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List of original publications

I J. Lellep, T. Kraav, Optimization of stepped columns under compression. *In: Continuous Optimization and Knowledge-Based Technologies. EUROPT 2008.* Editors: L. Sakalauskas, G.W. Weber, E.K. Zavadskas, Vilnius: VGTU Press "Technika", 2008, pp. 273-278.

II J. Lellep, T. Kraav, Elastic buckling of stepped beams with cracks. *In: Proceedings of the 6th IASME International Conference on Continuum Mechanics (CM '11): Recent Researches in Hydrology, Geology & Continuum Mechanics.* Editors: Z. Bojkovic, J. Kacprzyk, N. Mastorakis, V. Mladenov, R. Revetria, L. Zadeh, Cambridge, UK, 2011, pp. 11–16.

III T. Kraav, J. Lellep, Buckling of beams and columns on elastic foundations. *In: Proceedings of the 2nd International Conference Optimization and Analysis of Structures.* Editors: J. Lellep, E. Puman, University of Tartu Press, 2013, pp. 52–58.

IV T. Kraav, J. Lellep, Stability of two-stepped beams with cracks. *In: Proceedings of the 3nd International Conference Optimization and Analysis of Structures.* Editors: J. Lellep, E. Puman, University of Tartu Press, 2015, pp. 33–38.

V J. Lellep, T. Kraav, Buckling of beams and columns with defects. *International Journal of Structural Stability and Dynamics*, 16 (8), 2016.

VI T. Kraav, J. Lellep, Elastic stability of uniform and hollow columns, *Procedia Engineering, Modern Building Materials, Structures and Techniques*, 172, 2017, pp. 570–577.

Author's contribution

The author of this dissertation is responsible for majority of the research in all phases (including writing, simulation and preparing of images) of the papers I–VI. The solution procedure was developed in co-operation with supervisor; the statement of the problem belongs to the supervisor.

1 Introduction

1.1 Historical review of literature

In the linear elastic fracture mechanics it is a well-known matter that in the case of repeated or cyclic loading, also under conditions of extreme loads, temperature etc. micro and macro cracks can occur in the structural material. The basics of the fracture mechanics are presented in the books by Anderson [3], Broberg [14], Broek [15], Wen [90], Williams [91] and others. The presence of a crack in a beam, plate or shell usually deteriorates its response to external loads. This involves the need to account for cracks and other defects in the analysis of thin-walled structures. As regards to thin-walled columns, arches, plates and shells then probably the most dangerous deformation mode is the loss of stability. Thus it is important to account for the influence of cracks and other defects on the behavior of structures subjected to axial compression.

The loss of stability of thin-walled structures is investigated by many researchers. The basic studies in this area can be found in the books by Alfutov [1], Atanackovic [7], Bazant [11], Bazant, Cedolin [12], Carrera *et al* [22], Farshad [31], Iyengar [35], Jones [37], Reddy [67, 68], Simitses [72, 73, 74], Thomsen [77], Timoshenko, Gere [78], Vlasov [81], Volmir [82], Wang *et al* [85, 86], Ziegler [97] and others. One of the first papers devoted to the investigation of stability of cracked beams under compression is the paper by Okamura *et al* [65]. Recognizing the matter that a crack or a defect usually reduces the flexural rigidity of a beam and its load carrying capacity and increases the lateral deflections of eccentrically compressed beams the authors of [65] employed the results of experiments conducted by Gross and Srawley [33] in order to calculate the value of the stress intensity factor at the crack tip. The latter is coupled with the compliance due to the crack. It was recognized that the bending of a cracked section causes a tensile mode stress field at the crack tip which can be assessed by the stress intensity factor. Here the idea of the concept of the rotational spring was suggested and employed in the case of beams hinged at both edges. Later this concept was extended by Anifantis, Dimarogonas [4, 5, 6], Dimarogonas [28, 29], Chondros *et al* [25, 26] to the case of beam elements with cracks subjected to various generalized stresses.

Nikpour [63] and Nikpour, Dimarogonas [64] developed similar approach to composite bodies and applied it for the buckling analysis of composite columns weakened by cracks.

Investigating the bending with stretching of an elastic plate containing a part-through surface crack Rice and Levy [69] introduced the concept of a distributed line-spring. According to this method the crack is modeled as a continuous line spring having both stretching and bending resistance; its compliance coefficients are to be matched those of an edge cracked strip in plane strain. Extending the ideas of the "distributed line spring method" Anifantis and Dimarogonas [4] introduced a 5×5 matrix in order to prescribe the influence of generalized stresses on the local compliance of the structure.

In the one-dimensional case when the dominating stress component is bending moment this approach is known as "massless rotational spring method". According to the rotational spring method the influence of a crack on the local flexibility can be modelled by a system consisting of two segments of the beam connected with the massless rotational spring of given stiffness. This concept was employed for investigation of stability of cracked columns by Caddemi *et al* [16]–[21], Challamel and Xiang [23], Skrinar [75], Li [57, 58] and Wang [87]. Lellep and Sakkov [56] studied the stability of elastic stepped columns with stable cracks at the re-entrant corners of steps making use of the shape function obtained by the interpolation of experimental data by Brown, Gross and Srawley (see Tada *et al* [76], Murakami [62]) by the polynomials of the fourth order. In [54] the method of the massless spring is used and the stress intensity coefficent is constructed with the help of the bending moment.

The dynamic behavior of beams with cracks was studied by Alsabbagh *et al* [2], Yang and Chen [92], Chondros *et al* [25], Shen and Pierre [71], Kukla [42], Viola and Marzani [80]. Recently Zheng and Fan [95] have developed a procedure for determination of buckling loads for hollow columns injured with cracks. Although the paper [95] deals with hollow-sectional beams of rectangular cross-sections the attention is paid to tubular beams with circular cross-sections also. The both, the stability and natural vibrations of tubular beams are studied and the local flexibility is calculated with the aid of the stress intensity coefficient. The shape function is taken in the form including trigonometrical functions. Tubular hollow elements are considered also by Lellep and Liyvapuu [50]. In [48, 49, 50] natural vibrations of curved beams and arches with cracks are studied with the aid of the similar method. In [47] this method is applied for determination of eigenfrequencies of free vibrations of nano-beams.

Jiki [36] presented the stability analysis of the cracked beam in the case of a following force based on the Liapunov's method. Zhou and Huang [96] investigated the case of eccentrically loaded columns, Cicirello and Palmere [27] studied the case of multiple cracks. Fan and Zheng [30] used for the stability analysis of Timoshenko beams the Fourier' series. The vibration of a beam with an internal hinge was studied by Wang and Wang [84], Wang and Chase [88] as well as Wang and Quek [89] developed analytical models for the analysis of columns with piezoelectric layers and repair of a cracked column under axially compressive load.

Euler-Bernoulli beams with jump discontinuities are studied with the help of generalized functions by Yavari and Sarkani [93].

The post-buckling analysis is undertaken by Ke *et al* [38]. Chen and Meguid [24] have concentrated at the buckling of initially curved microbeams; Li *et al* [59] studied thermally and electrically actuated microbeams.

Barkanov *et al* [8, 9, 10] developed efficient algorithms for numerical optimization of laminated and composite structures. The general theory of thin-walled composite beams was developed by Librescu and Song [60].

In previous papers it was assumed that the beams, plates and shells are subjected to a fixed system of loading. The case of a stochastic loading combined with the topology optimization is studied by Logo [61].

Yokoyama and Chen [94] used a 2×2 matrix for prescribing the local compliance caused by the crack. Lellep *et al* [43]–[56] extended this concept for determination of natural frequencies of axisymmetric vibrations of circular cylindrical shells, circular arches and rectangular plates resting on an elastic foundation.

Circular cylindrical shells made of homogeneous and composite material have been investigated in [51]–[55] in the case of stepped thickness. The tubes are weakened with circular cracks which are located at the cross-sections associated with the abrupt change of the thickness. Lellep and Liyvapuu [48, 49, 50] concentrated on the free vibrations of circular arches with stable surface cracks. The paper [43] is devoted to the determination of eigenfrequencies for plates resting on an elastic foundation. In [47] an approximate method is developed for the investigation of free vibrations of nano-beams.

In the current thesis an approximate method is developed for the assessment of critical buckling loads of stepped beams and columns weakened with cracks or crack-like defects.

1.2 Aim of the dissertation

The aim of the thesis is to determine critical buckling loads of stepped beams and to study the sensitivity of the critical load on the parameters of stable cracks, such as location and depth. Combining the methods of the elastic beam theory and of the linear elastic fracture mechanics an approximate method for the stability analysis of beams and columns subjected to the axial pressure is developed. Introducing the additional compliance matrix the flexibility of the beam in the vicinity of a crack is prescribed by means of the compliance of the structure. This, in turn, is coupled with the stress intensity factor which can be calculated by methods of the linear elastic fracture mechanics. Critical buckling loads of stepped columns subjected to the axial pressure and weakened with cracks emanating from re-entrant corners of steps are established. Numerical results are presented for uniform and hollow beams with single step of the cross section, also for two-stepped beams. The beams under consideration are cantilevers, elatically fixed or resting on elastic foundation.

1.3 Structure of the dissertation

The dissertation is organized as follows. Section 1 contains historic background of the stability analysis, the aim and the structure of the dissertation. In section 2 the concept of local flexibility is described in detail. In sections 3, 4, 5 and 6 the method is applied to partcular cases of beams.

2 The main concepts

2.1 Governing equations

Long beams of rectangular cross section are usually considered as particular cases of rectangular plates (see Vinson [79], Reddy [67, 68]). In the case of relatively long beams it is reasonable to assume that the constitutive equation can be presented as [56, 67, 85]

$$M = -E_j I_j v^{''}. \tag{2.1}$$

In (2.1) M stands for the bending moment, v is the displacement in y-direction. Prims denote the differentiation with respect to x, E_j is Young's modulus for the section $S_j = (a_j, a_{j+1})$ and $I_j = BH_j^3/12$ being the moment of inertia where B is the width and H_j the height of the beam. From the equilibrium of a beam element it follows that (here $N = -P$)

$$M'' - Pv'' = 0, \tag{2.2}$$

provided the second order bending theory is employed. Here P is axial loading. Combining the last equation with (2.1) yields

$$(E_j I_j v^{''})^{''} - Pv'' = 0. \tag{2.3}$$

Note that in the case of inhomogeneous and composite material the modulus E_j can be a function of the coordinate x. However, we can choose sections (a_j, a_{j+1}) relatively small and use the averaged value of the modulus E_j for the section S_j. Therefore, it is reasonable to assume that $E_j = E = const$ and $I_j = const$ for $x \in S_j$. Thus the equation (2.3) can be converted into

$$v^{''''} + \frac{\lambda_j^2}{l^2} v^{''} = 0 \tag{2.4}$$

where

$$\lambda_j^2 = \frac{Pl^2}{EI_j}. \tag{2.5}$$

General solution of (2.4) is

$$v = A_j \cos \lambda_j \xi + B_j \sin \lambda_j \xi + C_j \xi + D_j \tag{2.6}$$

for $\xi l \in S_j$; $j = 0, \ldots, n$. Here A_j, B_j, C_j, D_j, $j = 0, \ldots, n$ are arbitrary constants to be defined later and $\xi = x/l$.

2.2 The concept of local flexibility

As a rule, cracks and other defects deteriorate the mechanical behavior of structural elements. The influence of the crack on the buckling and vibration is modeled with the aid of a weightless rotating spring, as shown by Anifantis and Dimarogonas [4]. The stiffness of the spring K_T is reciprocal to the additional compliance C owing to the crack.

It is known from the linear elastic fracture mechanics that the energy release rate caused by the crack propagation

$$G = \frac{1}{2} M^2 \frac{dC}{dA} \tag{2.7}$$

where M is the bending moment applied to the beam and A is the area of the surface of the crack. It is known also that $G = K^2(1-\nu^2)/E$ where K is the stress intensity factor and ν is Poisson's modulus.

On the other hand, the energy release rate G is coupled with the stress intensity factor K_j as [3]

$$K_j = \sigma\sqrt{\pi c_j} \cdot F(s_j) \tag{2.8}$$

where F is so-called shape function to be determined experimentally. A lot of data can be found from literature about the shape functions for specimens of various type. According to results of experiments presented in the handbook by Tada *et al* [76] one can take

$$F(s_j) = 1.93 - 3.07s_j + 14.53s_j^2 - 25.11s_j^3 + 25.8s_j^4 \tag{2.9}$$

where

$$s_j = \frac{c_j}{H_j}. \tag{2.10}$$

Another approximation of results of Brown and Srawley for larger cracks can be presented as [34]

$$F(s_j) = \begin{cases} 1.99 - 2.47s_j + 12.97s_j^2 - 23.17s_j^3 + 24.8s_j^4, & 0 < s_j < 0.5; \\ 0.663(1-s_j)^{-\frac{3}{2}}, & 0.5 \leq s_j < 1. \end{cases} \tag{2.11}$$

Tada *et al* [76] have also presented an approximation based on the use of trigonometrical functions as

$$F(s_j) = \frac{\sqrt{\tan\frac{\pi}{2}s_j}}{\frac{\pi}{2}s_j} \cdot \frac{0.923 + 0.199(1-\sin\frac{\pi}{2}s_j)^4}{\cos\frac{\pi}{2}s_j}. \tag{2.12}$$

Note that different forms of the stress correction function $F(s_j)$ are suggested by Chondros *et al* [25], Freund and Hermann [32], Ostachowich and

Krawczuk [66] and others. A comparison of these functions was undertaken by Caddemi and Calio [16].

The approximation (2.9) was widely used for developing solution procedures of particular problems. Dimarogonas [29] studied buckling of rings and tubes whereas Alsabbagh *et al* [2], Binici [13], Rizos *et al* [70], Li [57, 58] and others employed the flexibility of compressed beams.

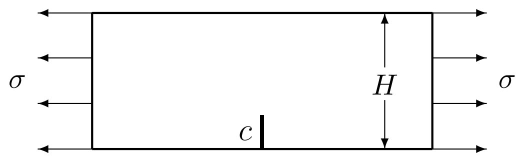

Figure 1: A specimen with an edge crack

In (2.8) σ is the normal stress at the edge of the specimen in the case of a streched specimen [70]; see Fig. 1. Note that for a specimen subjected to the bending moment M, one can take $\sigma = 6M/BH_j^2$. Combining (2.7) and (2.8) yields a differential equation

$$\frac{dC}{ds_j} = \frac{72\pi(1-\nu^2)}{EBH_j^2} s_j F^2(s_j). \tag{2.13}$$

Relations (2.7)–(2.13) hold good under the assumption that the only stress resultant to be taken into account is $P_1 = M$. Anifantis and Dimarogonas [6] have deduced the energy release rate for general case of loading of a cracked cross section with P_i $(i = 1, \dots, 5)$.

In this case

$$G = \frac{1}{E}\left[\left(\sum_{i=1}^{5} K_{1i}\right)^2 + \left(\sum_{i=1}^{5} K_{2i}\right)^2 + (1+\nu)\left(\sum_{i=1}^{5} K_{3i}\right)^2\right] \tag{2.14}$$

whereas

$$C_{ij} = \frac{\partial^2}{\partial P_i \partial P_j} \int_A G dA. \tag{2.15}$$

Here C_{ij} $(i = 1, \dots, 5)$ stand for elements of the compliance matrix $[C]$. By inversion of the compliance matrix one can obtain

$$[K] = [C]^{-1} \tag{2.16}$$

where $[K]$ is called local stiffness matrix. However in the present case, C is a scalar function depending on crack parameters.

Integrating (2.13) by the help of (2.9) with respect to s_j under the condition $C(0) = 0$ yields

$$C = \frac{72\pi(1-\nu^2)}{EBH_j^2} f(s_j) \tag{2.17}$$

where

$$\begin{aligned} f(s_j) &= 1.86s_j^2 - 3.95s_j^3 + 16.37s_j^4 - 34.23s_j^5 + 76.81s_j^6 \\ &- 126.93s_j^7 + 172s_j^8 - 143.97s_j^9 + 66.56s_j^{10}. \end{aligned} \tag{2.18}$$

Taking $K_{T_j} = \dfrac{1}{C(a_j)}$ one can state that

$$K_{T_j} = \frac{EI_j}{6\pi H_j f(s_j)(1-\nu^2)}. \tag{2.19}$$

It was mentioned above that the model of a massless rotational spring is used for modeling the local behavior of beam segments adjacent to a crack. In case of a spring the applied moment M is proportional to the rotational angle φ with coefficient of proportionality, called rigidity. On the other hand, the rigidity is reciprocal to the compliance C when speaking about elastic members, eg.

$$\frac{\varphi}{C} = M. \tag{2.20}$$

Evidently, in the latter relation φ and M can be replaced with appropriate generalized displacement q_{ij} and generalized force P_{ij}, respectively. In the case when at cross section $x = a_j$ $(j = 1, \dots, n)$ forces P_{ij} $(j = 1, \dots, k)$ are applied the concept of massless springs yields

$$P_{ij} = \frac{q_{ij}}{C_{ij}}. \tag{2.21}$$

Let us study the case when the moment M is dominating among other stresses in greater detail. Let

$$\varphi_j = v'(a_j + 0) - v'(a_j - 0) \tag{2.22}$$

be the angle of rotation due to the crack located at $x = a_j$. Thus, according to current concept one has

$$\varphi_j \cdot K_{T_j} = -M(a_j). \tag{2.23}$$

Substituting the bending moment in (2.23) yields the condition of the slope discontinuity

$$v'(a_j + 0) - v'(a_j - 0) = \frac{EI_j}{K_{T_j}} v''(a_j + 0) \tag{2.24}$$

where $K_{T_j} = K_T(a_j)$.

3 Elastic beams and columns with cracks

3.1 Formulation of the problem

We are studying the behaviour of stepped beams and columns in the case of axial compression and loss of stability. The effect of weakening cracks is taken into account. The column is of rectangular cross-section with piecewise constant thickness, so that $H = H_j$; $j = 0, \ldots, n$ for $x \in S_j$ where $S_j = (a_j, a_{j+1})$. The width B of the column is assumed to be constant and $a_0 = 0$, $a_{n+1} = l$. The column is clamped at $x = 0$ and free at $x = l$. At the free edge the axial load P is applied.

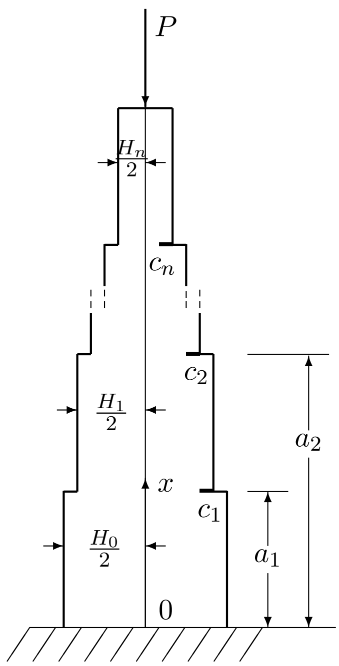

Figure 2: A stepped beam

An essential feature of the posed problem is that the stiffness of the column is weakened by flaws or cracks. Let crack of length c_j be located at the re-entrant corner of the step at $x = a_j$ (Fig. 2). Similarily to the papers by Chondros *et al* [25], Dimarogonas [29] it is assumed herein that the cracks are stable surface cracks during elastic buckling. On the other hand, the formation of a crack in an structural element accompanies considerable change of local flexibility due to the strain energy concentration in the vicinity of the crack tip. This effect was recognized a long ago by Irwin [34] and other investigators and many attempts were made to employ it in modeling of the influence of the crack on the behavior of beams and plates (see Dimarogonas [29]).

We are looking for the critical buckling load for stepped beams and columns subjected to the axial pressure. The aim of the study is to investigate the influence of crack parameters on the critical load.

3.2 Critical buckling load

The critical buckling load can be determined with the aid of (2.4) satisfying corresponding boundary and intermediate conditions at $\xi = \alpha_j$ ($j = 1, \dots, n$). Variables v, M and M' must be continuous whereas the slope v' has finite jumps as shown in (2.24). The boundary conditions at $\xi = 0$ and at $\xi = 1$ are (here and henceforth prims denote derivatives with respect to ξ)

$$\begin{aligned} v(0) &= 0, \\ v^{'}(0) &= 0, \\ v^{''}(1) &= 0, \\ v^{'''}(1) &= \lambda_n^2 v^{'}(1). \end{aligned} \tag{3.1}$$

Note that at $\xi = \alpha_j$ ($j = 1, \dots, n$) the displacement $v(\xi)$ must be continuous whereas the slope $v'(\xi)$ satisfies the requirement (2.24). Also, moment $M(\xi)$ and the shear force $M'(\xi)$ can not be discontinuous. The latter with (2.1) means that

$$\begin{aligned} EI_{j-1}v''(\alpha_j - 0) &= EI_j v''(\alpha_j + 0), \\ EI_{j-1}v'''(\alpha_j - 0) &= EI_j v'''(\alpha_j + 0). \end{aligned} \tag{3.2}$$

On grounds of physical consideration it is evident that the displacement v must be continuous at each point. Accounting for this matter and taking (2.24), (3.2) into account one can present the intermediate conditions at $\xi = \alpha_j$ ($j = 1, \dots, n$) as

$$\begin{aligned} v(\alpha_j - 0) &= v(\alpha_j + 0), \\ v'(\alpha_j - 0) &= v'(\alpha_j + 0) - \frac{EI_j}{K_{T_j} l} v''(\alpha_j + 0), \\ \frac{v''(\alpha_j - 0)}{\lambda_{j-1}^2} &= \frac{v''(\alpha_j + 0)}{\lambda_j^2}, \\ \frac{v'''(\alpha_j - 0)}{\lambda_{j-1}^2} &= \frac{v'''(\alpha_j + 0)}{\lambda_j^2}. \end{aligned} \tag{3.3}$$

Substituting the displacement v and its derivatives from (2.6) into (3.3) one can present the system (3.3) as

$$\begin{aligned}
&A_{j-1}\cos\lambda_{j-1}\alpha_j + B_{j-1}\sin\lambda_{j-1}\alpha_j + C_{j-1}\lambda_{j-1}\alpha_j + D_{j-1} \\
&\ = A_j\cos\lambda_j\alpha_j + B_j\sin\lambda_j\alpha_j + C_j\lambda_j\alpha_j + D_j, \\
&\lambda_{j-1}(-A_{j-1}\sin\lambda_{j-1}\alpha_j + B_{j-1}\cos\lambda_{j-1}\alpha_j) + C_{j-1}\lambda_{j-1} \\
&\ = \lambda_j(-A_j\sin\lambda_j\alpha_j + B_j\cos\lambda_j\alpha_j) + C_j\lambda_j \\
&\ -\lambda_j^2 k_j(A_j\cos\lambda_j\alpha_j + B_j\sin\lambda_j\alpha_j), \\
&A_{j-1}\cos\lambda_{j-1}\alpha_j + B_{j-1}\sin\lambda_{j-1}\alpha_j = A_j\cos\lambda_j\alpha_j + B_j\sin\lambda_j\alpha_j, \\
&\lambda_{j-1}(A_{j-1}\sin\lambda_{j-1}\alpha_j - B_{j-1}\cos\lambda_{j-1}\alpha_j) \\
&\ = \lambda_j(A_j\sin\lambda_j\alpha_j - B_j\cos\lambda_j\alpha_j)
\end{aligned} \tag{3.4}$$

for each $j = 1, \ldots, n$. Here $\alpha_j = a_j/l$; $k_j = EI_j/K_{T_j}l$. Boundary conditions (3.1) lead to the relations

$$\begin{aligned}
A_0 + D_0 &= 0, \\
B_0 + C_0 &= 0, \\
A_n\cos\lambda_n + B_n\sin\lambda_n &= 0, \\
C_n &= 0.
\end{aligned} \tag{3.5}$$

Equations (3.4), (3.5) serve for determination of arbitrary constants A_j, B_j, C_j, D_j ($j = 1, \ldots, n$). It is a linear homogeneous system of equations with respect to unknowns A_j, B_j, C_j, D_j where $j = 1, \ldots, n$. This system has a non-trivial solution only in the case when its determinant vanishes. Let Δ be the determinant of this system. Then the equation $\Delta = 0$ admits to define the critical buckling load.

3.3 Solutions for particular cases

Let us consider first the case when the non-dimensional thickness has the form

$$\frac{H}{H_0} = \begin{cases} 1, & \xi \in (0, \alpha); \\ \gamma, & \xi \in (\alpha, 1). \end{cases} \tag{3.6}$$

In this case according to (3.5)

$$D_0 = -A_0, \quad C_0 = -B_0 \tag{3.7}$$

and

$$B_1 = -A_1\cot\lambda_1, \quad C_1 = 0. \tag{3.8}$$

Substituting (3.7), (3.8) into (3.4) where $j = n = 1$ leads to the characteristic equation

$$\sin\lambda_0\alpha \sin\lambda_1(1-\alpha)\left[\sin\lambda_0\alpha - \frac{\lambda_0}{\lambda_1}\cos\lambda_0\alpha\cot\lambda_1(1-\alpha) + \frac{k_1\lambda_j^2}{\lambda_0}\right] = 0. \quad (3.9)$$

In the case of a two-stepped column (now $n = 2$) one has

$$v = A_0\cos\lambda_0\xi + B_0\sin\lambda_0\xi + C_0\xi + D_0 \quad (3.10)$$

for $\xi \in [0, \alpha]$,

$$v = A_1\cos\lambda_1\xi + B_1\sin\lambda_1\xi + C_1\xi + D_1 \quad (3.11)$$

for $\xi \in [\alpha, \beta]$,

$$v = A_2\cos\lambda_2\xi + B_2\sin\lambda_2\xi + C_2\xi + D_2 \quad (3.12)$$

for $\xi \in [\beta, 1]$. Here for the conciseness sake the notation $\alpha_1 = \alpha$, $\alpha_2 = \beta$ is used. Applying the continuity and jump conditions (3.3) to (3.10)–(3.12) at $\xi = \alpha$, $\xi = \beta$ and taking (3.7), (3.8) into account one obtains a linear homogeneous algebraic system with vanishing determinant Δ. After some algebraic transformations the equation $\Delta = 0$ may be expressed as

$$\begin{aligned}
&\lambda_1 R_2\Big\{k_2\lambda_2^2 + (k_1\lambda_2^2 + \lambda_0\sin\lambda_0\alpha)\cos\big[\lambda_1(\alpha-\beta)\big]\\
&\qquad\qquad + \lambda_1\cos\lambda_0\alpha\sin\big[\lambda_1(\alpha-\beta)\big]\Big\}\\
&+\lambda_2 S_2\Big\{(k_1\lambda_1^2 + \lambda_0\sin\lambda_0\alpha)\sin\big[\lambda_1(\alpha-\beta)\big]\\
&\qquad\qquad + \lambda_1\cos\lambda_0\alpha\cos\big[\lambda_1(\alpha-\beta)\big]\Big\} = 0.
\end{aligned} \quad (3.13)$$

In (3.13) the following notation

$$\begin{aligned}
R_2 &= -\cos\lambda_2\beta + \cot\lambda_2\sin\lambda_2\beta,\\
S_2 &= \sin\lambda_2\beta + \cot\lambda_2\cos\lambda_2\beta
\end{aligned} \quad (3.14)$$

is used.

Similarly in the case of a three-stepped column (now the steps are located at $\xi = \alpha$, $\xi = \beta$, $\xi = \delta$) one has

$$\begin{aligned}
&\lambda_0\lambda_1\lambda_2 R_3\Big\{k_3\lambda_3^2 + \big(k_2\lambda_2^3 + k_1\lambda_1^2\cos\big[\lambda_1(\alpha-\beta)\big]\big)\cos\big[\lambda_2(\beta-\delta)\big]\\
&\qquad\qquad + k_1\lambda_1\lambda_2\sin\big[\lambda_1(\alpha-\beta)\big]\sin\big[\lambda_2(\beta-\delta)\big]\Big\}\\
&+\lambda_0\lambda_1\lambda_3 S_3\Big\{\big(k_2\lambda_2^3 + k_1\lambda_1^2\cos\big[\lambda_1(\alpha-\beta)\big]\big)\sin\big[\lambda_2(\beta-\delta)\big]\\
&\qquad\qquad + k_1\lambda_1\lambda_2\sin\big[\lambda_1(\alpha-\beta)\big]\cos\big[\lambda_2(\beta-\delta)\big]\Big\} = 0.
\end{aligned} \quad (3.15)$$

For the conciseness sake in (3.15) the notation

$$R_3 = -\cos\lambda_3\delta + \cot\lambda_3 \sin\lambda_3\delta, \\ S_3 = \sin\lambda_3\delta + \cot\lambda_3 \cos\lambda_3\delta \tag{3.16}$$

is introduced.

3.4 Results and discussion

Results of calculations are presented in Fig. 3–8. Calculations are carried out for the column with dimensions $H_0 = 0.02$ m, $B = 0.02$ m, $l = 1$ m. The material of the column is mild steel with $E = 2.01$ GPa and $\nu = 0.3$. In Fig. 3, 4 the critical buckling load λ_0 for the beam with a single step is presented for different values of the crack length. Non-dimensional variables

$$\gamma = \frac{H_1}{H_0}, \quad \alpha = \frac{a_1}{l}, \quad s = \frac{c_1}{H_1} \tag{3.17}$$

are used whereas c_1 stands for the crack length located at $x = a_1$.

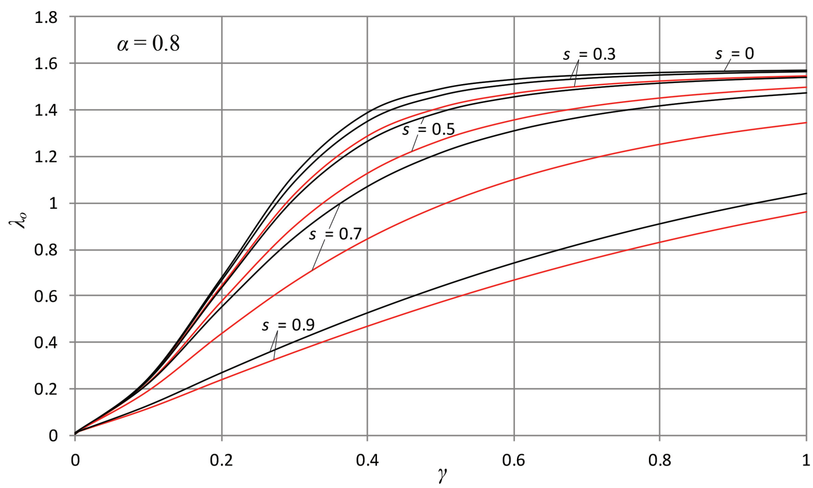

Figure 3: Critical buckling loads of a stepped beam ($\alpha = 0.8$)

In Fig. 3 $a_1 = 0.8l$ and in Fig. 4 $H_1 = 0.8H_0$. It can be seen from Fig. 4 that the buckling load increases when a_1/l increases for fixed thickness ratio $\gamma = 0.8$. Similarly, the critical buckling load increases with the non-dimensional thickness γ when the step location α is fixed, as might be expected. In the case $\alpha = 1$ or $\gamma = 1$ one obtains a uniform cantilever column. Corresponding buckling loads coincide with the classical solution for a cantilever column (see Wang *et al* [83, 84, 85]).

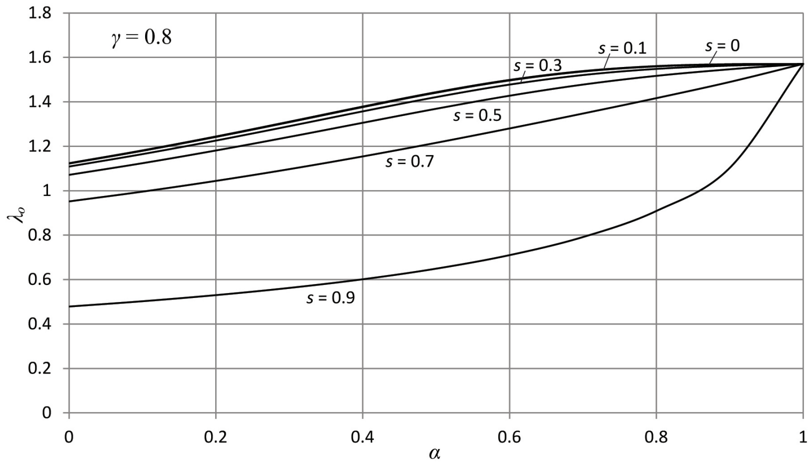

Figure 4: Critical buckling loads of a stepped beam ($\gamma = 0.8$)

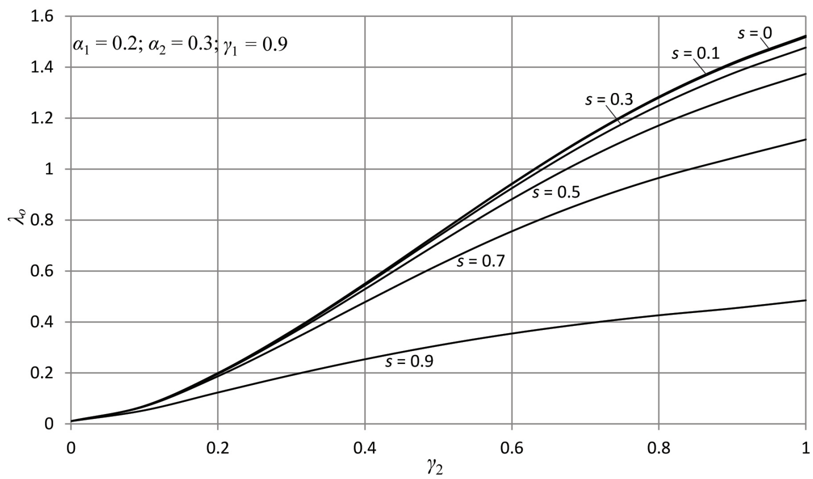

Figure 5: Critical buckling loads of a two-stepped beam ($\alpha_2 = 0.3$)

Taking $\alpha = 0$ in Fig. 4 one obtains critical buckling loads for a beam of constant thickness $H_1 = 0.8H_0$ with a crack located at the root section. For instance, the lowest curve in Fig. 4 gives for $\alpha = 0$ approximately $\lambda_0 = 0.48$. This is the buckling load for the beam with crack of length $c_1 = 0.9H_1$. The results presented in Fig. 3, 4 correspond to the form of the function F given by (2.12). This in turn is associated with the approximation of the experimental results presented in [76]. For comparison, red lines in Fig. 3 correspond to the form of the function F given by (2.9). Another approximations for three-point bend, compact and other types of specimens are presented in the handbook by Tada *et al* [76].

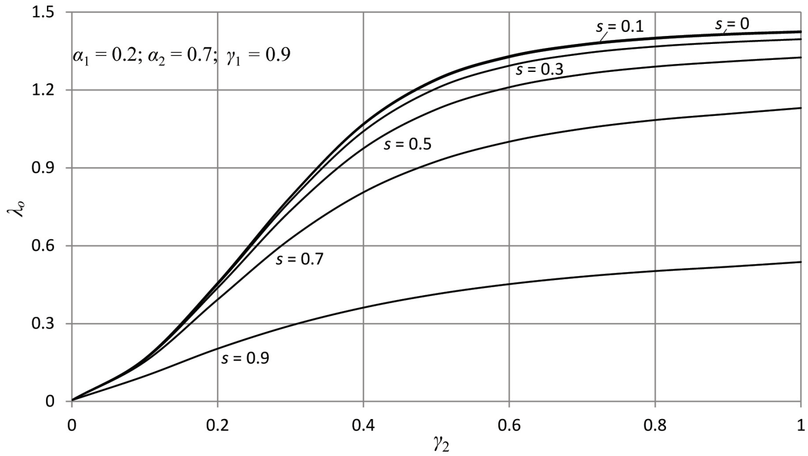

Figure 6: Critical buckling loads of a two-stepped beam ($\alpha_2 = 0.7$)

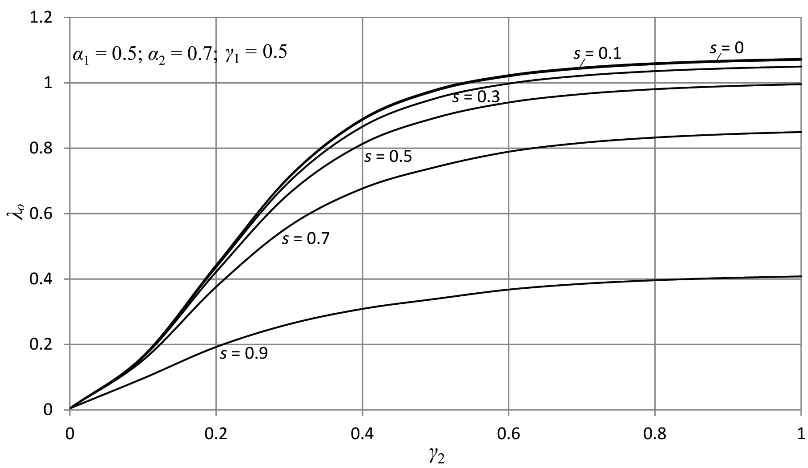

Figure 7: Critical buckling loads of a two-stepped beam ($\gamma_1 = 0.5$)

Critical buckling loads for two-stepped columns versus $\gamma_2 = H_2/H_0$ are depicted in Fig. 5–8. Here $\alpha_1 = a_1/l$, $\alpha_2 = a_2/l$, $s_2 = c_2/H_2$. Fig. 5 and 6 correspond to the case $\alpha_1 = 0.2$ and $\gamma_1 = 0.9$ whereas in Fig. 5 $\alpha_2 = 0.3$ and in Fig. 6 $\alpha_2 = 0.7$. It can be seen from Fig. 5, 6 that the results differ essentially in the range of large cracks and large values of the quantity γ_2. If, however, γ_2 is large and cracks are small then corresponding values of the buckling load compare favourably with each other.

The results presented in Fig. 7, 8 are associated with fixed locations of steps. Here $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$. In Fig. 7 $\gamma_1 = 0.5$ and in Fig. 8 $\gamma_1 = 0.9$. It reveals from Fig. 7, 8 that the critical buckling load is quite sensitive with

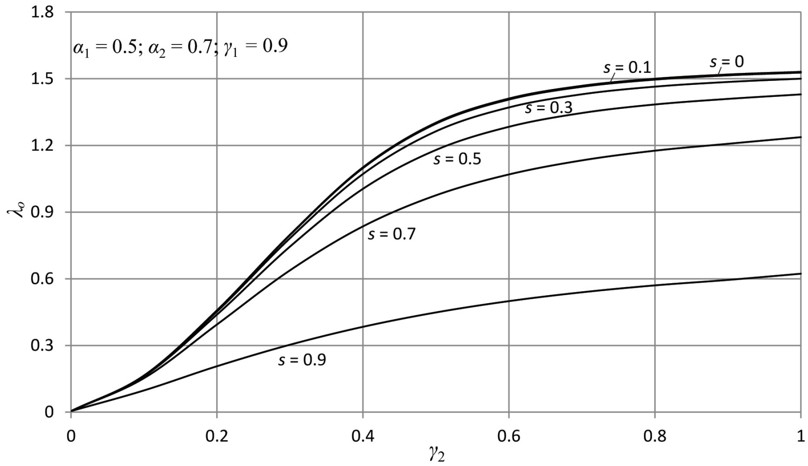

Figure 8: Critical buckling loads of a two-stepped beam ($\gamma_1 = 0.9$)

respect to changes of the thickness in the first section of the column. This regards especially the cases of large cracks.

4 Elastically fixed beams and columns with cracks

4.1 Formulation of the problem

In the present section the loss of stability of stepped beams with defects at the re-entrant corners of steps is studied under the condition that the cracks are stable surface cracks. The beams are clamped at an end and elastically fixed at the other end. The section is based on the papers by Kraav and Lellep [40, 46]. Let us consider a cantilever beam elastically fixed at the free end (Fig. 9). It is assumed that the beam or column is subjected to the axial pressure loading P. Let the origin of the coordinates be located at the center of the bottom of the beam. The beam under consideration has stepped cross section with piece wise constant dimensions.

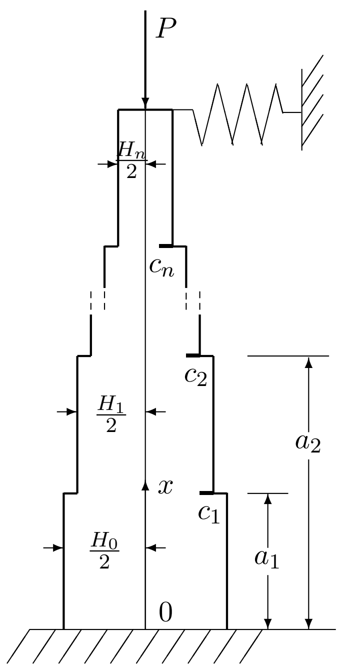

Figure 9: Elastically fixed stepped beam

In the following we shall concentrate on beams with rectangular cross section with dimensions B (width) and H (height). However, the analysis can be easily extended to other types of cross sections. In what follows we assume that $B = const$ and

$$H = H_j \tag{4.1}$$

for $x \in (a_j, a_{j+1})$ $(j = 0, \dots, n)$ where a_j and H_j $(j = 0, \dots, n)$ are constant values. It is reasonable to denote $a_0 = 0$, $a_{n+1} = l$, l being the length of the beam.

It is assumed that at the re-entrant corners of steps at $x = a_j$ $(j = 0, \dots, n)$ defects or flaws of depth c_j are located. It is known that, as a rule, defects will deteriorate the structural stiffness. That is why it is important to account for the influence of cracks in the structural analysis.

The aim of the section is to determine critical buckling loads of stepped beams and to study the sensitivity of the critical load on the location and depth of stable cracks.

4.2 Critical buckling load

Similarily to the section 3 of the current work one can present the displacements of the central line of the column as

$$v = A_j \cos \lambda_j \xi + B_j \sin \lambda_j \xi + C_j \xi + D_j \tag{4.2}$$

where $\xi = x/l$. Note that the solution (4.2) holds good for $x \in (a_j, a_{j+1})$, $j=0, \dots, n$. In (4.2) A_j, B_j, C_j, D_j stand for arbitrary constants to be defined from the boundary and intermediate conditions.

Boundary conditions depend on the type of support conditions. In the case of a beam clamped at $\xi = 0$

$$v(0) = 0, \quad v'(0) = 0. \tag{4.3}$$

However, at the elastically fixed end the boundary conditions are

$$v''(1) = 0 \tag{4.4}$$

and

$$v'''(1) + \lambda_j^2 v'(1) = \mu v(1) \tag{4.5}$$

where μ is the elastic modulus of the support. Note that (4.5) expresses the equilibrium of external forces with the shear force whereas (4.4) means that the bending moment must vanish at $\xi = 1$.

It was shown above that the displacement $v(\xi)$ is defined by (4.2) in each section of the column. Evidently, the displacement $v(\xi)$ is continuous everywhere, in particular at points $\xi = \alpha_j$ $(j = 1, \dots, n)$. However, the slope of the deflection according to the present concept has finite jumps at $x = a_j$, as shown in (2.24), e.g.

$$v'(\alpha_j + 0) - v'(\alpha_j - 0) = \frac{EI_j}{K_{T_j} l} v''(\alpha_j + 0). \tag{4.6}$$

Due to their physical background the generalized stresses are continuous, as well. Evidently, the bending moment $M(\xi)$ and the shear force $Q(\xi)$ are continuous at $\xi = \alpha_j$, if

$$\begin{aligned} EI_{j-1} v''(\alpha_j - 0) &= EI_j v''(\alpha_j + 0), \\ EI_{j-1} v'''(\alpha_j - 0) &= EI_j v'''(\alpha_j + 0) \end{aligned} \tag{4.7}$$

for $j = 1, \ldots, n$. Summarizing the results obtained above one can write

$$\begin{aligned} v(\alpha_j - 0) &= v(\alpha_j + 0), \\ v'(\alpha_j - 0) &= v'(\alpha_j + 0) - \frac{EI_j}{K_{T_j} l} v''(\alpha_j + 0), \\ \frac{v''(\alpha_j - 0)}{\lambda_{j-1}^2} &= \frac{v''(\alpha_j + 0)}{\lambda_j^2}, \\ \frac{v'''(\alpha_j - 0)}{\lambda_{j-1}^2} &= \frac{v'''(\alpha_j + 0)}{\lambda_j^2}. \end{aligned} \tag{4.8}$$

The equalitics (4.8) hold good for each $j = 1, \ldots, n$. Making use of (4.2) one can present the intermediate conditions (4.8) as

$$\begin{aligned} & A_{j-1} \cos \lambda_{j-1} \alpha_j + B_{j-1} \sin \lambda_{j-1} \alpha_j + C_{j-1} \lambda_{j-1} \alpha_j + D_{j-1} \\ & \quad = A_j \cos \lambda_j \alpha_j + B_j \sin \lambda_j \alpha_j + C_j \lambda_j \alpha_j + D_j, \\ & A_{j-1} \lambda_{j-1} \sin \lambda_{j-1} \alpha_j - \lambda_{j-1} (B_{j-1} \cos \lambda_{j-1} \alpha_j + C_{j-1}) \\ & \quad = \lambda_j (A_j \sin \lambda_j \alpha_j - B_j \cos \lambda_j \alpha_j - C_j) \\ & \quad + \lambda_j^2 k_j (A_j \cos \lambda_j \alpha_j + B_j \sin \lambda_j \alpha_j), \\ & A_{j-1} \cos \lambda_{j-1} \alpha_j + B_{j-1} \sin \lambda_{j-1} \alpha_j = A_j \cos \lambda_j \alpha_j + B_j \sin \lambda_j \alpha_j, \\ & A_{j-1} \lambda_{j-1} \sin \lambda_{j-1} \alpha_j - B_{j-1} \lambda_{j-1} \cos \lambda_{j-1} \alpha_j \\ & \quad = \lambda_j (A_j \sin \lambda_j \alpha_j - B_j \cos \lambda_j \alpha_j). \end{aligned} \tag{4.9}$$

Note that the system (4.9) is to be solved together with equations following from boundary conditions (4.3)–(4.5).

This system consists of linear homogeneous algebraic equations. It is well known that a linear homogeneous system has a non-trivial solution if and only if its determinant Δ equals to zero. The equation $\Delta = 0$ presents an equation with respect to the critical load P.

Let us consider in the following the case of the beam with a single step in a greater detail. The system of intermediate conditions takes the form

$$
\begin{aligned}
&A_0(\cos\lambda_0\alpha - 1) + A_1\cos\lambda_1\alpha + B_0(\sin\lambda_0\alpha - \lambda_0\alpha) - B_1\sin\lambda_1\alpha - \\
&\quad -C_1\alpha - D_1 = 0, \\
&A_0\lambda_0\sin\lambda_0\alpha - B_0\lambda_0(\cos\lambda_0\alpha - 1) + A_1\lambda_1(k_1\lambda_1\cos\lambda_1\alpha - \sin\lambda_1\alpha) + \\
&\quad +B_1\lambda_1(k_1\lambda_1\sin\lambda_1\alpha + \cos\lambda_1\alpha) + C_1 = 0, \\
&A_0\cos\lambda_0\alpha - A_1\cos\lambda_1\alpha + B_0\sin\lambda_0\alpha - B_1\sin\lambda_1\alpha = 0, \\
&A_0\lambda_0\sin\lambda_0\alpha - A_1\lambda_1\sin\lambda_1\alpha - B_0\lambda_0\cos\lambda_0\alpha + B_1\lambda_1\cos\lambda_1\alpha = 0.
\end{aligned}
\tag{4.10}
$$

Calculating the determinant of the amplified system and equalizing it to zero results in

$$
\begin{aligned}
&k_1\lambda_0\lambda_1^2\sin[\lambda_1(\alpha-\beta)](\mu(1-\alpha) - \lambda_1^2) + \mu\lambda_0\cos\lambda_0\alpha\sin[\lambda_1(\alpha-\beta)] + \\
&\quad +\mu\lambda_1\sin\lambda_1\alpha\cos[\lambda_1(\alpha-\beta)] + \lambda_0^2\sin\lambda_0\alpha\sin[\lambda_1(\alpha-\beta)](\mu-\lambda_1^2) - \\
&\quad -\lambda_0\lambda_1\cos\lambda_0\alpha\cos[\lambda_1(\alpha-\beta)](\mu-\lambda_1^2) = 0.
\end{aligned}
\tag{4.11}
$$

4.3 Results and discussion

The results of calculations are presented in Fig. 10–15. Calculations are carried out for the column with dimensions $H_0 = 0.02$ m, $B = 0.02$ m, $l = 1$ m. The material of the column is mild steel with $E = 2.01$ GPa and $\nu = 0.3$. Due to the matter that the results will be presented for beams and columns with a single step it is reasonable to introduce the following notation:

$$\alpha = \frac{a_1}{l}, \quad \gamma = \frac{H_1}{H_0}. \tag{4.12}$$

Evidently, the eigenvalues λ_0, λ_1 are related to each other by the equality

$$\lambda_0^2 = \gamma^3\lambda_1^3. \tag{4.13}$$

Since the non-dimentional roots λ_0, λ_1 of the characteristic equation are coupled by the relation (4.13) in the following we will confine our attention to the root λ_0 only.

In Fig. 10 the eigenvalue λ_0 (or non-dimensional critical buckling load) is presented versus the ratio of thicknesses for $a = 0.8l$ and $\mu = 1$. Red lines in Fig. 10 correspond to the stress correction function (2.9) suggested by Dimarogonas resorting to the results obtained by Gross and Srawley [33].

Black lines in Fig. 10–15 correspond to the function F suggested by Tada *et al* [76]. It can be seen from Fig. 10 that in the case of small cracks when $c < 0.1H_1$ the results obtained by different methods are quite close to each other. It reveals from Fig. 10 that the larger the thickness H_1 (or the ratio

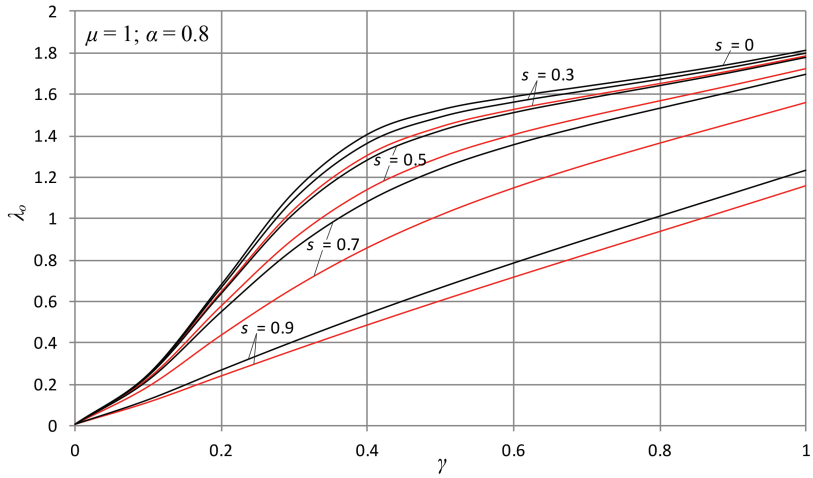

Figure 10: Comparison of results for different correction functions

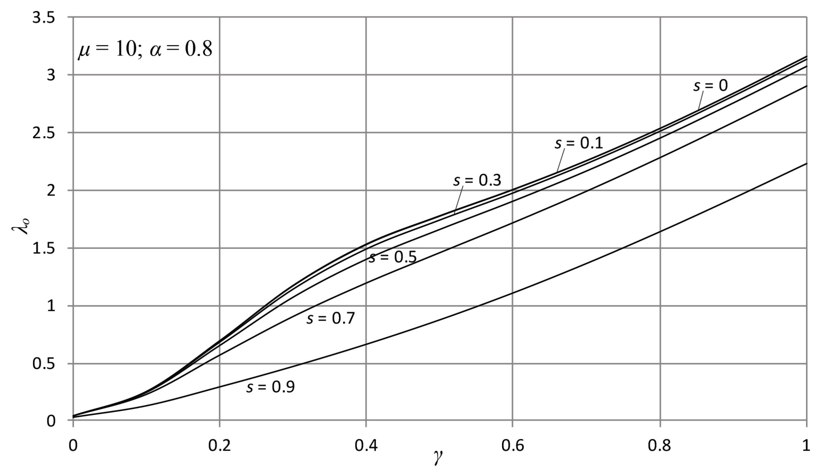

Figure 11: Eigenvalue λ_0 versus ratio of thicknesses

γ) the higher is the critical buckling load as might be expected. Calculations showed that this tendency remains valid for each value of the step location a/l. Note that the case $\mu = 0$ corresponds to the beam with absolutely free edge.

Similar results corresponding to the rigidity of the support $\mu = 10$ are depicted in Fig. 11. The step location is $a = 0.8l$, as in the previous case.

It can be seen from Fig. 12 and 13 that the larger is the crack length the lower is the critical buckling load for fixed value of the ratio of thicknesses. It is somewhat unexpected that in the range of small values of the crack length the buckling load is relatively weakly sensitive with respect to the

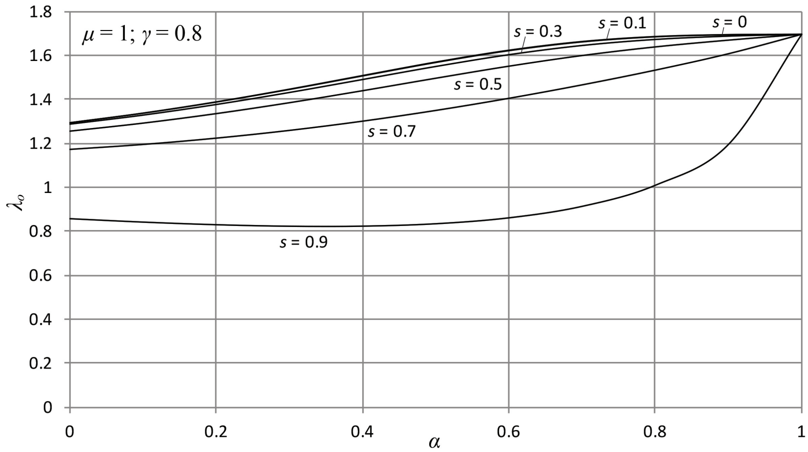

Figure 12: Eigenvalue λ_0 versus step location

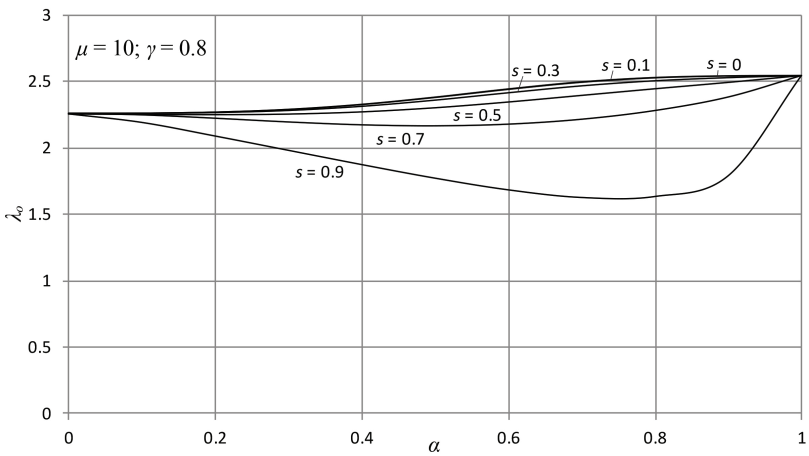

Figure 13: Eigenvalue λ_0 versus step location

crack length (the upper curves in Fig. 12–13 are quite close each other). Moreover, the elasticity of the support μ has weak influence on the buckling load in the cases of small cracks as the upper curves in Fig. 10–12 do not differ drastically for different values of μ. However, in the range of large cracks the difference between these is more obvious. For instance, the maximal value of λ_0 in the case $\mu = 1$, $s = 0.9$ is approximately 1.7. However, in the case $\mu = 10$ and $s = 0.9$ one has $\lambda_0 = 2.5$.

Comparing Fig. 13 with Fig. 12 one can see that in the case when $\mu = 10$ values of the critical buckling load are much higher than these corresponding to lower values of μ.

In Fig. 13 all curves cross the common point at $\alpha = 0$. This reflects the

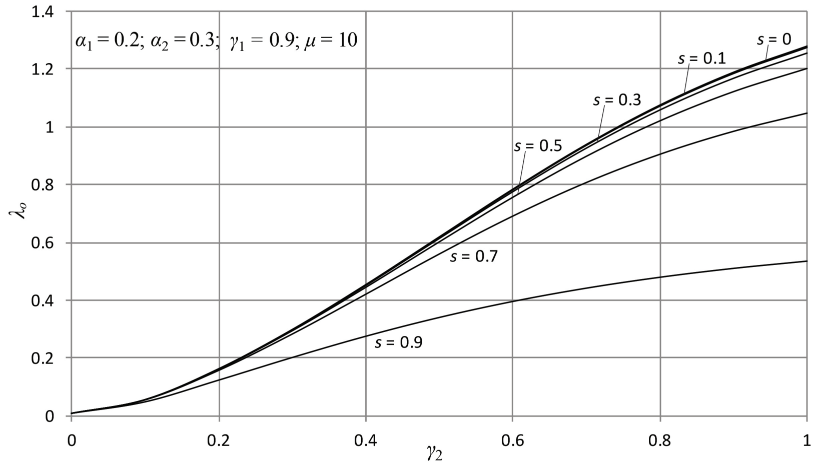

Figure 14: Eigenvalue λ_0 versus ratio of thicknesses

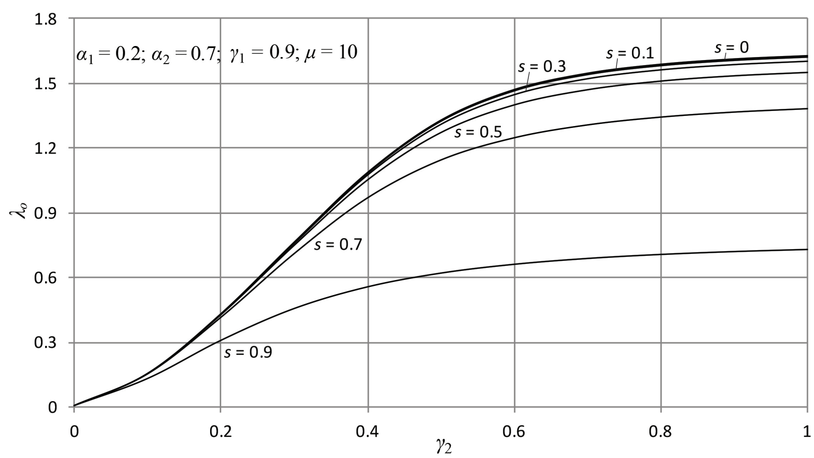

Figure 15: Eigenvalue λ_0 versus ratio of thicknesses

matter that any crack in the root section of the beam do not influence on the critical buckling load.

The results of calculations implemented for two-stepped columns are presented in Fig. 14, 15. In Fig. 14 $\alpha_1 = 0.2l$, $\alpha_2 = 0.3l$ and in Fig. 15 $\alpha_1 = 0.2l$, $\alpha_2 = 0.7l$ whereas in both cases $H_1 = 0.9H_0$ and $\mu = 10$. It reveals from Fig. 14, 15 that large cracks diminish essentially the critical buckling load.

5 Buckling of beams and columns on an elastic foundation

Beams and columns subjected to the axial pressure are studied. The beams under consideration have constant thickness and they are resting on an elastic foundation. The beams are weakened by cracks. Critical buckling loads are established for beams clamped at one end and free at the other end.

5.1 Problem formulation and governing equations

Let us consider a cantilever beam of piece wise constant thickness resting on an elastic foundation. The beam is fixed at one end and it is free at other end (Fig. 16). It has rectangular cross section with dimensions B (width), H (height) that are constant values. We assume the beams or columns to be subjected to the axial pressure loading P. Let the origin of the coordinates be located at the center of the bottom of the beam. Also, let us assume the beams are weakened by cracks with the length c_j as shown in Fig. 16.

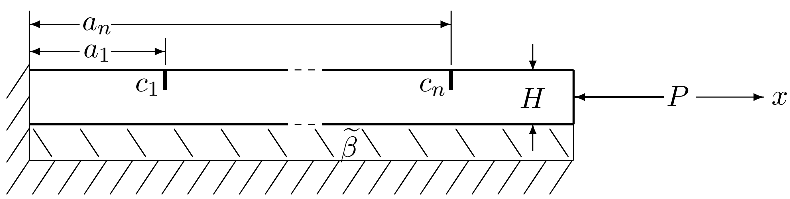

Figure 16: A beam on an elastic foundation

Let us assume that the material of the beam is purely elastic. It can be shown that the equilibrium equation of a beam resting on an elastic foundation can be presented as [31, 67, 68]

$$M'' - Pv'' + \widetilde{\beta}v = 0. \tag{5.1}$$

Here prims denote the differentiation with respect to the coordinate x and v stands for the displacement in the direction of the axis y and $\widetilde{\beta}$ is the modulus of the foundation. Since

$$M = EI_j\kappa \tag{5.2}$$

where

$$\kappa = -v'' \tag{5.3}$$

and

$$I_j = \frac{H_j^3 B}{12} \quad (j = 0, \dots, n) \tag{5.4}$$

one can write (5.1) as

$$(EI_j v'')'' - Pv'' + \widetilde{\beta} v = 0. \tag{5.5}$$

The formula (5.5) can be converted into the form

$$v'''' + \frac{\lambda_j^2}{l^2} v'' + \left(\frac{\pi}{l}\right)^4 \beta_j v = 0 \tag{5.6}$$

where

$$\beta_j = \frac{12\widetilde{\beta} l^4}{\pi^4 EBH_j^3} \tag{5.7}$$

and

$$\lambda_j^2 = \frac{12Pl^2}{EBH_j^3}. \tag{5.8}$$

Since the fourth order linear equation (5.6) has with constant coefficients one has to solve the characteristic equation

$$r^4 + \frac{\lambda_j^2}{l^2} r^2 + \beta_j = 0. \tag{5.9}$$

It can be shown that the solution of (5.9) is given by

$$\begin{aligned} r^2 &= -\frac{\lambda_j^2}{2l^2} \pm \frac{1}{2}\sqrt{\frac{\lambda_j^4}{l^4} - 4\beta_j} \\ &= \sqrt{\beta_j}\left(-\frac{\lambda_j^2}{2l^2\sqrt{\beta_j}} \pm \sqrt{\left(\frac{\lambda_j^2}{2l^2\sqrt{\beta_j}}\right)^2 - 1}\right). \end{aligned} \tag{5.10}$$

It follows from the equation (5.9) that the four roots of the characteristic equations are

$$\begin{aligned} r_{1j} &= i\beta_j^{\frac{1}{4}}\left(\varphi_j - \sqrt{\varphi_j^2 - 1}\right)^{\frac{1}{2}}, \qquad r_{2j} = -i\beta_j^{\frac{1}{4}}\left(\varphi_j - \sqrt{\varphi_j^2 - 1}\right)^{\frac{1}{2}}, \\ r_{3j} &= i\beta_j^{\frac{1}{4}}\left(\varphi_j + \sqrt{\varphi_j^2 - 1}\right)^{\frac{1}{2}}, \qquad r_{4j} = -i\beta_j^{\frac{1}{4}}\left(\varphi_j + \sqrt{\varphi_j^2 - 1}\right)^{\frac{1}{2}} \end{aligned} \tag{5.11}$$

where

$$\varphi_j = \frac{\lambda_j^2}{2l^2\sqrt{\beta_j}}. \tag{5.12}$$

It can be easily seen from (5.11) that there are three cases that we should consider: $\varphi_j > 1$, $\varphi_j = 1$, $\varphi_j < 1$.

5.2 Solution for a cantilever beam

Firstly, let us consider the case of $\varphi_j > 1$. If we denote the real positive parameters s_1 and s_2 as

$$s_{1j} = \beta_j^{\frac{1}{4}} \left(\varphi_j - \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \qquad s_{2j} = \beta_j^{\frac{1}{4}} \left(\varphi_j + \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}} \tag{5.13}$$

we can write the four roots of characteristic equation (5.9) as

$$r_{1j} = is_{1j}, \qquad r_{2j} = -is_{1j}, \quad r_{3j} = is_{2j}, \qquad r_{4j} = -is_{2j}. \tag{5.14}$$

Now the real part of the general solution of the equation (5.6) is

$$v_{1j} = A_{1j} \cos s_{1j}\xi + B_{1j} \sin s_{1j}\xi + C_{1j} \cos s_{2j}\xi + D_{1j} \sin s_{2j}\xi \tag{5.15}$$

where $\xi = x/l$. In (5.15) A_{1j}, B_{1j}, C_{1j}, D_{1j}, $j=0,\ldots,n$ stand for arbitrary constants to be defined from the boundary and intermediate conditions.

As boundary conditions depend on the type of support conditions, then in the case of a beam clamped at the left end and free at the right end one has at $\xi = 0$

$$v(0) = 0, \quad v'(0) = 0 \tag{5.16}$$

and at $\xi = 1$

$$v''(1) = 0, \quad v'''(1) + \lambda_j^2 v'(1) = 0. \tag{5.17}$$

For the case of $\varphi_j = 1$ let us denote the real positive parameter s_{3j} as

$$s_{3j} = \beta_j^{\frac{1}{4}} \tag{5.18}$$

and write the four roots of characteristic equation (5.9) as follows:

$$r_{1j} = is_{3j}, \qquad r_{2j} = -is_{3j}, \quad r_{3j} = is_{3j}, \qquad r_{4j} = -is_{3j}. \tag{5.19}$$

Here the real part of the general solution of the equation (5.6) is

$$v_{2j} = A_{2j} \cos s_{3j}\xi + B_{2j} \sin s_{3j}\xi + C_{2j} \cos s_{3j}\xi + D_{2j} \sin s_{3j}\xi. \tag{5.20}$$

Finally, if we consider the case of $\varphi_j < 1$ we write the four roots of characteristic equation (5.9) as

$$\begin{aligned} r_{1j} &= i\beta_j^{\frac{1}{4}} \left(\varphi_j - i\sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \qquad r_{2j} = -r_{1j}, \\ r_{3j} &= i\beta_j^{\frac{1}{4}} \left(\varphi_j + i\sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \qquad r_{4j} = -r_{3j} \end{aligned} \tag{5.21}$$

or

$$r_{1j} = t_j + iu_j = \eta_j, \qquad r_{2j} = -r_{1j},$$
$$r_{3j} = -t_j + iu_j = \omega_j, , \qquad r_{4j} = -r_{3j} \tag{5.22}$$

where

$$t_j = \beta_j^{\frac{1}{4}} \sqrt{\frac{1-\varphi_j}{2}}, \qquad u_j = \beta_j^{\frac{1}{4}} \sqrt{\frac{1+\varphi_j}{2}}. \tag{5.23}$$

Here the real part of the general solution of the equation (5.6) is

$$v_{3j} = A_{3j} \cosh \eta_j \xi + B_{3j} \cosh \omega_j \xi + C_{3j} \sinh \eta_j \xi + D_{3j} \sinh \omega_j \xi. \tag{5.24}$$

5.3 Results and discussion

The results of calculations in the case of a beam with no cracks are presented in Fig. 17. Here the critical buckling load versus β is portrayed. Calculations showed that the stability of the beam essentially depends of the parameters of the foundation.

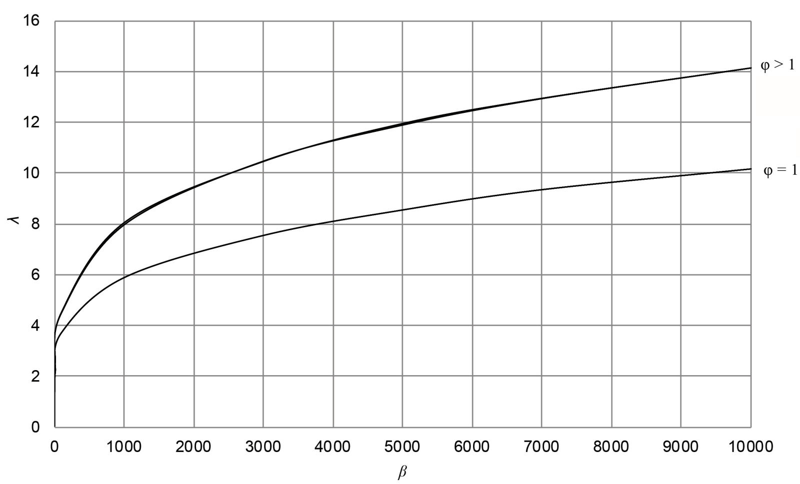

Figure 17: Critical buckling load versus the quantity β

6 Buckling of beams and columns with hollow cross-sections

The stability of elastic beams and columns is investigated. The beams are subjected to the axial pressure at the elastically supported end whereas the other end of the beam is clamped. The attention is confined to the case of the beam of rectangular cross-section of piece wise constant thickness. The hollow-sectional beams are considered. It is assumed that the beams are weakened by stable cracks. The influence of crack parameters on the critical buckling load is assessed making use of the method of massless rotational spring. The aim of this section is to reveal the influence of cracks on the critical buckling load of the weakened beam. The knowledge about the sensitivity of the buckling load on the crack parameters is essential in the practice as in many cases the operation of structural members is admitted in the stage of small cracks.

6.1 An hollow-sectional beam

Let us consider the elastic buckling of a stepped beam subjected to the axial load P. It is assumed that the loaded end is elastically fixed and the other end is fully fixed (Fig. 18).

It is assumed that the beam has hollow-sectional cross sections (Fig. 18). Let the external dimensions of the cross section be B (width) and H (height). The internal dimensions are b (width) and h (height). It is assumed that

$$H = H_j \tag{6.1}$$

for $x \in (a_j, a_{j+1})$, $j = 0, \ldots, n$. For the sake of simplicity it is assumed that $h = const$, $b = const$, $B = const$. The stepped beams are weakened by cracks or crack-like defects located at the re-entrant corners of steps at $x = a_j$ ($j = 0, \ldots, n$). Let the crack or flaw located at $x = a_j$ be with length c_j. The area of the crack is denoted by $A_C = A_{0j}$, $j = 0, \ldots, n$. It is assumed that

$$h_j = h, \tag{6.2}$$

also

$$B_j = B, \tag{6.3}$$

and

$$b_j = b \tag{6.4}$$

for $j = 0, \ldots, n$.

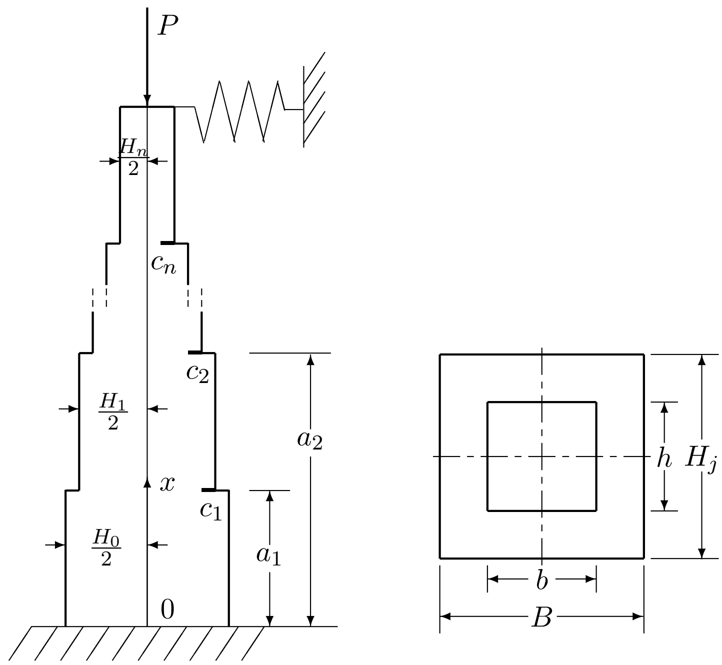

Figure 18: Elastically fixed hollow beam

6.2 The deflected shape of the beam

The displacements of curved beams are presented by (2.6) where $\xi = x/l$; A_j, B_j, C_j, D_j are arbitrary constants and

$$\lambda_j^2 = \frac{Pl^2}{EI_j} \tag{6.5}$$

where $j = 0, \ldots, n$ whereas I_j is the moment of inertia of the current cross section defined as

$$I_j = \iint_{\Omega_j} y^2 dydz. \tag{6.6}$$

In (6.6) Ω_j ($j = 0, \ldots, n$) stands for the configuration of the cross section of the beam occupied by the material. For the configuration presented in Fig. 18 one easily obtains

$$I_j = \frac{1}{12}\left(BH_j^3 - bh_j^3\right). \tag{6.7}$$

Thus, in the cases where H and h are not constants (6.5) could be put onto the form

$$\lambda_j = \sqrt{\frac{12Pl^2}{E(BH_j^3 - bh_j^3)}} \tag{6.8}$$

for $j=0,\ldots,n$. For determinations of the deflected shape of the beam one has to determine the arbitrary constants in (2.6). For this purpose one can use the boundary requirements at $\xi = 0$

$$v(0) = 0, \quad v'(0) = 0 \tag{6.9}$$

and at the elastically fixed end at $\xi = 1$

$$v''(1) = 0, \quad v'''(1) + \lambda_j^2 v'(1) = \mu v(1). \tag{6.10}$$

In (6.10) μ stands for the modulus of elasticity of the support. At the intermediate cross sections one has to satisfy the continuity and jump conditions (see Lellep and Liyvapuu [48])

$$\begin{aligned} [v(\alpha_j)] &= 0, \\ [M(\alpha_j)] &= 0, \\ [M'(\alpha_j)] &= 0 \end{aligned} \tag{6.11}$$

and

$$[v'(\alpha_j)] = \frac{EI_j}{K_{T_j} l} v''(\alpha_j + 0), \tag{6.12}$$

where for arbitrary function $g(x)$

$$[g(a_j)] = g(a_j + 0) - g(a_j - 0). \tag{6.13}$$

The quantity k_j in (6.12) can be presented in the form

$$k_j = 6\pi H_j f\left(\frac{c_j}{H_j}\right)(1 - \nu^2) \tag{6.14}$$

in the case of homogeneous beams with rectangular cross section (see Lellep, Kraav [46], Lellep, Liyvapuu [48]). However, in the case of rectangular tubes with external dimensions of the cross section B, H_j and internal dimensions b, h one has in the case of small cracks with $c_j < 0.5(H_j - h)$

$$k_j = \frac{6\pi B H_j^4}{B H_j^3 - b h_j^3} f(s_j)(1 - \nu^2) \tag{6.15}$$

as shown by Lellep and Liyvapuu [48]. In (6.14) and (6.15) (here $s_j = c_j/H_j$)

$$f(s_j) = \int_0^{s_j} x F^2(x) dx \tag{6.16}$$

where according to Tada *et al* [76]

$$F(s_j) = \frac{\sqrt{\tan \frac{\pi}{2} s_j}}{\frac{\pi}{2} s_j} \cdot \frac{0.923 + 0.199(1 - \sin \frac{\pi}{2} s_j)^4}{\cos \frac{\pi}{2} s_j}. \tag{6.17}$$

It is worthwhile to mention that (6.17) presents the shape correction function obtained by the approximation of appropriate experimental data. It is shown that the correction function in the form (6.17) leads to results which are quite close to exact ones, if the crack is not extended more than 60 % of the thickness.

6.3 Solutions for particular cases

Consider the case $n = 1$ in greater detail. Now the displacement can be presented as

$$v = A_0 \cos \lambda_0 \xi + B_0 \sin \lambda_0 \xi + C_0 \xi + D_0 \tag{6.18}$$

for $\xi \in (0, \alpha)$ and

$$v = A_1 \cos \lambda_1 \xi + B_1 \sin \lambda_1 \xi + C_1 \xi + D_1 \tag{6.19}$$

for $\xi \in (\alpha, 1)$. Here $\alpha_1 = \alpha$, $\alpha_2 = l$ and $\xi = x/l, \alpha = a/l$.

It immediately follows from (6.9) and (6.18) that

$$D_0 = -A_0, \quad C_0 = -B_0. \tag{6.20}$$

Thus, according to (6.18) and (6.20)

$$v = A_0 (\cos \lambda_0 \xi - 1) + B_0 (\sin \lambda_0 \xi - 1) \tag{6.21}$$

for $\xi \in (0, \alpha)$. Taking (6.18)–(6.21) into account one can present intermediate conditions (6.11)–(6.12) as

$$\begin{aligned}
&A_1 \cos \lambda_1 \alpha + B_1 \sin \lambda_1 \alpha + C_1 \alpha + D_1 \\
&\quad = A_0 (\cos \lambda_0 \alpha - 1) + B_0 (\sin \lambda_0 \alpha - 1), \\
&A_1 \lambda_1 \sin \lambda_1 \alpha - B_1 \lambda_1 \cos \lambda_1 \alpha - C_1 \\
&\quad = A_0 \lambda_0 \sin \lambda_0 \alpha - B_0 \lambda_0 \cos \lambda_0 \alpha + k_1 \lambda_1^2 (A_1 \cos \lambda_1 \alpha + B_1 \sin \lambda_1 \alpha), \\
&I_1 \lambda_1^2 (A_1 \cos \lambda_1 \alpha + B_1 \sin \lambda_1 \alpha) = I_0 \lambda_0^2 (A_0 \cos \lambda_0 \alpha + B_0 \sin \lambda_0 \alpha), \\
&I_1 \lambda_1^3 (-A_1 \sin \lambda_1 \alpha + B_1 \cos \lambda_1 \alpha) = I_0 \lambda_0^3 (-A_0 \sin \lambda_0 \alpha + B_0 \cos \lambda_0 \alpha).
\end{aligned} \tag{6.22}$$

The system (6.22) with boundary conditions (6.10) presents a linear system for determination of unknown constants A_1, B_1, C_1, D_1 and A_0, B_0. Equalizing its determinant Δ to zero leads to the equation for the determination of the eigenvalue λ_0. It infers from (6.5) that

$$\lambda_1 = \lambda_0 \sqrt{\frac{I_0}{I_1}}. \tag{6.23}$$

6.4 Results and discussion

The obtained equation is solved numerically. The results of calculations are presented in Fig. 19–22.

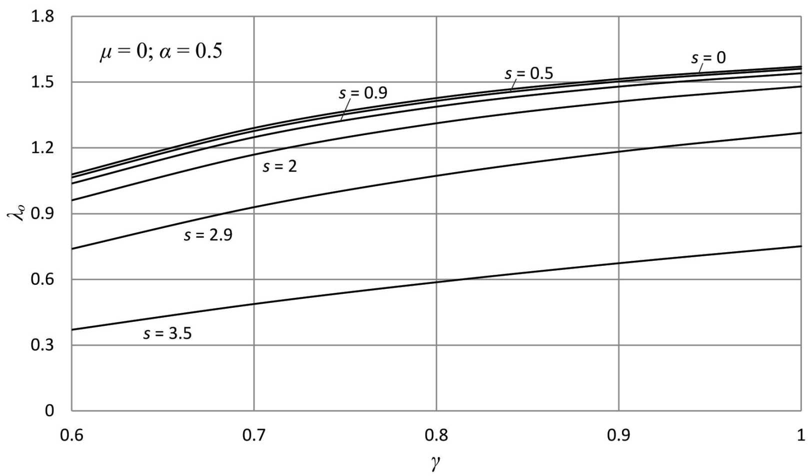

Figure 19: Critical buckling loads for $\mu = 0$, $\alpha = 0.5$

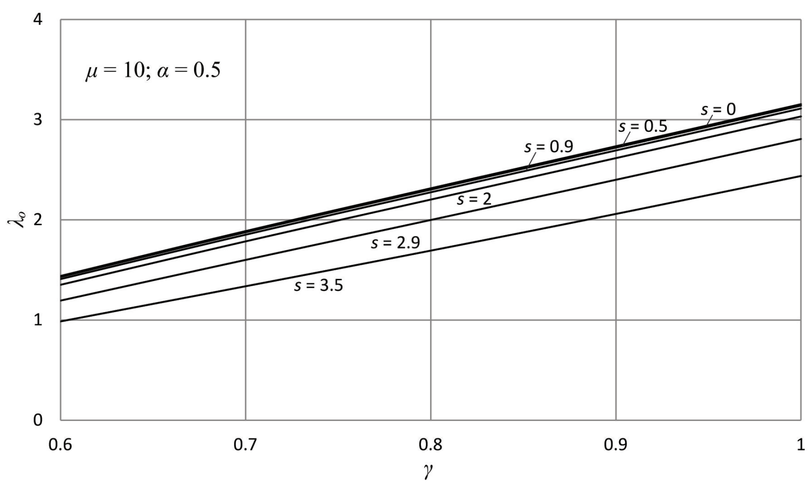

Figure 20: Critical buckling loads for $\mu = 10$, $\alpha = 0.5$

Calculations are carried out for the column with dimensions $H_0 = 0.02$ m, $B = 0.02$ m, $h = 0.01$ m, $b = 0.01$ m, $l = 1$ m. The material of the column is mild steel with $E = 2.01$ GPa and $\nu = 0.3$. In Fig. 19–22 $\alpha = a_1/l$, $\gamma = H_1/H_0$, $s = 2c_1/(H_1 - h)$.

In Fig. 19, 20 the eigenvalue (critical buckling load) λ_0 versus the ratio of thicknesses is presented for different values of the crack length. The case

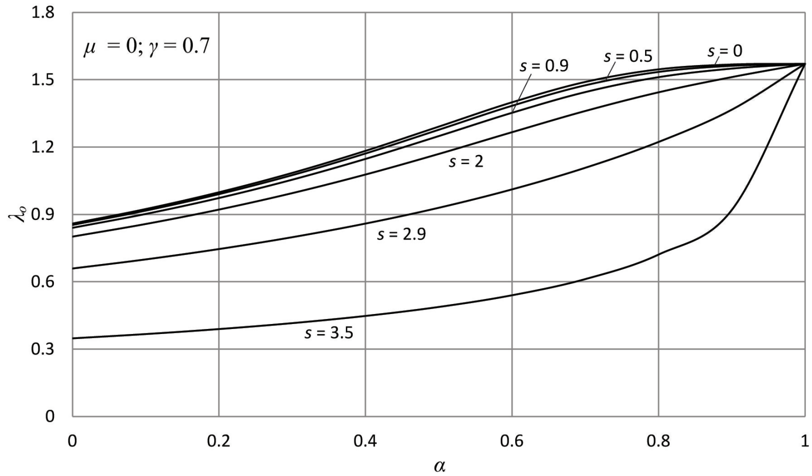

Figure 21: Critical buckling loads for $\mu = 0$ $\gamma = 0.7$

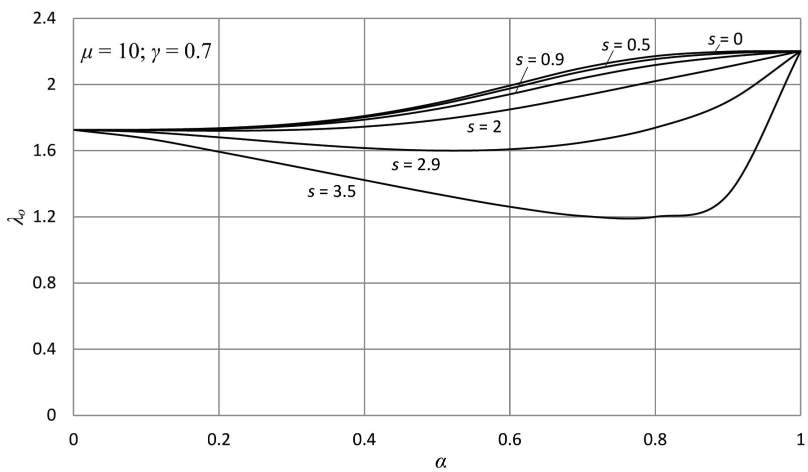

Figure 22: Critical buckling loads for $\mu = 10$, $\gamma = 0.7$

$\mu = 0$ corresponds to the column with free edge at $x = l$ (in this case no restriction is imposed at the free edge). The detailed analysis shows that the results of the current study coincide with those obtained by Zheng, Fan [95] in the case $\gamma = 1$.

The critical buckling load versus the step location α is depicted in Fig. 21, 22 for different crack extensions. Fig. 21 corresponds to the column with unconstrained free edge and Fig. 22 to the elastic support with $\mu = 10$. In both cases $H_1 = 0.7H_0$. It can be seen from Fig. 21, 22 that the elastic support makes the column stronger in comparison to the column with unconstrained edge. It also reveals from Fig. 19–22 that the highest critical buckling load corresponds to the column without any injuries as might be expected. The another observation to be mentioned is that the critical buckling load is not sensitive with respect to small cracks as the curves in Fig. 19–22 practically coincide if $s < 0.5$.

7 Conclusions

A method for determination of critical buckling loads for elastic non-uniform columns has been developed. The columns are weakened with cracks located at the internal corners of re-entrant steps. Combining the methods of the elastic beam theory and of the linear elastic fracture mechanics the influence of a crack on the stability of the beam is modelled as a change of the local flexibility of the beam.

Numerical analysis implemented for beams clamped at one end and elastically supported at the other end revealed the matter that the cracks have essential influence on the critical buckling load in the case of large cracks. However, calculations showed that small cracks which have penetrated less than 10 % of the thickness do not aggravate essentially the stability of the structure.

References

[1] N.A. Alfutov, *Stability of Elastic Structures*, Springer, Berlin, 2000.

[2] A.S. Alsabbagh, O.M. Abuzeid and M.H. Dado, *Simplified stress correction factor to study the dynamic behavior of a cracked beam*, Applied Mathem. Modelling **33** (2009), 127–139.

[3] T.L. Anderson, *Fracture Mechanics*, CRC Press, Boca Raton, 1995.

[4] N. Anifantis and A. Dimarogonas, *Stability of columns with a single crack subjected to follower and vertical loads*, Int. J. Solids Struct. **19** (1983), 281–291.

[5] N. Anifantis and A. Dimarogonas, *Imperfection post-buckling analysis of cracked columns*, Eng. Fracture Mech. **18**(3) (1983), 693–702.

[6] N. Anifantis and A. Dimarogonas, *Post buckling behavior of transverse cracked columns*, Comp. Struct. **18**(2) (1984), 351–356.

[7] T.M. Atanackovic, *Stability Theory of Elastic Rods*, World Scientific, Singapore, 1997.

[8] E. Barkanov, O. Ozolins, E. Eglitis, F. Almeido, M.C. Bowering and G. Watson, *Optimal design of composite lateral wing upper covers. Part 1. Linear buckling analysis*, Aerospace Science and Technology **38** (2014), 1–8.

[9] E. Barkanov, O. Ozolins, E. Eglitis, F. Almeido, M.C. Bowering and G. Watson, *Optimal design of composite lateral wing upper covers. Part 2. Nonlinear buckling analysis*, Aerospace Science and Technology **51** (2016), 87–95.

[10] E. Barkanov, R. Rikards and A. Chate, *Numerical optimization of sandwich and laminated composite structures*, In: Structural Optimization. Editors: S. Hernandez, M. El-Sayed, C.A. Brebbia, Comp. Mech. Publ., Southampton (1995), 311–318.

[11] Z.P. Bazant, *Stability of elastic, anelastic and disintegrating structures: A conspectus of main results*, ZAMM **80**(11/12) (2008), 709–732.

[12] Z.P. Bazant and L. Cedolin, *Stability of Structures. Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press, New York, 1991.

[13] B. Binici, *Vibration of beams with multiple open cracks subjected to axial force*, J. Sound Vibration **287** (2009), 277–295.

[14] K.B. Broberg, *Cracks and Fracture*, Academic Press, New York, 1999.

[15] D. Broek, *Practical Use of Fracture Mechanics*, Kluwer, Dordrecht, 1990.

[16] S. Caddemi and I. Calio, *Exact solution of the multi-cracked Euler-Bernoulli column*, Int. J. Solids Struct. **45** (2008), 1332–1351.

[17] S. Caddemi, I. Calio and F. Cannizzaro, *The influence of multiple cracks on tensile and compressive buckling of shear deformable beams*, Int. J. Solids Struct. **50**(20-21) (2013), 3166–3183.

[18] S. Caddemi, I. Calio and F. Cannizzaro, *Closed form solutions for stepped Timoshenko beams with internal singularities and along-axis external supports*, Arch. Appl. Mech. **83**(4) (2013), 559–577.

[19] S. Caddemi, I. Calio and F. Cannizzaro, *Tensile and compressive buckling of columns with shear deformation singularities*, Meccanica **50**(3) (2015), 707–720.

[20] S. Caddemi, I. Calio and F. Cannizzaro, *Influence of an elastic end support on the dynamic stability of Beck's column with multiple weak sections*, Int. J. Non-Linear Mech. **69** (2015), 14–28.

[21] S. Caddemi, I. Calio and F. Cannizzaro, *On the dynamic stability of shear deformable beams under a tensile load*, J. Sound Vibration **373** (2016), 89–103.

[22] E. Carrera, G. Giunta and M. Petrolo, *Beam Structures, Classical and Advanced Theories*, Wiley, 2011.

[23] N. Challamel and Y. Xiang, *On the influence of the unilateral damage behavior in the stability of cracked beam/columns*, Eng. Fracture Mech **77** (2010), 1467–1478.

[24] X. Chen and S.A. Meguid, *On the parameters which govern the symmetric snap-through behaviour of an initially curved microbeam*, Int. J. Solids Struct. **66** (2015), 77–87.

[25] T.J. Chondros, A.D. Dimarogonas and J. Yao, *A continuous cracked beam vibration theory*, J. Sound Vibration **215**(1) (1998), 17–34.

[26] T.J. Chondros, A.D. Dimarogonas and J. Yao, *Vibration of a beam with a breathing crack*, J. Sound Vibration **239** (2001), 57–67.

[27] A. Cicirello and A. Palmeri, *Static analysis of Euler-Bernoulli beams with multiple unilateral cracks under combined axial and transverse loads*, Int. J. Solids Struct. **51** (2014), 1020–1029.

[28] A.D. Dimarogonas, *Buckling of rings and tubes with longitudinal cracks*, Mech. Research Communications **8** (1991), 179–186.

[29] A.D. Dimarogonas, *Vibration of cracked structures: a state of the art review*, Eng. Fracture Mech. **55**(5) (1996), 831–857.

[30] S.C. Fan and D.Y. Zheng, *Stability of cracked Timoshenko beam column by modified Fourier series*, J. Sound Vibration **264** (2003), 475–484.

[31] M. Farshad, *Stability of Structures*, Elsevier, 1994.

[32] L.B. Freund and G. Hermann, *Dynamic fracture of a beam or plate in plane bending*, Trans. ASME (1976), 112–116.

[33] B. Gross and J.E. Srawley, *Stress intensity factors for single-edge notch specimens on bending or combined bending and tension by boundary collocation of a stress function*, NASA Techn. Note **D-2603** (1965).

[34] G.R. Irwin, *Fracture Mechanics. In: J.N. Goodier, J.N. Hoff (Eds), Structural Mechanics*, Pergamon Press, Oxford, 1960.

[35] N.G.R. Iyengar, *Structural Stability of Columns and Plates*, Wiley, New York, 1988.

[36] P.N. Jiki, *Buckling analysis of pre-cracked beam-columns by Liapunov's second method*, European J. Mechanics, A/Solids **26** (2007), 503–518.

[37] R. Jones, *Buckling of Bars, Plates, and Shells*, Taylor & Francis, Washington, 1999.

[38] L.-L. Ke, J. Yang and S. Kitipornchai, *Postbuckling analysis of edge cracked functionally graded Timoshenko beams under end shortening*, Composite Structures **90** (2009), 152–160.

[39] T. Kraav and J. Lellep, *Buckling of beams and columns on elastic foundations*, In: Proceedings of the 2nd International Conference Optimization and Analysis of Structures. Editors: J. Lellep, E. Puman, University of Tartu Press, (2013), 52–58.

[40] T. Kraav and J. Lellep, *Stability of two-stepped beams with cracks*, In: Proceedings of the 3nd International Conference Optimization and Analysis of Structures. Editors: J. Lellep, E. Puman, University of Tartu Press, (2015), 33–38.

[41] T. Kraav and J. Lellep, *Elastic stability of uniform and hollow columns*, Procedia Engineering, Modern Building Materials, Structures and Techniques **172** (2017), 570–577.

[42] S. Kukla, *Free vibrations and stability of stepped columns with cracks*, J. Sound Vibration **319** (2009), 1301–1311.

[43] E. Kägo and J. Lellep, *Free vibrations of stepped plates on elastic foundation*, Procedia Eng. **57** (2013), 489–496.

[44] J. Lellep and T. Kraav, *Optimization of stepped columns under compression*, In: Continuous Optimization and Knowledge-Based Technologies. EUROPT 2008. Editors: L. Sakalauskas, G.W. Weber, E.K. Zavadskas, Vilnius: VGTU Press "Technika", (2008), 273-278.

[45] J. Lellep and T. Kraav, *Elastic buckling of stepped beams with cracks*, In: Proceedings of the 6th IASME/Int. Conf. on Cont. Mech. (CM '11). Editors: Z. Bojkovic, J. Kacprzyk, N. Mastorakis, V. Mladenov, R. Revetria, L. Zadeh, Cambridge, UK, (2011), 11–16.

[46] J. Lellep and T. Kraav, *Buckling of beams and columns with defects*, Int. J. Struct. Stab. and Dynamics **16**(8) (2016).

[47] J. Lellep and A. Liyvapuu, *Natural vibrations of a nano-beam with cracks*, Int. J. Theor. Appl. Mech. **1** (2016), 247–252.

[48] J. Lellep and A. Liyvapuu, *Natural vibration of stepped arches*, In: Proceedings of the 3rd Int. Conf. Advances in Mech. and Autom. Eng. Editor: R. Kumar, Rome, (2015), 68–72.

[49] J. Lellep and A. Liyvapuu, *Natural vibrations of stepped arches with cracks*, Agron. Research **14**(3) (2016), 821–830.

[50] J. Lellep and A. Liyvapuu, *Free vibrations of uniform and hollow-sectional elastic arches*, Procedia Engineering, Modern Building Materials, Structures and Techniques **172** (2017), 640–648.

[51] J. Lellep, E. Puman, L. Roots and E. Tungel, *Optimization of rotationally symmetric shells*, In: Proc. Int. Conf. Appl. Mathem. Ed. C. Bulucea. Tenerife (2009), 233–238.

[52] J. Lellep and L. Roots, *Vibration of cylindrical shells with circumferential cracks*, WSEAS Trans. Mathem. **9**(9) (2010), 689–699.

[53] J. Lellep and L. Roots, *Vibration of stepped cylindrical shells with cracks*, In: 3rd Int. Conf. Eng. Mech., Struct., Engin., Geol. Editors: O. Martin, X. Zheng. Corfu (2010), 116–121.

[54] J. Lellep and L. Roots, *Axisymmetric vibrations of orthotropic cylindrical shells with cracks*, Mech. Composite Materials **49**(1) (2013), 59–68.

[55] J. Lellep and L. Roots, *Axisymmetric vibrations of stepped cylindrical shells made of composite materials*, Mech. Composite Materials **49**(2) (2013), 171–180.

[56] J. Lellep and E. Sakkov, *Buckling of stepped composite columns*, Mech. Composite Materials **42**(1) (2006), 63–72.

[57] Q.S. Li, *Buckling of multi-step non-uniform beams with elastically restrained boundary conditions*, J. Constr. Steel Res. **57** (2001), 753–777.

[58] Q.S. Li, *Buckling of multi-step non-uniform beams*, Adv. Struct. Eng. **3** (2000), 139–144.

[59] Y. Li, S.A. Meguid, Y. Fu and D. Xu, *Nonlinear analysis of thermally and electrically actuated functionally graded material microbeam*, Proc. Royal Soc. **A 470** (2014), 20130473.

[60] L. Librescu and O. Song, *Thin-Walled Composite Beams*, Springer, Dordrecht, 2006.

[61] J. Lógó, *New type of optimality criteria method in case of probabilistic loading conditions*, Mech. Based Design Struct. Mach. **35** (2007), 147–162.

[62] Y. Murakami, *Stress Intensity Factors Handbook*, Pergamon Press, New York, 1987.

[63] K. Nikpour, *Buckling of cracked composite columns*, Int. J. Solids Struct. **26** (1990), 1371–1386.

[64] K. Nikpour and A.D. Dimarogonas, *Local compliance of composite cracked bodies*, Compos. Sci. Solids Technol. **32** (1988), 209–223.

[65] H. Okamura, H.W. Liu, C.-S. Chu and H. Liebowitz, *A cracked column under compression*, Eng. Fracture Mech. **1** (1969), 547–564.

[66] W.M. Ostachowich and M. Krawczuk, *Analysis of cracks on the natural frequencies of a cantilever beam*, J. Sound Vibration **150**(2) (1991), 191–200.

[67] J.N. Reddy, *Mechanics of Laminated Composite Plates and Shells*, CRC Press, Boca Raton, 2007.

[68] J.N. Reddy, *Theory and Analysis of Elastic Plates and Shells*, CRC Press, Boca Raton, 2007.

[69] J. Rice and N. Levy, *The part-through surface crack in an elastic plate*, J. Applied Mechanics **39** (1972), 185–197.

[70] P.E. Rizos, N. Aspragathos and A.D. Dimarogonas, *Identification of crack location and magnitude in a cantilever beam from the vibration modes*, J. Sound Vibration **138** (1990), 381–388.

[71] M.H. Shen and C. Pierre, *Natural modes of Bernoulli-Euler beams with symmetric cracks*, J. Sound Vibration **138** (1990), 115–134.

[72] G.J. Simitses, *An Introduction to the Elastic Stability of Structures*, Publishing House, 2009.

[73] G.J. Simitses, *Elastic Stability of Structures*, Prentice-Hall, Englewood Cliffs, 1976.

[74] G.J. Simitses and D.H. Hodges, *Fundamentals of Structural Stability*, Elsevier, 2006.

[75] M. Skrinar, *On critical buckling load estimation for slender transversely cracked beam-columns by the application of a simple computational model*, Comput. Materials Science **43** (2008), 190–198.

[76] M. Tada, P.C. Paris and G.R. Irwin, *Stress Analysis of Cracks Handbook*, ASME, New York, 2000.

[77] J.J. Thomsen, *Vibrations and Stability. Advanced Theory, Analysis and Tools*, Springer, 2003.

[78] S.P. Timoshenko and J. Gere, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

[79] J. Vinson, *Plate and Panel Structures of Isotropic, Composite and Piezoelectric Materials, Including Sandwich Construction*, Springer, Dordrecht, 2005.

[80] E. Viola and A. Marzani, *Crack effect on dynamic stability of beams under conservative and nonconservative forces*, Eng. Fracture Mech. **71** (2004), 699–718.

[81] V.Z. Vlasov, *Thin-Walled Elastic Beams*, Israel, Jerusalem, 1961.

[82] A.S. Volmir, *Stability of Deformable Systems (in Russian)*, Nauka, Moscow, 1967.

[83] C.Y. Wang, C.M. Wang and T.M. Aung, *Buckling of a weakened column*, J. Eng. Mech. Proc. ASCE (2004), 1373–1376.

[84] C.Y. Wang and C.M. Wang, *Vibration of a beam with an internal hinge*, Int. J. Struct. Stab. Dynam.**1**(1) (2001), 163–167.

[85] C.M. Wang, J.N. Reddy and K.H. Lee, *Shear Deformable Beams and Plates*, Elsevier, Amsterdam, 2000.

[86] C.M. Wang, C.Y. Wang and J.N. Reddy, *Exact Solution for Buckling of Structural Members*, CRC Press, Boca Raton, 2005.

[87] Q.Wang, *A comprehensive stability analysis of a cracked beam subjected to follower compression*, Int. J. Solids and Structures **41** (2004), 4875–4888.

[88] Q. Wang and J.G. Chase, *Buckling analysis of cracked column structures and piezoelectric-based repair and enhancement of axial load capacity*, Int. J. Struct. Stab. Dynam. **3**(1) (2003), 17–33.

[89] Q. Wang and S.T. Quek, *Repair of cracked column under axially compressive load via piezoelectric patch*, Comput. Struct. **83** (2005), 1355–1363.

[90] P.H. Wen, *Dynamic Fracture Mechanics*, CMP, Southampton, 1996.

[91] J.B. Williams, *Fracture Mechanics of Polymers*, Halsted, New York, 1984.

[92] J. Yang and Y. Chen, *Free vibration and buckling analyses of functionally graded beams with edge cracks*, Comput. Struct. **83** (2008), 48–60.

[93] A. Yavari and S. Sarkani, *On applications of generalized functions to the analysis of Euler-Bernoulli beam-columns with jump discontinuities*, Int. J. Mech. Sci. **43**(6) (2001), 1543–1562.

[94] T. Yokoyama and M.-C. Chen, *Vibration analysis of edge-cracked beams using a line-spring model*, Eng. Fracture Mech. **59**(3) (1998), 403–409.

[95] D.Y. Zheng and S.C. Fan, *Vibration and stability of cracked hollow-sectional beams*, J. Sound Vibrations **267** (2003), 933–954.

[96] L. Zhou and Y. Huang, *Crack effect on the elastic buckling behavior on axially and eccentrically loaded columns*, Struct. Eng. Mech. **22**(2) (2006), 169–184.

[97] H. Ziegler, *Principles of Structural Stability*, Blaisdell, Massachussetts, 1968.

Summary

In the present thesis critical buckling loads of stepped beams are studied and the sensitivity of the critical load on the parameters of stable cracks as location and depth is analysed. Combining the methods of the elastic beam theory and of the linear elastic fracture mechanics an approximate method for the stability analysis of beams and columns subjected to the axial pressure is developed. Introducing the additional compliance matrix the flexibility of the beam in the vicinity of a crack is prescribed by means of the compliance of the structure. This, in turn, is coupled with the stress intensity factor which can be calculated by methods of the linear elastic fracture mechanics. Critical buckling loads of stepped columns subjected to the axial pressure and weakened with cracks emanating from re-entrant corners of steps are established. Numerical results are presented for uniform and hollow beams with single step of the cross section, also for two-stepped beams. The beams under consideration are simply supported or clamped at the ends, also cantilevers, elastically fixed. The case of beams resting on elastic foundation is studied separately. The dissertation is based on the six papers of the author (two of these are published during the last two years). The dissertation consists of the review of the obtained results, the copies of the papers, the list of literature and CV of the author.

The dissertation is organized as follows. Section 1 contains historic background of the stability analysis, the aim and the structure of the dissertation. In section 2 the concept of local flexibility is described in detail. In sections 3, 4, 5 and 6 the method is applied to partcular cases of beams. The first case concerns elastic beams that are clamped at one end and free at another end. Secondly elastically fixed beams are studied in greater detail. In section 5 beams resting on elastic foundation are considered. Finally, in section 6 beams with hollow cross sections are studied. The influence of crack length and step location on the stability of the beams has been analyzed.

Kokkuvõte

Pragudega elastsete astmeliste talade stabiilsus

Käesolevas väitekirjas vaadeldakse elastsete astmeliste talade stabiilsust. Töö aluseks on autori kuus teaduslikku publikatsiooni, millest kolm on avaldatud viimase kolme aasta jooksul.

Väitekiri koosneb neljast osast: kokkuvõtvast osast ehk kokkuvõtteartiklist, publikatsioonide koopiatest, kirjanduse ülevaatest ja autori elulookirjeldusest.

Antud töös uuritakse elastseid talasid, millele mõjub teljesuunaline koormus. Talad on astmelised ning astme kohtades asuvad defektid ehk praod, mis antud uurimuses on stabiilsed. Pragude sügavus ja asukoht mõjutab talade stabiilsust ning stabiilsuse tundlikkust antud parameetrite suhtes on analüüsitud kombineerides elastsusteooria ja lineaarse purunemismehaanika meetodeid. Esimeses peatükis tuuakse ajalooline ülevaade kirjandusest. Teises peatükis esitatakse uurimuse põhialused prao mõju analüüsiks. Praoga tala uurimiseks kasutatakse nn. kaalutu väändevedru mudelit. Selle mudeli kohaselt asendatakse praoga tala konstruktsiooniga, mis koosneb kahest tala tükist (elemendist). Need elemendid on omavahel ühendatud kaalutu väändevedruga, mille jäikus on võrdeline pinge intensiivsuse koefitsendiga prao tipu juures. Järgnevas neljas peatükis uuritakse kriitilise koormuse sõltuvust prao parameetritest erinevate talade ja kinnitustingimuste korral. Esimesel juhul on vaatluse all konsooltala, teisel juhul on vabale otsale lisatud elastne kinnitus. Kolmandaks uuritakse konsooltala, mis asub elastsel alusel ning lõpetuseks tala, mis on seest õõnes (nelinurkne toru).

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II J. Lellep, T. Kraav, Elastic buckling of stepped beams with cracks. *In: Proceedings of the 6th IASME International Conference on Continuum Mechanics (CM '11): Recent Researches in Hydrology, Geology & Continuum Mechanics.* Editors: Z. Bojkovic, J. Kacprzyk, N. Mastorakis, V. Mladenov, R. Revetria, L. Zadeh, Cambridge, UK, 2011, pp. 11–16.

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IV T. Kraav, J. Lellep, Stability of two-stepped beams with cracks. *In: Proceedings of the 3nd International Conference Optimization and Analysis of Structures.* Editors: J.Lellep, E.Puman, University of Tartu Press, 2015, pp. 33–38.

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