^1$ are the right and left multiplication of elements of $\mathscr{A}$-bimodule $\mathscr{M}$ by elements of an algebra $\mathscr{A}$. It can be verified that the homomorphisms $f_0, f_1, \ldots, f_{n+1}$ satisfy the coface homomorphisms relations (1.1.6). Consequently $C(\mathscr{A}, \mathscr{M})$ is the pre-cosimplicial module and equipping the positive graded module $C(\mathscr{A}, \mathscr{M})$ with simplicial differential d defined in (1.1.7) we get the positive cochain complex $C(\mathscr{A}, \mathscr{M})$ with differential d which is called in this case a *Hochschild differential*. The cohomologies of this cochain complex are called *Hochschild cohomologies* of an associative algebra $\mathscr{A}$. The pre-cosimplicial module of Hochschild cochains $C(\mathscr{A}, \mathscr{M})$ becomes the cosimplicial module if for any integer $i \in \{0, 1, 2, \ldots, n-1\}$ we define the codegeneracy homomorphisms $s_i : C^n(\mathscr{A}, \mathscr{M}) \longrightarrow C^{n-1}(\mathscr{A}, \mathscr{M})$ as follows

$$
s_i(\omega)(x_1, x_2, \ldots, x_{n-1}) = \omega(x_1, x_2, \ldots, x_i, \mathbb{1}, x_{i+1}, \ldots, x_{n-1}), \qquad (1.1.11)
$$

where $\omega \in C^n(\mathscr{A}, \mathscr{M})$.

Let $\mathfrak{g}$ be a Lie algebra over a commutative ring K and V be a left K-module. The left K-module $\operatorname{End} V$ is the Lie algebra over a commutative ring K if we endow it with the Lie commutator $[A, B] = A \circ B - B \circ A$, where $A, B \in \operatorname{End} V$. Let ϕ be a representation of Lie algebra $\mathfrak{g}$ in V which means that $\phi : \mathfrak{g} \longrightarrow \operatorname{End} V$ is a homomorphism of Lie algebras, i.e. for any $x, y \in \mathfrak{g}$ it holds $\phi([x, y]) = [\phi(x), \phi(y)]$. An V-valued n-cochain ω is a skew-symmetric K-linear mapping $\omega : \underbrace{\mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{n} \longrightarrow V$ and the left K-module of V-valued n-cochains will be denoted by $C^n(\mathfrak{g}, V)$. The direct sum $C(\mathfrak{g},V) = \otimes_{n\in\mathbb{N}} C^n(\mathfrak{g},V)$, where $C^0(\mathfrak{g},V) \equiv V$, is the positive graded K-module. This module is a positive cochain complex if for any $n \in \mathbb{N}$ one defines the coboundary operator $d : C^n(\mathfrak{g},V) \longrightarrow C^{n+1}(\mathfrak{g},V)$ by

$$\begin{aligned} d\omega(x_1,x_2,\dots,x_{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i+1}\phi(x_i)\omega(x_1,x_2,\dots,\hat{x}_i,\dots,x_{n+1}) \\ &\quad+\sum_{i<j}(-1)^{i+j}\omega([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1}), \end{aligned}$$

where $\hat{x}_i$ stands for omitted element. The cochain complex

$$C^0(\mathfrak{g},V) \xrightarrow{d} C^1(\mathfrak{g},V) \xrightarrow{d} \dots \xrightarrow{d} C^n(\mathfrak{g},V) \xrightarrow{d} C^{n+1}(\mathfrak{g},V) \xrightarrow{d} \dots, \qquad (1.1.12)$$

is called *the Chevalley-Eilenberg complex* of a Lie algebra $\mathfrak{g}$ with values in V. The cohomology of this cochain complex $H(\mathfrak{g},V)$ is called the V-valued cohomology of a Lie algebra $\mathfrak{g}$.

1.2 N-differential module and homologies

A concept of cohomology of a differential module or of a cochain complex with coboundary operator d is based on the quadratic nilpotency condition $d^2 = 0$. It is obvious that one can construct a generalization of a concept of cohomology of a cochain complex if the quadratic nilpotency $d^2 = 0$ is replaced by a more general nilpotency condition $d^N = 0$, where N is an integer satisfying $N \geq 2$. For the first time the question why we construct a cohomology theory taking $d^2 = 0$ and not $d^N = 0$, where N is any integer greater than one, was posed in the paper [34], where the author developed the structures based on a general nilpotency condition $d^N = 0$ and applied those structures to construct a generalization of the de Rham complex on a smooth manifold and generalization of characteristic classes of a vector bundle. The same idea to consider a more general nilpotency condition $d^N = 0$ instead of quadratic one was independently proposed in [29], where the authors elaborated a generalization of a cochain complex and its cohomologies, and later these generalizations were studied in the series of papers [19, 20, 22, 36]. In this section we describe a notion of an N-differential module, which can be considered as a generalization of a notion of differential module, define the generalized homologies of an N-differential module and state the conditions which ensure the triviality of these generalized homologies. Since several structures related with a notion of N-differential complex and its homologies are based on a calculus of q-numbers we begin this section with brief description of q-numbers and of corresponding notations.

Let K be a commutative ring with a unit. Fixing an element $q \in K$ of this commutative ring one defines the mapping $[\ ]_q : n \in \mathbb{N} \longrightarrow [n]_q \in K$ by setting $[0]_q = 0$ and

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \sum_{k=0}^{n-1} q^k, \quad n \geq 1.$$

The q-factorial of $[n]_q \in K$, where $n \in \mathbb{N}$, is defined by

$$[0]_q! = 1, \quad [n]_q! = [1]_q\,[2]_q \ldots [n]_q = \prod_{k=1}^{n} [k]_q, \ \ n \geq 1.$$

If k, n are integers satisfying $0 \leq k \leq n, n \geq 1$ then the Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

The Gaussian q-binomial coefficients satisfy the recursion relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \tag{1.2.1}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q. \tag{1.2.2}$$

Let us fix an integer N such that $N \geq 1$. Following the paper [35] we state two assumptions on a commutative ring K and on an element q of this ring:

($\mathfrak{N}_1$) $[N]_q = 0$,

($\mathfrak{N}_2$) $[N]_q = 0$ and the elements $[2]_q, \ldots, [N-1]_q$ are invertible.

It is easy to see that if an element q of a ring K satisfies the condition ($\mathfrak{N}_1$) then $q^N = 1$ and this implies that q is an invertible element of a ring K. It is worth mentioning that the Gaussian q-binomial coefficients in the case of an element $q \in K$ satisfying the assumption $(\mathfrak{N}_2)$ have very useful property which we shall often use in what follows in order to prove several propositions and theorems. This property is

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = 0, \quad k \in \{1, 2, \ldots, N-1\}, \tag{1.2.3}$$

where $q \in K$ satisfies $(\mathfrak{N}_2)$.

The most important example of the above described structure, which will be used throughout this thesis, is the field of complex numbers, i.e. if we take $K = \mathbb{C}$ then a complex number q satisfying the assumption $(\mathfrak{N}_1)$ is an Nth root of unity different from 1 and a complex number q satisfying the assumption $(\mathfrak{N}_2)$ is a primitive Nth root of unity, for instance $q = \exp(2\pi i/N)$.

Now we turn to a generalization of a differential module. Let E be a left K-module.

Definition 1.2.1. A left K-module E is said to be an *N-differential module* if it is equipped with an endomorphism $d : E \longrightarrow E$ which satisfies $d^N = 0$. An endomorphism d will be referred to as an N-differential of N-differential module E. If K is a field then N-differential module E will be referred to as an *N-differential vector space.*

It is clear that according to the definition (1.1.1) given in the previous section and to the above definition an 2-differential module is a differential module which means that a notion of N-differential module can be considered as a generalization of notion of differential module.

Let us fix a positive integer $m \in \{1, 2, \ldots, N-1\}$ and split up the Nth power of N-differential d as follows $d^N = d^m \circ d^{N-m}$. Then the nilpotency condition for N-differential can be written in the form $d^N = d^m \circ d^{N-m} = 0$ and this leads to possible generalization of a concept of cohomology. For each integer $1 \leq m \leq N-1$ one can define the submodules

$$Z_m(E) = \{x \in E : d^m x = 0\} \subset E, \tag{1.2.4}$$

$$B_m(E) = \{x \in E : \exists y \in E,\ x = d^{N-m} y\} \subset E. \tag{1.2.5}$$

From $d^N = 0$ it follows that $B_m(E) \subset Z_m(E)$.

Definition 1.2.2. For each $m \in \{1, 2, \ldots, N-1\}$ the quotient module $H_m(E) := Z_m(E)/B_m(E)$ is said to be a *generalized homology of order* m of N-differential module E.

It should be mention that in the case of classical theory of homology with $d^2 = 0$ (i.e. $N = 2$) there is only one choice for a value of m in the formulae (1.2.4), (1.2.5) which is $m = 1$ and in this case we have only homologies of order one which are easily identified with the ordinary homologies of differential module by $Z_1(E) \equiv Z(E), B_1(E) \equiv B(E), H_1(E) \equiv H(E) = Z(E)/B(E)$. In what follows we shall denote by $[x] \in H_m(E)$ the generalized homology class of order m generated by an element $x \in Z_m(E)$.

Let E, F be N-differential modules with N-differentials correspondingly d, d'. The definition 1.1.2 of a homomorphism of differential modules given in the previous section is easily generalized to N-differential modules as follows: a homomorphism of modules $\phi \in \mathrm{Hom}\,(E, F)$ is said to be a homomorphism of N-differential modules E, F if it satisfies $\phi \circ d = d' \circ \phi$. As in the case of differential modules a homomorphism ϕ of N-differential modules E, F induces the homomorphism of their homologies. Indeed if we fix an integer $m \in \{1, 2, \ldots, N-1\}$ and consider a homomorphism $\phi : E \longrightarrow F$ of N-differential modules respectively with differentials d, d' then for each element $x \in Z_m(E)$ it holds $(d')^m(\phi(x)) = (d')^m \circ \phi(x) = \phi \circ d^m(x) = \phi(0) = 0'$, where $0, 0'$ are respectively the zeroes of differential modules E, E'. Hence we have $\phi\,(Z_m(E)) \subset Z_m(F)$ and analogously $\phi\,(B_m(E)) \subset B_m(F)$. Consequently the mapping $\phi_* : H_m(E) \longrightarrow H_m(F)$ defined by

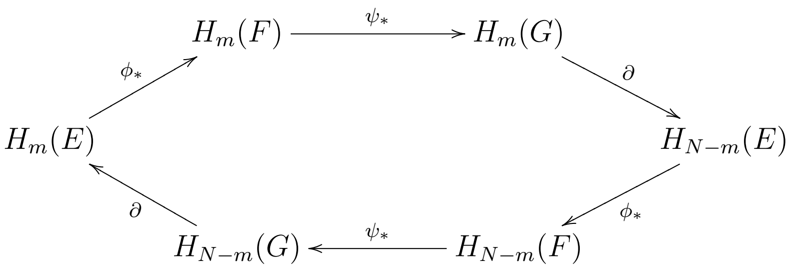
$$\phi_*([x]) = [\phi(x)], \quad x \in Z_m(E) \tag{1.2.6}$$

where $[x] \in H_m(E), [\phi(x)] \in H_m(F)$ are the homology classes of order M of x and $\phi(x)$, is the homomorphism of homologies of N-differential modules E, F. Thus a homomorphism $\phi : E \longrightarrow F$ of N-differential modules induces the homomorphism $\phi_* : H_m(E) \longrightarrow H_m(F)$ of their homologies of order m.

It turns out that in the case of N-differential modules one can prove a proposition [20] which is similar to the exact triangle proposition (1.1.3) for differential modules proved in the previous section.

Proposition 1.2.3. *If* $0 \to E \overset{\phi}{\to} F \overset{\psi}{\to} G \to 0$ *is an exact sequence of* N*-differential modules then for every* $m \in \{1, 2, \ldots, N-1\}$ *there are homomor-*

phisms $\partial : H_m(G) \longrightarrow H_{N-m}(E)$ such that the following hexagon diagram



is exact.

Proof. We begin the proof of this proposition with the exactness of the above diagram at the vertex $H_m(F)$, i.e. we will show $\mathrm{Im}\,\phi_* = \mathrm{Ker}\,\psi_*$. Let d_E, d_F, d_G be N-differentials correspondingly of N-differential modules E, F, G. For any $[y] \in \mathrm{Im}\,\phi_* \subset H_m(F)$ there exists $[x] \in H_m(E)$ such that $\phi_*([x]) = [y]$. From (1.2.6) it follows $[y] = [\phi(x)]$ which implies $y - \phi(x) = z$, where $z \in B_m(F)$. Obviously $\psi(z) \in B_m(G)$. Applying a homomorphism ψ to the both sides of $y - \phi(x) = z$ and taking into account the exactness of the sequence $E \xrightarrow{\phi} F \xrightarrow{\psi} G$ we get $\psi(y) = \psi(z)$. Hence $\psi(y) \in B_m(G)$ and $[\psi(y)] = 0$ in homologies of order m of a N-differential module G. Consequently $[\psi(y)] = \psi_*([y]) = 0$ which gives $[y] \in \mathrm{Ker}\,\psi_*$. Thus $\mathrm{Im}\,\phi_* \subset \mathrm{Ker}\,\psi_*$ and proving similarly $\mathrm{Ker}\,\psi_* \subset \mathrm{Im}\,\phi_*$ we obtain $\mathrm{Im}\,\phi_* = \mathrm{Ker}\,\psi_*$.

The connecting homomorphism $\partial : H_m(G) \longrightarrow H_{N-m}(E)$ of homologies can be constructed as follows: let $[x] \in H_m(G)$, i.e. $d_G^m x = 0$. As ψ is surjective homomorphism there exists $y \in F$ such that $\psi(y) = x$. We have $\psi(d_F^m(y)) = d_G^m x = 0$. Hence $d_F^m(y) \in \mathrm{Ker}\,\psi$ but $\mathrm{Ker}\,\psi = \mathrm{Im}\,\phi$. Thus there exists $z \in E$ such that $\phi(z) = d_F^m(y)$. Differentiating both sides of this relation $N - m$ times with respect to d_F we obtain

$$d_F^{N-m}(\phi(z)) = d_F^N(y) = 0 \Rightarrow \phi(d_E^{N-m}(z)) = 0.$$

As ϕ is injective we conclude $d_E^{N-m}(z) = 0$. Now let us define the homomorphism $\partial : H_m(G) \longrightarrow H_{N-m}(E)$ by $\partial([x]) = [z]$, where $[z] \in H_{N-m}(E)$. It is easy to prove $\mathrm{Im}\,\partial \subset \mathrm{Ker}\,\phi_*$ and $\mathrm{Im}\,\psi_* \subset \mathrm{Ker}\,\partial$. Indeed the former is equivalent to $\phi_* \circ \partial = 0$ and the latter is equivalent to $\partial \circ \psi_* = 0$. We have

$$\phi_* \circ \partial([x]) = \phi_*([z]) = [\phi(z)] = [d_F^m(y)] = 0 \in H_{N-m}(F).$$

If $[x] = \psi_*([y])$, where $x \in Z_m(G), y \in Z_m(F)$ and $\partial([x]) = [z]$, where $z \in Z_m(E)$, then $\phi(z) = d_F^m(y)$ but $d_F^m(y) = 0$ which implies $\phi(z) = 0$, and because of injectivity of ϕ we obtain $z = 0$ and $[z] = 0$. Similarly one can show $\operatorname{Ker}\phi_* \subset \operatorname{Im}\partial$ and $\operatorname{Ker}\partial \subset \operatorname{Im}\psi_*$, and this ends the proof. $\square$

It is obvious that each element x of the submodule $Z_m(E)$ also satisfies $d^{m+1}x = d(d^m x) = 0$ and consequently $Z_m(E) \subset Z_{m+1}(E)$. Similarly for each element y of the submodule $B_m(E)$ we have $y = d^{N-m}x = d^{N-(m+1)}(dx)$ which means $B_m(E) \subset B_{m+1}(E)$. Hence one can define the inclusion $i : Z_m(E) \longrightarrow Z_{m+1}(E)$ and $i : B_m(E) \longrightarrow B_{m+1}(E)$ which induces the inclusion of homologies $i_* : H_m(E) \longrightarrow H_{m+1}(E)$, where $i_*([x]) = [i(x)]$ and $x \in Z_m(E)$. Thus we have the sequence

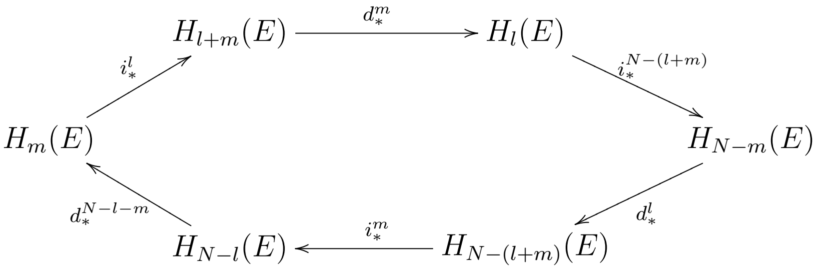
$$H_1(E) \xrightarrow{i_*} H_2(E) \xrightarrow{i_*} \ldots \xrightarrow{i_*} H_{N-1}(E).$$

If $x \in Z_{m+1}(E)$ then $d^m x = 0$. Consequently the element $dx \in dZ_{m+1}(E)$ satisfies $d^m x = 0$ which means $dx \in Z_m(E)$. Hence $dZ_{m+1}(E) \subset Z_m(E)$. Analogously for each $y \in B_{m+1}(E)$ there exists $x \in E$ such that $y = d^{N-(m+1)}x$ which implies $dy = d^{N-m}x$. Hence $dB_{m+1}(E) \subset B_m(E)$. From this it follows that if $x \in Z_{m+1}(E)$ then one can define $d_*([x]) = [dx]$ where $d_* : H_{m+1}(E) \longrightarrow H_m(E)$. Thus the N-differential d generates the sequence

$$H_{N-1}(E) \xrightarrow{d_*} H_{N-2}(E) \xrightarrow{d_*} \ldots \xrightarrow{d_*} H_1(E).$$

It can be shown [29] that the generalized homologies of different order of an N-differential module E are not independent.

Proposition 1.2.4. *For any integers l, m satisfying $l \geq 1, m \geq 1, l + m \leq N - 1$ the following diagram*



is exact.

Let us remind a reader that in the previous section a homomorphism $\phi : E \longrightarrow E'$ of differentials modules E, E' with differentials d, d' is defined as a homomorphism of modules which commutes with differentials $\phi \circ d = d' \circ \phi$. This definition is applicable to N-differential modules, i.e. if E, E' are N-differential modules with N-differentials d, d' then $\phi : E \longrightarrow E'$ is said to be a homomorphism of N-differentials modules if in addition to the requirement to be a homomorphism of modules it satisfies $\phi \circ d = d' \circ \phi$. It is evident that $\phi(Z_m(E)) \subset Z_m(E'), \phi(B_m(E)) \subset B_m(E')$ and hence a homomorphism φ induces the homomorphism of homologies of order m of N-differential modules E, E' which will be denoted just as in the case of differential modules by ϕ_*.

For applications of the theory of homologies of differential modules it is important to find the conditions which guarantee that the generalized homologies of an N-differential module is trivial. One criteria for generalized homologies to be trivial is stated in the following proposition [34].

Proposition 1.2.5. *Let E be an N-differential module with N-differential d. If there exists an integer $p \in \{1, 2, \ldots, N-1\}$ such that a generalized homology of order p of an N-differential module E is trivial, i.e. $H_p(E) = 0$, then generalized homology of any order of E is trivial, i.e. $H_m(E) = 0$ for any $m \in \{1, 2, \ldots, N-1\}$.*

The next very useful criteria for generalized homologies of an N-differential module to be trivial is related with suitable generalization of homotopy given in [34]. Here we give this criteria in the form proposed in [22].

Proposition 1.2.6. *Let E be an N-differential module with N-differential d. If there exist endomorphisms $h_0, h_1, \ldots, h_{N-1} \in End\, E$ which satisfy the relation*

$$d^{N-1}\, h_0 + d^{N-2}\, h_1\, d + \ldots + d^{N-1-k}\, h_k\, d^k + \ldots + h_{N-1} d^{N-1} = \mathrm{Id}_E,$$

where $\mathrm{Id}_E : E \longrightarrow E$ is the identity mapping, then the generalized homologies of any order $p \in \{1, 2, \ldots, N-1\}$ of N-differential module E is trivial, i.e. $H_p(E) = 0$.

The following proposition [22] can be used in order to show that the generalized homologies of an N-differential module are zeros in the case of a commutative ring K and its element $q \in K$ which satisfy the assumption $(\mathfrak{N}_2)$.

Proposition 1.2.7. *Let K and $q \in K$ satisfy the assumption ($\mathfrak{N}_2$). Then for any integer $k \in \{1, 2, \ldots, N-1\}$ a generalized homology $H_k(E)$ of an N-differential K-module is trivial if there exists an endomorphism $h : E \longrightarrow E$ which obeys the relation*

$$h \circ d - q\, d \circ h = \mathrm{Id}_E,$$

where d is an N-differential of E.

1.3 N-complexes and cohomologies

We start this section with a definition of N-complex [34] which is a generalization of the notion of cochain complex given in Section 1.1.

Definition 1.3.1. An N-differential module E with N-differential d is said to be a *cochain N-complex of modules* or simply *N-complex* if E is a graded module $E = \oplus_{k \in \mathbb{Z}} E^k$ and its N-differential d has degree 1 with respect to a graded structure of E, i.e. $d : E^k \longrightarrow E^{k+1}$.

It is worth mentioning that if N-differential of a graded N-differential module E has degree -1, i.e. $d : E^{k+1} \longrightarrow E^k$, then E is called a *chain N-complex of modules* [34]. Since in this thesis our main concern is cochain N-complexes in what follows N-complex stands for cochain N-complex of modules. According to this terminology we will call $H_m(E)$, where $m \in \{1, 2, \ldots, N-1\}$ a cohomology of order m of N-complex E.

It is easy to see that the cohomologies of order m of an N-complex E, where $m \in \{1, 2, \ldots, N-1\}$, inherit a graded structure of module E. Indeed let us fix $m \in \{1, 2, \ldots, N-1\}, k \in \mathbb{Z}$ and define the submodules

$$\begin{aligned} Z^k_m(E) &= \{x \in E^k : d^m x = 0\} \subset E^k, \\ B^k_m(E) &= \{x \in E^k : \exists\, y \in E^{k+m-N}, x = d^{N-m} y\} \subset Z^k_m(E). \end{aligned}$$

Then $H_m(E) = \oplus_{k \in \mathbb{Z}} H^k_m(E)$, where $H^k_m(E) = Z^k_m(E)/B^k_m(E)$.

Let $E = \oplus_{k \in \mathbb{Z}} E^k, F = \oplus_{k \in \mathbb{Z}} F^k$ be two N-complexes with N-differentials respectively d, d'. A homomorphism of N-complexes $\phi : E \longrightarrow F$ is a homomorphism of N-differential modules E, F which is of degree 0 with respect to graded structures of E and F. This can be illustrated by the following commutative diagram

$$\begin{array}{ccccccc}
\cdots \xrightarrow{d} & E^k & \xrightarrow{d} & E^{k+1} & \xrightarrow{d} \cdots \\
 & \downarrow \phi & & \downarrow \phi & \\
\cdots \xrightarrow{d'} & F^k & \xrightarrow{d'} & F^{k+1} & \xrightarrow{d'} \cdots
\end{array}$$

From the hexagon diagram of Proposition 1.2.3 it follows that a short exact sequence of N-complexes

$$0 \rightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} G \rightarrow 0$$

induces the exact sequence in cohomologies of these N-complexes

$$\begin{aligned} \ldots \xrightarrow{\phi_*} \quad H^k_m(F) \quad & \xrightarrow{\psi_*} H^k_m(G) \xrightarrow{\partial} H^{k+m}_{N-m}(E) \xrightarrow{\phi_*} \\ & \xrightarrow{\phi_*} H^{k+m}_{N-m}(F) \xrightarrow{\psi_*} H^{k+m}_{N-m}(G) \xrightarrow{\partial} H^{k+N}_m(E) \xrightarrow{\phi_*} \ldots \end{aligned}$$

Similarly the exact hexagon of Proposition 1.2.4 gives rise to the following exact sequence in cohomologies of an N-complex E

$$\begin{aligned} \ldots \rightarrow \quad H^k_m \quad (E) & \xrightarrow{i^l_*} H^k_{l+m}(E) \xrightarrow{d^m_*} H^{k+m}_l(E) \xrightarrow{i^{N-(l+m)}_*} H^{k+m}_{N-m}(E) \xrightarrow{d^l_*} \\ & \xrightarrow{d^l_*} H^{k+m+l}_{N-(l+m)}(E) \xrightarrow{i^m_*} H^{k+m+l}_{N-l}(E) \xrightarrow{d^{N-(l+m)}_*} H^{k+N}_m(E) \rightarrow \ldots \end{aligned}$$

An N-complex can be constructed [19] with the help of pre-cosimplicial K-module $E = \oplus_{n \in \mathbb{Z}} E^n$ whose coface endomorphisms are $f_0, f_1, \ldots, f_n \ldots$. Let us assume that $q \in K$ satisfies the assumption $(\mathfrak{N}_1)$, i.e. $[N]_q$=0, and for any integer $m \geq 0$ define the endomorphism d_m of a module E by

$$d_m = \delta_{m+1} + q^{n-m+1} \sum_{r=0}^{m} (-1)^r f_{n-m+r+1} : E^n \longrightarrow E^{n+1}, \tag{1.3.1}$$

where

$$\delta_m = \sum_{i=0}^{n-m+1} q^i f_i. \tag{1.3.2}$$

For instance if $m = 0, 1$ then

$$d_0 \quad = \quad \sum_{i=0}^{n+1} q^i f_i : E^n \longrightarrow E^{n+1},$$

$$d_1 = \sum_{i=0}^{n} q^i f_i - q^n f_{n+1} : E^n \longrightarrow E^{n+1}.$$

Obviously an endomorphism d_m has degree one with respect to graded structure of a module E, i.e. $d_m : E^n \longrightarrow E^{n+1}$.

Proposition 1.3.2. *If E is a pre-cosimplicial module with coface endomorphisms $f_0, f_1, \ldots, f_n, \ldots$ and $q \in K$ satisfies the assumption ($\mathfrak{N}_1$) then for any integer $m \geq 0$ the endomorphism d_m of a module E defined by (1.3.1) is an N-differential, i.e. $d_m^N = 0$, and a graded module E endowed with d_m is a positive N-complex.*

We can construct an N-complex by means of a graded associative unital algebra and this kind of N-complex is very important for the present thesis because we will use it the following sections to construct an analog of differential forms with exterior differential $d^N = 0$ on a reduced quantum plane and to construct a generalization of connection form and connection. This structure is proposed and studied in the papers [2, 3, 9].

From now on and until the end of this section we assume that

- K is the field of complex numbers $\mathbb{C}$,
- $q \in \mathbb{C}$ is a primitive Nth root of unity,
- graded structure stands for $\mathbb{Z}_N$-graded structure.

Let $\mathscr{A} = \oplus_{k \in \mathbb{Z}_N} \mathscr{A}^k = \mathscr{A}^0 \oplus \mathscr{A}^1 \oplus \ldots \oplus \mathscr{A}^{N-1}$ be a $\mathbb{Z}_N$-graded associative unital algebra whose identity element is denoted by $\mathbb{1}$. It is worth mentioning that in order to avoid cumbersome notations we use simplified notations for gradings (which are the elements of the group of residue classes modulo N) of an algebra $\mathscr{A}$ denoting them simply by $0, 1, \ldots, N-1$ instead of $\overline{0}, \overline{1}, \ldots, \overline{N-1}$. We remind a reader that a notion of graded associative unital algebra includes in addition to usual axioms (vector space over $\mathbb{C}$, associativity, identity element) a requirement for a law of composition to be consistent with a graded structure of an algebra in the sense that for any homogeneous elements $u, v \in \mathscr{A}$ it holds

$$|u\,v| = |u| + |v|, \tag{1.3.3}$$

where $|u|, |v|, |u+v| \in \mathbb{Z}_N$ are the gradings. Let us list few basic facts concerning a structure of a graded algebra. From (1.3.3) it follows immediately that the subspace $\mathscr{A}^0 \subset \mathscr{A}$ of elements of grading zero is the subalgebra of an algebra $\mathscr{A}$. Since this subalgebra plays an important role in several structures related to a graded algebra $\mathscr{A}$ we will denote it by $\mathfrak{A}$, i.e. $\mathfrak{A} \equiv \mathscr{A}^0$. It is easy to see that each subspace $\mathscr{A}^i \subset \mathscr{A}$ of homogeneous elements of grading i is the $\mathfrak{A}$-bimodule. Hence in the case of a graded algebra $\mathscr{A}$ we have the set of $\mathfrak{A}$-bimodules $\mathscr{A}^0, \mathscr{A}^1, \mathscr{A}^2, \ldots, \mathscr{A}^{N-1}$. The graded subspace $\mathscr{Z}(\mathscr{A}) \subset \mathscr{A}$ generated by homogeneous elements $u \in \mathscr{A}^k$, which for any $v \in \mathscr{A}^l$ satisfy $u\,v = (-1)^{kl} v\,u$, is called a *graded center* of a graded algebra $\mathscr{A}$.

Definition 1.3.3. Let $\mathscr{A} = \oplus_{k \in \mathbb{Z}_N} \mathscr{A}^k$ be a graded associative unital algebra over $\mathbb{C}$ and $u \in \mathscr{A}^k, v \in \mathscr{A}^l$ be homogeneous elements. The *graded commutator* $[\ ,\ ] : \mathscr{A}^k \otimes \mathscr{A}^l \longrightarrow \mathscr{A}^{k+l}$ is defined by

$$[u, v] = u\,v - (-1)^{kl} v\,u. \tag{1.3.4}$$

Definition 1.3.4. *A graded derivation of degree* m of a graded algebra $\mathscr{A}$ is a linear mapping $\delta : \mathscr{A} \longrightarrow \mathscr{A}$ of degree m with respect to graded structure of $\mathscr{A}$, i.e. $\delta : \mathscr{A}^i \longrightarrow \mathscr{A}^{i+m}$, which satisfies

$$\delta(u\,v) = \delta(u)\,v + (-1)^{ml} u\,\delta(v), \tag{1.3.5}$$

where u is a homogeneous element of grading l, i.e. $u \in \mathscr{A}^l$. If m is even then δ is a derivation of an algebra $\mathscr{A}$, and if m is odd then δ is called an *antiderivation* of a graded algebra $\mathscr{A}$. The property (1.3.5) of a graded derivation is called *graded Leibniz rule.*

Given a homogeneous element $v \in \mathscr{A}^m$ of grading m one associates to it a graded derivation of degree m, which is denoted by ad_v, as follows

$$\mathrm{ad}_v(u) = [v, u] = v\,u - (-1)^{ml} u\,v, \tag{1.3.6}$$

where $v \in \mathscr{A}^l$. The graded derivation ad_u is called an *inner graded derivation* of an algebra $\mathscr{A}$.

The notions of graded commutator and graded derivation of a graded algebra can be generalized within the framework of noncommutative geometry and the theory of quantum groups with the help of q-deformations. In general q may be any complex number different from 1 but for our purpose we need q to be a primitive Nth root of unity.

Definition 1.3.5. Let $\mathscr{A} = \oplus_{k\in\mathbb{Z}_N}\mathscr{A}^k$ be a graded associative unital algebra over $\mathbb{C}$ and $u \in \mathscr{A}^k, v \in \mathscr{A}^l$ be homogeneous elements. The *graded q-commutator* $[\ ,\ ]_q : \mathscr{A}^k \otimes \mathscr{A}^l \longrightarrow \mathscr{A}^{k+l}$ is defined by

$$[u, v]_q = u\,v - q^{kl} v\,u, \tag{1.3.7}$$

where q is a primitive Nth root of unity.

Definition 1.3.6. *A graded q-derivation of degree* m of a graded algebra $\mathscr{A}$ is a linear mapping $\delta : \mathscr{A} \longrightarrow \mathscr{A}$ of degree m with respect to graded structure of $\mathscr{A}$, i.e. $\delta : \mathscr{A}^i \longrightarrow \mathscr{A}^{i+m}$, which satisfies the graded q-Leibniz rule

$$\delta(u\,v) = \delta(u)\,v + q^{ml} u\,\delta(v), \tag{1.3.8}$$

where u is a homogeneous element of grading l, i.e. $u \in \mathscr{A}^l$.

In analogy with an inner graded derivation one defines an inner graded q-derivation of degree m of a graded algebra $\mathscr{A}$ associated to an element $v \in \mathscr{A}^m$ by the formula

$$\mathrm{ad}_v^q(u) = [v, u]_q = v\,u - q^{ml} u\,v, \tag{1.3.9}$$

where $u \in \mathscr{A}^l$. It is easy to verify that an inner graded q-derivation is a graded q-derivation.

The following theorem [2] can be used to construct a cochain N-complex for a certain class of graded associative unital algebras. It is worth mentioning that in [2] the author only suggests that this theorem can be proved by means of mathematical induction. In the present thesis we give a complete proof of the following theorem and prove a proposition which asserts that the generalized cohomologies of a cochain N-complex, constructed in the following theorem, are trivial.

Theorem 1.3.7. *Let $\mathscr{A} = \oplus_{k\in\mathbb{Z}_N}\mathscr{A}^k$ be a graded associative unital algebra and q be a primitive Nth root of unity. If there exists an element $v \in \mathscr{A}^1$ of grading one which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{A})$ then the inner graded q-derivation $d = \mathrm{ad}_v^q$ of degree 1 is an N-differential and the sequence*

$$\mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{A}^{N-1} \tag{1.3.10}$$

is the cochain N-complex.

Proof. We begin the proof with a power expansion of d^k, where $1 \leq k \leq N$. Let u be a homogeneous element of an algebra $\mathscr{A}$ whose grading will be denoted by $|u|$. For the first values of $k = 1, 2, 3$ a straightforward computation gives

$$\begin{aligned}
du &= [v, u]_q = v\,u - q^{|u|} u\,v, \\
d^2u &= [v, [v, u]_q]_q = v^2 u - q^{|u|}[2]_q\, v\,u\,v + q^{2|u|+1} u\,v^2, \\
d^3u &= v^3 u - q^{|u|}[3]_q v^2\,u\,v + q^{2|u|+1}[3]_q v\,u\,v^2 - q^{3|u|+3} u\,v^3.
\end{aligned}$$

We state that for any $k \in \{1, 2, \ldots, N\}$ and any homogeneous $u \in \mathscr{A}$ a power expansion of d^k has the form

$$d^k u = \sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i}\, u\, v^i, \tag{1.3.11}$$

where $p_i = q^{i|u|+\sigma(i)}$ and $\sigma(i) = \frac{i(i-1)}{2}$. We proof this statement by means of mathematical induction assuming that the above power expansion (1.3.11) for d^k is true and then showing that it has the same form for $k+1$. Indeed we have

$$\begin{aligned}
d^{k+1}u \; &= d(d^k u) = d\Big(\sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i} u v^i\Big) \\
&= \sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q (v^{k+1-i} u v^i - q^{|u|+k} v^{k-i} u v^{i+1}) \\
&= \sum_{i=0}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k+1-i} u v^i - \sum_{i=0}^{k} (-1)^i q^{|u|+k} p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i} u v^{i+1} \\
&= v^{k+1} u + \sum_{i=1}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k+1-i} u v^i \\
&\quad - \sum_{i=0}^{k-1} (-1)^i q^{|u|+k} p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k-i} u v^{i+1} - (-1)^k q^{|u|+k} p_k\, u v^{k+1} \\
&= v^{k+1} u + \sum_{i=1}^{k} (-1)^i p_i \begin{bmatrix} k \\ i \end{bmatrix}_q v^{k+1-i} u v^i \\
&\quad + \sum_{i=1}^{k} (-1)^i q^{|u|+k} p_{i-1} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q v^{k-i} u v^{i+1} + (-1)^{k+1} q^{|u|+k} p_k\, u v^{k+1}
\end{aligned}$$

$$= \quad v^{k+1}u + \sum_{i=1}^{k}(-1)^i\Big(p_i \left[\begin{array}{c} k \\ i \end{array}\right]_q + q^{|u|+k}p_{i-1}\left[\begin{array}{c} k \\ i-1 \end{array}\right]_q\Big)v^{k+1-i}uv^i$$
$$+(-1)^{k+1}q^{|u|+k}p_k\, uv^{k+1}.$$

Now the coefficients in the last expansion we can write as follows

$$p_i \left[\begin{array}{c} k \\ i \end{array}\right]_q + q^{|u|+k}p_{i-1}\left[\begin{array}{c} k \\ i-1 \end{array}\right]_q = p_i\Big(\left[\begin{array}{c} k \\ i \end{array}\right]_q + q^{k+\sigma(i-1)-\sigma(i)}\left[\begin{array}{c} k \\ i-1 \end{array}\right]_q\Big),$$

and making use of

$$\sigma(i-1) - \sigma(i) = \frac{(i-1)(i-2)}{2} - \frac{i(i-1)}{2} = 1 - i$$

together with the recurrent relation for q-binomial coefficients (1.2.2) we get

$$\left[\begin{array}{c} k \\ i \end{array}\right]_q + q^{k+1-i}\left[\begin{array}{c} k \\ i-1 \end{array}\right]_q = \left[\begin{array}{c} k+1 \\ i \end{array}\right]_q.$$

As $p_{k+1} = q^{|u|+k}p_k$ we finally obtain

$$d^{k+1}u = \sum_{i=0}^{k+1}(-1)^i p_i \left[\begin{array}{c} k+1 \\ i \end{array}\right]_q v^{k-i}uv^i,$$

and this ends the proof of the formula for power expansion of d^k.

Now our aim is to show that the power expansion (1.3.11) implies $d^N u = 0$ for any $u \in \mathscr{A}$. Indeed making use of (1.3.11) we can express the Nth power of d as follows

$$d^N u = \sum_{i=0}^{N}(-1)^i p_i \left[\begin{array}{c} N \\ i \end{array}\right]_q v^{k-i}uv^i. \qquad (1.3.12)$$

Now we take into account that q is a primitive Nth root of unity. In this case we can apply (1.2.3) which gives

$$\left[\begin{array}{c} N \\ i \end{array}\right]_q = 0, \quad i \in \{1, 2, \ldots, N-1\}.$$

Hence the terms in (1.3.12), which are numbered with $i = 1, 2, \ldots, N-1$, vanish, and we are left with two terms

$$d^N u = v^N\, u + (-1)^N q^{\sigma(N)} u\, v^N.$$

As v^N is the element of grading zero (modulo N) of the graded center $\mathscr{Z}(\mathscr{A})$ we can rewrite the above formula as follows

$$d^N u = (1 + (-1)^N q^{\sigma(N)})\, u\, v^N, \quad \sigma(N) = \frac{N(N-1)}{2}.$$

In order to show that the multiplier in the above formula vanish for any $N \geq 2$ we consider separately two cases for N to be an odd or even positive integer. If N is an odd positive integer then the multiplier $1 + (-1)^N q^{\sigma(N)}$ vanish because in this case

$$1 + (-1)^N q^{\sigma(N)} = 1 - (q^N)^{\frac{N-1}{2}} = 0.$$

If N is an even positive integer then

$$1 + (-1)^N q^{\sigma(N)} = 1 + (q^{\frac{N}{2}})^{N-1} = 1 + (-1)^{N-1} = 0.$$

Hence for any $N \geq 2$ we have $d^N = 0$, and this ends the proof of the theorem. □

Theorem 1.3.8. *Let q be a primitive Nth root of unity, $\mathscr{A} = \oplus_{i\in\mathbb{Z}_N}\mathscr{A}^i$ be a graded associative unital algebra with an element $v \in \mathscr{A}^1$ satisfying $v^N = \lambda\, \mathbb{1}$, where $\lambda \neq 0$ and $\mathbb{1}$ is the identity element of an algebra $\mathscr{A}$. Then the generalized cohomologies $H_n(\mathscr{A})$ of the cochain N-complex of Theorem 1.3.7*

$$\mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{A}^{N-1} \qquad (1.3.13)$$

with N-differential $d = \mathrm{ad}^q_v$, induced by an element v, are trivial, i.e. for any $n \in \{1, 2, \ldots, N-1\}$ we have $H_n(\mathscr{A}) = 0$.

Proof. Let us define the endomorphism h of the vector space of $\mathscr{A}$ as follows

$$h(u) = \frac{1}{(1-q)\,\lambda}\, v^{N-1}\, u,$$

where u is an element of an algebra $\mathscr{A}$. If u is a homogeneous element of a graded algebra $\mathscr{A}$ then $|h(u)| = |u| + N - 1$, where $|u|$ is the grading of an element u. For any homogeneous $u \in \mathscr{A}$ we have

$$\begin{aligned}(h\circ d - q\, d\circ h)(u) &= h(du) - q\, d(h(u)) \\ &= h(\mathrm{ad}^q_v(u)) - \frac{q}{(1-q)\,\lambda}\, \mathrm{ad}^q_v(v^{N-1}u)\end{aligned}$$

$$
\begin{aligned}
&= h([v,u]_q) - \frac{q}{(1-q)\,\lambda}\,[v, v^{N-1}]_q \\
&= h\,(v\,u - q^{|u|}uv) - \frac{q}{(1-q)\,\lambda}\,(v^N u - q^{|u|+N-1} v^{N-1} u\,v) \\
&= \frac{1}{(1-q)\,\lambda}\,v^N u - \frac{q^{|u|}}{(1-q)\,\lambda} v^{N-1} u\,v - \\
&\qquad \frac{q}{(1-q)\,\lambda}\,v^N u + \frac{q^{|u|}}{(1-q)\,\lambda} v^{N-1} u\,v \\
&= \frac{(1-q)\,\lambda}{(1-q)\,\lambda}\,u = \mathrm{Id}_{\mathscr{A}}(u).
\end{aligned}
$$

The endomorphism $h : \mathscr{A} \longrightarrow \mathscr{A}$ of the vector space of an algebra $\mathscr{A}$ satisfies $h \circ d - q\,d \circ h = \mathrm{Id}_{\mathscr{A}}$ and applying Proposition 1.2.7 we conclude that the generalized cohomology of the cochain N-complex

$$
\mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{A}^{N-1}
$$

are trivial. □

Let us mention that there is a class of algebras which satisfy the conditions formulated in Theorem 1.3.7, and we can use this kind of algebras to construct cochain N-modules. These algebras are called generalized Clifford algebras.

Definition 1.3.9. A *generalized Clifford algebra* $\mathfrak{C}_p^N$, where p, N are integers satisfying $p \geq 1, N \geq 2$, is an associative unital algebra over the complex numbers $\mathbb{C}$ generated by a set of p canonical generators $\{x_1, x_2, \ldots, x_p\}$ which are subjected to the relations

$$
x_i\,x_j = q^{\mathrm{sg}(j-i)} x_j\,x_i, \quad x_i^N = \mathbb{1}, \quad i,j = 1,2,\ldots,p \tag{1.3.14}
$$

where $\mathrm{sg}(k)$ is the usual sign function, and $\mathbb{1}$ is the identity element of an algebra.

Since Theorem 1.3.7 deals with a graded algebra we can endow a generalized Clifford algebra with an $\mathbb{Z}_N$-gradation. There are several ways to equip a generalized Clifford algebra with $\mathbb{Z}_N$-gradation. One possible way of doing this is to assign the grading zero to the identity element $\mathbb{1}$, the grading one to each generator x_i and to define the grading of any product of generators as the sum of gradings of its factors modulo N. From the definition of generalized

Clifford algebra it follows that each generator x_i, $i \in \{1, 2, \ldots, p\}$ satisfies the condition of Theorem 1.3.7, i.e. $x_i^N = \mathbb{1} \in \mathscr{Z}(\mathfrak{C}_p^N)$, where $\mathscr{Z}(\mathfrak{C}_p^N) \subset \mathfrak{C}_p^N$ is the graded center of a generalized Clifford algebra. Hence we can use each generator to construct an N-differential d, and then a generalized Clifford algebra endowed with an appropriate $\mathbb{Z}_N$-gradation becomes a cochain N-complex. A more general way to construct an N-differential by means of an element v of grading one satisfying $v^N \in \mathscr{Z}(\mathfrak{C}_p^N)$ is to take the linear combination of generators $x_1, x_2, \ldots, x_p$ with complex coefficients $\lambda_1, \lambda_2, \ldots, \lambda_p$, i.e.

$$v = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_p x_p. \tag{1.3.15}$$

In the next chapter we will give a more detailed description of this construction proving that

$$v^N = (\sum_{i=0}^{p} \lambda_i x_i)^N = \prod_{i=0}^{p} \lambda_i \mathbb{1} \in \mathscr{Z}(\mathfrak{C}_p^N).$$

Hence the element v defined in (1.3.15) induces the N-differential, and a generalized Clifford algebra $\mathfrak{C}_p^N$ becomes the cochain N-complex. In the next chapter we will show that this kind of cochain N-complexes can be used to construct a generalized exterior calculus with exterior differential $d^N = 0$ on a reduced quantum plane.

It should be mentioned that a matrix cochain N-complex proposed in [22] is a particular case of the above described cochain N-complex. Indeed let $M_N(\mathbb{C})$ be the algebra of square matrices of order N. The matrices E_l^k, where $k, l \in \{1, 2, \ldots, N\}$ defined by $(E_l^k)_j^i = \delta_j^k \delta_l^i$ form the basis for the algebra $M_N(\mathbb{C})$. The structure constants with respect to this basis are determined by $E_l^k E_s^r = \delta_s^k E_r^l$. The matrix algebra $M_N(\mathbb{C})$ becomes an $\mathbb{Z}_N$-graded algebra if one attributes the grading $k-l$ (modulo N) to the matrix E_l^k. Then $M_N(\mathbb{C}) = \oplus_{i \in \mathbb{Z}_N} M_N^i(\mathbb{C})$, where $M_N^i(\mathbb{C})$, is the subspace of homogeneous elements of grading i. Evidently the matrix $V = \sum_{k=1}^{N-1} \lambda_k E_k^{k+1} + \lambda_N E_N^1$, where λ_k are arbitrary complex numbers, has grading one, i.e. $V \in M_N^1(\mathbb{C})$. It can be verified that $V^N = \lambda_1 \lambda_2 \ldots \lambda_N \mathbb{1}$. Consequently according to Theorem 1.3.7 the inner q-derivation d induced by matrix V is the N-differential and the graded matrix algebra $M_N(\mathbb{C})$ endowed with d is the cochain N-complex.

Chapter 2

Graded differential algebras and exterior calculus

2.1 Graded differential algebras

Throughout this chapter K will be either the field of real or complex numbers. We begin this section with a series of definitions.

Definition 2.1.1. *A graded differential algebra* is a graded associative unital K-algebra $\mathscr{A} = \oplus_{i\in\mathbb{Z}}\mathscr{A}^i$ together with a linear mapping $d : \mathscr{A} \longrightarrow \mathscr{A}$, where

i) $d : \mathscr{A}^i \longrightarrow \mathscr{A}^{i+1}$ for any integer $i \in \mathbb{Z}$, i.e. d is of degree 1,

ii) $d(u\,v) = d(u)\,v + (-1)^{|u|}u\,d(v)$ for any homogeneous $u \in \mathscr{A}$ and any $v \in \mathscr{A}$, i.e. d satisfies the graded Leibniz rule,

iii) $d^2u = 0$ for any $u \in \mathscr{A}$.

A linear mapping $d : \mathscr{A} \longrightarrow \mathscr{A}$ is called a differential of a graded differential algebra $\mathscr{A}$. The properties i) and ii) show that differential d is an antiderivation of a graded algebra $\mathscr{A}$.

Definition 2.1.2. Let $\mathscr{A}, \mathscr{B}$ be two graded differential algebras with differentials correspondingly d, d'. A linear mapping $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *a homomorphism of graded differential algebras* if

i) $\phi : \mathscr{A}^i \longrightarrow \mathscr{B}^i$ for any integer $i \in \mathbb{Z}$,

ii) $\phi(u\,v) = \phi(u)\,\phi(v)$ for any $u, v \in \mathscr{A}$,

iii) $\phi \circ d = d' \circ \phi$.

Definition 2.1.3. A graded differential algebra $\mathscr{A}$ is said to be *commutative graded differential* if it is commutative graded algebra, i.e. the graded commutator of any two homogeneous elements $u, v \in \mathscr{A}$ vanishes $[u, v] = u\,v - (-1)^{|u||v|} = 0$.

Making use of the notions introduced in the previous chapter we can say that a graded differential algebra $\mathscr{A}$ with differential d is a cochain complex whose cohomology is the quotient space $H(\mathscr{A}) = Z(\mathscr{A})/B(\mathscr{A})$, where $Z(\mathscr{A}) = \{u \in \mathscr{A} : du = 0, \text{i.e. } u \in \operatorname{Ker} d\} \subset \mathscr{A}$ and $B(\mathscr{A}) = \{u \in \mathscr{A} : \exists v \in \mathscr{A}, u = dv, \text{i.e. } u \in \operatorname{Im} d\} \subset Z(\mathscr{A})$. The cohomology of a graded differential algebra inherits the graded structure of $\mathscr{A}$, i.e. $H(\mathscr{A}) = \oplus_{i\in\mathbb{Z}} H^i(\mathscr{A})$, where $H^i(\mathscr{A}) = Z_i(\mathscr{A})/B_i(\mathscr{A})$. Clearly $Z(\mathscr{A}) \subset \mathscr{A}$ is the graded subalgebra of $\mathscr{A}$, and $B(\mathscr{A}) \subset Z(\mathscr{A})$ is the graded bilateral ideal in $Z(\mathscr{A})$. Hence the cohomology $H(\mathscr{A})$ is a graded algebra. If $\mathscr{A}$ is commutative graded differential algebra then its cohomology $H(\mathscr{A})$ is also commutative graded algebra. A homomorphism $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ of graded differential algebras $\mathscr{A}, \mathscr{B}$ induces the homomorphism of their cohomologies which will be denoted by $\phi_* : H(\mathscr{A}) \longrightarrow H(\mathscr{B})$.

Graded differential algebras play an important role in modern differential geometry. If M is a smooth finite dimensional manifold then *the de Rham algebra of differential forms* $\Omega(M) = \oplus_p \Omega^p(M)$ together with exterior differential d is a commutative graded differential algebra. If M, N are two smooth finite dimensional manifolds then a smooth mapping $\phi : M \longrightarrow N$ induces with the help of pull-back of differential forms the homomorphism of graded differential algebras $\phi^* : \Omega(N) \longrightarrow \Omega(M)$. The cohomologies of the commutative graded differential algebra $\Omega(M)$ is called *the de Rham cohomologies* of a manifold M, and they play an important role in differential topology of manifolds.

The next important example of a graded differential algebra is based on the Chevalley-Eilenberg cochain complex of V-valued cochains on a Lie algebra $\mathfrak{g}$, where V is a vector space of representation of $\mathfrak{g}$. The description of this cochain complex is given at the end of the Section 1.1 of the previous chapter. Let G be a Lie group, $\mathfrak{g}$ be a Lie algebra of G, $C^n(\mathfrak{g})$ be the vector space of K-valued n-cochains, i.e. if $\omega \in C^n(\mathfrak{g})$ then ω is a skew-symmetric linear mapping $\omega : \mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}$ (n times) $\longrightarrow K$. Obviously the vector space of

K-valued n-cochains $C^n(\mathfrak{g})$ can be identified with $\wedge^n\mathfrak{g}^*$. In turn the vector space $\wedge^n\mathfrak{g}^*$ can be identified with the vector space $\Omega^n_{\mathrm{inv}}(G)$ of left-invariant differential n-forms on a Lie group G. Then the exterior differential d induces the differential on the vector space $C(\mathfrak{g}) = \oplus_n C^n(\mathfrak{g})$ and we get the cochain complex

$$C^0(\mathfrak{g}) \xrightarrow{d} C^1(\mathfrak{g}) \xrightarrow{d} \ldots \xrightarrow{d} C^n(\mathfrak{g}) \xrightarrow{d} C^{n+1}(\mathfrak{g}) \xrightarrow{d} \ldots, \tag{2.1.1}$$

Obviously $C(\mathfrak{g})$ equipped with the differential d is the graded differential algebra whose multiplication is induced by the exterior multiplication of left-invariant differential forms on a Lie group G. It is worth mentioning that in the case of a compact Lie group G the cohomology of $C(\mathfrak{g})$ is isomorphic to the de Rham cohomology of G.

If $\mathscr{A}, \mathscr{B}$ are graded differential algebras then we can consider the tensor product of cochain complexes $\mathscr{A} \otimes \mathscr{B}$ which is also the cochain complex. This cochain complex becomes a graded differential algebra if one endows it with multiplication

$$(u \otimes v)\,(u' \otimes v') = (-1)^{|v||u'|}(u\,u') \otimes (v\,v'). \tag{2.1.2}$$

The cochain complex $\mathscr{A} \otimes \mathscr{B}$ endowed with the multiplication (2.1.2) will be referred to as *the tensor product of graded differential algebras.*

Theorem 2.1.4. *Let $\mathscr{A} = \oplus_{i\in\mathbb{Z}}\mathscr{A}^n$ be a positive graded algebra which is also a cosimplicial module whose coface and degeneracy homomorphisms are denoted respectively by $f_0, f_1, \ldots, f_n, \ldots$ and $s_0, s_1, \ldots, s_n, \ldots$. If for any homogeneous elements $u, v \in \mathscr{A}$ and for any integer $i \in \{0, 1, \ldots, |u| + |v| + 1\}$ the multiplication $(u, v) \longrightarrow uv$ in $\mathscr{A}$ and coface homomorphisms are related by*

$$f_i(uv) = \begin{cases} f_i(u)\,v, & i \leq |u| \\ u\,f_{i-|u|}(v), & i > |u| \end{cases} \tag{2.1.3}$$

and

$$f_{|u|+1}(u)\,v = u\,f_0(v), \tag{2.1.4}$$

then a graded algebra $\mathscr{A}$ endowed with the simplicial differential d is a graded differential algebra. If in addition to relations (2.1.3),(2.1.4) for any $i \in \{0, 1, \ldots, |u| + |v| - 1\}$ the codegeneracy homomorphisms satisfy

$$s_i(uv) = \begin{cases} s_i(u)\,v, & i < |u| \\ u\,s_{i-|u|}(v), & i \geq |u| \end{cases} \tag{2.1.5}$$

then the subcomplex of normalized cochains $N(\mathscr{A}) \subset \mathscr{A}$ is the subalgebra of a graded differential algebra $\mathscr{A}$.

An important example of a graded differential algebra is based on Proposition 2.1.4. Let $\mathscr{A}$ be an associative unital K-algebra with the unit $\mathbb{1}$. Let us denote the tensor algebra of $\mathscr{A}$ by $T(\mathscr{A}) = \oplus_{i\in\mathbb{N}} T^i(\mathscr{A})$, where $T^i(\mathscr{A}) = \otimes^{i+1}\mathscr{A}$. We remind a reader that the multiplication $\mathscr{A}^i \times \mathscr{A}^j \longrightarrow A^{i+j}$ in $T(\mathscr{A})$ is given by

$$(u_1 \otimes u_2 \otimes \ldots \otimes u_i)(v_1 \otimes v_2 \otimes \ldots \otimes v_j) = u_1 \otimes u_2 \otimes \ldots \otimes u_i \otimes v_1 \otimes v_2 \otimes \ldots \otimes v_j,$$

and $T(\mathscr{A})$ is the associative unital algebra. For any integer $n \geq 0$ and $i \in \{1, 2 \ldots, n\}$ we define the homomorphisms of vector spaces $f_i : \mathscr{A}^n \longrightarrow \mathscr{A}^{n+1}$ and $s_{i-} : \mathscr{A}^n \longrightarrow \mathscr{A}^{n-1}$ as follows

$$\begin{aligned}
f_0(u_0 \otimes u_1 \otimes \ldots \otimes u_n) &= \mathbb{1} \otimes u_0 \otimes u_1 \otimes \ldots \otimes u_n, \\
f_i(u_0 \otimes u_1 \otimes \ldots \otimes u_n) &= u_0 \otimes u_1 \otimes \ldots \otimes u_{i-1} \otimes \mathbb{1} \otimes u_i \otimes \ldots \otimes u_n, \\
f_{n+1}(u_0 \otimes u_1 \otimes \ldots \otimes u_n) &= u_0 \otimes u_1 \otimes \ldots \otimes u_n \otimes \mathbb{1}, \\
s_{i-1}(u_0 \otimes u_1 \otimes \ldots \otimes u_n) &= u_0 \otimes u_1 \otimes \ldots \otimes u_{i-1} u_i \otimes \ldots \otimes u_n.
\end{aligned}$$

It can be checked that the homomorphisms f_i, s_j satisfy respectively the coface and degeneracy homomorphisms conditions together with the conditions (2.1.3),(2.1.4),(2.1.5) of the Theorem 2.1.4. Thus the tensor algebra $T(\mathscr{A})$ equipped with the homomorphisms f_i, s_j is the cosimplicial K-vector space and it is the graded differential algebra if we endow it with the simplicial differential d.

Definition 2.1.5. The graded differential subalgebra $N(T(\mathscr{A})) \subset T(\mathscr{A})$ of normalized cochains is called *the universal graded differential envelope* of a graded algebra $\mathscr{A}$ and is denoted by $\Omega(\mathscr{A})$.

Definition 2.1.6. Let $\mathscr{A}$ be an associative unital algebra and $\mathscr{M}$ be an $\mathscr{A}$-bimodule. A linear mapping $d : \mathscr{A} \longrightarrow \mathscr{M}$ which satisfies the Leibniz rule

$$d(u\,v) = d(u)\,v + u\,d(v), \quad u, v \in \mathscr{A}$$

is called *an $\mathscr{M}$-valued first order differential calculus* or simply *$\mathscr{M}$-valued differential calculus* over an algebra $\mathscr{A}$. A mapping d is referred to as differential of differential calculus. If $\mathscr{A}$ is freely generated by a set of generators

$\{x^i\}_{i\in I}$, $\mathscr{M}$ is a free right (or left) $\mathscr{A}$-module with the basis $\{e^i\}_{i\in I}$, and $\mathscr{A} \xrightarrow{d} \mathscr{M}$ is an $\mathscr{M}$-valued differential calculus, where differential d satisfies the condition $e^i = dx^i$, then a differential calculus is called *a coordinate differential calculus* over $\mathscr{A}$ with values in $\mathscr{M}$.

Let us mention that if $\mathscr{A}$ is an algebra of functions and $\mathscr{A} \xrightarrow{d} \mathscr{M}$ is a coordinate differential calculus over $\mathscr{A}$ then each x^i, where $i \in I$, can be viewed as a coordinate function. If $\mathscr{A}$ is a graded differential algebra with differential d and the subalgebra of elements of grading zero $\mathscr{A}^0$ denoted by $\mathfrak{A}$ then as it was mentioned before $\mathscr{A}^1$ is the $\mathfrak{A}$-bimodule and it is easy to see that $\mathfrak{A} \xrightarrow{d} \mathscr{A}^1$ is the $\mathscr{A}^1$-valued differential calculus over $\mathfrak{A}$.

Definition 2.1.7. Let $\mathfrak{A}$ be an associative unital algebra. If there exists a graded differential algebra $\mathscr{A}$ such that $\mathscr{A}^0$ is isomorphic to $\mathfrak{A}$ then $\mathscr{A}$ is called *an exterior calculus* over an algebra $\mathfrak{A}$. In this case the elements of $\mathscr{A}^i$ are called differentials forms of degree i of exterior calculus.

As an example of an exterior calculus over an algebra we can consider the universal graded differential envelope $\Omega(\mathscr{A})$ of an associative unital algebra $\mathscr{A}$. This exterior calculus is usually referred to as *the universal exterior calculus over an algebra* $\mathscr{A}$ or *the algebra of universal differential forms* on $\mathscr{A}$ [17]. It is worth mentioning that the universal exterior calculus over an algebra $\mathscr{A}$ is the biggest exterior calculus whose subalgebra of elements of degree zero is an algebra $\mathscr{A}$, and any other exterior calculus over an algebra is a quotient algebra by some (graded) differential ideal. For the practical purpose it is usually not convenient to use the algebra of universal differential forms and there are few approaches of how to construct a "smaller" exterior calculus [10, 30, 32, 39, 43, 48]. In this section we will give a brief description of one of these approaches [48] which yields the Wess-Zumino algebra Ω_{WZ} of differential forms on a reduced quantum plane.

Let $\mathscr{A} \xrightarrow{d} \mathscr{M}$ is a differential calculus over an algebra $\mathscr{A}$ such that $\mathscr{M}$ is a finite freely generated right $\mathscr{A}$-module with a basis $\{e_i\}_{i=1}^n$. Then the $\mathscr{A}$-bimodule structure on $\mathscr{M}$ is uniquely determined by the commutation relation

$$v\, e^i = e^j\, R(u)^i_j, \tag{2.1.6}$$

where $R : \mathscr{A} \longrightarrow \mathrm{Mat}_n(\mathscr{A})$ is a homomorphism from an algebra $\mathscr{A}$ to the algebra of $\mathscr{A}$-valued square matrices of order n. Here we assume that $R(u) =$

$(R(u)_i^j) \in \mathrm{Mat}_n(\mathscr{A})$ is a square matrix of order n whose entry $R(u)_i^j$ is at the intersection of its jth column and ith row. Thus for any $u, v \in \mathscr{A}$ we have

$$R(uv)_i^j = R(u)_k^j R(v)_i^k \Leftrightarrow R(uv) = R(u)R(v),$$

where $R(u)R(v)$ is the product of two matrices.

Definition 2.1.8. Let $\mathscr{A} \xrightarrow{d} \mathscr{M}$ be a differential calculus such that $\mathscr{M}$ is a finite freely generated right $\mathscr{A}$-module with a basis $\{e_i\}_{i=1}^n$. The mappings $\partial_k : \mathscr{A} \longrightarrow \mathscr{A}$, where $k \in \{1, 2, \ldots, n\}$, uniquely defined by

$$dv = e^k \partial_k(v), \quad v \in \mathscr{A} \tag{2.1.7}$$

are called *the right partial derivatives* of differential calculus $\mathscr{A} \xrightarrow{d} \mathscr{M}$.

Proposition 2.1.9. *If $\mathscr{A} \xrightarrow{d} \mathscr{M}$ is a differential calculus over an algebra $\mathscr{A}$ such that $\mathscr{M}$ is a finite freely generated right $\mathscr{A}$-module with a basis $\{e_i\}_{i=1}^n$ then the right partial derivatives $\partial_k : \mathscr{A} \longrightarrow \mathscr{A}$ of this differential calculus satisfy*

$$\partial_k(uv) = \partial_k(u)\, v + R(u)_k^i \partial_i(v). \tag{2.1.8}$$

The property (2.1.8) is called *the twisted (with homomorphism R) Leibniz rule* for partial derivatives.

Let x, y be two variables which obey the commutation relation

$$x\, y = q\, y\, x, \tag{2.1.9}$$

where $q \neq 0, 1$ is a complex number. These two variables generate the algebra of polynomials over the complex numbers which we denote by $\mathfrak{P}_q[x, y]$. This algebra is an associative algebra of polynomials over $\mathbb{C}$ and the identity element of this algebra will be denoted by $\mathbb{1}$. In noncommutative geometry and theoretical physics a polynomial $P \in \mathfrak{P}_q[x, y]$ is interpreted as a function on *a quantum plane* with two noncommuting coordinate functions x, y and the algebra of polynomials $\mathfrak{P}_q[x, y]$ is interpreted as the algebra of (polynomial) functions on a quantum plane [16, 40, 41]. If we fix an integer $N \geq 2$ and impose the additional condition

$$x^N = y^N = \mathbb{1}, \tag{2.1.10}$$

then a quantum plane is called *a reduced quantum plane*. Hence an algebra of functions on a reduced quantum plane is the algebra of polynomials generated by two variables x, y which obey the commutation relation (2.1.9) and the relation (2.1.10). Let us mention that from an algebraic point of view an algebra of functions on a reduced quantum plane may be identified with the generalized Clifford algebra $\mathfrak{C}_2^N$ with two generators x, y (see Definition 1.3.9). In order not to make the notations very complicated we will denote the algebra of functions on a reduced quantum plane by the same symbol $\mathfrak{C}_2^N$. It is well known that the generalized Clifford algebras have matrix representations [45], and in the particular case of the algebra $\mathfrak{C}_2^N$ the generators of this algebra x, y can be identified with the square matrices of order N

$$x = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & q^{-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & q^{-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q^{-(N-2)} & 0 \\ 0 & 0 & 0 & \dots & 0 & q^{-(N-1)} \end{pmatrix}, \tag{2.1.11}$$

$$y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{2.1.12}$$

where q is a primitive Nth root of unity. As the matrices (2.1.11),(2.1.12) generate the algebra $\mathrm{Mat}_N(\mathbb{C})$ of square matrices of order N we can identify the algebra of functions on a reduced quantum plane with the algebra of matrices $\mathrm{Mat}_N(\mathbb{C})$.

It is obvious that if we wish to construct differential-geometric structures on a reduced quantum plane then we should have an analog of a group acting on a reduced quantum plane. This analog of a group is called *a reduced quantum unimodular group* at a primitive Nth root of unity and it is constructed as follows: let $\alpha, \beta, \gamma, \delta$ be noncommuting symbols such that the left and right coaction of the second order matrix, whose entries are $\alpha, \beta, \gamma, \delta$, on a reduced quantum plane defined by

$$\Delta_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{2.1.13}$$

$$\Delta_R \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x' & y' \end{pmatrix}. \tag{2.1.14}$$

will preserve the basic relations (2.1.9),(2.1.10), i.e. left and right coaction satisfy the relations

$$\Delta_{L,R}(xy - q\,yx) = 0, \tag{2.1.15}$$

$$\Delta_{L,R}(x^N - \mathbb{1}) = 0, \tag{2.1.16}$$

$$\Delta_{L,R}(y^N - \mathbb{1}) = 0. \tag{2.1.17}$$

Evidently the matrix relations (2.1.13),(2.1.14) are equivalent to

$$\begin{aligned} \Delta_L(x) &= \alpha \otimes x + \beta \otimes y, \\ \Delta_L(y) &= \gamma \otimes x + \delta \otimes y, \\ \Delta_R(x) &= x \otimes \alpha + y \otimes \beta, \\ \Delta_R(y) &= x \otimes \gamma + y \otimes \delta, \end{aligned}$$

and we extend ∇_L, ∇_R as homomorphisms $\Delta_L : \mathfrak{C}_2^N \longrightarrow \mathfrak{F} \otimes \mathfrak{C}_2^N$ and $\Delta_R : \mathfrak{C}_2^N \longrightarrow \mathfrak{C}_2^N \otimes \mathfrak{F}$, i.e. from the algebra of functions on a reduced quantum plane $\mathfrak{C}_2^N$ to the algebra $\mathfrak{F} \otimes \mathfrak{C}_2^N$ (left coaction) or to the algebra $\mathfrak{C}_2^N \otimes \mathfrak{F}$ (right coaction), where $\mathfrak{F}$ is the algebra generated by the symbols $\alpha, \beta, \gamma, \delta$ which satisfy certain relations. These relations can be found by means of (2.1.15), (2.1.16), (2.1.17), and a straightforward computation gives

$$\alpha\beta = q\,\beta\alpha \qquad \beta\delta = q\,\delta\beta, \tag{2.1.18}$$

$$\alpha\gamma = q\,\gamma\alpha, \qquad \gamma\delta = q\,\delta\gamma, \tag{2.1.19}$$

$$\beta\gamma = \gamma\beta, \qquad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma. \tag{2.1.20}$$

The element $\mathfrak{d} = \alpha\delta - q\,\beta\gamma \in \mathscr{Z}(\mathfrak{F})$ belongs to the center of the algebra $\mathfrak{F}$ and it is called *a q-determinant* of the second order matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Since the aim is to construct an analog of unimodular group one imposes the additional relation $\mathfrak{d} = \mathbb{1}$. The generator δ can be expressed in terms of other generators as follows

$$\delta = \alpha^{N-1}(\mathbb{1} + q\,\beta\gamma),$$

which implies that the algebra $\mathfrak{F}$ is spanned by the monomials $\alpha^i\beta^j\gamma^k$, where $i, j, k \in \{0, 1, \ldots, N-1\}$. Hence the dimension of the vector space of the algebra $\mathfrak{F}$ is 3^N.

It is very important that the algebra $\mathfrak{F}$ is a Hopf algebra. Let us remind a reader the definition of Hopf algebra.

Definition 2.1.10. An associative unital algebra H over a field K with multiplication $m : H \otimes H \longrightarrow H$ is said to be *a Hopf algebra* if it is endowed with *comultiplication* $\Delta : H \longrightarrow H \otimes H$, *counit* $\epsilon : H \longrightarrow K$ and *antipode* $S : H \longrightarrow H$ that satisfy

1. $(\mathrm{id}_H \otimes \Delta)\,\Delta = (\Delta \otimes \mathrm{id}_H)\,\Delta : H \longrightarrow H \otimes H \otimes H$ (coassociativity),
2. $m(\mathrm{id}_H \otimes \epsilon)\Delta = m(\epsilon \otimes \mathrm{id}_H)\Delta = \mathrm{id}_H$ (counitarity),
3. $m(\mathrm{id}_H \otimes S)\Delta = m(S \otimes \mathrm{id}_H)\Delta = \iota\epsilon$,

where $\iota : K \longrightarrow H$ is the unit map for H.

The algebra $\mathfrak{F}$ becomes the Hopf algebra if one defines the comultiplication

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, \\ \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, \\ \Delta(\delta) &= \gamma \otimes \delta + \delta \otimes \delta, \end{aligned}$$

the antipode

$$S(\alpha) = \delta, \;\; S(\beta) = -q^{-1}\beta, \;\; S(\gamma) = -q\,\gamma, \;\; S(\delta) = \alpha,$$

and the counit

$$\epsilon(\alpha) = 1, \;\; \epsilon(\beta) = 0, \;\; \epsilon(\gamma) = 0, \;\; \epsilon(\delta) = \alpha.$$

The algebra of differential forms on a reduced quantum plane Ω_{WZ} with the differential d is called *the reduced Wess-Zumino algebra* [17, 48]. The reduced Wess-Zumino algebra is the $\mathbb{Z}_3$-graded differential algebra

$$\Omega_{WZ} = \Omega^0_{WZ} \oplus \Omega^1_{WZ} \oplus \Omega^2_{WZ},$$

where the subalgebra of forms of degree zero Ω^0_{WZ} is the algebra of functions on a reduced quantum plane $\mathfrak{C}^N_2$ (algebra of polynomials generated by the coordinate functions x, y which are subjected to the relations (2.1.9), (2.1.10)), the bimodule of differential forms of degree one Ω^1_{WZ} is the bimodule over the algebra of functions $\mathfrak{C}^N_2$ freely generated by the differentials of coordinate functions dx, dy, and the bimodule of differentials forms of degree two Ω^2_{WZ} is the bimodule over $\mathfrak{C}^N_2$ freely generated by $dx\,dy$. In order to be more exact we can describe the structure of the reduced Wess-Zumino algebra in a more explicit way. As it was mentioned before any function $f(x,y) \in \mathfrak{C}^N_2 = \Omega^0_{WZ}$ on a reduced quantum plane is a polynomial

$$f(x,y) = \sum_{\mu,\nu} f_{\mu\nu} x^\mu y^\nu.$$

The bimodule structure of Ω^1_{WZ} is determined by the relations

$$x\,dx = q^2\,dx\,x, \quad x\,dy = q\,dy\,x + (q^2-1)\,dx\,y, \tag{2.1.21}$$

$$y\,dx = q\,dx\,y, \quad y\,dy = q\,dy\,y. \tag{2.1.22}$$

The differential $d : \Omega^0_{WZ} \longrightarrow \Omega^1_{WZ}$ is defined by $d(x) = dx, d(y) = dy$ and it satisfies the Leibniz rule. Hence $\Omega^0_{WZ} \xrightarrow{d} \Omega^1_{WZ}$ is the first order coordinate differential calculus over the algebra Ω^0_{WZ} (see Definition 2.1.6). From the bimodule structure of Ω^1_{WZ} determined by the relations (2.1.21), (2.1.22) it follows that the homomorphism $R : \Omega^0_{WZ} \longrightarrow \mathrm{Mat}_2(\Omega^0_{WZ})$ induced by the bimodule structure has the form

$$R(x) = \begin{pmatrix} q^2\,x & (q^2-1)\,y \\ 0 & q\,x \end{pmatrix}, R(y) = \begin{pmatrix} q\,y & 0 \\ 0 & q^2\,y \end{pmatrix}. \tag{2.1.23}$$

It was explained earlier in this section that a coordinate differential calculus over an algebra induces the right partial derivatives which satisfy the twisted Leibniz rule. In the case of a reduced quantum plane we have the first order coordinate differential calculus $\Omega^0_{WZ} \xrightarrow{d} \Omega^1_{WZ}$ and this calculus induces the right partial derivatives as follows

$$d(f) = dx\,\frac{\partial f}{\partial x} + dy\,\frac{\partial f}{\partial y},$$

where f is a function on a reduced quantum plane. Taking into account that these right partial derivatives satisfy the twisted Leibniz rule which is twisted with the help of matrices (2.1.23) we can easily calculate

$$\frac{\partial}{\partial x}\,x = 1 + q^2 x\frac{\partial}{\partial x} + (q^2-1)\,y\frac{\partial}{\partial y}, \tag{2.1.24}$$

$$\frac{\partial}{\partial x}\,y = q\,y\frac{\partial}{\partial y}, \tag{2.1.25}$$

$$\frac{\partial}{\partial y}\,x = q\,x\,\frac{\partial}{\partial y}, \tag{2.1.26}$$

$$\frac{\partial}{\partial y}\,y = 1 + q^2\,y\,\frac{\partial}{\partial y}. \tag{2.1.27}$$

The structure of graded differential algebra of Ω_{WZ} is determined by the relations

$$(dx)^2 = (dy)^2 = 0, \quad dx\,dy + q^2\,dy\,dx = 0, \quad d^2 = 0. \tag{2.1.28}$$

From this it follows immediately that any 2-form on a reduced quantum plane, i.e. an element of Ω^2_{WZ}, can be written in the form

$$\sum_{\mu,\nu} f_{\mu\nu}(x,y)x^\mu y^\nu\,dx\,dy. \tag{2.1.29}$$

2.2 Graded q-differential algebras

In this section we will describe a natural generalization of the notion of graded differential algebra which was introduced in [20] and studied in the series of paper[2, 21, 22, 29]. The key idea of this generalization is the same as in the case of a differential module and a cochain complex (described in the previous chapter), where the basic property $d^2 = 0$ of a differential d is given a more general form $d^N = 0$, $N \geq 2$. However in the case of a graded differential algebra we also have, in addition to a structure of cochain complex, a structure of graded associative unital algebra, and consequently a more general property $d^N = 0$ of differential should be consistent with this structure which means that the graded Leibniz rule should be represented in a more general form. It is clear that if our aim is to define a graded algebra which is more general than a graded differential algebra by means of replacing the basic property of differential $d^2 = 0$ by a more general one $d^N = 0,\ N \geq 2$ then we should also replace the graded Leibniz rule by a more general graded q-Leibniz rule. In this section we also use the cochain N-complex described in the Theorem 1.3.7 to construct a graded q-differential algebra. Then we will describe a graded q-differential polynomial algebra which arises in relation with a connection form. In the next chapter this algebra will be used for calculation the curvature of connection form. Finally making use of a generalized Clifford algebra we will construct an algebra of differential forms on a reduced quantum plane.

In this section K is the field of complex numbers $\mathbb{C}$ and q is a primitive Nth root of unity, where $N \geq 2$.

Definition 2.2.1. *A graded q-differential algebra* is a graded associative unital algebra $\mathscr{A} = \oplus_{k\in\mathbb{Z}}\mathscr{A}^k$ endowed with a linear mapping d of degree one such that the sequence

$$\ldots \xrightarrow{d} \mathscr{A}^{k-1} \xrightarrow{d} \mathscr{A}^{k} \xrightarrow{d} \mathscr{A}^{k+1} \xrightarrow{d} \ldots$$

is an N-complex with N-differential d satisfying the graded q-Leibniz rule

$$d(u\,v) = d(u)\,v + q^k u\,d(v), \tag{2.2.1}$$

where $u \in \mathscr{A}^k, v \in \mathscr{A}$.

In the previous chapter it is shown (1.3.6) that a homogeneous element of grading one $v \in \mathscr{A}^1$ of a graded algebra $\mathscr{A}$ induces the inner graded q-derivation by means of a graded q-commutator as follows

$$v \mapsto \mathrm{ad}_v^q(u) = [v,u]_q = vu - q^k uv, \tag{2.2.2}$$

where $u \in \mathscr{A}^k$.

Taking into account the following facts:

- an inner graded q-derivation satisfies the graded q-Leibniz rule (1.3.8),

- the inner graded q-derivation $d = \mathrm{ad}_v^q$, induced by an element of grading one $v \in \mathscr{A}^1$ such that $v^N \in \mathscr{Z}(\mathscr{A})$, is the N-differential of the cochain complex (Theorem 1.3.7)

$$\ldots \xrightarrow{d} \mathscr{A}^{k-1} \xrightarrow{d} \mathscr{A}^k \xrightarrow{d} \mathscr{A}^{k+1} \xrightarrow{d} \ldots$$

we conclude

Corollary 2.2.2. *Let $\mathscr{A}$ be a graded associative unital algebra $\mathscr{A} = \oplus_k \mathscr{A}^k$, and q be a primitive Nth root of unity. If there exists an element of grading one $v \in \mathscr{A}^1$ which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{A})$, where $\mathscr{Z}(\mathscr{A})$ is the graded center of $\mathscr{A}$, then a graded algebra $\mathscr{A}$ endowed with the inner graded q-derivation $d = \mathrm{ad}_v^q$ is the graded q-differential algebra and d is its N-differential.*

Later in this section we will use the Corollary 2.2.2 in order to construct an algebra of differential forms on a reduced quantum plane with N-differential d. It is worth noting that the Corollary 2.2.2 can be generalized within the framework of the theory of (σ, τ)-derivations developed by S. Silvestrov and his collaborators, and this generalization is proposed in [38]. We will briefly describe the main result proved in [38]. Let K be a commutative associative ring with unity, and $\mathscr{A}$ be an associative unital K-algebra. Let $\mathscr{G} \subset \mathscr{A}$ be a subset of $\mathscr{A}$ which generates the K-algebra $K[\mathscr{G}]$. Let $\sigma : K[\mathscr{G}] \longrightarrow K[\mathscr{G}]$ be an K-algebra endomorphism defined by $\sigma(u) = \phi(v, u)\, u$ for a fixed $v \in K[\mathscr{G}]$ and any $u \in K[\mathscr{G}]$, where ϕ is a mapping $\phi : \{v\} \times K[\mathscr{G}] \longrightarrow \mathscr{Z}(K[\mathscr{G}])$. Here $\mathscr{Z}(K[\mathscr{G}])$ is the center of $K[\mathscr{G}]$. Let us define $\Delta(u) = v\, u - \sigma(u)\, v$, where $u \in K[\mathscr{G}]$. This is an σ-derivation on $K[\mathscr{G}]$ with values in $\mathscr{A}$. If $K[\mathscr{G}]$ is a two-sided ideal in $\mathscr{A}$, then $\Delta(K[\mathscr{G}]) \subset K[\mathscr{G}]$, and hence Δ becomes a σ-derivation of $K[\mathscr{G}]$.

Theorem 2.2.3. ([38]) *Let $v \in K[\mathscr{G}]$ be such that $v^N \in \mathscr{Z}(K[\mathscr{G}])$ for some $N \geq 2$. Suppose that $\phi(v, v)$ is a primitive Nth root of unity and that $(\phi(v, u))^N = \mathbb{1}$ for any $u \in K[\mathscr{G}]$. Then $\Delta^N(u) = 0$ for any $u \in K[\mathscr{G}]$.*

Let $\mathscr{A}$ be a graded q-differential algebra with N-differential d and the subspace of elements of grading zero denoted by $\mathfrak{A}$. As it was mentioned before the subspace $\mathfrak{A} \subset \mathscr{A}$ of elements of grading zero is the subalgebra of an algebra $\mathscr{A}$, and each subspace $\mathscr{A}^i \subset \mathscr{A}$ of elements of grading i can be considered as the $\mathfrak{A}$-bimodule. Hence we have the following sequence

$$\ldots \xrightarrow{d} \mathscr{A}^{i-1} \xrightarrow{d} \mathscr{A}^{i} \xrightarrow{d} \mathscr{A}^{i+1} \xrightarrow{d} \ldots \tag{2.2.3}$$

of the $\mathfrak{A}$-bimodules. It is important that the part $\mathfrak{A} \xrightarrow{d} \mathscr{A}^1$ of this sequence is the differential calculus over the algebra $\mathfrak{A}$ because in this case d satisfies the Leibniz rule.

The next graded q-differential algebra arises in relation with an algebraic model of a connection form and this algebraic model is based on exterior calculus with differential satisfying $d^N = 0$. A graded q-differential algebra of polynomials related to connection form was introduced and studied in [9]. In the next chapter we will show in what way this algebra can be used in order to calculate the curvature of a connection form.

Let $\mathbb{N}_1 = \{i \in \mathbb{Z} : i \geq 1\}$ be the set of integers greater than or equal to one and $\{\mathfrak{d}, a_i\}_{i\in\mathbb{N}_1}$ be the set of variables. We consider the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ over $\mathbb{C}$ generated by the set of variables $\{\mathfrak{d}, a_i\}_{i\in\mathbb{N}_1}$ which are subjected to the commutation relations

$$\mathfrak{d}a = q^i\, a_i\mathfrak{d} + a_{i+1}, \qquad \forall i \in \mathbb{N}_1 \tag{2.2.4}$$

where q is any complex number different from zero. We denote the identity element of this algebra by $\mathbb{1}$. Obviously we can split up the set of variables of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ into two subsets $\{\mathfrak{d}\}$, $\{a_i\}_{i\in\mathbb{N}_1}$ which generate respectively the subalgebras $\mathfrak{P}_q[\mathfrak{d}] \subset \mathfrak{P}_q[\mathfrak{d}, a]$ and $\mathfrak{P}_q[a] \subset \mathfrak{P}_q[\mathfrak{d}, a]$. Hence the subalgebra $\mathfrak{P}_q[\mathfrak{d}]$ is generated by a single variable $\mathfrak{d}$, and the subalgebra $\mathfrak{P}_q[a]$ is freely generated by the set of variables $\{a_i\}_{i\in\mathbb{N}_1}$ because we do not assume any relation between variables a_i.

Now our aim is to equip the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ with a graded structure so that $\mathfrak{P}_q[\mathfrak{d}, a]$ will become a graded algebra. This can be done as follows: we assign grading zero to the identity element $\mathbb{1}$ of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$, grading one to the generator $\mathfrak{d}$ and grading i to a generator a_i, where $i \in \mathbb{N}_1$. Thus making use of previously defined notations we can describe the graded structure of generators of $\mathfrak{P}_q[\mathfrak{d}, a]$ by the formulae

$$|\mathbb{1}| = 0, \quad |\mathfrak{d}| = |a_1| = 1, \quad |a_i| = i, \quad i \geq 2. \tag{2.2.5}$$

As usual we extend this graded structure to the whole algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ by defining the grading of any product of variables $\{\mathfrak{d}, a_i\}_{i\in\mathbb{N}_1}$ as the sum of gradings of its factors. It is easy to see that the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ becomes the positively graded algebra. Hence we can write

$$\mathfrak{P}_q[\mathfrak{d}, a] = \oplus_{k\in\mathbb{N}} \mathfrak{P}_q^k[\mathfrak{d}, a],$$

where $\mathfrak{P}_q^k[\mathfrak{d}, a]$ is the subspace of homogeneous polynomials of grading k. It should be mentioned that the graded structure of $\mathfrak{P}[\mathfrak{d}, a]$ induces the graded structures of the subalgebras $\mathfrak{P}_q[\mathfrak{d}], \mathfrak{P}_q[a]$ which are positively graded algebras as well. Clearly the positively graded algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ becomes the $\mathbb{Z}_N$-graded algebra, where N any integer greater than 1, if we slightly modify the above described gradation by taking all gradings modulo N. Let us denote by $\mathrm{Lin}\,\mathfrak{P}_q[a]$ the algebra of $\mathbb{C}$-endomorphisms of the vector space of the algebra $\mathfrak{P}_q[a]$. Obviously $\mathrm{Lin}\,\mathfrak{P}_q[a]$ is the graded algebra with gradation induced by the gradation of $\mathfrak{P}_q[a]$. Having defined the positively graded structure of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ we can apply the notions of graded commutator and inner graded q-derivation described in the previous chapter to study the structure of $\mathfrak{P}_q[\mathfrak{d}, a]$. First of all we observe that the commutation relations (2.2.4) can be written by means of graded commutator and inner graded q-derivation in the form

$$[\mathfrak{d}, a_i]_q = a_{i+1}, \qquad \text{or} \qquad \mathrm{ad}_{\mathfrak{d}}^q(a_i) = a_{i+1}, \tag{2.2.6}$$

where $i \in \mathbb{N}_1$. This form of commutation relations suggests us to introduce the inner graded q-derivation $\mathrm{ad}_{\mathfrak{d}}^q$ associated with a variable $\mathfrak{d}$, which we will denote by d and consider it as the inner graded q-derivation of the algebra $\mathfrak{P}_q[a]$ (generated by the set of variables $\{a_i\}_{i\in\mathbb{N}_1}$), i.e.

$$d := \mathrm{ad}_{\mathfrak{d}}^q, \qquad d : \mathfrak{P}_q[a] \longrightarrow \mathfrak{P}_q[a]. \tag{2.2.7}$$

Obviously d is the inner graded q-derivation of grading one of the $\mathbb{Z}_N$-graded algebra $\mathfrak{P}_q[a]$. From the commutation relations (2.2.6) it follows that

$$d(\mathbb{1}) = 0, \quad d(a_i) = a_{i+1},$$

for any $i \geq 1$. Let us define $D, \nabla \in \mathrm{Lin}\,\mathfrak{P}_q[a]$ of grading one and the polynomials $f_k \in \mathfrak{P}_q[a]$, where k is an integer greater than or equal to zero, by the formulae

$$D(P) \;=\; d(P) + a_1\, P, \tag{2.2.8}$$

$$\begin{aligned} \nabla(P) &= d(P) + [a_1, P]_q, & (2.2.9)\\ f_0 &= \mathbb{1}, \\ f_1 &= a_1, \\ f_k &= D(f_{k-1}), & (2.2.10) \end{aligned}$$

where $P \in \mathfrak{P}_q[a]$ is a homogeneous polynomial. We can write the linear mapping ∇ in the form $\nabla = \mathrm{ad}^q_{\mathfrak{d}+a_1}$ which clearly shows that ∇ is the inner graded q-derivation of the algebra $\mathfrak{P}_q[a]$. Hence for any polynomials $P, Q \in \mathfrak{P}_q[a]$, where P is a homogeneous, it holds

$$\begin{aligned} D(PQ) &= D(P)\,Q + q^{|P|} P\, d(Q), & (2.2.11)\\ \nabla(PQ) &= \nabla(P) \,+ q^{|P|} P\, \nabla(Q). & (2.2.12) \end{aligned}$$

For the first values of k we calculate by means of the recurrent relation (2.2.10)

$$\begin{aligned} f_2 &= a_2 + a_1^2, \\ f_3 &= a_3 + a_2\, a_1 + [2]_q\, a_1\, a_2 + a_1^3, \\ f_4 &= a_4 + a_3\, a_1 + [3]_q\, a_1 a_3 + [3]_q\, a_2^2 \\ &\quad + a_2 a_1^2 + [3]_q\, a_1^2 a_2 + [2]_q\, a_1 a_2 a_1 + a_1^4, & (2.2.13)\\ f_5 &= a_5 + a_4 a_1 + [4]_q\, a_1 a_4 + [4]_q\, a_3 a_2 \\ &\quad + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q a_2 a_3 + a_3 a_1^2 + [3]_q\, a_2^2 a_1 + [4]_q\, a_2 a_1 a_2 \\ &\quad + [2]_q\, [4]_q\, a_1 a_2^2 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q a_1^2 a_3 + [3]_q\, a_1 a_3 a_1 \\ &\quad + [2]_q\, a_1 a_2 a_1^2 + [3]_q\, a_1^2 a_2 a_1 + a_2 a_1^3 + [4]_q\, a_1^3 a_2 + a_1^5. \end{aligned}$$

Getting a bit ahead we would like to point out that the polynomials f_k may be interpreted as the curvature of a connection if we view the generator a_1 as an algebraic model for a connection one form. Let us remind that if k is a positive integer then *a composition of an integer* k is a way of writing k as the sum of strictly positive integers. For example if $k = 3$ then

$$3 = 3,\ 3 = 2 + 1,\ 3 = 1 + 2,\ 3 = 1 + 1 + 1.$$

Let Ψ_k be the set of all compositions of an integer k. We will write a composition of an integer k in the form of a sequence of strictly positive integers

$\sigma = (i_1, i_2, \ldots, i_r)$, where $i_1 + i_2 + \ldots + i_r = k$. Let us denote

$$\begin{aligned} k_1 &= i_1, \\ k_2 &= i_1 + i_2, \\ k_3 &= i_1 + i_2 + i_3, \\ &\ldots \\ k_{r-1} &= i_1 + i_2 + \ldots + i_{r-1}. \end{aligned}$$

It is well known that the number of elements in the set Ψ_k is 2^{k-1}. The following proposition gives an explicit formula for the polynomials f_k:

Theorem 2.2.4. *For any integer $k \geq 2$ we have the following expansion of power of the operator D and the expansion of a polynomial f_k in terms of generators a_i:*

$$\begin{aligned} D^k &= \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i \, d^{k-i}, \\ f_k &= \sum_{\sigma \in \Psi_k} \begin{bmatrix} k_2 - 1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3 - 1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k - 1 \\ k_{r-1} \end{bmatrix}_q a_{i_1} a_{i_2} \ldots a_{i_r}, \end{aligned}$$

where $\sigma = (i_1, i_2, \ldots, i_r)$ is a composition of an integer k.

Proof. We will prove the expansion formulae of this theorem by the method of mathematical induction. In order to prove the expansion of power of the operator D by means of mathematical induction we begin with the base case and show that this formula holds when k is equal to 1. This is true because

$$D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q f_0 \, d + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q f_1 = d + a_1.$$

Next step in the proof is an inductive step, i.e. we assume that the expansion formula holds for some integer $k > 1$ and show that it also holds when $k + 1$ is substituted for k. Indeed we have

$$D^{k+1} = D(D^k) = D\Big(\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q f_i \, d^{k-i}\Big)$$

$$
\begin{aligned}
&= \sum_{i=0}^{k} \left[\begin{array}{c} k \\ i \end{array} \right]_q \Big(D(f_i)\, d^{k-i} + q^i\, f_i d^{k+1-i} \Big) \\
&= \sum_{i=0}^{k} \left[\begin{array}{c} k \\ i \end{array} \right]_q \Big(f_{i+1}\, d^{k-i} + q^i\, f_i d^{k+1-i} \Big) \\
&= f_{k+1} + \sum_{i=0}^{k-1} \left[\begin{array}{c} k \\ i \end{array} \right]_q f_{i+1}\, d^{k-i} + q^i \sum_{i=1}^{k} \left[\begin{array}{c} k \\ i \end{array} \right]_q f_i\, d^{k+1-i} + d^{k+1} \\
&= f_{k+1} + \sum_{i=1}^{k} \left[\begin{array}{c} k \\ i-1 \end{array} \right]_q f_i\, d^{k+1-i} + q^i \sum_{i=1}^{k} \left[\begin{array}{c} k \\ i \end{array} \right]_q f_i\, d^{k+1-i} + d^{k+1} \\
&= f_{k+1} + \sum_{i=1}^{k} \Big(\left[\begin{array}{c} k \\ i-1 \end{array} \right]_q + q^i \left[\begin{array}{c} k \\ i \end{array} \right]_q \Big) f_i\, d^{k+1-i} + d^{k+1} \\
&= f_{k+1} + \sum_{i=1}^{k} \left[\begin{array}{c} k+1 \\ i \end{array} \right]_q f_i\, d^{k+1-i} + d^{k+1} \\
&= \sum_{i=0}^{k+1} \left[\begin{array}{c} k+1 \\ i \end{array} \right]_q f_i\, d^{k+1-i}.
\end{aligned}
$$

Thus the expansion of power of the operator D is proved. Now if we apply the both sides of the proved formula to a_1 we obtain

$$
f_{k+1} = \sum_{i=0}^{k} \left[\begin{array}{c} k \\ i \end{array} \right]_q f_i\, a_{k+1-i}, \tag{2.2.14}
$$

and this is the recurrent formula for the polynomials f_k which we will use in the second part of the present proof in order to prove the expansion formula for f_k.

We start the proof of the expansion formula for a polynomial f_k with the base case when $k = 2$. In this case there are two compositions $2 = 2,\ 2 = 1 + 1$. Hence we have

$$
f_2 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]_q a_2 + \left[\begin{array}{c} 1 \\ 0 \end{array} \right]_q \left[\begin{array}{c} 1 \\ 1 \end{array} \right]_q a_1^2 = a_2 + a_1^2.
$$

Comparing this result with the first formula in (2.2.13) we see that in the case when $k = 2$ the expansion formula for f_k is correct. The next step is an inductive step, i.e. we assume that the expansion formula holds for some positive integer $k > 2$ and show that it also holds when $k+1$ is substituted for k. Let us consider the sum

$$\sum_{\sigma\in\Psi_{k+1}} \left[\begin{array}{c} k_2-1 \\ k_1 \end{array}\right]_q \left[\begin{array}{c} k_3-1 \\ k_2 \end{array}\right]_q \cdots \left[\begin{array}{c} k \\ k_r \end{array}\right]_q a_{i_1}a_{i_2}\ldots a_{i_{r+1}}, \qquad (2.2.15)$$

where $\sigma = (i_1, i_2, \ldots, i_r, i_{r+1})$ is a composition of an integer $k+1$. Hence $i_1 + \ldots + i_r + i_{r+1} = k+1$. Our aim is to show that this sum is equal to the polynomial f_{k+1}. Let us fix an integer $i \in \{0, 1, \ldots, k\}$ and a generator a_{k+1-i}. It is clear that if we select the compositions of an integer $k+1$ which have the form $(i_1, i_2, \ldots, i_r, k+1-i)$, i.e. the last integer of each composition is previously fixed integer $k+1-i$, and we remove in each composition the last integer then the set of compositions $(i_1, i_2, \ldots, i_r)$ is the set of all compositions of an integer i, i.e. $\{(i_1, i_2, \ldots, i_r)\} = \Psi_i$. Indeed we have

$$i_1 + i_2 + \ldots + i_r + k + 1 - i = k + 1,$$

which implies $i_1 + i_2 + \ldots + i_r = i$. Consequently if we select in the sum (2.2.15) all terms with $i_{r+1} = k+1-i$ (i.e. containing a generator a_{k+1-i} at the end of a product of generators) then we get the sum

$$\sum_{\sigma\in\Psi_{k+1}} \left[\begin{array}{c} k_2-1 \\ k_1 \end{array}\right]_q \left[\begin{array}{c} k_3-1 \\ k_2 \end{array}\right]_q \cdots \left[\begin{array}{c} k \\ i \end{array}\right]_q a_{i_1}a_{i_2}\ldots a_{i_r}a_{k+1-i}, \qquad (2.2.16)$$

where the sum is taken over the compositions of integer $k+1$ which have the form $\sigma = (i_1, i_2, \ldots, i_r, k+1-i) \in \Psi_{k+1}$. We would like to point out that the product of binomial coefficients of each term in this sum contains the factor

$$\left[\begin{array}{c} k \\ i \end{array}\right]_q.$$

Hence we can write the sum (2.2.16) as follows

$$\left[\begin{array}{c} k \\ i \end{array}\right]_q \Big(\sum_{\tau\in\Psi_i} \left[\begin{array}{c} k_2-1 \\ k_1 \end{array}\right]_q \left[\begin{array}{c} k_3-1 \\ k_2 \end{array}\right]_q \cdots \left[\begin{array}{c} i-1 \\ k_{r-1} \end{array}\right]_q a_{i_1}a_{i_2}\ldots a_{i_r}\Big)\, a_{k+1-i},$$

where $\tau = (i_1, i_2, \ldots, i_r) \in \Psi_i$ and the sum is taken over all compositions of integer i. Now we make use of the assumption of an inductive step that the expansion formula for a polynomial f_m holds for each integer $m \in \{1, 2, \ldots, k\}$.

Hence the sum in the previous formula is equal to f_i, i.e

$$\sum_{\tau\in\Psi_i}\begin{bmatrix} k_2-1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3-1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} i-1 \\ k_{r-1} \end{bmatrix}_q a_{i_1}a_{i_2}\dots a_{i_r} = f_i.$$

Thus the sum (2.2.16) is equal to

$$\begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, a_{k+1-i},$$

and summing up all these terms with respect to i we get the sum (2.2.15). Consequently the sum (2.2.15) we started with is equal to the sum

$$\sum_{i=0}^{k}\begin{bmatrix} k \\ i \end{bmatrix}_q f_i\, a_{k+1-i},$$

which in turn is equal to f_{k+1} (see the recurrent relation (2.2.14)). This ends the proof. □

We remind a reader that the parameter q which plays an important role in the structure of an the algebra $\mathfrak{P}_q[\mathfrak{d},a]$ is any complex number different from zero. Now we are going to study the structure of the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d},a]$ at a primitive Nth root of unity, i.e. we assume q to be a primitive Nth root of unity. We may expect that in this case the infinite set of variables $\{\mathfrak{d},a_1,a_2,\dots\}$ is "cut off" and we get an algebra whose vector space is finite dimensional. Indeed we can prove the following proposition:

Proposition 2.2.5. *Let $\mathfrak{P}_q[\mathfrak{d},a]$ be the algebra of polynomials generated by the set of variables $\{\mathfrak{d},a_i\}_{i\in\mathbb{N}_1}$ which obey the commutation relations (2.2.4). If we assume that q is a primitive Nth root of unity and the variable $\mathfrak{d}$ is subjected to the additional relation $\mathfrak{d}^N=\lambda\cdot\mathbb{1}$, where λ is any complex number, then for any integer $k\geq N$ a variable a_k vanishes, i.e. the algebra $\mathfrak{P}_q[\mathfrak{d},a]$ is generated by the finite set of variables $\{\mathfrak{d},a_k\}_{k=1}^{N-1}$ which obey the relations*

$$\begin{aligned}
\mathfrak{d}a_1 &= q\,a_1\mathfrak{d}+a_2,\\
\mathfrak{d}a_2 &= q^2\,a_2\mathfrak{d}+a_3,\\
&\cdots\\
\mathfrak{d}a_{N-2} &= q^{N-2}\,a_{N-2}\mathfrak{d}+a_{N-1},
\end{aligned}\qquad(2.2.17)$$

$$\begin{aligned} \mathfrak{d} a_{N-1} &= a_{N-1}\mathfrak{d}, \\ \mathfrak{d}^N &= \lambda \cdot \mathbb{1}. \end{aligned}$$

The inner graded q-derivation $d = \mathrm{ad}^q_{\mathfrak{d}} : \mathfrak{P}_q[a] \longrightarrow \mathfrak{P}_q[a]$ *associated to variable* $\mathfrak{d}$ *is the N-differential, i.e.* $d^N = 0$, *and the sequence*

$$\ldots \xrightarrow{d} \mathfrak{P}^{i-1}_q[a] \xrightarrow{d} \mathfrak{P}^{i}_q[a] \xrightarrow{d} \mathfrak{P}^{i+1}_q[a] \xrightarrow{d} \ldots$$

is the cochain N-complex. The graded algebra $\mathfrak{P}_q[a]$ *equipped with the N-differential d is the graded q-differential algebra.*

Proof. We suppose that the algebra of polynomials is equipped with the $\mathbb{Z}_N$-gradation as it was explained earlier (2.2.5). It easily follows from the commutation relations of the algebra $\mathfrak{P}_q[\mathfrak{d}, a]$ that for any integer $k \geq 2$ we have

$$a_k = d^k(a_1),$$

where $d = \mathrm{ad}^q_{\mathfrak{d}}$ is the inner graded q-derivation associated with a variable $\mathfrak{d}$. Making use of the expansion of power of an inner graded q-derivation derived in the proof of the Theorem 1.3.7 we obtain

$$a_k = a_k = d^k(a_1) = (\mathrm{ad}^q_{\mathfrak{d}})^k(a_1) = \sum_{i=0}^{k} (-1)^i\, p_i \left[\begin{array}{c} k \\ i \end{array} \right]_q \mathfrak{d}^{k-i}\, u\, \mathfrak{d}^i.$$

Thus if q is a primitive Nth root of unity, $\mathfrak{d}$ satisfies $\mathfrak{d}^N = \lambda \cdot \mathbb{1}$ and $k = N$ all terms of the sum in the right-hand side of the above expansion formula vanish (see the proof of the Theorem 1.3.7). Consequently we have $a_N = a_{N+1} = \ldots = 0$ and this ends the proof. □

Let us denote by $\mathfrak{P}_q[\mathfrak{d}, a]$ the finite dimensional graded algebra generated by $\{\mathfrak{d}, a_k\}_{k=1}^{N-1}$ which obey relations (2.2.17) and by $\mathfrak{P}_q[a]$ the graded q-differential algebra generated by $\{a_k\}_{k=1}^{N-1}$ and equipped with N-differential d. Now we give the following definition:

Definition 2.2.6. The generator a_1 of the $\mathbb{Z}_N$-graded q-differential algebra $\mathfrak{P}_q[a]$ will be called the N-connection form and the algebra $\mathfrak{P}_q[a]$ will be called *the algebra of N-connection form.* The operator $D = d + a_1 : \mathfrak{P}_q[a] \longrightarrow \mathfrak{P}_q[a]$ will be called *the covariant N-differential*, and the the polynomial f_N of grading zero will be called *the curvature of N-connection form.*

Proposition 2.2.7. *If $\mathfrak{P}_q[a]$ is the algebra of N-connection form and d is its N-differential then the Nth power of the covariant N-differential D is the operator of multiplication by f_N.*

Proof. The proof of this proposition is based on the first expansion formula proved in the Theorem 2.2.4. Indeed we can expand an Nth power of the covariant N-differential D into the sum of products of polynomials f_i and the powers of the N-differential d as follows

$$D^N = \sum_{i=0}^{N} \left[\begin{array}{c} N \\ i \end{array} \right]_q f_i \, d^{N-i}.$$

As q is a primitive Nth root of unity this expansion can be essentially simplified in the case $k = N$ if we take into account that all q-binomial coefficients with $i \in \{1, 2, \ldots, N-1\}$ vanish. The first term in this expansion also vanishes because d is the N-differential. Hence for any polynomial $P \in \mathfrak{P}_q[a]$ we have

$$D^N(P) = f_N \cdot P,$$

and this ends the proof. □

Proposition 2.2.8. *If $\mathfrak{P}_q[a]$ is the algebra of connection form and f_N is the curvature of connection form then the curvature satisfies the identity*

$$\nabla(f_N) = 0. \tag{2.2.18}$$

Proof. Let us remind a reader that $\nabla = d + \mathrm{ad}^q_{a_1}$. We prove this proposition by means of the recurrent relation for polynomials f_k

$$f_{k+1} = \sum_{i=0}^{k} \left[\begin{array}{c} k \\ i \end{array} \right]_q f_i \, a_{k+1-i}.$$

Substituting N for k in the above relation we obtain

$$f_{N+1} = \sum_{i=0}^{N} \left[\begin{array}{c} N \\ i \end{array} \right]_q f_i \, a_{N+1-i}. \tag{2.2.19}$$

As q is a primitive Nth root of unity we have

$$\left[\begin{array}{c} N \\ i \end{array} \right]_q = 0,$$

for any integer $i \in \{1, 2, \ldots, N-1\}$. Consequently there are only two terms with non-zero q-binomial coefficients (labeled by $i = 0, N$) at the right-hand side of the relation (2.2.19) and

$$f_{N+1} = f_0\, a_{N+1} + f_N\, a_1.$$

The first term at the right-hand side of the above formula is also zero because of $a_{N+1} = 0$ (Proposition 2.2.5). Hence

$$\begin{aligned} 0 &= f_{N+1} - f_N\, a_1 = D(f_N) - f_N\, a_1 \\ &= d(f_N) + a_1\, f_N - f_N\, a_1 = d(f_N) + [a_1, f_N]_q = (d + \mathrm{ad}^q_{a_1})(f_N) = \nabla(f_N). \end{aligned}$$

□

We will call the identity (2.2.18) *the Bianchi identity for curvature* of N-connection form. It is worth mentioning that we can write the Bianchi identity in a different way if we consider the covariant N-differential D and the curvature f_N as the linear operators $D, f_N : \mathfrak{P}_q[a] \longrightarrow \mathfrak{P}_q[a]$, i.e. $D, f_N \in \mathrm{Lin}\,\mathfrak{P}_q[a]$, where f_N is the operator of multiplication by f_N (we denote it by the same symbol as the curvature f_N in order not to make the notations very complicated). Then the Bianchi identity may be written in the form

$$[D, f_N]_q = 0.$$

Indeed

$$\begin{aligned} [D, f_N]_q &= D \circ f_N - f_N \circ D \\ &= d(f_N) + f_N \circ d + a_1\, f_N - f_N \circ d - f_N\, a_1 \\ &= d(f_N) + [a_1, f_N]_q = \nabla(f_N) = 0. \end{aligned}$$

2.3 N-differential forms on reduced quantum plane

In this section we show that a graded q-differential algebra may be constructed by means of differential forms on a reduced quantum plane. This is important for a notion of N-connection which we will develop in Chapter 3 because our approach to a notion of N-connection is based on the algebraic structures such as a graded q-differential algebra, module over a graded q-differential algebra, and in order to give this algebraic approach a geometric meaning we should have an algebra of differential forms which is a graded q-differential algebra. Let us remind that a reduced quantum plane is described at the end of Section 2.1, and the reduced Wess-Zumino algebra Ω_{WZ} is the algebra of differential forms on a reduced quantum plane with differential satisfying $d^2 = 0$ (exterior differential). In this section we construct an algebra of differential forms on a reduced quantum plane which is the graded q-differential algebra with N-differential d satisfying $d^N = 0$, and this algebra may be viewed as an analog of the reduced Wess-Zumino algebra. Our approach is different from the one proposed in [17] and is based on the Theorem 1.3.7. A first order coordinate differential calculus in our approach is very similar to the one of the reduced Wess-Zumino algebra but the structure of the algebra with respect to higher degree differential forms is different. The reason is $d^k \neq 0$ for $k \in \{2, 3, \ldots, N-1\}$ which implies the presence of higher order differentials $d^2x, d^3x, \ldots, d^{N-1}x, d^2y, d^3y, \ldots, d^{N-1}y$ in our approach whereas in the case of the reduced Wess-Zumino algebra one has $d^2x = d^2y = 0$. This section is based on the paper [3].

Let us remind that a generalized Clifford algebra is an algebra over the complex numbers $\mathbb{C}$ generated by a set of canonical generators $\{x_1, x_2, \ldots, x_p\}$ which are subjected to the relations

$$x_i x_j = q^{\mathrm{sg}(j-i)} x_j x_i, \quad x_i^N = \mathbb{1}, \quad i, j = 1, 2, \ldots, p \tag{2.3.1}$$

where $\mathrm{sg}(k)$ is the usual sign function, and $\mathbb{1}$ is the identity element of an algebra. We will use the generalized Clifford algebra with four generators, i.e. $p = 4$, in order to construct an algebra of differential forms on a reduced quantum plane with N-differential d satisfying $d^N = 0$. In this section we will denote the generalized Clifford algebra with four generators x_1, x_2, x_3, x_4 which obey the relations (2.3.1) by $\mathfrak{C}_N$. We split the set of generators of this algebra into two pairs x_1, x_3 and x_2, x_4 denoting the generators of the first pair by x, y, i.e. $x = x_1, y = x_3$, and the generators of the second pair by u, v, i.e. $u = x_2, v = x_4$. From (2.3.1) it follows

$$x\,y = q\,y\,x, \qquad x^N = y^N = \mathbb{1}, \tag{2.3.2}$$

$$x\,u = q\,u\,x, \qquad x\,v = q\,v\,x, \tag{2.3.3}$$

$$y\,u = q^{-1}\,u\,y, \quad y\,v = q\,v\,y, \tag{2.3.4}$$

$$u\,v = q\,v\,u, \qquad u^N = v^N = \mathbb{1}. \tag{2.3.5}$$

Let $\mathfrak{P}_N$ be the subalgebra of the algebra $\mathfrak{C}_N$ generated by x, y. The relations (2.3.2) show that the generators x, y can be interpreted as coordinate functions on a reduced quantum plane [17], and thus the subalgebra $\mathfrak{P}_N$ can be interpreted as the algebra of (polynomial) functions on a reduced quantum plane. Our next step is to construct an N-differential d with the help of Theorem 1.3.7 and the algebra $\mathfrak{P}_N$ endowed with an N-differential d will become a graded q-differential algebra. Obviously for this we need to define a graded structure of $\mathfrak{P}_N$. We do this as follows: we assign the grading zero to the generators x, y and the grading one to the generators u, v. Hence denoting as before the grading of an element w by $|w|$ we can write

$$|x| = |y| = \bar{0},\ |u| = |v| = \bar{1}, \tag{2.3.6}$$

where $\bar{0}, \bar{1}$ are the residue classes of $0, 1$ modulo N. As usual the grading of any product of generators x, y, u, v is the sum of gradings of its factors. Obviously $\mathfrak{C}_N = \oplus_{i\in\mathbb{Z}_N}\mathfrak{C}_N^i$, where $\mathfrak{C}_N^i$ is the subspace of homogeneous elements of grading i, and $\mathfrak{C}_N^{\bar{0}} = \mathfrak{P}_N$.

Proposition 2.3.1. *For any $\lambda, \mu \in \mathbb{C}$ an element $\omega = \lambda\, u + \mu\, v \in \mathfrak{C}_N^{\bar{1}}$ satisfies $\omega^N \in Z(\mathfrak{C}_N)$, where $Z(\mathfrak{C}_N)$ is the center of the algebra $\mathfrak{C}_N$.*

Proof. For any $2 \leq k \leq N$ we have

$$\omega^k = \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_q \lambda^{k-l}\mu^l\, v^l u^{k-l}. \tag{2.3.7}$$

Since q is a primitive Nth root of unity we have $\begin{bmatrix} N \\ l \end{bmatrix}_q = 0$ for $1 \leq l \leq N-1$. Thus taking $k = N$ in (2.3.7) we obtain

$$\omega^N = \lambda^N\, u^N + \mu^N\, v^N = (\lambda^N + \mu^N)\,\mathbb{1} \in Z(\mathfrak{C}_N).$$

□

Now it follows from Theorem 1.3.7 that the inner graded q-derivation $d = \mathrm{ad}_\omega^q$ associated to an element $\omega = \lambda\, u + \mu\, v \in \mathfrak{C}_N^{\bar{1}}$ is the N-differential of the $\mathbb{Z}_N$-graded algebra $\mathfrak{C}_N$. Hence the generalized Clifford algebra $\mathfrak{C}_N$ endowed with the $\mathbb{Z}_N$-graded structure (2.3.6) and with the N-differential d is the graded q-differential algebra. It should be mentioned that in our approach ω is a fixed element of grading one, and hence the structure of the graded q-differential algebra $\mathfrak{C}_N$ depends on a choice of element ω. Consequently the numbers λ, μ can be considered as the free parameters of our approach.

The N-differential d induces the differentials of coordinate functions $x \xrightarrow{d} dx$, $y \xrightarrow{d} dy$, and later in this section we will show that any element of grading $k > \bar{0}$ of the graded q-differential algebra $\mathfrak{C}_N$ can be expressed in terms of the differentials of coordinates dx, dy. Since N-differential d satisfies $d^N = 0$ the sequence of vector spaces

$$\ldots \xrightarrow{d} \mathfrak{C}_N^{\bar{0}} \xrightarrow{d} \mathfrak{C}_N^{\bar{1}} \xrightarrow{d} \mathfrak{C}_N^{\bar{2}} \xrightarrow{d} \ldots \xrightarrow{d} \mathfrak{C}_N^{\overline{N-1}} \xrightarrow{d} \mathfrak{C}_N^{\bar{0}} \xrightarrow{d} \ldots, \qquad (2.3.8)$$

is the cochain N-complex. The N-differential d can be viewed as a higher order analog of exterior differential on a reduced quantum plane. Accordingly to this analogy we will call the graded q-differential algebra $\mathfrak{C}_N$ *the algebra of q-differential forms on the reduced quantum plane* and its elements of grading $k > \bar{0}$, written in terms of the differentials dx, dy, the q-differential k-forms. The cochain N-complex (2.3.8) will be called *the reduced quantum de Rham N-complex.* We will describe the cohomologies of this N-complex $H(\mathfrak{C}_N)$ with the help of notions of closed and exact q-differential forms. An q-differential form θ will be called *an m-closed q-differential form*, where $1 \leq m \leq N-1$, if $d^m\theta = 0$, and an q-differential n-form θ will be called *an q-differential l-exact form* if there exists an q-differential $(n-l)$-form ρ such that $d^l\rho = \theta$. It follows from $d^N = 0$ that each q-differential $(N-m)$-exact form is m-closed. Let us denote the vector space of m-closed q-differential forms by $Z^m(\mathfrak{C}_N)$, and the vector space of $(N-m)$-exact q-differential forms by $B^m(\mathfrak{C}_N)$. Then the cohomologies of the reduced quantum de Rham N-complex is $H^m(\mathfrak{C}_N) = Z^m(\mathfrak{C}_N)/B^m(\mathfrak{C}_N)$. Now we can prove an analog of *Poincaré lemma* for the reduced quantum de Rham N-complex.

Proposition 2.3.2. *The cohomologies of the reduced quantum de Rham N-complex are trivial, i.e. for any $m \in \{\bar{0}, \bar{1}, \bar{2} \ldots, \overline{N-1}\}$ we have $H^m(\mathfrak{C}_N) = 0$. Thus any m-closed q-differential form on a reduced quantum plane is exact.*

The statement of this proposition immediately follows from Theorem 1.3.8.

Our next aim is to describe the structure of the algebra of q-differential forms in terms of the differentials of coordinates dx, dy. We begin with the first order differential calculus $\mathfrak{P}_N \xrightarrow{d} \mathfrak{C}_N^{\bar{1}}$, where $\mathfrak{P}_N$ is the algebra of polynomial functions, d is the N-differential and $\mathfrak{C}_N^{\bar{1}}$ is the $(\mathfrak{P}_N, \mathfrak{P}_N)$-bimodule of q-differential 1-forms. Evidently $dx, dy \in \mathfrak{C}_N^{\bar{1}}$. Let us express the differentials dx, dy in terms of the generators of $\mathfrak{C}_N$. It is worth mentioning that in what follows we shall use the structure of the right $\mathfrak{P}_N$-module of $\mathfrak{C}_N^{\bar{1}}$ to write q-differential 1-forms in terms of differentials. We have the relations

$$x\,\omega = q\,\omega\,x, \qquad y\,\omega = \omega_1\,y, \tag{2.3.9}$$

where $\omega = \lambda\,u + \mu\,v,\ \omega_1 = q^{-1}\lambda\,u + q\,\mu\,v$. Using these relations we obtain

$$dx = [\omega, x]_q = (1-q)\,\omega\,x, \quad dy = [\omega, y]_q = (\omega - \omega_1)\,y, \tag{2.3.10}$$

where

$$\omega - \omega_1 = (1-q^{-1})\lambda\,u + (1-q)\mu\,v. \tag{2.3.11}$$

It is evident that the right $\mathfrak{P}_N$-module $\mathfrak{C}_N^{\bar{1}}$ is a free right module and $\{u, v\}$ is the basis for this right module.

Proposition 2.3.3. *For any integer $N \geq 3$ the right $\mathfrak{P}_N$-module of q-differential 1-forms $\mathfrak{C}_N^{\bar{1}}$ is freely generated by the differentials of coordinates dx, dy.*

Proof. Let $f, h \in \mathfrak{P}_N$ be functions on a reduced quantum plane. Making use of (2.3.10),(2.3.11) and taking into account that $\{u, v\}$ is the basis for the right $\mathfrak{P}_N$-module $\mathfrak{C}_N^{\bar{1}}$ we can show that the equality $dx\,f + dy\,h = 0$ is equivalent to the system of equations

$$\begin{aligned} (1-q)\,x\,f + (1-q^{-1})\,y\,h &= 0, \\ x\,f + y\,h &= 0. \end{aligned}$$

Multiplying the second equation by $q-1$ and adding it to the first equation we obtain $(q - q^{-1})\,h = 0$. As $q - q^{-1} \neq 0$ for $N \geq 3$ we conclude that $h = 0$. In the same way we show that $f = 0$, and this proves that the differentials dx, dy are linearly independent q-differential 1-forms.

In order to prove that any q-differential 1-form is a linear combination of differentials we find the transition matrix from the basis $\{u, v\}$ to the basis $\{dx, dy\}$. Let us denote the algebra of square matrices of order 2, whose entries are the elements of the algebra $\mathfrak{P}_N$, by $\mathrm{Mat}_2(\mathfrak{P}_N)$. Then $(dx\ \ dy) = (u\ \ v)\cdot A$, where $A \in \mathrm{Mat}_2(\mathfrak{P}_N)$, and from (2.3.10) we find

$$A = \begin{pmatrix} (1-q)\,\lambda\,x & (1-q^{-1})\,\lambda\,y \\ (1-q)\,\mu\,x & (1-q)\,\mu\,y \end{pmatrix}.$$

It should be noted that the transition matrix depends on the coordinates of a point of a reduced quantum plane. As the coordinates of a reduced quantum plane obey the relations $x^N = \mathbb{1}, y^N = \mathbb{1}$ they are invertible elements of the algebra $\mathfrak{P}_N$ and $x^{-1} = x^{N-1}, y^{-1} = y^{N-1}$. If $N \geq 3$ then the matrix A is an invertible matrix and

$$A^{-1} = \frac{1}{(1-q^2)} \begin{pmatrix} \lambda^{-1}\, qx^{-1} & \mu^{-1}\, x^{-1} \\ -\lambda^{-1}\, qy^{-1} & \mu^{-1}\, qy^{-1} \end{pmatrix}.$$

Consequently we have

$$u = \frac{q}{(1-q^2)\lambda}\,(dx\, x^{-1} - dy\, y^{-1}), \tag{2.3.12}$$

$$v = \frac{1}{(1-q^2)\mu}\,(dx\, x^{-1} + q\, dy\, y^{-1}). \tag{2.3.13}$$

Now any q-differential 1-form $\theta = u\, f + v\, h \in \mathfrak{C}^{\bar{1}}_N$, where $f, h \in \mathfrak{P}_N$, can be expressed in terms of the differentials, and this ends the proof. □

From Proposition 2.3.3 it follows that the first order differential calculus $\mathfrak{P}_N \xrightarrow{d} \mathfrak{C}^{\bar{1}}_N$ is the coordinate calculus with coordinate differential d [12, 13]. If we have a coordinate calculus then a coordinate differential of this calculus induces the right partial derivatives which satisfy the twisted Leibniz rule. The second term at the right hand side of the twisted Leibniz rule for right partial derivative depends on the twisting homomorphism $R : \mathfrak{P}_N \longrightarrow \mathrm{Mat}_2(\mathfrak{P}_N)$, and this homomorphism is determined by the relation between the right and left module structures of the bimodule $\mathfrak{C}^{\bar{1}}_N$. Hence for any function $f \in \mathfrak{P}_N$ we have

$$R(f) = \begin{pmatrix} r_{11}(f) & r_{12}(f) \\ r_{21}(f) & r_{22}(f) \end{pmatrix}, \tag{2.3.14}$$

where

$$f\, dx = dx\, r_{11}(f) + dy\, r_{21}(f), \tag{2.3.15}$$

$$f\, dy = dx\, r_{12}(f) + dy\, r_{22}(f). \tag{2.3.16}$$

Since $\mathfrak{P}_N$ is the algebra of polynomials generated by two variables x, y, which are subjected to the relations $xy = q\, yx,\ x^N = \mathbb{1},\ y^n = \mathbb{1}$, it is sufficient to find the explicit formula for the homomorphism R in the case of coordinate functions x, y. Taking $f = x$ and $f = y$ in (2.3.15), (2.3.16) we find

$$R(x) = \begin{pmatrix} qx & 0 \\ 0 & q^2x \end{pmatrix},\ R(y) = \begin{pmatrix} q^{-1}\, y & q^{-1}(q-1)\, x^{-1}y^2 \\ (q-1)\, x & q^{-1}(q^2-q+1)\, y \end{pmatrix}. \tag{2.3.17}$$

Putting the entries of these matrices into the relations (2.3.15), (2.3.16) we get

$$x\,dx = q\;dx\,x, \quad y\,dx = \frac{1}{q}\;dx\,y + (q-1)\;dy\,x, \tag{2.3.18}$$

$$x\,dy = q^2\;dy\,x, \quad y\,dy = \frac{q-1}{q}\;dx\,x^{-1}y^2 + \frac{q^2-q+1}{q}\;dy\,y. \tag{2.3.19}$$

The right partial derivatives induced by the N-differential d are defined by

$$df = dx\,\frac{\partial f}{\partial x} + dy\,\frac{\partial f}{\partial y}.$$

These right partial derivatives satisfy the twisted Leibniz rule

$$\begin{aligned}\frac{\partial (fh)}{\partial x} &= \frac{\partial f}{\partial x}\,h + r_{11}(f)\frac{\partial h}{\partial x} + r_{12}(f)\,\frac{\partial h}{\partial y},\\ \frac{\partial (fh)}{\partial y} &= \frac{\partial f}{\partial y}\,h + r_{21}(f)\,\frac{\partial h}{\partial x} + r_{22}(f)\frac{\partial h}{\partial y}.\end{aligned}$$

Using the twisted Leibniz rule and (2.3.17) we find

$$\begin{aligned}\frac{\partial x^k}{\partial x} &= [k]_q\,x^{k-1}, & \frac{\partial x^k}{\partial y} &= 0,\\ \frac{\partial y^l}{\partial x} &= \frac{[l]_q(q^{l-1}-1)}{q^{l-1}(q+1)}\,x^{-1}y^l, & \frac{\partial y^l}{\partial y} &= \frac{[l]_q(q^l+1)}{q^{l-1}(q+1)}\,y^{l-1}.\end{aligned}$$

Using these formulae we can calculate the partial derivatives of any function

$$f = \sum_{k,l=0}^{N-1} \zeta_{kl}\,x^k y^l \in \mathfrak{P}_N.$$

For instance the derivative with respect to x of f is

$$\frac{\partial f}{\partial x} = \sum_{k,l=0}^{N-1} ([k]_q + q^{k-l+1}[l]_q\frac{q^{l-1}-1}{q+1})\;\zeta_{kl}\,x^{k-1}y^l.$$

We remind that the set of generators $\{x,u,y,v\}$ of the generalized Clifford algebra $\mathfrak{C}_N$ has been split into two parts, where the first part $\{x,y\}$ generates the algebra of polynomials $\mathfrak{P}_N$, and the second part $\{u,v\}$ generates the N-differential calculus $\mathfrak{P}_N \xrightarrow{d} \mathfrak{C}^{\bar{1}}_N$. We have already proved that any q-differential 1-form $\theta = u\,f + v\,h \in \mathfrak{C}^{\bar{1}}_N$ can be uniquely written as a linear combination of the differentials dx, dy where coefficients of a linear combination are polynomials in coordinate functions of a reduced quantum plane x, y. If we consider x, y, dx, dy as the generators for the algebra $\mathfrak{C}_N$ then we may divide the algebraic relations between the new generators into three parts. The first part contains the relations

$$xy = q\,yx, \quad x^N = y^N = \mathbb{1},$$

which determine the structure of the algebra $\mathfrak{P}_N$ of polynomials on a reduced quantum plane. The second part determines the structure of the first order coordinate calculus $\mathfrak{P}_N \xrightarrow{d} \mathfrak{C}^{\bar{1}}_N$ and consists of the relations (2.3.18),(2.3.19) between coordinates x, y and their differentials

$$x\,dx = q\;dx\,x, \quad y\,dx = \frac{1}{q}\;dx\,y + (q-1)\;dy\,x,$$
$$x\,dy = q^2\;dy\,x, \quad y\,dy = \frac{q-1}{q}\;dx\,x^{-1}y^2 + \frac{q^2-q+1}{q}\;dy\,y.$$

The third part will contain the relations between the differentials dx, dy, and this part of relations originates from the structure of a graded q-differential algebra of $\mathfrak{C}_N$. It should be pointed out that unlike the reduced Wess-Zumino algebra of differential forms proposed in [17] we do not impose the relations $(dx)^2 = (dy)^2 = 0$. It is evident that the commutation relation $uv = q\,vu$ written in terms of differentials dx, dy will give us a quadratic relation for the differentials dx, dy, and the relations $u^N = v^N = \mathbb{1}$ will lead to two relations of degree N with respect to differentials.

Proposition 2.3.4. *The commutation relation $uv = q\,vu$ for the generators u, v written in terms of the differentials dx, dy takes on the form*

$$dx\,dy = \gamma_1\;dy\,dx + \gamma_2\;(dx)^2\,x^{-1}y + \gamma_3\;(dy)^2\,y^{-1}x, \tag{2.3.20}$$

where

$$\gamma_1 = \frac{1+q^4}{2}, \quad \gamma_2 = \frac{q-1}{2q}, \quad \gamma_3 = \frac{(1-q^3)q}{2}. \tag{2.3.21}$$

This proposition can be proved by straightforward computation with the help of the formulae (2.3.12), (2.3.13) and the relations (2.3.18), (2.3.19).

The relation (2.3.20) allows us to choose the ordered set of monomials $\mathcal{B}$, where

$$\mathcal{B} = \{(dx)^k, dy(dx)^{k-1}, \ldots, (dy)^{k-1}dx, (dy)^k\}, \quad 2 \le k \le N-1,$$

as the basis for the right $\mathfrak{P}_N$-module of q-differential k-forms $\mathfrak{C}_N^k$. For example the right $\mathfrak{P}_N$-module of q-differential 2-forms $\mathfrak{C}_N^{\bar{2}}$ is spanned by the monomials $(dx)^2, dy\,dx, (dy)^2$. Hence any q-differential k-form θ on the reduced quantum plane can be uniquely expressed as follows:

$$\theta = (dx)^k\, f_0 + dy(dx)^{k-1}\, f_1 + \ldots (dy)^{k-1}dx\, f_{k-1} + (dy)^k\, f_k,$$

where $f_0, f_1, \ldots, f_k \in \mathfrak{P}_N$ are polynomials. The peculiar property of our approach with differential d satisfying $d^N = 0$ is an appearance of the higher order differentials of coordinates, and this gives us a possibility to construct one more basis for the module of q-differential 2-forms. Indeed as $d^k \neq 0$ for k running integers from 2 to $N-1$ we have the set of higher order differentials of coordinates $d^2x, d^2y, , \ldots, d^{N-1}x, d^{N-1}y$, and we can use these higher order differentials to construct a basis for $\mathfrak{C}_N^k$. The elements $\omega, \omega_1, \omega - \omega_1$ can be written as q-differential 1-forms as follows:

$$\omega = \frac{1}{1-q}\, dx\, x^{-1}, \quad \omega_1 = \frac{1}{1-q}\, dx\, x^{-1} - dy\, y^{-1}, \quad \omega - \omega_1 = dy\, y^{-1}. \quad (2.3.22)$$

Differentiating ω we obtain

$$d\omega = [\omega, \omega]_q = \frac{1}{q(1-q)}\, (dx)^2\, x^{-2}. \quad (2.3.23)$$

where $d\omega$ is the q-differential 2-form. Now we can write the second order differential d^2x as a q-differential 2-form as follows:

$$d^2x = (1-q)d(\omega\, x) = (1-q)(d(\omega)\, x + q\, \omega\, dx) = \frac{1+q}{q}(dx)^2 x^{-1}. \quad (2.3.24)$$

Expressing the second order differential d^2y in terms of $(dx)^2, dy\, dx, (dy)^2$ we prove the following proposition:

Proposition 2.3.5. *The second order differential d^2y can be written as follows:*

$$d^2y = \frac{1}{1-q}(\frac{1}{q^2}\,dx\,dy - q\;dy\,dx)\,x^{-1}. \tag{2.3.25}$$

Proof. As $dy = (\omega-\omega_1)y,\;\; y\omega = \omega_1 y$ we have

$$d^2y = [\omega, (\omega-\omega_1)y] = (\omega(\omega-\omega_1) - q\;(\omega-\omega_1)\omega_1)\,y.$$

Now applying the formula (2.3.22) and the multiplication rules (2.3.18), (2.3.19) we get the expression (2.3.25). □

Propositions 2.3.4, 2.3.5 show that we can replace the basis $\mathcal{B}_2$ by the basis $\mathcal{B}_2^{'}$ in the right $\mathfrak{P}_N$-module of q-differential 2-forms $\mathfrak{C}^2_N$, where

$$\mathcal{B}_2 = \{(dx)^2, dy\,dx, (dy)^2\}, \quad \mathcal{B}_2^{'} = \{d^2x, dy\,dx, d^2y\}.$$

We point out that from Proposition 2.3.5 it follows that the relation (2.3.20) can be written by means of the second order differential d^2y in a more symmetric form

$$dx\,dy = q^3\;dy\,dx + q^2(1-q)\;d^2y\;x. \tag{2.3.26}$$

We end this section by considering the structure of algebra of q-differential forms on a reduced quantum plane at square root of unity and at cubic root of unity, i.e. in the case of $N=2$ and $N=3$. If $N=2$ then q is the primitive square root of unity, i.e. $q=-1$. In this case $d^2=0$, i.e. d is the exterior differential. It is interesting that in this particular case it follows from Proposition 2.3.5 that

$$d^2y = \frac{1}{2}\Big(dx\,dy + dy\,dx\Big)\;x = 0,$$

which implies $dx\,dy = -dy\,dx$. Hence this is a classical case of the algebra of differential forms with anticommuting differentials of coordinates.

If $N=3$ then we have the algebra of q-differentials forms on a reduced quantum plane at cubic root of unity with differential d satisfying $d^3=0$. It can be verified that now the right hand sides of the formulae (2.3.24), (2.3.25) are the 1-closed q-differential 2-forms, i.e.

$$d\left((dx)^2x^{-1}\right) = 0, \qquad d\,(\frac{1}{q^2}\,dx\,dy\;x^{-1} - q\;dy\,dx\;x^{-1}) = 0.$$

The last term in the relation (2.3.20) vanishes because q is a primitive cube root of unity and satisfies $q^3 - 1 = 0$. Making use of the relation $1 + q + q^2 = 0$ we can write the coefficient γ_1 as follows:

$$\gamma_1 = \frac{1+q^4}{2} = \frac{1+q}{2} = -\frac{q^2}{2}.$$

Hence the relation for the differentials (2.3.20) with respect to the basis $\mathcal{B}'_2$ takes on the form

$$dx\, dy = \frac{q}{2}\Big(-dy\, dx + (q-1) d^2x\, y\Big). \tag{2.3.27}$$

Comparing with the relation for differentials in the previous case, where $d^2 = 0$ and $dx\, dy = -dy\, dx$, we see that the peculiar property of the first non-classical case of exterior calculus, where the differential d satisfies $d^3 = 0$, is the appearance of second order differentials in the commutation relation for differentials dx, dy, which "deform" the classical anticommutativity of differentials.

Chapter 3

Generalization of the theory of connections

The general goal of this chapter is to describe a concept of N-connection on modules and N-connection form by means of the notion of graded q-differential algebra. The chapter is based on [1, 4, 6, 7, 8, 9].

3.1 Connection on vector bundles

This section has an introductory character and makes reader familiar with the main notions of differential geometry such as connection and its curvature in the context of vector bundles. The section is based heavily on [11, 47].

We begin with the geometric definition of a vector bundle.

Definition 3.1.1. Let M be a smooth manifold. A smooth manifold E together with a smooth surjection $\pi : E \longrightarrow M$ is called a *real vector bundle of rank* k, if the following conditions are satisfied:

i) $E_x = \pi^{-1}(x)$ has the structure of k-dimensional vector space over the field of real numbers for every $x \in M$;

ii) there exists an open cover $\{U_\alpha\}_{\alpha \in J}$, where J is the set of indexes, of manifold M, i.e. $U_\alpha \subset M$ and $M = \bigcup\limits_{\alpha \in J} U_\alpha$, and a diffeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ such that $\psi_\alpha(E_x) = \{x\} \times \mathbb{R}^k$ and map $\psi_\alpha^x : E_x \longrightarrow \{x\} \times \mathbb{R}^k \xrightarrow{\text{proj}} \mathbb{R}^k$ is a vector space isomorphism for each $x \in U_\alpha$.

The manifold E is said to be the total space of the vector bundle, M is the base space, and the vector spaces E_x are the fibers. It is possible equally well to consider the fiber E_x over the field $\mathbb{C}$ instead of $\mathbb{R}$, obtaining the notion of a complex vector bundle. We will often make use of the convention just calling the vector bundle E, letting the rest of data be implicit.

The open neighborhood U_α together with the diffeomorphism ψ_α is called a *local trivialization of the vector bundle*. For two local trivializations (U_α, ψ_α) and (U_β, ψ_β) we define a smooth composite map $\psi_\beta \circ \psi_\alpha^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \longrightarrow U_{\alpha\beta} \times \mathbb{R}^k$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, by

$$\psi_\beta \circ \psi_\alpha^{-1}(x, v) = (x, g_\alpha^\beta(v)).$$

For every fixed x the above composition is a linear isomorphism of $\mathbb{R}^k$. Thus, the composition map $\psi_\beta \circ \psi_\alpha^{-1}$ induces a smooth map

$$g_\alpha^\beta : U_{\alpha\beta} \longrightarrow \mathrm{GL}(k, \mathbb{R}).$$

These are called the *transition functions of the vector bundle* E. The transition functions g_β^α satisfy the following conditions:

$$\begin{aligned} g_\beta^\alpha g_\gamma^\beta g_\alpha^\gamma &= I \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma, \quad \text{cocycle condition}, \\ g_\alpha^\alpha &= I \qquad \text{on } U_\alpha, \end{aligned}$$

where the product is a matrix product and I is the identity matrix of order k.

A smooth *section of a vector bundle* E over M is a smooth map $s : M \longrightarrow E$ assigning to each $x \in M$ a vector $s(x)$ in the fiber E_x, i.e. $\pi \circ s = \mathrm{Id}_M$, where Id_M is the identity map of M. The set of all smooth sections, denoted by $\Gamma(M, E)$, is an infinite-dimensional real vector space, and is also a module over $C^\infty(M)$, the algebra of smooth functions on M, if we define $(f \cdot s)(x) = f(x) \cdot s(x)$ and $(s_1 + s_2)(x) = s_1(x) + s_2(x)$, where $x \in M$, $f \in C^\infty(M)$, $s, s_1, s_2 \in \Gamma(M, E)$. We will use the notation $\Gamma(U, E)$ to emphasize the vector space of smooth sections of $E|_U$ over an open subset U of the base manifold M. A *frame for the vector bundle* E over U is a set of k smooth sections $\{s_1, s_2, \ldots, s_k\}$, where $s_i : U \longrightarrow \pi^{-1}(U)$, such that $\{s_1(x), s_2(x), \ldots, s_k(x)\}$ is a basis for fiber E_x for any $x \in U$.

We give some examples of a vector bundles. The first example is a tangent bundle $TM = \bigcup\limits_{x \in M} T_xM$, where T_xM is a tangent space at x, of a smooth manifold M. Tangent bundle is a real vector bundle of rank n, where $\dim M = n$. The smooth sections of TM are the vector fields. Let $U \subset M$ be a neighborhood of x. Derivations $\left\{\frac{\partial}{\partial x^1}|_x, \frac{\partial}{\partial x^2}|_x, \ldots, \frac{\partial}{\partial x^n}|_x\right\}$ form a basis for T_xM at $x \in U$. The vector field $\mathscr{X} : M \longrightarrow TM$ can be represented locally as $\mathscr{X}|_U = \sum_{i=1}^{n} \mathscr{X}^i \frac{\partial}{\partial x^i}$ with coefficients $\mathscr{X}^i$ smooth functions. The second example is a cotangent bundle. The fiber T_x^*M at $x \in M$ of a cotangent bundle $T^*M = \bigcup_{x \in M} T_x^*M$ is the dual to T_xM. The smooth section of a cotangent bundle is a differential 1-form on a manifold M and its local expression is $\omega|_U = \sum_{i=1}^{n} w_i dx^i$, where $\{dx^1, dx^2, \ldots, dx^n\}$ is a dual basis to $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$, $(dx^i(\frac{\partial}{\partial x^j}) = \delta_j^i)$. The exterior algebra bundle $\wedge^p T^*M$, whose fiber at $x \in M$ is the antisymmetric tensor product of degree p of vector spaces T_x^*M and $\wedge T^*M = \bigoplus_{p=0}^{n} \wedge^p T^*M$. The smooth sections of the vector bundle $\wedge T^*M$ is a smooth differential forms and the space of differential forms is denoted by $\Omega(M)$. The elements of the space $\Omega^p(M) = \Gamma(M, \wedge^p T^*M)$ are the differential forms of degree p or briefly p-forms. In a neighborhood U of an arbitrary point $x \in M$ a differential p-form ω can be uniquely represented in local coordinates $x^1, x^2, \ldots, x^n$ by an expression

$$\omega|_U = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} \omega_{i_1 i_2 \ldots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p},$$

where $dx^I = \{dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}\}$, $I = (i_1, i_2, \ldots, i_p)$ and $1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n$, form a basis for $\wedge^p T^*M$. Suppose that $\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_p$ are the vector fields on the manifold M and ω is a p-form, then $\omega(\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_p)$ is a smooth function given by

$$\omega(\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_p)(x) := \omega_x(\mathscr{X}_1(x), \mathscr{X}_2(x), \ldots, \mathscr{X}_p(x)) \in \mathbb{R}$$

for $x \in U \subset M$. It means p-forms act on p-tuples of vector fields to give real-valued function.

Suppose that E is a vector bundle of rank k over M and $\{e_\alpha\}_{\alpha=1}^r$ is a frame over $U \subset M$ for E. Then the local representation of the smooth section $s \in \Gamma(U, E)$ is

$$s|_U = \sum_{\alpha=1}^{k} s^\alpha e_\alpha,$$

where s^α are smooth functions on U. For simplicity of notation we continue to write $s|_U = s^\alpha e_\alpha$ (Einstein summation convention assumed). It is worth mentioning that this assumption will be needed throughout the chapter.

Let us now consider differential forms with values in vector bundle E over M. Differential p-form ω is said to be E-valued p-form if

$$\omega_x(\mathscr{X}_1(x), \mathscr{X}_2(x), \ldots, \mathscr{X}_p(x)) \in \pi^{-1}(x) = E_x$$

and the section $\omega(\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_p) : U \longrightarrow \pi^{-1}(U)$, such that

$$x \mapsto \omega_x(\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_p),$$

is a smooth section of vector bundle E, where $\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_p$ are smooth vector fields on $U \subset M$ and $x \in U$. Therefore, E-valued differential form of degree p is a smooth section of the tensor product $\wedge^p T^*M \otimes E$. Hence the space of such forms is

$$\Omega^p(M, E) = \Gamma(M, \wedge^p T^*M \otimes E).$$

Following [47] it can be shown that there exists the isomorphism of spaces

$$\Omega^p(M, E) \cong \Omega^p(M) \otimes_{C^\infty(M)} \Gamma(M, E). \tag{3.1.1}$$

The image of $\omega \otimes s$ under the above isomorphism is denoted by $\omega \cdot s \in \Omega^p(M, E)$, for ω being the differential form of degree p and s being the smooth section of vector bundle E. Let us describe a local representation for a E-valued differential form. If $\{e_\alpha\}_{\alpha=1}^k$ is a frame for E over an open set U, then making use of the local representation of an arbitrary section $s \in \Gamma(U, E)$ and p-form $\omega \in \Omega^p(U)$ we get

$$\begin{aligned}(\omega \otimes s)|_U &= (\omega_{i_1 i_2 \ldots i_p} dx^{i_i} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}) \otimes (s^\alpha e_\alpha)\\ &= \omega_{i_1 i_2 \ldots i_p} s^\alpha (dx^{i_i} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}) \otimes e_\alpha\\ &\mapsto \omega_{i_1 i_2 \ldots i_p} s^\alpha (dx^{i_i} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}) \cdot e_\alpha,\end{aligned}$$

where $\mapsto$ stands for isomorphism 3.1.1. For the general case, a local representation for an arbitrary $\theta \in \Omega^p(U, E)$ is given by

$$\Omega^p(U, E) \cong [\Omega^p(U)]^k = \underbrace{\Omega^p(U) \times \ldots \times \Omega^p(U)}_{k\,\text{times}},$$

where

$$\theta|_U = \theta^\alpha_{i_1 i_2 \ldots i_p}(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}) \cdot e_\alpha.$$

Definition 3.1.2. A *connection* D on vector bundle E over manifold M is a linear map

$$D : \Gamma(M, E) \longrightarrow \Omega^1(M, E),$$

which satisfies

$$D(f \cdot s) = df \cdot s + f Ds, \tag{3.1.2}$$

where $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$.

Now we give a local description of a connection. Let $\{e_\alpha\}$ be a frame over U for a vector bundle E, equipped with a connection D. If $s \in \Gamma(U, E)$ is an arbitrary section of vector bundle E then the action of connection D on this section is

$$Ds = D(s^\alpha e_\alpha) = ds^\alpha e_\alpha + s^\alpha De_\alpha$$

We define the matrix of connection Θ by setting $\Theta = (\theta^\alpha_\beta)$, where $De_\beta = \theta^\alpha_\beta e_\alpha$. The entries θ^α_β are differential 1-forms. Therefore, the action of the connection D on sections of a vector bundle E can be represented

$$Ds = (ds^\beta + \theta^\beta_\alpha s^\alpha) e_\beta,$$

or, in the matrix form,

$$Ds = ds + \Theta \cdot s,$$

where the section s and its differential are considered as columns of k components, i.e.

$$s = \begin{pmatrix} s^1 \\ s^2 \\ \vdots \\ s^k \end{pmatrix}, \quad ds = \begin{pmatrix} ds^1 \\ ds^2 \\ \vdots \\ ds^k \end{pmatrix}.$$

The next aim is definition of the curvature of a connection. Let E be a vector bundle of rank k, equipped with a connection D. The curvature matrix Ψ associated with the connection matrix Θ is an $k \times k$ matrix of 2-forms defined as

$$\Psi = d\Theta + \Theta \wedge \Theta.$$

Componentwise, this is

$$\psi^\alpha_\beta = d\theta^\alpha_\beta + \theta^\alpha_\gamma \wedge \theta^\gamma_\beta.$$

The *curvature of connection* is defined to be the element $F \in \Omega^2(M, \mathrm{End}E)$, which locally have the form

$$\Psi = d\Theta + \Theta \wedge \Theta.$$

A connection D can be extended to E-valued differential forms of higher order $D : \Omega^p(M,E) \longrightarrow \Omega^{p+1}(M,E)$. Let $\{e_\alpha\}_{\alpha=1}^k$ be a frame over U for a vector bundle E. It is easy to see that operator $D^2 = D \circ D$ coincide with an operator of multiplication by F. Indeed,

$$\begin{aligned}(D^2 s)^\alpha &= d(ds^\alpha + \theta^\alpha_\beta s^\beta) + \theta^\alpha_\beta \wedge (ds^\beta + \theta^\beta_\gamma s^\gamma)\\ &= d\theta^\alpha_\beta s^\beta - \theta^\alpha_\beta \wedge ds^\beta + \theta^\alpha_\beta \wedge ds^\beta + (\theta^\alpha_\beta \wedge \theta^\beta_\gamma s^\gamma)\\ &= \underbrace{(d\theta^\alpha_\beta + \theta^\alpha_\gamma \wedge \theta^\gamma_\beta)}_{\psi^\alpha_\beta} s^\beta.\end{aligned}$$

Thus the curvature measures the failure of the sequence

$$\Gamma(M,E) \xrightarrow{D} \Omega^1(M,E) \xrightarrow{D} \Omega^2(M,E) \longrightarrow \dots$$

to be a cochain complex in the sense of de Rham cohomology.

Let Θ, Ψ be the local connection and its curvature forms with respect to the frame $\{e_\alpha\}_{\alpha=1}^r$ for the vector bundle E with a connection D. The curvature form satisfies the Bianchi identity $d\Psi = [\Psi, \Theta]$. Indeed,

$$\begin{aligned}d\psi^\alpha_\beta &= d(d\theta^\alpha_\beta + \theta^\alpha_\gamma \wedge \theta^\gamma_\beta) = d^2\theta^\alpha_\beta + d\theta^\alpha_\gamma \wedge \theta^\gamma_\beta - \theta^\alpha_\gamma \wedge d\theta^\gamma_\beta\\ &= (d\theta^\alpha_\gamma + \theta^\alpha_\sigma \wedge \theta^\sigma_\gamma) \wedge \theta^\gamma_\beta - \theta^\alpha_\sigma \wedge \theta^\sigma_\gamma \wedge \theta^\gamma_\beta\\ &- \theta^\alpha_\gamma \wedge (d\theta^\gamma_\beta + \theta^\gamma_\sigma \wedge \theta^\sigma_\beta) + \theta^\alpha_\gamma \wedge \theta^\gamma_\sigma \wedge \theta^\sigma_\beta\\ &= \psi^\alpha_\gamma \wedge \theta^\gamma_\beta - \theta^\alpha_\gamma \wedge \psi^\gamma_\beta = [\psi, \theta]^\alpha_\beta.\end{aligned}$$

It is necessary to examine how the connection matrix and the curvature matrix behave, if we pass from one frame to another. Suppose that $G = (g^\alpha_\beta) \in \mathrm{GL}(k, \mathbb{R})$ is a change of frame

$$\{e_\alpha\}_{\alpha=1}^k \longrightarrow \{(ge)_\alpha\}_{\alpha=1}^k,$$

i.e. $(ge)_\alpha = e_\beta g^\beta_\alpha = \tilde{e}_\alpha$, for a vector bundle E. Let Θ, Ψ and $\tilde{\Theta}$, $\tilde{\Psi}$ be the connection and curvature matrices in terms of the frames $\{e_\alpha\}_{\alpha=1}^k$ and

$\{(ge)_\alpha\}_{\alpha=1}^k = \{\tilde{e}_\alpha\}_{\alpha=1}^k$, respectively. Upon applying the connection D to the frames we get $De_\alpha = \theta_\alpha^\beta e_\beta$ and $D\tilde{e}_\alpha = \tilde{\theta}_\alpha^\beta \tilde{e}_\beta$. Therefore,

$$D(\tilde{e}_\alpha) = D(g_\alpha^\gamma e_\gamma) = \tilde{\theta}_\alpha^\beta g_\beta^\gamma e_\gamma,$$

on the other hand,

$$D(g_\alpha^\gamma e_\gamma) = dg_\alpha^\gamma e_\gamma + g_\alpha^\gamma \theta_\gamma^\sigma e_\sigma,$$

comparing coefficients, we get

$$\tilde{\theta}_\alpha^\beta g_\beta^\gamma e_\gamma = dg_\alpha^\gamma e_\gamma + g_\alpha^\beta \theta_\beta^\gamma e_\gamma$$

or in the matrix form

$$\tilde{\Theta} = G^{-1}\Theta G + G^{-1}dG.$$

It shows that the components of connection with respect to different frames are related by *gauge transformation*. Now we show that the curvature form transforms as

$$\tilde{\Psi} = G^{-1}\Psi G.$$

To this end, we use the fact that $dG^{-1}G + G^{-1}dG = 0$. We obtain

$$\begin{aligned}
\tilde{\Psi} &= d\tilde{\Theta} + \tilde{\Theta} \wedge \tilde{\Theta} = d(G^{-1}\Theta G + G^{-1}dG) \\
&+ (G^{-1}\Theta G G^{-1}dG) \wedge (G^{-1}\Theta G + G^{-1}dG) \\
&= (dG^{-1}) \wedge \Theta G + G^{-1}d\Theta G - G^{-1}\Theta \wedge dG + dG^{-1} \wedge dG \\
&+ G^{-1}\Theta \wedge \Theta G + G^{-1}\Theta \wedge dG + \\
&+ G^{-1}dG \wedge (G^{-1}\Theta G) + (G^{-1}dG) \wedge (G^{-1}dG) \\
&= -(G^{-1}dG) \wedge (G^{-1}\Theta G)G^{-1}(d\Theta + \Theta \wedge \Theta)G + dG^{-1} \wedge dG \\
&+ (G^{-1}dG) \wedge (G^{-1}\Theta G) - dG^{-1} \wedge dG = G^{-1}\Psi G.
\end{aligned}$$

3.2 N-connection on modules

In this section we propose a notion of N-connection, which is a generalization of a concept of Ω-connection on modules. A theory of Ω-connection can be found in an excellent survey [21] and this was a motivation of our generalization. We study the structure of an N-connection, define its curvature and prove the Bianchi identity [1, 4]. We define the dual N-connection and N-connection consistent with the Hermitian structure of a module. We prove that every projective module admits an N-connection. This section is based on [7, 8].

We begin this section by recalling the notion of Ω-connection given in [21]. Suppose that $\mathfrak{A}$ is an unital associative algebra over the field of complex numbers and $\mathcal{E}$ is a left module over $\mathfrak{A}$. Let Ω be a graded differential algebra with differential d, such that $\Omega^0 = \mathfrak{A}$, it means that the map $d : \mathfrak{A} \longrightarrow \Omega^1$ is a differential calculus over $\mathfrak{A}$. Since an subspace of elements of grading one can be viewed as a $(\mathfrak{A}, \mathfrak{A})$-bimodule, the tensor product $\Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ clearly has the structure of left $\mathfrak{A}$-module.

Definition 3.2.1. A linear map $\nabla : \mathcal{E} \longrightarrow \Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ is called an Ω-*connection* if it satisfies

$$\nabla(us) = du \otimes_{\mathfrak{A}} s + u\nabla(s)$$

for any $u \in \mathfrak{A}$ and $s \in \mathcal{E}$.

Similarly to the case of connections on vector bundles, this map has a natural extension $\nabla : \Omega \otimes_{\mathfrak{A}} \mathcal{E} \longrightarrow \Omega \otimes_{\mathfrak{A}} \mathcal{E}$ by setting

$$\nabla(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + (-1)^p \omega \nabla(s),$$

where $\omega \in \Omega^p$ and $s \in \mathcal{E}$.

Now our aim is to generalize a notion of Ω-connection taking graded q-differential algebra instead of graded differential algebra Ω. Let $\mathfrak{A}$ be an unital associative $\mathbb{C}$-algebra, Ω_q is a graded q-differential algebra with N-differential d and $\mathfrak{A} = \Omega_q^0$. Let $\mathcal{E}$ be a left $\mathfrak{A}$-module. Considering algebra Ω_q as the $(\mathfrak{A}, \mathfrak{A})$-bimodule we take the tensor product of left $\mathfrak{A}$-modules $\Omega_q \otimes_{\mathfrak{A}} \mathcal{E}$ which clearly has the structure of left $\mathfrak{A}$-module. To shorten the notation, we denote this left $\mathfrak{A}$-module by $\mathfrak{F}$. Taking into account that an algebra Ω_q can be viewed as the direct sum of $(\mathfrak{A}, \mathfrak{A})$-bimodules Ω_q^i we can split the left $\mathfrak{A}$-module $\mathfrak{F}$ into the direct sum of the left $\mathfrak{A}$-modules $\mathfrak{F}^i = \Omega_q^i \otimes_{\mathfrak{A}} \mathcal{E}$ [18], i.e. $\mathfrak{F} = \oplus_i \mathfrak{F}^i$, which means that $\mathfrak{F}$ inherits the graded structure of algebra Ω_q, and $\mathfrak{F}$ is the graded left $\mathfrak{A}$-module. It is worth noting that the left $\mathfrak{A}$-submodule $\mathfrak{F}^0 = \mathfrak{A} \otimes_{\mathfrak{A}} \mathcal{E}$ of elements of grading zero is isomorphic to a left $\mathfrak{A}$-module $\mathcal{E}$, where isomorphism $\varphi : \mathcal{E} \longrightarrow \mathfrak{F}^0$ can be defined for any $s \in \mathcal{E}$ by

$$\varphi(s) = e \otimes_{\mathfrak{A}} s, \tag{3.2.1}$$

where e is the identity element of algebra $\mathfrak{A}$. Since a graded q-differential algebra Ω_q can be viewed as the (Ω_q, Ω_q)-bimodule, the left $\mathfrak{A}$-module $\mathfrak{F}$ can be also considered as the left Ω_q-module [18] and we will use this structure to describe a concept of N-connection. Let us mention that multiplication by elements of Ω^i, where $i \neq 0$, does not preserve the graded structure of the left Ω_q-module $\mathfrak{F}$.

The tensor product $\mathfrak{F}$ has also the structure of the vector space over $\mathbb{C}$ where this vector space is the tensor product of the vector spaces Ω_q and $\mathcal{E}$. It is evident that $\mathfrak{F}$ is a graded vector space, i.e. $\mathfrak{F} = \oplus_i \mathfrak{F}^i$, where $\mathfrak{F}^i = \Omega_q^i \otimes_{\mathbb{C}} \mathcal{E}$. Due to the structure of vector space of $\mathfrak{F}$ we can introduce the notion of linear operator on $\mathfrak{F}$. We denote the vector space of linear operators on $\mathfrak{F}$ by $\mathfrak{L}(\mathfrak{F})$. The structure of the graded vector space of $\mathfrak{F}$ induces the structure of a graded vector space on $\mathfrak{L}(\mathfrak{F})$, and we shall denote the subspace of homogeneous linear operators of degree i by $\mathfrak{L}^i(\mathfrak{F})$.

Definition 3.2.2. An *N-connection* on the left Ω_q-module $\mathfrak{F}$ is a linear operator $\nabla_q : \mathfrak{F} \longrightarrow \mathfrak{F}$ of degree one satisfying the condition

$$\nabla_q(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + q^{|\omega|}\, \omega\, \nabla_q(s), \qquad (3.2.2)$$

where $\omega \in \Omega_q^k$, $s \in \mathcal{E}$, and $|\omega|$ is the degree of the homogeneous element of algebra Ω_q.

Making use of the previously introduced notations we can write $\nabla_q \in \mathfrak{L}^1(\mathfrak{F})$. It is worth pointing out that if $N = 2$ then $q = -1$, and in this particular case the Definition 3.2.2 gives us the algebraic analog of a classical connection. Indeed, making use of the Definition 3.1.2, we see that connection on vector bundle can be viewed as a linear map on a left module of sections of vector bundle, taking values a algebra of differential 1-forms with values in this vector bundle, which clearly has a structure of a left module over an algebra of smooth functions on a base manifold. In this case relation (3.1.2) is a particular case of (3.2.2) for $N = 2$. Hence a concept of a N-connection can be viewed as a generalization of a classical connection.

In the same manner as in Definition 3.2.2 one can define an N-connection on right modules. If $\mathcal{E}^R$ is a right $\mathfrak{A}$-module, a N-connection on $\mathfrak{G} = \mathcal{E}^R \otimes_{\mathfrak{A}} \Omega_q$ is a linear map $\nabla_q : \mathfrak{G} \longrightarrow \mathfrak{G}$ of degree one such that $\nabla_q(\xi \otimes_{\mathfrak{A}} \omega) = \xi \otimes_{\mathfrak{A}} d\omega + q^{\omega} \nabla_q(\xi)\omega$ for any $\xi \in \mathcal{E}^R$ and homogeneous element $\omega \in \Omega_q$.

Let $\mathcal{E}$ be a left $\mathfrak{A}$-module. The set of all homomorphisms of $\mathcal{E}$ into $\mathfrak{A}$ has the structure of the dual module of the left $\mathfrak{A}$-module $\mathcal{E}$ and is denoted by $\mathcal{E}^*$. It is evident that $\mathcal{E}^*$ is a right $\mathfrak{A}$-module.

Definition 3.2.3. A linear map $\nabla_q^*: \mathcal{E}^* \longrightarrow \mathcal{E}^* \otimes_{\mathfrak{A}} \Omega_q^1$ defined as follows

$$\nabla_q^*(\eta)(\xi) = d(\eta(\xi)) - \eta(\nabla_q(\xi)),$$

where $\xi \in \mathcal{E}$, $\eta \in \mathcal{E}^*$ and ∇_q is an N-connection on $\mathcal{E}$, is said to be the dual connection of ∇_q.

It is easy to verify that ∇_q^* has a structure of N-connection on the right module $\mathcal{E}^*$. Indeed, for any $f \in \mathfrak{A}$, $\eta \in \mathcal{E}^*$, $\xi \in \mathcal{E}$ we have

$$\begin{aligned} \nabla_q^*(\eta f)(\xi) &= d(\eta f(\xi)) - (\eta f)(\nabla_q \xi) = d(\eta(\xi) f) - \eta(\nabla_q \xi) f \\ &= d(\eta(\xi)) f + \eta(\xi) \otimes_{\mathfrak{A}} df - \eta(\nabla_q \xi) f = \eta(\xi) \otimes_{\mathfrak{A}} df + \nabla_q^*(\eta(\xi)) f. \end{aligned}$$

In order to define a Hermitian structure on a right $\mathfrak{A}$-module $\mathcal{E}$ we assume $\mathfrak{A}$ to be a graded q-differential algebra with involution $*$ such that the largest linear subset contained in the convex cone $C \in \mathfrak{A}$ generated by a^*a is equal to zero, i.e. $C \cap (-C) = 0$. The right $\mathfrak{A}$-module $\mathcal{E}$ is called a Hermitian module if $\mathcal{E}$ is endowed with a sesquilinear map $h: \mathcal{E} \times \mathcal{E} \longrightarrow \mathfrak{A}$ which satisfies

$$\begin{aligned} &h(\xi\omega, \xi\omega') = \omega^* h(\xi, \xi')\omega', \quad \forall \omega, \omega' \in \mathfrak{A}, \ \forall \xi, \xi' \in \mathcal{E}, \\ &h(\xi, \xi) \in C, \quad \forall \xi \in \mathcal{E} \ \text{ and } \ h(\xi, \xi) = 0 \Rightarrow \xi = 0. \end{aligned}$$

We have used the convention for sesquilinear map to take the second argument to be linear, therefore we define a Hermitian structure on right modules. In a similar manner one can define a Hermitian structure on left modules.

Definition 3.2.4. An N-connection ∇_q on a Hermitian right $\mathfrak{A}$-module $\mathcal{E}$ is said to be *consistent with a Hermitian structure of* $\mathcal{E}$ if it satisfies

$$dh(\xi, \xi') = h(\nabla_q(\xi), \xi') + h(\xi, \nabla_q(\xi')),$$

where $\xi, \xi' \in \mathcal{E}$.

Our next aim is to define a curvature of N-connection. Following [1] we start with

Proposition 3.2.5. *The N-th power of any N-connection ∇_q is the endomorphism of degree N of the left Ω_q-module $\mathfrak{F}$.*

Proof. It suffices to verify that for any homogeneous element $\omega \in \Omega_q$ an endomorphism $\nabla_q \in \mathfrak{L}^1(\mathfrak{F})$ satisfies the condition

$$\nabla_q^N(\omega \otimes_{\mathfrak{A}} s) = \omega\, \nabla_q^N(s).$$

We expand the k-th power of ∇_q as follows

$$\nabla_q^k(\omega \otimes_{\mathfrak{A}} s) = \sum_{m=0}^{k} q^{m|\omega|} \begin{bmatrix} k \\ m \end{bmatrix}_q d^{k-m}\omega\, \nabla_q^m(s), \tag{3.2.3}$$

where $\begin{bmatrix} k \\ m \end{bmatrix}_q$ are the q-binomial coefficients. Since d is the N-differential of a graded q-differential algebra Ω_q we have $d^N\omega = 0$. According to $\begin{bmatrix} N \\ m \end{bmatrix}_q = 0$ for $1 \leq m \leq N-1$, we see that in the case of $k = N$ the expansion (3.2.3) takes the following form

$$\nabla_q^N(\omega \otimes_{\mathfrak{A}} s) = q^{N|\omega|}\omega\, \nabla_q^N(s) = \omega\, \nabla_q^N(s) \tag{3.2.4}$$

and this clearly shows that ∇_q^N is the endomorphism of the left Ω_q-module $\mathfrak{F}$. □

This proposition allows us to define the curvature of N-connection as follows

Definition 3.2.6. The endomorphism $F = \nabla_q^N$ of degree N of the left Ω_q-module $\mathfrak{F}$ is said to be the *curvature of a N-connection* ∇_q.

Suppose that $\mathfrak{L}(\mathfrak{F})$ is the graded vector space. We proceed to show that $\mathfrak{L}(\mathfrak{F})$ has a structure of graded algebra. To this end, we take the product $A \circ B$ of two linear operators A, B of the vector space $\mathfrak{F}$ as an algebra multiplication. If $A : \mathfrak{F} \longrightarrow \mathfrak{F}$ is a homogeneous linear operator than we can extend it to the linear operator $L_A : \mathfrak{L}(\mathfrak{F}) \longrightarrow \mathfrak{L}(\mathfrak{F})$ on the whole graded algebra of linear operators $\mathfrak{L}(\mathfrak{F})$ by means of the graded q-commutator as follows

$$L_A(B) = [A, B]_q = A \circ B - q^{|A||B|} B \circ A, \tag{3.2.5}$$

where B is a homogeneous linear operator. It makes allowable to extend an N-connection ∇_q to the linear operator on the vector space $\mathfrak{L}(\mathfrak{F})$

$$\nabla_q(A) = [\nabla_q, A]_q = \nabla_q \circ A - q^{|A|}\, A \circ \nabla_q, \tag{3.2.6}$$

where A is a homogeneous linear operator. As it follows from the Definition (3.2.2), ∇_q is the linear operator of degree one on the vector space $\mathfrak{L}(\mathfrak{F})$, i.e. $\nabla_q : \mathfrak{L}^i(\mathfrak{F}) \longrightarrow \mathfrak{L}^{i+1}(\mathfrak{F})$, and ∇_q satisfies the graded q-Leibniz rule with respect to the algebra structure of $\mathfrak{L}(\mathfrak{F})$. Consequently the curvature F of an N-connection can be viewed as the linear operator of degree N on the vector space $\mathfrak{F}$, i.e. $F \in \mathfrak{L}^N(\mathfrak{F})$. Therefore one can act on F by N-connection ∇_q, and it holds that

Proposition 3.2.7. *For any N-connection ∇_q the curvature F of this connection satisfies the Bianchi identity $\nabla_q(F) = 0$.*

Proof. We have

$$\nabla_q(F) = [\nabla_q, F]_q = \nabla_q \circ F - q^N F \circ \nabla_q = \nabla_q^{N+1} - \nabla_q^{N+1} = 0.$$

□

The following theorem shows that not every left $\mathfrak{A}$-module admits an N-connection [8]. In analogy with the theory of Ω-connection [21] we can prove that there is an N-connection on every projective module. Let us first prove the following proposition.

Proposition 3.2.8. *If $\mathcal{E} = \mathfrak{A} \otimes V$ is a free $\mathfrak{A}$-module, where V is a $\mathbb{C}$-vector space, then $\nabla_q = d \otimes I_V$ is N-connection on $\mathcal{E}$ and this connection is flat, i.e. its curvature vanishes.*

Proof. Indeed, $\nabla_q : \mathfrak{A} \otimes V \longrightarrow \Omega_q^1 \otimes (\mathfrak{A} \otimes V)$ and

$$\begin{aligned} \nabla_q(f(g \otimes v)) &= (d \otimes I_V)(f(g \otimes v)) = d(fg) \otimes v = \\ &= (dfg) \otimes v + f(dg \otimes v) = df \otimes_{\mathfrak{A}} (g \otimes v) + f\nabla_q(g \otimes v), \end{aligned}$$

where $f, g \in \mathfrak{A}$, $v \in V$. As $d^N = 0$ and q is the primitive Nth root of unity, we get $\nabla_q^N(f(g \otimes v)) = \sum\limits_{k+m=N} \left[\begin{array}{c} N \\ m \end{array} \right]_q d^k f(d^m g \otimes v) = 0$, i. e. the curvature of such a N-connection vanishes. □

Theorem 3.2.9. *Every projective module admits an N-connection.*

Proof. Let $\mathcal{P}$ be a projective module. From the theory of modules it is known that a module $\mathcal{P}$ is projective if and only if there exists a module $\mathcal{N}$ such that $\mathcal{E} = \mathcal{P} \oplus \mathcal{N}$ is a free module [37]. A free left $\mathfrak{A}$-module $\mathcal{E}$ can be represented as the tensor product $\mathfrak{A} \otimes V$, where V is a $\mathbb{C}$-vector space. A linear map $\nabla_q = \pi \circ (d \otimes I_V) : \mathcal{P} \longrightarrow \Omega^1_q \otimes_{\mathfrak{A}} \mathcal{P}$ is a N-connection on a projective module $\mathcal{P}$, where $d \otimes I_V$ is a N-connection on a left $\mathfrak{A}$-module $\mathcal{E}$, π is the projection on the first summand in the direct sum $\mathcal{P} \oplus \mathcal{N}$ and $\pi(\omega \otimes_{\mathfrak{A}} (g \otimes v)) = \omega \otimes_{\mathfrak{A}} \pi(g \otimes v) = \omega \otimes_{\mathfrak{A}} m$, where $\omega \in \Omega^1_q$, $g \in \mathfrak{A}$, $v \in V$, $m \in \mathcal{P}$. Taking into account Proposition 3.2.8 we get

$$\begin{aligned}\nabla_q(fm) &= \pi((d \otimes I_V)(fm)) = \pi(df \otimes_{\mathfrak{A}} m + f dm) = \\ &= df \otimes_{\mathfrak{A}} \pi(m) + f\nabla_q(m) = df \otimes_{\mathfrak{A}} m + f\nabla_q(m),\end{aligned}$$

where $f \in \mathfrak{A}$, $m \in \mathcal{P}$. □

3.3 Generalization of superconnection

Superconnections were introduced by D. Quillen and V. Mathai [42] as geometric objects associated with graded vector bundles whereby the integer grading by differential form degree is replaced by a $\mathbb{Z}_2$-grading. In order to generalize the notion of a superconnection to any integer $N > 2$ we need a $\mathbb{Z}_N$-graded analog of an algebra of differential forms, assuming that a vector bundle has also a $\mathbb{Z}_N$-graded structure we can elaborate a generalization of superconnection following the scheme proposed by D. Quillen and V. Mathai. This section is based on [1, 6] where $\mathbb{Z}_N$-connection was defined and studied.

Following [11] we begin this section with the brief description of Quillen's concept of superconnection. Suppose that the vector bundle E over M has a $\mathbb{Z}_2$-graded structure $E = E^+ \oplus E^-$, i.e. the fibers of this vector bundle $E_x = E^+_x \oplus E^-_x$, $x \in M$, are $\mathbb{Z}_2$-graded complex vector spaces. Let $\Omega(M) = \oplus_p \Omega^p(M)$ be a graded algebra of smooth differential forms on the base manifold M with a natural $\mathbb{Z}$-graded structure defined by the degree of differential form. Let us also mention that $\Omega(M)$ is a $\mathbb{Z}_2$-graded algebra, where the grading of a homogeneous differential form equals to its degree modulo 2. The space $\Omega(M, E)$ of smooth E-valued differential forms on M has a $\mathbb{Z} \times \mathbb{Z}_2$-grading, but we will be concerned with its total $\mathbb{Z}_2$-grading, which will be denoted by

$$\Omega(M, E) = \Omega^+(M, E) \oplus \Omega^-(M, E),$$

where

$$\Omega^{\pm}(M,E) = \oplus_p\Big(\Omega^{2p}(M,E^{\pm}) \oplus \Omega^{2p+1}(M,E^{\mp})\Big). \tag{3.3.1}$$

Keeping in mind the isomorphism of spaces (3.1.1), we see that the total grading of a homogeneous differential form with values in E is the sum of two gradings, where the first grading is determined by the graded structure of the algebra of differential forms and the second comes from the graded structure of $\mathbb{Z}_2$-graded bundle E. For instance, 1-form $\theta \in \Omega^1(M,E)$ such that $\theta(\mathscr{X}) \in \Gamma(M,E^-)$, $\forall \mathscr{X} \in \Gamma(M,TM)$, is a form of even grading under the total grading of the differential form θ.

A *superconnection* on a vector bundle E is a linear operator of odd degree

$$\mathbb{A} : \Omega^{\pm}(M,E) \longrightarrow \Omega^{\mp}(M,E)$$

which satisfies Leibniz rule

$$\mathbb{A}(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{|\omega|}\omega \wedge \mathbb{A}(\theta), \tag{3.3.2}$$

where $\omega \in \Omega(M)$ and $\theta \in \Omega(M,E)$.

We now in a position to show how the generalization of superconnection can be constructed. To do this, we use the notion of N-connection, introduced in the previous section. Our approach is based on the algebraic structures such as differential algebras and modules. Let $\mathfrak{A}$ be an unital associative $\mathbb{C}$-algebra and $d : \mathfrak{A} \longrightarrow \Omega_q$ be an N-differential calculus over $\mathfrak{A}$. The basic difference from the concept of an N-connection is that we suppose here that there is a graded structure on a left $\mathfrak{A}$-module $\mathcal{E}$. Let $\mathcal{E} = \oplus_{k\in\mathbb{Z}_N}\mathcal{E}^k$ be a $\mathbb{Z}_N$-graded left $\mathfrak{A}$-module. In the same manner like it was done in the previous section we construct the tensor product $\mathfrak{F} = \Omega_q \otimes_{\mathfrak{A}} \mathcal{E}$, which may be considered as a left Ω_q-module. Since a graded q-differential algebra Ω_q can also be viewed as a $(\mathfrak{A},\mathfrak{A})$-bimodule, the tensor product $\mathfrak{F}$ has the left $\mathfrak{A}$-module structure. It should be mentioned that $\mathfrak{F}$, being the tensor product of two vector spaces over $\mathbb{C}$, has also a structure of a $\mathbb{C}$-vector space. The space of endomorphisms of the vector space $\mathfrak{F}$, denoted by $\mathrm{End}_{\mathbb{C}}(\mathfrak{F})$, becomes a graded differential algebra if the multiplication is defined by a product of two endomorphisms.

The left Ω_q-module $\mathfrak{F}$ becomes a graded left Ω_q-module if we construct $\mathbb{Z}_N$-graded structure on it as follows: given two homogeneous elements $\omega \in$

$\Omega_q, s \in \mathcal{E}$ one defines the total grading of the element $\omega \otimes_{\mathfrak{A}} s \in \mathfrak{F}$ by

$$|\omega \otimes_{\mathfrak{A}} s| = |\omega| + |s|.$$

Then

$$\mathfrak{F} = \oplus_{k\in\mathbb{Z}_N}\mathfrak{F}^k, \quad \mathfrak{F}^k = \oplus_{m+l=k}\mathfrak{F}^{m,l} = \oplus_{m+l=k}\, \Omega^m \otimes_{\mathfrak{A}} \mathcal{E}^l,$$

where $k, l, m \in \mathbb{Z}_N$. It should be mentioned that if we consider the tensor product $\mathfrak{F}$ as the left Ω_q-module then multiplication by a homogeneous element $\omega \in \Omega_q$ of grading k maps an element $\zeta \in \mathfrak{F}^{m,l}$ into the element $\omega\zeta \in \mathfrak{F}^{m+k,l}$. If we consider the tensor product $\mathfrak{F}$ as the left $\mathfrak{A}$-module then multiplication by any element $u \in \mathfrak{A}$ preserves the $\mathbb{Z}_N$-graded structure of $\mathfrak{F}$. Hence if $m + l = k$ then $\mathfrak{F}^{m,l}$ is the left $\mathfrak{A}$-submodule of a left $\mathfrak{A}$-module $\mathfrak{F}^k$. The $\mathbb{Z}_N$-graded left $\mathfrak{A}$-module $\mathfrak{F}^{0,l} = \oplus_l\Omega_q^0 \otimes_{\mathfrak{A}} \mathcal{E}^l$ is isomorphic to a left $\mathfrak{A}$-module $\mathcal{E}$. We define the isomorphism $\varphi : \mathcal{E} \longrightarrow \mathfrak{F}^{0,l}$ analogously with 3.2.1 by $\varphi(s) = e \otimes_{\mathfrak{A}} s$, where e is the identity element of $\mathfrak{A}$. We emphasize that this isomorphism preserves the graded structures of the left $\mathfrak{A}$-modules $\mathcal{E}$ and $\mathfrak{F}^{0,l}$, i.e. $\varphi : \mathcal{E}^l \longrightarrow {\Omega_q}^0 \otimes_{\mathfrak{A}} \mathcal{E}^l$.

Definition 3.3.1. An $\mathbb{Z}_N$-*connection* on the graded left Ω_q-module $\mathfrak{F}$ is an endomorphism $\mathbb{A}_{\mathbb{Z}_N}$ of degree 1 of the vector space $\mathfrak{F}$ satisfying the condition

$$\mathbb{A}_{\mathbb{Z}_N}(\omega\, \zeta) = d(\omega)\, \zeta + q^{|\omega|}\omega\, \mathbb{A}_{\mathbb{Z}_N}(\zeta),$$

where $\omega \in \Omega_q$, $\zeta \in \mathfrak{F}$, and d is the N-differential of a graded q-differential algebra Ω_q.

An $\mathbb{Z}_N$-connection $\mathbb{A}_{\mathbb{Z}_N}$ can be extended to act on the $\mathbb{Z}_N$-graded algebra $\mathrm{End}_{\mathbb{C}}(\mathfrak{F})$ in a way consistent with the graded q-Leibniz rule if we define for any $A \in \mathrm{End}_{\mathbb{C}}(\mathfrak{F})$

$$\mathbb{A}_{\mathbb{Z}_N}(A) = [\mathbb{A}_{\mathbb{Z}_N}, A]_q = \mathbb{A}_{\mathbb{Z}_N} \circ A - q^{|A|} A \circ \mathbb{A}_{\mathbb{Z}_N}. \tag{3.3.3}$$

Consequently $\mathbb{A}_{\mathbb{Z}_N} : \mathrm{End}^k_{\mathbb{C}}(\mathfrak{F}) \longrightarrow : \mathrm{End}^{k+1}_{\mathbb{C}}(\mathfrak{F})$ and

$$\mathbb{A}_{\mathbb{Z}_N}(AB) = \mathbb{A}_{\mathbb{Z}_N}(A) \circ B + q^{|A|} A \circ \mathbb{A}_{\mathbb{Z}_N}(B).$$

We proceed to show that in the case of $N = 2$ an $\mathbb{Z}_N$-connection can be realized as a superconnection. Geometrically speaking, suppose that E is a superbundle with a base M. In this case $\Omega_q = \oplus_p \Omega^p_q(M)$ is the algebra of differential forms on a manifold M and d is the exterior differential of this algebra. Let $\mathfrak{A} = \Gamma^0(M,E) = C^\infty(M)$ be the algebra of smooth functions on a manifold M, and $\Gamma(M,E)$ is the left $\mathbb{Z}_2$-graded $\Gamma^0(M,E)$-module of smooth sections of a superbundle E. The tensor product $\mathfrak{F} = \Omega_q \otimes_{\mathfrak{A}} \mathcal{E}$ is the space of E-valued smooth differential forms on M. The space $\mathrm{End}_{\mathbb{C}}(\mathfrak{F})$ is the space of differential forms on a manifold M with the values in the superbundle $\mathrm{End}(E)$, the q-commutator becomes the supercommutator. Summarizing, the definition of an $\mathbb{Z}_N$-connection coincides in this special case with the definition of a superconnection.

In order to better understand what an $\mathbb{Z}_N$-connection consists of, we use an algebraic analog of a covariant derivative. The left $\mathfrak{A}$-modules $\mathfrak{F}^k$, $\mathfrak{F}^{k+1}$ can be split into the direct sums

$$\begin{aligned} \mathfrak{F}^k &= \bigoplus_{m+l=k} \mathfrak{F}^{m,l} = \mathfrak{F}^{0,k} \oplus \mathfrak{F}^{1,k-1} \oplus \mathfrak{F}^{2,k-2} \oplus \ldots \oplus \mathfrak{F}^{N-1,k+1}, \\ \mathfrak{F}^{k+1} &= \bigoplus_{m+l=k+1} \mathfrak{F}^{m,l} = \mathfrak{F}^{0,k+1} \oplus \mathfrak{F}^{1,k} \oplus \mathfrak{F}^{2,k-1} \oplus \ldots \oplus \mathfrak{F}^{N-1,k+2}. \end{aligned}$$

Let us introduce following projections of the left $\mathfrak{A}$-modules onto their $\mathfrak{A}$-submodules

$$p_{i,j} : \mathfrak{F} \longrightarrow \mathfrak{F}^{i,j},\ p_i : \mathfrak{F} \longrightarrow \oplus_l \mathfrak{F}^{i,l},\ \pi_k : \Omega_q \longrightarrow \Omega_q^k, \rho_l : \mathcal{E} \longrightarrow \mathcal{E}^l.$$

Each projection is the homomorphism of the corresponding left $\mathfrak{A}$-modules, $p_{k,l} = \pi_k \otimes_{\mathfrak{A}} \rho_l$ and

$$p_{k,l}(\omega \otimes_{\mathfrak{A}} s) = \pi_k(\omega) \otimes_{\mathfrak{A}} \varphi_l(s), \quad \forall \omega \in \Omega, s \in \mathcal{E}.$$

The pair (Ω_q^1, d) is the differential calculus over an algebra $\mathfrak{A}$ and $\mathcal{E}$ is a left $\mathfrak{A}$-module.

Proposition 3.3.2. *The linear map $D = p_1 \circ \mathbb{A}_{\mathbb{Z}_N} \circ \varphi$ is the covariant derivative on a left $\mathfrak{A}$-module $\mathcal{E}$ with respect to the differential calculus (Ω_q^1, d). The covariant derivative D preserves the $\mathbb{Z}_N$-graded structures of the left $\mathfrak{A}$-modules $\mathcal{E}$ and $\Omega_q^1 \otimes_{\mathfrak{A}} \mathcal{E}^k$, i.e. $D : \mathfrak{F}^k \longrightarrow \mathfrak{F}^{1,k}$.*

Proof. The proof is based on the following observation. For any $f \in \mathfrak{A}$, $s \in \mathcal{E}$ we have

$$D(fs) = p_1(\mathbb{A}_{\mathbb{Z}_N}(\varphi(fs))) = p_1(\mathbb{A}(f\varphi(s))) = p_1(df\varphi(s) + f\mathbb{A}\varphi(s))$$

$$
\begin{aligned}
&= p_1(df(e \otimes_{\mathfrak{A}} s)) + fp_1(\mathbb{A}\varphi(s)) = p_1(df \otimes_{\mathfrak{A}} s) + fD(s) = \\
&= \sum_l p_{1,l}(du \otimes_{\mathfrak{A}} s) + fD(s) = \pi_1(df) \otimes_{\mathfrak{A}} \sum_l \varphi_l(s) + fD(s) \\
&= df \otimes_{\mathfrak{A}} s + fD(s). \qquad (3.3.4)
\end{aligned}
$$

□

The task is now to define a curvature of $\mathbb{Z}_N$-connection. Analogously to Proposition 3.2.5 one can show that the N-th power of an endomorphism $\mathbb{A}_{\mathbb{Z}_N} \in \mathrm{End}^1_{\mathbb{C}}(\mathfrak{F})$ is the grading zero endomorphism of the left $\mathfrak{A}$-module $\mathfrak{F}$.

Definition 3.3.3. The *curvature F of an $\mathbb{Z}_N$-connection* $\mathbb{A}_{\mathbb{Z}_N}$ is the endomorphism $\mathbb{A}^N_{\mathbb{Z}_N}$ of grading zero of the left $\mathbb{Z}_N$-graded Ω_q-module $\mathfrak{F}$:

$$
F = \mathbb{A}^N_{\mathbb{Z}_N} \in \mathrm{End}^0_{\Omega_q}(\mathfrak{F}).
$$

At the end of this section we consider the example of $\mathbb{Z}_N$-connection [1]. We extend the N-differential of a graded q-differential algebra Ω_q to the $\mathbb{Z}_N$-graded left Ω_q-module $\mathfrak{F}$ in a way consistent with the graded q-Leibniz rule by putting $d(\omega \otimes_{\mathfrak{A}} s) = d(\omega) \otimes_{\mathfrak{A}} s$, where $\omega \in \Omega_q$, $s \in \mathcal{E}$. It is evident that $d \in \mathrm{End}_{\mathbb{C}}(\mathfrak{F})$. Let L be an endomorphism of grading 1 of a left $\mathfrak{A}$-module $\mathcal{E}$, i.e. $L \in \mathrm{End}^1_{\mathfrak{A}}(\mathcal{E})$. This endomorphism can be extended to the Ω_q-endomorphism of the left Ω_q-module $\mathfrak{F}$ in a way consistent with the $\mathbb{Z}_N$-graded structure of $\mathfrak{F}$ by means of $L(\omega \otimes_{\mathfrak{A}} s) = q^{|\omega|}\, \omega \otimes_{\mathfrak{A}} L(s)$. Indeed if $\zeta = \omega \otimes_{\mathfrak{A}} s \in \mathfrak{F}$ then

$$
L(\theta\zeta) = L(\theta(\omega \otimes_{\mathfrak{A}} s)) = L((\theta\omega) \otimes_{\mathfrak{A}} s) = q^{|\theta|+|\omega|}(\theta\omega) \otimes_{\mathfrak{A}} L(s) \qquad (3.3.5)
$$

Obviously $L \in \mathrm{End}^1_{\Omega_q}(\mathfrak{F}) \subset \mathrm{End}^1_{\mathbb{C}}(\mathfrak{F})$. The endomorphism $\mathbb{A}_{\mathbb{Z}_N} = d + L$ of grading 1 of the vector space $\mathfrak{F}$ is a $\mathbb{Z}_N$-connection. Indeed for any $\omega \in \Omega_q, \zeta \in \mathfrak{F}$ we have

$$
\begin{aligned}
\mathbb{A}_{\mathbb{Z}_N}(\omega\zeta) &= (d+L)(\omega\zeta) = d(\omega\zeta) + L(\omega\zeta) \\
&= d(\omega)\,\zeta + q^{|\omega|}\omega\, d(\zeta) + q^{|\omega|}\omega L(\zeta) \\
&= d(\omega)\zeta + q^{|\omega|}\omega\, \mathbb{A}_{\mathbb{Z}_N}(\zeta).
\end{aligned}
$$

We can decompose L into the homogeneous parts $L_{ij}, i, j \in \mathbb{Z}_N$ with respect to the $\mathbb{Z}_N$-graduation of $\mathcal{E}$, where $L_{ij} : \mathcal{E}^j \longrightarrow \mathcal{E}^i$. This components form the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & L_{0,N-1} \\ L_{1,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & L_{2,1} & 0 & \dots & 0 & 0 \\ 0 & 0 & L_{3,2} & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & L_{N-2,N-1} & 0 \end{pmatrix}.$$

Let us denote by $\{d^m, L_1\, L_2 \dots L_k\}$, where m, k are non-negative integers, the sum of all possible products made up of the mappings $d, L_1, L_2, \dots, L_k$, where each product contains m-times the differential d and k mappings $L_1, L_2, \dots, L_k$ succeeding in the same order. For $m = 2, k = 1$ we have $\{d^2, L\} = d^2\, L + d\, L\, d + L\, d^2$. The curvature of the $\mathbb{Z}_N$-connection $\mathbb{A}_{\mathbb{Z}_N}$ can be written as follows $F = \mathbb{A}^N_{\mathbb{Z}_N} = \sum_{m+k=N}\{d^m, L^k\}$. Using the matrix associated to L we obtain the $N \times N$-matrix corresponding to the curvature F, where the entry F_{ij} of this matrix can be written as follows

$$F_{ij} = \sum_{m+k=N, i,j\in\mathbb{Z}_N} \{d^m, L_{i,i-1}\, L_{i-1,i-2} \dots L_{j+1,j}\},$$

where m, k are non-negative integers running $m, k = 0, 1, \dots, N$, and each product in $\{d^m, L_{i,i-1}\, L_{i-1,i-2} \dots L_{j+1,j}\}$ contains k entries of the matrix associated to L which means that $i - j = k$. For instance if $N = 2$ we obtain the matrix of a superconnection $D = d + L$ and the matrix of its curvature which can be written in the standard notations of the supergeometry $\mathcal{E}_{\bar{0}} = \mathcal{E}^+, \mathcal{E}_{\bar{1}} = \mathcal{E}^-, L^+ = L_{\bar{0}\bar{1}}, L^- = L_{\bar{1}\bar{0}}$ as follows

$$L \longrightarrow \begin{pmatrix} 0 & L^- \\ L^+ & 0 \end{pmatrix}, \quad F \longrightarrow \begin{pmatrix} L^-L^+ & dL^- \\ dL^+ & L^+L^- \end{pmatrix}. \tag{3.3.6}$$

3.4 Local structure of an N-connection

In this section we consider an algebraic analog of the local structure of a vector bundle. Connection on the vector bundle of finite rank over a finite dimensional smooth manifold can be studied locally by choosing a local trivialization of the vector bundle and this leads to the basis for the module of sections of this vector bundle. This section is based on [7, 9]

Let us concentrate now on an algebraic analog of the local structure of an N-connection ∇_q (3.2.2). For this purpose we assume $\mathcal{E}$ to be a finitely generated free left $\mathfrak{A}$-module. Let $\mathfrak{e} = \{\mathfrak{e}_\mu\}_{\mu=1}^r$ be a basis for a left module $\mathcal{E}$. Any element $s \in \mathcal{E}$ can be expressed as $s = s^\mu \mathfrak{e}_\mu$. As it was mentioned above $\mathcal{E} \cong \mathfrak{F}^0$ (3.2.1). Therefore the basis for a module $\mathcal{E}$ induces the basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, where $\mathfrak{f}_\mu = e \otimes_{\mathfrak{A}} \mathfrak{e}_\mu$, for the left $\mathfrak{A}$-module $\mathfrak{F}^0$. For any $\xi \in \mathfrak{F}^0$ we have $\xi = \xi^\mu \mathfrak{f}_\mu$. Taking into account that $\mathfrak{F}^0 \subset \mathfrak{F}$ and $\mathfrak{F}$ is the left Ω_q-module we can multiply the elements of the basis $\mathfrak{f}$ by elements of a graded q-differential algebra Ω_q. It is easily seen that if $\omega \in \Omega_q^i$ then for any μ we have $\omega \mathfrak{f}_\mu \in \mathfrak{F}^i$. Consequently we can express any element of the $\mathfrak{F}^i$ as a linear combination of $\mathfrak{f}_\mu$ with coefficients from Ω_q^i. Indeed let $\omega \otimes_{\mathfrak{A}} s$ be an element of $\mathfrak{F}^i = \Omega^i \otimes_{\mathfrak{A}} \mathcal{E}$. Then

$$\begin{aligned} \omega \otimes_{\mathfrak{A}} s &= (\omega\, e) \otimes_{\mathfrak{A}} (s^\mu \mathfrak{e}_\mu) = (\omega\, e\, s^\mu) \otimes_{\mathfrak{A}} \mathfrak{e}_\mu \\ &= (\omega s^\mu\, e) \otimes_{\mathfrak{A}} \mathfrak{e}_\mu = \omega s^\mu\, (e \otimes_{\mathfrak{A}} \mathfrak{e}_\mu) = \omega^\mu \mathfrak{f}_\mu, \end{aligned}$$

where $\omega^\mu = \omega s^\mu \in \Omega_q^i$.

Let $\mathfrak{F}^0$ be a finitely generated free module with a basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, and $s = s^\mu \mathfrak{f}_\mu \in \mathfrak{F}^0$, where $s^\mu \in \mathfrak{A}$. Since N-connection ∇_q is a linear operator of degree one, it follows that $\nabla_q(s) \in \mathfrak{F}^1$, and making use of q-Leibniz rule (3.2.2) we can express the element $\nabla_q(s)$ as follows

$$\nabla_q(s) = \nabla_q(s^\mu \mathfrak{f}_\mu) = ds^\mu \otimes_{\mathfrak{A}} \mathfrak{f}_\mu + s^\mu\, \nabla_q(\mathfrak{f}_\mu). \tag{3.4.1}$$

Let $\mathrm{Mat}_r(\Omega_q)$ be the vector space of square matrices of order r whose entries are the elements of a graded q-differential algebra Ω_q. If each entry of a matrix $\Theta = (\theta^\nu_\mu)$ is an element of a homogeneous subspace Ω_q^i, i.e. $\theta^\nu_\mu \in \Omega_q^i$ then Θ will be refereed to as a homogeneous matrix of degree i and we shall denote the vector space of such matrices by $\mathrm{Mat}_r^i(\Omega_q)$. Obviously $\mathrm{Mat}_r(\Omega_q) = \oplus_i \mathrm{Mat}_r^i(\Omega_q)$. The vector space $\mathrm{Mat}_r(\Omega_q)$ of $r \times r$-matrices becomes the associative unital graded algebra if we define the product of two matrices $\Theta = (\theta^\nu_\mu), \Theta' = (\theta'^\nu_\mu) \in \mathrm{Mat}_r(\Omega_q)$ as follows

$$(\Theta\, \Theta')^\nu_\mu = \theta^\sigma_\mu\, \theta'^\nu_\sigma. \tag{3.4.2}$$

If $\Theta, \Theta' \in \mathrm{Mat}_r(\Omega_q)$ are homogeneous matrices then we define the graded q-commutator by $[\Theta, \Theta']_q = \Theta\, \Theta' - q^{|\Theta||\Theta'|} \Theta'\, \Theta$. We extend the N-differential

d of a graded q-differential algebra Ω_q to the algebra $\mathrm{Mat}_r(\Omega_q)$ as follows $d\Theta = d(\theta^\nu_\mu) = (d\theta^\nu_\mu)$.

Since any element of a left $\mathfrak{A}$-module $\mathfrak{F}^1$ can be expressed in terms of the basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$ with coefficients from Ω^1_q, we have

$$\nabla_q(\mathfrak{f}_\mu) = \theta^\nu_\mu\, \mathfrak{f}_\nu, \tag{3.4.3}$$

where $\theta^\nu_\mu \in \Omega^1_q$. In analogy with the classical theory of connections on vector bundles we introduce a notion of a matrix of N-connection.

Definition 3.4.1. An $r \times r$-matrix $\Theta = (\theta^\nu_\mu)$, whose entries θ^ν_μ are the elements of Ω^1_q i.e. $\Theta \in \mathrm{Mat}^1_r(\Omega_q)$, is said to be a *matrix of an N-connection* ∇_q with respect to the basis $\mathfrak{f}$ of the left $\mathfrak{A}$-module $\mathfrak{F}^0$.

Combining (3.4.1) with (3.4.3) we obtain

$$\nabla_q(s) = (ds^\mu + s^\nu \theta^\mu_\nu)\, \mathfrak{f}_\mu. \tag{3.4.4}$$

Let $\mathfrak{f}' = \{\mathfrak{f}'_\mu\}_{\mu=1}^r$ be another basis for the left $\mathfrak{A}$-module $\mathfrak{F}^0$ with the same number of elements (this will always be the case if $\mathfrak{A}$ is a division algebra or if $\mathfrak{A}$ is commutative). Then $\mathfrak{f}'_\mu = g^\nu_\mu \mathfrak{f}_\nu$, where $G = (g^\nu_\mu) \in \mathrm{Mat}^0_r(\Omega_q)$ is a transition matrix from the basis $\mathfrak{f}$ to the basis $\mathfrak{f}'$. It is well known [33] that in the case of finitely generated free module transition matrix is an invertible matrix. If we denote by θ'^μ_ν the coefficients of ∇_q with respect to a basis $\mathfrak{f}'$ and $\tilde{g}^\mu_\nu$ are the entries of the inverse matrix G^{-1} then

$$\theta'^\mu_\nu = dg^\sigma_\nu \tilde{g}^\mu_\sigma + g^\sigma_\nu \theta^\tau_\sigma \tilde{g}^\mu_\tau,$$

and this clearly shows that the components of ∇_q with respect to different bases of module $\mathfrak{F}^0$ are related by the gauge transformation.

Our next aim is to express the components of the curvature F of a N-connection ∇_q in the terms of the entries of the matrix Θ of a N-connection ∇_q. Computation in successive steps allows us to introduce polynomials $\psi^{l,\mu}_\nu \in \Omega^l_q$ on the entries of the matrix of N-connection and their differentials. We have

$$\begin{aligned} \nabla_q(s) &= (ds^\mu + s^\nu \theta^\mu_\nu)\, \mathfrak{f}_\mu, \\ \psi^{1,\mu}_\nu &:= \theta^\mu_\nu, \end{aligned}$$

$$
\begin{aligned}
\nabla_q^2(s) &= (d^2 s^\mu + [2]_q ds^\nu \theta_\nu^\mu + s^\nu (d\theta_\nu^\mu + q\theta_\nu^\sigma \theta_\sigma^\mu))\mathfrak{f}_\mu, \\
\psi_\nu^{2,\mu} &:= d\theta_\nu^\mu + q\,\theta_\nu^\sigma \theta_\sigma^\mu,
\end{aligned} \tag{3.4.5}
$$

$$
\begin{aligned}
\nabla_q^3(s) &= \Big(d^3 s^\mu + [3]_q d^2 s^\nu \theta_\nu^\mu + [3]_q ds^\nu (d\theta_\nu^\mu + q\theta_\nu^\sigma \theta_\sigma^\mu) \\
&+ s^\nu (d^2\theta_\nu^\mu + (q+q^2) d\theta_\nu^\sigma \theta_\sigma^\mu + q^2 \theta_\nu^\sigma d\theta_\sigma^\mu + q^3 \theta_\nu^\tau \theta_\tau^\sigma \theta_\sigma^\mu)\Big)\mathfrak{f}_\mu, \\
\psi_\nu^{(3,k)\mu} &:= d^2\theta_\nu^\mu + (q+q^2)\, d\theta_\nu^\sigma \theta_\sigma^\mu + q^2\, \theta_\nu^\sigma\, d\theta_\sigma^\mu + q^3\, \theta_\nu^\tau \theta_\tau^\sigma \theta_\sigma^\mu
\end{aligned} \tag{3.4.6}
$$

Therefore, the kth power of N-connection ∇_q has the following form

$$
\begin{aligned}
\nabla_q^k(s) &= \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_q d^{k-l} s^\mu\, \psi_\mu^{l,\nu}\, \mathfrak{f}_\nu \\
&= (d^k s^\mu\, \psi_\mu^{0,\nu} + [k]_q\, d^{k-1} s^\mu\, \psi_\mu^{1,\nu} + \ldots + s^\mu\, \psi_\mu^{k,\nu})\, \mathfrak{f}_\nu,
\end{aligned} \tag{3.4.7}
$$

We can calculate the polynomials $\psi_\mu^{l,\nu}$ by means of the following recursion formula

$$
\psi_\mu^{l,\nu} = d\psi_\mu^{l-1,\nu} + q^{l-1}\, \psi_\mu^{l-1,\sigma}\, \theta_\sigma^\nu, \tag{3.4.8}
$$

or in the matrix form

$$
\Psi^l = d\Psi^{l-1} + q^{l-1}\, \Psi^{l-1}\, \Theta, \tag{3.4.9}
$$

We begin with the polynomial $\psi_\mu^{0,\nu} = \delta_\mu^\nu\, e \in \mathfrak{A}$, and e is the identity element of $\mathfrak{A} \subset \Omega_q$. From (3.4.7) it follows that if $k = N$ then the first term $d^N \xi^\mu\, \psi_\mu^{(0,N)\nu}$ in this expansion vanishes because of the N-nilpotency of the N-differential d, and the next terms corresponding to the l values from 1 to $N-1$ also vanish because of the property of q-binomial coefficients. Hence if $k = N$ then the formula (3.4.7) takes on the form

$$
\nabla_q^N(s) = s^\mu\, \psi_\mu^{(N,N)\nu}\, \mathfrak{f}_\nu. \tag{3.4.10}
$$

In order to simplify the notations and assuming that N is fixed we shall denote $\psi_\mu^\nu = \psi_\mu^{(N,N)\nu}$.

Definition 3.4.2. An $(r \times r)$-matrix $\Psi = (\psi_\mu^\nu)$, whose entries are the elements of degree N of a graded q-differential algebra Ω_q, is said to be the *curvature matrix* of a N-connection ∇_q.

Obviously $\Psi \in \mathrm{Mat}_r^N(\Omega_q)$. In new notations the formula (3.4.10) can be written as follows $\nabla_q^N(s) = s^\mu\, \psi_\mu^\nu\, \mathfrak{f}_\nu$, and it shows that ∇_q^N is the endomorphism of degree N of the left Ω_q-module $\mathfrak{F}$.

Let us consider the expressions for curvature in two cases when $N = 2$ and $N = 3$. If $N = 2$ then $q = -1$, and a graded q-differential algebra Ω_q is a graded differential algebra with differential d satisfying $d^2 = 0$. This is a classical case, and if we assume that Ω_q is the algebra of differential forms on a smooth manifold M with exterior differential d and exterior multiplication $\wedge$, $\mathcal{E}$ is the module of smooth sections of a vector bundle E over M, ∇_q is a connection on E, $\mathfrak{e}$ is a local frame of a vector bundle E then Θ is the matrix of 1-forms of a connection ∇_q and we have for the components of curvature $\psi_\mu^\nu = d\theta_\mu^\nu - \theta_\mu^\sigma\theta_\sigma^\nu$. In this case Ω_q is super-commutative algebra and we can put the expressions for components of curvature into the form $\psi_\mu^\nu = d\theta_\mu^\nu + \theta_\sigma^\nu\theta_\mu^\sigma$. or by means of matrices $\Psi = d\Theta + \Theta \cdot \Theta$ in which we recognize the classical expression for the curvature.

If $N = 3$ then $q = \exp(\frac{2\pi i}{3})$ is the cubic root of unity satisfying the relations $q^3 = 1$, $1 + q + q^2 = 0$. This is a first non-classical case of a q-connection, and the formula (3.4.6) gives the following expression for the curvature of a N-connection

$$\begin{aligned}
\psi_\mu^\nu &= d^2\theta_\mu^\nu + (q+q^2)\, d\theta_\mu^\sigma\theta_\sigma^\nu + q^2\,\theta_\mu^\sigma\, d\theta_\sigma^\nu + q^3\,\theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\nu \\
&= d^2\theta_\mu^\nu - d\theta_\mu^\sigma\theta_\sigma^\nu + q^2\,\theta_\mu^\sigma\, d\theta_\sigma^\nu + \theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\nu \\
&= d^2\theta_\mu^\nu - (d\theta_\mu^\sigma\theta_\sigma^\nu - q^2\,\theta_\mu^\sigma\, d\theta_\sigma^\nu) + \theta_\mu^\tau\theta_\tau^\sigma\theta_\sigma^\nu
\end{aligned}$$

It is useful to write the above expression for the curvature in a matrix form

$$\Psi = d^2\Theta - [d\Theta, \Theta]_q + \Theta^3. \tag{3.4.11}$$

From Proposition 3.2.7 it follows that the curvature of a N-connection satisfies the Bianchi identity. If θ_ν^μ, ψ_ν^μ are the components of an N-connection ∇_q and its curvature F with respect to a basis $\mathfrak{f}$ for the module $\mathfrak{F}$ then the Bianchi identity takes on the form

$$d\psi_\nu^\mu = \theta_\mu^\sigma\psi_\sigma^\nu - \psi_\mu^\sigma\theta_\sigma^\nu.$$

The last part of this section is devoted to the structure of N-connection forms and their curvature. We apply the algebra of polynomials $\mathfrak{P}_q[\mathfrak{d}, a]$ over $\mathbb{C}$, constructed in the Section 2.2 to study the structure of N-curvature. Let Ω_q be a graded q-differential algebra.

Definition 3.4.3. We will call an element of degree one $\Theta \in \Omega_q^1$ an N-connection form in a graded q-differential algebra Ω_q. The linear operator of degree one $\nabla_q = d + \Theta$ will be referred to as a covariant N-differential induced by a N-connection form Θ.

We remind that d is an N-differential which means that $d^k \neq 0$ for $1 \leq k \leq N-1$ and if we successively apply it to an N-connection form Θ we get the sequence of elements $\Theta, d\Theta, d^2\Theta, \ldots, d^{N-1}\Theta$, where $d^k\Theta \in \Omega_q^{k+1}$. Let us denote

$$\begin{aligned}\Theta_1 &= \Theta\\ \Theta_2 &= d\Theta\\ &\vdots\\ \Theta_N &= d^{N-1}\Theta.\end{aligned}$$

We denote by $\Omega_q[\Theta]$ the graded subalgebra of Ω_q generated by elements $\Theta_1, \Theta_2, \ldots, \Theta_N$. For any integer $k = 1, 2, \ldots, N$ we define the polynomial $F_k \in \Omega_q[\Theta]$ by the formula $F_k = \nabla_q^{k-1}(\Theta)$. Evidently the subalgebra $\Omega_q[\Theta]$ is isomorphic to the graded q-differential algebra $\mathfrak{P}_q[a]$ of Section 2.2. if we identify $\Theta_i \longrightarrow a_i$. Then the polynomials F_k are identified with the polynomials f_k and we can apply all formulae proved in the case of $\mathfrak{P}_q[a]$ to study the structure of $\Omega_q[\Theta]$.

Definition 3.4.4. Let $[\Omega_q, \Omega_q]_q$ be the subspace spanned by graded q-commutators $[u, v] = uv - q^{|u||v|}vu$. A *trace* on Ω_q is a degree zero homomorphism τ from the N-complex Ω_q to an N-complex V with N-differential d', i.e. $d'\tau = \tau d$, which satisfy $\tau[\Omega_q, \Omega_q]_q = 0$

Definition 3.4.5. The *Chern character form* of a graded q-differential algebra Ω_q is

$$\mathrm{ch}(\Theta, \tau) = \tau(\frac{F_N^n}{n!}).$$

Proposition 3.4.6. *The Chern character form is closed, i.e.*

$$d(ch(\Theta, \tau)) = 0.$$

The proof follows from Bianchi identity.

It follows from Theorem 2.2.4 that for any integer $1 \leq k \leq N$ the kth power of the covariant N-differential ∇_q can be expanded as follows

$$(\nabla_q)^k = \sum_{i=0}^{k} \left[\begin{array}{c} k \\ i \end{array} \right]_q F_{(i)}\, d^{k-1} = d^k + [k]_q F_1 d^{k-1} + \ldots + [n]_q F_{k-1} d + F_k,$$

where $F_k = (\nabla_q)^{k-1}(\Theta)$. Particularly if $k = N$ then the Nth power of the covariant N-differential ∇_q is the operator of multiplication by the element F_N of grading zero. It makes possible to define the curvature of an N-connection form Θ as follows

Definition 3.4.7. The N-curvature form of an N-connection form Θ is the element of grading zero $F_N \in \mathfrak{A}$.

Theorem 2.2.4 gives the explicit power expansion formula for N-curvature form of an N-connection

$$F_k \;=\; \sum_{\sigma\in\Upsilon_k} \left[\begin{array}{c} k_2-1 \\ k_1 \end{array} \right]_q \left[\begin{array}{c} k_3-1 \\ k_2 \end{array} \right]_q \cdots \left[\begin{array}{c} k-1 \\ k_{r-1} \end{array} \right]_q \Theta_{i_1}\Theta_{i_2}\ldots\Theta_{i_r},$$

where Υ_k is the set of all compositions of an integer $1 \leqslant k \leqslant N$, $\sigma = (i_1, i_2, \ldots, i_r)$ is composition of an integer k in the form of a sequence of strictly positive integers , where $i_1 + i_2 + \ldots + i_r = N$, and

$$\begin{aligned} k_1 &= i_1, \\ k_2 &= i_1 + i_2, \\ k_3 &= i_1 + i_2 + i_3, \\ &\cdots \\ k_{r-1} &= i_1 + i_2 + \ldots + i_{r-1}. \end{aligned}$$

Hence we obtain the expressions for N-curvature form

$$F_2 = d\Theta + \Theta^2, \tag{3.4.12}$$

$$F_3 = d^2\Theta + d\Theta\,\Theta + [2]_q\,\Theta\, d\Theta + \Theta^3, \tag{3.4.13}$$

$$\begin{aligned} F_4 = {} & d^3\Theta + (d^2\Theta)\,\Theta + [3]_q\,\Theta\,(d^2\Theta) + [3]_q\,(d\Theta)^2 \\ & + d\Theta\,\Theta^2 + [3]_q\,\Theta^2\, d\Theta + [2]_q\,\Theta\, d\Theta\,\Theta + \Theta^4, \end{aligned} \tag{3.4.14}$$

$$
\begin{aligned}
F_5 \;=\;& d^4\Theta + (d^3\Theta)\,\Theta + [4]_q\,\Theta\,(d^3\Theta) + [4]_q\,(d^2\Theta)\,d\Theta \\
& + \left[\begin{array}{c} 4 \\ 2 \end{array}\right]_q d\Theta\,(d^2\Theta) + (d^2\Theta)\,\Theta^2 + [3]_q\,(d\Theta)^2\,\Theta \\
& + [4]_q d\Theta\,\Theta\,d\Theta + [2]_q\,[4]_q\Theta(d\Theta)^2 + \left[\begin{array}{c} 4 \\ 2 \end{array}\right]_q \Theta^2\,d^2\Theta \\
& + [3]_q\Theta\,d^2\Theta\,\Theta + [2]_q\Theta\,d\Theta\,\Theta^2 + [3]_q\Theta^2\,d\Theta\,\Theta \\
& + d\Theta\,\Theta^3 + [4]_q\Theta^3\,d\Theta + \Theta^5. \qquad (3.4.15)
\end{aligned}
$$

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Kokkuvõte

Gradueeritud q-diferentsiaalalgebrad ja algebralised mudelid mittekommutatiivses geomeetrias

Diferentsiaalmooduli mõiste võtsid kasutusele H. Cartan ja S. Eilenberg [14]. Olgu E diferentsiaalmoodul üle kommutatiivse ringi ja d selle mooduli diferentsiaal, st d on mooduli E endomorfism, mis rahuldab tingimust $d^2 = 0$. Tingimusest $d^2 = 0$ järeldub, et $\mathrm{Im}d \subset \mathrm{Ker}d$. Faktormoodulit $\mathrm{Ker}d/\mathrm{Im}d$ nimetatakse diferentsiaalmooduli homoloogiaks. Diferentsiaalmooduli homoloogia mõõdab, kui palju jada $E \xrightarrow{d} E \xrightarrow{d} E$ erineb täpsest jadast. Tundub loomulik üldistada diferentsiaalmooduli mõistet, kirjutades tingimuse $d^2 = 0$ üldisemal kujul $d^N = 0$, kus $N \geqslant 2$. Sellise üldistuse idee pakuti välja sõltumatult töödes [34] ja [29], ning seda uuriti põhjalikult artiklites [19, 20, 22, 36]. Antud üldistuse uurimise tulemuseks said järgmised struktuurid: N-diferentsiaalkompleks, koahelate N-kompleks ja selle üldistatud kohomoloogiad, gradueeritud q-diferentsiaalalgebra, kus q on N-inda astme algjuur.

Käesoleva väitekirja uurimisobjektideks on gradueeritud N-diferentsiaalmoodulid, gradueeritud q-diferentsiaalalgebrad, selle üldistatud kohomoloogiad, gradueeritud q-diferentsiaalalgebra abil konstrueeritud diferentsiaalvormide algebra taandatud kvanttasandil, seostuse vormi poolt tekitatud gradueeritud q-diferentsiaalalgebra, seostuse üldistus.

Väitekirja esimeses peatükis tutvustatakse lugejale N-kompleksi ja selle kohomoloogia teooriat. Peatüki esimene paragrahv on pühendatud koahelate kompleksidele ja kohomoloogiatele. On toodud järgmised mõisted: diferentsiaalmoodul, diferentsiaalmoodulite homomorfism, diferentsiaalmooduli homoloogia, gradueeritud diferentsiaalmoodul, koahelate kompleks. On tutvustatud tulemused, mis on järgnevates osades vajalikud: diferentsiaalmoodulite täpsete jadade homoloogiate täpsed kolmnurgad, koahelate komplekside tensorkorrutise kohomoloogiate Künnethi valem. On toodud ka koahelate komplekside näited, nendehulgas, Hochschildi koahelate kompleks, Chevalley-Eilenbergi kompleks. Teises paragrahvis kirjeldatakse N-diferentsiaalmooduli struktuuri. Paragrahv algab q-arvutusega, seejärel defineeritakse N-diferentsiaalmoodul ja selle üldistatud homoloogiad. Tuuakse kriteeriumid selleks, et üldistatud homoloogiad oleksid triviaalsed. Kolmandas paragrahvis keskendutakse N-kompleksi struktuurile. Defineeritakse gradueeritud q-kommutaatori ja gradueeritud q-derivatsiooni mõisted. Näidatakse, et N-kompleks võib olla konstrueeritud gradueeritud assotsiatiivse algebra abil; sellised N-kompleksid mängivad olulist rolli edaspidistes struktuurides, sest nende abil konstrueeritakse diferentsiaalvormide analoogid taandatud kvanttasandil ning seostuse ja seostuse vormi üldistused. Tõestatakse kriteerium koahelate N-kompleksi konstrueerimiseks teatud liiki gradueeritud assotsiatiivse ühikuga algebra jaoks. Tuleb mainida, et teoreem esineb tõestamata kujul juba artiklis [2]. Väitekirjas esitatakse teoreemi põhjalik tõestus, lisaks sellele tõestatakse järeldus, mis väidab, et sellise koahelate N-kompleksi üldistatud kohomoloogiad on triviaalsed. Paragrahvi lõpuosas tuuakse üldistatud Cliffordi algebra definitsioon ning näidatakse, kuidas antud algebra võib olla varustatud koahelate N-kompleksi struktuuriga.

Väitekirja teine peatükk on pühendatud gradueeritud diferentsiaalalgebratele ja nende üldistustele. Esimeses paragrahvis vaadeldakse gradueeritud diferentsiaalalgebra olulisi näiteid: diferentsiaalvormide de Rhami algebra, Lie rühma Lie algebral koahelate gradueeritud diferentsiaalalgebra, assotsiatiivse ühikuga algebra universaalne gradueeritud hõlmav algebra. Kirjeldatakse taandatud Wess-Zumino diferentsiaalvormide algebra taandatud kvanttasandil. Teises paragrahvis käsitletakse gradueeritud q-diferentsiaalalgebra struktuuri, mis kujutab endast gradueeritud diferentsiaalalgebra üldistust juhul, kui diferentsiaal rahuldab tingimust $d^N = 0$, $N \geqslant 2$, ja gradueeritud Leibnizi reegel on asendatud gradueeritud q-Leibnizi reegliga. Konstrueeritakse N-seostuse vormiga seotud polünoomide gradueeritud q-diferentsiaalalgebra struktuur. Tõestatakse valem, mis näitab, kuidas N-kõveruse vormi algebraline analoog astmes k avaldub polünoomide algebra moodustajate kaudu. Tõestatakse, et N-seostuse vormi kõverus rahuldab Bianchi samasust. Need tulemused on avaldatud artiklis [9]. Viimases paragrahvis, tuginedes artiklile [3], kirjeldatakse N-diferentsiaalvormide algebrat taandatud kvanttasandil.

Kolmas peatükk algab selliste oluliste diferentsiaalgeomeetria struktuuridega nagu seostus vektorkihtkonnal ja selle kõverus. Teises paragrahvis konstrueeritakse seostuse üldistus, mis tugineb q-diferentsiaalalgebra mõistele. N-seostuse idee pakkus välja V. Abramov artiklis [1], kus N-seostus defineeritakse seostuse teooria algebralise formalismi raames. Väitekirjas uuritakse N-seostuse struktuuri ja tõestatakse, et igal projektiivsel moodulil eksisteerib N-seostus. Tuuakse sisse mooduli Hermite'i struktuuriga kooskõlalise N-seostuse mõiste. Tõestatakse Bianchi samasus seostuse kõveruse jaoks. Need tulemused on ilmunud artiklites [7, 8]. Kolmandas paragrahvis tuginedes artiklile [6], on kirjeldatud superseostust üldistav $\mathbb{Z}_N$-seostus; selleks eeldatakse, et moodulil on olemas ka gradueeritud struktuur. Peatüki viimane osa on pühendatud N-seostuse lokaalsele kirjeldusele.

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