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CAPM model and its extensions: an overview and applications

Actuarial and Financial Engineering

Master's Thesis (30 ECTS)

Supervisor: Prof. Kalev Pärna

Tartu 2022

CAPM MODEL AND ITS EXTENSIONS: AN OVERVIEW AND APPLICATIONS

Master thesis

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Abstract

The thesis examines the validity of the Capital Asset Pricing Model (CAPM). CAPM includes components to quantify the systematic risk of assets/portfolios and evaluate the performance of assets concerning the related market. The basis of this method is rooted in the analysis of mean-variance (return and risk), which is part of Modern Portfolio Theory (MPT). The two main components of this model are beta and Jensen's alpha. Based on the degree of risk aversion of investors, beta helps investors construct a well-diversified or less risky portfolio, the most challenging aspect of this model. Alpha evaluates the performance of assets, even portfolio managers' performance. We first present the concepts and mathematical foundation of CAPM and then explore the validity of the model in two different markets: the Tehran Stock Exchange (TSE), 30 selected companies, and the New York Stock Exchange (NYSE), 30 companies constituted in the Dow Jones Industrial Average (DJIA). The behavior of these markets was opposite of each other, but they both confirmed CAPM. To improve our estimation, we used the Fama-French three-factor model, which improved asset pricing in both data sets, and finally, we added the illiquidity factor to the Fama-French three-factor model, which added a bit more improvement to the Fama-French model.

CERCS research specialisation: P160 Statistics, operations research, programming, financial and actuarial mathematics.

Key Words: Asset Pricing, Modern Portfolio Theory, CAPM, Fama-French, Illiquidity.

CAPM MUDEL JA SELLE LAIENDUSED: ÜLEVAADE JA RAKENDUSED

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Lühikokkuvõte

Lõputöö uurib varade hinnastamise CAPM mudeli (Capital Asset Pricing Model) kehtivust. CAPM mudel sisaldab komponente varade/portfellide süstemaatilise riski kvantifitseerimiseks ja varade tootluse hindamiseks. Meetod põhineb vara tulususe keskväärtuse ja dispersiooni (tulu ja risk) analüüsil, mis on osa kaasaegsest portfelliteooriast. CAPM mudeli kaks põhikomponenti on beeta ja (Jenseni) alfa. Investorite riskikartlikkuse määra alusel aitab beeta kordaja investoril luua hästi hajutatud või vähem riskantne portfell, mis on selle mudeli kõige keerulisem aspekt. Alfa kordaja hindab varade, aga ka portfellihaldurite toimimise edukust. Töös tutvustame esmalt CAPM-i mõisteid ja matemaatilisi aluseid ning seejärel uurime mudeli kehtivust kahel erineval turul: Teherani aktsiaturg (Tehran Stock Exchange TSE) koos 30 sealt valitud ettevõttega ning New Yorgi aktsiabörs (NYSE) koos Dow Jonesi tööstusindeksisse DJIA kuuluva 30 ettevõttega. Nende kahe turu käitumine oli teineteisele vastandlik, kuid kumbki eraldi kinnitas CAPM mudeli kasutatavust. Omavahel on võrreldud 'keskväärtus-dispersiooni' analüüsi abil ja teiselt poolt CAPM mudeli abil tuletatud riskinäitajaid ning ettevõtted on järjestatud nende mõõdikute alusel. Varade hinnastamise mudeli tulemuste parandamiseks kasutasime Fama-French kolme faktori mudelit, mis parandas varade hinnakujundust mõlemas andmekogumis. Lõpuks lisasime Fama-French mudelile neljandana juurde mittelikviidsusfaktori, mis pisut parandas mudelit.

CERCS teaduseriala: P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika.

Märksõnad: varade hindamine, portfelliteooria, CAPM, Fama-French mudel

Contents

Introduction

Return on an investment is a crucial concern for almost all investors. Since various factors affect the behavior of financial markets, which is often unpredictable, the rate of return estimation is always accompanied by uncertainty. Experience has shown that even in highly volatile markets, recessions, or negative markets, some assets or portfolios can be profitable or at least perform better than some other assets. The first problem is deciding how to quantify the level of risk of an asset (Roman, 2004, p 1) and how to connect the risk with the return by examining historical data. For the first time, the concept of expected return versus taken risk was introduced by Harry Markowitz in Modern Portfolio Theory (MPT). Haim Levy, Dean of the School of Business, Hebrew University about this theory says:
"Most people praise the Modern Portfolio Theory (MPT) paradigm innovated by Nobel Laureate Harry Markowitz, and a few people criticize it, but all share one thing in common: they use it intensively in their academic research and practical investments alike. What makes the MPT so immense is the amazing optimal combination of three elements: a profound analytical basis, strong intuition, and a simplicity that makes it easy to implement. No wonder it is still a pillar of modern finance in its publication, and I have no doubt it will be the center of modern finance theory for many more years to come." (M. Markowitz and Blay, 2013). CAPM operationalized MPT, as it drastically simplified the calculations necessary. Markowitz's MPT required a risk calculation for every investment entity, whereas CAPM did not. It was argued that although it was inaccurate, it was accurate enough. In this sense, CAPM is, like MPT, performative: People use the model to price securities (or, at least as a starting point to do so), and so it has power because it is used, not because it is correct. The Capital Asset Pricing Model (CAPM) is derived from MPT. This model can help capture, quantify, and present market risk, and translate it to an expected return. It is safe to say that the CAPM model is one of the most challenging topics in the financial field. The model introduced by Treynor, 1961, Sharpe, 1964, Lintner, 1965, and Mossin, 1966 independently, building on the earlier work of Harry Markowitz on diversification and modern portfolio theory. Sharpe, Markowitz, and Merton Miller jointly received the Nobel Memorial Prize in Economics for

this contribution to the field of financial economics. This model answers the questions in a way that first, the risk of each asset is determined based on the degree of dependence of its return on market return, and secondly, the relationship between risk and expected return will be a simple and direct linear relationship. In other words, traditional CAPM is a static model of portfolio selection in conditions of uncertainty and risk aversion (Misra, Vishnani, and Mehrotra, 2019).

Despite the popularity and widespread use of this model, many studies have shown that CAPM has different validity in different markets (Chen et al., 2022). One of the strongest critics of this model includes Fama and French, who based their multi-factor models on CAPM. Beside two other factors that Fama-French added to CAPM, illiquidity is an influencing factor. In the second step, we check the three-factor model of Fama-French to see each model's performance. And finally we add illiquidity risk factor to Fama-French three factor model to check improvement of pricing assets models.

Levine and Zervos, 1998, find that stock market development plays an important role in predicting future economic growth. It is one of the financial sectors that determines the extent of measuring economical power and growth (of a country or economic unit).

Attracting stagnant savings and directing them to production units, firms and economic companies is one of the most important tasks of the stock exchange. Facilitating public participation in development and creation is another task. Access to information and investing in different projects/shares is convenient and easy on this platform, even for individuals with a minimal level of investment (*Advantages of bourse* 2020). Although there are shortcomings in this market, especially in emerging markets that are more affected by macroeconomic factors, this type of investment can be considered a win-win business (especially in long run). All these explanation, led us to examine CAPM and its extensions on two stock markets. The first data set comes from the Tehran stock market and consists of 30 stocks selected from the whole list of companies. The second data set contains the stock prices of 30 companies on the New York Stock Exchange (NYSE) that are constituents of the Dow Jones Industrial Average (DJIA). The latter provides a good benchmark for our analysis. The risk parameters (betas) we obtain can be compared with the results of previous studies (still with different time windows) of the same market. In the next step, we applied three other factors to the model's expected return for the TSE market.

In the following we first provide a brief history of previous studies related to this topic. Then explain importance of this study and our main goals. Since we need to be familiar with some definition we have dedicated a section to clarify them. In second chapter, we present basis of the models we are using in this dissertation. Including Markowitz portfolio theory, derivation of CAPM, explaining Fama-French three factors model and, illiquidity risk factor. In third chapter stock market and our data sets will be introduced. In addition, methodology of this study will be explained. Finally our conclusion will be reported.

1 Literature review

In the following, the studies that form the basis of this thesis's model are described. As well, some of the most relevant research has been briefly explained. In the next step, some definitions and concepts are added. Finally, we will discuss the importance of the stock exchange market in the economy, and the reasons for choosing this market.

1.1 Thesis background

Portfolio selection modeling dates back to the development of the Markowitz mean-variance analysis. Markowitz, in 1952, published "Portfolio Selection", one of the most influential articles in the history of finance and investing (Markowitz, 1952). A key component of the MPT theory is diversification. Most investments are either high risk and high return or low risk and low return. Markowitz argued that investors could achieve their best results by choosing an optimal mix of the two based on an assessment of their risk tolerance (Chen et al., 2022). Here, for the first time, was an analytical construct showing the relationship between risk and return, and demonstrating that to achieve a greater return, you would likely have to endure greater risk, defined as variance around your expected outcome. If you were content to accept less return, you would benefit from having less overall movement in the portfolio. In this view, risk and return were correlated, and there was an ideal combination of assets for any given toleration of risk or expectation of return. The CAPM was introduced by Jack Treynor, William F. Sharpe, John Lintner, and Jan Mossin independently, building on the earlier work of Harry Markowitz on diversification and modern portfolio theory, has long shaped the way academics and practitioners think about average returns and risk. The central prediction of the model is that the market portfolio of invested wealth is mean-variance efficient in the sense of Markowitz, 1959 (Fama and French, 1992). This model opened the door for the development of more models like the Fama-French multi-factors model.

In asset pricing and portfolio management, the Fama–French three-factor model is a statistical model designed by Eugene Fama and Kenneth French to describe stock returns. In

2013, Fama shared the Nobel Memorial Prize in Economic Sciences for his empirical analysis of asset prices (EconomicScience, 2022). The three factors are (1) market excess return, (2) the outperformance of small versus big companies, and (3) the outperformance of high book/market versus low book/market companies. In 2015, Fama and French extended the model, adding two further factors — profitability and investment. Defined analogously to the HML[1] factor, the profitability factor RMW[2] which is the difference between the returns of firms with robust (high) and weak (low) operating profitability; and the investment factor CMA[3] is the difference between the returns of firms that invest conservatively and firms that invest aggressively. There is academic debate about the last two factors (Petkova, 2006). In the US (1963-2013), adding these two factors makes the HML factors redundant since the time series of HML returns are completely explained by the other four factors (most notably CMA which has a 0.7 correlation with HML) (Fama and French, 2015). Whilst the model still fails the Shanken, Gibbons, and Ross, 1989, tests whether the factors fully explain the expected returns of various portfolios. The test suggests that the five-factor model improves the explanatory power of the returns of stocks relative to the three-factor model. The failure to fully explain all portfolios tested is driven by the particularly poor performance (i.e. large negative five-factor alpha) of portfolios made up of small firms that invest a lot despite low profitability (i.e. portfolios whose returns co-vary positively with SMB[4] and negatively with RMW and CMA). If the model fully explains stock returns, the estimated alpha should be statistically indistinguishable from zero. Foye, 2018, tested the five-factor model in the UK and raises some serious concerns. Firstly, he questions how Fama and the French measure profitability. Furthermore, he shows that the five-factor model is unable to offer a convincing asset pricing model for the UK (Wikipedia, 2022d).

Pastor and Stambaugh, 2003, used 34 years data. They studied whether market liquidity is an important variable for asset pricing. They stated that liquidity risk is the profit or loss

[1] High Minus Low
[2] Robust Minus Weak
[3] Conservative Minus Aggressive
[4] Small Minus Big

that is experienced by investors as a result of market liquidity variations. Therefore, the deficiency of liquidity results in formation of a sensitivity that has a negative effect on stock value, and results in investors leaving the market. Gibson and Mougeot, 2004, examined whether aggregate market liquidity risk is priced in the US stock market. They defined a bivariate GARCH (1,1) in the mean specification for the market portfolio excess returns and changes in the standardized number of shares in the S&P 500 Index, the aggregate market liquidity proxy. Their findings suggest that systematic liquidity risk is priced in the US over the period January 1973–December 1997. The liquidity premium represents a non-negligible, negative, and time-varying component of the total market risk premium whose magnitude is not influenced by the October '87 Crash (Gibson and Mougeot, 2004). Acharya and Pedersen, 2005, provided a liquidity-adjusted CAPM, which explained the data better than the standard CAPM. They concluded that their model had a reasonably fit for portfolio sorted by liquidity, liquidity variation, and size, but failed to explain the book-to-market. Kim and Lee, 2014, investigated the pricing implication of liquidity risks in the liquidity-adjusted capital asset pricing model of Acharya and Pedersen, 2005, using multiple liquidity measures and their principal component. They found that the empirical results are sensitive to the liquidity measure used in the test, and they found strong evidence of pricing of liquidity risks when estimating liquidity risks based on the first principal component across eight measures of liquidity. Their finding implies that the the liquidity factor is an undiversifiable source of risk. Fallah Shams et al., 2014, investigated the relationship between liquidity risk

[4] Black Monday is the name commonly given to the global, sudden, severe, and largely unexpected (Gencay and Gradojevic, 2010), stock market crash on October 19, 1987. All of the twenty-three major world markets experienced a sharp decline in October 1987. When measured in United States dollars, eight markets declined by 20 to 29%, three by 30 to 39% (Malaysia, Mexico, and New Zealand), and three by more than 40% (Hong Kong, Australia, and Singapore)(Roll, 1988). The least affected was Austria (a fall of 11.4%) while the most affected was Hong Kong with a drop of 45.8%. Out of twenty-three major industrial countries, nineteen had a decline greater than 20% (Sornette, 2003). Worldwide losses were estimated at US$1.71 trillion (Schaede, 1991). The degree to which the stock market crashes spread to the wider economy (or "real economy") was directly related to the monetary policy each nation pursued in response. The central banks of the United States, West Germany, and Japan provided market liquidity to prevent debt defaults among financial institutions, and the impact on the real economy was relatively limited and short-lived (Wikipedia, 2022a).

and market risk with non-ordinary[5] return at the Fama-French three-factor model at the Tehran Stock Exchange. They analyze correlation through regression analysis for study patterns and relationships between statistical variables and extended Pastor and Stambaugh, 2003, model by adding information quality variable to Pastor and Stambaugh model. They concluded, investors "In their decisions, they always prefer investment in high liquidity securities, and seek risk premium in accepting high illiquidity securities. Given shareholders expected return increases as liquidity risk increases, therefore, in order to reduce risk, firms must consider profit quality as well as considering factors affecting risk."

1.2 Concepts and definitions

This section provides definitions and notations that will be used extensively throughout the thesis.

Return: A financial return, in its simplest terms, is the money made or lost on an investment over a defined time period, which may be represented in terms of the price change or percentage change (Heys, 2021b). $r_{i,T}$, or return of asset i, at the end of our investing period, T, is equal to:

$$r_{i,T} = \frac{P_{i,T} - P_{i,0} + D_{i,T}}{P_{i,0}}, \tag{1}$$

which, $P_{i,0}$ is the price of the asset i, at the start of the time interval, $P_{i,T}$, is the price of asset i, at the end of the investing time horizon, and $D_{i,T}$ [6], shows the dividends.

The expected return for a portfolio with n assets:

$$e(r)^{7} = \bar{r} = e(\sum_{i=1}^{n} w_i{}^{8} r_i). \tag{2}$$

[5]Nonordinary factors include accounting information and market information.

[6]A dividend is a distribution of profits by a corporation to its shareholders (O'Sullivan and M.Sheffrin, 2003, p273).

[7]Notation for expected return in this study is e, which is equal to the weighted sum of return. In the next chapters for an asset, we also use α as expected return.

[8]Weights of assets in portfolio

Risk: Risk is defined in financial terms as the chance that an outcome or investment's actual gains will differ from an expected outcome or return. Quantifiably, the risk is usually assessed by considering historical behaviors and outcomes and measuring the dispersion of outcomes around the expected value (Chen, 2020b).

It is generally accepted that a good measure of assets' risk is the variance or standard deviation (Roman, 2004). Since standard deviation has the same scale of expected return, it is more common to use.

Generally, individual returns are not independent, so the risk of a portfolio's return is obtained by:

$$\sigma^2 = Var(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov(ri, rj). \quad (3)$$

Standard deviation captures the effect of all types of risks. Some of these risks can not be controlled by investors, while a group of risks can be mitigated to some extent. So, in this perspective, we can define risk in two categories, systematic and idiosyncratic risk.

- **Systematic risk (Undiversifiable/ Unique/ market risk):** kind of risk inherent compared to the entire market or a market segment. It is difficult to avoid this type of risk (Chen, 2020a). Most of the time is unpredictable, such as changes in interest rates, inflation, natural disasters, war, oil price, and all macroeconomic factors.

- **Unsystematic risk (Specific/idiosyncratic risk):** risk associated with a company or asset (e.g., entry of new competitor, employee strike, confiscation of company property, etc.)

Risk Premium: The portion of the average holding period return above the riskless interest rate is called a risk premium (Agarwal, 2014).

Excess Return: Returns which achieved beyond the return of the proxy. Alpha is one type of excess return metric that focuses on performance return over a closely comparable benchmark (Chen, 2021).

Market Index: Market Index plays a vital role in CAPM, it is the benchmark of our calculation to obtain our portfolios/assets risk. The total Index market indicates the general level of prices of listed companies in the whole market. To make it easier to understand the concept of the total market index, suppose you have a portfolio composed of shares of all listed companies with their *weight* in the total index. In this case, the change in the market index will be equal to the return on your portfolio. They calculate all changes in stock prices and annual profits that companies pay you. According to this definition the larger the market capitalization of the companies, the more weight they will have on the total index (*Khaneye sarmayeh* 2017).

Risk-free rate of return: The risk-free rate of return is the theoretical rate of return of an investment with zero risk (Heys, 2021c). It is usually presented annually. For our monthly data, we have to calculate it by the following formula.

$$r_{f-monthly} = (1 + r_{f-yearly})^{(1/12)} - 1. \tag{4}$$

Floating stock: refers to the number of shares a company has available to trade in the open market (Mitchell, 2021).

Company Size: In this study, by the term "size," we mean the market capitalization of a company (or simply market cap). It is equal to the multiplication of the number of total outstanding shares to the price of each share at the considered time. It refers to how much a company is worth as determined by the stock market (Fernando, 2022).

Book Value: In the traditional way, which I have used for TSE market data, is equal to the equity. (In the balance sheet, it would be equal to assets minus liabilities.)

Book to Market Ratio (BV/MV) : Or BE/ME, is calculated in two ways: *i)* division of the book value per share to stock price or *ii)* division of book value to market capitalization.

1.3 Stock Exchange Market

The purpose of forming a stock exchange is to collect funds and direct them toward economic activities. Levine and Zervos, 1998, showed that stock market liquidity positively predicts growth, capital accumulation, and productivity improvements when entered together in regressions. In addition, controlling money supply, liquidity, and inflation by selling stocks and issuing bonds; collecting small and scattered capital; utilizing stagnant savings in production and financing of government and institutions, and making a fully competitive market are some of the other benefits of this market for the economic growth of a country (EMofidlearning, 2020). From an investor's point of view, capital preservation, high liquidity, transparency, security in investment, earning income, risk control, low transaction cost, and low tax cost are some of the advantages of investing in the stock market. Though stocks are often perceived as risky investments, over time they have performed better than almost all other types of security, even gold. For long-term real returns, you really can not beat the stock market. (See Figure 1) (Cagan, 2016).

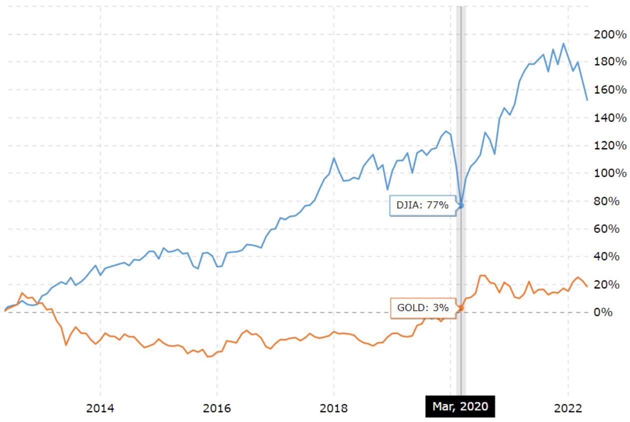

Figure 1: The chart compares the historical percentage return for the Dow Jones Industrial Average with the return for gold prices over the last 10 years[10](Macrotrend, 2022).

[10]The first case of COVID 19 in the U.S state of New York during the pandemic was confirmed on March 1, 2020. The Federal Reserve, cut interest rates to essentially zero on Sunday and launched a massive $700 billion quantitative easing program to shelter the economy from the effects of the virus. The new fed funds rate, used as a benchmark both for short-term lending for financial institutions and consumer rates, will now be targeted at 0% to 0.25% down from a previous target range of 1% to 1.25%.

2 Theoretical Framework

We briefly review the theoretical framework of the models in this chapter. Our presentation is mainly based on Markowitz portfolio selection articles (Markowitz, 1952, Markowitz, 1959, and Markowitz, 1970) which represent a mathematical framework for assembling a portfolio of assets such that the risk is minimized for a given level of return (or maximize return for a given level of risk). It is a formalization and extension of diversification in investing (Wikipedia, 2022f).

2.1 Markowitz theory

Modern Portfolio Theory, or mean-variance analysis, is the basis of the models we use here. A key insight of MPT is that an asset's risk and return should not be assessed by itself, but by how it contributes to a portfolio's overall risk and return. It uses the variance of asset prices as a proxy for risk[11].

Another key component of the MPT is diversification (Chen et al., 2022). MPT quantified the concept of diversification by introducing the statistical notion of covariance, or correlation. In essence, putting all your money in investments that may all go broke at the same time, i.e., whose returns are highly correlated, is not a very prudent investment strategy—no matter how small the chance is that any one single investment will go broke. This is because if any single investment goes broke, it is very likely, due to its high correlation with the other investments, that the other investments are also going to go broke, leading to the entire portfolio going broke (Fabozzi, Gupta, and Markowitz, 2002). In the following, we will demonstrate that as long as all assets in a portfolio were not perfectly correlated, there would be benefits from diversification.

[11]Markowitz also acknowledges that risk can be imagined in many ways. He defines six of them, all based on the total return performance—income plus asset price movement—in the public marketplace: standard deviation, semivariance, the expected value of the loss, expected absolute deviation, probability of loss, and maximum loss. He then mathematically proceeds to show that the standard deviation of returns is the most appropriate measure of risk (Kim and Francis, 2013)

2.1.1 Diversification effect

If we have n assets in a portfolio, we can use the following equations to earn the expected return and variance by considering the below notations:

$\mathbf{r} = (r_1, ..., r_n)'$: returns of n assets.

$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)'$: expected returns of assets, $\alpha_i = e(r_i)$.

$\mathbf{w} = (w_1, ..., w_n)'$: weights or portions of asset. And $\Sigma_{i=1}^{n} w_i = 1$.

$\mathbf{1} = (1, ..., 1)'$: $n \times 1$ vector of ones.

$\mathbf{0} = (0, ..., 0)'$: $n \times 1$ vector of zeros.

$\boldsymbol{\Sigma} = (\sigma_{ij})_{n\times n}$: covariance matrix, which diagonal elements represents variances.

$\sigma_{ij} = cov(r_i, r_j)$.

$\sigma_i^2 = \sigma_{ii}$: variance of r_i.

$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$: correlation coefficient.

We have simple relationship for the expected return and variance for a portfolio p_i.

$$\alpha_p = e(r_p) = e(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} w_i e(r_i) = \sum_{i=1}^{n} w_i \alpha_i = \mathbf{w}'\boldsymbol{\alpha}.$$

$$\sigma^2(r_p) = e[(\sum_{i=1}^{n} w_i(r_i - \alpha_i))^2] = e[\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j (r_i - \alpha_i)(r_j - \alpha_j)],$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} e[w_i w_j (r_i - \alpha_i)(r_j - \alpha_j)] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j cov(r_i, r_j),$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}.$$

Which $\mathbf{w}$ and $\mathbf{w}'$ are vectors of weights and transposed vector of weights respectively. $\boldsymbol{\Sigma}$ is covariance matrix. The above in short is:

$$e(r_p) = \mathbf{w}'\boldsymbol{\alpha},$$

$$\sigma(r_p) = \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}.$$

Assume that variance of each asset is a finite number, and that they are bounded, Namely, $\sigma_i^2 \leq k$. Then we can see the effect of diversification:

In general, the variance of a set of random variables depends on the variance of each of them and the covariance between them in pairs.

Case I: Considering n *uncorrelated* assets which are *equally weighted* (all are $\frac{1}{n}$ portion of portfolio):

$$\sigma^2(r_p) = \sum_{i=1}^{n} \frac{1}{n^2}\sigma_i^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma_i^2}{n} = \frac{nk}{n^2} = \frac{k}{n} \Longrightarrow \lim_{n\to+\infty}\frac{k}{n} \to 0. \tag{5}$$

we see that when number of assets increases, then risk (volatility) reduces.

Case II: If we have assets which may be *correlated*, again with assumption above that we do not have infinite variance and all are less than a constant number like k, we can consider the worst case which all have the highest possible correlation $\rho = +1$,we will have:

$$\sigma^2(r_p) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{n^2}\sigma_{ij} = \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma_i^2}{n} + \frac{n-1}{n}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{\sigma_{ij}}{n(n-1)} = \frac{nk}{n^2} + \frac{n-1}{n}\overline{\sigma_{ij,i\neq j}}$$

$$lim_{n\to+\infty}\frac{nk}{n^2} + \frac{n-1}{n}\overline{\sigma_{ij,i\neq j}} = \overline{\sigma_{ij,i\neq j}}, \tag{6}$$

Where the limit, $\overline{\sigma_{ij,i\neq j}}$, is the average covariance which is a measure of risk.

2.1.2 Efficient Frontier

An efficient investment has either (1) more return than any other investment in its risk class (that is, any other security with the same variability of return), or (2) less risk than any other security with the same level of return. The efficient frontier of the opportunity set dominates all other investments in the opportunity set. These investments are said to be Markowitz efficient frontier, referring to Harry Markowitz, the Nobel prize winner who created two-parameter portfolio theory (Kim and Francis, 2013). To show mathematically, at first we analyze two-asset portfolio behavior, then we continue for n risky assets.

Efficient frontier of two assets portfolio: Let us have a portfolio of two risky assets, n=2.

$$w_1 + w_2 = 1, r_p = w_1 r_1 + w_2 r_2.$$

$$\sigma_p^2 = var(r_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + +2\rho w_1w_2\sigma_1\sigma_2.$$

$$\sigma_p = (w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12})^{1/2}.$$

Let us consider three different values for correlation coefficient: $\{-1, 0, +1\}$

$$\rho = +1 \Rightarrow \sigma_p^2 = (w_1\sigma_1 + w_2\sigma_2)^2 \Rightarrow \sigma_p = w_1\sigma_1 + w_2\sigma_2,$$

$$\rho = -1 \Rightarrow \sigma_p^2 = (w_1\sigma_1 - w_2\sigma_2)^2 \Rightarrow \sigma_p = |w_1\sigma_1 - w_2\sigma_2|,$$

$$\rho = 0 \Rightarrow \sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2}.$$

We see that in case of perfectly positive correlated relation between two assets, we can not benefit from diversity affect very much. However, in case of perfectly negative correlated we can make the portfolio risk equal to zero. Indeed, optimal weights of two-asset portfolio with $\rho = -1$ that have $\sigma_p = 0$ are, $w_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}$ and $w_2 = \frac{\sigma_1}{\sigma_1+\sigma_2}$.
Let's now calculate optimal weights w_1 and w_2 to obtain minimum variance of portfolio return for general case.
Since $\sigma_p = (w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12})^{\frac{1}{2}}$ and $w_2 = 1 - w_1$ we have:

$$\sigma_p = (w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_{12})^{\frac{1}{2}}.$$

To minimize σ_p we need to calculate derivation of σ_p with respect to w_1, and put it equal to zero:

$$\frac{\partial\sigma_p}{\partial w_1} = \frac{1}{2}(w_1^2\sigma_1^2+(1-w_1)^2\sigma_2^2+2w_1(1-w_1)\sigma_{12})^{-\frac{1}{2}}(2w_1\sigma_1^2-2\sigma_2^2+2w_1\sigma_2^2+2\sigma_{12}-4w_1\sigma_{12}) = 0,$$

$$\Rightarrow \quad w_1\sigma_1^2 - \sigma_2^2 + w_1\sigma_2^2 + \sigma_{12} - 2w_1\sigma_{12} = 0,$$

$$w_1(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}) = \sigma_2^2 - \sigma_{12}.$$

Hence optimal weights are :

$$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \frac{\sigma_2^2 - 2\rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

$$w_2 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} = \frac{\sigma_1^2 - 2\rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Figure 2, depicts the relationship between portfolio risk σ_p and its expected return when the weights w_1 varies from 0 to 1 (and w_2 varies from 1 to 0) for these different correlation coefficients.

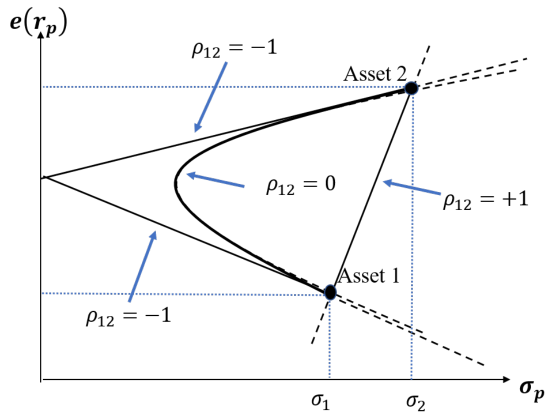

Figure 2: Combination of two-asset portfolios with three correlation values.

We see that this curve (efficient frontier) is a hyperbola, points on the upper branch of the hyperbola are the combination of optimal assets and none of that points dominates the other one. Markowitz two-moment analysis evaluate different portfolios $\mathbf{w}$ using the mean variance pair of the portfolio, with our preferences,

- Higher expected return α_p,
- Lower variance σ_p^2.

To achieve these preferences we can solve three types of problems, minimizing risk with a given level of expected return, maximizing return with a given level of risk, and risk aversion optimization (Kempthorne, 2013). In this study, we focus on the first method.

2.1.3 Risk minimization: Markowitz theory for general n

For a given level of target mean (expected return of portfolio) such as α_o, we will choose the portfolio such that optimal weights are shown by $\mathbf{w_o}$[12]:

$$Minimize: \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w}, \tag{7}$$

under conditions:

$$\mathbf{w}'\boldsymbol{\alpha} = \alpha_{\mathbf{o}},$$

$$\mathbf{w}'\mathbf{1} = 1.$$ [13]

Lagrangian multiplier method for convex optimization (minimizing problems) subject to linear constraints (quadratic programming) helps to solve this question:
1- Define the Lagrangian equation:

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} + \lambda_1(\alpha_o - \mathbf{w}'\boldsymbol{\alpha}) + \lambda_2(1 - \mathbf{w}'\mathbf{1}). \tag{8}$$

2- Derive first order conditions (FOC). See (4.1):

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{0} = \mathbf{\Sigma}\mathbf{w} - \lambda_{\mathbf{1}}\boldsymbol{\alpha} - \lambda_{\mathbf{2}}\mathbf{1}. \tag{9}$$

$$\frac{\partial L}{\partial \lambda_1} = 0 = \alpha_o - \mathbf{w}'\boldsymbol{\alpha}. \tag{10}$$

$$\frac{\partial L}{\partial \lambda_2} = 0 = 1 - \mathbf{w}'\mathbf{1}. \tag{11}$$

[12] w_o and α_o show optimal weights and desired return respectively.
[13] Bold 1 here means vector of ones with dimension of number of assets in portfolio

If we take second order Lagrangian, it will be $\mathbf{\Sigma}$ which is non-negative, and confirms our minimizing problem.

$$\frac{\partial^2 L}{\partial \mathbf{w} \partial \mathbf{w'}} = \mathbf{\Sigma} \geq 0,$$

3- Solve for $\mathbf{w}$ in terms of λ_1 and λ_2.

$$\mathbf{w_o} = \lambda_1 \mathbf{\Sigma^{-1}} \boldsymbol{\alpha} + \lambda_2 \mathbf{\Sigma}^{-1} \mathbf{1}. \tag{12}$$

4- Solve for λ_1 and λ_2 by substituting in $\mathbf{w_o}$:

$$e(r_p) = \alpha_o = \mathbf{w'_o} \boldsymbol{\alpha} = \boldsymbol{\alpha}' \mathbf{w_o} = \lambda_1 (\boldsymbol{\alpha'} \Sigma^{-1} \boldsymbol{\alpha}) + \lambda_2 (\boldsymbol{\alpha}' \Sigma^{-1} \mathbf{1}), \tag{13}$$

$$1 = \mathbf{w'_o 1} = \lambda_{\mathbf{1}} (\boldsymbol{\alpha}' \mathbf{\Sigma^{-1} 1}) + \lambda_{\mathbf{2}} (\mathbf{1}' \mathbf{\Sigma^{-1} 1}). \tag{14}$$

If we set :

$$a = \boldsymbol{\alpha'} \mathbf{\Sigma^{-1}} \boldsymbol{\alpha}, b = \boldsymbol{\alpha'} \mathbf{\Sigma^{-1} 1}, c = \mathbf{1}' \mathbf{\Sigma^{-1} 1}, \tag{15}$$

then we can solve two last equation :

$$\begin{bmatrix} \alpha_o \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{16}$$

For a given level of return, equal to α_o, and given values of λ_1 and λ_2 by obtained optimal weights, would be equal to:

$$\sigma_o^2 = \mathbf{w'_o} \mathbf{\Sigma} \mathbf{w_o} = \lambda_1^2 (\alpha' \mathbf{\Sigma^{-1}} \alpha) + \mathbf{2} \lambda_{\mathbf{1}} \lambda_{\mathbf{2}} (\alpha' \mathbf{\Sigma^{-1} 1}) + \lambda_{\mathbf{2}}^{\mathbf{2}} (\mathbf{1}' \mathbf{\Sigma^{-1} 1}). \tag{17}$$

$$= [\lambda_1 \quad \lambda_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{18}$$

From (16) we can write:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} \alpha_o \\ 1 \end{bmatrix} {}^{14}, \tag{19}$$

$$\sigma_o^2 = \begin{bmatrix} \alpha_o \\ 1 \end{bmatrix}' \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} \alpha_o \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_o \\ 1 \end{bmatrix}' \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} \alpha_o \\ 1 \end{bmatrix}, \tag{20}$$

$$= \frac{1}{ac - b^2}(c\alpha_o^2 - 2b + a). \tag{21}$$

[15] The last equation (21) shows that our graph of a feasible portfolio is a hyperbola, and the upper branch of the hyperbola shows an optimum portfolio (Kempthorne, 2013 and Kim and Francis, 2013).

2.2 Mean-Variance Optimization with Risk-Free assets

Markowitz theory considered the case of only risky assets in the portfolio. In the next step of two-moments analysis we add a riskless asset into the portfolio (with a little change in notation).

- Assume there is a risk-free asset (i=0) such that $e(r_0) = r_0$, $\sigma_0^2 = 0$.
- Suppose the investor invests the fraction $\mathbf{w'1} \leq 1$ in n risky assets and $1 - \mathbf{w'1}$ is invested in risk-free asset. (Note that in Markowitz portfolio we had $\mathbf{w'1} = 1$).
- If borrowing is allowed, $(1 - \mathbf{w'1})$ can be negative.

Now portfolio $r_p = \mathbf{w'r} + (1 - \mathbf{w'1})r_0$ has expected return and variance equal to:

$$\alpha_w = \mathbf{w}'\alpha + (1 - \mathbf{w'1})r_0 \ , \quad \sigma_w^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w}.$$

When the risk (variance) of a risk-free asset is zero, its covariance and correlation with other (risky) assets would be zero. As it is shown in figure 3, risk-free asset located on the vertical axis, point $(0, r_0)$, we can draw many lines from this point crossing feasible portfolio sets. The interesting thing is we can combine risk-free assets with risky ones such that return a portfolio includes a higher return with less risk in comparison to a minimum

[15] $\lambda_1 = \frac{c\alpha_o - b}{ac - b^2}$, $\lambda_2 = \frac{a - b\alpha_o}{ac - b^2}$

variance portfolio [16], like point B in figure 3. This means we can improve Markowitz's mean-variance analysis. So we can solve the above problem in a new version with a risk-free asset (Kempthorne, 2013).

2.2.1 Risk Minimization with Risk-Free Asset

For a given level of target mean return α_o, we minimize :

$$Minimize : \frac{1}{2}\mathbf{w'\Sigma w}, \tag{22}$$

$$s.t, \ \ \mathbf{w'}\boldsymbol{\alpha} + (1 - \mathbf{w'1})r_0 = \alpha_o.$$

In order to solve the problem (22), we use standard Lagrange's method:

$$L(w, \lambda_1) = \frac{1}{2}\mathbf{w'\Sigma w} + \lambda_1[(\alpha_o - r_0) - \mathbf{w'}(\boldsymbol{\alpha} - \mathbf{1}r_0)],$$

we equate first order derivatives to zero:

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{0} = \mathbf{\Sigma w} - \lambda_1(\boldsymbol{\alpha} - \mathbf{1}r_0), \ \frac{\partial L}{\partial \lambda_1} = 0 = (\alpha_o - r_0) - \mathbf{w'}(\boldsymbol{\alpha} - \mathbf{1}r_0).$$

This gives us optimal weight vector $\mathbf{w_o}$ of risky assets:

$$\mathbf{w_o} = \lambda_1\mathbf{\Sigma^{-1}}(\boldsymbol{\alpha} - \mathbf{1}r_0),$$

and

$$\lambda_1 = \frac{(\alpha_o - r_0)}{[(\boldsymbol{\alpha} - \mathbf{1}r_0)'\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha} - \mathbf{1}r_0)]}.$$

What varies depends on our target return α_o, which only affect on λ_1 and weights of risky assets is a simple multiplication of a fixed vector to λ_1.

[16] Or global minimum variance portfolio, means a portfolio that has the lowest risk of any feasible portfolio (Elton et al., 2014).

Portfolio return is

$$r_p = \mathbf{w_o'r} + (1 - \mathbf{w_o'1})\mathbf{r_0},$$

and portfolio variance is equal to

$$\sigma^2(r_p) = \sigma^2(\mathbf{w_o'r} + (1 - \mathbf{w_o'1})r_0) = \sigma^2(\mathbf{w_o'r}) = \mathbf{w_o'\Sigma w_o} = \frac{(\alpha_o - r_0)^2}{[(\boldsymbol{\alpha} - \mathbf{1}r_0)'\mathbf{\Sigma^{-1}}(\boldsymbol{\alpha} - \mathbf{1}r_0)]}.$$

Again, we see that the variance is a quadratic function of our target return α_o (Kempthorne, 2013).

2.2.2 Deriving Capital Market Line

If we consider a fully invested optimal portfolio (nothing is in cash , every asset is risky, and call it market portfolio (M), we have:

$$\mathbf{w'}_M\mathbf{1} = 1, \quad \mathbf{w}_M = \lambda_1\mathbf{\Sigma^{-1}}[\boldsymbol{\alpha} - \mathbf{1}r_0],$$

by multiplying both sides of $\mathbf{w}_M$ formula we can write:

$$\mathbf{1'w}_M = \mathbf{1'}\lambda_1\Sigma^{-1}[\boldsymbol{\alpha} - \mathbf{1}r_0],$$

$$1 = \mathbf{1'}\lambda_1\Sigma^{-1}[\boldsymbol{\alpha} - \mathbf{1}r_0],$$

$$\lambda_1 = \lambda_1(M) = (\mathbf{1'\Sigma^{-1}}[\boldsymbol{\alpha} - \mathbf{1}r_0])^{-1}.$$

Market portfolio return: $r_M = \mathbf{w'}_M\mathbf{r} + 0.r_0 = \mathbf{w'}_M\mathbf{r}$.

$$e(r_M) = e(\mathbf{w'}_M\mathbf{r}) = \mathbf{w'}_M\boldsymbol{\alpha} = \frac{(\boldsymbol{\alpha}'\mathbf{\Sigma^{-1}}[\boldsymbol{\alpha} - \mathbf{1}r_0])}{(\mathbf{1'\Sigma^{-1}}[\boldsymbol{\alpha} - \mathbf{1}r_0])},$$

$$= r_0 + \frac{[\boldsymbol{\alpha} - \mathbf{1}r_0]'\mathbf{\Sigma^{-1}}[\boldsymbol{\alpha} - \mathbf{1r_0}]}{(\mathbf{1'\Sigma^{-1}}[\boldsymbol{\alpha} - \mathbf{1}r_0])},$$

$$\sigma^2(r_M) = \mathbf{w'}_M \Sigma \mathbf{w}_M,$$

$$= \frac{(e(r_M) - r_0)^2}{[(\boldsymbol{\alpha} - \mathbf{1}r_0)'\boldsymbol{\Sigma}^{-\mathbf{1}}(\boldsymbol{\alpha} - \mathbf{1}r_0)]} = \frac{[(\boldsymbol{\alpha} - \mathbf{1r_0})'\boldsymbol{\Sigma}^{-\mathbf{1}}(\boldsymbol{\alpha} - \mathbf{1}r_0)]}{(\mathbf{1'}\boldsymbol{\Sigma}^{-\mathbf{1}}[\boldsymbol{\alpha} - \mathbf{1}r_0])^2}.$$

For the risky assets, we have some feasible portfolios, and for a riskless asset we have point $(0, r_0)$, the optimal portfolio including both risky and riskless assets obtained by a simple tangent line crosses r_0 point in the vertical axis and touch efficient frontier.

Based on Tobin's separation theorem (Tobin, 1958)[17], every optimal portfolio invests in a combination of the risk-free asset and the market portfolio (M in our case). Let P be the notation of optimal portfolio for target of expected return α_o with risky investment weights $\mathbf{w}_p$ as specified above. P invests in the same risky assets as the market portfolio in the same proportions, the only difference is the total weight, $w_M = \mathbf{w'}_P\mathbf{1}$, equal to:

$$w_M = \frac{\lambda_1(P)}{\lambda_1(M)} = \frac{(\alpha_o - r_0)/[(\boldsymbol{\alpha} - \mathbf{1}r_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \mathbf{1}r_0)]}{(\mathbf{1'}\Sigma^{-1}[\boldsymbol{\alpha} - \mathbf{1}r_0])^{-1}},$$

$$= (\alpha_o - r_0)\frac{(\mathbf{1'}\boldsymbol{\Sigma}^{-1}[\alpha - \mathbf{1}r_0])}{[(\boldsymbol{\alpha} - \mathbf{1}r_0)\boldsymbol{\Sigma}^{-\mathbf{1}}(\boldsymbol{\alpha} - \mathbf{1}r_0)]} = \frac{(\alpha_o - r_0)}{e(r_M) - r_0}.$$

From

$$r_P = (1 - w_M)r_0 + w_M r_M,$$

we obtain

$$\sigma_p^2 = \sigma^2(r_p) = \sigma^2(w_M r_M) = w_M^2 \sigma^2(r_M) = w_M^2 \sigma_M^2,$$

and therefore

$$w_M = \frac{\sigma_P}{\sigma_M},$$

we also have

$$e(r_p) = r_0 + w_M(e(r_M) - r_0) = r_0 + \sigma_P(\frac{e(r_M) - r_0}{\sigma_M}).$$

[17] Tobin's Separation Theorem says you can separate the problem by first finding the optimal combination of risky securities and then deciding whether to lend or borrow, depending on your attitude toward risk.

These leads to Capital Market Line (CML) which is the efficient frontier of optimal portfolios as represented on the (σ_M, α_p)-plane of return expectation (α_P) versus standard deviation (σ_P) for all portfolios.

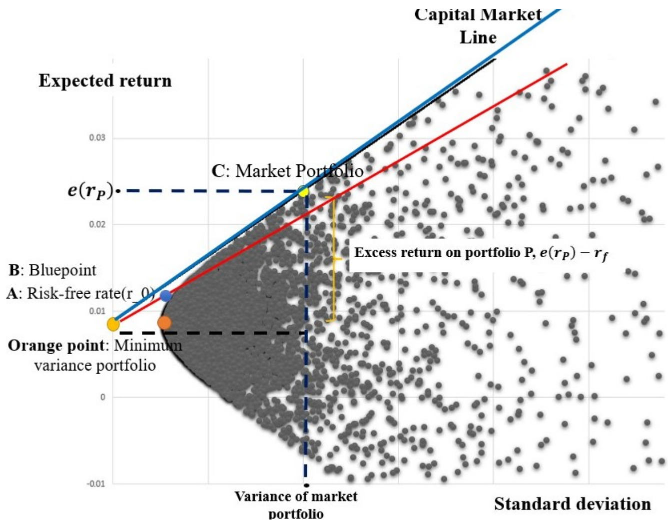

Figure 3: Capital Market Line.[18]

[19] The slope, $\frac{e(r_M)-r_0}{\sigma_M}$, of CML is the "Market Price of risk", and expected return of portfolio p increases linearity with risk σ_P. As shown in figure 3, $\frac{e(r_M)-r_0}{\sigma_M}$ is the return per risk of the market portfolio, which is equal to the Sharpe ratio for the market portfolio. If we want to invest in the market, our decision depends on how much risk versus reward we are taking. Points on CML, which are located right side of the market portfolio, have a higher return and higher volatility. These points indicate we can borrow and leverage our portfolio (Kempthorne, 2013).

2.3 Capital Asset Pricing Model (CAPM)

To better understand the Capital Market Line, we can consider an equilibrium point. Which at this point investors have both the risk free asset and the market portfolio. At this point

[19]This graph is plotted by simulating 100000 possible portfolios which composed by four hypothetical stocks' return and risk.

we buy an asset, like j with expected return equal to $e(r_j) = \alpha_j$, standard deviation σ_j and weight equals to w. Weight of market portfolio would be $1 - w$. So:

$$r_P = wr_j + (1 - w)r_M.$$

$$\alpha_P = w\alpha_j + (1 - w)\alpha_M.$$

$$\sigma_P^2 = w^2\sigma_j^2 + (1 - w)^2\sigma_M^2 + 2w(1 - w)\sigma_{jM}.$$

2.3.1 Asset's Beta

We are going to see the ratio of expected return versus risk change. (Ratio of changes in expected return vs. standard deviation in point w =0). As it can be seen in the figure 4, deviation direction of w from zero would be below the capital market line, just in point M, it goes through M. Otherwise, it means there is a new dominant portfolio that is not included in the present CML. Consequently, we need to find the new efficient frontier and new equilibrium point. As is shown in figure 4, the slope of CML in point M is equal to: $\frac{\alpha_M - r_f}{\sigma_M}$.

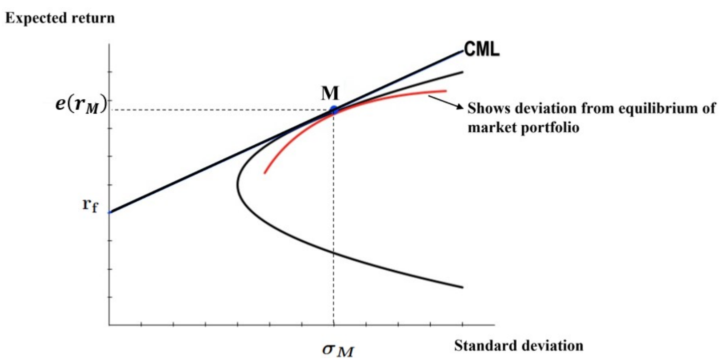

Figure 4: Equilibrium in Market portfolio (Ireland, 2022).

This value should be equal to changes of return to risk in w=0, because in tangent point slopes would be equal. We know return of portfolio is function of standard deviation so we can write following equation:

$$e(r_P) = f(\sigma_P). \tag{23}$$

We also can write expected return and standard deviation, as a function of weights like below:

$$e(r_p) = g(w) \Longrightarrow g(w) = w\alpha_j + (1-w)\alpha_M. \tag{24}$$

$$\sigma_P = h(w) \Longrightarrow h(w) = (w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM})^{\frac{1}{2}}. \tag{25}$$

Equation (26) can be established because we know, that the expected return of a portfolio is a function of the weights of assets, and also weights affect risk (standard deviation).

$$g(w) = f(h(w)). \tag{26}$$

To get the slope of the curve (red curve in figure 4), we need to use the chain rule. Since we look at these changes in proportion to the change in weight (w), it means we use a derivative with respect to w. Here, instead of partial derivative notation, we have used prime notation.

$$g'(w) = f'(h(w))h'(w) = f'(\sigma_P)h'(w) \Rightarrow f'(\sigma_P) = \frac{g'(w)}{h'(w)}.$$

Deriving from (24) and (25) :

$$g'(w) = \alpha_j - \alpha_M.$$

$$h'(w) = \frac{w\sigma_j^2 - (1-w)\sigma_M^2 + (1-2w)\sigma_{jM}}{(w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM})^{1/2}}.$$

$$f'(\sigma_P) = \frac{g'(w)}{h'(w)}|_{w=0} = \frac{(\alpha_j - \alpha_M)\sigma_M}{\sigma_{jM} - \sigma_M^2}. \tag{27}$$

On the other hand we know slope of CML which passes through points, $(0, r_f)$ and (σ_M, α_M) is equal to, $\frac{\alpha_M - r_f}{\sigma_M}$. So the (27) and slope of CML should be equal (in w=0):

$$\frac{\alpha_j - \alpha_M}{\sigma_{jM} - \sigma_M^2} = \frac{\alpha_M - r_f}{\sigma_M},$$

$$\alpha_j = r_f + \frac{\sigma_{jM}}{\sigma_M^2}(e(r_M) - r_f), \tag{28}$$

$$\alpha_j - r_f = \frac{cov(r_j, r_M)}{\sigma_M^2}(e(r_M) - r_f) \tag{29}$$

The quantity $\frac{cov(r_j,r_M)}{\sigma_M^2}$ is called β_j. We see that β_j is the slope of simple linear regression $r_j - r_f = \beta_j(e(r_M) - r_f)$ which can be estimated from return data (M&B1 and Pricing, 2017).

2.3.2 Interpretation of Beta

Two equivalent formula for beta of stock are:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} \ ,$$

$$\beta_i = \rho_{iM}\frac{\sigma_i}{\sigma_M},$$

where ρ_{iM} is the correlation coefficient between the individual company's stock return and the market. In practice, β is between zero and two, and in rare cases we had β higher than two[20], or negative betas. Beta is a measure of systematic risk. Investors are only **compensated for bearing systematic risk, not risks that can be diversified away**. In many applications, beta compares returns on a single asset or an asset class with returns on the tradable universe. It is less comprehensive than standard deviation, which captures both systematic and idiosyncratic risk. Beta is arguably more helpful in signaling success in portfolio management because it can isolate forecasting skills from dumb luck or a manager's risk appetite. Unlike correlation, beta does not range strictly between -1 and 1. Zero betas indicate an absence of correlation between an asset and its benchmark. Negative beta indicates an inverse correlation. Positive market movement for an asset with a negative beta generates a loss in value. By contrast, such an asset gains value when the broader market declines. Most applications of beta involve positive values. This resets the analytical baseline to $\beta \approx +1$. A

[20] Tesla's beta in May of 2022 is 2.12

beta of $+1$ indicates an asset whose systematic volatility responds precisely to movements in the broader market. Where $0 < \beta < 1$, the relevant asset moves in the same direction as the market, though with less sensitivity. Beta exceeding 1 indicates greater sensitivity to market-wide movements (Chen et al., 2022). By expressing CAPM in the canonical form of a linear equation $y = b + mx$ and plotting assets' return versus the beta of assets, we will have a security market line (SML), which is the visual presentation of this model. With slope $r_M - rf$ and intercept r_f, by this line we can easily find by increasing one unit in the beta how much risk premium we expect to have. Obviously, the validity of the CAPM depends on the increase in the slope of the SML as our betas increase.

2.3.3 Representation of Capital Asset Pricing Model

As we have seen, under the capital asset pricing model, there is a linear relationship between systematic risk and the asset's return. Line that relates these measures is called the Security Market Line (SML). SML or Characteristic Line is a graphic representation of a market's risk and asset's return at a given time (based on Kenton, 2022). To confirm the CAPM model, the slope of SML should be positive when the market is trending upward. Security Market Line and Capital Market Line differ mainly in the type of risk involved. The CML represents the risk premium of efficient portfolios as a function of portfolio's standard deviation, while SML depicts individual security risk premium as a function of security risk. If security return has a perfect correlation with return on the market portfolio, CML coincides with SML. Other differences are listed in table 1 (*SML vs CML - Meaning and Differences* 2022). We also can use the security market line to check overvalued and undervalued assets (see figure 5, red points are overvalued assets, green points undervalued ones, and black points have Jensen's alpha equal to zero.)

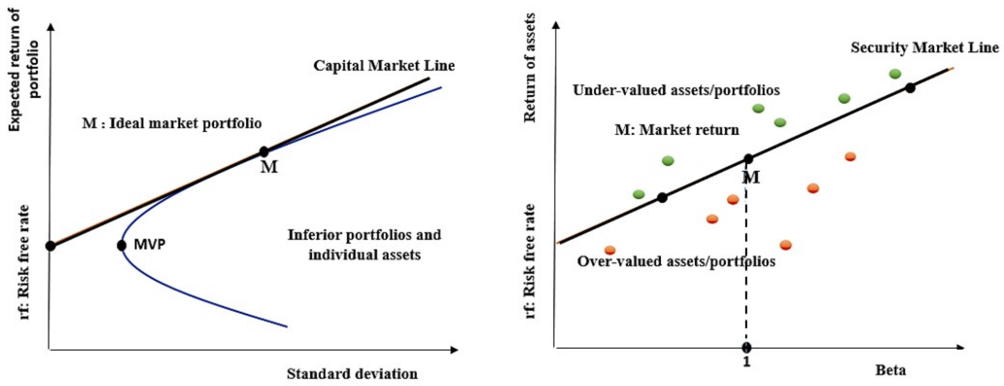

Figure 5: Capital Market Line and Security Market Line.

Basis	Capital Market Line	Security Market Line
Definition	Determine mean rate of success/loss	Determine the market risk in investment
Portfolios	Defines only efficient portfolios	Defines both efficient & nonefficient ones
Functioning	More efficient	Less efficient
Objective	Describes only market portfolios & r_f	Illustrate all security factors
Type of risk	Total risk (SD)	Only systematic risk
Graph	SD vs expected return	Beta vs. return of assets
Slope	$(r_M - r_f)/\sigma_M$	$(r_M - r_f)$

Table 1: Comparison of CML and SML.

2.4 Single-Parameter portfolio performance measures based on CAPM

To evaluate a portfolio's performance, we need to introduce a performance measure that has the same nature to be able to be compared. The coefficient beta represents the systematic risk that is due to exposition to the market variations. There are three different measures rooted in CAPM which can be considered portfolio evaluation metrics.

2.4.1 Sharpe ratio

The classic measure of excess return (return of the portfolio subtracted by the risk-free rate) relative to its risk is suggested by Sharpe, 1966.

$$S_r = \frac{\alpha_p - r_f}{\sigma_p}. \tag{30}$$

It is used to help investors understand the return of an investment compared to its risk. If the portfolio is well-diversified, the Sharpe ratio is close to the market portfolio. The higher the Sharpe ratio the higher the excess return relative to the volatility. If the performance is evaluated in a period, mostly the mean of the Sharpe ratio is calculated. The problem with this is that if the returns are strongly skewed or non-normal, the mean Sharpe ratio would be misleading. That's why Leland, 1999 suggests higher orders of the moment be used to evaluate the performance of a portfolio (Salehi, 2013). In addition to being able to use this ratio as a measure of performance comparison, it can also be used to add hedge funds. If adding a hedge fund increase the Sharpe ratio it would be a good choice to add, but recall that, an increase in the Sharpe ratio may not mean an increase in return. It shows the reward-to-risk ratio.

2.4.2 Jensen's Alpha

Jensen's measure, or Jensen's alpha, is another performance measure suggested by Jensen, 1968, represents the average return on a portfolio or investment, above or below that predicted by the CAPM model, given the portfolio's or investment's beta and the average market return. Evaluation by Jensen's α is based on the deviation of the return of the portfolio from the benchmark. Assume:

$$r_p = \alpha_p + \beta_p r_M + \epsilon, \tag{31}$$

where the excess return of the portfolio at time t is decomposed into an intercept, benchmark return, and error term with a mean equal to zero and a standard deviation of one. Based on CAPM the expected value of α_p must be equal to zero. Jensen's measure is the difference in how much a portfolio returns vs. the overall market (Salehi, 2013, p 22). If α_p is positive means the investor (portfolio manager) performs better than the market. Unlike the other ratios, the Jensen measure contains the benchmark itself.

2.4.3 Treynor ratio

The Treynor ratio is similar to the Sharpe ratio, but uses the portfolio's $\boldsymbol{\beta}$ to adjust the portfolio returns:

$$T_r = \frac{\alpha_p - r_f}{\beta_p}. \tag{32}$$

Also known as the reward-to-volatility ratio, is a performance metric for determining how much excess return was generated for each unit of risk taken on by a portfolio. The Treynor ratio allows us to evaluate the performance of a well-diversified portfolio since it only involves systematic risk. It can be used to examine the performance of a portfolio which is only a part of the investor's assets. (Risk in the Treynor ratio refers to systematic risk as measured by a portfolio's beta.) A higher Treynor ratio result means a portfolio is a more suitable investment.

2.5 Assumptions of CAPM and portfolio performance measures

All the following assumptions are combined to create the capital market line (CML) and the security market line (SML), and definitely, they should behold for other measures as well (Kim and Francis, 2013,p 438). We have to consider the assumptions of Markowitz's mean-variance analysis. Therefore, we are faced with a large number of hypotheses, including:

- In the classical approach Markowitz, 1952, returns are assumed to follow a normal distribution, which is symmetric around the mean, and minimizing the variance is equivalent to minimizing the variability below the mean.

- The market is efficient[21], so an investor can continuously make superior returns either by predicting past behavior of stocks through technical analysis or by fundamental analysis of internal company management or by finding out the intrinsic value of shares. Hence, all investors are in an equal category.
- Investors are rational, risk-averse, and would like to earn the maximum rate of return that they can achieve from their investments.
- Investors assume greater return positively correlated with higher risk. On the contrary, when risks are low, the return can also be expected to be small (This happens on portfolios located on the efficient frontier).
- All market assets are infinitely divisible, (fractional shares of every asset may be purchased)
- Homogeneous expectations, (idealized uncertainty) same expected return, risk, and correlation statistics for every asset.
- An infinite amount of money can be borrowed or lent at one risk-free rate of interest.
- Friction-less markets assumption, which means taxes, commission expenses, and all the other transaction charges that are commonly charged for buying and selling securities do not exist and use the same one-period investment horizon when making their investment decisions.
- Inflation is zero and no change in the level of market interest rates is anticipated. Stated differently, the nation's central bank always pursues optimal monetary policies.
- The capital markets are in a static equilibrium in which supply equals demand for every asset and every liability (Kim and Francis, 2013 p 438).

[21] The efficient-market hypothesis (EMH) is a hypothesis in financial economics that states that asset prices reflect all available information. A direct implication is that it is impossible to "beat the market" consistently on a risk-adjusted basis since market prices should only react to new information. Because the EMH is formulated in terms of risk adjustment, it only makes testable predictions when coupled with a particular model of risk (Fama, 1970, Wikipedia, 2022c).

The assumption of CAPM led to many criticisms of this model. In the next section, we will discuss its weaknesses.

2.6 Criticisms on the CAPM

Fama and French have conducted many studies on asset pricing models and have commented on the CAPM. They stated that there are two groups of critics of CAPM. The first group, behavioralists, believes that "sorting firms on book-to-market ratios exposes investors to overreaction to good and bad times. Investors over-extrapolate past performance.[22] The second group for explaining the empirical contradictions of the CAPM is that they point to the need for a more complicated asset pricing model. The CAPM is based on many unrealistic assumptions. For example, the assumption that investors care only about the mean and variance of one-period portfolio returns is extreme. It is reasonable that investors also care about how their portfolio return co-varies with labor income and future investment opportunities, so a portfolio's return variance misses important dimensions of risk. If so, market beta is not a complete description of an asset's risk, and we should not be surprised to find that differences in expected return are not completely explained by differences in beta. In this view, the search should turn to asset pricing models that do a better job explaining average returns." (Fama and French, 2004). On the other hand testability of CAPM is difficult, Roll, 1977, stated that testing CAPM may be infeasible. Tests of the CAPM focus on three implications of the relation between expected return and market beta as implied by the model:

- Linear relation between expected returns on all assets and their betas.
- The beta premium is positive which means that the expected return on the market portfolio is more than that of assets that are uncorrelated with the market return.
- Assets whose returns are uncorrelated with the market have expected returns equal

[22] Proponents of this view include DeBondt and Thaler (1987), Lakonishok, Shleifer and Vishny (1994) and Haugen (1995) (Fama and French, 2004).

to the risk-free rate, and the beta premium is equal to the expected return on the market portfolio minus the risk-free rate.

However, in testing these implications empirically, researchers face many obstacles (Abdou, 2019, p29). Generally, to check the validity of CAPM, the slope of SML is considered. When it has an upward slope in a positive market, it means a relationship between expected return and beta. Some factors added to CAPM to improve and develop it. We will see two of them.

2.7 Multi-factor model of asset pricing

Fama and French presented their three-factor model based on their findings and CAPM in 1992. Their findings suggest that the size and ratio of book value to market value may be proxies for systematic risk, which are not considered in the beta of the CAPM model. They stated market equity (ME or MV, stock price times shares outstanding) and the ratio of book equity to market equity (BV/MV) capture much of the cross-section of average stock returns. With rational pricing, size and BE/ME must proxy for sensitivity to common risk factors in returns (Fama and French, 1995). They presented a three-factor regression model, equal to:

$$r_{it} - r_f = \alpha_i + \beta_{im}(R_{mt} - r_{it}) + \beta_{iSMB}SMB_t + \beta_{iHML}HML_t + \epsilon_{it}, \tag{33}$$

which,

- $\mathbf{r_{it}}$: return of asset i at time t.
- $\mathbf{R_{mt}} - \mathbf{r_{it}}$: Market excess return.
- **SMB**: (small minus big) is the difference between the returns on small-stock and big-stock portfolios with about the same weighted average book-to-market equity.
- **HML**: (high minus low) is the difference between the returns on high and low book-to-market equity portfolios with about the same weighted average size (Fama and French, 1993).

Fama and French, 1993, by using two factors, the size of the company (three groups; small, Medium, and big) and BV/MV (concerning the book value in three groups; Low, Nuetral, and High), divided data into 6 portfolios. And calculated SMB and HML based on the below formula.

$$SMB = \frac{1}{3}(\frac{S}{L} + \frac{S}{M} + \frac{S}{H}) - \frac{1}{3}(\frac{B}{L} + \frac{B}{M} + \frac{B}{H}) \tag{34}$$

$$HML = \frac{1}{2}(\frac{S}{H} + \frac{B}{H}) - \frac{1}{2}(\frac{S}{L} + \frac{B}{L}) \tag{35}$$

In equation (34) and (35), S/L means small size companies with low BV/MV; S/M refers to small size and medium BV/MV companies and S/H stands for small companies with high BV/MV. This order is the same for large companies in the second fraction.
- ϵ_{it}: errors of model.

2.8 Illiquidity risk

By the term "liquidity" we mean the ability of the market to absorb a large number of transactions without causing excessive fluctuations in the price. Liquidity risk means at a certain time a given financial asset cannot be traded quickly enough in the market without impacting the market price.
Risk-averse investors naturally require a higher expected return as compensation for liquidity risk. The liquidity-adjusted CAPM pricing model states that, the higher an asset's market-liquidity risk, the higher its required return (Acharya and Pedersen, 2005). There are different measures to calculate liquidity. These criteria can be based on transaction costs, price or volume of transactions. Some liquidity proxies are:

- **Trading volume**: this measure is widely used. It is simple and easy to get the needed information.
- **Trading frequency**: equals to the number of transactions during a time interval.
- **Percentage of trading days**: equals to the division of the number of days that at

least one transaction/trade occur, to the total number of trade in a year (that market is active).

- **Trading value**: or size measure, calculated by the number of transactions times to the price of the stock.
- **Percent of Floating shares**: it is usually reported quarterly.
- **Floating stock turnover**: calculated by volume of traded stocks to float stock.
- **Trading waiting time:** equals to the number of days on which market is active, divided by the frequency of trading.
- **Amihud criteria**:

$$ILLIQ_t = \Sigma_{i=1}^{t} \frac{|r_t|}{VOL_t} \tag{36}$$

VOL_t, is the dollar (currency) volume of stock at time t, and $|r_t|$ is the absolute return of the asset at time t. The liquidity value is inverse of the illiquidity value.

- **Amivest ratio**: Obtained by division of the number of trading to return.
- **Rank of trading**: Usually reported by investing companies.
- **Bid-Ask Spread**: Comes from the difference between selling and buying prices.
- **and so on** (IOSCO Emerging Markets Committee, 2007).

Liquidity is a broad concept and many studies have been done to define it. In this thesis, we use the well-known, Amihude liquidity criterion, based on Amihud, 2002 and Barardehi et al., 2021 papers. Amihude's key advantage stems from the fact that its simple construction only requires return and dollar (currency) volume data that is available for many markets and countries over long periods. The logic of the Amihude liquidity measure is to relate the absolute return of stocks to their daily value. It quantifies, how much a company's stock needs to be traded to initiate a price movement. Obviously, the more sensitive is the absolute return to changes in volume, the more illiquid the asset is.[23] Amihude's original measure uses the ratio of absolute daily (monthly, ...) close-to-close return to dollar volume as a proxy for price impacts of trading, i.e., the amount a given trading volume moves market prices. Data and institutional details indicate that while there is nearly no trading volume outside regular trading hours, the corresponding overnight price movements make up a large share of close-to-close absolute returns. The literature has established that overnight (after-hours) price movements are typically driven by information arrivals that are unrelated to the daily trading volume used in the denominator. This means that overnight returns in a proxy of price impacts creates a fundamental time mismatch between inputs. So, Barardehi et al. used an open-to-close price to calculate the return (Yashar H. Barardehi, 2019).

By adding the illiquidity factor (showing by ILIQ)we will have the below regression model:

$$r_{it} - r_f = \alpha_i + \beta_{im} R_{mt} + \beta_{iSMB} SMB_t + \beta_{iHML} HML_t + \beta_{ILIQ} ILIQ + \epsilon_{it}, \tag{37}$$

[23]For example, if you want to sell a billion dollar worth of a company's shares, you might trigger a movement in price that would decrease the value of stock. It means the market can not absorb it. And the injection of one billion dollar shares leads you to sell your shares with a loss. And vice versa, buying a large number of shares causes price increases.

3 Cases of Study and methodology of research

We were primarily interested in investigating the Tehran stock market. On the other hand, information on the New York stock market is widely available to the public and has been the subject of many studies. Therefore, we can use it as a proxy to check the correctness of this work. In addition, the behavior of TSE during the last years was a bit opposite of other markets. So a comparison between these markets will be interesting.

In this chapter, we briefly introduce the two markets we are about to review, then explain the methods of this study. Finally, we will report our findings.

3.1 NYSE 30 selected companies

The Dow Jones Industrial Average (DJIA), also known as Dow Jones, or simply the Dow, is a price-weighted measurement stock market index of 30 prominent companies listed on stock exchanges in the United States (Wikipedia, 2022b). All major sectors of the economy are represented, except transportation and utilities (Hall, 2022)(see 4.2). It has introduced by Charles Dow, a co-founder of Dow Jones & Company, on May 26, 1896, the DJIA index originally contained only 12 stocks. The index membership expanded to 20 stocks in 1916 and 30 stocks in 1928. Since its inception in 1896, the DJIA index has changed its composition 49 times (Biktimirov and Xu, 2019). The newest member of DJIA is Dow Inc. which is added in March 2019. In a price-weighted index, such as the Dow Jones Industrial Average, constituent weights are determined solely by the prices of the constituent stocks. Shares outstanding are set to a uniform number throughout the index. Indices using this methodology will adjust the index divisor for any price impacting corporate action on one of its member stocks; this includes price adjustments, special dividends, stock splits[24], and rights offerings[25]. The index

[24] A stock split is when a company's board of directors issues more shares of stock to its current shareholders without diluting the value of their stakes. A stock split increases the number of shares outstanding and lowers the individual value of each share (Marquit and Schmidt, 2022).

[25] A rights offering (rights issue) is a group of rights offered to existing shareholders to purchase additional stock shares, known as subscription warrants, in proportion to their existing holdings. These are considered to be a type of option since it gives a company's stockholders the right, but

divisor will also adjust in the event of an addition to or deletion from the index. Formula of calculation index for Dow Jones(S&P 500, ...) is :

$$Index\ Level = \frac{\Sigma_{i=1}^{n} p_i * q_i}{Divisor}. \tag{38}$$

(Wikipedia, 2022b).

Sum of the stock prices of the companies included in the index, divided by a factor which is currently (as of November 2021) approximately 0.152. The factor is changed whenever a constituent company undergoes a stock split so that the value of the index is unaffected by the stock split (Wikipedia, 2022b). You can see movement of these stocks in figure 6 and compare it with our market index (DJIA index).

3.1.1 Collecting and Analyzing data

We have used Yahoo Finance (*Yahoo-Finance* ,2022) and Fama French Data Library (*FamaFrench data library* ,2022), to collect the necessary data for 5 years monthly data and information of Dow & companies. And to read, clean, analyze, and visualize data, we have used Excel, R, and Power BI. In theoretical part we discuss CAPM model and two extension model on it which are:

- Single factor model (CAPM):

$$r_i - r_f = \alpha_i + \beta_i(R_m - r_f) + \epsilon_i, \tag{39}$$

- Three factors model of Fama and French:

$$r_i - r_f = \alpha_i + \beta_{im}(R_m - r_f) + \beta_{iSMB}SMB + \beta_{iHML}HML + \epsilon_i, \tag{40}$$

not the obligation, to purchase additional shares in the company (Heys, 2020).

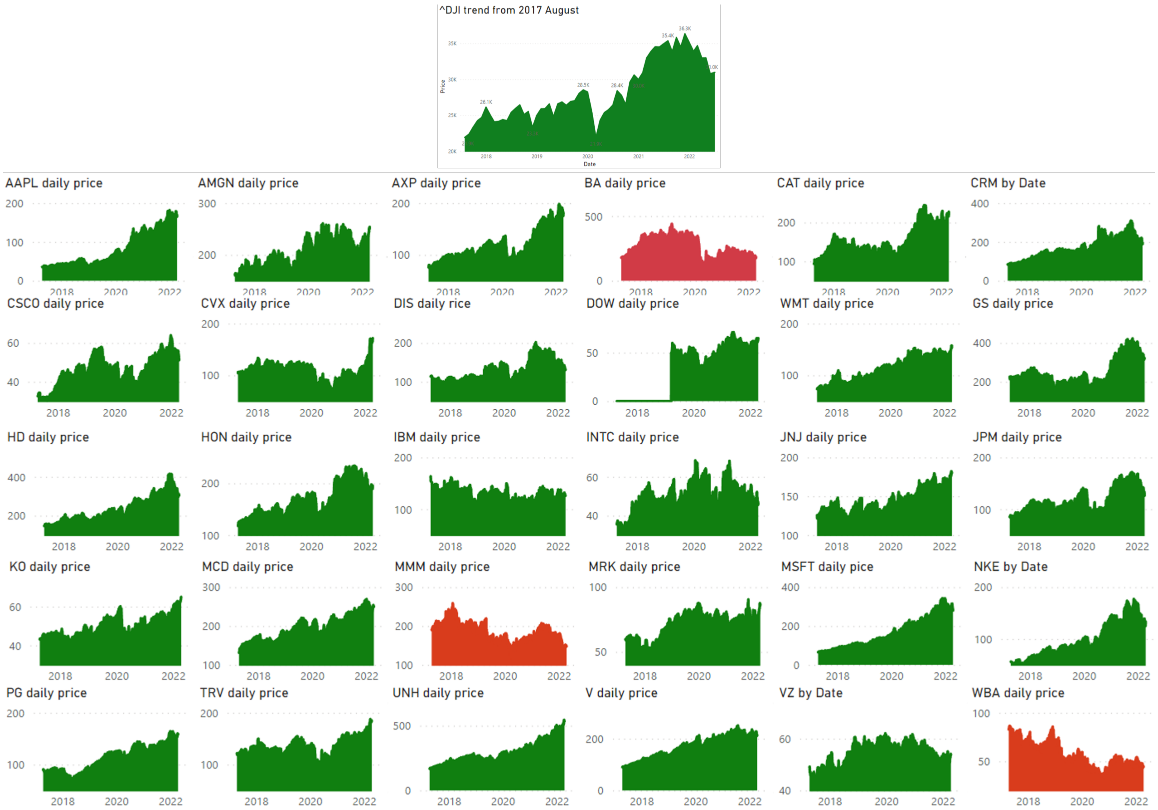

Figure 6: NYSE 30 companies price behavior.

- And four factor model by adding illiquidity factor to Fama -French three-factor model:

$$r_i - r_f = \alpha_i + \beta_{im}(R_m - r_f) + \beta_{iSMB}SMB + \beta_{iHML}HML + \beta_{IILIQ}ILLIQ + \epsilon_i. \quad (41)$$

In our models we generally have four explanatory variables. In the following the calculation of each one is explained one by one.

3.1.2 Explanatory variables calculations

1. **Market excess return** ($R_m - rf$): This factor calculated by monthly return of market minus monthly risk free return. Prices of assets and index are reported in Yahoo finance website. Using (1) we can obtain return values of index and stocks.

And r_f reported in Fama-french data library and Fred website.

2. **Size factor, SMB:** To determine SMB factor values, we first used the market capitalization of each company at the beginning of each financial year, then sorted them from the smallest to the biggest and classified them into three groups. In this case, 25% of companies are listed in the first group (smallest ones), which in our case contains eight companies; 50% in the second group (median group), which includes 14 companies; and the last 25% of companies, which includes companies with the largest market value. Considering the BV/MV ratio of companies in each group (they should be close to each other), and finally, subtract the monthly mean return of the big group from the small group.

   SMB = Return of small market cap companies - Return of big market cap companies.

3. **Value factor, HML:** The working method is that we sort companies based on the ratio of book value to market value (BV/MV) in each year for all the investigated companies, from small to large, and then, based on this rate, we divide the companies into three groups. 25% small, 50% medium, and 25% big ones. Finally, to calculate HML, we have subtracted the returns of small companies from the returns of large companies each month.

   HML = Return of companies with large $\frac{BV}{MV}$ ratio - Return of companies with small $\frac{BV}{MV}$ ratio.

4. **Illiquidity measure:** This measure calculation have three steps:

   - Calculation of $(|\frac{close.price}{open.price} - 1|) \times 100$
   - Dollar volume: $\frac{Volume\ of\ shares \times\ open.price}{10^9}$, divide by 10^9 (report in Billion dollar).
   - **Illiquidity** = $\frac{Average\ of\ absolute\ returns}{Dollar\ volume}$. To calculate monthly illiquidity, We have used above steps for each month.

Based on our calculations, the CAPM plot for NYSE is illustrated in figure 7. It confirms CAPM model, since the Security Market Line has an upward trend, which means tolerating more systematic risk we will earn more. (In positive market such as our cases)

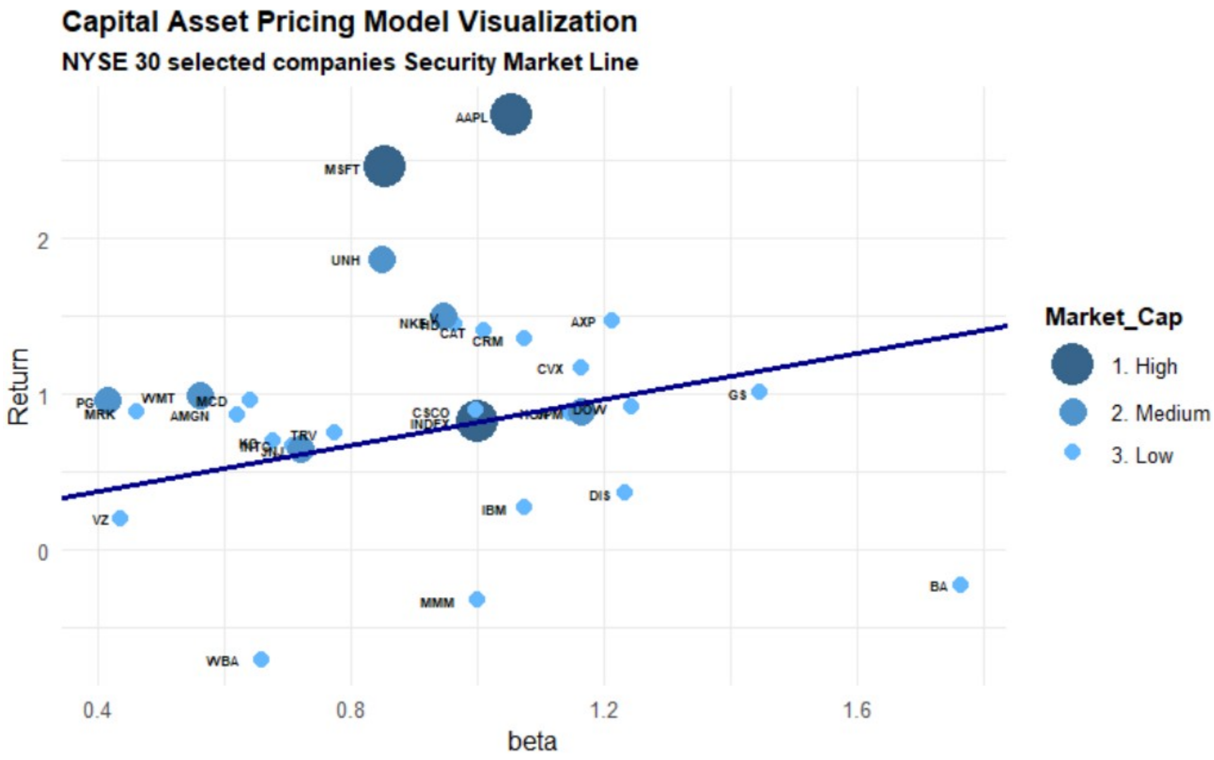

Figure 7: CAPM visualization for Dow & company.

Calculated performance and risk measures of single factor model of NYSE are reported in table 2. Tickers are arranged based on each values. In the fifth column, a 5-year monthly beta from the Yahoo finance website is also reported. The differences happen because the market index that is used in yahoo finance website, is S&P 500 index. Our calculation based on S&P 500 index, were very close to reported beta of yahoo finance. For example, reported beta of apple Inc, is 1.19, and what we have obtained is 1.187.

Ticker	α	Ticker	β	$\beta_{y.f}$	Ticker	Treynor.r	Ticker	Sharpe.r	Ticker	Mean	Ticker	Std.dev	Ticker	MC(B$)
AAPL	1.93	BA	1.76	1.37	MSFT	2.78	MSFT	0.41	AAPL	2.79	BA	12.70	INDEX	10350
MSFT	1.75	GS	1.45	1.39	AAPL	2.57	AAPL	0.31	MSFT	2.46	CRM	8.91	AAPL	2390
UNH	1.16	DOW	1.24	1.41	UNH	2.10	UNH	0.26	UNH	1.87	AAPL	8.78	MSFT	2000
V	0.72	DIS	1.23	1.24	PG	2.10	V	0.23	V	1.50	GS	8.68	UNH	487.92
NKE	0.68	AXP	1.21	1.14	MRK	1.74	HD	0.20	AXP	1.47	DIS	8.67	JNJ	470.55
HD	0.66	JPM	1.17	1.14	WMT	1.61	NKE	0.19	NKE	1.47	CVX	8.67	V	423.67
CAT	0.58	CVX	1.16	1.13	V	1.50	PG	0.18	HD	1.46	WBA	8.46	PG	348.57
PG	0.57	HON	1.15	1.08	NKE	1.46	AXP	0.18	CAT	1.41	CAT	7.81	WMT	344.45
AXP	0.49	CRM	1.08	1.08	HD	1.42	MCD	0.18	CRM	1.35	AXP	7.68	JPM	337.53
WMT	0.49	IBM	1.07	0.93	MCD	1.37	CAT	0.17	CVX	1.17	CSCO	7.55	HD	294.87
CRM	0.48	AAPL	1.05	1.19	CAT	1.31	DOW	0.16	GS	1.01	INTC	7.51	CVX	282.17
MRK	0.46	CAT	1.01	0.99	AMGN	1.27	WMT	0.16	WMT	0.99	IBM	7.30	KO	274.45
MCD	0.41	MMM	1.00	0.95	CRM	1.18	INDEX	0.15	MCD	0.96	NKE	7.28	MRK	235.58
AMGN	0.33	INDEX	1.00		AXP	1.15	CRM	0.14	PG	0.96	JPM	7.20	VZ	212.71
CVX	0.23	CSCO	1.00	0.96	CVX	0.93	MRK	0.14	DOW	0.92	HD	6.90	MCD	187.94
KO	0.12	HD	0.96	0.99	KO	0.91	CVX	0.13	CSCO	0.90	UNH	6.75	CSCO	180.05
TRV	0.10	NKE	0.95	1.01	TRV	0.86	HON	0.13	JPM	0.89	AMGN	6.48	DIS	175.32
CSCO	0.08	V	0.95	0.90	INTC	0.83	AMGN	0.12	MRK	0.89	HON	6.37	CRM	174.41
INTC	0.07	MSFT	0.86	0.93	CSCO	0.82	KO	0.12	HON	0.88	V	6.24	NKE	169.92
JNJ	0.03	UNH	0.85	0.77	JNJ	0.78	TRV	0.11	AMGN	0.87	MMM	6.18	INTC	155.20
INDEX	0.00	TRV	0.77	0.70	INDEX	0.74	JPM	0.11	INDEX	0.82	MRK	5.94	AMGN	133.17
HON	-0.05	JNJ	0.72	0.63	DOW	0.74	JNJ	0.11	TRV	0.75	TRV	5.89	AXP	129.08
JPM	-0.05	INTC	0.71	0.67	HON	0.69	CSCO	0.11	KO	0.70	WMT	5.81	IBM	126.56
GS	-0.14	KO	0.68	0.56	JPM	0.69	GS	0.11	INTC	0.67	DOW	5.81	HON	118.34
VZ	-0.20	WBA	0.66	0.59	GS	0.64	INTC	0.08	JNJ	0.65	MSFT	5.80	GS	102.04
DOW	-0.48	MCD	0.64	0.55	VZ	0.27	DIS	0.03	DIS	0.37	KO	5.24	CAT	96.02
IBM	-0.60	AMGN	0.62	0.58	DIS	0.23	VZ	0.03	IBM	0.27	JNJ	5.09	BA	82.43
DIS	-0.63	WMT	0.56	0.52	IBM	0.18	IBM	0.03	VZ	0.20	MCD	5.00	MMM	73.41
MMM	-1.15	MRK	0.46	0.36	BA	-0.17	BA	-0.02	BA	-0.23	PG	4.82	TRV	40.66
WBA	-1.27	VZ	0.44	0.38	MMM	-0.40	MMM	-0.07	MMM	-0.32	INDEX	4.77	DOW	37.58
BA	-1.61	PG	0.42	0.39	WBA	-1.19	WBA	-0.09	WBA	-0.70	VZ	4.55	WBA	32.59

Table 2: Dow & companies' order based on risk measures.

By examining two other models we see improvement in pricing of assets. statistical result of this three models are shown in table 3.

	CAPM	Three Factors model	Four Factors model
$\boldsymbol{\beta}_{mkt}$	$0.03_{(0.428)}$	$0.967^{*}_{(0.405)}$	$0.951_{(0.560)}$
$\boldsymbol{\beta}_{smb}$		$-0.552_{(0.421)}$	$-0.359_{(0.46)}$
$\boldsymbol{\beta}_{hml}$		$-0.742^{**}_{(0.316)}$	$-0.75^{**}_{(0.32)}$
$\boldsymbol{\beta}_{illiq}$			$0.4057_{(0.423)}$
Adj R^2	-0.03435	0.3	0.2204
Residual std.error	0.7	0.63	0.633
F-statistic	0.003754	3.953**	3.121**
AIC	72.4023	65.36	66.25

Table 3: Summary of three asset pricing model of Dow & companies.(*p<0.05; **p<0.01, ***p<0.001) , numbers in brackets are variance of each measure.

3.2 Tehran 30 selected companies

Tehran Stock Exchange (TSE) is Iran's largest stock exchange market, which first opened in 1967. The concept of stock industrialization dates to 1936, when Melli Bank (national bank of Iran) together with Belgian experts, issued a report to plan for an operational exchange in Iran. However, the plan was not implemented before the outbreak of World War II and did not gain traction until 1967. TSE began operation in 1967, trading only in corporate and government bonds. Iran's rapid economic expansion in the 1970s, coupled with a popular desire to participate in the country's economic growth through the financial markets, led to a demand for equity. Everything came to a standstill after the Islamic Revolution leading to a prohibition against interest-based activities and the nationalization of major banks and industrial giants. Mobilization of all resources towards the war effort during the 8-year Iran–Iraq imposed War did not help matters (Wikipedia, 2022g). At present more than 700 active companies are in TSE. During the last 3-4 years many different firms are added to this market. 30 companies are introduced by TSE as top corporations which are presented in 4.5 section. Selection criteria are based on :

- At least 3 months have passed since their admission and trading in TSE.
- Monthly value of their transactions in the last 6 months should be more than 25 percent of the average monthly value of 100 large companies' transactions in the same period.
- Percent of their float share should be more than 10 percent.

There are different indices in the Tehran stock market(TEPIX, TEDPIX, TEFIX,...). We are using the TEDPIX index, which is more common and adjusted. Changes in this indicator reflect the total return of the stock market, considering price and dividends (cash return paid). All accepted companies in this market are included in this index. If a ticker is closed or if it is closed for a while, its latest price is included in the indicators. Its calculation is based on the Laspeyres index.[26] The market cap of each company is the criteria of their

[26] An index formula used in price statistics for measuring the price development of the basket of

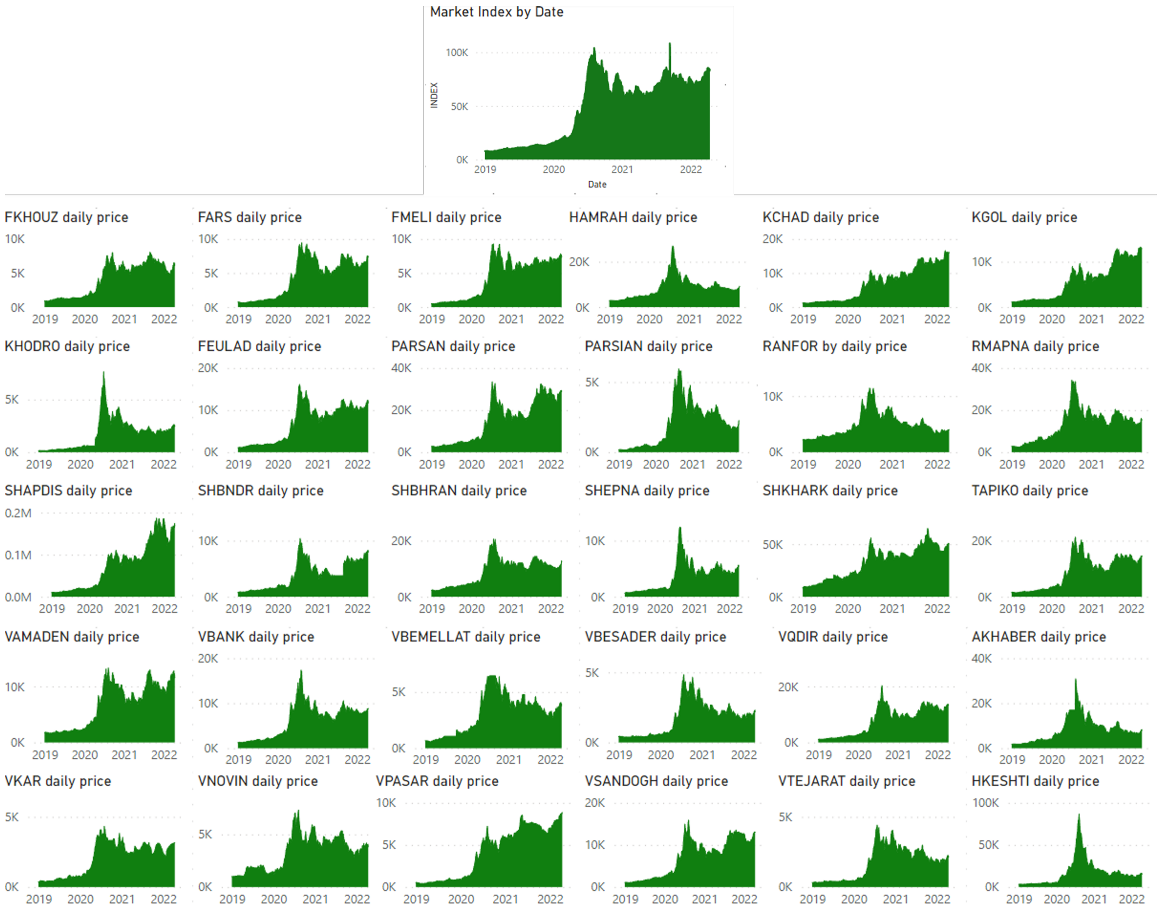

Figure 8: Daily Price behavior of TSE.

weight.

Generally, the below factors uses to adjust the market index:

- Increase capital through cash inflows.
- Change in the number of included companies (increase/ decrease)
- Merge or decomposition of companies.

goods and services consumed in the base period. The question it answers is how much a basket that consumers bought in the base period would cost in the current period (Eurostat, 2022).

TEDPIX formula is:

$$TEDPIX_t = \frac{\Sigma_{i=1}^{n} p_{it} q_{it}}{RD_t} \times 100, \tag{42}$$

Adjustment of index is done by:

$$RD_{t+1} = \frac{\Sigma_{i=1}^{n} p_{it} q_{it} - \Sigma_{i=1}^{n} DPS_{it+1}}{\Sigma_{i=1}^{n} RD_t} + \frac{RD_t}{D_t} \times (D_{t+1} - D_t). \tag{43}$$

p_{it} = Price of i^{th} company at time t.

q_{it} = Number of issued shares of i^{th} company at time t.

RD_{t+1} = Basis of price and cash return index at time t+1 (After adjustment).

RD_t = Basis of price and return (in cash) at time t (Before Adjustment).

DPS_{it+1} = Dividends paid in cash by i^{th} company at time t+1.

D_{t+1} = Basis of total index at time t (After adjustment).

D_t = Basis of total index at time t (Before adjustment) (sena, 2020).

3.2.1 Collecting and analyzing data

We have extracted monthly data of TSE market from the 2019-01 to 2022-06. Data processing for TSE is similar to NYSE, making three regression models and defining explanatory variables are the same. While to gather information about the TSE market we used, the Tehran Stock Exchange market (tse, 2022, tsetmc, 2022), Stock market fundamental analysis system (bourseview, 2022), MofidSecurities, 2022, Rahavard Novin, 2022, and to extract financial statements and balanced sheets, *codal.ir* (2022) website has been used.

We exactly used models that applied for NYSE market, three models with one, three and four explanatory variables respectively. Calculation of each variable values, done like steps that explained in 3.1.2. For TSE market there were 2 different months for three companies which did not have trading days, we used last month values instead. To calculate SMB and HML we consider financial year (June by June) values, while when we used monthly values the results accuracy increased. And finally, we have calculated the CMA and RMW factors of Fama-French five factors model. Since the correlation between variables were high (see table 4). Although using five-factor Fama-French and six-factor model using illiquidity factor had smaller RMSE, for colinearity issue,we did not proceed five and six factor models anymore.

	Column1	β_{mkt}	β_{smb}	β_{hml}	β_{iliq}	β_{rmw}	β_{cma}
1	β_{mkt}	1	-0.37	-0.13	-0.32	0.17	-0.70
2	β_{smb}		1.00	0.43	0.54	-0.65	0.77
3	β_{hml}			1.00	0.21	0.09	0.30
4	β_{iliq}				1.00	-0.47	0.63
5	β_{rmw}					1.00	-0.47
6	β_{cma}						1.00

Table 4: Correlation between five factors of Fama-French model.

Using adjusted total index, return based adjusted monthly close prices, during 2019-01 to 2022-06, risk measures have been calculated and reported in table 5.

Tickers	α	Tickers	β	Tickers	Tr_r	Tickers	Sh_r	Tickers	Mean	Tickers	AdjR^2	Tickers	Std	Tickers	MC
VPASAR	3.66	KHODRO	1.79	VPASAR	10.84	VPASAR	0.54	KHODRO	10.73	INDEX	1.00	KHODRO	34.69	INDEX	32788.50
SHPDIS	2.57	SHPNA	1.70	SHPDIS	9.47	SHPDIS	0.53	VPASAR	8.94	PARSAN	0.85	SHPNA	27.93	FARS	4082.40
PARSIAN	2.10	HKSHTI	1.49	VKAR	8.85	KCHAD	0.48	SHPNA	8.83	VSNDOQ	0.82	HKSHTI	27.26	FULAD	3448.60
VKAR	1.97	VTJARAT	1.43	PARSIAN	8.76	VSNDOQ	0.48	FMLI	8.33	FULAD	0.80	SHBNDR	24.47	FMLI	3000.00
KGOL	1.82	SHBNDR	1.40	VBMLAT	8.67	KGOL	0.46	SHBNDR	8.32	VTJARAT	0.78	VNOVIN	22.91	KGOL	2510.00
KCHAD	1.70	VBSDR	1.36	KGOL	8.49	INDEX	0.46	SHPDIS	7.91	VBANK	0.78	VTJARAT	22.55	KCHAD	1906.38
VBMLAT	1.58	FULAD	1.33	KCHAD	8.28	VQADIR	0.43	PARSIAN	7.82	TAPIKO	0.74	VBSDR	22.03	RMPNA	1794.00
VSNDOQ	0.95	AKHBR	1.26	VSNDOQ	7.35	PARSIAN	0.43	VQADIR	7.71	VBSDR	0.74	AKHBR	20.85	TAPIKO	1348.80
VQADIR	0.90	TAPIKO	1.23	VQADIR	7.25	FMLI	0.41	FULAD	7.63	FARS	0.73	FULAD	20.77	SHPNA	1293.90
FMLI	0.54	PARSAN	1.22	SHKHRK	7.08	PARSAN	0.39	KCHAD	7.48	SHPNA	0.72	FMLI	20.33	SHBNDR	1229.23
SHKHRK	0.46	FMLI	1.22	SHBHRAN	6.88	FARS	0.37	TAPIKO	7.43	AKHBR	0.71	TAPIKO	19.90	PARSAN	1224.70
SHBHRAN	0.38	FARS	1.20	FMLI	6.84	VBMLAT	0.37	KGOL	7.39	VMDEN	0.71	VKAR	19.63	VPASAR	1172.50
VNOVIN	0.33	VBANK	1.13	VNOVIN	6.81	TAPIKO	0.37	VSNDOQ	7.34	VQADIR	0.70	FARS	19.52	VNOVIN	1171.50
FKHUZ	0.05	RMPNA	1.11	FKHUZ	6.46	FKHUZ	0.37	FARS	7.32	FMLI	0.70	RMPNA	19.12	SHPDIS	1047.40
INDEX	-0.00	VQADIR	1.06	INDEX	6.40	SHKHRK	0.37	HKSHTI	7.32	KCHAD	0.65	PARSAN	18.53	VQADIR	1013.76
FARS	-0.33	VMDEN	1.05	FARS	6.12	FULAD	0.37	PARSAN	7.28	RMPNA	0.65	PARSIAN	18.35	VMDEN	977.70
TAPIKO	-0.44	INDEX	1.00	TAPIKO	6.04	VKAR	0.36	VKAR	7.11	FKHUZ	0.63	VBANK	17.89	VBMLAT	763.60
PARSAN	-0.56	VSNDOQ	1.00	KHODRO	5.99	VBANK	0.36	VTJARAT	6.67	SHBNDR	0.63	VQADIR	17.71	KHODRO	730.00
SHBNDR	-0.63	KCHAD	0.90	SHBNDR	5.95	VMDEN	0.35	VBANK	6.42	SHPDIS	0.61	VMDEN	17.33	FKHUZ	697.20
VMDEN	-0.64	PARSIAN	0.89	PARSAN	5.95	SHBNDR	0.34	INDEX	6.40	HKSHTI	0.57	VPASAR	16.65	VSNDOQ	574.00
RMPNA	-0.71	KGOL	0.87	VMDEN	5.79	SHBHRAN	0.34	RMPNA	6.37	KGOL	0.56	VBMLAT	16.16	AKHBR	479.40
KHODRO	-0.74	FKHUZ	0.85	RMPNA	5.76	RMPNA	0.33	VMDEN	6.05	SHKHRK	0.52	KGOL	16.16	HMRH	456.00
RNFOR	-0.81	SHPDIS	0.83	FULAD	5.72	SHPNA	0.32	VBMLAT	6.04	KHODRO	0.51	SHBHRAN	16.12	HKSHTI	406.20
VBANK	-0.84	VPASAR	0.82	VBANK	5.66	KHODRO	0.31	VBSDR	5.72	VPASAR	0.47	HMRH	15.57	VTJARAT	391.65
HMRH	-0.87	VNOVIN	0.82	HMRH	5.27	VTJARAT	0.30	VNOVIN	5.55	SHBHRAN	0.46	KCHAD	15.54	VBSDR	385.25
FULAD	-0.91	VKAR	0.80	SHPNA	5.19	HKSHTI	0.27	FKHUZ	5.50	HMRH	0.46	VSNDOQ	15.41	SHKHRK	301.60
SHPNA	-2.05	SHBHRAN	0.80	HKSHTI	4.92	VBSDR	0.26	SHBHRAN	5.47	PARSIAN	0.45	FKHUZ	14.88	VBANK	274.13
HKSHTI	-2.19	HMRH	0.77	RNFOR	4.78	HMRH	0.26	AKHBR	5.18	VBMLAT	0.35	SHPDIS	14.84	RNFOR	165.38
VTJARAT	-2.49	VBMLAT	0.70	VTJARAT	4.66	AKHBR	0.25	SHKHRK	4.74	VKAR	0.31	INDEX	14.01	SHBHRAN	162.04
AKHBR	-2.91	SHKHRK	0.67	VBSDR	4.20	VNOVIN	0.24	HMRH	4.04	RNFOR	0.26	RNFOR	13.33	VKAR	129.00
VBSDR	-2.99	RNFOR	0.50	AKHBR	4.10	RNFOR	0.18	RNFOR	2.39	VNOVIN	0.23	SHKHRK	12.85	PARSIAN	43.78

Table 5: CAPM risk measures of TSE.

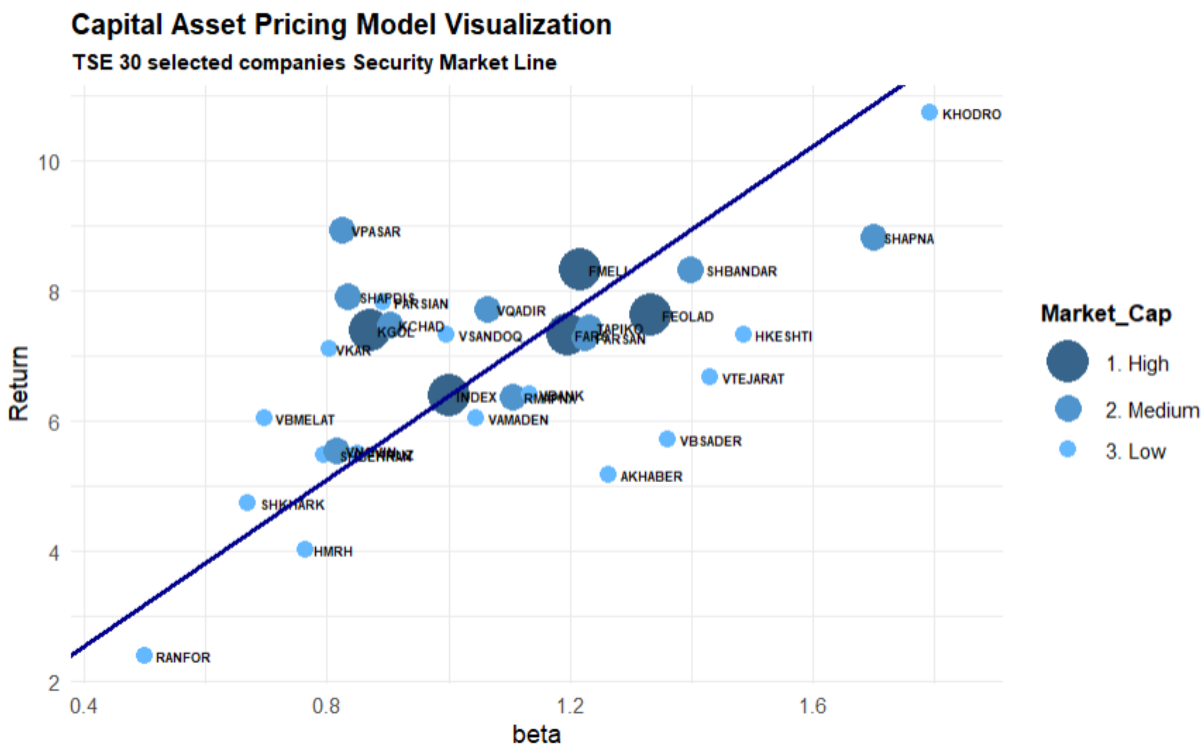

Figure 9: CAPM visualization of Tehran market.

The statistical results of the used models are shown in the table 6. One can see improvement of asset pricing, by adding factors we have used.

	CAPM	Three Factor	Four Factor model
$\boldsymbol{\beta}_{mkt}$	3.285***$_{(0.994)}$	2.965***$_{(0.961)}$	3.340***$_{(0.969)}$
$\boldsymbol{\beta}_{smb}$		-0.794**$_{(0.388)}$	-1.023**$_{(.385)}$
$\boldsymbol{\beta}_{hml}$		-1.849$_{(5.794)}$	3.903 $_{(6.175)}$
$\boldsymbol{\beta}_{illiq}$			0.551$_{(0.508)}$
Adj R^2	0.26	0.3087	0.379
F-statistic	11.38	7.02	5.584
Residual Std.Error	1.387	1.33	1.26
AIC	112.1903	111.3875	108.8298

Table 6: Summary of three asset pricing model of TSE market.('p<0.1; *p<0.05; **p<0.01, ***p<0.001) , numbers in brackets are variance of each measure.

3.3 Results

Results show:

- Both market we reviewed are highly volatile, especially TSE market with 18.53 standard deviation.
- SML has an upward trend, in both models, which confirms the CAPM concept.
- TSE 30 selected companies are conservative stocks. NYSE's selected companies are diverse, and this portfolio is almost representative of the whole market. The slope of SML in the TSE plot is steeper than the NYSE selected portfolio. It is a reconfirmation to CAPM.
- CAPM performed poorly in both markets, specially in NYSE market to price the assets. And adding extra factors improved our estimations. Generally, our models had better pricing power in TSE market.
- Seems the size of the company is an important factor in the price of asset. In both figure 7 and 9 we see companies with higher market capitalization have beta near to one. In NYSE market Apple and Microsoft are in top of table of mean return, Jensen's alpha, Treynor and Sharpe ratio and have the highest market cap with a large margin.

In TSE market we see companies with high market cap are less scattered, and mostly located in the center of plot, while companies with low capital tend to have a broader spectrum of beta changes.

- In TSE market adding illiquidity risk factor, improved our model. While in NYSE market Fama-French three factor model had better explanatory power.
- Normality test results (Shapiro-Wilk's and Kolmogrov-Smirnov tests) returned almost all p-values less than 1.0E-05. So normality assumption of our data is violated. The assets' returns show Leptokurtic distributions.[27]
- And in practice, we see the excitement and biases of investors, especially in distress, take the investors away from a rational state, which is in conflict with our first assumptions.

[27] Leptokurtic distributions are distributions with positive kurtosis larger than that of a normal distribution (Heys, 2021a)

Conclusion

Investors' most essential concern is achieving an acceptable return on investment. Return estimation and pricing of assets involve many challenges, and the results are open to question as a reliable benchmark. This thesis aims to evaluate the power of three asset pricing models used in two distinct markets: the Tehran Stock Exchange market monthly data from 2019-01 up to 2022-06 as an emerging market and the 5-year monthly price of the NYSE as a developed one. These asset pricing models' basis is risk-return concepts. At the first stage of mean-variance analysis, the risk is more general and includes all types of risk. The efficient frontier, in modern portfolio theory, has a parabola shape. The upper branch of this parabola includes optimal portfolios. The capital asset pricing model simplifies it by adding a risk-free asset and changing the efficient frontier to a straight line. It also made the risk specified to the systematic risk. CAPM introduces two measures, Jensen's alpha and beta, to evaluate asset values. Our results from two markets are consistent with the CAPM concept. Still, problems with assumptions remained, and CAPM performed poorly in the estimation of assets' prices. So, in the next step, we use the three-factor Fama-French model, which significantly improves the results. The illiquidity of stocks is one of the other influential features. In the last step, we add this factor to evaluate all four elements, and the errors of the models are reduced. While the best model is the Fama-French three-factor model, it is still not a satisfactory pricing method. We should also consider the time interval we have chosen as one of the most volatile (pandemic situations, war, and, in the case of the Tehran market, severe economic sanctions, unprecedented inflation, and other macroeconomic factors). Generally, we can take advantage of this model and get a better view of the portfolio we have composed, making it personalized and more diversified based on our degree of risk aversion, the time interval of investment, and market conditions.

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4 Appendix

4.1 Derivative of vectors and matrix

- Derivative of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to vector $\mathbf{x} \in \mathbb{R}^n$ is defined by:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{pmatrix}. \tag{44}$$

- Derivative of a vector function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ with respect to vector $\mathbf{x} \in \mathbb{R}^n$ is defined by:

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_1} \\ \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_n}. \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial f_n(\mathbf{x})}{\partial x_1} \\ \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial f_n(\mathbf{x})}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_1(\mathbf{x})}{\partial x_n} & \frac{\partial f_2(\mathbf{x})}{\partial x_n} & \dots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{pmatrix}. \tag{45}$$

To find minimum $\mathbf{w}'\mathbf{\Sigma}\mathbf{w}$ we can use derivation definition, which gives:

$$\frac{\partial \mathbf{w}'\mathbf{\Sigma}\mathbf{w}}{\partial \mathbf{w}} = 2\mathbf{\Sigma}\mathbf{w}, \tag{46}$$

(Wikipedia, 2022e).

4.2 List of NYSE's selected companies

	Ticker	Name	Section
1	AAPL	Apple Inc.	Information Technology
2	AMGN	Amgen Inc.	Biotechnology
3	AXP	American Express Company	Consumer Finance
4	BA	The Boeing Company	Capital Goods
5	CAT	Caterpillar Inc	Gas turbines worldwide
6	CRM	Sales Force, Inc	Financial Services
7	CSCO	Cisco Systems, Inc	Communications Equipment
8	CVX	Chevron Corporation	Energy
9	DIS	Walt Disney	203.24
10	GS	Goldman Sachs Group	Consumer Discretionary
11	HD	Home Depote Inc.	Consumer Discretionary
12	HON	Honeywell international Inc.	Industrial
13	IBM	International Business Machines Corporation	Technology
14	INTC	Intel Corporation	178.226
15	JNJ	Johnson and Johnson	Health care
16	JPM	JPMorgan Chase and Co.	Financials
17	KO	The Coca-Cola Company	Consumer Staples
18	MCD	McDonald's Corporation	Consumer Discretionary
19	MMM	3M Company	Consumer Electronics/Appliances
20	MRK	Merk and Co. Inc.	Health care
21	MSFT	Microsoft Corporation	Information Technology
22	NKE	Nike, Inc.	Consumer Discretionary
23	PG	The Procter and Gamble Company	Consumer Discretionary
24	TRV	The Travelers Companies, Inc.	Property-Casualty Insurers
25	UNH	United Health Group Incorporated	Life Insurance
26	V	Visa Inc.	Business Services
27	VZ	Verizon Communication Inc.	Telecommunications Equipment
28	WBA	Walgreens Boots Alliance, Inc.	Consumer Staples
29	WMT	Walmart Inc.	Consumer Discretionary
30	DOW	Dow Inc.	Industrial

Table 7: NYSE 30 selected companies (Dow & companies).

4.3 Results of three factor Fama-French model (NYSE market).

	Tickers	Alpha	β_{mkt}	β_{smb}	β_{hml}	$Adj\ R^2$
1	INDEX	0.00	1***	-5.60E-17*	-1.78E-17	1.00
2	AAPL	1.82	1.163139***	-0.08711	-0.71017***	0.32
3	AMGN	0.26	0.669644***	-0.17762	-0.1914	0.20
4	AXP	0.59	1.139043***	0.201344	0.368282**	0.56
5	BA	-1.38	1.604791***	0.823854*	0.42352	0.43
6	CAT	0.67	0.948689***	0.269217	0.219724	0.37
7	CRM	0.39	1.171395***	0.074606	-0.76228***	0.32
8	CSCO	0.10	0.999557***	0.192481	-0.17941	0.39
9	CVX	0.45	0.994158***	0.528042**	0.76617***	0.40
10	DIS	-0.55	1.197166***	0.422746	-0.10327	0.45
11	GS	-0.00	1.353358***	0.487801**	0.245797’	0.63
12	HD	0.61	1.007251***	-0.11198	-0.21046	0.43
13	HON	-0.04	1.133901***	-0.00592	0.100196’	0.73
14	IBM	-0.61	1.068993***	-0.12686	0.151802	0.48
15	INTC	0.18	0.643062***	0.548111	-0.01098	0.19
16	JNJ	-0.04	0.76478***	-0.31997*	-0.02751	0.45
17	JPM	0.10	1.045593***	0.378828**	0.530196***	0.59
18	KO	0.01	0.724765***	-0.6323***	0.223423**	0.37
19	MCD	0.29	0.702842***	-0.56257***	0.055317	0.36
20	MMM	-1.13	0.993438***	0.063111	0.006852’	0.59
21	MRK	0.35	0.5326***	-0.48341*	-0.07406	0.12
22	MSFT	1.64	0.951014***	-0.12878	-0.57452***	0.49
23	NKE	0.67	0.994088***	0.265401	-0.54414***	0.38
24	PG	0.46	0.481154***	-0.47432**	-0.04328	0.16
25	TRV	0.16	0.711737***	0.056815	0.408351***	0.38
26	UNH	1.08	0.904509***	-0.24711	-0.1769	0.35
27	V	0.63	1.009665***	-0.24724	-0.23191**	0.52
28	VZ	-0.27	0.469779***	-0.35381**	0.085252	0.20
29	WBA	-1.15	0.572005**	0.356085	0.322015	0.12
30	WMT	0.32	0.679691***	-0.58553***	-0.3295**	0.20
31	DOW	-0.53	1.186***	0.1782	0.9756***	0.68

Table 8: Three factor Fama-French model coefficients (Dow & companies).

4.4 Results of four factor model (NYSE market)

	Tickers	**Alpha**	β_{mkt}	β_{smb}	β_{hml}	β_{iliq}	$Adj\,R^2$
1	INDEX	0.00	1***	-5.08E-17	-1.67E-17	5.32E-17	1.00
2	AAPL	1.84	1.167745367***	-0.070676351	-0.70655062***	0.169546218	0.32
3	AMGN	0.26	0.667891881***	-0.18386936	-0.192774577	-0.064476275	0.20
4	AXP	0.62	1.146357538***	0.227442311	0.374027731**	0.269216755’	0.56
5	BA	-1.26	1.63235004***	0.922176981**	0.445167139	1.014267183*	0.43
6	CAT	0.62	0.938049851***	0.231261131	0.211367142	-0.391542795	0.37
7	CRM	0.37	1.167759481***	0.06163694	-0.76513292***	-0.133781768	0.32
8	CSCO	0.13	1.007344721***	0.220266364	-0.17329768	0.286621053	0.39
9	CVX	0.48	1.002845971***	0.559037984**	0.772994077***	0.319742893’	0.40
10	DIS	-0.50	1.20912445***	0.465410399	-0.093879097	0.440108391’	0.45
11	GS	0.03	1.36087373***	0.514615405**	0.251700471	0.276604971	0.63
12	HD	0.55	0.994145323***	-0.15874242	-0.220753983’	-0.482350365	0.43
13	HON	-0.05	1.130682028***	-0.0174069	0.097667135	-0.118476196	0.73
14	IBM	-0.59	1.072751676***	-0.1134455	0.154754804	0.138345744	0.48
15	INTC	0.28	0.665486868***	0.628116964*	0.006629904	0.825317865**	0.19
16	JNJ	-0.04	0.764011207***	-0.3227175*	-0.028113452	-0.028308211	0.45
17	JPM	0.07	1.0393695***	0.3566239*	0.525307784***	-0.2290457’	0.59
18	KO	0.00	0.723306211***	-0.6375030***	0.2222775**	-0.05368528	0.37
19	MCD	0.30	0.704748671***	-0.5557698***	0.056814217	0.070167265	0.36
20	MMM	-1.18	0.982634878***	0.0245668	-0.00163373	-0.39760577	0.59
21	MRK	0.34	0.531224004***	-0.488321968*	-0.075142126	-0.05063371	0.12
22	MSFT	1.62	0.947282747***	-0.142089142	-0.577454351***	-0.137320064	0.49
23	NKE	0.67	0.99511751***	0.26907233	-0.543332045***	0.03787452	0.38
24	PG	0.46	0.480601144***	-0.476295825**	-0.04371881	-0.020341504	0.16
25	TRV	0.16	0.710809554***	0.05350562	0.407622751***	-0.034141965	0.38
26	UNH	1.06	0.899053965***	-0.266576596	-0.181189244	-0.200777591	0.35
27	V	0.65	1.013674783***	-0.232928945’	-0.228763652*	0.14759068	0.52
28	VZ	-0.28	0.467525567***	-0.36185061**	0.083482199	-0.082920291	0.20
29	WBA	-1.14	0.573850965***	0.362672273	0.32346475	0.0679475	0.12
30	WMT	0.33	0.682996823***	-0.573731935**	-0.326904756**	0.121665754	0.20
31	DOW	-0.53	1.1724***	0.1822	1.0045***	-0.3299	0.68

Table 9: Four factor model coefficients (Dow & companies).

4.5 List of TSE's 30 selected companies

	Ticker	Name	Sector
1	SHBANDAR	B.A Oil Refine	Petrochemical
2	VBANK	Bank Melli Inv	Investing
3	SHBEHRAN	Behran Oil	Petrochemical
4	KCHAD	Chadormalu	Mining and Industrial
5	VGHADIR	Ghadir Inv	Investing
6	KGOL	Gol gohar	Mining
7	RANFOR	Inf. Services	Information Technology
8	KHODRO	Iran Khodro	Industry
9	HAMRAH	Iran Mobile Tele	Telecommunication
10	AKHABER	Iran Tele Co.	Telecommunication
11	VKAR	Karafarin Bank	Investing
12	FARS	Khalij Fars	Petrochmical Industry
13	SHKHARK	Khark Petro	Industry
14	FKHOUZ	Khouz Steel	Metal Industry
15	RMAPNA	MAPNA	Power Plant Industry
16	SHAPDIS	Pardis Petr	Petrochemical Industry
17	VPARS	Parsian Bank	Investing
18	VSANDOGH	Pension Fund	Investing
19	VNOVIN	S EN Bank	Investing
20	FMELI	SI.N.C.Ind	Industry (Copper)
21	HKESHTI	S IRI Marine	Marine Industry
22	SHEPNA	S Isf.Oil Ref.Co.	Refine Industry
23	VBEMELLAT	S Mellat Bank	Investing
24	VMAADEN	S Metals and Min	Mining Industry
25	MOBARAKE	S Mobarake Steel	Steel Industry
26	PARSAN	Parsian Insurance	Insurance
27	VPASAR	S Pasargad Bank	Investing
28	VTEJARAT	S Tejarat Bank	Investing
29	VBESADER	Saderat Bank	Investing
30	TAPIKO	Tamin Petro	Petrochemical

Table 10: TSE 30 selected companies.

4.6 Results of three factor Fama-french model (TSE market market)

	Tickers	Alpha	β_{mkt}	β_{smb}	β_{hml}	$Adj\ R^2$
1	INDEX	0.00	1***	-1.83E-17	7.89E-18	1.00
2	AKHABER	-0.04	1.375631888***	0.609810287**	-0.013860537	0.71
3	FARS	0.01	1.028117955***	-0.831548686***	0.001887545	0.73
4	FKHUZ	0.01	0.823817604***	-0.281184104***	0.039733077**	0.63
5	FMELI	0.02	1.106059346***	-0.681484465**	0.035007122	0.70
6	FEOLAD	0.01	1.196363772***	-0.801567412***	0.033918977**	0.80
7	HMRH	-0.03	0.845990208***	0.557691787**	-0.043487553**	0.46
8	HKESHTI	-0.04	1.303692432***	-0.526438423'	-0.099991786***	0.57
9	KCHAD	0.03	0.875901949***	-0.223443465	0.024358615	0.65
10	KGOL	0.03	0.780453268***	-0.49480851**	0.013397227	0.56
11	KHODRO	-0.04	1.77380761***	0.363119713	-0.122690088**	0.51
12	PARSAN	-0.00	1.176396374***	-0.241973081	0.001330666	0.85
13	PARSIAN	-0.00	1.062648449***	1.019923741**	-0.0485947**	0.45
14	RANFOR	-0.02	0.698211739***	1.052324375***	-0.020140551	0.26
15	RMAPNA	-0.03	1.15701517***	0.502828082**	-0.068188206***	0.65
16	SHAPDIS	0.02	0.846962523***	0.132652676	-0.019616673	0.61
17	SHAPNA	0.00	1.588859112***	-0.792995982**	0.067942815**	0.72
18	SHBANDAR	0.00	1.309863453***	-0.463113291*	0.00656471'	0.63
19	SHBEHRAN	-0.01	1.000984389***	1.074871193***	-0.016775316	0.46
20	SHKHARK	0.00	0.671474604***	0.067394858	-0.014940991	0.52
21	TAPIKO	-0.01	1.135187908***	-0.300308435	-0.045756774**	0.74
22	VBANK	-0.02	1.239708848***	0.603299577**	-0.022314468*	0.78
23	VBMELAT	0.03	0.866449485***	0.588885158**	0.067190668***	0.35
24	VBSADER	-0.02	1.363436793***	-0.140202242	0.041307728**	0.74
25	VQADIR	0.00	1.07129416***	0.101884492	-0.016841525	0.70
26	VKAR	0.02	1.046817636***	1.038749971***	0.043564581*	0.31
27	VAMADEN	0.01	0.994515933***	-0.442523374**	0.052173977**	0.71
28	VNOVIN	-0.04	1.189795741***	2.178177514***	-0.089386561**	0.23
29	VPASAR	0.04	0.984534961***	0.678099649**	0.030249681	0.47
30	VSANDOQ	0.01	1.042339414***	0.195082913*	0.007113631*	0.82
31	VTJARAT	-2.16	1.49***	0.20568782	0.0.021955108'	0.82

Table 11: Three factors model coefficients (TSE).

4.7 Results of four factors model (TSE market market)

	Tickers	Alpha	β_{mkt}	β_{smb}	β_{hml}	β_{iliq}	$Adj\ R^2$
1	INDEX	0.00	1***	1.50353E-16	-4.3728E-18	0	1.00
2	AKHABER	-3.92	1.37***	0.616344415**	-0.01389139	-0.01607336	0.73
3	FARS	0.53	1.02***	-0.80933148**	0.001782619	-0.05465230	0.81
4	FKHUZ	1.17	0.82***	-0.22855602	0.039484528**	-0.1294603	0.65
5	FMELI	2.26	1.11***	-0.71848623*	0.035181871*	0.09102101	0.73
6	FEOLAD	0.20	1.17***	-0.61803499**	0.033052202**	-0.4514732*	0.87
7	HMRH	-2.10	0.87***	0.41943614	-0.04283460**	0.340096418	0.51
8	HKESHTI	-4.28	1.3***	-0.48102898	-0.10020624***	-0.11170314	0.69
9	KCHAD	2.47	0.87***	-0.20210921	0.024257859	-0.05248034	0.64
10	KGOL	3.08	0.8***	-0.62016261*	0.013989241	0.308359795	0.58
11	KHODRO	-2.10	1.87***	-0.290151	-0.11960486**	1.60698717*	0.61
12	PARSAN	-0.34	1.17***	-0.217564332	0.001215390	-0.06004332	0.85
13	PARSIAN	0.72	1.11***	0.73288340**	-0.04723908**	0.706093380**	0.59
14	RANFOR	-1.71	0.73***	0.8281971**	-0.0190820	0.551332891**	0.55
15	RMAPNA	-2.71	1.17***	0.428855870	-0.0678388**	0.181964977	0.70
16	SHAPDIS	2.22	0.86***	0.043180556	-0.01919412	0.220093354	0.60
17	SHAPNA	-0.77	1.53***	-0.400336919	0.066088391**	-0.96590592**	0.78
18	SHBANDAR	-1.52	1.24***	0.017344667	0.004295635	-1.18188329***	0.69
19	SHBEHRAN	-0.40	1.04***	0.83193419***	-0.01562799	0.59760312**	0.68
20	SHKHARK	-0.35	0.65***	0.18019553	-0.015473718	-0.27747949	0.50
21	TAPIKO	-1.50	1.12***	-0.229710743	-0.046090187*	-0.173663962	0.79
22	VBANK	-1.50	1.26***	0.434894395*	-0.021519135'	0.414261576**	0.83
23	VBMELAT	2.58	0.86***	0.60550053*	0.0671121983**	-0.04087233	0.57
24	VBSADER	-2.57	1.33***	0.095318944	0.0401954241*	-0.57936090*	0.76
25	VQADIR	0.33	1.07***	0.113881827	-0.016898185	-0.02951236	0.68
26	VKAR	1.82	1.04***	1.07815965**	0.04337846	-0.09694427	0.51
27	VAMADEN	1.06	0.99***	-0.41975056**	0.052066426**	-0.05601906	0.74
28	VNOVIN	-4.25	1.19***	2.18613389***	-0.089424133**	-0.01957197	0.58
29	VPASAR	4.38	1.02***	0.46315467'	0.031264808'	0.528745284*	0.61
30	VSANDOQ	0.91	1.04***	0.19704002	0.007104389	-0.00481430	0.82
31	VTEJARAT	-2.58	1.47***	0.33409467	0.021348676	-0.31586986	0.79

Table 12: Four factors model coefficients (TSE).

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