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# Large Induced Forests in Planar Graphs

Masters's Thesis (30 ECTS)

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## Suured indutseeritud metsad tasandilistes graafides

**Lühikokkuvõte:** Tasandilised graafid on graafid, mida saab joonistada tasandile nii, et tema servad ei lõiku üksteisega mujal kui tippudes. Selles töös me uurime, kui suuri indutseeritud metsi on alati võimalik tasandilistes graafides leida. Praegu parim teadaolev tulemus on pärit Borodinilt ([Bor79]), mille kohaselt igas tasandilises graafis leidub indutseeritud mets, mis sisaldab vähemalt $\frac{2}{5}$ tema tippudest. Selles tööd anname me osalise tulemuse selle hinnangu parandamise suunas. Täpsemalt, me näitame, et minimaalne vastunäide meie parandatud tulemusele ei sisalda tippe, mille aste on väiksem kui 4, ja et selles sisalduvad astmega 4 tipud saavad olla vaid üht kindlat tüüpi.

**Võtmesõnad:** tasandiline graaf, indutseeritud mets

**CERCS:** P170 (Arvutiteadus, arvutusmeetodid, süsteemid, juhtimine )

## Large Induced Forests in Planar Graphs

**Abstract:** Planar graphs are graphs that can be drawn on a plane so that its edges do not cross each other (other that at their endpoints). In this thesis, we study the size of induced forests in planar graphs. The current best result by Borodin ([Bor79]) states that every planar graph has an induced forest that contains at least $\frac{2}{5}$ of its vertices. In this thesis, we give partial results towards improving this bound. Specifically, we show that a minimal counterexample to an improved bound has minimal degree 3 and can contain only a specific type of vertices with degree 4.

**Keywords:** planar graph, induced forest

**CERCS:** P170 (Computer science, numerical analysis, systems, control)

# Contents

# 1 Introduction

Graphs are a way of modeling objects with some kind of connections between them, e.g. people who know each other, cities that have direct flights between them, etc. They arise naturally in many different problems, and therefore the study of their properties is an important part of mathematics. Planar graphs are an important type of graphs that can be drawn onto a plane so that their edges (the connections between objects) do not cross each other. Planar graphs arise maybe the most naturally when thinking about maps, e.g. roads between intersections, subway systems, or neighboring countries.

In this thesis, we study a particular property of planar graphs, namely, the size of their largest possible induced forest, that is, the largest number of vertices such that any cycle in the graph contains at least one vertex not among them. This study is motivated by a conjecture by Albertson and Berman [AB79], which states that every planar graph has an induced forest with at least half of its vertices. This conjecture is as yet unproved, with the current best known result stating that every planar graph with $n$ vertices has an induced forest with at least $\frac{2}{5}n$ vertices ([Bor79]). Our aim was to improve this bound, but we have not managed to complete the proof yet, so we give some partial results towards that goal in the last part of the thesis.

In the first section of the thesis, we give the definitions of terms and results about graphs and planar graphs that are used in the rest of this thesis. In the second section, we present the background of the problem along with an overview of previous works and an example of the methodology we are using to solve the problem. In the last section, we define the result we are trying to prove, and prove some properties of a minimal counterexample to that proof.

# 2 Preliminaries

This section gives an overview of the necessary graph theory terms and results that are required to understand the rest of the thesis. Readers who are already familiar with planar graphs may safely skip this section. The definitions and theorems on graphs, along with the terminology, have been taken from [MN08]

## 2.1 General graphs

**Definition 2.1.** A (simple undirected) graph $G$ is an ordered pair $(V, E)$, where $V$ is some set and $E$ is a set of 2-element subsets of $V$. $V$ is called the *vertex set* of $G$ and elements of $V$ are called *vertices* of the graph, while $E$ is called the *edge set* of $G$ and the elements of $E$ are called the *edges* of the graph. The *size* of a graph is the number of elements in its vertex set.

When talking about a known graph $G$, we denote its vertex set as $V(G)$ and its edge set as $E(G)$. While there are other types of graphs, in the rest of the thesis, the term *graph* is used to refer to a simple undirected graph as defined above.

**Definition 2.2.** Let $G = (V, E)$ be a graph. If for two vertices $u, v \in V$ we have $u, v \in E$, we say that $u$ and $v$ are *adjacent* in $G$, or that $u$ is a *neighbor* of $v$.

**Definition 2.3.** Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $d(v)$, is the number of neighbors it has in $G$ (alternately, the number of edges $e \in E$ such that $v \in e$).

**Definition 2.4.** Let $G$ and $G_1$ be graphs. We say that $G_1$ is a *subgraph* of $G$ if $V(G_1) \subseteq V(G)$ and $E(G_1) \subseteq E(G)$. We say that $G_1$ is an *induced subgraph* of $G$ if $V(G_1) \subseteq V(G)$ and $E(G_1) = \{\{u, v\} : u, v \in V(G_1) \text{ and } \{u, v\} \in E(G)\}$ (that is, for any two vertices $u, v \in V(G_1)$, we have $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G)$).

**Definition 2.5.** Let $G = (V, E)$ be a graph. A *path* in the graph $G$ is a sequence $(v_0, e_1, v_1, \ldots, e_t, v_t)$, where $v_0, v_1, \ldots, v_t$ are mutually distinct vertices and for each $e_i$, $i \in \{1, 2, \ldots, t\}$, we have $e_i = \{v_{i-1}, v_i\} \in E$. If the first vertex of a path $P$ is $u$ and the last vertex is $v$, we say that $P$ is *a path from $u$ to $v$*. The *length* of a path is the number of edges in it.

**Definition 2.6.** Let $G = (V, E)$ be a graph. A *cycle* in the graph $G$ is a sequence $(v_0, e_1, v_1, \ldots, e_t, v_0)$, where $v_0, v_1, \ldots, v_{t-1}$ are mutually distinct vertices and for each $e_i$, $i \in \{1, 2, \ldots, t-1\}$, we have $e_i = \{v_{i-1}, v_i\} \in E$, and also $e_t = \{v_{t-1}, v_0\} \in E$. The *length* of a cycle is the number of edges (or vertices) in it.

**Definition 2.7.** The *girth* of a graph $G$ is the length of the shortest cycle in $G$. If $G$ does not contain any cycles, its girth is defined to be infinity.

**Definition 2.8.** Let $G = (V, E)$ be a graph. We say that $G$ is *connected* if for every two vertices $u, v \in V$, $G$ contains a path from $u$ to $v$.

**Definition 2.9.** Let $G$ be a graph, and let $G_1$ be an induced subgraph of $G$. We say that $G_1$ is a *connected component* of $G$ if $G_1$ is connected and there is no edge $\{u, v\} \in E(G)$ such that $u \in V(G_1)$, but $v \notin V(G_1)$.

It is easy to see that if $G_1$ and $G_2$ are connected components of $G$ with $V(G_1) \cap V(G_2) \neq \emptyset$, then $G_1 = G_2$. Hence every graph can be represented as a disjoint union of its connected components.

**Definition 2.10.** Let $G$ be a graph. We say that an edge $e \in E(G)$ is a *bridge* in $G$ if removing the edge $e$ increases the number of connected components of $G$.

**Definition 2.11.** Let $G$ be a graph. We say that a vertex $v \in V(G)$ is a *cut-vertex* in $G$ if removing $v$ from $G$ increases the number of connected components of $G$.

**Definition 2.12.** Let $G$ be a connected graph. We say that $G$ is a *tree* if $G$ contains no cycles.

**Proposition 2.1.** *A tree with $n$ vertices has exactly $n - 1$ edges.*

*Proof.* Let $G = (V, E)$ be a tree with $n = |V|$. Remove all edges from $G$, then we have $n$ connected components (the vertices). Start adding back edges one by one, then each edge decreases the number of connected components by one (an edge cannot connect more than two components, and if an edge does not connect two different components, then both of its endpoints were in the same component and already had a path between them, so adding the edge would create a cycle, which is impossible, because $G$ is a tree). Since $G$ is connected (has one connected component), we need exactly $n - 1$ edges to join all the components. $\square$

**Definition 2.13.** Let $G$ be a graph. We say that $G$ is a forest if all of its connected components are trees. If $G_1$ is an induced subgraph of $G$, then we say that $G_1$ is an *induced forest* in $G$ if $G_1$ is a forest.

**Definition 2.14.** Let $G$ be a graph. We say that $G$ is *bipartite* if its vertex set can be divided into two disjoint sets $S, T$ such that every edge in $E(G)$ has one endpoint in the set $S$ and the other endpoint in the set $T$ (in other words, the subgraphs induced by $S$ and $T$ have no edges).

*Remark.* Let $G$ be a graph. In the following thesis, we use the following notation: if $S \subseteq V(G)$, then we use $G - S$ to denote the subgraph of $G$ that arises when we remove all vertices in $S$ from $G$ (along with all the edges connected to them). If $F$ is an induced forest in $G$ and $S \subseteq V(G) \backslash V(F)$, then we use $F + S$ to denote the subgraph of $G$ induced by $V(F) \cup S$. If $F_1, F_2$ are induced forests in $G$, we use $F_1 + F_2$ to denote the subgraph of $G$ induced by $V(F_1) \cup V(F_2)$. If $a, b \in V(G$ are vertices of $G$, we use $ab$ to denote the edge $\{a, b\}$.

## 2.2 Planar graphs

Graphs are often represented visually by drawing them onto some flat surface, with dots representing vertices and lines connecting the dots representing edges between the vertices. Ideally, we would like to draw a graph in such a way that the lines representing edges do not cross each other (other than at their endpoints), otherwise we might mistake such intersections for vertices. This gives rise to the notion of planar graphs: a graph is *planar* if it is possible to draw it on a plane in such a way that the lines representing edges intersect only at their endpoints (vertices). This drawing is also called an *embedding* of the graph.

**Definition 2.15.** Let $G = (V, E)$ be a graph. We say that $G$ is *planar* if there exists a mapping $f$ that maps every vertex $v \in V$ to a (different) point on the plane, and maps every edge $\{u, v\} \in E$ to a continuous arc with endpoints $f(u)$ and $f(v)$, such that for every two edges $e_1, e_2 \in E$, either $f(e_1) \cap f(e_2) = \emptyset$ or $e_1$ and $e_2$ have a common endpoint $v \in V$ and $f(e_1) \cap f(e_2) = \{f(v)\}$. Any such mapping is called a *planar embedding* (or just *embedding*, if planarity is understood from context) of $G$.

**Definition 2.16.** Let $G$ be a planar graph together with some embedding. Then the edges of the embedding divide the plane into a finite amount of connected regions. These regions are called the *faces* of the graph (note that faces are defined with respect to a fixed embedding).

**Proposition 2.2** (Euler's formula)**.** *Let $G = (V, E)$ be a connected planar graph, and let $f$ be the number of faces in a given embedding of $G$. Then we have*

$$|V| - |E| + f = 2.$$

*Proof.* We use induction by the number of edges in $E$. If $E = \emptyset$, then $|V| = 1$ (otherwise would not be connected) and $f = 1$, hence the formula holds. Now assume $|E| > 0$. If $G$ is a tree, then $|V| = |E| + 1$, and any embedding has only one face, so $|V| - |E| + f = |E| + 1 - |E| + 1 = 2$. If $G$ is not a tree, then there exists an $e \in E$ that is part of a cycle. Removing this edge results in a graph with one less edge and one less face (we consider the embedding that results by erasing the edge representing $e$ from the embedding of $G$), because the cycle $e$ is part of divided the plane into two parts, and hence the faces on either side of $e$ could not be the same face. By our inductive hypothesis, the formula holds for this smaller graph, so we have $2 = |V| - (|E| - 1) + (f - 1) = |V| - |E| + f$, so the formula holds for $G$ as well. □

**Proposition 2.3.** *For every planar graph with $n \geqslant 3$ vertices and $m$ edges, we have $m \leqslant 3n - 6$. Equality holds for any maximal planar graph, i.e. a planar graph such that adding any new edge between its vertices makes it nonplanar.*

*Proof.* Let $G$ be a planar graph with $n \geqslant 3$ vertices and $m$ edges with a given embedding with $f$ faces. If $G$ is not maximal planar, then we can add edges until it becomes maximal planar, so it is sufficient to show that the equality $m = 3n-6$ holds for any maximal planar graph. Now suppose that $G$ is maximal planar. We show that then every face of $G$ is bounded by a triangle, i.e. a cycle with length 3.

$G$ is connected, because otherwise we could add an edge connecting two connected components. There are also no cut-vertices in $G$: suppose $v$ is a cut-vertex, then removing $v$ will separate $G$ into at least two different connected components. Choose two edges $e_1$, $e_2$ such that their one endpoint is $v$, their other endpoints would be in different components after removing $v$, and they are drawn next to each other, then it is always possible to add an edge between their other endpoints. Then every face boundary must be a cycle: since $G$ is connected, every face boundary is in one piece, i.e. can be represented as some sequence $(v_0, e_1, v_1, \ldots, v_{t-1}, e_t, v_0)$ (with $v_0, \ldots, v_{t-1} \in V(G), e_1, \ldots, e_t \in E(G)$ and $e_i = \{v_{i-1}, v_i\}, 1 \leqslant i \leqslant t-1$, ant $e_t = \{v_{t-1}, v_0\}$), with possibly some repeating vertices. But if the boundary contains repeating vertices, then there exist $i, j \in \{0, 1, \ldots, t-1\}$ such that $i \neq j$, $v_i = v_j$ and the sequence $C = (v_i, e_{i+1}, \ldots, e_j, v_j)$ is a cycle. But then $v_i$ would be a cut-vertex, because any path connecting a vertex in $C$ with a vertex in the face boundary, but not in $C$, would have to contain $v_i$. Hence any face boundary cannot contain repeating vertices, and therefore is a cycle.

Suppose now that there exists a face $F$ whose boundary is a cycle of length at least 4. Let $v_1, v_2, v_3, v_4$ be four consecutive vertices in that cycle. If $\{v_1, v_3\} \notin E(G)$, we can obviously add this edge. If $\{v_1, v_3\} \in E(G)$, then this edge must be outside the face $F$, but then $\{v_2, v_4\} \notin E(G)$, because it would likewise have to be outside the face, and hence would have to cross the edge $\{v_1, v_3\}$. Then we can add $\{v_2, v_4\}$ to $G$. Hence every face of $G$ is bounded by a cycle of length 3.

Now, since every edge belongs to exactly two faces, and every face is bounded by three edges, we get $3f = 2m$. Using Euler's formula, we get $n - m + \frac{2}{3}m = 2$, from which $m = 3n - 6$. □

## 2.3 Graph colorings

**Definition 2.17.** Let $G = (V, E)$ be a graph. For a $k \in \mathbb{N}$, a mapping $c : V \to \{1, 2, \dots, k\}$ is called a *coloring* (specifically, a *$k$-coloring*) of $G$ if for every edge $\{u, v\} \in E$, $c(u) \neq c(v)$, i.e. no neighboring vertices have the same color. A graph is called *$k$-colorable* if there exists a $k$-coloring of $G$.

**Proposition 2.4.** *Every tree is 2-colorable.*

*Proof.* Let $T$ be a tree with $n$ vertices. Since $T$ has $n-1$ edges, at least two of its vertices have degree 1 (called *leaves*). Let $u, v$ be leaves of $T$. Color the vertices on the path from $u$ to $v$ with alternating colors. If there are no more leaves, then all vertices are colored. Otherwise let $t$ be another leaf in $T$. Consider the path from $t$ to its closest vertex $s$ in the colored part, then none of the vertices on that path are colored except for $s$, so we can color this path with alternating colors starting from $s$ as well. Continue until no leaves are left. $\square$

One of the most famous theorems in graph theory is the Four Color Theorem:

**Theorem 2.1.** *Every planar graph is 4-colorable.*

# 3 Problem description and previous work

In the first part of this section, we give an overview of the problem we are trying to solve along with a summary of the current best results for different kinds of graphs. In the second part, we explain the method we are trying to and give an example of this method using an article by Salavatipour [Sal06].

## 3.1 Problem overview

Let $G$ be a planar graph. The problem we are interested in is estimating the size of the maximal induced forest in $G$. Note that a set $F \subseteq V(G)$ is an induced forest in $G$ if and only if the graph formed by removing the vertices $V(G) - F$ from $G$ has no cycles, hence finding the size of the maximal induced forest in $G$ is equivalent to determining the smallest number of vertices that need to be removed from $G$ to result in a graph with no cycles. In [Kar72], it is shown that the problem of determining this smallest number is NP-hard even when restricted to planar graphs, so determining the exact size of the maximal induced forest in a planar graph is NP-hard. Hence we do not try to determine the exact size of the maximal induced forest in $G$, but instead try to find a lower bound for its size that is as large as possible.

In 1979 ([AB79]), Albertson and Berman gave the following conjecture on the size of induced forests in planar graphs:

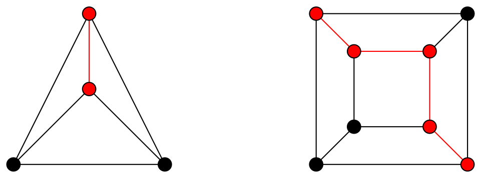
**Conjecture 3.1.** *Every planar graph with $n$ vertices has an induced forest with size at least $\frac{n}{2}$.*

If this conjecture holds, then it follows directly, independently of the Four Color Theorem, that every planar graph has an independent set (a set of vertices that induces a subgraph with no edges) of size at least $\frac{n}{4}$: for each tree in the maximal induced forest, color it with two colors as per Proposition 2.4, and all vertices of one color (if the number of vertices of each color are different, choose the one with more vertices), then each such set is obviously an independent set, and since the induced forest has size at least $\frac{n}{2}$ and we choose at least half of the vertices from each tree in the forest, we have an independent set of size at least $\frac{n}{4}$.

In [AW87], Akiyama and Watanabe gave another conjecture for bipartite planar graphs:

**Conjecture 3.2.** *Every bipartite planar graph with $n$ vertices has an induced forest with size at least $\frac{5}{8}n$.*

If true, the bounds in both of these conjectures is tight, as seen by the complete graph on 4 vertices for Conjecture 3.1 and the graph with 8 vertices with every vertex having degree 3 and every face being bounded by a 4-cycle (also called the cube graph) for Conjecture 3.2, see Figure 1 (in [AW87], Akiyama and Watanabe give whole families of graphs for which these conjectures are tight).



$K_4$ Cube graph

Figure 1: Examples of graphs for which the conjectures are tight: $K_4$ has maximal induced forest of size 2, and the cube graph has maximal induced forest of size 5 (possible maximal forests drawn in red)

Unfortunately, neither of these conjectures has yet been proven. There have been a number of results for planar graphs with certain properties, for example, Alon et al. proved in [AMT01] that every triangle-free planar graph with $n$ vertices and $m$ edges has forest of size at least $n - \frac{m}{4}$ (from which it follows directly that Conjecture 3.2 holds for triangle-free bigraphs), and Dross et al. proved in [DMP14] that every triangle-free planar graph with $n$ vertices has an induced forest of size at least $\frac{6n+7}{11}$, and every planar graph with $n$ vertices and girth $g \geqslant 5$ has an induced forest of size at least $n - \frac{(5n-10)g}{23(g-2)}$. But for general planar graphs, the best known bound comes from [Bor79], where Borodin proved that every planar graph is acyclically 5-colorable, that is, every planar graph is 5-colorable so that every cycle contains at least 3 colors. This means that the set of vertices that results in combining the vertices of two colors is an induced forest in the graph, and so by choosing the vertices of the two colors with the most vertices, we get an induced forest of size at least $\frac{2}{5}n$, where $n$ is the size of the graph. Trying to improve this bound is the main focus of this thesis.

## 3.2 Discharging method example

Our aim is to improve on the bound given by Borodin's result ([Bor79]) using a method called the discharging method. This method is used to show that every graph (or some interesting subset of graphs, like planar graphs) has some desirable property, and generally has the following steps:

1. It is shown that a minimal counterexample (a minimal graph that does not have the desired property) cannot have any of a number of configurations, e.g. vertices with a certain degree, bridges, large faces etc.

2. Every vertex (and sometimes face, if we have a planar graph) of the counterexample is assigned some number, called *charge*, so that the sum of all charges is negative.

3. It is shown (using the fact that the counterexample does not have the configurations considered in step 1) that the charges can be redistributed ("discharged", e.g. every vertex with a high degree gives a part of its charge to their neighbors etc.) in such a way that all charges are non-negative, which gives a contradiction.

Since our proof is not yet complete (we are still at the step of disproving configurations), we present here a short summary of the article "Large Induced Forests in Triangle-Free Planar Graphs" by Salavatipour ([Sal06]), concentrating on the discharging part, to illustrate how a full proof of the existence of large induced forests using the discharging method might look like. The main result of their paper is as follows:

**Theorem 3.1.** *Every triangle-free planar graph with $n$ vertices and $m$ edges has an induced forest of size at least $\frac{29n-6m}{32}$.*

Suppose that $G$ is a counterexample to this theorem with the minimal number of vertices with $n$ vertices and $m$ edges together with some planar embedding. For the first step of the discharging method, the following configurations are shown not to exist:

**Lemma 3.1.** *$G$ is connected.*

*Proof.* If $G$ is not connected, there exist induced forests of size at least $\frac{29n_i-6m_i}{32}$ for each of its connected component $G_i$ with $n_i$ vertices and $m_i$ edges, because $G$ is a minimal counterexample. Combining the vertex sets of these forests gives us a vertex set that is an induced forest in $G$ with size at least $\frac{29n-6m}{32}$, because all of the forests are pairwise disjoint. □

**Lemma 3.2.** *$G$ does not have a vertex with degree at least* 5.

*Proof.* Assume that $t$ is a vertex in $G$ with $d(t) \geqslant 5$. Then the graph $G - \{t\}$ has a forest $F$ of size at least $\frac{29(n-1)-6(m-5)}{32} = \frac{29n-6m}{32} + \frac{1}{32}$, so $F$ is an induced forest with size at least $\frac{29n-6m}{32}$ in $G$, so $G$ is not a counterexample. □

**Lemma 3.3.** *$G$ does not have bridges.*

*Proof.* Suppose that $e \in E(G)$ is a bridge. Then deleting $e$ gives us two connected components $G_1$ and $G_2$. If $G_1$ has $n_1$ vertices and $m_1$ edges, then $G_1$ has an induced forest of size at least $\frac{29n_1-6m_1}{32}$ and $G_2$ has an induced forest of size at least $\frac{29(n-n_1)-6(m-m_1-1)}{32}$. Combining these two forests gives us an induced forest in $G$ (because $e$ is a bridge, so it is not part of any cycles in $G$) with size at least $\frac{29n-6m}{32} + \frac{6}{32}$. □

**Corollary 3.1.** *$G$ does not have 1-vertices.*

*Proof.* If $t$ is a 1-vertex in $G$, then its only edge would be a bridge, which contradicts the previous lemma. □

**Lemma 3.4.** *$G$ does not have a 2-vertex that is neighbors with a 4-vertex.*

*Proof.* Suppose that $t$ is a 2-vertex in $G$ that has a neighbor $a$ with $d(a) \geqslant 4$. Then the graph $G - \{t, a\}$ has an induced forest of size at least $\frac{29(n-2)-6(m-5)}{32}$. We can add $t$ to this forest to get an induced forest in $G$, because $t$ has at most one neighbor in the forest, and hence cannot create a cycle. Therefore $G$ has an induced forest of size at least $\frac{29(n-2)-6(m-5)}{32} + 1 = \frac{29n-6m}{32} + \frac{4}{32}$. □

We will not give the proofs for the next three lemmas here (they are quite long, but follow the same general idea: remove some vertices to get a smaller graph, find an induced forest in the smaller graph, show that you can add enough vertices to the forest to get a large enough induced forest in $G$), refer to [Sal06] for the proofs:

**Lemma 3.5.** *$G$ does not have 2-vertices.*

**Lemma 3.6.** *No 5-face in $G$ has more than three 3-vertices.*

**Lemma 3.7.** *No 4-face in $G$ has a 3-vertex.*

This concludes step 1 of the discharging method for this proof.

For the second step of the discharging method, we assign each vertex $v \in V(G)$ the charge $d(v) - 6$, and each face $f \in F$ (where $F$ is the set of faces in the planar embedding of $G$) the charge $2|f| - 6$, where $|f|$ is the number of edges of the face boundary. Note that since according to Lemma 3.3, $G$ does not have bridges, every edge in $G$ belongs to exactly two faces, and also every edge has two endpoints, so

$$\sum_{v \in V(G)} d(v) = \sum_{f \in F} |f| = 2m.$$

Therefore, the sum of all charges is (using Euler's formula)

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F} (2|f| - 6) = 2m - 6n + 4m - 6|F| = -6(n - m + |F|) = -12.$$

This concludes step 2 of the discharging method. For step 3, redistribute the charges according to the following rule: every face gives charge 1 to every 3-vertex on its boundary, and charge $\frac{1}{2}$ to every 4-vertex on its boundary. Then after discharging, the following lemmas hold:

**Lemma 3.8.** *Every vertex $v \in V(G)$ has non-negative charge.*

*Proof.* Recall that according to the configuration disproved in step 1, $G$ has only 3-vertices and 4-vertices. Before discharging, a 3-vertex had charge $-3$, and it received charge 1 from each of the three faces it belongs to in the discharging step, so after discharging, every 3-vertex has charge 0. Analogously, before discharging, a 4-vertex had charge $-2$, and it received charge $\frac{1}{2}$ from each of the four faces it belongs to in the discharging step, so after discharging, every 4-vertex has charge 0. $\square$

**Lemma 3.9.** *Every face $f \in F$ has non-negative charge.*

*Proof.* Let $f \in F$ be a face in the embedding of $G$. If $|f| \geqslant 6$, then its initial charge is $2|f| - 6 \geqslant |f|$, and in the discharging step, it can lose at most charge $|f|$ (if its every vertex is a 3-vertex), so after discharging, $f$ has non-negative charge. If $|f| = 5$, then according to Lemma 3.6, it has at most three 3-vertices, so during the discharging, it can lose at most $3 \cdot 1 + 2 \cdot \frac{1}{2} = 4$ charge. Since the initial charge of $f$ was 4, its charge after discharging is non-negative. If $|f| = 4$, then according to Lemma 3.7, it has only 4-vertices, so it loses 2 charge during discharging. Since the initial charge of $f$ was 2, this results in charge 0 after discharging. The case $|f| = 3$ is not possible, because $G$ is triangle-free. $\square$

Therefore after discharging, all vertices and faces have non-negative charge, which contradicts the fact that their sum should be negative (since we only move charges around, the sum of the charges does not change). Therefore no counterexample exists.

# 4 Our work

In this section, we present our partial results towards the goal of improving Borodin's result ([Bor79]). Our aim is to show that every planar graph with $n$ vertices and $m$ edges has an induced forest of size at least $\alpha n - \beta m$, where $\alpha$ and $\beta$ are some non-negative constants such that $\alpha - 3\beta > \frac{2}{5}$ (since for every planar graph, $m \leqslant 3n - 6$, such a result would mean that every planar graph has a forest of size at least $\alpha n - \beta m \geqslant \alpha n - \beta(3n-6) = (\alpha - 3\beta)n + 6\beta > \frac{2}{5}n$, which would improve on Borodin's result). Since our proof is not yet complete, we have not yet fixed the values of $\alpha$ and $\beta$, so most of our results are of the form "If $\alpha$ and $\beta$ satisfy some constraint, then we can show that...".

Let $G$ be a minimal (in the number of vertices) counterexample to our result, i.e. $G$ is a planar graph with $m$ edges and $n$ vertices such that the maximal induced forest of $G$ is smaller than $\alpha n - \beta m$, and for all graphs with less than $n$ vertices, it is possible to find an induced forest such that our result holds. Then we can prove that $G$ has the following properties:

**Lemma 4.1.** *$G$ is connected.*

*Proof.* Suppose that $G$ is not connected, and let $G_1$ be a connected component in $G$ with $n_1$ vertices and $m_1$ edges. Since $G$ is not connected, $n_1 < n$, and hence there exists an induced forest of size at least $\alpha n_1 - \beta m_1$ in $G_1$, and also there exists a forest of size $\alpha(n - n_1) - \beta(m - m_1)$ in the graph $G - G_1$. Combining these two forests gives us an induced forest in $G$ (there are no possible edges between the two forests, because $G_1$ was a connected component) with size at least $\alpha n_1 - \beta m_1 + \alpha(n - n_1) - \beta(m - m_1) = \alpha n - \beta m$, which is a contradiction (since $G$ was a counterexample). □

**Lemma 4.2.** *There are no bridges in $G$.*

*Proof.* Suppose there exists a bridge with endpoints $a$ and $b$ in $G$. Denote the two connected components that form by removing the edge $ab$ by $G_1$ and $G_2$ (since by Lemma 4.1, $G$ is connected, there are exactly two connected components). If $G_1$ has $n_1$ vertices and $m_1$ edges, then since $G_1$ and $G_2$ are both smaller than $G$, there exists an induced forest $F_1$ in $G_1$ with size at least $\alpha n_1 - \beta m_1$ and an induced forest $F_2$ in $G_2$ with size at least $\alpha(n - n_1) - \beta(m - m_1 - 1)$. Then $F_1 + F_2$ is an induced forest in $G$, since there is at most one edge connecting $F_1$ and $F_2$, and has size at least $\alpha n_1 - \beta m_1 + \alpha(n - n_1) - \beta(m - m_1 - 1) = \alpha n - \beta m + \beta \geqslant \alpha n - \beta m$. □

**Lemma 4.3.** *There are no cut-vertices in $G$.*

*Proof.* Suppose there exists a cut-vertex $t$ in $G$. Let $G_1$ and $G_2$ be the connected components in $G - \{t\}$ (since by Lemma 4.1, $G$ is connected, there are exactly two connected components), and let $n_1$ be the number of vertices in $G_1$, $m_1$ be the number of edges in $G_1$, $d_1$ the number of neighbors (in $G$) of $t$ in $G_1$, and $d_2$ the number of neighbors of $t$ in $G_2$. Let us look at the maximal induced forests of $G_3 = G_1 + \{t\}$ and $G_4 = G_2 + \{t\}$.

*Case* 1. There exist maximal induced forests $F_3$ and $F_4$ in $G_3$ and $G_4$ respectively such that $t \notin F_3$ and $t \notin F_4$. Then $F_3 + F_4$ is an induced forest in $G$ (no edges between them) with size at least $\alpha(n_1+1)-\beta(m_1+d_1)+\alpha(n-n_1)-\beta(m-m_1-d_1) = \alpha n - \beta m + \alpha > \alpha n - \beta m$.

*Case* 2. There exists a maximal induced forest $F_3$ in $G_3$ such that $t \notin F_3$, but every maximal forest of $G_4$ contains the vertex $t$. Let $F_2$ be a maximal induced forest in $G_2$, then $F_3 + F_2$ is an induced forest in $G$ (no possible edges between them) with size at least $\alpha(n_1+1) - \beta(m_1+d_1) + \alpha(n-n_1-1) - \beta(m-m_1-d_1-d_2) = \alpha n - \beta m + \beta d_2 \geqslant \alpha n - \beta m$.

*Case* 3. There exists a maximal induced forest $F_4$ in $G_4$ such that $t \notin F_4$, but every maximal forest of $G_3$ contains the vertex $t$. Symmetric to above case.

*Case* 4. Every maximal induced forest in $G_3$ and $G_4$ contains the vertex $t$. Let $F_2, F_3, F_4$ be maximal forests in $G_2, G_3, G_4$ respectively. Then we have $|F_2| = |F_4| - 1$, because since every maximal forest in $G_4$ contains the vertex $t$, $|F_4| > |F_2|$ (otherwise $F_2$ would be a maximal forest in $G_4$ that does not contain $t$), and if $|F_2| < |F_4| - 1$, then $F_4 - \{t\}$ would be a larger forest than $F_2$ in $G_2$. Then $F_3 + F_4$ is a forest in $G$, because any path from $F_3$ to $F_4$ would have to pass through $t$, and hence cannot form a cycle that would not be contained in either $F_3$ or $F_4$, and this forest has size $|F_3| + |F_4| - 1 = |F_3| + |F_2| \geqslant \alpha(n_1+1) - \beta(m_1+d_1) + \alpha(n - n_1 - 1) - \beta(m - m_1 - d_1 - d_2) = \alpha n - \beta m + \beta d_2 \geqslant \alpha n - \beta m$.

□

**Lemma 4.4.** *There are no* 1*-vertices in* $G$ *if* $\alpha - \beta \leqslant 1$.

*Proof.* Suppose there exists a 1-vertex $t$ in $G$. Then the graph $G - \{t\}$ has an induced forest of size at least $\alpha(n-1) - \beta(m-1)$, and adding $t$ to this forest creates an induced forest in $G$, because $t$ has at most one edge in this forest and hence cannot be part of a cycle, and this forest has size $\alpha(n-1) - \beta(m-1) + 1 = \alpha n - \beta m - (\alpha - \beta) + 1 \geqslant \alpha n - \beta m$ if $1 \geqslant \alpha - \beta$. □

**Lemma 4.5.** *There are no* 2*-vertices in* $G$ *if* $2\alpha - 3\beta \leqslant 1$.

*Proof.* Suppose there exists a 2-vertex $t$ in $G$ with neighbors $a$ and $b$. If $2\alpha - 3\beta \leqslant 1$, then also $\alpha - \beta \leqslant 1$, because from $\alpha - 3\beta > \frac{2}{5}$ we get $1 \geqslant 2\alpha - 3\beta > \alpha + \frac{2}{5}$, from which $\alpha < \frac{3}{5}$, and hence also $\alpha - \beta \leqslant 1$ (since $\beta \geqslant 0$), so by Lemma 4.4, there are no 1-vertices in $G$. Then the graph $G - \{t, b\}$ has $n - 2$ vertices and at most $m - 3$ edges (it is missing the edges $ta, bt$ and at least one more edge connected to

$b$, because $b$ is not a 1-vertex), and hence has an induced forest $F$ of size at least $\alpha(n-2)-\beta(m-3)$ (if there were less edges, then the forest would be even larger). Then $F+\{t\}$ is a forest in $G$, because $t$ has at most one edge in $F+\{t\}$, and it has size at least $\alpha(n-2)-\beta(m-3)+1 \geqslant \alpha n-\beta m$ if $2\alpha-3\beta \leqslant 1$. □

**Lemma 4.6.** *There are no* 3*-vertices in* $G$ *if* $2\alpha-3\beta \leqslant 1$ *and* $3\alpha-13\beta \leqslant 1$.

*Proof.* Suppose there exists a 3-vertex $t$ in $G$ with neighbors $a, b$ and $c$. If one of the edges $ab, bc, ac$ is not in $G$ (assume without loss of generality that $ab \notin E(G)$), then let $G_1$ be the graph $G-\{t,c\}$ with the edge $ab$ added. $G_1$ is planar, because the edge $ab$ can take the path that in $G$ was taken by edges $bt$ and $at$. Let $F_1$ be a maximal induced forest in $G_1$, then $|F_1| \geqslant \alpha(n-2)-\beta(m-4)$, because by Lemmas 4.4 and 4.5 the degree of vertex $c$ is at least 3, so we removed at least 5 edges and added 1 edge. Then $F_1+\{t\}$ is an induced forest in $G$, because if either $a$ or $b$ are not in $F_1$, then $t$ has degree at most 1 in this forest, and if $a$ and $b$ are both in $F_1$, then the edge $ab$ was also in $F_1$, and adding $t$ is equivalent to dividing the edge $ab$, which cannot create a cycle. Then the size of this forest is at least $\alpha(n-2)-\beta(m-4)+1=\alpha n-\beta m-(2\alpha-3\beta)+1+\beta \geqslant \alpha n-\beta m$, since $2\alpha-3\beta \leqslant 1$. Hence in the following we can assume that all of the edges $ab, bc, ac$ are in $G$.

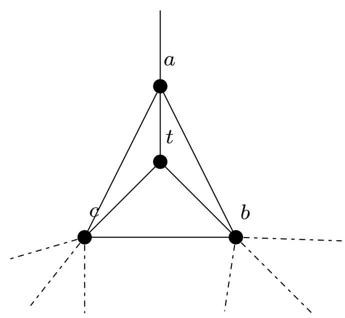



Figure 2: 3-vertex with a neighbor with degree at most 4

*Case* 1. One of the vertices $a, b, c$ has degree at most 4.

Assume w.l.o.g. that $d(a) \leqslant 4$ (See Figure 2). Let $F$ be a maximal forest in the graph $G-\{a,b,c,t\}$, then $|F| \geqslant \alpha(n-4)+\beta(m-6)$, since we removed at least 6 edges by deleting the 4 vertices. Then $F+\{a,t\}$ is a forest in $G$, because $a$ has at most one neighbor in $F$ and $t$ has no neighbors in $F$ (its only neighbor in the new forest is $a$), and it has size at least $\alpha(n-4)+\beta(m-6)+2 \geqslant \alpha n - \beta m + 2(1-(2\alpha-3\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

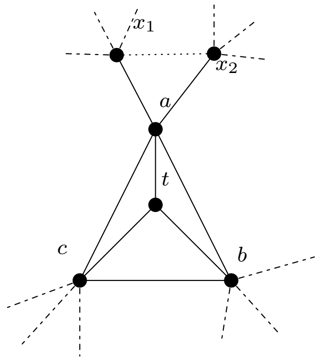



Figure 3: 3-vertex with a neighbor with degree 5 with no edge between the neighbors

*Case* 2. One of the vertices $a, b, c$ (assume w.l.o.g $a$) has degree 5 and no edge between the neighbors that are not $b, c, t$. Let the two neighbors of $a$ that are not $b, c, t$ be $x_1$ and $x_2$, then the edge $x_1x_2 \notin E(G)$ (Figure 3). Let $G_1$ be the graph $G - a, b, c, t$ with the edge $x_1x_2$ added, and let $F_1$ be a maximal induced forest in $G_1$. Then $|F_1| \geqslant \alpha(n-4) - \beta(m-11)$ because by the previous case, all of $a, b, c$ have degree at least 5, so we removed at least 12 edges and added 1 edge. Then $F_1 + \{a, t\}$ is a forest in $G$, because adding $a$ to $F_1$ cannot create a cycle, since $x_1x_2 \notin E(G)$, so $F_1 + \{a\}$ is a forest, and since $t$ has only one neighbor in $F_1 + \{a\}$, adding $t$ cannot create a cycle either. This forest has size $\alpha(n-4) - \beta(m-11) + 2 = \alpha n - \beta m + 2(1-(2\alpha-3\beta)) + 5\beta \geqslant \alpha n - \beta m$, because $2\alpha - 3\beta \leqslant 1$.

*Case* 3. There are (at least) two vertices with degree 5 among $a, b, c$ (assume w.l.o.g. $a, b$, with the neighbors of $a$ being $x_1, x_2, b, t, c$, and the neighbors of $b$ being $y_1, y_2, a, t, c$), and they have at least one neighbor not in common (w.l.o.g., assume $x_1 \notin \{y_1, y_2\}$) (Figure 4). Let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $x_1y_1$ and $x_1y_2$ added, if they were not in $G$. $G_1$ is still planar, because we can follow the paths $x_1aby_1$ and $x_1aby_2$ arbitrarily closely to add the edges $x_1y_1$ and $x_1y_2$. Let $F_1$ be a maximal induced forest in $G_1$, then it cannot contain all of $x_1, y_1, y_2$, because by the previous case, $y_1y_2 \in E(G)$, and hence the three vertices induce a cycle in $G_1$. $F_1$ has size at least $\alpha(n-4) - \beta(m-10)$, because we deleted at least 12 edges with the four vertices (since $d(c) \geqslant 5$), and added at most two edges. If $x_1 \in F_1$,

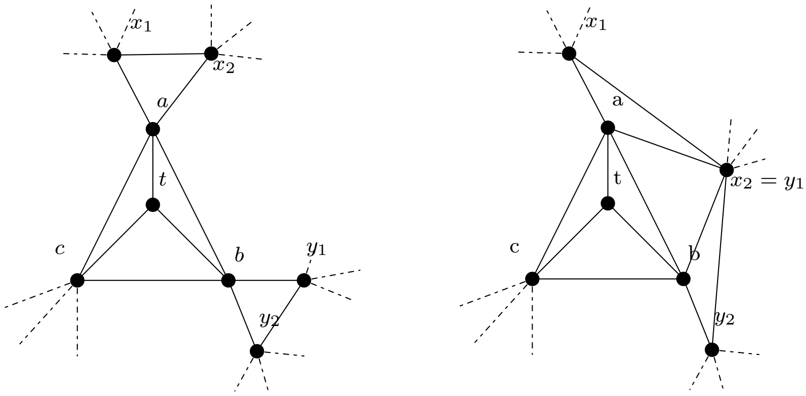



Figure 4: 3-vertex with two neighbors with degree 5 with at least one neighbor not in common

then at least one of $y_1, y_2$ is not in $F_1$, and hence $F_1+\{b\}$ is an induced forest in $G$, because $b$ has only one neighbor in $F_1$, and hence $F_1+\{b,t\}$ also is an induced forest in $G_1$, because $t$ has only one neighbor in $F_1+\{t\}$. So $G$ has an induced forest of size at least $\alpha(n-4)-\beta(m-10)+2 = \alpha n-\beta m+2(1-(2\alpha-3\beta))+4\beta \geqslant \alpha n-\beta m$, because $2\alpha-3\beta \leqslant 1$. If $x_1 \notin F_1$, then we can analogously add $a$ and $t$ to $F_1$ to get an induced forest in $G$ with the same size.

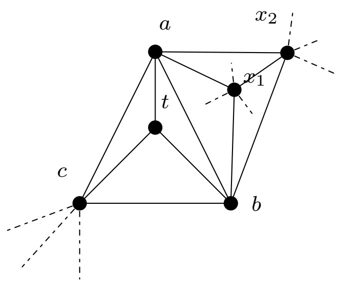



Figure 5: 3-vertex with two neighbors with degree 5 with all neighbors in common

*Case* 4. There are (at least) two vertices with degree 5 among $a, b, c$ (assume w.l.o.g. $a, b$) that have all neighbors in common, i.e. if the neighbors of $a$ are

$x_1, x_2, b, c, t$, then the neighbors of $b$ are $x_1, x_2, a, c, t$ (Figure 5). Note that at least one of $x_1, x_2$ is not a neighbor of $c$, because the paths $atb, ax_1b, ax_2b$ divide the plane into three pieces, and any point within one piece can only have two neighbors in $\{x_1, x_2, t\}$, and by definition $c$ is a neighbor of $t$. Assume w.l.o.g. that $x_1$ is not a neighbor of $c$. If $d(c) = 5$, then we can use the previous case. Hence we can assume $d(c) \geqslant 6$.

If $d(x_1) \leqslant 4$, then let $F$ be a maximal induced forest in the graph $G - \{a, b, c, t, x_1, x_2\}$. Then $|F| \geqslant \alpha(n-6) - \beta(m-14)$, because $d(c) \geqslant 6$ and $x_1x_1 \in E(G)$. $F + \{x_1, a, t\}$ is a forest in $G$, because $x_1$ has at most one neighbor in $F$, $a$ has one neighbor in $F + \{x_1\}$ and $t$ has one neighbor in $F + \{x_1, a\}$, hence $G$ has an induced forest of size at least $\alpha(n-6) - \beta(m-14) + 3 = \alpha n - \beta m + 3(1 - (2\alpha - 3\beta)) + 5\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If $d(x_1) \geqslant 5$, then let $F$ be a maximal induced forest in the graph $G - \{a, b, c, t, x_1\}$. Then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $d(c) \geqslant 6$, $x_1x_1 \in E(G)$, $d(x_1) \geqslant 5$ and $x_1$ is not a neighbor of $c$. $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor ($x_2$) in $F$, and $t$ has one neighbor in $F + \{a\}$, hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

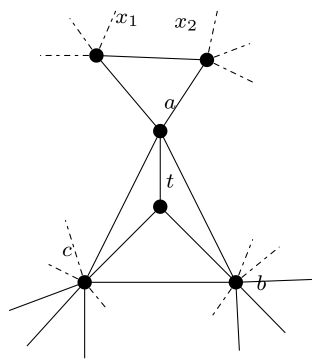



Figure 6: 3-vertex with one neighbor with degree 5 and two neighbors with degree at least 6

*Case* 5. One of the vertices $a, b, c$ has degree 5 (assume w.l.o.g. $a$), the other two have degree at least 6 (Figure 6). Let the two neighbors of $a$ that are not $b, c, t$ be $x_1$ and $x_2$.

If one of $x_1, x_2$ (assume w.l.o.g. $x_1$) has at most one neighbor not in $\{x_2, a, b, c\}$, then let $F$ be a maximal induced forest in the graph $G - \{a, b, c, t, x_1, x_2\}$. Then

$|F| \geqslant \alpha(n-6) - \beta(m-15)$, because $x_1x_2 \in E(G)$. $F + \{x_1, a, t\}$ is an induced forest in $G$, because $x_1$ has at most one neighbor in $F$ by our assumption, $a$ has one neighbor in $F+\{x_1\}$, and $t$ has one neighbor in $F+\{x_1, a\}$. Hence $G$ has an induced forest of size at least $\alpha(n-6)-\beta(m-15)+3 = \alpha n-\beta m+3(1-(2\alpha-3\beta))+6\beta \geqslant \alpha n - \beta m$, because $2\alpha - 3\beta \leqslant 1$.

If $x_1$ has at least two neighbors not in $\{x_2, a, b, c\}$, then let $F$ be a maximal induced forest in the graph $G - \{a, b, c, t, x_1\}$. Then $|F| \geqslant \alpha(n-5) - \beta(m-17)$, because $x_1$ has at least two edges not connected to $\{x_2, a, b, c\}$. $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor ($x_2$) in $F$, and $t$ has one neighbor in $F + a$, hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-17) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) + \beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

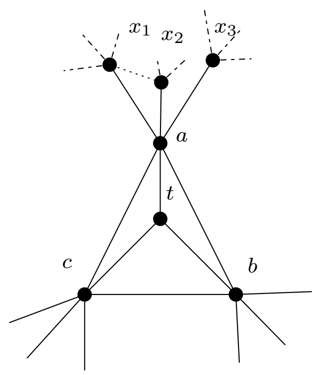



Figure 7: 3-vertex with all neighbors with degree 6 and at least one edge missing between neighbors

*Case* 6. All of $a, b, c$ have degree 6 and one of them (assume w.l.o.g. $a$) has at least one edge missing between its three neighbors that are not $b, c, t$, i.e. if the three neighbors of $a$ that are not in $\{b, c, t\}$ are $x_1, x_2$ and $x_3$, then at least one of the edges $x_1x_2, x_2x_3, x_1x_3$ is not in $G$ (Figure 7).

If none of the edges $x_1x_2, x_2x_3, x_1x_3$ is in $G$, then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the additional edges of $x_1x_2, x_2x_3, x_1x_3$. $G_1$ is planar, because $x_1, x_2, x_3$ were all neighbors of $a$, and hence after deletion of $a$, they all belong to the same face. Let $F_1$ be a maximal induced forest in $G_1$, then $|F_1| \geqslant \alpha(n-4) - \beta(m-12)$, because we deleted 15 edges and added 3. Then $F_1 + \{a, t\}$ is an induced forest in $G$, because $a$ has at most two neighbors in $F_1$ ($F_1$ cannot contain all of $x_1, x_2, x_3$ due to the edges we added), and if $a$ has two neighbors in $F$, then adding $a$ is equivalent to dividing an edge in the forest $F_1$, and $t$ has one neighbor in the forest $F_1 + \{a\}$. Hence $G$ has a forest of size at least $\alpha(n-4) - \beta(m-12) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 6\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If exactly one of the edges $x_1x_2, x_2x_3, x_1x_3$ is in $G$, assume w.l.o.g. that $x_2x_3 \in E(G)$. Let $G_1$ be either the graph $G - \{a, b, c, t, x_2\}$ with the edge $x_1x_3$ added or the graph $G - \{a, b, c, t, x_3\}$ with the edge $x_1x_2$ added, whichever has the smaller number of edges. Let us consider how many edges $G_1$ can have. If either $x_2$ or $x_3$ has degree 3, then by all the previous cases, the other one must have degree at least 6, and choosing the graph with the vertex with degree at least 6 removed gives us at most $m - 17$ edges (two of the six edges can be connected to $b$ and $c$, and we added one edge). If both $x_2$ and $x_3$ have degree 4, then at least one of them has at most one neighbor among $\{b, c\}$, otherwise the subgraph induced by the vertices $\{a, b, c, t, x_2, x_3\}$ would have 6 vertices and 13 edges, which cannot be planar (since $13 > 3 \cdot 6 - 6 = 12$). Then choosing the graph with the vertex that has at most one neighbor among $\{b, c\}$ removed gives us a graph with at most $m - 16$ edges (since at least one of the four edges is not $x_2x_3$ and not connected to any of $\{a, b, c\}$, and we added one edge). If one of $x_2$ and $x_3$ has degree at least 5, then choosing the graph with this vertex removed gives us at most $m - 16$ edges (two of the five edges can be connected to $b$ and $c$, and we added one edge). So $G_1$ has at most $m - 16$ edges, and hence, if $F_1$ is a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$. Assume w.l.o.g. that $G_1$ is the graph $G - \{a, b, c, t, x_3\}$ with the edge $x_1x_2$ added. Then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most two neighbors in $F$, and if it does have two neighbors, then the forest induced by $F$ in $G_1$ contained the edge $x_1x_2$, and so adding $a$ is equivalent to dividing the edge $x_1x_2$, and $t$ has one neighbor in $F + \{a\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

If exactly one of the edges $x_1x_2, x_2x_3, x_1x_3$ is not in $G$, assume w.l.o.g. that $x_1x_2 \notin E(G)$. Let $G_1$ be the graph $G - \{a, b, c, t, x_3\}$ with the edge $x_1x_2$ added. Let $F_1$ be a maximal induced forest in $G_1$, then $|F_1| \geqslant \alpha(n-5) - \beta(m-16)$, since we removed at least 17 edges and added one. Then analogously to the previous subcase, $F + \{a, t\}$ is an induced forest in $G$ with size at least $\alpha n - \beta m$.

*Case* 7. All of $a, b, c$ have degree 6 and all of the possible edges between their three neighbors that are not in $\{a, b, c, t\}$ are in $G$ (Figure 8). Let $x_1, x_2$ and $x_3$ be the three neighbors of $a$ that are not $b, c, t$. Then at least one of $x_1, x_2, x_3$ is not connected to neither of $\{b, c\}$, because the cycles $ax_1x_2$, $ax_2x_3$, $ax_1x_3$ and $x_1x_2x_3$ divide the plane into four pieces, and any point within one of these pieces can be connected to at most three of $\{a, x_1, x_2, x_3\}$, and since $b$ and $c$ are neighbors with each other, they must be in the same piece, and since they are also neighbors of

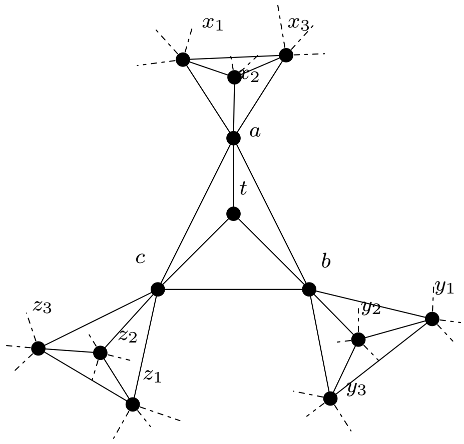



Figure 8: 3-vertex with all neighbors having degree 6 and all edges existing between neighbors

$a$, they cannot be in the piece bounded by $x_1x_2x_3$. Assume w.l.o.g. that $x_2$ is not connected to neither $b$ nor $c$.

If $d(x_2) = 3$, then by all the previous cases, $d(x_1) \geqslant 6$ and $d(x_3) \geqslant 6$ (since they are neighbors of $x_2$). Let $F$ be a maximal induced forest in the graph $G - \{a, c, t, x_1, x_3\}$, then $|F| \geqslant \alpha(n-5)-\beta(m-19)$, because $x_1$ and $x_3$ can be connected to $c$. Then $F + \{x_2, t\}$ is an induced forest in $G$, because $x_2$ has no neighbors in $F$, and $t$ has one neighbor ($b$) in $F + \{x_2\}$, so $G$ has an induced forest of size at least $\alpha(n-5)-\beta(m-19)+2 = \alpha n-\beta m+(1-(2\alpha-3\beta))+(1-(3\alpha-13\beta))+3\beta \geqslant \alpha n-\beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

If $d(x_2) = 4$, then let $F$ be a maximal induced forest in the graph $G - \{a, b, c, t, x_1, x_2, x_3\}$. Then $|F| \geqslant \alpha(n-7) - \beta(m-19)$, since $x_2$ is not connected to neither $b$ nor $c$. $F + \{x_2, a, t\}$ is an induced forest in $G$, because $x_2$ has at most one neighbor in $F$, $a$ has one neighbor in $F + \{x_2\}$, and $t$ has one neighbor in $F + \{x_2, a\}$. Hence $G$ has an induced forest of size at least $\alpha(n-7)-\beta(m-19)+3 = \alpha n-\beta m+2(1-(2\alpha-3\beta))+(1-(3\alpha-13\beta)) \geqslant \alpha n-\beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

If $d(x_2) \geqslant 5$, then at least one of $x_1, x_3$ is not connected to both $b$ and $c$, otherwise the subgraph induced by $\{a, b, c, t, x_1, x_3\}$ would have 6 vertices and 13 edges, which is not planar. Assume w.l.o.g. that $x_1$ is not a neighbor of $b$, and let the neighbors of $b$ that are not $a, c, t$ be $y_1, y_2, y_3$. Let $G_1$ be the graph $G - \{a, b, c, t, x_2\}$ with the edges $x_1y_1, x_1y_2, x_1y_3$ added if they did not exist. $G_1$ is still planar, because after the removal of $a$ and $b$, $x_1, y_1, y_2$ and $y_3$ belong to the same face. Let $F_1$ be a maximal induced forest in $G_1$, then $|F_1| \geqslant \alpha(n-5)-\beta(m-16)$, because $x_2$ is not connected to neither $b$ nor $c$, so we removed 19 edges, and added at most 3. If $x_1 \in F_1$, then at most one of $y_1, y_2, y_3$ is in $F_1$, because the edges $y_1y_2, y_1y_3, y_2y_3$ are all in $G$ (and so also in $G_1$). Then $F_1 + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F_1$, and $t$ has one neighbor in $F_1 + \{b\}$. If $x_1 \notin F_1$, then $F_1 + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor ($x_3$) in $F_1$, and $t$ has one neighbor in $F_1 + \{a\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

*Case* 8. There exist two vertices among $a, b, c$ such that the sum of their degrees is at least 13. Assume w.l.o.g. that $d(a) + d(b) \geqslant 13$. Let $F$ be a maximal induced forest in the graph $G - \{a, b, t\}$, then $|F| \geqslant \alpha(n-3) - \beta(m-13)$, because we removed at least 13 edges. $F + \{t\}$ is an induced forest in $G$, because $t$ has at most one neighbor in $F$. Hence $G$ has an induced forest of size at least $\alpha(n-3) - \beta(m-13) + 1 = \alpha n - \beta m + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $3\alpha - 13\beta \leqslant 1$.

Since this covers all possibilities for the degrees of the neighbors of a 3-vertex, there are no 3-vertices in $G$ if $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

□

**Lemma 4.7.** *Every 4-vertex in $G$ has at least one edge between any three of its neighbors if* $2\alpha - 3\beta \leqslant 1$.

*Proof.* Assume $t$ is a 4-vertex in $G$ with neighbors $a, b, c, d$ such that for some three neighbors (w.l.o.g. $a, b, c$) there are no edges between them, i.e. $ab \notin E(G), bc \notin E(G)$ and $ac \notin E(G)$. Let $G_1$ be the graph $G - \{d, t\}$ with the edges $ab, bc, ac$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-2) - \beta(m-3)$, because we removed at least 6 edges (since $2\alpha - 3\beta \leqslant 1$, by Lemma 4.5, $d(d) \geqslant 3$) and added 3. Then $F + \{t\}$ is an induced forest in $G$, because there are at most two of $a, b, c$ in $F$ and the edge between those two does not exist in $G$, so adding $t$ cannot create a cycle. Hence $G$ has an induced forest with size at least $\alpha(n-2) - \beta(m-3) + 1 = \alpha n - \beta m + 1 - (2\alpha - 3\beta) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$. □

Let us consider what this lemma means for 4-vertices in $G$ (assuming $2\alpha - 3\beta \leqslant 1$). Let $t$ be a 4-vertex in $G$ with neighbors $a, b, c, d$ in clockwise order in the embedding. Then we have to have at least one edge between every triple of the neighbors. Since $G$ is planar, we cannot have both "diagonals" $ac$ and $bd$ (if we had e.g. the edge $ac$, then the cycle $atc$ would divide the plane into two, with $b$ in one part and $d$ in another). Since there are four possible triples of neighbors and any edge belongs to two triples (i.e. there are only two triples that contain both endpoints of the edge), there has to be at least two edges overall between the neighbors, so there has to be at least one "non-diagonal" ($ab, bc, cd, ad$) edge. Assume w.l.o.g. that $ab \in E(G)$. If $cd$ is also in $G$, then the conditions of the lemma are satisfied. Analogously if $ad$ and $bc$ are both in $G$, the conditions of the lemma are satisfied. If $cd \notin E(G)$ and also one of $ad$, $bc$ not in $G$ (assume $ad \notin E(G)$), then the triple $a, d, c$ has two edges not in $G$, so the third edge $ac$ must be in $G$. Since $ac \in E(G), bd \notin E(G)$, from the triple $b, d, c$ we get that $bc \in E(G)$. Hence there can be two types of 4-vertices in $G$: 4-vertices with a triple of neighbors that have all three possible edges between them (call this type of vertex *Type I*), and 4-vertices that have a pair of "opposite" edges ($ab$ and $cd$, or $ad$ and $bc$) (call this *Type II*) (Figure 9). Note that these types are not mutually exclusive.

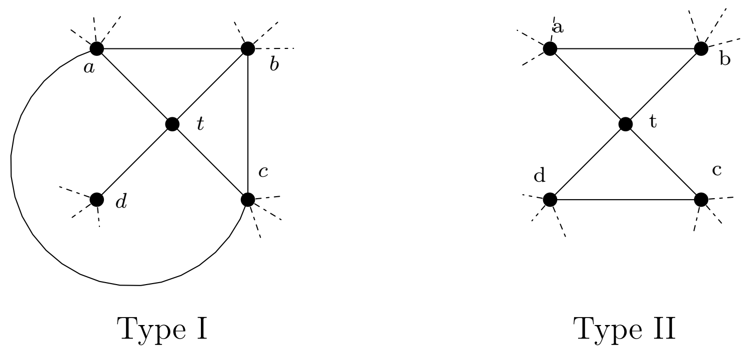



Figure 9: Possible types of 4-vertices

**Lemma 4.8.** *There are no Type I 4-vertices in G if* $2\alpha - 3\beta \leqslant 1$ *and* $3\alpha - 13\beta \leqslant 1$.

*Proof.* Since we have $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$, according to Lemmas 4.4, 4.5, 4.6, the minimum degree of any vertex in $G$ is at least 4. Suppose $t$ is a 4-vertex in $G$ with neighbors $a, b, c, d$ in clockwise order in the embedding such that $ab, bc, ac$ are all in $G$. Then the cycle $atc$ divides the plane into two parts, with $b$ in one part and $d$ in the other. In the following proof, we say that a point is *inside* the cycle $atc$ if it is in the part with $d$, and *outside* the cycle $atc$ if it is in the part with $b$.

*Case* 1. The sum of the degrees of $a$ and $c$ is at least 12. Let $F$ be a forest in the graph $G - \{a, c, t\}$, then $|F| \geqslant \alpha(n-3) - \beta(m-13)$. $F + \{t\}$ is an induced forest in $G$, because $t$ has only at most two neighbors($b, d$) in $F$ and since $b$ and $d$ are on different sides of the cycle $atc$, there is no possible path from $b$ to $d$ in $G$ that does not pass through either $t$, $a$ or $c$, and none of them are in $F$. Hence $G$ has an induced forest of size at least $\alpha(n-3) - \beta(m-13) + 1 = \alpha n - \beta m + 1 - (3\alpha - 13\beta) \geqslant \alpha n - \beta m$, since $3\alpha - 13\beta \leqslant 1$.

*Case* 2. The degree of $b$ is 4. Let $F$ be a forest in the graph $G - \{a, b, c, t\}$, then $|F| \geqslant \alpha(n-4) - \beta(m-10)$, since $d(a), d(c) \geqslant 4$. Then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has only at most one neighbor in $F$, and since there is no possible path from $b$ to $d$ in $F + \{b\}$, adding $t$ cannot create a cycle. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-10) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 4\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

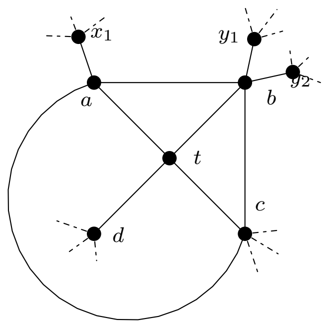



Figure 10: Type I 4-vertex with $d(b) = 5$ and $d(a) = 4$

*Case* 3. The degree of $b$ is 5, and the degree of either $a$ or $c$ (assume w.l.o.g. $a$) is 4. Let the neighbor of $a$ that is not $b, c, t$ be $x_1$, and the two neighbors of $b$ that are not $a, c, t$ be $y_1$ and $y_2$ (Figure 10).

If $x_1$ is inside the cycle $atc$, then $x_1 \notin \{y_1, y_2\}$, because $y_1, y_2$ are outside the cycle $atc$. Let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $x_1y_1, x_1y_2, y_1y_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-8)$, because we removed at least 11 edges and added at most 3. Since $x_1y_1y_2$ is a triangle in $G$, at most two of $x_1, y_1, y_2$ are in $F$. If $x_1 \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because $t$ has at most one neighbor in $F$, and $a$ has one neighbor in $F + \{t\}$. If one of $y_1, y_2$ is not in the forest, then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$, and there is no possible path from $b$ to $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-8) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 2\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If $x_1$ is outside the cycle $atc$, then let $F$ be a forest in the graph $G - \{a, b, c, t\}$. Then $|F| \geqslant \alpha(n-4) - \beta(m-11)$, and $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$, and there is no path between $a$ and $d$ in $F + \{a\}$ (because $x_1$ is outside the cycle $atc$). Hence $G$ has an induced forest of size at least

$\alpha(n-4) - \beta(m-11) + 2 = \alpha n - \beta m + 2(1-(2\alpha-3\beta)) + 5\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

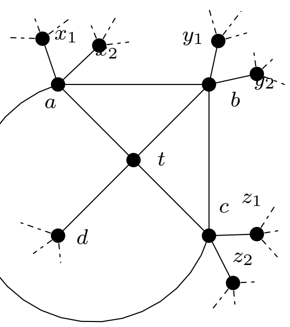



Figure 11: Type I 4-vertex with $d(a) = d(b) = d(c) = 5$

*Case* 4. All of $a, b, c$ have degree 5. Let the two neighbors of $a$ that are not $b, c, t$ be $x_1, x_2$, the two neighbors of $b$ that are not $a, c, t$ be $y_1, y_2$, and the two neighbors of $c$ that are not $a, b, c$ be $z_1, z_2$ (Figure 11).

If $x_1, x_2, z_1, z_2$ are all inside the cycle $act$, then $b$ is a cut-vertex: if $b$ is removed, there is no possible path between e.g. $t$ and $y_1$, because $a$ and $c$ are not connected to anything that is outside the cycle $act$ as $b$. This contradicts Lemma 4.3, which says that there are no cut-vertices in $G$.

If $y_1y_2 \notin E(G)$, then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edge $y_1y_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-12)$, because we removed 13 edges and added 1. $F + \{b, t\}$ is an induced forest in $G$, because $b$ divides the edge $y_1y_2$ if both $y_1, y_2$ are in $F$, and otherwise $b$ has at most one neighbor in $F$, and there is no possible path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest with size at least $\alpha(n-4) - \beta(m-12) = \alpha n - \beta m + 2(1-(2\alpha-3\beta)) + 6\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If $y_1y_2 \in E(G)$ and for either the pair $\{x_1, x_2\}$ or $\{z_1, z_2\}$, one of the pair is outside the cycle $act$ and the other is not in $\{y_1, y_2\}$ (assume w.l.o.g. $x_1$ is outside the cycle $act$, and $x_2 \notin \{y_1, y_2\}$), then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $x_2y_1, x_2y_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $F$ cannot contain all three of $x_2, y_1, y_2$, and $|F| \geqslant \alpha(n-4) - \beta(m-11)$, because we removed at least 13 edges and added at most 2. If $x_2 \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$ and there is no possible path between $a$ and $d$ in $F + \{a\}$, because $x_1$ is outside the cycle $act$.

If $x_2 \in F$, then one of $y_1, y_2$ is not in the forest, and $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no possible path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest with size at least $\alpha(n-4) - \beta(m-11) = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 5\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If $y_1y_2 \in E(G)$ and neither the pair $\{x_1, x_2\}$ nor $\{z_1, z_2\}$ satisfies the conditions of the previous subcase, then one pair must be inside the cycle *act* and one pair must equal the pair $\{y_1, y_2\}$ (not all of them can be outside of the cycle *act*, otherwise $t$ would be a cut-vertex, if only one of a pair would be inside the cycle *act*, the pair would satisfy the conditions of the previous subcase, so one pair must be inside and one pair outside the cycle *act*, and if one of the outside pair was not in $\{y_1, y_2\}$, it would satisfy the conditions of the previous subcase). Assume w.l.o.g. that $\{x_1, x_2\} = \{y_1, y_2\}$, and $z_1, z_2$ are inside the cycle *act*. Let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $dy_1, dy_2, y_2z_1, y_2z_2, z_1z_2$ added, and let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-8)$, because we removed at least 13 edges and added at most 5. Since we have the triangles $dy_1y_2$ and $y_2z_1z_2$ in $G_1$, either one of $y_1, y_2$ is not in $F$ or $d$ and one of $z_1, z_2$ is not in $F$. In the first case, $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no possible path from $b$ to $d$ in $F + \{b\}$, and in the second case, $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most one neighbor in $F$ and $t$ has one neighbor in $F + \{t\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-8) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 2\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

*Case* 5. The degree of $b$ is 5, and $d(a) + d(b) = 11$. Let $y_1, y_2$ be the neighbors of $b$ that are not $a, c, t$.

If $y_1y_2 \notin E(G)$, then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edge $y_1y_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-13)$, because we removed 14 edges and added 1. $F + \{b, t\}$ is an induced forest in $G$, because $b$ divides the edge $y_1y_2$ if both $y_1, y_2$ are in $F$, and otherwise $b$ has at most one neighbor in $F$, and there is no possible path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest with size at least $\alpha(n-4) - \beta(m-13) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 7\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If $y_1y_2 \in E(G)$, then at least one of $y_1, y_2$ has at least two neighbors not in $\{a, b, c, t\}$: since the cycles $atc, btc, atb, abc$ divide the plane into four pieces bordered with one of these cycles and any point in one of the pieces can only be neighbors with three of $\{a, b, c, t\}$, if one of $y_1, y_2$ has degree at least 5, they have at least two neighbors not in $\{a, b, c, t\}$. If both $y_1$ and $y_2$ have degree 4, then both of them cannot be neighbors with all three of the vertices of the cycle bordering the piece of the plane they are in (they are in the same piece, because $y_1y_2 \in E(G)$), otherwise the subgraph induced by $y_1, y_2$ and the three vertices of the bordering cycle would have 5 vertices and 10 edges, which is not planar ($10 > 3*5-6=9$), hence one of them is neighbors with at most two of $\{a,b,c,t\}$ and has two neighbors not in $\{a,b,c,t\}$.

Assume w.l.o.g. that $y_1$ has at least two neighbors not in $\{a,b,c,t\}$. Let $F$ be a maximal induced forest in the graph $G-\{a,b,c,t,y_1\}$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $y_1$ has two neighbors not in $\{a,b,c,t\}$. Then $F+\{b,t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$, and there is no path between $b$ and $d$ in $F+\{b\}$. Hence $G$ has an induced forest with size at least $\alpha(n-5)-\beta(m-16) = \alpha n - \beta m + (1-(2\alpha-3\beta)) + (1-(3\alpha-13\beta))$, because $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

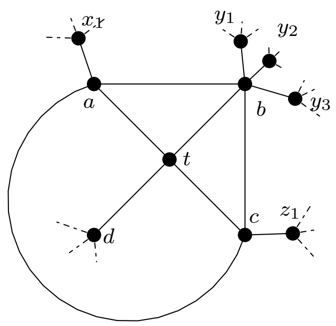



Figure 12: A Type I 4-vertex with $d(b)=6$ and $d(a)=d(c)=4$

*Case* 6. The degree of $b$ is at least 6, and $d(a)=d(c)=4$). Let $x_1$ be the neighbor of $a$ that is not $b,c,t$, and let $z_1$ the the neighbor of $c$ that is not $a,b,t$ (Figure 12). If $x_1, z_1$ are both inside the cycle $act$, then $b$ is a cut-vertex, which contradicts Lemma 4.3. Assume w.l.o.g. that $x_1$ is outside the cycle $act$. Let $F$ be a maximal induced forest in the graph $G-\{a,b,c,t\}$, then $|F| \geqslant \alpha(n-4)-\beta(m-12)$, and $F+\{a,t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$, and there is no possible path between $a$ and $d$ in $F+\{a\}$, because $x_1$ is outside the cycle $act$. Hence $G$ has an induced forest with size at least $\alpha(n-4)-\beta(m-12)+2 = \alpha n - \beta m + 2(1-(2\alpha-3\beta)) + 6\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

*Case* 7. The degree of $b$ is 6, and $d(a)+d(c)=9$ (assume w.l.o.g. $d(a)=4$ and $d(c)=5$. Let the neighbor of $a$ not in $\{b,c,t\}$ be $x_1$, the three neighbors of $b$ not in $\{a,c,t\}$ be $y_1,y_2,y_3$ and the two neighbors of $c$ be $z_1,z_2$ (Figure 13).

If $x_1$ is outside the cycle $act$, let $F$ be a maximal induced forest in the graph $G-\{a,b,c,t\}$, then $|F| \geqslant \alpha(n-4)-\beta(m-13)$, and $F+\{a,t\}$ is an induced forest

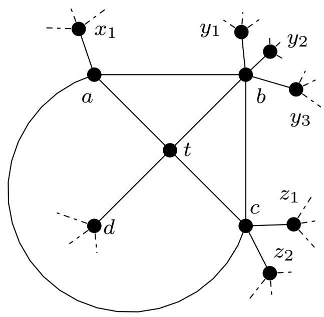



Figure 13: A Type I 4-vertex with $d(b) = 6$, $d(a) = 4$ and $d(c) = 5$

in $G$, because $a$ has at most one neighbor in $F$, and there is no path between $a$ and $d$ in $F + \{a\}$, because $x_1$ is outside the cycle $act$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-13) + 2 = \alpha n - \beta m + 2(1-(2\alpha-3\beta)) + 7\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If $x_1$ is inside the cycle $act$, but $x_1 \neq d$, then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $x_1d, y_1x_1, y_1d, y_2y_3, y_2d, y_3d$ added (since $x_1$ is inside and $y_1, y_2, y_3$ are outside the cycle $act$, $x_1 \notin \{y_1, y_2, y_3\}$). $G_1$ is planar, because after the deletion of $\{a, b, c, t\}$, $y_1, y_2, y_3, x_1, d$ all belong to the same face, and $x_1, d$ are in a separate component from $y_1, y_2, y_3$. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-7)$, because we added at most 6 edges. Since $G_1$ contains the triangles $y_1x_1d$ and $y_2y_3d$, either $d$ or $x_1$ are not in the forest, or at least two of $y_1, y_2, y_3$ are not in $F$. If $d \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$ and $t$ has one neighbor in $F + \{a\}$. If $x_1 \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because $t$ has at most one neighbor in $F$ and $a$ has one neighbor in $F + \{t\}$. If at least two of $y_1, y_2, y_3$ are not in $F$, then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-7) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + \beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If $x_1 = d$, then if $z_1, z_2$ are both inside the cycle $act$, then $b$ would be a cut-vertex. Assume w.l.o.g. that $z_1$ is outside the cycle $act$. If $z_2 = d$, then $d$ is a cut-vertex (since $d(d) \geqslant 4$, $d$ has at least one neighbor not in $\{a, c, t\}$, and this neighbor would not have any connection to $a, c, t$ after $d$ is removed). If $z_2$ is also outside the cycle $act$, then $d$ is still a cut-vertex. If $z_2 \neq d$ and $z_2$ is inside the cycle $act$, then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $y_1y_2, y_1d, y_2d, y_3d, y_3z_2, z_2d$

added (if necessary, renumber the $y$ vertices so that $y_1, y_2, d$ would remain in the same face after adding the edge $y_3z_2$). Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-7)$, because we added at most 6 edges. Since $G_1$ contains the triangles $y_1y_2d$ and $y_3z_2d$, either $d$ or $z_2$ are not in the forest, or two of $y_1, y_2, y_3$ are not in $F$. If $d \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because neither $a$ nor $t$ have any neighbors in $F$. If $z_2 \notin F$, then $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most one neighbor in $F$, and there is no path between $c$ and $d$ in $F + \{c\}$, since $z_1$ is outside the cycle $act$. If two of $y_1, y_2, y_3$ are not in $F$, then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-7) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + \beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

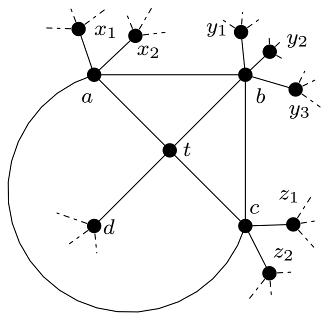



Figure 14: A Type I 4-vertex with $d(6) = 5$ and $d(a) = d(c) = 5$

*Case* 8. The degree of $b$ is 6, and $d(a) = d(c) = 5$. Let the two neighbors (not in $\{a, b, c, t\}$) of $a$ be $x_1, x_2$, the three neighbors of $b$ be $y_1, y_2, y_3$, and the two neighbors of $c$ be $z_1, z_2$ (Figure 14).

If all of $x_1, x_2, z_1, z_2$ are inside the cycle $act$, then $b$ would be a cut-vertex, which are not in $G$ according to Lemma 4.3. If one of the pairs $\{x_1, x_2\}$, $\{z_1, z_2\}$ has one of the pair inside and one outside the cycle $act$ (assume w.l.o.g. that $x_1$ is outside and $x_2$ inside the cycle $act$), then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $y_1y_2, y_1y_3, y_2y_3, y_1d, y_2d, y_3d$ added (this is possible while retaining planarity because $d$ and $y_1, y_2, y_3$ are in separate components in $G_1$, so it is possible to go "around" the connected component containing $d$ to add the third edge between $y$-vertices). Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-8)$, since we removed 14 edges and added at most 6. Since the vertices $y_1, y_2, y_3, d$ have all possible edges between them in $G_1$, $F$ can contain at most two of them (any three would induce a cycle). If $d \notin F$, $F+\{a,t\}$ is an induced forest in $G$, because $a$ has at most two neighbors in $F$ with no possible path between them (because $x_1$ and $x_2$ are on different sides of the cycle $act$), and $t$ has one neighbor in $F+\{a\}$. If $d \in F$, then at least two of $y_1, y_2, y_3$ are not in the forest, and $F+\{b,t\}$ is an induced forest in $G$, since $b$ has at most one neighbor in $F$ and there is no path between $b$ and $d$ in $F+\{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4)-\beta(m-8)+2 = \alpha n - \beta m + 2(1-(2\alpha-3\beta)) + 2\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If neither pair $\{x_1, x_2\}$, $\{z_1, z_2\}$ has one of the pair inside and one outside the cycle $act$, then one pair is inside the cycle $act$ and one pair is outside (both pairs cannot be outside, otherwise $d$ would be a cut-vertex). Assume w.l.o.g. that $x_1, x_2$ are outside the cycle $act$. Let $F$ be a maximal induced forest in the graph $G - \{a,b,c,t,x_2\}$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $d(x_2) \geqslant 4$ and $x_2$ is not neighbors with $t$ or $c$ (because $z_1, z_2$ are inside the cycle $act$, and $x_2$ is outside). Then $F+\{a,t\}$ is an induced forest in $G$, since $a$ has at most one neighbor in $F$ and there is no possible path between $a$ and $d$ in $F+\{a\}$, since $x_1$ is outside the cycle $act$. Hence $G$ has an induced forest of size at least $\alpha(n-5)-\beta(m-16)+2 = \alpha n - \beta m + (1-(2\alpha-3\beta)) + (1-(3\alpha-13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

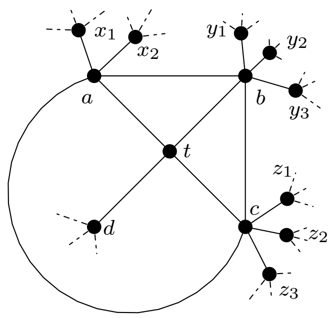



Figure 15: A Type I 4-vertex with $d(b) = 6$, $d(a) = 5$ and $d(c) = 6$

*Case* 9. The degree of $b$ is 6, and $d(a)+d(c) = 11$ (assume w.l.o.g. $d(a) = 5, d(c) = 6$). Let the two neighbors (not in $\{a,b,c,t\}$) of $a$ be $x_1, x_2$, the three neighbors of $b$ be $y_1, y_2, y_3$, and the three neighbors of $c$ be $z_1, z_2, z_3$ (Figure 15).

If $x_1$ and $x_2$ are both outside the cycle $act$, let $F$ be a maximal induced forest in the graph $G - \{b, c, t\}$. Then $|F| \geqslant \alpha(n-3) - \beta(m-13)$ and $F + \{t\}$ is an induced forest in $G$, since there is no path between $a$ and $d$ in $F$ (because both $x_1$ and $x_2$ are outside the cycle $act$). Hence $G$ has an induced forest of size $\alpha(n-3) - \beta(m-13) + 1 \geqslant \alpha n - \beta m$, since $3\alpha - 13\beta \leqslant 1$.

If $x_1, x_2$ are on different sides of the cycle $act$ (assume w.l.o.g. $x_1$ is outside), then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $y_1y_2, y_1y_3, y_2y_3, y_1d, y_2d, y_3d$ added (this is possible while retaining planarity because $d$ and $y_1, y_2, y_3$ are in separate components in $G_1$, so it is possible to go "around" the connected component containing $d$ to add the third edge between $y$-vertices). Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-9)$, since we removed 15 edges and added at most 6. Since the vertices $y_1, y_2, y_3, d$ have all possible edges between them in $G_1$, $F$ can contain at most two of them (any three would induce a cycle). If $d \notin F$, $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most two neighbors in $F$ with no possible path between them (because $x_1$ and $x_2$ are on different sides of the cycle $act$), and $t$ has one neighbor in $F + \{a\}$. If $d \in F$, then at least two of $y_1, y_2, y_3$ are not in the forest, and $F + \{b, t\}$ is an induced forest in $G$, since $b$ has at most one neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-9) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 3\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$. Hence $x_1, x_2$ are both inside the cycle $act$.

If one of the edges between $y_1, y_2, y_3$ is not in $G$ (assume w.l.o.g. that $y_1y_2 \notin E(G)$, then let $G_1$ be the graph $G - \{a, b, c, t, y_3\}$ with the edge $y_1y_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $d(y_3) \geqslant 4$ and $y_3$ is not neighbors with $t$ and $a$ (because $x_1, x_2$ are both inside the cycle $act$, and $y_3$ is outside), so we removed at lest 17 edges, and added 1. Then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most two neighbors in $F$, and if both $y_1, y_2 \in F$, then adding $b$ is equivalent to dividing the edge $y_1y_2$, and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$. Hence all three edges $y_1y_2, y_1y_3, y_2y_3$ are in $G$ (Figure 16).

If $d \in \{x_1, x_2\}$, then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $y_1d, y_2d, y_3d$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-12)$, because we removed 15 edges and added 3. Since $y_1, y_2, y_3, d$ have all possible edges between them in $G_1$, $F$ contains at most two of them. If $d \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$, and $t$ has one neighbor in $F + \{a\}$. If $d \in F$ then at least two of $y_1, y_2, y_3$ are not in $F$, so $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no possible path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-12) + 2 = \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 6\beta \geqslant \alpha n - \beta m$,

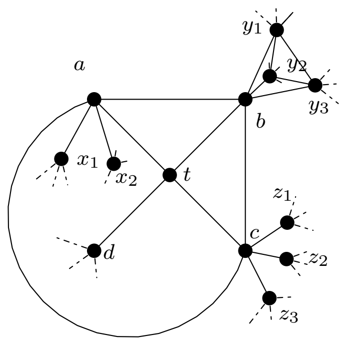



Figure 16: $x_1$ and $x_2$ are inside the cycle *act* and all three edges $y_1y_2, y_1y_3, y_2y_3$ are in $G$

since $2\alpha - 3\beta \leqslant 1$. Hence $d \notin \{x_1, x_2\}$.

If $x_1x_2 \notin E(G)$, then let $G_1$ be the graph $G - \{a, b, c, d, t\}$ with the edge $x_1x_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $d(d) \geqslant 4$ and $d$ is not neighbors with $a$ or $b$, so we removed at least 17 edges, and added one. Then $F + \{a, t\}$ is an induced forest in $G$, because adding $a$ is equivalent to dividing the edge $x_1x_2$ in $F$ if both $x_1, x_2$ are in $F$, and otherwise, $a$ has only one neighbor in $F$, and $t$ has only on neighbor in $F + \{a\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$. Hence $x_1x_2 \in E(G)$.

If $z_1, z_2, z_3$ are all outside the cycle *act* and at least one edge between them is not in $G$ (assume w.l.o.g. $z_1z_2 \notin E(G)$), then let $G_1$ be the graph $G - \{a, b, c, t, z_3\}$ with the edge $z_1z_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $d(z_3) \geqslant 4$ and $z_3$ is not neighbors with $t$ and $a$ (because $x_1, x_2$ are both inside the cycle *act*, and $z_3$ is outside), so we removed at lest 17 edges, and added 1. Then $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most two neighbors in $F$, and if both $z_1, z_2 \in F$, then adding $c$ is equivalent to dividing the edge $z_1z_2$, and there is no path between $c$ and $d$ in $F + \{c\}$, because $z_1, z_2, z_3$ are all outside the cycle *act*. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

If $z_1, z_2, z_3$ are all outside the cycle *act* and all the edges between them are in $G$, then

- if $\{z_1, z_2, z_3\} = \{y_1, y_2, y_3\}$, then the subgraph induced by $\{b, c, z_1, z_2, z_3\}$

would have 5 vertices and 10 edges, which is not planar.

- if $\{z_1, z_2, z_3\}$ and $\{y_1, y_2, y_3\}$ have two elements in common, assume w.l.o.g. that $z_1 = y_1, z_2 = y_2$. Let $G_1$ be the graph $G - \{a, b, c, t, z_1\}$ with the edge $z_3y_3$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-5) - \beta(m-17)$, because we removed 15 edges by removing $\{a, b, c, t\}$ and the three edges $z_1y_3, z_1z_2, z_1z_3$, and added one. Since $G_1$ contains the triangle $z_3y_3z_2$, not all of them are in $F$. If either $z_3$ or $z_2$ is not in $F$, then $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most one neighbor in $F$, and there is no path between $c$ and $d$ in $F + \{c\}$, since all the $z$ vertices are outside the cycle $act$. If $y_3$ is not in $F$, then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-17) + 2 = \alpha n - \beta m + (1-(2\alpha - 3\beta)) + (1-(3\alpha - 13\beta)) + \beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

- if $\{z_1, z_2, z_3\}$ and $\{y_1, y_2, y_3\}$ have one element in common, assume w.l.o.g. that $z_1 = y_1$. Let $G_1$ be the graph $G - \{a, b, c, t, z_1\}$ with the edges $z_2y_2, z_3y_3$ added (if necessary, swap the labels of $y_2, y_3$ so the edges do not cross in the planar embedding). Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-5) - \beta(m-17)$, because we removed 15 edges by removing $\{a, b, c, t\}$ and the four edges $z_1y_2, z_1y_3, z_1z_2, z_1z_3$, and added two. Since $G_1$ contains the cycle $y_2y_3z_3z_2$, not all of them are in $F$. If either $z_2$ or $z_3$ is not in $F$, then $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most one neighbor in $F$, and there is no path between $c$ and $d$ in $F + \{c\}$, since all the $z$ vertices are outside the cycle $act$. If either $y_2$ or $y_3$ is not in $F$, then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-17) + 2 = \alpha n - \beta m + (1-(2\alpha - 3\beta)) + (1-(3\alpha - 13\beta)) + \beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

- if $\{z_1, z_2, z_3\}$ and $\{y_1, y_2, y_3\}$ have no elements in common, then at least one of $z_1, z_2, z_3$ has degree at least 5 (if they all had degree 4, we would have either Case 2 or Case 6 for any of the $z$ vertices). Assume w.l.o.g. $d(z_1) \geqslant 5$. Let $G_1$ be the graph $G - \{a, b, c, t, z_1\}$ with the edges $z_2y_1, z_2y_2, z_2y_3$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $z_1$ is not neighbors with $a, b, t$, so we removed at least 19 edges, and added at most 3. Since $G_1$ contains all possible edges between $z_2, y_1, y_2, y_3$, at most two of these are in $F$. If $z_2 \notin F$, then $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most one neighbor in $F$, and there is no path between $c$ and $d$ in $F + \{c\}$, since $z_3$ is outside the cycle $act$. If two of $y_1, y_2, y_3$ are not in $F$, then $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most one neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

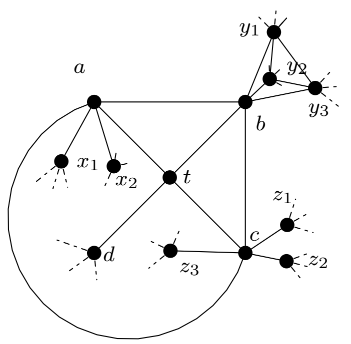



Figure 17: $z_1, z_2$ are outside the cycle $act$ and $z_3$ is inside

If two of $z_1, z_2, z_3$ are outside the cycle $act$ and one is inside (assume w.l.o.g. that $z_1, z_2$ are outside and $z_3$ is inside, see Figure 17), then let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $y_1d, y_2d, y_3d, x_1x_2, x_2z_2, x_1z_1, z_1z_2$ added (if necessary, swap the labels of $x_1, x_2$, so the edges of the cycle $x_1x_2z_2z_1$ would not cross).Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-8)$, because we removed at least 15 edges and added at most 7. Since $G_1$ contains all the possible edges between $y_1, y_2, y_3, d$, at most two of these are in $F$, and since $G_1$ contains the cycle $x_1x_2z_2z_1$ (since $x_1, x_2 \notin \{z_1, z_2\}$, because they are on different sides of the cycle $act$, we actually do have a cycle), at least one of these is not in $F$. If $d$ is not in $F$ and either $x_1$ or $x_2$ is not in $F$, then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$ and $t$ has one neighbor in $F + \{t\}$. Similarly, if $d$ is not in $F$ and one of $z_1, z_2$ is not in $F$, $F + \{c, t\}$ is an induced forest in $G$. If $d \in F$, then at least two of $y_1, y_2, y_3$ are not if $F$, and so $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most on neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-8) + 2 \geqslant \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 2\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If two of $z_1, z_2, z_3$ are inside the cycle $act$ and one is outside (assume w.l.o.g. that $z_1, z_2$ are inside and $z_3$ is outside, see Figure 18), then

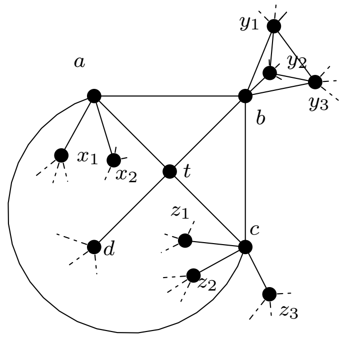



Figure 18: $z_1, z_2$ are inside the cycle *act* and $z_3$ is outside

- if $\{z_1, z_2\} = \{x_1, x_2\}$, let $F$ be a maximal induced forest in the graph $G - \{b, c, t\}$. Then $|F| \geqslant \alpha(n-3) - \beta(m-13)$. If $z_1, z_2$ are both in $F$, then $a$ is not in $F$, because $x_1x_2 = z_1z_2 \in E(G)$, and hence $F + \{t\}$ is an induced forest in $G$, because $t$ has at most one neighbor in $F$. If one of $z_1, z_2$ is not in $F$, then $F + \{c\}$ is an induced forest in $G$, because $c$ has at most two neighbors on different sides of the cycle *act* in $F$. Hence $G$ has an induced forest of size at least $\alpha(n-3)-\beta(m-13)+1 \geqslant \alpha n - \beta m$, since $3\alpha - 13\beta \leqslant 1$.

- if at least one of $x_1, x_2$ is not a neighbor of $c$ (assume w.l.o.g. $x_1 \notin \{z_1, z_2\}$), let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $y_1d, y_2d, y_3d, x_1z_1, x_1z_2, z_1z_2$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-9)$, because we removed 15 edges and added ad most 6. Since $G_1$ contains all possible edges between $y_1, y_2, y_3, d$, at most two of these are in $F$, and since $G_1$ contains the triangle $x_1z_1z_2$, at least one of these is not in $F$. If $d \notin F$ and $x_1 \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$ and $t$ has one neighbor in $F + \{a\}$. If $d \notin F$ and one of $z_1, z_2$ not in $F$, then $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most two neighbors that are on different sides of the cycle *act* in $F$, and $t$ has at most one neighbor in $F + \{c\}$. If $d \in F$, then at least two of $y_1, y_2, y_3$ are not if $F$, and so $F + \{b, t\}$ is an induced forest in $G$, because $b$ has at most on neighbor in $F$ and there is no path between $b$ and $d$ in $F + \{b\}$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-9) + 2 \geqslant \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 3\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

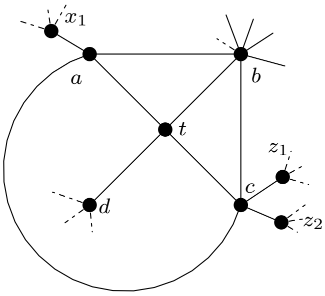



Figure 19: A Type I 4-vertex with $d(b) \geqslant 7$, $d(a) = 4$ and $d(c) = 5$

*Case* 10. The degree of $b$ is at least 7, one of $a, c$ has degree 4 (assume w.l.o.g. $a$), and the other has degree 5. Let the neighbor of $a$ that is not $b, c, t$ be $x_1$, and let the two neighbors of $c$ that are not $a, b, t$ be $z_1, z_2$ (Figure 19).

If $x_1, z_1, z_2$ were all inside the cycle $act$, $b$ would be a cut-vertex, which contradicts Lemma 4.3.

If $x_1$ is outside the cycle $act$, let $F$ be a maximal induced forest in the graph $G - \{a, b, c, t\}$. Then $|F| \geqslant \alpha(n-4) - \beta(m-14)$ and $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $G$, and there is no path between $a$ and $d$ on $F + \{a\}$, since $x_1$ is outside the cycle $act$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-14) + 2 \geqslant \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 8\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

If one of $z_1, z_2$ is inside the cycle $act$ and one outside (assume w.l.o.g. that $z_1$ is inside and $z_2$ is outside), then if $z_1 = x_1 = d$, then $d$ is a cut-vertex, so $d$ is neighbors with at most one of $a, c$. Let $F$ be a maximal induced forest in $G - \{a, b, c, d, t, \}$, then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $d(d) \geqslant 4$ and $d$ has at most two neighbors in $\{a, b, c, t\}$. Then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$, and $t$ has one neighbor in $F + \{a\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

If both $z_1, z_2$ are outside the cycle $act$, let $G_1$ be the graph $G - \{a, b, c, t\}$ with the edges $z_1 d, z_2 d, z_1 z_2$ added ($d \notin \{z_1, z_2\}$, because $z_1, z_2$ are outside the cycle $act$). Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-4) - \beta(m-11)$, because we removed 14 edges and added at most 3. Since $G_1$ contains the triangle $z_1 z_2 d$, not all of them are in $F$. If $d \notin F$, then $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$, and $t$ has one neighbor in $F + \{a\}$.

If one of $z_1, z_2$ not in $F$, then $F + \{c, t\}$ is an induced forest in $G$, because $c$ has at most one neighbor in $F$ and there is no path between $c$ and $d$ in $F + \{c\}$, because $z_1, z_2$ are outside the cycle $act$. Hence $G$ has an induced forest of size at least $\alpha(n-4) - \beta(m-11) + 2 \geqslant \alpha n - \beta m + 2(1 - (2\alpha - 3\beta)) + 5\beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$.

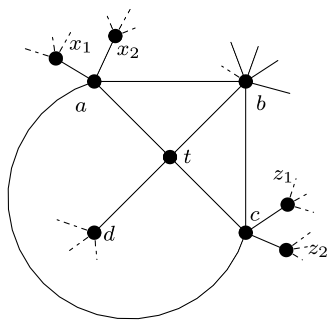



Figure 20: A Type I 4-vertex with $d(b) \geqslant 7$ and $d(a) = d(c) = 5$

*Case* 11. The degree of $b$ is at least 7, and $d(a) = d(c) = 5$. Let the two neighbors of $a$ that are not $b, c, t$ be $x_1, x_2$, and let the two neighbors of $c$ that are not $a, b, t$ be $z_1, z_2$ (Figure 20).

If $x_1, x_2, z_1, z_2$ were all inside the cycle $act$, $b$ would be a cut-vertex, which contradicts Lemma 4.3.

If at least one of them is outside the cycle $act$ (assume w.l.o.g. $x_1$), then let $F$ be a maximal induced forest in $G - \{a, b, c, t, x_2\}$. Then $|F| \geqslant \alpha(n-5) - \beta(m-16)$, because $d(x_2) \geqslant 4$ and $x_2$ has at most three neighbors in $\{a, b, c, t\}$, and $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$, and there is no possible path between $a$ and $d$ in $F + \{a\}$, because $x_1$ is outside the cycle $act$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-16) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

*Case* 12. The degree of $b$ is at least 7, and $d(a) + d(c) = 11$ (assume w.l.o.g. $d(a) = 5, d(c) = 6$). Let the two neighbors of $a$ that are not $b, c, t$ be $x_1, x_2$ (Figure 21).

If $d \in \{x_1, x_2\}$, let $F$ be a maximal induced forest in $G - \{a, b, c, d, t\}$. Then $|F| \geqslant \alpha(n-5) - \beta(m-17)$, because $d(d) \geqslant 4$, and $F + \{a, t\}$ is an induced forest in $G$, because $a$ has at most one neighbor in $F$, and $t$ has at most one neighbor in

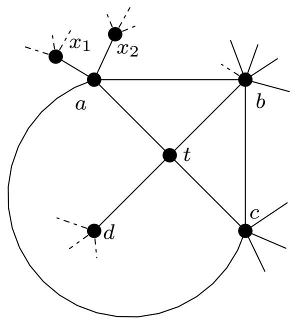



Figure 21: A Type I 4-vertex with $d(b) \geqslant 7$, $d(a) = 5$ and $d(c) = 6$

$F + \{a\}$. Hence $G$ has an induced forest of size at least $\alpha(n-5) - \beta(m-17) + 2 = \alpha n - \beta m + (1 - (2\alpha - 3\beta)) + (1 - (3\alpha - 13\beta)) + \beta \geqslant \alpha n - \beta m$, since $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$.

If $d \notin \{x_1, x_2\}$, then $ad \notin E(G)$. Let $G_1$ be the graph $G - \{b, c, t\}$ with the edge $ad$ added. Let $F$ be a maximal induced forest in $G_1$, then $|F| \geqslant \alpha(n-3) - \beta(m-13)$, because we removed 14 edges and added one. Then $F + \{t\}$ is an induced forest in $G$, because if $a, d$ are both in $F$, adding $t$ is equivalent to dividing the edge $ad$ in $F$, and otherwise $t$ has at most one neighbor in $F$. Hence $G$ has an induced forest of size at least $\alpha(n-3) - \beta(m-13) + 1 \geqslant \alpha n - \beta m$, since $3\alpha - 13\beta \leqslant 1$.

□

In summary, we have proved that if the constants $\alpha$ and $\beta$ satisfy the constraints $\alpha - 3\beta > \frac{2}{5}$, $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$ (it is possible to find such $\alpha, \beta$, e.g. $\alpha = \frac{10}{17}, \beta = \frac{1}{17}$), then any graph $G$ with the minimum number of vertices that does not have an induced forest of size $\alpha|V(G)| - \beta|E(G)|$, then $G$ does not contain cut-vertices or bridges, its minimum degree is at least 4 and any 4-vertex must be a Type II 4-vertex.

# 5 Conclusion

In this work, we gave an overview of the current results concerning the size of induced forests in planar graphs. We also described the method we are trying to use to improve the current best known bound, the discharging, method, and presented a summary of an article by Salavatipour ([Sal06]) to illustrate its use in a related context. Lastly, we proved that if non-negative real numbers $\alpha, \beta$ satisfy the inequalities $\alpha - 3\beta > \frac{2}{5}$, $2\alpha - 3\beta \leqslant 1$ and $3\alpha - 13\beta \leqslant 1$, and $G$ is a graph with the minimal number of vertices such that it does not contain an induced forest of size at least $\alpha|V(G)| - \beta|E(G)|$, then the minimum degree of $G$ is at least 4, $G$ is connected, contains no bridges or cut-vertices, and for every 4-vertex $t$ in $G$ with the neighbors $a, b, c, d$ in clockwise order (in the embedding), either $ab, cd \in E(G)$ or $ad, bc \in E(G)$ (or both).

To continue our proof, the next step would be to finish proving that the minimal counterexample contains no 4-vertices, and then consider possible properties of 5-vertices. If we would manage to prove that there are no 5-vertices in the counterexample either, then the proof would be complete, because there is no planar graph with minimum degree 6. Otherwise, we might try to prove that every 5-vertex belongs to at least some 4-faces, or that 5-vertices cannot be neighbors, etc.

# 6 Suured indutseeritud metsad tasandilistes graafides

Magistritöö(30 EAP)
Kairi Hennoch
Resümee

Tasandilised graafid on graafid, mida saab joonistada tasandile nii, et nende servad lõikuksid vaid nende tippudes. Ehk kõige loomulikumalt tekivad tasandilised graafid siis, kui mõelda (maa)kaartidest, näiteks ristmikud linnas ja neid ühendavad tänavad, metrooliinid ja peatused, või riigid ühendatuna naaberriikidega.

Käesolevas töös uuritakse tasandiliste graafide ühte omadust, nimelt neis leiduvate indutseeritud metsade (indutseeritud alamgraafide, milles pole ühtegi tsüklit) suurust. Aastal 1979 püstitasid Albertson ja Berman hüpoteesi ([AB79]), mille kohaselt igas tasandilises graafis on võimalik leida indutseeritud mets, mis sisaldab vähemalt pooli graafi tippudest. See hüpotees on seni tõestamata, seni parima tulemuse kohaselt on tõestatud ([Bor79]), et igas tasandilises graafis leidub indutseeritud mets, mis sisaldab vähemalt $\frac{2}{5}$ selle graafi tippudest. Käesolev töö on motiveeritud soovist seda tulemust parandada.

Töös esitatakse kõigepealt ülevaade antud probleemi tagamaast ja eelnevatest tulemustest. Seejärel kirjeldatakse meetodit, mida me soovime oma tulemuse tõestamiseks kasutada, ja esitatakse selle näide olemasoleva artikli ([Sal06]) baasil. Töö viimases osas näidatakse kõigepealt, et kui mittenegatiivsed reaalarvud $\alpha, \beta$ rahuldavad tingimust $\alpha - 3\beta > \frac{2}{5}$, siis olemasoleva tulemuse parandamiseks piisab näidata, et igas tasandilises graafis, millel on $n$ tippu ja $m$ serva, leidub indutseeritud mets, milles on vähemalt $\alpha n - \beta m$ tippu. Seejärel tõestatakse, et minimaalses vastunäites, s.t. minimaalse tippude arvuga graafis $G$, millel on $n$ tippu ja $m$ serva, aga milles ei leidu indutseeritud metsa suurusega vähemalt $\alpha n - \beta m$, ei ole tippe, mille aste oleks väiksem kui 4, ja ühelgi tipul astmega 4 ei leidu kolme naabrit, mille vahel oleks kõik kolm serva.

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