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Diameter 2 Properties

Master's Thesis

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Tartu 2012

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Acknowlegdements

First of all I would like to thank both of my supervisors—Rainis Haller and Olav Nygaard. I am very grateful for their advice, comments and support they have given during the writing of this thesis.

Special thanks go to Märt Põldvere for his help and suggestions.

I would like to thank Eve Oja for introducing me to the beauty of Functional Analysis in the first place.

Finally, I thank my family and friends for all of their support.

1 Preface

1.1 Introduction

Nygaard and Werner showed that for any infinite-dimensional uniform algebra, every nonempty relatively weakly open subset of its closed unit ball has diameter equal to 2 [16]. If a Banach space satisfies the above condition, then it has the diameter 2 property (see, e.g., [7] or [3]).

In addition to the diameter 2 property Abrahamsen, Lima and Nygaard [1] consider two other formally different diameter 2 properties—the local diameter 2 property and the strong diameter 2 property.

Let X be a Banach space. By a slice of B_X we mean a set of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

A nontrivial Banach space X has the

(i) *local diameter 2 property* if every slice of B_X has diameter 2;

(ii) *diameter 2 property* if every nonempty relatively weakly open subset of B_X has diameter 2;

(iii) *strong diameter 2 property* if every convex combination of slices of B_X has diameter 2.

The following implications hold in general (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The first implication is a consequence of Bourgain's lemma: every nonempty relatively weakly open subset of B_X contains a convex combination of slices. The second implication holds because slices are relatively weakly open.

1 Preface

Standard examples of Banach spaces with diameter 2 properties include ℓ_∞, c_0, $C[0,1]$, and $L_1[0,1]$.

One of the main questions in [1] is whether these three diameter 2 properties differ from each other. We will show that there exist Banach spaces with the diameter 2 property but lacking the strong diameter 2 property. So far, there is no known example of a Banach space with the local diameter 2 property but lacking the diameter 2 property.

The main aim of this thesis is to give an overview of results on diameter 2 properties, and to provide new results. Our starting points are the survey [1], and articles [16], [14] and [18].

1.2 Synopsis

The thesis consists of a preliminary part and a main part, which has been organized as follows.

Chapter 1 contains an introduction, where we explain our motivation and the goal of the thesis, and present a brief overview of our starting points. In addition to this narrative summary section, we describe the notation.

In chapter 2, we recall some basic definitions and initial results. The first section deals with the weak topology of a normed space and the weak* topology of its dual space. We added this section because a student with a solid first course in functional analysis may not have seen some results mentioned here. This is followed by a section where we introduce the notion of a slice. The essential concept of this master thesis is based on slices of the unit ball. In the third section, we recall the term of an extreme point and the Krein–Milman theorem. The Choquet lemma is presented next, this is used in our fifth section to prove the main result in this chapter—Bourgain's lemma.

Chapter 3 is the main part of this thesis. We start with the definitions of the diameter 2 properties under consideration, and establish them for classical spaces ℓ_∞, c_0, $L_1[0,1]$, and $C_0(K)$. It is known that Banach spaces with the Daugavet property have the strong diameter 2 property. We will verify this following the main idea but modifying slightly some details to our liking.

Next we study how the diameter 2 properties are preserved by projective tensor products and ℓ_p-sums of Banach spaces. A detailed proof is given to the fact that the projective tensor product $X\hat{\otimes}_\pi Y$ of Banach spaces X and Y has the local diameter 2 property whenever X or Y has the local diameter 2 property. It is known that the (local) diameter 2 property is stable by taking ℓ_p-sums for all $1 \leq p \leq \infty$. On the other hand, we show that, for nontrivial Banach spaces X and Y, for all $1 < p < \infty$, the Banach space $X \oplus_p Y$ cannot enjoy the strong diameter 2 property whether or not

X and Y have it.

We end this chapter by establishing the diameter 2 properties for M-ideals. In fact, if Y is a strict M-ideal in X, then both Y and X have the strong diameter 2 property. Thus, if X is an M-ideal in X^{**}, then both X and X^{**} have the strong diameter 2 property. Finally, we show that if Y is an M-ideal in X, then any diameter 2 property of Y is carried to X.

1.3 Notation

Our notation is standard in the theory of Banach spaces.

We will consider vector spaces only over the field of real numbers. In a Banach (or normed) space X, we denote the unit sphere by S_X and the closed unit ball by B_X. A Banach space X is sometimes regarded as a subspace of its bidual X^{**} under the canonical embedding $j_X : X \to X^{**}$. For a set $A \subset X$, its diameter is denoted by $\operatorname{diam}(A)$, its closure is denoted by $\overline{A}$, and its convex hull by $\operatorname{conv}(A)$; the closure of the latter set is denoted by $\overline{\operatorname{conv}}(A)$. For closures with respect to other topologies, we mark the topology separately, such as $\overline{A}^{w^*}$. For Banach spaces X and Y, we denote the Banach space of all bounded linear operators from X to Y by $\mathcal{L}(X, Y)$. For $1 \leq p \leq \infty$, we denote the ℓ_p-sum of Banach spaces X and Y by $X \oplus_p Y$. For an operator $T \colon X \to Y$, we denote $\ker T = \{x \in X \colon Tx = 0\}$.

2 Preliminaries

In this chapter, we recall some basic definitions and initial results needed for the main chapter. These include the weak and the weak* topology together with Goldstine's theorem and the Banach–Alaoglu theorem, extreme points, and the Krein–Milman theorem. We also introduce the notion of a slice. The essential concept of this master's thesis is based on slices of the unit ball. At the end of this chapter, we present two important tools—Choquet's lemma and Bourgain's lemma. These two results form the basis for our subsequent results in Chapter 3.

Throughout this chapter X denotes a normed space unless specifically stated otherwise.

2.1 Weak and weak* topology

One can find the information presented in this section in almost any Banach space textbook. We use [8] and [15] as references. But to keep our treatment self-contained and for the sake of completeness, we will add this review.

The main purpose of this section is to study two extremely important locally convex Hausdorff topologies, used through the thesis, and unavoidable in the Banach space theory—the weak topology, and the weak* topology on a dual space. These topologies are in general weaker than the norm topology and not always induced by metrics.

Definition. The *weak topology* on X is the topology whose neighborhood basis at $x_0 \in X$ consists of all sets

$$\{x \in X\colon\ |x_i^*(x) - x_i^*(x_0)| < 1, \quad i = 1, \dots, n\},$$

where $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in X^*$.

The *weak* topology* on X^* is the topology whose neighborhood basis at $x_0^* \in X^*$ consists of all sets

$$\{x^* \in X^*\colon\ |x^*(x_i) - x_0^*(x_i)| < 1, \quad i = 1, \dots, n\},$$

where $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$.

In fact, the weak topology for X is the coarsest topology on X that makes every element of X^* continuous; the weak* topology for X^* is the coarsest topology on X^* that makes every element of X continuous. The weak* topology of X^* is included in the weak topology of X^*.

Note that nonempty weakly open sets or weak* open sets in an infinite-dimensional normed space are unbounded with respect to the norm (see, e.g., [8, Proposition 3.89]).

It turns out that, (the canonical image of) X must be weak* dense in X^{**}. In fact, the following says even more.

Theorem 2.1 (Goldstine, 1938; see, e.g., [8, Theorem 3.96])**.** *The weak* closure of B_X in X^{**} is $B_{X^{**}}$.*

Recall that the topology induced by a topological space X on a subset B is called the *relative topology on B*. The open sets of B in the relative topology are the intersections $B \cap U$, where U is an open subset of X.

Lemma 2.2. *Every nonempty relatively weak* open subset of $B_{X^{**}}$ contains a nonempty relatively weakly open subset of B_X.*

Proof. Let U be a nonempty relatively weak* open subset of $B_{X^{**}}$ containing an element x_0^{**}. We may assume that

$$\{x^{**} \in B_{X^{**}}\colon\ |(x^{**} - x_0^{**})(x_i^*)| < 1, \quad i = 1, \dots, n\} \subset U,$$

for some $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in X^*$. By Goldstine's theorem (see Theorem 2.1), there is an element x_0 in B_X such that

$$|(x_0 - x_0^{**})(x_i^*)| < \frac{1}{2}$$

for all $i = 1, \dots, n$. Obviously,

$$V = \{x \in B_X \colon\ |x_i^*(x - x_0)| < \frac{1}{2}, \quad i = 1, \dots, n\}$$

is relatively weakly open in B_X.

We will show that V is contained in U. Let $x \in V$. For every $i = 1, \dots, n$,

$$\begin{aligned} |(x - x_0^{**})(x_i^*)| &\le |x_i^*(x - x_0)| + |(x_0 - x_0^{**})(x_i^*)| \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Hence $x \in U$.

$\square$

One of the major results of the theory of normed spaces is that B_{X^*} is weak* compact.

Theorem 2.3 (Banach, 1932; Alaoglu, 1940; see, e.g., [15, Theorem 2.6.18])**.** *The unit ball B_{X^*} is weak* compact.*

Note that weakly closed sets are always norm closed. It turns out that the converse holds for convex sets.

Theorem 2.4 (Mazur, 1933; see, e.g., [8, Theorem 3.45])**.** *The weak closure and the norm closure of any convex subset of X coincide.*

We end this section by recalling that the norm is weakly lower semicontinuous, and the norm on the dual space is weak* lower semicontinuous.

Theorem 2.5 (see, e.g., [15, Theorem 2.5.21, Theorem 2.6.14])**.**

(i) *If a net (x_α) in X converges to an element $x \in X$ in the weak topology, then*

$$\|x\| \le \liminf_\alpha \|x_\alpha\|\,;$$

(ii) *If a net* (x_α^*) *in* X^* *converges to an element* $x^* \in X^*$ *in the weak* topology, then*

$$\|x^*\| \leq \liminf_\alpha \|x_\alpha^*\| \,.$$

2.2 Slices

As we mentioned before, slices are the key concept of this thesis.

Definition. Let B be a nonempty bounded subset of X. A *slice* of B is a set of the form

$$S_B(x^*, \alpha) = \{x \in B\colon x^*(x) > \sup_{y \in B} x^*(y) - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

If B is the unit ball of X, then we write $S(x^*, \alpha)$ instead of $S_B(x^*, \alpha)$.

If X is a dual space, then slices of B whose defining functional comes from (the canonical image of) the predual of X are called *weak* slices* of B. We denote a weak* slice of B by adding a *-symbol;

$$S_B^*(\tilde{x}, \alpha) = \{x \in B\colon x(\tilde{x}) > \sup_{y \in B} y(\tilde{x}) - \alpha\},$$

for some $\tilde{x}$ in the unit sphere of the predual $\tilde{X}$ of X and $\alpha > 0$.

A slice $S_B(x^*, \alpha)$ is clearly a nonempty intersection of B with the open half-space $\{x \in X\colon x^*(x) > \sup_{y \in B} x^*(y) - \alpha\}$. Therefore a slice is always relatively weakly open, moreover a weak* slice is always relatively weak* open.

By a convex combination of slices of B_X we mean a set of the form

$$\sum_{i=1}^{n} \lambda_i S(x_i^*, \alpha_i),$$

where $n \in \mathbb{N}$, $S(x_1^*, \alpha_1), \dots, S(x_n^*, \alpha_n)$ are slices of B_X, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$.

In the thesis, we sometimes refer to the Radon–Nikodým property. Although it has many equivalent formulations, it also can be characterized by slices.

Definition (see, e.g., [8, Theorem 11.15])**.** A Banach space X is said to have the *Radon–Nikodým property* if every nonempty bounded subset of X

has slices of arbitrarily small diameter, that is, for every bounded subset B of X and for every $\varepsilon > 0$, there is an $x^* \in S_{X^*}$ and $\alpha > 0$ such that $S_B(x^*, \alpha)$ has diameter less than ε.

We remark that all reflexive spaces enjoy the Radon–Nikodým property (see, e.g., [8, Corollary 11.10]).

2.3 Extreme points

We now proceed to the principal notion of the next section.

Definition. Let X be a vector space and C be a nonempty convex subset of X. A point in C is called an *extreme point of* C if it does not lie inside an interval with endpoints in C, that is, $x \in C$ is an extreme point of C if

$$y, z \in C, \quad \lambda \in (0,1), \quad x = \lambda y + (1-\lambda)z \quad \Rightarrow \quad x = y = z.$$

We denote by $\operatorname{ext}(C)$ the set of all extreme points of C.

We describe now the extreme points of the unit ball in ℓ_∞. This will be used in Chapter 3 (see Proposition 3.5).

Lemma 2.6 (see, e.g., [8, Exercise 3.129]). *A sequence $x = (x_k)_{k=1}^\infty$ is an extreme point of B_{ℓ_∞} if and only if $|x_k| = 1$ for every $k \in \mathbb{N}$.*

Proof. Necessity. Let $x = (x_k)_{k=1}^\infty \in \operatorname{ext}(B_{\ell_\infty})$. Suppose that there is an index $K \in \mathbb{N}$ with $|x_K| < 1$. Consider $y = (y_k)_{k=1}^\infty$ and $z = (z_k)_{k=1}^\infty$, where

$$y_k = \begin{cases} 1, & \text{if } k = K; \\ x_k, & \text{if } k \neq K; \end{cases}$$

and

$$z_k = \begin{cases} -1, & \text{if } k = K; \\ x_k, & \text{if } k \neq K. \end{cases}$$

Since $x = \lambda y + (1-\lambda)z$ for some $\lambda \in (0,1)$, but $y \neq z$, we get a contradiction.

Sufficiency. Let $x = (x_k)_{k=1}^\infty$ where $|x_k| = 1$ for all $k \in \mathbb{N}$. Suppose that $x = \lambda y + (1-\lambda)z$ for some $y = (y_k)_{k=1}^\infty$, $z = (z_k)_{k=1}^\infty \in B_{\ell_\infty}$, and $\lambda \in (0,1)$. If $y_k \neq z_k$ for some $k \in \mathbb{N}$, then

$$1 = |x_k| = |\lambda y_k + (1-\lambda)z_k| < 1,$$

a contradiction. Thus, $y_k = z_k$ for all $k \in \mathbb{N}$, and hence $x = y = z$. $\square$

Proposition 2.7 (see [8, Exercise 3.127]). *Let X be a vector space, C a convex subset of X, and $e \in \operatorname{ext}(C)$. If $e = \sum_{i=1}^{n} \lambda_i c_i$ for some $n \in \mathbb{N}$, $c_1, \ldots, c_n \in C$, and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, then there exists an index $i \in \{1, \ldots, n\}$ such that $e = c_i$.*

Proof. The proof is by induction on n. The assertion clearly holds for $n = 1$.

Assume now that the assertion holds if $n = m$. We are going to show that it holds for $n = m + 1$. Suppose that $e = \sum_{i=1}^{m+1} \lambda_i c_i$. We may clearly assume that $0 < \lambda_{m+1} < 1$. Indeed, if $\lambda_{m+1} = 0$, then the assertion follows by the assumption; if $\lambda_{m+1} = 1$, then $x = c_{m+1}$, and we are also done. Set

$$c = \frac{\lambda_1}{\sum_{i=1}^{m} \lambda_i} c_1 + \cdots + \frac{\lambda_m}{\sum_{i=1}^{m} \lambda_i} c_m.$$

Then $c \in C$ by the convexity of C, and

$$e = (1 - \lambda_{m+1})c + \lambda_{m+1} c_{m+1}.$$

Since e is an extreme point of C, it immediately follows that $e = c = c_{m+1}$. The proof is now complete. □

We would like to use the Krein–Milman theorem in case X is equipped with the weak topology, and in case we consider X^* with the weak* topology. To do this, we need to assume that X is a locally convex topological vector space.

Theorem 2.8 (Krein–Milman, 1940; see, e.g., [8, Theorem 3.65]). *Let X be a locally convex topological vector space and let K be a nonempty convex and compact subset of X. Then*

(i) *K possesses an extreme point;*

(ii) $K = \overline{\operatorname{conv}}(\operatorname{ext}(K))$.

2.4 Choquet's lemma

There are two versions of the Choquet lemma presented in this section—the weak and the weak* case, although only the second one is put to use in the next section to prove Bourgain's result. In fact, in the following section, we will present also two versions of the Bourgain lemma, whereas we will derive one version from another; the weak case of the Choquet lemma can be used to provide its direct proof.

In turn we use the following.

Theorem 2.9 (see, e.g., [8, Theorem 3.32 and Corollary 3.34]).

(i) *If C is a nonempty closed convex set in X and $x_0 \notin C$, then there exists an element $x^* \in S_{X^*}$ and $\alpha \in \mathbb{R}$ such that*

$$x^*(x_0) > \alpha > \sup\{x^*(x)\colon x \in C\}.$$

(ii) *If C is a nonempty weak* closed convex set in X^* and $x_0^* \notin C$, then there exists an element $x \in S_X$ and $\alpha \in \mathbb{R}$ such that*

$$x_0^*(x) > \alpha > \sup\{x^*(x)\colon x^* \in C\}.$$

Proposition 2.10 (see, e.g., [8, Exercise 3.14]). *Let X be a locally convex topological vector space. If A and B are two convex compact subsets of X, then* $\operatorname{conv}(A \cup B)$ *is compact.*

Proof. Consider the map $T\colon \mathbb{R} \times X \times X \to X$ defined by

$$T(\lambda, x, y) = \lambda x + (1 - \lambda)y.$$

Observe that $\operatorname{conv}(A \cup B) = \{\lambda a + (1-\lambda)b\colon a \in A, b \in B, \lambda \in [0, 1]\}$. Clearly, T is continuous and by this

$$T([0, 1] \times A \times B) = \operatorname{conv}(A \cup B).$$

Since the continuous image of a compact set is compact, it follows that $\operatorname{conv}(A \cup B)$ is a compact set in X. □

Now we are prepared to present the Choquet lemma.

Lemma 2.11 (Choquet, 1969; see, e.g., [8, Lemma 3.69])**.** *Let C be a weakly compact convex set in X and e an extreme point of C. Then slices of C containing e form a neighborhood basis at e in the relative weak topology of C.*

Remark. Observe that, for every $c \in C$, slices of C that contain c form a neighborhood subbasis for the relative weak topology of C at c; finite intersections of slices therefore form a basis for the relative weak topology. Choquet's lemma says that under the assumptions for C at any extreme point $e \in C$ the latter subbasis is, in fact, a basis.

Proof. Let V be a neighborhood of e in the relative weak topology of C. We assume that $V = \bigcap_{i=1}^{n} S_i$, where $n \in \mathbb{N}$ and $S_1, \dots, S_n$ are slices of C. We will find a slice S of C such that $e \in S \subset V$. Denote by

$$K = \bigcup_{i=1}^{n} (C \setminus S_i).$$

Then clearly $e \notin K$. By Proposition 2.7, it follows that $e \notin \operatorname{conv}(K)$. In fact, $e \notin \overline{\operatorname{conv}}(K)$. Indeed, we note that, by Proposition 2.10, $\operatorname{conv}(K)$ is relatively weakly compact since $K = \bigcup_{i=1}^{n} (C \setminus S_i)$ is relatively weakly compact. Thus, $\operatorname{conv}(K)$ is weakly closed, hence, by the Mazur theorem (see Theorem 2.4), $\operatorname{conv}(K) = \overline{\operatorname{conv}}(K)$.

Now, by Theorem 2.9, there is a functional x^* in S_{X^*} and $\alpha \in \mathbb{R}$ such that

$$x^*(e) > \alpha > \sup\{x^*(x) \colon x \in \overline{\operatorname{conv}}(K)\}.$$

Notice that the slice $S = C \cap \{x \in X \colon x^*(x) > \alpha\}$ contains e and is contained in V.

□

Similarly, we have the result for the relative weak* topology.

Lemma 2.12 (Choquet)**.** *Let C be a weak* compact convex set in X^* and e^* an extreme point of C. Then weak* slices of C containing e^* form a neighborhood basis at e^* in the relative weak* topology of C.*

Remark. Observe that, for every $c^* \in C$, weak* slices of C that contain c^* form a neighborhood subbasis for the relative weak* topology of C at c^*; finite intersections of weak* slices therefore form a basis for the relative weak* topology. Choquet's lemma says that under the assumptions for C at any extreme point $e^* \in C$ the latter subbasis is, in fact, a basis.

Proof. Let V be a neighborhood of e^* in the relative weak* topology of C. We assume that $V = \bigcap_{i=1}^{n} S_i^*$, where $n \in \mathbb{N}$ and $S_1^*, \ldots, S_n^*$ are weak* slices of C. We will find a weak* slice S^* of C such that $e^* \in S^* \subset V$. Denote by

$$K = \bigcup_{i=1}^{n} (C \setminus S_i^*).$$

Then clearly $e^* \notin K$. By Proposition 2.7, it follows that $e^* \notin \operatorname{conv}(K)$. In fact, $e^* \notin \overline{\operatorname{conv}}^{w^*}(K)$. Indeed, we note that, by Proposition 2.10, $\operatorname{conv}(K)$ is relatively weak* compact since $K = \bigcup_{i=1}^{n}(C \setminus S_i^*)$ is relatively weak* compact. Thus, $\operatorname{conv}(K) = \overline{\operatorname{conv}}^{w^*}(K)$.

Now, by Theorem 2.9, there is an x in S_X and $\alpha \in \mathbb{R}$ such that

$$e^*(x) > \alpha > \sup\{x^*(x) \colon x^* \in \overline{\operatorname{conv}}^{w^*}(K)\}.$$

Notice that the weak* slice $S^* = C \cap \{x^* \in X^* \colon x^*(x) > \alpha\}$ contains e^* and is contained in V.

□

2.5 Bourgain's lemma

Now we are ready to prove the main result in this chapter.

Lemma 2.13 (Bourgain, 1979; cf. [9, Lemma II.1 p. 26])**.** *Let C be a bounded convex set in X^* and let U be a nonempty relatively weak* open subset of C. Then there exists $n \in \mathbb{N}$, weak* slices $S_1^*, \dots, S_n^*$ of C, and scalars $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that*

$$\sum_{i=1}^{n} \lambda_i S_i^* \subset U.$$

Proof. Let U be a relatively weak* open subset of C containing an element x^*. Find a weak* convex neighbourhood V of zero such that $(x^* + 2V) \cap C \subset U$.

By the Banach–Alaoglu theorem (see Theorem 2.3), $\overline{C}^{w^*}$ is weak* compact. Therefore, by the Krein–Milman theorem (see Theorem 2.8), we have that $\overline{C}^{w^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{ext}(\overline{C}^{w^*}))$.

Denote by $E = \operatorname{ext}(\overline{C}^{w^*})$. Then clearly $x^* \in \overline{\operatorname{conv}}^{w^*}(E)$. Thus, there are $n \in \mathbb{N}$, $e_1^*, \dots, e_n^* \in E$, and scalars $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i e_i^* \in x^* + V.$$

By Lemma 2.12, there is a weak* slice $\tilde{S}_i^*$ of $\overline{C}^{w^*}$ with $\tilde{S}_i^* \subset e_i^* + V$ for every $i = 1, \dots, n$. We take $S_i^* = \tilde{S}_i^* \cap C$ for every $i = 1, \dots, n$. Then $S_1^*, \dots, S_n^*$ are weak* slices of C satisfying

$$\sum_{i=1}^{n} \lambda_i S_i^* \subset \sum_{i=1}^{n} \lambda_i (e_i^* + V) \cap C \subset (x^* + 2V) \cap C \subset U.$$

$\square$

Lemma 2.14 (Bourgain)**.** *If U is a nonempty relatively weakly open subset of B_X, then there exists $n \in \mathbb{N}$, slices $S_1, \dots, S_n$ of B_X, and scalars $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that*

$$\sum_{i=1}^{n} \lambda_i S_i \subset U.$$

Proof. Let U be a nonempty relatively weakly open subset of B_X. Observe that U is a relatively weak* open subset of $B_X \subset X^{**}$. By Lemma 2.13, there exists $n \in \mathbb{N}$, weak* slices $S_1^*, \ldots, S_n^*$ of B_X, and scalars $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i S_i^* \subset U.$$

Notice that the weak* slices of B_X are precisely the weak slices of B_X. This proves the result. $\square$

3 Diameter 2 properties

In Banach spaces with the Radon–Nikodým property one can always find arbitrary small slices in any nonempty bounded set. But some of the classical Banach spaces which fail to have the Radon–Nikodým property actually have the "opposite property" that all slices of its unit ball have diameter 2.

For example, Nygaard and Werner [16] showed that for any infinite-dimensional uniform algebra, every nonempty relatively weakly open subset of its unit ball has diameter 2. The latter property is now known as the diameter 2 property (see, e.g., [7] or [3]).

In addition to the diameter 2 property Abrahamsen, Lima and Nygaard [1] consider two other formally different diameter 2 properties—the local diameter 2 property and the strong diameter 2 property. One of their main questions is whether these three diameter 2 properties differ from each other.

We will now study these three diameter 2 properties in more detail. At the end, we will show that the strong diameter 2 property is not equivalent to the diameter 2 property (see Theorem 3.23).

3.1 Definitions and prerequisites

Definition (see [1], cf. [7])**.** A nontrivial Banach space X has the

(i) *local diameter 2 property* if every slice of B_X has diameter 2;

(ii) *diameter 2 property* if every nonempty relatively weakly open subset of B_X has diameter 2;

(iii) *strong diameter 2 property* if every convex combination of slices of B_X has diameter 2.

We first observe that the following implications hold for these properties:

- the diameter 2 property implies the local diameter 2 property;
- the strong diameter 2 property implies the diameter 2 property.

The first implication is clear since any slice of the unit ball is also relatively weakly open. The second implication follows directly from Bourgain's lemma (see Lemma 2.14).

Remark. Since isometric isomorphisms preserve distances, it follows that they preserve all three diameter 2 properties.

Notice that a nonempty relatively weakly open subset of B_X always intersects S_X since it contains a finite number of intersections of open half-spaces with B_X.

Example. A convex combination of slices need not be relatively weakly open (it might be contained in the open unit ball or even in some ball $B(0, r)$, where $r < 1$ (see, e.g., [9, Remark IV.5 p. 48] or the proof of Theorem 3.23 and the remark after that)).

We have the following example. Let X be an infinite-dimensional strictly convex space (e.g., ℓ_2) and fix a functional x^* in S_{X^*}. Consider the slices $S(x^*, 1)$ and $S(-x^*, 1)$ of B_X. Although the set $1/2 \cdot S(x^*, 1) + 1/2 \cdot S(-x^*, 1)$ has diameter 2, it is not relatively weakly open in B_X since it obviously does not contain a point from the unit sphere.

We remark that in a strictly convex space the convex combination of slices might not be contained in some ball $B(0, r)$, where $r < 1$. Indeed, on page 168 in [11] there exists a strictly convex space with the strong diameter 2 property. By Lemma 3.1, convex combinations of slices in that space cannot sit inside some open ball with radius less than 1.

Lemma 3.1. *If a Banach space X has the strong diameter 2 property, then every convex combination of nonempty relatively weakly open subsets of B_X has diameter 2.*

Proof. Assume that X has the strong diameter 2 property. Let $W = \sum_{i=1}^n \lambda_i W_i$, where $n \in \mathbb{N}$, $W_1, \dots, W_n$ are nonempty relatively weakly open subsets of B_X, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. We will show that the diameter of W equals 2.

By Bourgain's lemma (see Lemma 2.14) any W_i under consideration contains some convex combination of slices of B_X, say

$$\sum_{j=1}^{m_i} \mu_{i,j} S_{i,j},$$

where $m_i \in \mathbb{N}$, $S_{i,1}, \dots, S_{i,m_i}$ are slices of B_X, and $\mu_{i,1}, \dots, \mu_{i,m_i} \geq 0$ with $\sum_{j=1}^{m_i} \mu_{i,j} = 1$.

The set

$$\sum_{\substack{i=1,\dots,n \\ j=1,\dots,m_i}} (\lambda_i \mu_{i,j}) S_{i,j}$$

is clearly a convex combination of slices of B_X that is contained in W. Since X has the strong diameter 2 property, it follows that $\operatorname{diam}(W) = 2$. □

A similar result holds for weak* slices.

Lemma 3.2. *If every convex combination of weak* slices of B_{X^*} has diameter 2, then every convex combination of nonempty relatively weak* open sets in B_{X^*} has diameter 2.*

Proof. Assume that every convex combination of weak* slices of B_{X^*} has diameter 2. Let $W = \sum_{i=1}^n \lambda_i W_i$, where $n \in \mathbb{N}$, $W_1, \dots, W_n$ are nonempty relatively weak* open subsets of B_{X^*}, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. We will show that the diameter of W equals 2.

By Bourgain's lemma (see Lemma 2.13) any W_i under consideration contains some convex combination of weak* slices of B_{X^*}, say

$$\sum_{j=1}^{m_i} \mu_{i,j} S^*_{i,j},$$

where $m_i \in \mathbb{N}$, $S^*_{i,1}, \ldots, S^*_{i,m_i}$ are weak* slices of B_{X^*}, and $\mu_{i,1}, \ldots, \mu_{i,m_i} \geq 0$ with $\sum_{j=1}^{m_i} \mu_{i,j} = 1$.

The set

$$\sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,m_i}} (\lambda_i \mu_{i,j}) S^*_{i,j}$$

is clearly a convex combination of weak* slices of B_{X^*} that is contained in W. Since every convex combination of weak* slices of B_{X^*} has diameter 2, it follows that $\operatorname{diam}(W) = 2$.

□

Proposition 3.3. *A Banach space X has the local diameter 2 property (resp. the diameter 2 property, the strong diameter 2 property) if and only if every weak* slice (resp. nonempty relatively weak* open subset, every convex combination of weak* slices) of $B_{X^{**}}$ has diameter 2.*

Proof. Assume first that X has the local diameter 2 property. Let $S^*(x^*, \alpha)$ be a weak* slice of $B_{X^{**}}$. Clearly, $S(x^*, \alpha) \subset S^*(x^*, \alpha)$, where $S(x^*, \alpha)$ is the corresponding slice of B_X. By the assumption,

$$2 = \operatorname{diam}(S(x^*, \alpha)) \leq \operatorname{diam}(S^*(x^*, \alpha)) \leq 2.$$

Suppose now that every weak* slice of $B_{X^{**}}$ has diameter 2. Let $S(x^*, \alpha)$ be a slice of B_X. Then $S(x^*, \alpha)$ is weak* dense in the corresponding weak* slice $S^*(x^*, \alpha)$ of $B_{X^{**}}$. Indeed, fix $x^{**} \in S^*(x^*, \alpha)$. By Goldstine's theorem (see Theorem 2.1), there is a net (x_α) in B_X which converges to x^{**} in the weak* topology. Since

$$1 - \alpha < x^{**}(x^*) = \lim_\alpha x^*(x_\alpha),$$

there is an index α_0 such that $x_\alpha \in S(x^*, \alpha)$ whenever $\alpha \geq \alpha_0$. This proves our claim.

Let $\varepsilon > 0$. By the assumption, there exist $x^{**}, \tilde{x}^{**} \in S^*(x^*, \alpha)$ such that $\|x^{**} - \tilde{x}^{**}\| > 2 - \varepsilon$. Since $S(x^*, \alpha)$ is weak* dense in $S^*(x^*, \alpha)$, there are nets $(x_\alpha), (\tilde{x}_\alpha) \subset S(x^*, \alpha)$ such that the net $(x_\alpha - \tilde{x}_\alpha)$ converges to $x^{**} - \tilde{x}^{**}$ in the weak* toplogy. We have

$$2 - \varepsilon < \|x^{**} - \tilde{x}^{**}\| \leq \liminf_{\alpha} \|x_\alpha - \tilde{x}_\alpha\|,$$

because the norm on X^{**} is weak* lower semicontinuous (see Theorem 2.5). Thus, the diameter of $S(x^*, \alpha)$ is equal to 2. The proof of Proposition 3.3 is now complete for the local diameter 2 case.

Assume first that X has the diameter 2 property. Let U be a nonempty relatively weak* open subset of $B_{X^{**}}$. By Lemma 2.2, there is a nonempty relatively weakly open set V of B_X with $V \subset U$. By the assumption,

$$2 = \operatorname{diam}(V) \leq \operatorname{diam}(U) \leq 2.$$

Suppose now that every nonempty relatively weak* open subset of $B_{X^{**}}$ has diameter 2. Let U be a relatively weakly open set containing x_0. We may assume that U is of the form

$$U = \{x \in B_X \colon\ |x_i^*(x - x_0)| < 1, \quad i = 1, \dots, n\},$$

for some $n \in \mathbb{N}$ and $x_1^*, \dots, x_n^* \in X^*$. Consider the set

$$V = \{x^{**} \in B_{X^{**}} \colon\ |x_i^*(x^{**} - x_0)| < 1, \quad i = 1, \dots, n\}.$$

Then V is a nonempty relatively weak* open subset of $B_{X^{**}}$. We claim that U is weak* dense in V. Fix $x^{**} \in V$. By Goldstine's theorem, there is a net (x_α) in B_X which converges to x^{**} in the weak* topology. Since

$$\lim_{\alpha} |x_i^*(x_\alpha - x_0)| = |x_i^*(x^{**} - x_0)| < 1$$

for every $i = 1, \dots, n$, there is an index α_0 such that $x_\alpha \in U$ whenever $\alpha \geq \alpha_0$. This proves our claim. From the assumption and the weak* lower semicontinuity of the norm on X^{**}, it follows that the diameter of U is 2. The proof of Proposition 3.3 is now complete for the diameter 2 case.

Assume first that X has the strong diameter 2 property. Let S^* be a convex combination of weak* slices of $B_{X^{**}}$. Clearly, S^* contains a convex combination of corresponding slices of B_X. By the assumption, the diameter of S^* is 2.

Suppose now that every convex combination of weak* slices of $B_{X^{**}}$ has diameter 2. Let S be a convex combination of slices of B_X. Denote by S^* the convex combination of corresponding weak* slices of $B_{X^{**}}$. Since every slice of B_X is weak* dense in the corresponding slice of $B_{X^{**}}$, it follows that S is weak* dense in S^*. Again, using the assumption and the weak* lower semicontinuity of the norm on X^{**}, we deduce that the diameter of S is 2. Proposition 3.3 is proved now.

$\square$

From Proposition 3.3 we conclude that X inherits all three diameter 2 properties from its bidual X^{**}.

Corollary 3.4. *If X^{**} has the local diameter 2 property (resp. the diameter 2 property, the strong diameter 2 property), then X has the local diameter 2 property (resp. the diameter 2 property, the strong diameter 2 property).*

3.2 Examples

In this section, we will establish the diameter 2 properties for classical spaces ℓ_∞, c_0, and $L_1[0,1]$, and improve the known result for $C_0(K)$. With this we solve a challenging exercise from the introduction of [1].

We begin with an example that essentially is an exercise from [8].

Proposition 3.5 (cf. [8, Exercise 3.147])**.** *The Banach space ℓ_∞ has the local diameter 2 property.*

Proof. Consider a slice $S(x^*, \alpha)$ of B_{ℓ_∞}. It is enough to show that the slice $S(x^*, \alpha)$ contains at least two distinct extreme points of B_{ℓ_∞} (see Lemma 2.6).

In fact, every slice contains some extreme point of the unit ball. This is because of the Krein–Milman theorem (see Theorem 2.8), we have that a convex combination of some extreme points lie in the slice (which is relatively weakly open subset) and this is immediate that at least one of those extreme points must belong to the slice.

Therefore $S(x^*, \alpha)$ certainly contains an extreme point e of B_{ℓ_∞}.

We will now show the existence of another extreme point in $S(x^*, \alpha)$ different from e. Denote by A the set of all those functionals in $B_{\ell_\infty^*}$ which do not attain their norms. We observe that A is dense in $B_{\ell_\infty^*}$. Otherwise, there is an element x_0^* in $B_{\ell_\infty^*}$ and $r > 0$ such that

$$B(x_0^*, r) \subset B_{\ell_\infty^*} \setminus \overline{A}.$$

Which would imply that all functionals in ℓ_∞^* would attain their norms and ℓ_∞ would be reflexive (see, e.g., [8, Corollary 3.131]), a contradiction.

Let $y^* \in A \cap S_{\ell_\infty^*}$ be such that $\|x^* - y^*\| < \alpha/2$. Then, for $x \in S(y^*, \frac{\alpha}{2})$, we have

$$x^*(x) = y^*(x) - (y^*(x) - x^*(x)) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha.$$

Thus, $S(y^*, \alpha/2) \subset S(x^*, \alpha)$, and we may assume that x^* does not attain its norm. If not, then we can pass to the slice $S(y^*, \alpha/2)$.

If e were the only extreme point in $S(x^*, \alpha)$, then $e \in S(x^*, \beta)$ for every positive member $\beta < \alpha$, meaning that $x^*(e) = 1 = \|x^*\|$. Thus, x^* attains its norm, a contradiction. $\square$

More generally, uniform algebras (e.g., ℓ_∞) have the strong diameter 2 property (see [1, Theorem 4.2]).

Proposition 3.6. *The Banach space c_0 has the diameter 2 property.*

Proof. Let U be a nonempty relatively weakly open subset of B_{c_0}. First, fix an element $u = (u_k)_{k=1}^\infty$ of U. Then U contains a set of the form

$$\{x \in B_{c_0} \colon |x_i^*(x-u)| < 1, \quad i = 1, \ldots, n\},$$

for some $n \in \mathbb{N}$ and functionals $x_1^*, \ldots, x_n^* \in c_0^*$.

The dual space c_0^* is identified with ℓ_1 in the usual way. Thus, for every $i = 1, \ldots, n$, we identify the functional x_i^* with an element $(\alpha_k^i)_{k=1}^\infty$ from ℓ_1, where

$$x_i^*(x) = \sum_{k=1}^\infty \alpha_k^i x_k \qquad \text{for all } x = (x_k)_{k=1}^\infty \in c_0.$$

Choose a $K \in \mathbb{N}$ such that $|\alpha_k^i| < 1/2$ for all $k \geq K$ and for all $i = 1, \ldots, n$.

To show that c_0 has the diameter 2 property we will pick elements x and $\tilde{x}$ of U such that $\|x - \tilde{x}\| = 2$ as follows.

By setting

$$x_k = \begin{cases} 1, & \text{if } k = K; \\ u_k, & \text{if } k \neq K; \end{cases}$$

and

$$\tilde{x}_k = \begin{cases} -1, & \text{if } k = K; \\ u_k, & \text{if } k \neq K; \end{cases}$$

we take $x = (x_k)_{k=1}^\infty$ and $\tilde{x} = (\tilde{x}_k)_{k=1}^\infty$. It is clear that x and $\tilde{x}$ are in U. In fact,

$$|x_i^*(x-u)| = \left|\sum_{k=1}^\infty \alpha_k^i (x_k - u_k)\right| = \left|\alpha_K^i (1 - u_K)\right| \leq 2\left|\alpha_K^i\right| < 1,$$

and

$$|x_i^*(\tilde{x}-u)| = \left|\sum_{k=1}^{\infty} \alpha_k^i(\tilde{x}_k - u_k)\right| = \left|\alpha_K^i(-1-u_K)\right| \le 2\left|\alpha_K^i\right| < 1,$$

for all $i = 1, \ldots, n$. Obviously, $\|x - \tilde{x}\| = 2$. Consequently, c_0 has the diameter 2 property. □

More generally, Becerra Guerrero, López Pérez, and Rodríguez Palacios (see [6, Lemma 2.2]) have shown that $C_0(K)$ has the diameter 2 property whenever K is an infinite locally compact Hausdorff space. Our next result shows that even a further improvement is possible.

Proposition 3.7. *If K is an infinite locally compact Hausdorff space, then the Banach space $C_0(K)$ has the strong diameter 2 property.*

Proof. Since $(C_0(K))^{**} = C(\Omega)$ for a suitable compact Hausdorff topological space Ω (see [13, (7.1)]), and $C(\Omega)$ has the strong diameter 2 property (see [1, Theorem 4.2]), we deduce by Corollary 3.4 that $C_0(K)$ has the strong diameter 2 property.

□

Since $c_0 = C_0(\mathbb{N})$, we have the following corollary.

Corollary 3.8. *The Banach space c_0 has the strong diameter 2 property.*

We remark that this result can also be deduced by using the fact that $c_0^{**} = \ell_\infty$ and Corollary 3.4.

Proposition 3.9. *The Banach space $L_1[0,1]$ has the strong diameter 2 property.*

In order to prove this proposition, we need the following lemma.

Lemma 3.10. *Let $\alpha > 0$, $n \in \mathbb{N}$, and $f_1, \ldots, f_n \in S_{L_\infty[0,1]}$. Then there are pairwise disjoint subsets $E_1, \ldots, E_n \subset [0,1]$ with positive measure such that, for every $i = 1, \ldots, n$,*

$$|f_i(t)| \ge 1 - \alpha \qquad \text{for all } t \in E_i.$$

Proof. We will show the existence of such subsets $E_1, \ldots, E_n$ by induction. The existence of a set E_1 is immediate from $\|f_1\| = 1$.

Suppose that we can find sets $E_1, \ldots, E_m$ as needed for some $m \in \mathbb{N}$, where $m < n$. We will show the existence of a suitable E_{m+1}. Denote by

$$D_{m+1} = \{t \in [0,1] \colon |f_{m+1}(t)| \geq 1 - \alpha\}.$$

If $\mu(D_{m+1} \setminus \bigcup_{i=1}^m E_i) > 0$, then we may take $E_{m+1} = D_{m+1} \setminus \bigcup_{i=1}^m E_i$. Otherwise, there is an index $i_0 \in \{1, \ldots, m\}$ such that $\mu(D_{m+1} \cap E_{i_0}) > 0$. Choose disjoint subsets $\tilde{E}_{i_0}, E_{m+1} \subset D_{m+1} \cap E_{i_0}$ with positive measure (see, e.g., [4, Theorem 10.52]) and redefine $E_{i_0} = \tilde{E}_{i_0}$. $\square$

Proof of Proposition 3.9. Let $\sum_{i=1}^n \lambda_i S(f_i, \alpha_i)$ be a convex combination of slices of $B_{L_1[0,1]}$, where $n \in \mathbb{N}$, $f_1, \ldots, f_n \in S_{L_\infty[0,1]} = S_{(L_1[0,1])^*}$, $\alpha_1, \ldots, \alpha_n > 0$, and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. We will show that $\operatorname{diam}(\sum_{i=1}^n \lambda_i S(f_i, \alpha_i)) = 2$.

We take $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}/2$. By Lemma 3.10, there are pairwise disjoint subsets $E_1, \ldots, E_n \subset [0,1]$ with positive measure such that, for every $i = 1, \ldots, n$,

$$|f_i(t)| \geq 1 - \alpha \qquad \text{for all } t \in E_i.$$

We shall split every E_i, $i = 1, \ldots, n$, further into two disjoint subsets F_i and G_i such that $E_i = F_i \cup G_i$ and $\mu(F_i) = \mu(G_i)$ (see, e.g., [4, Theorem 10.52]).

We take

$$x = \sum_{i=1}^n \lambda_i \frac{\operatorname{sgn} f_i \cdot \chi_{F_i}}{\mu(F_i)} \quad \text{and} \quad \tilde{x} = \sum_{i=1}^n \lambda_i \frac{\operatorname{sgn} f_i \cdot \chi_{G_i}}{\mu(G_i)}.$$

Notice that $\|x\| = 1$, because

$$\int_0^1 |x(t)| \, dt = \sum_{i=1}^n \frac{\lambda_i}{\mu(F_i)} \mu(F_i) = \sum_{i=1}^n \lambda_i = 1,$$

and, for every $i = 1, \ldots, n$, one has

$$\int_0^1 f_i(t) \frac{\operatorname{sgn} f_i(t) \cdot \chi_{F_i}(t)}{\mu(F_i)} \, dt = \int_{F_i} \frac{|f_i(t)|}{\mu(F_i)} \, dt \geq 1 - \alpha > 1 - \alpha_i.$$

Thus, x is an element in $\sum_{i=1}^n \lambda_i S(f_i, \alpha_i)$. Similarly one shows that $\|\tilde{x}\| = 1$ and $\tilde{x}$ is an element in $\sum_{i=1}^n \lambda_i S(f_i, \alpha_i)$.

Therefore

$$\operatorname{diam}\Big(\sum_{i=1}^n \lambda_i S(f_i, \alpha_i)\Big) \geq \|x - \tilde{x}\| = \|x\| + \|\tilde{x}\| = 2.$$

$\square$

More generally, all Banach spaces with the Daugavet property (e.g., $L_1(\mu)$ and $L_\infty(\mu)$, where μ is a nonatomic measure (see, e.g., [20])) have the strong diameter 2 property (see Theorem 3.13).

3.3 The Daugavet property

The main task in this section is to show that the Daugavet property implies the strong diameter 2 property. We follow the idea from [1]; our approach, however, is slightly different.

Definition (see, e.g., [20])**.** A Banach space X has the *Daugavet property* if
$$\|I+T\| = 1 + \|T\|$$
for every rank-1 operator $T : X \to X$.

In fact, it is enough to check the Daugavet property for rank-1 operators with norm 1 (cf. discussion in [20]). Indeed, if $\|I+T\| = 1+\|T\|$ holds for some T, then the nonnegative function $\varphi(\lambda) = \lambda + (1-\lambda)\|T\| - \|\lambda I + (1-\lambda)T\|$ is concave on $[0,1]$ with $\varphi(1/2) = 0$; therefore $\varphi \equiv 0$ on $[0,1]$ which implies that $\|I+\mu T\| = 1+\mu\|T\|$ for all $\mu \geq 0$.

The definition of the Daugavet property modestly involves only rank-1 operators, but it is well known that then automatically the latter norm identity also holds for all compact and even for all weakly compact operators.

Banach spaces with the Daugavet property have been studied since Daugavet in his 1963 paper established a remarkable result that the norm identity
$$\|I+T\| = 1 + \|T\|$$
holds for compact operators T on $C[0,1]$.

The class of Banach spaces with the Daugavet property include the spaces $C(K)$, whenever K is a compact Hausdorff space without isolated points, and the spaces $L_1(\mu)$ and $L_\infty(\mu)$, when μ is a nonatomic measure.

If a Banach space X has the Daugavet property, then X fails to have the Radon–Nikodým property. For more details we refer the interested reader to a survey paper [20] by Werner.

Our starting point is the following basic geometric description of the Daugavet property.

Lemma 3.11 (cf. [18, Lemma 2] or see [20, Lemma 2.2])**.** *The following assertions are equivalent:*

(i) *X has the Daugavet property;*

(ii) *For every slice $S(x^*, \alpha)$ of B_X, every $x \in S_X$ and every $\varepsilon > 0$ there exists a vector $\tilde{x} \in S(x^*, \alpha)$ such that $\|x + \tilde{x}\| \geq 2 - \varepsilon$.*

Remark. *1)* For the sake of completeness, we present the full proof of the above lemma in detail; we remark that in [20] only the idea of the proof was given.

2) After we established the detailed verification of the equivalence in Lemma 3.11, we learned that [12] also contains the proof.

3) Condition (ii) implies the local diameter 2 property. In fact, (ii) immediately implies that $\operatorname{diam}(S(x^*, \alpha) \cup \{x\}) = 2$ for all $x \in S_X$. It is straightforward that if $U \subset B_X$ satisfies $\operatorname{diam}(U \cup \{x\}) = 2$ for all $x \in S_X$, then $\operatorname{diam}(U) = 2$. Therefore, in particular, every slice of B_X has diameter 2.

Proof of Lemma 3.11. Suppose first that X has the Daugavet property. Consider a slice $S(x^*, \alpha)$ of B_X, an $x \in S_X$ and $\varepsilon > 0$. Without loss of generality we can assume that $\varepsilon/2 < \alpha$. Since the rank-1 operator T defined by $Ty = x^*(y)x$, $y \in X$, has norm 1, we have $\|I + T\| = 2$. Hence there is a vector $\tilde{x} \in S_X$ such that $\|\tilde{x} + T\tilde{x}\| \geq 2 - \varepsilon/2$ and $x^*(\tilde{x}) \geq 0$. It follows that $x^*(\tilde{x}) \geq 1 - \varepsilon/2$, so $\tilde{x} \in S(x^*, \alpha)$, and

$$\|x + \tilde{x}\| \geq \|\tilde{x} + T\tilde{x}\| - \|x - T\tilde{x}\| \geq 2 - \varepsilon/2 - 1 + x^*(\tilde{x}) \geq 2 - \varepsilon.$$

This proves the implication (i) $\Rightarrow$ (ii).

Let us now turn to the converse implication. Let T be a rank-1 operator on X. We may assume without loss of generality that $\|T\| = 1$. Hence T is defined by $Ty = x^*(y)x$ for some $x^* \in S_{X^*}$ and $x \in S_X$. We have to show that $\|I + T\| = 2$.

Let us fix an arbitrary $\varepsilon > 0$. Choose a vector $\tilde{x} \in S(x^*, \varepsilon/2)$ such that $\|x + \tilde{x}\| \geq 2 - \varepsilon/2$. Then we have

$$\begin{aligned}\|I + T\| \geq \|\tilde{x} + T\tilde{x}\| = \|x + \tilde{x} - (x - T\tilde{x})\| \geq \|x + \tilde{x}\| - 1 + x^*(\tilde{x})\\ > 2 - \varepsilon/2 - 1 + 1 - \varepsilon/2 = 2 - \varepsilon.\end{aligned}$$

Thus, $\|I + T\| = 2$. □

The following is our essential tool to complete the main task of this section.

Corollary 3.12. *If X has the Daugavet property, then for every slice $S(x^*, \alpha)$ of B_X, for every $x \in X$, and every $\varepsilon > 0$ there exists a vector $\tilde{x} \in S(x^*, \alpha)$ such that*

$$\|x + \lambda\tilde{x}\| \geq (\|x\| + \lambda)(1 - \varepsilon) \qquad \text{for all } \lambda \geq 0.$$

We base the proof on the following elementary fact (cf., e.g, [2, Problem 11.1.2]).

If two elements x and $\tilde{x}$ in the unit ball of a normed space satisfy $\|x + \tilde{x}\| \geq 1 + c$ for some c, then $\|\kappa x + \lambda\tilde{x}\| \geq (\kappa + \lambda)c$ for all $\kappa, \lambda \geq 0$. Indeed, note that $c \leq 1$ and

$$\begin{aligned}\|\kappa x + \lambda\tilde{x}\| &\geq \max\{\kappa, \lambda\} \|x + \tilde{x}\| - \|(\kappa - \lambda)x\|\\ &\geq \max\{\kappa, \lambda\}(1 + c) - \max\{\kappa, \lambda\} + \min\{\kappa, \lambda\}\\ &\geq \max\{\kappa, \lambda\}c + \min\{\kappa, \lambda\}c = (\kappa + \lambda)c.\end{aligned}$$

Proof of Corollary 3.12. We may assume that $x \neq 0$. Find a vector $\tilde{x} \in S(x^*, \alpha)$ such that $\|x/\|x\| + \tilde{x}\| \geq 2 - \varepsilon$. Then $\|x + \lambda\tilde{x}\| \geq (\|x\| + \lambda)(1 - \varepsilon)$ for any $\lambda \geq 0$.

□

Now we are ready to prove the main result in this section.

Theorem 3.13 (see [1, Theorem 4.4])**.** *A Banach space with the Daugavet property has the strong diameter 2 property.*

Proof. Let X be a Banach space with the Daugavet property and fix a vector $x \in S_X$. We only have to show that $\operatorname{diam}(S \cup \{x\}) = 2$, where S is any convex combination of slices of B_X (cf. Remark *3*) after Lemma 3.11). Consider, for some $n \in \mathbb{N}$, slices $S_1, \dots, S_n$ of B_X, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. In order to show that $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i \cup \{x\}) = 2$ we fix an $0 < \varepsilon < 1$.

By Corollary 3.12, find one after another, elements $x_1 \in S_1, \dots, x_n \in S_n$ such that

$$
\begin{aligned}
&\|-x + \lambda_1 x_1\| \geq (\|x\| + \lambda_1)(1-\varepsilon),\\
&\|(-x + \lambda_1 x_1) + \lambda_2 x_2\| \geq (\|-x + \lambda_1 x_1\| + \lambda_2)(1-\varepsilon),\\
&\qquad\qquad\vdots\\
&\|(-x + \sum_{i=1}^{n-1} \lambda_i x_i) + \lambda_n x_n\| \geq (\| - x + \sum_{i=1}^{n-1} \lambda_i x_i\| + \lambda_n)(1-\varepsilon).
\end{aligned}
$$

Substituting all previous estimations in the last one it implies that

$$
\left\| -x + \sum_{i=1}^{n} \lambda_i x_i \right\| \geq (\|x\| + \sum_{i=1}^{n} \lambda_i)(1-\varepsilon)^n = 2(1-\varepsilon)^n.
$$

Therefore $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i \cup \{x\}) = 2$, and we are done.

□

We end this section with a note from [1] that the Daugavet property can be weakened so that it still implies the local diameter 2 property (cf. [20, Problem (7)]).

Proposition 3.14 (see [1, Proposition 2.2]). *A Banach space X has the local diameter 2 property provided it enjoys the property that $x \in \overline{\operatorname{conv}}\Delta_\varepsilon(x)$ for every $x \in S_X$ and $\varepsilon > 0$, where $\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}$.*

Proof. Let $S(x^*, \alpha)$ be a slice of B_X. Find an $x \in S(x^*, \alpha)$ with $\|x\| = 1$ and $\delta > 0$ such that $x^*(x) > 1 - \alpha + \delta$.

By the assumption, there are $n \in \mathbb{N}$, $x_1, \dots, x_n \in \Delta_\varepsilon(x)$, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $\|x - \sum_{i=1}^n \lambda_i x_i\| < \delta$. It follows that

$$x^*(\sum_{i=1}^{n} \lambda_i x_i) > x^*(x) - \delta > 1 - \alpha.$$

Therefore there is at least one index $i_0 \in \{1, \dots, n\}$ such that $x^*(x_{i_0}) > 1 - \alpha$. Indeed, if $x^*(x_i) \leq 1 - \alpha$ for every $i = 1, \dots, n$, then we would have that $x^*(\sum_{i=1}^{n} \lambda_i x_i) \leq 1 - \alpha$. Now x and x_{i_0} are elements in $S(x^*, \alpha)$ with $\|x - x_{i_0}\| \geq 2 - \varepsilon$. Hence X has the local diameter 2 property. □

It remains open whether one can conclude the diameter 2 or even the strong diameter 2 property in Proposition 3.14 (cf. Problem (a) in [1]).

We remark that the reverse implication does not hold in Proposition 3.14 ($X = \ell_\infty$, and $x = (1, 0, 0, \dots)$ would provide a counterexample).

3.4 Projective tensor products

In the current section, we will prove that the projective tensor product of two Banach spaces has the local diameter 2 property whenever one of these Banach spaces has the local diameter 2 property.

We begin by collecting preliminary material to understand the notion of the projective tensor product. We encourage the reader unfamiliar with this subject to consult [17]. To this end, let X and Y be Banach spaces.

Definition. A mapping $B\colon X \times Y \to \mathbb{R}$ is called a *bilinear form*, if it is linear in each variable, that is,

(i) $B(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 B(x_1, y) + \lambda_2 B(x_2, y)$,

(ii) $B(x, \mu_1 y_1 + \mu_2 y_2) = \mu_1 B(x, y_1) + \mu_2 B(x, y_2)$

for all $x_1, x_2, x \in X$, $y_1, y_2, y \in Y$ and all scalars $\lambda_1, \lambda_2, \mu_1, \mu_2$.

The vector space of all bilinear forms from $X \times Y$ is denoted by $B(X \times Y)$. Its algebraic dual is denoted by $B(X \times Y)^\sharp$. For $x \in X$ and $y \in Y$, we denote by $x \otimes y$ the linear functional on $B(X \times Y)$ given by

$$(x \otimes y)(B) = B(x, y)$$

for each $B \in B(X \times Y)$.

Definition. The *algebraic tensor product* $X \otimes Y$ of X and Y is the subspace of $B(X \times Y)^\sharp$ spanned by all functionals of the form $x \otimes y$, where $x \in X$ and $y \in Y$,

$$X \otimes Y = \operatorname{span}\{x \otimes y \colon x \in X, y \in Y\}.$$

Definition. A norm $\|\cdot\|$ on the algebraic tensor product $X \otimes Y$ is called a *crossnorm*, if

$$\|x \otimes y\| = \|x\| \, \|y\| \qquad \text{for all } x \in X, y \in Y.$$

Definition. The *projective norm* $\|\cdot\|_\pi$ on the algebraic tensor product $X \otimes Y$ is given by

$$\|u\|_\pi = \inf\{\sum_{i=1}^{n} \|x_i\| \, \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

The projective norm is a crossnorm. Denote by

$$X \otimes_\pi Y = (X \otimes Y, \|\cdot\|_\pi).$$

Unless X and Y are finite-dimensional, the space $X \otimes_\pi Y$ is not complete.

Definition. The completion of $X \otimes_\pi Y$ is called the *projective tensor product* of Banach spaces X and Y, and is denoted by $X \hat{\otimes}_\pi Y$.

The next result is stated in [1] (see Theorem 2.7 (i)) and in [3] (see introduction) without a proof. We will present it with a detailed proof.

Proposition 3.15. *Let X and Y be Banach spaces. If X or Y has the local diameter 2 property, then the projective tensor product $X \hat{\otimes}_\pi Y$ has the local diameter 2 property.*

Proof. Since $X \hat{\otimes}_\pi Y$ is isometrically isomorphic to $Y \hat{\otimes}_\pi X$, we may assume without loss of generality that Y has the local diameter 2 property. To this end, it is convenient to denote by Z the projective tensor product $X \hat{\otimes}_\pi Y$. Let $S(z^*, \alpha)$ be a slice of B_Z and $\varepsilon > 0$. We will show that there are elements z and $\tilde{z}$ in $S(z^*, \alpha)$ such that $\|z - \tilde{z}\| \geq 2 - \varepsilon$.

It is known that Z^* is isometrically isomorphic to $\mathcal{L}(X, Y^*)$ (see, e.g., [17, Theorem 2.9]), where every $A \in \mathcal{L}(X, Y^*)$ is identified with the functional

$$x \otimes y \mapsto (Ax)(y) \qquad \text{for all } x \otimes y \in Z.$$

Let A be the functional in $\mathcal{L}(X, Y^*)$ that corresponds to z^*. Find $x \in S_X$ with $(1 - \alpha/2) \|Ax\| \geq 1 - \alpha$. We take $y^* = Ax / \|Ax\|$. Then clearly $y^* \in S_{Y^*}$. Consider the slice $S(y^*, \alpha/2)$ of B_Y. Since Y has the local diameter 2 property, we can find elements y and $\tilde{y}$ in $S(y^*, \alpha/2)$ with $\|y - \tilde{y}\| \geq 2 - \varepsilon$.

We take $z = x \otimes y$ and $\tilde{z} = x \otimes \tilde{y}$. Then $z, \tilde{z} \in B_Z$ and $\|z - \tilde{z}\| \geq 2 - \varepsilon$, because π is a crossnorm. Note that

$$z^*(z) = (Ax)(y) = \|Ax\| \, y^*(y) > \|Ax\| \, (1 - \frac{\alpha}{2}) \geq 1 - \alpha.$$

Similarly, $z^*(\tilde{z}) > 1 - \alpha$. Thus, $z, \tilde{z} \in S(z^*, \alpha)$, and we have proved the Proposition 3.15.

□

It is unclear to us whether similar result to Proposition 3.15 holds for the (strong) diameter 2 property.

In [3], it is shown that $X \hat{\otimes}_\pi Y$ has the diameter 2 property under some special assumptions for X and Y.

3.5 ℓ_p-sums

In this section, we study how the diameter 2 properties are preserved by taking ℓ_p-sums of Banach spaces.

In [14, Lemma 2.1], it is shown that $X \oplus_\infty Y$ has the diameter 2 property whenever X or Y has the diameter 2 property (see also [5, Lemma 2.2], or the remark after Lemma 3.17). If we assume that one of the spaces X or Y has the strong diameter 2 property, then this is true even for the strong diameter 2 property [1, Proposition 4.6].

Proposition 3.16 (see [1, Proposition 4.6]). *Let X and Y be Banach spaces. If X has the strong diameter 2 property, then $X \oplus_\infty Y$ has the strong diameter 2 property.*

In fact, we give two different proofs to Proposition 3.16—first, we follow the idea from [1], and present a direct proof at the end. In the next section we generalize the result (see Proposition 3.28).

Lemma 3.17 (see [1, Lemma 4.5]). *Let X and Y be Banach spaces, W a nonempty weakly open subset in $Z = X \oplus_\infty Y$, and $(x_0, y_0) \in W$. There exist weakly open subsets U of X and V of Y such that $(x_0, y_0) \in U \times V \subset W$. Moreover, if W is a relatively weakly open subset of B_Z, then U and V can be chosen to be relatively weakly open subsets of B_X and B_Y respectively.*

Remark. It is immediate from Lemma 3.17 that $Z = X \oplus_\infty Y$ has the diameter 2 property whenever X or Y has the diameter 2 property.

Proof of Lemma 3.17. We may assume that

$$W_0 = \{(x, y) \in Z\colon\ |z_i^*(x, y) - z_i^*(x_0, y_0)| < 1, i = 1, \dots, n\} \subset W$$

for some $n \in \mathbb{N}$ and $z_1^* = (x_1^*, y_1^*), \dots, z_n^* = (x_n^*, y_n^*) \in X^* \oplus_1 Y^*$.

Set

$$U = \{x \in X\colon\ |x_i^*(x) - x_i^*(x_0)| < \frac{1}{2}, i = 1, \dots, n\}$$

and
$$V = \{y \in Y\colon\ |y_i^*(y) - y_i^*(y_0)| < \frac{1}{2}, i = 1, \ldots, n\}.$$
Then U and V are weakly open in X and Y respectively, and $(x_0, y_0) \in U \times V \subset W_0$. In this part of the proof, we have not used the fact that Z is equipped with the supremum norm.

For the last part, notice that because of the supremum norm $B_Z = B_X \times B_Y$, and just redefine $U = U \cap B_X$ and $V = V \cap B_Y$.

□

Proof of Proposition 3.16. Our proof is a slight modification of the proof in [1].

Let $Z = X \oplus_\infty Y$ and let $P\colon Z \to X$ be the natural projection onto X. Let $S = \sum_{i=1}^n \lambda_i S_i$, where $n \in \mathbb{N}$, $S_1, \ldots, S_n$ are slices of B_Z, and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$.

We recall that slices $S_1, \ldots, S_n$ are relatively weakly open in B_Z. It follows by Lemma 3.17 that for every $i = 1, \ldots, n$ one can find relatively weakly open subsets U_i of B_X and V_i of B_Y such that $U_i \times V_i \subset S_i$.

We have now
$$P(S) = \sum_{i=1}^n \lambda_i P(S_i) \supset \sum_{i=1}^n \lambda_i U_i.$$
By Lemma 3.1, $\operatorname{diam}(\sum_{i=1}^n \lambda_i U_i) = 2$. Since $\|P\| = 1$, we must have
$$\operatorname{diam}(P(S)) = \operatorname{diam}(S) = 2.$$

□

In a similar fashion, we can prove the weak* version of Proposition 3.16 (cf. [1, Proposition 4.9]).

Proposition 3.18. *Let Z be a Banach space, X and Y closed subspaces of Z such that $Z = X \oplus_1 Y$. If every convex combination of weak* slices of B_{X^*} has diameter 2, then every convex combination of weak* slices of B_{Z^*} has diameter 2.*

To prove Proposition 3.18, we need the following lemma.

Lemma 3.19. *Let Z be a Banach space, and let X and Y be closed subspaces of Z such that $Z = X \oplus_1 Y$. Let W be a nonempty weak* open subset in Z^*, and $(x_0^*, y_0^*) \in W$. There exist weak* open subsets U of X^* and V of Y^* such that $(x_0^*, y_0^*) \in U \times V \subset W$. Moreover, if W is a relatively weak* open subset of B_{Z^*}, then U and V can be chosen to be relatively weak* open subsets of B_{X^*} and B_{Y^*} respectively.*

Proof. Denote by $z_0^* = (x_0^*, y_0^*)$. We may assume that

$$W_0 = \{z^* \in Z^* \colon\ |(z^* - z_0^*)(z_i)| < 1, i = 1, \dots, n\} \subset W,$$

for some $n \in \mathbb{N}$ and $z_1 = (x_1, y_1), \dots, z_n = (x_n, y_n) \in X \oplus_1 Y$.

Set

$$U = \{x^* \in X^* \colon\ |x^*(x_i) - x_0^*(x_i)| < \frac{1}{2}, i = 1, \dots, n\}$$

and

$$V = \{y^* \in Y^* \colon\ |y^*(y_i) - y_0^*(y_i)| < \frac{1}{2}, i = 1, \dots, n\}.$$

Then U and V are weak* open in X^* and Y^* respectively, and $(x_0^*, y_0^*) \in U \times V \subset W_0$. In this part of the proof, we have not used the fact that Z^* is equipped with the supremum norm.

For the last part, notice that because of the supremum norm $B_{Z^*} = B_{X^*} \times B_{Y^*}$, and just redefine $U = U \cap B_{X^*}$ and $V = V \cap B_{Y^*}$.

□

Proof of Proposition 3.18. Notice first that $Z^* = X^* \oplus_\infty Y^*$. Let $P \colon Z^* \to X^*$ be the natural projection onto X^*. Let $S^* = \sum_{i=1}^n \lambda_i S_i^*$, where $n \in \mathbb{N}$, $S_1^*, \dots, S_n^*$ are weak* slices of B_{Z^*}, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$.

We recall that weak* slices $S_1^*, \dots, S_n^*$ are relatively weak* open in B_{Z^*}. It follows by Lemma 3.19 that for every $i = 1, \dots, n$ one can find relatively weak* open subsets U_i of B_{X^*} and V_i of B_{Y^*} such that $U_i \times V_i \subset S_i^*$.

We have now

$$P(S^*) = \sum_{i=1}^n \lambda_i P(S_i^*) \supset \sum_{i=1}^n \lambda_i U_i.$$

By Lemma 3.2, $\operatorname{diam}(\sum_{i=1}^{n} \lambda_i U_i) = 2$. Since $\|P\| = 1$, we must have

$$\operatorname{diam}(P(S^*)) = \operatorname{diam}(S^*) = 2.$$

□

Now we want to present a direct proof of Proposition 3.16. (Similarly, one can give a direct proof to Proposition 3.18.)

First, observe that if $Z = X \oplus_\infty Y$, then for every slice $S(z^*, \alpha)$ of B_Z there exists a slice S of B_X and $y \in B_Y$ such that

$$S(z^*, \alpha) \supset S \times \{y\}.$$

Indeed, let $z^* = (x^*, y^*) \in S_{Z^*} = S_{X^* \oplus_1 Y^*}$ and let $\alpha > 0$. If $x^* = 0$, then $S(z^*, \alpha) \supset B_X \times \{y\}$ for any y in the slice $S(y^*, \alpha)$ of B_Y. This proves our result since B_X can also be considered as a slice.

Assume now that $x^* \neq 0$. Choose an $y \in B_Y$ such that $y^*(y) > \|y^*\| - \alpha/2$. It is straightforward to verify that

$$S(z^*, \alpha) \supset S(x^*/\|x^*\|, \beta/\|x^*\|) \times \{y\},$$

where $\beta = \alpha + \|x^*\| + y^*(y) - 1$.

This observation clearly implies that if X has the local diameter 2 property, then also $X \oplus_\infty Y$ has the local diameter 2 property.

Second proof of Proposition 3.16. Let $Z = X \oplus_\infty Y$ and let $P\colon Z \to X$ be the natural projection onto X. Let $S = \sum_{i=1}^{n} \lambda_i S_i$, where $n \in \mathbb{N}$, $S_1, \ldots, S_n$ are slices of B_Z, and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$.

By the observation above, there exist slices $\tilde{S}_1, \ldots, \tilde{S}_n$ of B_X and elements $y_1, \ldots, y_n \in B_Y$ such that

$$\sum_{i=1}^{n} \lambda_i S_i \supset \sum_{i=1}^{n} \lambda_i (\tilde{S}_i \times \{y_i\}).$$

We have now that

$$P(S) = \sum_{i=1}^{n} \lambda_i P(S_i) \supset \sum_{i=1}^{n} \lambda_i \tilde{S}_i.$$

Since $\|P\| = 1$, it follows that

$$2 = \operatorname{diam}(P(S)) = \operatorname{diam}(S).$$

$\square$

Next, we will show that if X and Y have the strong diameter 2 property, then $X \oplus_1 Y$ has the strong diameter 2 property. The proof is essentially due to Becerra Guerrero and López Pérez in [5, proof of Lemma 2.1 (ii)]. It is remarkable that in [5] their proof does not fit to the corresponding statement—if X or Y has the diameter 2 property, then $X \oplus_1 Y$ has the diameter 2 property. Later, we will see that the above-mentioned result in [5] is also true (see Theorem 3.22).

Proposition 3.20 (cf. [5, Lemma 2.1]). *Let X and Y be Banach spaces. If X and Y have the strong diameter 2 property, then $X \oplus_1 Y$ has the strong diameter 2 property.*

Proof. Denote by $Z = X \oplus_1 Y$. Let $S = \sum_{i=1}^n \lambda_i S(z_i^*, \alpha_i)$ be a convex combination of slices of B_Z, where $n \in \mathbb{N}$, $z_1^* = (x_1^*, y_1^*), \dots, z_n^* = (x_n^*, y_n^*) \in S_{Z^*}$, $\alpha_1, \dots, \alpha_n > 0$, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. We will show that the diameter of S is arbitrarily close to 2.

Split the set $\{1, \dots, n\}$ into two disjoint subsets I and J, such that $\|x_i^*\| = 1$ for every $i \in I$ and $\|y_j^*\| = 1$ for every $j \in J$. For every $i \in I$ consider the slice $S(x_i^*, \alpha_i)$ of B_X and for every $j \in J$ consider the slice $S(y_j^*, \alpha_j)$ of B_Y. Observe that $S(x_i^*, \alpha_i) \times \{0\} \subset S(z_i^*, \alpha_i)$ for every $i \in I$ and $\{0\} \times S(y_j^*, \alpha_j) \subset S(z_j^*, \alpha_j)$ for every $j \in J$.

Denote by $\lambda_I = \sum_{i \in I} \lambda_i$ and $\lambda_J = \sum_{j \in J} \lambda_j$. Assume first that $\lambda_I = 0$ or $\lambda_J = 0$. To be more specific, suppose that $\lambda_J = 0$, then $\lambda_I = 1$. Let $\varepsilon > 0$. Since X has the strong diameter 2 property, there are elements $x, \tilde{x} \in \sum_{i \in I} \lambda_i S(x_i^*, \alpha_i)$ such that $\|x - \tilde{x}\| > 2 - \varepsilon$. Note that $(x, 0), (\tilde{x}, 0)$ are elements in S. Finally,

$$\operatorname{diam}(S) \geq \|x - \tilde{x}\| > 2 - \varepsilon.$$

Suppose now that $\lambda_I \neq 0$ and $\lambda_J \neq 0$. We have that

$$\sum_{i\in I} \frac{\lambda_i}{\lambda_I} S(x_i^*, \alpha_i) \times \{0\} \subset \sum_{i\in I} \frac{\lambda_i}{\lambda_I} S(z_i^*, \alpha_i)$$

and

$$\{0\} \times \sum_{j\in J} \frac{\lambda_j}{\lambda_J} S(y_j^*, \alpha_j) \subset \sum_{j\in J} \frac{\lambda_j}{\lambda_J} S(z_j^*, \alpha_j).$$

Let $\varepsilon > 0$. Since X and Y both have the strong diameter 2 property, there are elements $x, \tilde{x} \in \sum_{i\in I} \frac{\lambda_i}{\lambda_I} S(x_i^*, \alpha_i)$ and $y, \tilde{y} \in \sum_{j\in J} \frac{\lambda_j}{\lambda_J} S(y_j^*, \alpha_j)$ such that $\|x - \tilde{x}\| > 2 - \varepsilon$ and $\|y - \tilde{y}\| > 2 - \varepsilon$. Note that $(\lambda_I x, \lambda_J y)$ is an element in S, because

$$\begin{aligned}
(\lambda_I x, \lambda_J y) &= (\lambda_I x, 0) + (0, \lambda_J y) \\
&\in \sum_{i\in I} \lambda_i S(x_i^*, \alpha_i) \times \{0\} + \{0\} \times \sum_{j\in J} \lambda_j S(y_j^*, \alpha_j) \\
&\subset \sum_{i\in I} \lambda_i S(z_i^*, \alpha_i) + \sum_{j\in J} \lambda_j S(z_j^*, \alpha_j) = S.
\end{aligned}$$

Similarly, $(\lambda_I \tilde{x}, \lambda_J \tilde{y})$ is in S. Finally,

$$\operatorname{diam}(S) \geq \lambda_I \|x - \tilde{x}\| + \lambda_J \|y - \tilde{y}\| > 2 - \varepsilon.$$

□

The next result is stated in [1] without a proof. We will present it with a detailed proof.

Theorem 3.21 (see [1, Theorem 3.2]). *Let X and Y be Banach spaces and $1 \leq p \leq \infty$. If X and Y have the local diameter 2 property, then $X \oplus_p Y$ has the local diameter 2 property.*

Proof. To begin with, note that the case $p = \infty$ is already done (see the observation before the second proof of Proposition 3.16). We assume therefore that $p < \infty$.

Denote by $Z = X \oplus_p Y$. Let q be such that $1/p + 1/q = 1$, if $p > 1$; and $q = \infty$, if $p = 1$. Consider a slice $S(z^*, \alpha)$ of B_Z, where $z^* = (x^*, y^*) \in$

$S_{Z^*} = S_{X^*\oplus_q Y^*}$ and $\alpha > 0$. Without loss of generality we may assume that $\alpha \leq 1$. Fix an arbitrary $\varepsilon > 0$. We will show the existence of elements in $S(z^*, \alpha)$ with distance arbitrarily close to 2.

Assume that $x^* = 0$ or $y^* = 0$. To be more specific, suppose that $y^* = 0$, then $x^* \in S_{X^*}$. The other case is similar. Consider the slice $S(x^*, \alpha)$ of B_X. By the assumption, we can find $x, \tilde{x} \in S(x^*, \alpha)$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$. We take $z = (x, 0)$ and $\tilde{z} = (\tilde{x}, 0)$. Clearly, z and $\tilde{z}$ are in $S(z^*, \alpha)$ with $\|z - \tilde{z}\| \geq 2 - \varepsilon$.

Consider now the case $x^* \neq 0$ and $y^* \neq 0$. Find an element $z_0 = (x_0, y_0) \in S(z^*, \alpha/4)$ with $\|z_0\| = 1$.

Choose $x, \tilde{x} \in S(x^*/\|x^*\|, \alpha/2)$ and $y, \tilde{y} \in S(y^*/\|y^*\|, \alpha/2)$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$ and $\|y - \tilde{y}\| \geq 2 - \varepsilon$. We take $z = (\|x_0\| x, \|y_0\| y)$ and $\tilde{z} = (\|x_0\| \tilde{x}, \|y_0\| \tilde{y})$. Observe that $z, \tilde{z} \in S(z^*, \alpha)$. In fact,

$$\|z\|^p = \|x_0\|^p \|x\|^p + \|y_0\|^p \|y\|^p \leq \|x_0\|^p + \|y_0\|^p = 1,$$

and

$$\begin{aligned} z^*(z) &= \|x_0\| x^*(x) + \|y_0\| y^*(y) > (\|x_0\| \|x^*\| + \|y_0\| \|y^*\|)(1 - \alpha/2) \\ &\geq z^*(z_0)(1 - \alpha/2) > (1 - \alpha/4)(1 - \alpha/2) > 1 - \alpha. \end{aligned}$$

Thus, $z \in S(z^*, \alpha)$. Similarly we have $\tilde{z} \in S(z^*, \alpha)$. Finally,

$$\begin{aligned} \|z - \tilde{z}\|^p &= \|x_0\|^p \|x - \tilde{x}\|^p + \|y_0\|^p \|y - \tilde{y}\|^p \\ &\geq (2 - \varepsilon)^p (\|x_0\|^p + \|y_0\|^p) = (2 - \varepsilon)^p. \end{aligned}$$

□

In [1], it is shown that the diameter 2 property is stable by taking ℓ_p-sums of Banach spaces for all $1 \leq p \leq \infty$. Our proof, however, is slightly different from [1]; we like to think that our approach is more direct.

Theorem 3.22 (see [1, Theorem 3.2])**.** *Let X and Y be Banach spaces and $1 \leq p \leq \infty$. If X and Y have the diameter 2 property, then $X \oplus_p Y$ has the diameter 2 property.*

Proof. To begin with, note that the case $p = \infty$ is already done (see the remark after Lemma 3.17). We assume therefore that $p < \infty$.

Denote by $Z = X \oplus_p Y$. Let W be a nonempty relatively weakly open subset of B_Z. Fix $z_0 = (x_0, y_0) \in W \cap S_Z$. We may assume that

$$W \supset \{z \in B_Z \colon |z_i^*(z - z_0)| < 1, \quad i = 1, \dots, n\}$$

for some $n \in \mathbb{N}$, $z_1^* = (x_1^*, y_1^*), \dots, z_n^* = (x_n^*, y_n^*) \in Z^*$. Fix an arbitrary $\varepsilon > 0$. We will show the existence of elements in W with distance arbitrarily close to 2.

Assume that $x_0 = 0$ or $y_0 = 0$. To be more specific, suppose that $y_0 = 0$, then $x_0 \in S_X$. The other case is similar. The set

$$U = \{x \in B_X \colon |x_i^*(x - x_0)| < 1, \quad i = 1, \dots, n\}$$

is a nonempty relatively weakly open subset of B_X. By assumption, we can find $x, \tilde{x} \in U$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$. We take $z = (x, 0)$ and $\tilde{z} = (\tilde{x}, 0)$. Clearly, z and $\tilde{z}$ are elements in W with $\|z - \tilde{z}\| \geq 2 - \varepsilon$.

Suppose now that $x_0 \neq 0$ and $y_0 \neq 0$. Consider the sets

$$U = \{x \in B_X \colon |x_i^*(x - \frac{x_0}{\|x_0\|})| < \frac{1}{2\,\|x_0\|}, \quad i = 1, \dots, n\},$$

and

$$V = \{y \in B_Y \colon |y_i^*(y - \frac{y_0}{\|y_0\|})| < \frac{1}{2\,\|y_0\|}, \quad i = 1, \dots, n\}.$$

Clearly, U and V are nonempty relatively weakly open subsets of B_X and B_Y respectively.

By the assumption, we can find $x, \tilde{x} \in U$ and $y, \tilde{y} \in V$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$ and $\|y - \tilde{y}\| \geq 2 - \varepsilon$. We take $z = (\|x_0\|\, x, \|y_0\|\, y)$ and $\tilde{z} = (\|x_0\|\, \tilde{x}, \|y_0\|\, \tilde{y})$. Observe that $z, \tilde{z} \in W$. In fact

$$\|z\|^p = \|x_0\|^p \|x\|^p + \|y_0\|^p \|y\|^p \leq \|x_0\|^p + \|y_0\|^p = 1,$$

and

$$\begin{aligned}
|z_i^*(z - z_0)| &= |x_i^*(\|x_0\| \, x - x_0) + y_i^*(\|y_0\| \, y - y_0)| \\
&\le \|x_0\| \, |x_i^*(x - \frac{x_0}{\|x_0\|})| + \|y_0\| \, |y_i^*(y - \frac{y_0}{\|y_0\|})| \\
&< \|x_0\| \frac{1}{2\,\|x_0\|} + \|y_0\| \frac{1}{2\,\|y_0\|} = 1,
\end{aligned}$$

for fixed $i = 1, \ldots, n$. Thus, $z \in W$. Similarly, $\tilde{z} \in W$. Finally,

$$\begin{aligned}
\|z - \tilde{z}\|^p &= \|x_0\|^p \, \|x - \tilde{x}\|^p + \|y_0\|^p \, \|y - \tilde{y}\|^p \\
&\ge (2 - \varepsilon)^p (\|x_0\|^p + \|y_0\|^p) = (2 - \varepsilon)^p.
\end{aligned}$$

□

We end this section by showing that if X and Y are nontrivial Banach spaces, then the Banach space $X \oplus_p Y$ does not have the strong diameter 2 property for each $1 < p < \infty$. With this we are going to present a negative answer to question (c) in [1]. Moreover, it is conjectured in [1] that all three diameter 2 properties are in general different. Our next result also shows that at least the diameter 2 property differs from the strong diameter 2 property.

Theorem 3.23. *Let X and Y be nontrivial Banach spaces and let $1 < p < \infty$. The Banach space $Z = X \oplus_p Y$ does not have the strong diameter 2 property.*

To prove this theorem, we will need the following lemma.

Lemma 3.24. *Let $1 < q < \infty$ be such that $1/p + 1/q = 1$. If $z^* = (x^*, y^*)$ is an element in $S_{Z^*} = S_{X^* \oplus_q Y^*}$, then for every $\varepsilon > 0$ there exists $\alpha > 0$ such that*

$$\Big| \, \|x\| - \|x^*\|^{q-1} \Big| + \Big| \, \|y\| - \|y^*\|^{q-1} \Big| < \varepsilon,$$

whenever $z = (x, y)$ is an element in $S(z^*, \alpha)$.

Proof. Note that if $z = (x, y)$ is an element in $S(z^*, \alpha)$, then $(\|x\|, \|y\|)$ and $(\|x^*\|^{q-1}, \|y^*\|^{q-1})$ are both elements of the slice $S((\|x^*\|, \|y^*\|), \alpha)$ of $B_{\ell_p^2}$. Obviously, when α tends to 0, then $\operatorname{diam}(S((\|x^*\|, \|y^*\|), \alpha))$ tends to 0 as well. This proves the result. □

Proof of Theorem 3.23. We will show that, for every $\lambda \in (0, 1)$, there exists $\alpha > 0$ and $\beta > 0$ such that

$$\lambda S(z^*, \alpha) + (1-\lambda)S(\tilde{z}^*, \alpha) \subset (1-\beta)B_Z,$$

where $S(z^*, \alpha)$ and $S(\tilde{z}^*, \alpha)$ are two suitable slices of B_Z.

Let $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$. We take $z^* = (x^*, 0)$ and $\tilde{z}^* = (0, y^*)$. Then z and $\tilde{z}$ are elements in S_{Z^*}. Fix $\lambda \in (0, 1)$. Denote by

$$\varepsilon = 1 - \Big(\lambda^p + (1-\lambda)^p\Big)^{1/p}.$$

Clearly, $\varepsilon > 0$. By Lemma 3.24, there exists $\alpha > 0$ such that

$$\begin{aligned}
&\left(\Big(\lambda\|x\| + (1-\lambda)\|\tilde{x}\|\Big)^p + \Big(\lambda\|y\| + (1-\lambda)\|\tilde{y}\|\Big)^p\right)^{1/p} \\
&\le \left(\Big(\lambda \cdot 1 + (1-\lambda)\cdot 0\Big)^p + \Big(\lambda \cdot 0 + (1-\lambda)\cdot 1\Big)^p\right)^{1/p} + \frac{\varepsilon}{2} \\
&= \Big(\lambda^p + (1-\lambda)^p\Big)^{1/p} + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2},
\end{aligned}$$

whenever $z = (x, y) \in S(z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(\tilde{z}^*, \alpha)$.

One may take $\beta = \varepsilon/2$. Indeed, for $z = (x, y) \in S(z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(\tilde{z}^*, \alpha)$, we have now

$$\begin{aligned}
&\|\lambda z + (1-\lambda)\tilde{z}\| = \Big(\|\lambda x + (1-\lambda)\tilde{x}\|^p + \|\lambda y + (1-\lambda)\tilde{y}\|^p\Big)^{1/p} \\
&\le \left(\Big(\lambda\|x\| + (1-\lambda)\|\tilde{x}\|\Big)^p + \Big(\lambda\|y\| + (1-\lambda)\|\tilde{y}\|\Big)^p\right)^{1/p} \\
&\le 1 - \frac{\varepsilon}{2}.
\end{aligned}$$

□

Remark. In the proof of Theorem 3.23, the constructed convex combination of slices is, in particular, not relatively weakly open. In fact, we can point out a lot of convex combinations of slices in $X \oplus_p Y$, which are not relatively weakly open.

We will show that, if functionals z^* and $\tilde{z}^*$ in S_{Z^*} are different enough, then, for every $\lambda \in (0,1)$, there exists $\alpha > 0$ and $\beta > 0$ such that

$$\lambda S(z^*, \alpha) + (1-\lambda) S(\tilde{z}^*, \alpha) \subset (1-\beta) B_Z,$$

where $S(z^*, \alpha)$ and $S(\tilde{z}^*, \alpha)$ are two suitable slices of B_Z. Hence $\lambda S(z^*, \alpha) + (1-\lambda) S(\tilde{z}^*, \alpha)$ cannot be relatively weakly open.

Let $z^* = (x^*, y^*), \tilde{z}^* = (\tilde{x}^*, \tilde{y}^*) \in S_{Z^*}$ be such that $(\|x^*\|, \|y^*\|)$ and $(\|\tilde{x}^*\|, \|\tilde{y}^*\|)$ are linearly independent vectors in $\mathbb{R} \times \mathbb{R}$. Fix $\lambda \in (0,1)$. Denote by

$$\varepsilon = 1 - \left(\left(\lambda \|x^*\|^{q-1} + (1-\lambda) \|\tilde{x}^*\|^{q-1} \right)^p + \left(\lambda \|y^*\|^{q-1} + (1-\lambda) \|\tilde{y}^*\|^{q-1} \right)^p \right)^{1/p}.$$

Using the Minkowski inequality, we have that

$$\begin{aligned}
&\left(\left(\lambda \|x^*\|^{q-1} + (1-\lambda) \|\tilde{x}^*\|^{q-1} \right)^p + \left(\lambda \|y^*\|^{q-1} + (1-\lambda) \|\tilde{y}^*\|^{q-1} \right)^p \right)^{1/p} \\
&< \left((\lambda \|x^*\|^{q-1})^p + (\lambda \|y^*\|^{q-1})^p \right)^{1/p} \\
&\quad + \left(((1-\lambda) \|\tilde{x}^*\|^{q-1})^p + ((1-\lambda) \|\tilde{y}^*\|^{q-1})^p \right)^{1/p} \\
&= \left(\lambda^p \|x^*\|^q + \lambda^p \|y^*\|^q \right)^{1/p} + \left((1-\lambda)^p \|\tilde{x}^*\|^q + (1-\lambda)^p \|\tilde{y}^*\|^q \right)^{1/p} \\
&= \lambda + 1 - \lambda = 1.
\end{aligned}$$

Hence $\varepsilon > 0$. By Lemma 3.24, there exists $\alpha > 0$ such that

$$\begin{aligned}
&\Big(\Big(\lambda\|x\| + (1-\lambda)\|\tilde{x}\|\Big)^p + \Big(\lambda\|y\| + (1-\lambda)\|\tilde{y}\|\Big)^p\Big)^{1/p} \\
&\leq \Big(\Big(\lambda\|x^*\|^{q-1} + (1-\lambda)\|\tilde{x}^*\|^{q-1}\Big)^p \\
&\quad + \Big(\lambda\|y^*\|^{q-1} + (1-\lambda)\|\tilde{y}^*\|^{q-1}\Big)^p\Big)^{1/p} + \frac{\varepsilon}{2},
\end{aligned}$$

when $z = (x, y) \in S(z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(\tilde{z}^*, \alpha)$.

Finally, we will show that one may take $\beta = \varepsilon/2$. Indeed, for $z = (x, y) \in S(z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(\tilde{z}^*, \alpha)$, we have

$$\begin{aligned}
&\|\lambda z + (1-\lambda)\tilde{z}\| = \Big(\|\lambda x + (1-\lambda)\tilde{x}\|^p + \|\lambda y + (1-\lambda)\tilde{y}\|^p\Big)^{1/p} \\
&\leq \Big(\Big(\lambda\|x\| + (1-\lambda)\|\tilde{x}\|\Big)^p + \Big(\lambda\|y\| + (1-\lambda)\|\tilde{y}\|\Big)^p\Big)^{1/p} \\
&\leq \Big(\Big(\lambda\|x^*\|^{q-1} + (1-\lambda)\|\tilde{x}^*\|^{q-1}\Big)^p \\
&\quad + \Big(\lambda\|y^*\|^{q-1} + (1-\lambda)\|\tilde{y}^*\|^{q-1}\Big)^p\Big)^{1/p} + \frac{\varepsilon}{2} \\
&= 1 - \varepsilon + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2}.
\end{aligned}$$

3.6 M-ideals

The question whether M-ideals have the diameter 2 property was probably first considered in [14], further investigation was carried out in [1].

For example, it was shown in [1] that if Y is a strict M-ideal in X, then both Y and X have the strong diameter 2 property.

We denote the *annihilator* of a subspace Y of a Banach space X by

$$Y^{\perp} = \{x^* \in X^* \colon x^*(y) = 0 \quad \text{for all } y \in Y\}.$$

Definition (cf., e.g., [11]). Let X be a Banach space. A closed subspace $Y \subset X$ is called an *M-ideal* if there exists a norm-1 projection P on X^* with $\ker P = Y^{\perp}$ and

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \qquad \text{for all } x^* \in X^*.$$

Definition (see, e.g., [8, p. 160]). Let X be a Banach space. A subspace Z of X^* is called 1-*norming* if

$$\|x\| = \sup\{|z(x)| : z \in B_{X^*} \cap Z\} \qquad \text{for all } x \in X.$$

By the Hahn–Banach theorem, X^* is 1-norming.

Definition (cf., e.g., [10]). An M-ideal is called a *strict M-ideal* if the range of the corresponding projection is 1-norming.

The following result is inspired from [14, Proposition 2.3].

Proposition 3.25 (cf. [14, Proposition 2.3]). *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X, that is $X^* = Z \oplus_1 Y^{\perp}$ for some subspace Z of X^*. Then every nonempty convex combination of $\sigma(X, Z)$-slices of B_X which intersects B_Y has diameter 2.*

Proof. Our proof is inspired by the proofs of [14, Proposition 2.3] and [1, Theorem 4.10].

Consider $S = \sum_{i=1}^{n} \lambda_i S_i$, where

$$S_i = \{x \in B_X : z_i(x) > 1 - \alpha_i\} \cap B_Y,$$

$n \in \mathbb{N}$, $z_1, \ldots, z_n \in S_Z$, $\alpha_1, \ldots, \alpha_n > 0$, and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. We will show that the diameter of S is equal to 2.

Let $\delta > 0$. Due to Riesz' lemma (see, e.g., [8, Lemma 1.37]), there exists an $x \in S_X$ such that $d(x, Y) > 1 - \delta$. By [19, Proposition 2.3], there is a net (y_β) in Y such that (y_β) converges to x in the $\sigma(X, Z)$-topology and

$$\limsup_\beta \|y \pm (x - y_\beta)\| \leq 1 \qquad \text{for all } y \in B_Y.$$

Fix an element $\sum_{i=1}^{n} \lambda_i y_i$ in S. Let $\varepsilon > 0$ be such that $z_i(y_i) > 1 - \alpha_i + \varepsilon$ for every $i = 1, \ldots, n$. Choose an index β_1 such that $(1 - \varepsilon/2) \|y_i \pm (x - y_\beta)\| \leq 1$ for every $i = 1, \ldots, n$ and $\beta \geq \beta_1$. Since the net (y_β) converges to x in the $\sigma(X, Z)$-topology, there is an index β_2 such that $|z_i(x) - z_i(y_\beta)| \leq \varepsilon/2$ for every $i = 1, \ldots, n$ and $\beta \geq \beta_2$. If $\beta \geq \beta_1$ and $\beta \geq \beta_2$, then $(1 - \varepsilon/2)(y_i \pm (x - y_\beta)) \in S_i$ for all $i = 1, \ldots, n$. Indeed,

$$\begin{aligned}
(1 - \frac{\varepsilon}{2}) z_i(y_i \pm (x - y_\beta)) &\geq z_i(y_i) - \frac{\varepsilon}{2} z_i(y_i) - (1 - \frac{\varepsilon}{2}) |z_i(x) - z_i(y_\beta)| \\
&> 1 - \alpha_i + \varepsilon - \frac{\varepsilon}{2} - |z_i(x) - z_i(y_\beta)| \\
&\geq 1 - \alpha_i + \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \alpha_i.
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{diam}(S) &\geq \left\| \sum_{i=1}^{n} \lambda_i (1 - \frac{\varepsilon}{2})(y_i + (x - y_\beta)) - \sum_{i=1}^{n} \lambda_i (1 - \frac{\varepsilon}{2})(y_i - (x - y_\beta)) \right\| \\
&= 2(1 - \frac{\varepsilon}{2}) \|x - y_\beta\| > 2(1 - \frac{\varepsilon}{2})(1 - \delta).
\end{aligned}$$

Since δ and ε can be arbitrarily small, we obtain that the diameter of S is 2.

$\square$

Now we are ready to show that if Y is a strict M-ideal in X, then both Y and X have the strong diameter 2 property.

Theorem 3.26 (see [1, Theorem 4.10])**.** *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X, that is $X^* = Z \oplus_1 Y^\perp$ for some subspace Z of X^*. If Z is 1-norming for X, then both Y and X have the strong diameter 2 property.*

Proof. Our proof is slightly different from [1], it is inspired by the proof of [14, Theorem 2.4].

We will present the proof in two parts. First we will show that Y has the strong diameter 2 property, and then we will deduce the property for X.

Part 1. We identify Y^* and Z (see [11, Proposition I.1.12]). Every slice $S(y^*, \alpha)$ of B_Y therefore corresponds to some $\sigma(X,Z)$-slice $S(z,\alpha) = \{x \in B_X : z(x) > 1-\alpha\}$ of B_X. In fact, $S(y^*,\alpha)$ is dense in $S(z,\alpha)$ with respect to $\sigma(X,Z)$-topology. To see this, consider an $x \in S(z,\alpha)$. Since $X^{**} = Y^{\perp\perp} \oplus_\infty Z^\perp$ (see, e.g., [11]), there exists $u \in B_{Y^{\perp\perp}}$ and $v \in B_{Z^\perp}$ such that $x = u + v$. We also identify $Y^{\perp\perp}$ and Y^{**} (see, e.g., [15, Proposition 1.11.14]). By Goldstine's theorem (see Theorem 2.1), one can find a net (y_β) in B_Y which converges to u in the weak* topology. Hence, for every $z \in Z$, we have

$$z(x) = z(u+v) = z(u) = \lim_\beta z(y_\beta).$$

Since $z(x) > 1-\alpha$, it follows that there exists an index β_0 such that

$$y^*(y_\beta) = z(y_\beta) > 1-\alpha \qquad \text{for all } \beta \geq \beta_0.$$

Thus, $(y_\beta)_{\beta\geq\beta_0}$ is a net in $S(y^*,\alpha)$ that converges to x in the $\sigma(X,Z)$-topology; therefore $S(y^*,\alpha)$ is dense in $S(z,\alpha)$.

Let $S = \sum_{i=1}^n \lambda_i S(y_i^*, \alpha_i)$ be a convex combination of slices of B_Y, where $n \in \mathbb{N}$, and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$. We will show that the diameter of S is 2.

By our observation, every slice $S(y_i^*, \alpha_i)$ is $\sigma(X,Z)$-dense in the corresponding $\sigma(X,Z)$-slice $S(z_i, \alpha_i)$ of B_X. It follows that S is $\sigma(X,Z)$-dense in $T = \sum_{i=1}^n \lambda_i S(z_i, \alpha_i)$. Note that, by Proposition 3.25, the diameter of T is 2.

We next show that the norm on X is $\sigma(X,Z)$-lower semicontinuous. To see this, consider a net (x_γ) in X that converges to some element x in X in the $\sigma(X,Z)$-topology. If $z \in Z$, then $|z(x)| = \lim_\gamma |z(x_\gamma)| \leq \liminf_\gamma \|z\| \, \|x_\gamma\|$. Since Z is 1-norming, it follows that

$$\begin{aligned} \|x\| &= \sup\{|z(x)| : z \in B_{X^*} \cap Z\} \\ &\leq \liminf_\gamma \|x_\gamma\| . \end{aligned}$$

Now we are ready to accomplish that $\operatorname{diam}(S) = 2$. Let $\varepsilon > 0$. We recall that the diameter of T is 2. Find elements x and $\tilde{x}$ in T such that $\|x - \tilde{x}\| > 2 - \varepsilon$. Consider nets (y_γ) and $(\tilde{y}_\gamma)$ in S converging to x and $\tilde{x}$ respectively in the $\sigma(X,Z)$-topology. By the $\sigma(X,Z)$-lower semicontinuity of the norm on X, we have

$$2 - \varepsilon < \|x - \tilde{x}\| \leq \liminf_\gamma \|y_\gamma - \tilde{y}_\gamma\| \leq \operatorname{diam}(S).$$

Since ε can be arbitrarily small, we obtain that the diameter of S is 2.

Part 2. Firstly, we will show that every convex combination of weak* slices of $B_{Y^{\perp\perp}}$ has diameter 2. Since we identify Y^{**} and $Y^{\perp\perp}$, it is enough to show that every convex combination of weak* slices of $B_{Y^{**}}$ has diameter 2. Since Y has the strong diameter 2 property, by Proposition 3.3, every convex combination of weak* slices of $B_{Y^{**}}$ has diameter 2.

Proposition 3.18 and $X^{**} = Y^{\perp\perp} \oplus_\infty Z^\perp$ implies that every convex combination of weak* slices of $B_{X^{**}}$ has diameter 2. Hence, by Proposition 3.3, X has the strong diameter 2 property. □

If X is an M-ideal in X^{**}, then $X^{***} = X^* \oplus_1 X^\perp$. Since X^* is 1-norming, it follows that X is a strict M-ideal in X^{**}, and we derive the following immediate corollary.

Corollary 3.27. *Let X be a nonreflexive Banach space. If X is an M-ideal in X^{**}, then both X and X^{**} have the strong diameter 2 property.*

In our next proposition we will demonstrate an alternative version for the second part of the proof of Theorem 3.26. Note that our proof makes no use of second duals, and thus we derive the result more directly.

Proposition 3.28. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the strong diameter 2 property, then X has the strong diameter 2 property.*

Proof. Let $\sum_{i=1}^{n} \lambda_i S(x_i^*, \alpha_i)$ be a convex combination of slices of B_X, where $n \in \mathbb{N}$, and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\alpha_1, \ldots, \alpha_n\}/3$.

We will show the existence of $x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2 \in B_X$ such that $x_i^k \in S(x_i^*, \alpha_i)$ for every $i = 1, \ldots, n$, for every $k = 1, 2$, and

$$\left\| \sum_{i=1}^{n} \lambda_i (x_i^1 - x_i^2) \right\| > \frac{2-\varepsilon}{1+\varepsilon}.$$

Denote by P the M-ideal projection on X^* with $\ker P = Y^\perp$. For every $i = 1, \ldots, n$, we take

$$y_i^* = \frac{Px_i^*}{\|Px_i^*\|} \text{ and } \beta_i = \frac{\varepsilon - \varepsilon \|Px_i^*\| + \varepsilon^2}{\|Px_i^*\|}.$$

Therefore $\sum_{i=1}^{n} \lambda_i S(y_i^*, \beta_i)$ is a convex combination of slices of B_Y. Since Y has the strong diameter 2 property, we can find $y_1^1, \ldots, y_n^1, y_1^2, \ldots, y_n^2 \in B_Y$ such that

$$Px_i^*(y_i^k) > (\|Px_i^*\| - \varepsilon)(1+\varepsilon), \qquad k = 1, 2, \quad i = 1, \ldots, n,$$

and

$$\left\| \sum_{i=1}^{n} \lambda_i (y_i^1 - y_i^2) \right\| > 2 - \varepsilon.$$

There are $x_1, \ldots, x_n \in B_X$ such that

$$(x_i^* - Px_i^*)(x_i) > (\|x_i^* - Px_i^*\| - \varepsilon)(1+\varepsilon),$$

for every $i = 1, \dots, n$.

Since Y is an M-ideal in X, then by [19, Proposition 2.3], we can, for every $i = 1, \dots, n$, choose $z_i \in B_Y$ such that

$$\left\| y_i^k + x_i - z_i \right\| < 1 + \varepsilon, \qquad k = 1, 2,$$

and

$$|Px_i^*(x_i - z_i)| < \varepsilon.$$

We take

$$x_i^k = \frac{y_i^k + x_i - z_i}{1 + \varepsilon}, \qquad k = 1, 2, \quad i = 1, \dots, n.$$

Now, for every $i = 1, \dots, n$, for every $k = 1, 2$, x_i^k is an element in $S(x_i^*, \alpha_i)$, because

$$\begin{aligned} x_i^*(x_i^k) &= \frac{x_i^*(y_i^k + x_i - z_i)}{1 + \varepsilon} \\ &= \frac{Px_i^*(y_i^k) + (x_i^* - Px_i^*)(x_i) + Px_i^*(x_i - z_i)}{1 + \varepsilon} \\ &> \|Px_i^*\| - \varepsilon + \|x_i^* - Px_i^*\| - \varepsilon - \varepsilon \\ &= \|x_i^*\| - 3\varepsilon > 1 - \alpha_i. \end{aligned}$$

Finally, observe that

$$\left\| \sum_{i=1}^n \lambda_i (x_i^1 - x_i^2) \right\| = \frac{1}{1 + \varepsilon} \left\| \sum_{i=1}^n \lambda_i (y_i^1 - y_i^2) \right\| > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

□

Remark. Proposition 3.28 generalizes Proposition 3.16. In fact, if X and Y are closed subspaces of a Banach space Z, then (see, e.g., [11, I.1])

$$Z = X \oplus_\infty Y \quad \Longleftrightarrow \quad Z^* = X^\perp \oplus_1 Y^\perp.$$

We will finish with the local diameter 2 and the diameter 2 versions of Proposition 3.28.

Proposition 3.29. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the local diameter 2 property, then X has the local diameter 2 property.*

Proof. Take $n = 1$ in the proof of Proposition 3.28. $\square$

The next result is obtained in the proof of [14, Theorem 2.4], but not stated explicitly. We will give a direct proof to this result.

Proposition 3.30. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the diameter 2 property, then X has the diameter 2 property.*

Proof. Let U be a nonempty relatively weakly open subset of B_X containing an element x_0. We may assume that

$$\{x \in B_X \colon\ |x_i^*(x - x_0)| < \gamma, \quad i = 1, \ldots, n\} \subset U,$$

for some $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in S_{X^*}$, and $\gamma > 0$.

Denote by P the M-ideal projection on X^* with $\ker P = Y^\perp$, and by $\delta = \max\{\|Px_i^*\| : i = 1, \ldots, n\}$. Let $\varepsilon > 0$ be such that $\frac{\varepsilon(1+\delta)}{1+\varepsilon} < \gamma$. We will show the existence of elements x and $\tilde{x}$ in U such that

$$\|x - \tilde{x}\| > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

Since Y is an M-ideal in X, by [19, Proposition 2.3], there is an element $y_0 \in B_Y$ such that

$$\|y + x_0 - y_0\| < 1 + \varepsilon \qquad \text{for all } y \in B_Y.$$

Consider the set

$$V = \{y \in B_Y \colon\ |Px_i^*(y - y_0)| < \varepsilon\delta, \quad i = 1, \ldots, n\}.$$

Clearly V is a nonempty relatively weakly open subset of B_Y. By the assumption, there are $y, \tilde{y} \in V$ with $\|y - \tilde{y}\| > 2 - \varepsilon$. We take

$$x = \frac{y + x_0 - y_0}{1 + \varepsilon} \qquad \text{and} \qquad \tilde{x} = \frac{\tilde{y} + x_0 - y_0}{1 + \varepsilon}.$$

Now, for every $i = 1, \dots, n$, we have

$$\begin{aligned}|x_i^*(x - x_0)| &= \frac{1}{1+\varepsilon}\,|x_i^*(y - \varepsilon x_0 - y_0)| \\ &\leq \frac{1}{1+\varepsilon}\Big(\,|Px_i^*(y - y_0)| + \varepsilon\,|x_i^*(x_0)|\,\Big) \\ &< \frac{1}{1+\varepsilon}(\varepsilon\delta + \varepsilon) < \gamma.\end{aligned}$$

Thus, $x \in U$. Similarly one can show that $\tilde{x} \in U$. Finally, observe that

$$\|x - \tilde{x}\| = \frac{1}{1+\varepsilon}\,\|y - \tilde{y}\| > \frac{2-\varepsilon}{1+\varepsilon}.$$

$\square$

Diameeter 2 omadused

Magistritöö
Johann Langemets

Kokkuvõte

Kui Banachi ruumi ühikkera iga mittetühja suhteliselt nõrgalt lahtise ühikkera alamhulga, nt. viilu, diameeter on 2, siis öeldakse, et sellel Banachi ruumil on d2 omadus (vt. [7] või [3]).

D2 omaduse uuringute lähtekohaks peetakse Nygaardi ja Werneri artiklit [16], kus tõestati, et lõpmatumõõtmelistel ühtlastel algebratel on d2 omadus. Veidi varem oli teada, et Daugaveti ruumidel on d2 omadus, vastav artikkel [18] ilmus siiski artiklist [16] hiljem. Nygaardi ja Werneri tähelepanek oli aluseks artiklite seeriale (vt. [6], [5] ja [14]), kus mitmed matemaatikud selgitasid samasuguse geomeetrilise fenomeni esinemist erinevates konkreetsetes Banachi ruumides. Muuhulgas uuriti selle omaduse ülekandumist Banachi ruumide abil konstrueeritud ruumidele, nt. päranduvust komponentidelt otsekorrutistele.

Põhjalikum ülevaade on ilmumas artiklis [1].

Suhteliselt nõrgalt lahtise alamhulga erijuhuks on viil, kusjuures on (Bourgain'i lemma (vt. lemma 2.14) põhjal) teada, et ühikkera iga mittetühi suhteliselt nõrgalt lahtine alamhulk sisaldab teatud viilude kumerat kombinatsiooni. Seda asjaolu silmas pidades vaatlevad Abrahamsen, Lima ja Nygaard artiklis [1] d2 omaduse kõrval selle omaduse kahte formaalselt erinevat versiooni – tugevat d2 omadust ja lokaalset d2 omadust. Artiklis

[1] on püstitatud hüpotees, et need omadused on üksteisest erinevad.

Üldiselt on Banachi ruumi d2 omadus tarvilik tugeva d2 omaduseks ja piisav lokaalse d2 omaduseks.

$$\text{tugev d2 omadus} \quad \Rightarrow \quad \text{d2 omadus} \quad \Rightarrow \quad \text{lokaalne d2 omadus}$$

Käesoleva magistritöö põhieesmärk oli anda ülevaade d2 omadustega ruumidest ja saada uusi tulemusi.

Töö koosneb sissejuhatavast osast ja põhiosast, kus esmalt teeme kindlaks, millised d2 omadused on Banachi ruumidel ℓ_∞, c_0 ja $L_1[0,1]$. Artikli [1] eeskujul näitame, et Daugaveti ruumidel on tugev d2 omadus. Detailne tõestus esitatakse artiklis [1] sõnastatud tulemusele – Banachi ruumide projektiivsel tensorkorrutisel on lokaalne d2 omadus, kui ühel komponendil on lokaalne d2 omadus. Abrahamsen, Lima ja Nygaard [1, teoreem 3.2] näitasid, et kui Banachi ruumidel X ja Y on d2 omadus, siis ka $X \oplus_p Y$ on d2 omadusega iga $1 \leq p \leq \infty$ korral. Kas analoogiline väide jääb kehtima tugeva d2 omaduse korral (vt. [1, küsimus (c)])? Me vastame sellele küsimusele eitavalt. Nimelt, näitame, et mittetriviaalsete Banachi ruumide X ja Y ning suvalise $1 < p < \infty$ korral ei ole ruumil $X \oplus_p Y$ tugevat d2 omadust (vt. teoreem 3.23). Ühtlasi oleme sellega saanud oma magistritöö põhitulemuse, et d2 omadus ja tugev d2 omadus on üldiselt erinevad. Seni ei ole teada, kas lokaalne d2 omadus ja d2 omadus erinevad. Töö lõpus uurime d2 omadusi M-ideaalidel. Artiklitest [14] ja [1] inspireerituna tõestame, et kui Y on range M-ideaal Banachi ruumis X, siis ruumidel Y ja X on tugev d2 omadus. Viimasena näitame, et kui Y on M-ideaal ruumis X, siis vastav d2 omadus ruumil Y kandub üle ka ruumile X (vt. laused 3.28–3.30).

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