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STABILITY OF THE SPLINE COLLOCATION METHOD FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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CONTENTS

Chapter 1

INTRODUCTION

1.1 A brief history of Volterra integro-differential equations

The theory of integral equations has been an active research field for many years and is based on analysis, function theory, and functional analysis.

An application arises on considering population dynamics involving a gestation period. Immune response and the heart-lung mechanism provides examples from medicine. The control of a satellite from an earth-based control system provides another example. Another application area is economics.

The theory of integral equations is interesting not only in itself, but its results are essential for the analysis of numerical methods. Besides existence and uniqueness statements, the theory concerns, in particular, questions of regularity and stability.

An integral equation is a functional equation in which the unknown function appears under one or several integral signs; if, in addition, the equation contains a derivative of this function we call the equation an integro-differential equation. In an integral or integro-differential equation of Volterra type the integrals containing the unknown function are characterized by a variable upper limit of integration. To be more precise, let $I := [0, T]$ denote a given closed and bounded interval, with $0 < T$, and set $S := \{(t, s) : 0 \le s \le t \le T\}$.

The functional equation (for the unknown function y) of the form

$$y'(t) = F(t, y(t), z(t)), \quad t \in I,$$

with

$$z(t) = \int_0^t \mathcal{K}(t, s, y(t))ds$$

is called a *first order Volterra integro-differential equation.* Here, one usually looks for a solution which satisfies the initial condition $y(0) = y_0$.

The name "Volterra integral equation" was first coined by Rumanian mathematician Traian Lalesco in 1908, seemingly following a suggestion by his teacher French mathematician Emile Picard. The terminology "integral equation of the first (second, third) kind" was first used by German mathematician David Hilbert in connection with his study of Fredholm integral equations, while the name "integral equation" is due to German mathematician Paul Du Bois-Reymond.

The origins of the quantitative theory of integral equations with variable (upper) limits of integration go back to the early 19th century. Norwegian mathematician Niels Hendrik Abel in his works in 1823 and in 1826 considered the problem of determining the equation of a curve in a vertical plane such that the time taken by a mass point to slide, under the influence of gravity, along this curve from a given positive height to the horizontal axis is equal to a prescribed (monotone) function of the height. He showed that this problem can be described by a first kind integral equation of the form

$$\int_0^t (t-s)^{-\alpha} y(s)ds = g(t), \quad t < 0, \tag{1.1}$$

with $\alpha = 1/2$, and then he proved that, for any $\alpha \in (0,1)$, the solution of (1.1) is given by the "inversion formula",

$$y(t) = c_\alpha \frac{d}{dt}\left\{\int_0^t (t-s)^{-\alpha-1} g(s)ds\right\}, \quad t < 0, \tag{1.2}$$

with $c_\alpha = \sin(\alpha\pi)/\pi = 1/(\Gamma(\alpha)\Gamma(1-\alpha))$.

Three years after Abel's death, in 1832, the problem of inverting (1.1) was also studied by French mathematician Joseph Liouville (who seems to have been unaware of Abel's work), again in a purely formal manner. The discovery of the inversion formula (1.2) was the starting point for the systematic development of what is known now as *Fractional Calculus.*

In 1896 Italian mathematician Vito Volterra published his general theory of the inversion of first kind integral equation. He transformed

$$\int_0^t \mathcal{K}(t,s)y(s)ds = g(t), \quad t \in T, \quad g(0) = 0 \tag{1.3}$$

into an integral equation of the second kind whose kernel and forcing functions are, respectively,

$$\mathcal{K}(t,s) = -\frac{\partial \mathcal{K}(t,s)}{\partial t} \quad \text{and} \quad \widetilde{g}(t) = \frac{g'(t)}{\mathcal{K}(t,t)}.$$

If $\mathcal{K}(t,t)$ does not vanish on I, and if the derivates of $\mathcal{K}$ and g are continuous, then the (unique) solution of (1.3) is given by the "inversion formula"

$$y(t) = g(t) + \int_0^t \widetilde{\mathcal{R}}(t,s)g(s)ds, \quad t \in I.$$

Here, $\widetilde{\mathcal{R}}(t,s)$ denotes the so-called *resolvent kernel* of $\widetilde{\mathcal{K}}(t,s)$; it is defined in terms of the iterated kernels $\widetilde{\mathcal{K}}_n(t,s)$ of $\widetilde{\mathcal{K}}(t,s)$,

$$\widetilde{\mathcal{K}}_n(t,s) = \int_0^t \widetilde{\mathcal{K}}(t,u)\widetilde{\mathcal{K}}_{n-1}(u,s)du, \quad n \geq 2, \quad \widetilde{\mathcal{K}}_1(t,s) = \widetilde{\mathcal{K}}(t,s).$$

Volterra proved that the sequence $\widetilde{\mathcal{K}}_n$ converges absolutely and uniformly on S for any kernel $\widetilde{\mathcal{K}}$ in (1.3).

Even though Volterra's result was new, his way of attack was not entirely a novel one. In his thesis in 1894, French mathematician Joel Le Roux had already studied the problem of inverting the "definite integral" (1.3), using the same approach. But second kind integral equation with variable limit of integration occurred already in the work of French mathematician Joseph Liouville in 1837.

The notion of the iterated kernels and the associated "Neumann series" were first used by French mathematician Joseph Caqué in 1864. Generalizing Liouville's idea, he studied the solution of the $(p+1)$-st order linear differential equation

$$y^{(p+1)} = \sum_{j=0}^{p} A_j(t)y^{(j)} + A(t),$$

by rewriting the equation as a second kind integral equation of Volterra type with the kernel

$$\mathcal{K}(t,s) = \sum_{j=0}^{p} \frac{A_j(s)(t-s)^{p-j}}{(p-j)!}.$$

The existence of a solution was then established formally by introducing the iterated kernels and the corresponding Neumann series. At about the same time, in 1865, German mathematician August Beer used the same concepts, still in a purely formal way, in connection with the study of second kind integral equations with fixed limits of integration which arise in the analysis of Poisson's equation in Potential Theory. It was left to German mathematician Carl Gottfried Neumann to furnish the rigorous convergence analysis for the series of iterated kernels (associated with a second kind integral equation of Fredholm type), now named after him.

In another paper in the year of 1896, Volterra extended his idea to linear integral equation of the first kind with weakly singular kernels. Using the approach employed by Abel to establish the inversion formula (1.2), he showed that

$$\int_0^t (t-s)^{-\alpha}\mathcal{K}(t,s)y(s)ds = g(t), \quad t \in I, \quad 0 < \alpha < 1,$$

can be transformed into a first kind equation with regular kernel, to which the theory of his first work applies. The remaining two papers of Volterra from 1896 are concerned with the analysis of integral equation of the third kind.

The next forty years mainly a consolidation of Volterra's work took place. During this time the center stage belonged to the study of Fredholm integral equations and their implications for the development of Functional Analysis.

Since 1970 there has been renewed interest in study of qualitative and asymptotic properties of solutions of Volterra equations.

It is known that the Cauchy problem for ordinary differential equation is equivalent to a Volterra integral equation (VIE), the first order Volterra integro-differential equation(VIDE) can be written as VIE and the second order VIDE as first order VIDE. Thus, all approximate methods for solving VIDE could be applied to Cauchy problem and to integral equations as well.

The presented brief history of Volterra equations is mainly based on [7].

One of the most natural methods for solving VIE and VIDE is the standard step-by-step collocation method with polynomial splines. The collocation method with piecewise polynomials is well studied for different kind of equations under various assumptions and, as a rule, the convergence results are positive, see, e.g., [9]. General case of collocation method can be found in [7] and [20], see also [17].

Discretization methods in practical solving of differential and integral equations are applicable only if they are stable, which we will mean as the boundedness of approximate solutions when the number of knots increases. In general such stability is necessary for convergence and it is also sufficient in the case of a certain test equation. Convergence theory for collocation is well developed for polynomial splines without any continuity conditions in the knots or which are only continuous (see, e.g., [7], [9]). Let us mention that general convergence theorems with two-sided error estimates and stability analysis for VIDE are established in [26], see also [1]. They use orthogonal projectors in Hilbert spaces which is not the case for spline collocation methods.

Closely related problems have been investigated by several authors. The stability of the numerical solutions obtained when applying very general Runge Kutta methods to VIE and VIDE with degenerate kernels is analysed in [12]. The authors show that, under certain assumptions, the numerical solution is bounded; this is the numerical analogue of the boundedness of exact solution. The given results are generalizations of other results of the authors of [13] for exact collocation methods applied to this type of equation. Investigations of stability properties of exact and discretized collocation methods for VIDE with degenerate kernel is continued in [15]. Some linear stability results for the repeated spline-collocation method applied to the linear VIDE of first order is obtained in the paper [19]. For the stability condition introduced in [20] is proved that the repeated collocation method is stable for any choice of collocation parameters and degree of the spline function. Investigation of the convergence and the stability of collocation method for VIDE with weakly singular kernels can be found in [11]. Stability properties of reducible linear multistep methods and modified multilag methods, which are based on the test convolution equation is investigated in [6].

Using the Lyapunov method for solving VIDE, stability criterias are well studied. (see, e.g., [14] and [29]). Stability analysis of reducible quadrature methods for VIDE and necessary conditions for the method to be absolutely stable for given parameters of a test equation are derived in [10].

The authors of [5] consider the stability properties of certain integral equation type numerical methods when applied to the certain test equation. The simplest results are those obtained for a class of methods which may be derived on applying an appropriate method to a system of integral equations derived from the integro-differential equation. Results in [5] are similar to those obtained for integral equations in [4], from which they may be derived, and they are complementary to or consistent with earlier results of [8].

The first results about stability of the collocation method by polynomial splines for VIE are given in [21] and the most adequate ones seem to be in [24]. Investigation in [24] shows that in the case of piecewise polynomials (without continuity) the collocation method is stable for any order of spline and any choice of collocation parameters. Special case of smooth splines is treated in [25]. The most systematic attempt to study the numerical stability for VIDE seems to be [18]. It should be remarked that the proof of the main result of [18] (Theorem 2.3) is not correct. In [18] this Theorem 2.3 is also applied to the particular cases and there are obtained stability conditions. These results are disproved in our work.

The collocation with multiple collocation nodes coinciding with spline knots for the Cauchy problem of ordinary differential equations is studied in [23]. In particular, depending on order of the polynomial and multiplicity of the nodes, it is proved when the method is convergent and when divergent.

In the following we give a brief overview of the work by chapters. The present work consists of 8 chapters.

In present Chapter 1 we already gave an overview of history of integral equations. There is a standard reduction of 1st order VIDE to VIE considering the derivative of the solution as a new unknown solution. This connection between VIDE and VIE is shown in Section 1.2. There will be also shown that the certain test equation, which we use in studying the stability of collocation method, with constant kernel, transforms into an equation with nonconstant kernel and the results obtained for VIE are not directly extendable to the 1st order VIDE. Similar phenomena takes place if we try to reduce the problem of stability for 2nd order VIDE to that for 1st order VIDE.

In Chapter 2 the standard step-by-step spline collocation method is described.

In Chapter 3 we give a short overview of results containing numerical stability conditions of spline collocation method for VIE. In addition, some auxiliary results from Linear Algebra, which will be used in the sequel, is given in Section 3.2.

In Chapter 4 we show the connection between stability conditions for integral and 1st order integro-differential equations, when the splines to be used are at least continuous. In some cases we get explicit formulae showing the dependence of the stability on collocation parameters.

In Chapter 5 we investigate the numerical stability of the spline collocation method by piecewise polynomials for 1st order VIDE. In this special case we will see that there is also dependence on the parameters of a certain test equation.

Chapter 6 treats the numerical stability of the spline collocation method for 2nd order VIDE. We also show the connection between stability conditions for 1st order VIDE and the 2nd order VIDE.

Chapter 7 deals with investigations of stability of spline collocation method with multiple nodes for 1st order VIDE. We consider the collocation method with only one collocation point per subinterval of the grid, with given multiplicity.

There is also given some examples in several cases.

In Chapter 8, a series of numerical tests is given to support the theoretical results.

1.2 Connection with integral equations

In this section we will show the connection between linear Volterra integro-differential equations and Volterra integral equations.

Let us consider the linear integro-differential equation in the form

$$y'(t) = p(t)y(t) + q(t) + \int_0^t \mathcal{K}(t,s)y(s)ds, \quad t \in [0,T], \tag{1.4}$$

with initial condition

$$y(0) = y_0.$$

Here p, q and $\mathcal{K}$ are supposed to be real-valued and continuous on $[0, T]$ and S, respectively. Integration of (1.4) yields

$$y(t) = \int_0^t p(s)y(s)ds + \int_0^t q(s)ds + \int_0^t \int_0^\tau \mathcal{K}(\tau,s)y(s)dsd\tau + y_0, \quad t \in [0,T]. \tag{1.5}$$

Using the Dirichlet's formula which states

$$\int_0^t \int_0^\tau \Phi(\tau,s)dsd\tau = \int_0^t \int_s^t \Phi(\tau,s)d\tau ds, \quad (t,s) \in S,$$

provided the integral exists, we may rewrite equation (1.5) as

$$y(t) = \int_0^t q(s)ds + \int_0^t \left[p(s) + \int_s^\tau \mathcal{K}(\tau, s)d\tau \right] y(s)ds + y_0, \quad t \in [0, T],$$

or as

$$y(t) = g(t) + \int_0^t \mathcal{Q}(t, s)y(s)ds, \quad t \in [0, T], \tag{1.6}$$

where $g(t)$ and $\mathcal{Q}(t, s)$ are the functions

$$g(t) = y_0 + \int_0^t q(s)ds, \quad t \in [0, T],$$

and

$$\mathcal{Q}(t, s) = p(s) + \int_s^t \mathcal{K}(\tau, s)d\tau, \quad (t, s) \in S.$$

An alternative to this approach is to consider an integro-differential equation as a system of two Volterra integral equations of the second kind. For the linear case (1.4), let

$$z(s) := q(s) + \int_0^s \mathcal{K}(s, u)y(u)du, \quad s \in [0, T].$$

This allows us to rewrite (1.5) in the form

$$\begin{aligned} y(t) =& y_0 + \int_0^t p(s)y(s)ds + \int_0^t \left[q(s) + \int_0^s \mathcal{K}(s, u)y(u)du \right] ds \\ =& y_0 + \int_0^t p(s)y(s)ds + \int_0^t z(s)ds, \quad t \in [0, T]. \end{aligned}$$

Thus, the equation (1.4) is reduced to the system

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} y_0 \\ q(t) \end{pmatrix} + \int_0^t \begin{pmatrix} p(s) & 1 \\ \mathcal{K}(t, s) & 0 \end{pmatrix} \begin{pmatrix} y(s) \\ z(s) \end{pmatrix} ds, \quad t \in [0, T].$$

Example 1. Let us consider the first order VIDE having constant kernel

$$y'(t) = \alpha y(t) + \lambda \int_0^t y(s)ds + f(t), \quad t \in [0, T], \tag{1.7}$$

with $y(0) = y_0$. Equation (1.7) is called the basis test equation and it was suggested by Brunner and Lambert in 1974 (see [8]). It has been extensively used for investigating stability properties of several methods.

The transformation which we considered at the beginning of this section leads now to the equation (1.6), where

$$g(t) = y_0 + \int_0^t f(s)ds, \quad t \in [0, T],$$

and

$$\mathcal{Q}(t, s) = \alpha + \int_s^t \lambda d\tau = \alpha + \lambda(t - s), \quad (t, s) \in S.$$

Thus, equation (1.7) can be rewritten as

$$y(t) = g(t) + \int_0^t (\alpha + \lambda(t - s))y(s)ds, \quad t \in [0, T].$$

We see that the equation is not any more with a constant kernel, and later on when we will investigate stability, results obtained for VIE with a constant kernel are not extendable to the VIDE in form (1.7).

Let us now consider the second order Volterra integro-differential equation

$$y''(t) = p(t)y'(t) + q(t)y(t) + f(t) + \int_0^t \mathcal{K}(t, s)y(s)ds, \quad t \in [0, T], \quad (1.8)$$

$$y(0) = y_0,\ y'(0) = y_1$$

with p, q, f and $\mathcal{K}$ to be real-valued and continuous on $[0, T]$ and S, respectively. Integrating equation (1.8) and using Dirichlet's formula, we get

$$\begin{aligned} y'(t) &= \int_0^t p(s)y'(s)ds + \int_0^t q(s)y(s)ds + \int_0^t f(s)ds \\ &+ \int_0^t \int_0^\tau \mathcal{K}(\tau, s)y(s)dsd\tau + y_1 \\ &= \int_0^t p(s)y'(s)ds + \int_0^t f(s)ds \\ &+ \int_0^t \left[q(s) + \int_s^t \mathcal{K}(\tau, s)d\tau\right] y(s)ds + y_1, \quad t \in [0, T]. \end{aligned} \quad (1.9)$$

Assume, in addition, the continuous differentiability of p. Then, using integration by parts in

$$\int_0^t p(s)y'(s)ds = p(s)\left(y(s)+y_0\right)\Big|_0^t - \int_0^t \left(y(s)+y_0\right)p'(s)ds$$

$$= p(t)y(t) - p(0)y(0) + (p(t)-p(0))y_0 - \int_0^t \left(y(s)+y_0\right)p'(s)ds,$$

we obtain first order VIDE

$$y'(t) = p(t)y(t) + g(t) + \int_0^t \mathcal{Q}(t,s)y(s)ds, \quad t \in [0,T],$$

where

$$g(t) = -p(0)y(0) + \left(p(t)-p(0)\right)y_0 + y_1 + \int_0^t f(s)ds - \int_0^t y_0 p'(s)ds,\ t \in [0,T],$$

and

$$\mathcal{Q}(t,s) = q(s) - p'(s) + \int_s^t \mathcal{K}(\tau,s)d\tau, \quad (t,s) \in S.$$

An easier way is to present second order VIDE as a system consisting of two first order VIDEs. First, transform (1.8) to (1.9). Now taking $z(t) = y'(t)$, i.e.,

$$y(t) = \int_0^t z(s)ds + y_0$$

and setting

$$g(t) = \int_0^t f(s)ds + y_1, \quad t \in [0,T],$$

$$\mathcal{Q}(t,s) = q(s) + \int_s^t \mathcal{K}(\tau,s)d\tau, \quad (t,s) \in S,$$

equation (1.8) reduces to the system

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} y_0 \\ g(t) \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ \mathcal{Q}(t,s) & p(s) \end{pmatrix} \begin{pmatrix} y(s) \\ z(s) \end{pmatrix} ds, \quad t \in [0,T]. \tag{1.10}$$

Example 2. Let us look at the second order VIDE with a constant kernel

$$y''(t) = \alpha y(t) + \beta y'(t) + \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0,T],$$

$$y(0) = y_0,\ y'(0) = y_1$$

which we will write as a system of two first order VIDEs. Using notations given in (1.10), we have

$$g(t) = \int_0^t f(s)ds + y_1, \quad t \in [0, T],$$

$$\mathcal{Q}(t, s) = \alpha + \lambda(t - s), \quad (t, s) \in S,$$

and the system

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} y_0 \\ g(t) \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ \mathcal{Q}(t, s) & \beta \end{pmatrix} \begin{pmatrix} y(s) \\ z(s) \end{pmatrix} ds, \quad t \in [0, T].$$

As in Example 1, we have got an equation with a nonconstant kernel.

Chapter 2

THE SPLINE COLLOCATION METHOD

2.1 Description of the method

Consider the first order Volterra integro-differential equation

$$y'(t) = f(t, y(t)) + \int_0^t \mathcal{K}(t, s, y(s))ds, \quad t \in [0, T], \tag{2.1}$$

with the initial condition $y(0) = y_0$. Here the functions $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{K}\colon S \times \mathbb{R} \to \mathbb{R}$ (where $S = \{(t,s)\colon 0 \le s \le t \le T\}$) with number y_0 are supposed to be given.

In order to describe this method, let $0 = t_0 < t_1 < \ldots < t_N = T$ (with t_n depending on N) be a mesh on the interval $[0, T]$.

Denote

$$h_n = t_n - t_{n-1}, n = 1, \ldots, N,$$

$$\sigma_n = (t_{n-1}, t_n], n = 1, \ldots, N,$$

$$\Delta_N = \{t_1, \ldots, t_{N-1}\}.$$

Let $\mathcal{P}_k$ denote the space of polynomials of degree not exceeding k.

Definition 2.1 *For given integers $m \ge 1$ and $d \ge -1$ the space of polynomial spline functions of degree $m + d$ and continuity class d, possessing the knots Δ_N, is the set*

$$S^d_{m+d}(\Delta_N) = \{u\colon u_n := u|_{\sigma_n} \in \mathcal{P}_{m+d},\ n = 1, \ldots, N,\ u^{(j)}_{n-1}(t_n) = u^{(j)}_n(t_n), \\ t_n \in \Delta_N,\ j = 0, 1, \ldots, d\}.$$

If $d = -1$, then the elements of $S^{-1}_{m-1}(\Delta_N)$ may have jump discontinuities at the knots Δ_N.

An element $u \in S^d_{m+d}(\Delta_N)$ as a polynomial spline of degree not greater than $m + d$ for all $t \in \sigma_n$, $n = 1, \ldots, N$, can be represented in the form

$$u_n(t) = \sum_{k=0}^{m+d} b_{nk}(t - t_{n-1})^k. \tag{2.2}$$

From (2.2) we have that an element $u \in S^d_{m+d}(\Delta_N)$ is well defined, when we know the coefficients b_{nk} for all $n = 1, ..., N$ and $k = 0, ..., m + d$. In order to compute these coefficients we consider the set of collocation parameters

$$0 < c_1 < \ldots < c_m \leq 1,$$

and we define the set of collocation points by

$$X(N) = \bigcup_{n=1}^{N} X_n,$$

with

$$X_n := \{t_{nj} = t_{n-1} + c_j h_n, j = 1, .., m\}, n = 1, ..., N.$$

So, the approximate solution $u \in S^d_{m+d}(\Delta_N)$ of the equation (2.1) will be determined imposing the condition that u satisfies the integro-differential equation (2.1) on set $X(N)$, i.e.,

$$u'(t) = f(t, u(t)) + \int_0^t \mathcal{K}(t, s, u(s))ds, \quad t \in X(N). \tag{2.3}$$

Starting the calculations by this method we assume also that we can use the initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \ldots, d$, which is justified by the requirement $u \in C^d[0, T]$. Another possible approach is to use only $u_1(0) = y(0)$ and more collocation points (if $d \geq 1$) to determine u_1. Thus, on every interval σ_n we have $d + 1$ conditions of smoothness and m collocation conditions to determine $m+d+1$ parameters b_{nk}. This allows us to implement the method step-by-step going from an interval σ_n to the next one.

In the case $d = -1$, to be able to use initial condition on $\sigma_1 = [0, t_1]$, one collocation condition should be dropped.

In the case of second order VIDE

$$y''(t) = f(t, y(t), y'(t)) + \int_0^t \mathcal{K}(t, s, y(s), y'(s))ds, \quad t \in [0, T], \tag{2.4}$$

with initial conditions

$$y(0) = y_0,\ y'(0) = y_1$$

description of the collocation method is similar. To calculate approximate solution $u \in S^d_{m+d}(\Delta_N)$ of equation (2.4) we impose the following collocation condition

$$u''(t) = f(t, u(t), u'(t)) + \int_0^t \mathcal{K}(t, s, u(s), u'(s))ds, \quad t \in X(N). \tag{2.5}$$

Here, starting calculation by collocation method, other approach, additional to use initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \ldots, d$, is to use initial conditions $u_1(0) = y(0)$, $u_1'(0) = y'(0)$ and more collocation points (if $d \geq 2$) to determine u_1.

Remark 2.1 As the description of the collocation method for nonlinear equations is not more complicated than for linear ones, we presented here the method in general case. Moreover, the research practice shows that convergence theorems for linear equations also hold for some nonlinear equations without any additional requirements on the method.

Chapter 3

AUXILIARY RESULTS

3.1 An overview of numerical stability conditions for VIE

In this section we review some results about stability conditions for VIE. A thorough treatment of the numerical stability of the polynomial spline collocation method for VIE of the second kind is presented in [22] with equidistant collocation points (i.e. $c_j = j/m$, $j = 1, \dots, m$). The method for general setting of collocation points is considered in [16], but the proof of the main result (Theorem 3.3 of [16]) is not correct. This result is also applied to the particular cases, and stability conditions are obtained. Note that several results of [16] are disproved in [24].

Consider the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t, s, y(s))ds + f(t), \quad t \in [0, T], \tag{3.1}$$

with given functions $f\colon [0, T] \times \mathrm{I\!R} \to \mathrm{I\!R}$ and $\mathcal{K}\colon S \times \mathrm{I\!R} \to \mathrm{I\!R}$ (where $S = \{(t, s)\colon 0 \le s \le t \le T\}$).

The step-by-step collocation method for VIE is supposed to determine the approximate solution $u \in S^d_{m+d}(\Delta_N)$ by the collocation conditions at the points t_{nj}

$$u(t) = \int_0^t \mathcal{K}(t, s, u(s))ds + f(t), \quad t \in X(N). \tag{3.2}$$

The spline collocation method for the test equation

$$y(t) = \lambda \int_0^t y(s)ds + f(t), \quad t \in [0, T], \tag{3.3}$$

where λ may be any complex number, leads to the iteration process

$$\alpha_{n+1} = (\overline{M} + W)\alpha_n + r_n,\ n = 1, \ldots, N, \tag{3.4}$$

with $W = O(h)$ and $r_n = O(h)$. Here $\overline{M} = U_0^{-1}U$, where U_0 and U are $(m+d+1) \times (m+d+1)$ matrices as follows:

$$U = \begin{pmatrix} \begin{array}{c|c} I & 0 \\ \hline \multicolumn{2}{c}{G} \end{array} \end{pmatrix}, \qquad U_0 = \begin{pmatrix} A \\ G \end{pmatrix},$$

A being a $(d+1) \times (m+d+1)$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & \ldots & 1 \\ 0 & 1 & 2 & 3 & \ldots & \ldots & m+d \\ 0 & 0 & 1 & \binom{3}{2} & \ldots & \ldots & \binom{m+d}{2} \\ \multicolumn{7}{c}{\dotfill} \\ 0 & \ldots & \ldots & \ldots & 1 & \ldots & \binom{m+d}{d} \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & c_1 & \ldots & c_1^{m+d} \\ \multicolumn{4}{c}{\dotfill} \\ 1 & c_m & \ldots & c_m^{m+d} \end{pmatrix},$$

and I being the $(d+1) \times (d+1)$ identity matrix.

Denote $d_0 = \max\{d, 0\}$, $d_1 = \max\{d, 1\}$ for the method with initial values and $d_1 = 1$ for the method with additional initial collocation.

Definition 3.1 *We say that the spline collocation method is stable if for any $\lambda \in \mathbb{C}$ and any $f \in C^{d_1}[0, T]$ the approximate solution u of* (3.1) *remains bounded in $L_\infty(0, T)$ in the process $h \to 0$.*

Proposition 3.1 *Matrix $\overline{M}$ has eigenvalue $\mu = 1$ with geometric multiplicity m.*

Proposition 3.2 *If all eigenvalues of $\overline{M}$ are in the closed unit disk and if those which lie on the unit circle have equal algebraic and geometric multiplicities, then the spline collocation method is stable. If $\overline{M}$ has an eigenvalue outside of the closed unit disk, then the method is not stable(u has exponential growth: $\| u \|_\infty \geq \mathrm{const}\, e^{KN}, K > 0$).*

Proposition 3.3 *If all eigenvalues of $\overline{M}$ are in the closed unit disk and there is an eigenvalue on the unit circle with different algebraic and geometric multiplicities, then the method is weakly unstable (u may have polynomial growth: $\| u \|_{\infty} \sim \text{const}\, N^k, k \in N$).*

Propositions 3.1 - 3.3 are proved in [24].

3.2 Behaviour of linear iteration process

In this section we will review some well-known results from Linear Algebra, which will be used in the sequel.

1. Let M be a given $m \times m$ matrix. The polynomial $f_M(\lambda) = \det(\lambda I - M)$ is called the *characteristic polynomial* of M. The *eigenvalues* of M are the roots of the characteristic polynomial $f_M(\lambda)$. Denote by $\lambda_{max}(M)$ the maximal by modulus eigenvalue of the matrix M. The *spectral radius* of M is $|\lambda_{max}(M)|$. If $f_M(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$, then λ_0 has *algebraic multiplicity* k. The algebraic multiplicity counts the number of times, an eigenvalue occurs. The dimension of the eigenspace $\text{Ker}(\lambda I - M)$ of an eigenvalue λ is called the *geometric multiplicity* of λ.

2. The geometric multiplicity is smaller or equal than the algebraic multiplicity.

3. There exists a vector norm such that the corresponding matrix norm is equal to the spectral radius of the matrix, i.e., $\sup_{\|x\| \leq 1} \|Mx\| = |\lambda_{max}(M)|$ if and only if all eigenvalues with maximal modulus have equal algebraic and geometric multiplicities.

Let us look at the following iteration process

$$\alpha_{n+1} = (M + W)\alpha_n + r_n, \quad n = 1, \ldots, N-1,$$

where $\alpha_1, r_1, \ldots, r_{N-1}$ are supposed to be given, M is a fixed matrix, not depending on $h = T/N$, $r_n = O(h)$ and $W = O(h)$. We calculate

$$\begin{aligned} \alpha_{n+1} &= (M + W)\alpha_n + r_n \\ &= (M + W)((M + W)\alpha_{n-1} + r_{n-1}) + r_n \\ &= (M + W)^n \alpha_1 + (M + W)^{n-1} r_1 + \ldots + r_n. \end{aligned} \tag{3.5}$$

If additionally, all $\lambda_{max}(M)$ having equal algebraic and geometric multiplicities, we assume that $|\lambda_{max}(M)| \leq 1$, then there is a vector norm such that the corresponding matrix norm $\|M\| \leq 1$. Thus, (3.5) yields

$$\begin{aligned}\|\alpha_{n+1}\| &\leq \|(M+W)\|^n \|\alpha_1\| + \|(M+W)\|^{n-1} \|r_1\| + \ldots + \|r_n\| \\ &\leq (1+K_1h)^n \|\alpha_1\| + ((1+K_1h)^{n-1} + \ldots + 1) \max_{1\leq i\leq n} \|r_i\| \\ &\leq (1+K_1h)^n \|\alpha_1\| + \frac{(1+K_1h)^n - 1}{(1+K_1h)-1} K_2 h,\end{aligned}$$

with some positive constants K_1 and K_2. Using the inequality

$$(1+K_1h)^n \leq (1+K_1h)^N$$

and the convergence

$$(1+K_1h)^N \to e^{K_1T},$$

we get that α_n is bounded uniformly in n.

4. The eigenvalues of a matrix depend continuously on the coefficients of a matrix.

If $|\lambda_{max}(M)| > 1$, then $|\lambda_{max}(M)| \geq 1+\delta$, $\delta > 0$. Thus,

$$|\lambda_{max}(M+W)| \geq 1 + \frac{\delta}{2} = 1+\epsilon,\ \epsilon > 0 \ \text{ if } \ 0 < h \leq h_0$$

for sufficiently small h_0. Take $r_1 = \ldots = r_{N-1} = 0$ and α_1 such that $(M+W)\alpha_1 = \lambda_{max}(M+W)\alpha_1$, $\|\alpha_1\| = 1$. Then

$$\begin{aligned}\|\alpha_{n+1}\| &= \|(M+W)^n\alpha_1\| = \|(\lambda_{max}(M+W))^n \alpha_1\| \\ &= |\lambda_{max}(M+W)|^n \|\alpha_1\| \geq (1+\epsilon)^n \to \infty \quad \text{as} \quad n \to \infty.\end{aligned}$$

So, if $|\lambda_{max}(M)| > 1$ then the sequence α_n is not bounded.

5. If some of the eigenvalues of M have different geometric and algebraic multiplicity, then the matrix $M \in \mathbb{R}^{m\times m}$ can be decomposed into the form

$$M = PJP^{-1}, \tag{3.6}$$

where P is an $m\times m$ invertible matrix, having eigenvectors of M as columns, and J is a block-diagonal matrix having the form

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_p \end{pmatrix},$$

with J_i as follows

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}.$$

Representation (3.6) gives us

$$M^n = (PJP^{-1})^n = PJ^nP^{-1},$$

where J^n is

$$J^n = \begin{pmatrix} J_1^n & & & 0 \\ & J_2^n & & \\ & & \ddots & \\ 0 & & & J_p^n \end{pmatrix},$$

with

$$J_i^n = \begin{pmatrix} \lambda_i^n & n\lambda_i^{n-1} & \frac{n(n-1)}{2!}\lambda_i^{n-2} & \cdots & \cdots \\ & \lambda_i^n & n\lambda_i^n & \ddots & \cdots \\ & & \lambda_i^n & \ddots & \frac{n(n-1)}{2!}\lambda_i^{n-2} \\ & & & \ddots & n\lambda_i^{n-1} \\ 0 & & & & \lambda_i^n \end{pmatrix}.$$

If $|\lambda_i| = 1$, then $|\lambda_i^n| = 1$, but $|n\lambda_i^n| = n$. Therefore, the matrix J^n (such is also M) is not bounded. Choosing α_1 and r_n as in the previous case we get that α_n is not bounded.

Chapter 4

STABILITY OF THE SPLINE COLLOCATION METHOD FOR FIRST ORDER VIDE

In this chapter we will analyze the stability of the spline collocation method where the splines are at least continuous, i.e., we suppose that $d \geq 0$.

4.1 Method in the case of test equation

Consider the test equation

$$y'(t) = \alpha y(t) + \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0, T], \tag{4.1}$$

where, in general, λ and α may be any complex numbers.

Assume that the mesh sequence $\{\Delta_N\}$ is uniform, i.e., $h_n = h = T/N$ for all n. Representing $t \in \sigma_n$ as $t = t_{n-1} + \tau h$, $\tau \in (0, 1]$, we have on σ_n

$$u_n(t_{n-1} + \tau h) = \sum_{k=0}^{m+d} a_{nk}\tau^k, \qquad \tau \in (0, 1], \tag{4.2}$$

where we passed to the parameters $a_{nk} = b_{nk}h^k$.

The smoothness conditions (for any $u \in S^d_{m+d}(\Delta_N)$)

$$u_n^{(j)}(t_n - 0) = u_{n+1}^{(j)}(t_n + 0), \; j = 0, \ldots, d, \; n = 1, \ldots, N-1,$$

can be expressed in the form

$$a_{n+1,j} = \sum_{k=j}^{m+d} \frac{k!}{(k-j)!j!} \, a_{nk}, \quad j = 0, \ldots, d, \, n = 1, \ldots, N-1. \tag{4.3}$$

The collocation conditions (2.3), applied to the test equation (4.1), give

$$u'(t_{nj}) = \alpha u(t_{nj}) + \lambda \int_0^{t_{nj}} u(s)ds + f(t_{nj}),$$
$$j = 1, \ldots, m,\ n = 1, \ldots, N. \quad (4.4)$$

From (4.2) we get

$$u_n(t_{nj}) = \sum_{k=0}^{m+d} a_{nk} c_j^k$$

and

$$u_n'(t_{nj}) = \frac{1}{h} \sum_{k=1}^{m+d} a_{nk} k c_j^{k-1}.$$

Now the equation (4.4) becomes

$$\frac{1}{h} \sum_{k=0}^{m+d} a_{nk} k c_j^{k-1} = \alpha \sum_{k=0}^{m+d} a_{nk} c_j^k + \sum_{r=1}^{n-1} \lambda \int_{t_{r-1}}^{t_r} u_r(s)ds$$
$$+ \lambda \int_{t_{n-1}}^{t_{nj}} u_n(s)ds + f(t_{nj}).$$

Using notations $s = t_{r-1} + \tau h$ or $s = t_{n-1} + \tau h$, we have $ds = hd\tau$. The new limits of integration for $s = t_{r-1}$ or $s = t_{n-1}$ is $\tau = 0$, for $s = t_r$ is $\tau = 1$ and for $s = t_{n-1} + c_j h$ is $\tau = c_j$.
So, we get that

$$\frac{1}{h} \sum_{k=0}^{m+d} a_{nk} k c_j^{k-1} = \alpha \sum_{k=0}^{m+d} a_{nk} c_j^k + \sum_{r=1}^{n-1} \lambda h \int_0^1 \Big(\sum_{k=0}^{m+d} a_{rk} \tau^k \Big) d\tau$$
$$+ \lambda h \int_0^{c_j} \Big(\sum_{k=0}^{m+d} a_{nk} \tau^k \Big) d\tau + f(t_{nj})$$
$$= \alpha \sum_{k=0}^{m+d} a_{nk} c_j^k + \sum_{r=1}^{n-1} \lambda h \Big(\sum_{k=0}^{m+d} \frac{1}{k+1} a_{rk} \Big)$$
$$+ \lambda h \sum_{k=0}^{m+d} a_{nk} \frac{c_j^{k+1}}{k+1} + f(t_{nj}). \quad (4.5)$$

Using the notation $\alpha_n = (a_{nk})_{k=0}^{m+d}$, we write (4.5) as follows:

$$\sum_{k=0}^{m+d} a_{nk} k c_j^{k-1} - \alpha h \sum_{k=0}^{m+d} a_{nk} c_j^k - \lambda h^2 \sum_{k=0}^{m+d} a_{nk} \frac{c_j^{k+1}}{k+1}$$

$$= \lambda h^2 \big\langle q, \sum_{r=1}^{n-1} \alpha_r \big\rangle + h f(t_{nj}), \tag{4.6}$$

where $q = (1, 1/2, \ldots, 1/(m+d+1))$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^{m+d+1}$. The difference of the equations (4.6) with n and $n+1$ yields

$$\sum_{k=0}^{m+d} a_{n+1,k} k c_j^{k-1} - \alpha h \sum_{k=0}^{m+d} a_{n+1,k} c_j^k - \lambda h^2 \sum_{k=0}^{m+d} a_{n+1,k} \frac{c_j^{k+1}}{k+1}$$

$$= \sum_{k=0}^{m+d} a_{nk} k c_j^{k-1} - \alpha h \sum_{k=0}^{m+d} a_{nk} c_j^k - \lambda h^2 \sum_{k=0}^{m+d} a_{nk} \frac{c_j^{k+1}}{k+1} + \lambda h^2 \langle q, \alpha_n \rangle$$

$$+ h f(t_{n+1,j}) - h f(t_{nj}), \quad j = 1, \ldots, m, \ n = 1, \ldots, N-1. \tag{4.7}$$

Now we may write together the equations (4.3) and (4.7) in matrix form

$$(V - \alpha h V_1 - \lambda h^2 V_2)\alpha_{n+1} = (V_0 - \alpha h V_1 - \lambda h^2 (V_2 - V_3))\alpha_n + h g_n,$$

$$n = 1, \ldots, N-1, \tag{4.8}$$

with $(m+d+1) \times (m+d+1)$ matrices V, V_0, V_1, V_2, V_3 as follows:

$$V = \left(\begin{array}{c|c} I & 0 \\ \hline \multicolumn{2}{c}{C} \end{array} \right), \qquad V_0 = \begin{pmatrix} A \\ C \end{pmatrix},$$

I being the $(d+1) \times (d+1)$ unit matrix,

$$C = \begin{pmatrix} 0 & 1 & 2c_1 & \ldots & (m+d)c_1^{m+d-1} \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 1 & 2c_m & \ldots & (m+d)c_m^{m+d-1} \end{pmatrix},$$

A being defined as in Section 3.1,

$$V_1 = \begin{pmatrix} & & 0 & & \\ 1 & c_1 & c_1^2 & \ldots & c_1^{m+d} \\ \multicolumn{5}{c}{\dotfill} \\ 1 & c_m & c_m^2 & \ldots & c_m^{m+d} \end{pmatrix},$$

$$V_2 = \begin{pmatrix} & & 0 & \\ c_1 & c_1^2/2 & \dots & c_1^{m+d+1}/(m+d+1) \\ \dots & \dots & \dots & \dots \\ c_m & c_m^2/2 & \dots & c_m^{m+d+1}/(m+d+1) \end{pmatrix},$$

V_3 having first $d+1$ rows 0 and last m rows the vector q, and, finally, the $m+d+1$ dimensional vector

$$g_n = (0, \dots, 0, f(t_{n+1,1}) - f(t_{n1}), \dots, f(t_{n+1,m}) - f(t_{nm})).$$

Thus $g_n = O(h)$ for $f \in C^1$.

Proposition 4.1 *The matrix $V - \alpha h V_1 - \lambda h^2 V_2$ is invertible for sufficiently small h.*

Proof. Since $d \geq 0$, we have

$$\det V = \begin{vmatrix} (d+1)c_1^d & \dots & (m+d)c_1^{m+d-1} \\ \dots & \dots & \dots \\ (d+1)c_m^d & \dots & (m+d)c_m^{m+d-1} \end{vmatrix}$$

$$= (d+1)c_1^d \dots (m+d)c_m^d \begin{vmatrix} 1 & c_1 & \dots & c_1^{m-1} \\ \dots & \dots & \dots & \dots \\ 1 & c_m^d & \dots & c_m^{m-1} \end{vmatrix} \neq 0,$$

so the matrix V is invertible. Such is also $V - \alpha h V_1 - \lambda h^2 V_2$ for small h, which completes the proof.

Let us now take a look at $(V - \alpha h V_1 - \lambda h^2 V_2)^{-1}$. Denote $B = \alpha V_1 + \lambda h V_2$, $B_1 = hV^{-1}B$ and observe that $\|B\| \leq const$, $\|B_1\| \leq const$. Then

$$\begin{aligned} (V - \alpha h V_1 - \lambda h^2 V_2)^{-1} &= (V - hB)^{-1} \\ &= (V(I - hV^{-1}B))^{-1} \\ &= (I - B_1)^{-1} V^{-1} \\ &= (I + B_1 + B_1^2 + ...)V^{-1} \\ &= V^{-1} + B_1(I + B_1 + B_1^2 + ...)V^{-1} \\ &= V^{-1} + B_1(I - B_1)^{-1}V^{-1} \\ &= V^{-1} + hB_2, \end{aligned}$$

where $B_2 = V^{-1}B(I - B_1)^{-1}V^{-1}$ is such that $\|B_2\| \leq const$.

Again, denoting $B_3 = \alpha V_1 + \lambda h(V_2 - V_3)$ and having $\|B_3\| \leq const$, the equation (4.8) becomes

$$\begin{aligned}\alpha_{n+1} &= (V - \alpha h V_1 - \lambda h^2 V_2)^{-1}(V_0 - \alpha h V_1 - \lambda h^2(V_2 - V_3))\alpha_n \\ &+ (V - \alpha h V_1 - \lambda h^2 V_2)^{-1} h g_n \\ &= (V^{-1} + hB_2)(V_0 - hB_3)\alpha_n + (V^{-1} + hB_2)hg_n \\ &= (V^{-1}V_0 + W)\alpha_n + r_n,\end{aligned}$$

where $W = O(h)$ and $r_n = O(h^2)$ because of $g_n = O(h)$ for $f \in C^1$. Note that $W = 0$ if $\alpha = 0$ and $\lambda = 0$.

Set $M = V^{-1}V_0$, then the equation (4.8) takes the form

$$\alpha_{n+1} = (M + W)\alpha_n + r_n. \tag{4.9}$$

4.2 Stability of the method

We have seen that the spline collocation method (2.3) for the test equation (4.1) leads to the iteration process

$$\alpha_{n+1} = (V^{-1}V_0 + W)\alpha_n + r_n, \quad n = 1, \ldots, N-1, \tag{4.10}$$

with $W = O(h)$ and $r_n = O(h^2)$.

We distinguish the method with initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \ldots, d$, and another method which uses only $u_1(0) = y(0)$ and additional collocation points $t_{0j} = t_0 + c_{0j}h$, $j = 1, \ldots, d$, with fixed $c_{0j} \in (0, 1] \setminus \{c_1, \ldots, c_m\}$ on the first interval σ_1.

Denote $d_0 = \max\{d-1, 0\}$ for the method with initial values and $d_0 = 0$ for the method with additional initial collocation.

Definition 4.1 *We say that the spline collocation method is stable if for any $\alpha, \lambda \in \mathbb{C}$ and any $f \in C^{d_0}[0,T]$ the approximate solution u of (4.1) remains bounded in $C[0,T]$ in the process $h \to 0$.*

Let us notice that the boundedness of $||u||_{C[0,T]}$ is equivalent to the boundedness of $||\alpha_n||$ in n and h in any fixed norm of $\mathbb{R}^{m+d+1}$.

The principle of uniform boundedness allows to establish

Proposition 4.2 *The spline collocation method is stable if and only if*

$$||u||_{C[0,T]} \leq const||f||_{C^{d_0}[0,T]} \quad \forall f \in C^{d_0}[0,T], \tag{4.11}$$

where the constant may depend only on T, α, λ and on parameters c_j and c_{0j}.

Proposition 4.3 *Matrix M has eigenvalue $\mu = 1$ with geometric multiplicity m.*

Proof. Since $\det(M - \mu I) = 0$ is equivalent to $\det(V_0 - \mu V) = 0$, then $\mathrm{Ker}(M - \mu I) = \mathrm{Ker}(V_0 - \mu V)$. The geometric multiplicity of $\mu = 1$ is $\dim \mathrm{Ker}(V_0 - V)$. But $\dim \mathrm{Ker}(V_0 - V) = m + d + 1 - \mathrm{rank}(V_0 - V)$. As $\mathrm{rank}(V_0 - V) = d + 1$, we get the assertion.

Theorem 4.1 *For fixed c_j the eigenvalues of M for VIDE in the case m and $d+1$ and eigenvalues of $\overline{M}$ for VIE in the case m and d coincide and have the same algebraic and geometric multiplicities, except $\mu = 1$ whose algebraic multiplicity for VIDE is greater by one than for VIE.*

Proof. The eigenvalue problem for M is equivalent to the generalized eigenvalue problem for V_0 and V, i.e., $(M - \mu I)v = 0$ for $v \neq 0$ if and only if $(V_0 - \mu V)v = 0$ and $(M - \mu I)w = v$ takes place if and only if $(V_0 - \mu V)w = Vv$. Denote $\nu = 1 - \mu$. Then for VIDE with the parameters m and $d + 1$ we have

$$V_0 - \mu V = \begin{vmatrix} \nu & 1 & 1 & 1 & \dots & \dots & 1 \\ 0 & \nu & 2 & 3 & \dots & \dots & m+d+1 \\ 0 & 0 & \nu & \binom{3}{2} & \dots & \dots & \binom{m+d+1}{2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \nu & \dots & \binom{m+d+1}{d+1} \\ 0 & \nu & \nu \cdot 2c_1 & \dots & \dots & \dots & \nu(m+d+1)c_1^{m+d} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \nu & \nu \cdot 2c_m & \dots & \dots & \dots & \nu(m+d+1)c_m^{m+d} \end{vmatrix}. \quad (4.12)$$

Let $I_{i,p}$ be the diagonal matrix obtained from an identity matrix, replacing the i-th diagonal element by the number p. Thus, the products $I_{i,p}A$ and $AI_{i,p}$ mean the multiplication of i-th row and i-th column of A, respectively, by p. Consider also the matrices U_0 and U, defined in Section 3.1, with the parameters m and d. A direct calculation and the observation that $\binom{p}{q}\frac{q}{p} = \binom{p-1}{q-1}$, allows us to get from (4.12)

$$I_{d+2,d+1} \dots I_{3,2}(V_0 - \mu V)I_{3,1/2} \dots I_{d+m+2,1/(m+d+1)}$$

$$= \begin{pmatrix} \nu & 1 \quad 1/2 \quad \dots \quad 1/(m+d+1) \\ 0 & U_0 - \mu U \end{pmatrix}$$

or

$$S(V_0 - \mu V)S^{-1} = R \begin{pmatrix} \nu & 1 & 1/2 & \dots & 1/(m+d+1) \\ 0 & & & U_0 - \mu U & \end{pmatrix}, \qquad (4.13)$$

where

$$S = I_{d+2,d+1} \dots I_{3,2}$$

and

$$R = I_{d+m+2,d+m+1} \dots I_{d+3,d+2}.$$

Now (4.13) gives

$$\det(V_0 - \mu V) = (d+2) \dots (d+m+1)\nu \det(U_0 - \mu U)$$

which permits to get the assertion about algebraic multiplicities of eigenvalues of M and $\overline{M}$. By Propositions 4.3 and 3.1 the eigenvalue $\mu = 1$ of M and $\overline{M}$ has geometric multiplicity m.

It remains to consider the geometric multiplicity of eigenvalues $\mu \neq 1$. Thus, suppose $\nu \neq 0$. Using (4.13), the equation $(V_0 - \mu V)v = 0$ can be written as

$$\begin{pmatrix} \nu & 1 & \dots \\ 0 & U_0 - \mu U & \end{pmatrix} Sv = 0$$

or, denoting $w = Sv$, equivalently

$$\nu w_1 + w_2 + \dots + w_{m+d+2}/(m+d+1) = 0, \qquad (4.14)$$

$$(U_0 - \mu U)\overline{w} = 0 \qquad (4.15)$$

with $\overline{w} = (w_2, \dots, w_{m+d+2})$.

Let $\overline{w}^1, \dots, \overline{w}^k$ be linearly independent solutions of (4.15). Extending these vectors with the first components defined by (4.14), we get vectors $w^1, \dots, w^k$ and then $S^{-1}w^1, \dots, S^{-1}w^k$ as linearly independent solutions of $(V_0 - \mu V)v = 0$.

Conversely, consider $v^1, \dots, v^k$ as linearly independent solutions of $(V_0 - \mu V)v = 0$. Dropping the first components of the vectors $w^i = Sv^i$ we get the solutions $\overline{w}^1, \dots, \overline{w}^k$ of (4.15). Suppose $\gamma_1\overline{w}^1 + \dots + \gamma_k\overline{w}^k = 0$ with at least one $\gamma_i \neq 0$. Then, (4.14) allows to get $\gamma_1 w^1 + \dots + \gamma_k w^k = 0$ or $\gamma_1 v^1 + \dots + \gamma_k v^k = 0$. This contradiction shows that the geometric multiplicities of $\mu \neq 1$ as an eigenvalue of M and $\overline{M}$ coincide. The proof is complete.

Proposition 4.4 *If M has an eigenvalue outside of the closed unit disk, then the spline collocation method is not stable with possible exponential growth of approximate solution.*

Proof. Consider an eigenvalue μ of $M+W$ such that $|\mu| \geq 1+\delta$ with some fixed $\delta > 0$ for any sufficiently small h. For $\alpha_1 \neq 0$, being an eigenvector of $M+W$, we have here

$$(V - \alpha h V_1 - \lambda h^2 V_2)\alpha_1 = h g_0, \tag{4.16}$$

where

$$g_0 = (\alpha_{10}, \ldots, \alpha_{1d}, f(t_{11}), \ldots, f(t_{1m}))$$

and

$$\alpha_{1j} = h^j y^{(j)}(0)/j!, \quad j = 0, \ldots, d.$$

Because of

$$y'(0) = \alpha y(0) + f(0),$$

$$y^{(j)}(0) = \alpha y^{(j-1)}(0) + \lambda y^{(j-2)}(0) + f^{(j-1)}(0), \quad j = 2, \ldots, d, \tag{4.17}$$

the vector α_1 determines via (4.16) and (4.17) the values $f^{(j)}(0)$, $j = 0, \ldots, d-1$, $f(t_{11}), \ldots, f(t_{1m})$.

We take f on $[0, h]$ as the polynomial interpolating the values $f^{(j)}(0)$, $j = 0, \ldots, d-1$, $f(t_{1j})$, $j = 1, \ldots, m$, and $f^{(j)}(h) = 0$, $j = 0, \ldots, d_0$ (if $c_m = 1$ then $f^{(j)}(h) = 0$, $j = 1, \ldots, d_0$).

In the case of the method of additional knots let f be on $[0, h]$ the interpolating polynomial by the data $f(0), f(t_{0j}), j = 0, \ldots, d$, $f(t_{1j})$, $j = 1, \ldots, m$, and $f^{(j)}(h) = 0$ (here $d_0 = 0$ and if $c_m = 1$, then $f(t_{1m}) = f(h)$ is already given and we drop the requirement $f(h) = 0$).

In both cases we ask f to be on $[nh, (n+1)h]$, $n \geq 1$, the interpolating polynomial by the values $f^{(j)}(nh) = 0$ and $f^{(j)}((n+1)h) = 0$, $j = 0, \ldots, d_0$ (if $c_m = 1$, then for $j = 1, \ldots, d_0$), and also $f(t_{n+1,j}) = f(t_{1j})$, $j = 1, \ldots, m$. This guarantees that $f \in C^{d_0}[0, T]$ and $r_n = 0$, $n \geq 1$.

To represent function f, we introduce Newton's divided difference interpolation formula. Let

$$\pi_k(x) = \prod_{j=0}^{k} (x - x_j), \; k = 0, \ldots, n.$$

Then Newton's formula is

$$f(x) = f(x_0) + \sum_{k=1}^{n} \pi_{k-1}(x) f[x_0, \ldots, x_k] + R_n(x), \tag{4.18}$$

where $f[x_0, \ldots, x_k]$ is divided difference, and the remainder is

$$R_n(x) = \pi_n(x) f[x_0, \ldots, x_n, x] = \pi_n(x) \frac{f^{(n+1)}(\xi)}{(n+1)!}, \quad x_0 < \xi < x_n.$$

The divided differences $f[x_0,\dots,x_n]$ on $n+1$ points $x_0,\dots,x_n$ of a function $f(x)$ are defined by $f[x_0]=f(x_0)$ and for $n\ge 1$

$$f[x_0,\dots,x_n]=\frac{f[x_0,\dots,x_{n-1}]-f[x_1,\dots,x_n]}{x_0-x_n}. \tag{4.19}$$

In fact, Newton's formula (4.18) holds also for multiple knots. Then the divided differences could be represented, basing on the formula (4.19), by the divided differences of the form $f[x_i,\dots,x_i]$ which, in turn, may be written as

$$f[x_i,\dots,x_i]=\frac{f^{(l)}(x_i)}{l!},$$

where $l+1$ is the multiplicity of the knot x_i.

So, considering previous discussion, the interpolant f can be represented on $[t_n,t_{n+1}]$ by the formula:

$$f(t)=f(t_n+\tau h)=\sum_{i=0}^{\kappa}\left(\sum_{l=0}^{k_i}h^{s_l}p_{il}f^{(s_l)}(\xi_l)\right)\prod_{r=0}^{i-1}(\tau-b_r) \tag{4.20}$$

with b_r being c_j or c_{0j}, ξ_l being t_{nj} or t_j, $0\le s_l\le d_0$, $k_i\le i$, constants p_{il} depending on c_j and c_{0j}.

In the case of initial conditions $\kappa=m+d+d_0$ ($\kappa=m+d+d_0-1$, if $c_m=1$), in the case of additional knots $\kappa=m+d+1$ ($\kappa=m+d$, if $c_m=1$) on the interval $[0,h]$ and $\kappa=m+2d_0+1$ ($\kappa=m+2d_0$, if $c_m=1$) on the interval $[nh,(n+1)h]$, $n\ge 1$.

Replacing h by h/k, $k=1,2,\dots$, and keeping $||\alpha_1||=h/k$, we have

$$\begin{aligned}||g_0||_\infty &= ||\frac{k}{h}(V-\alpha hV_1-\lambda h^2V_2)\alpha_1||_\infty\\ &\le \frac{k}{h}||V-\alpha hV_1-\lambda h^2V_2||_\infty||\alpha_1||_\infty.\end{aligned}$$

So, $||g_0||_\infty$ is bounded which means that $f(t_{1j})$, $j=1,\dots,m$, and $h^jy^{(j)}(0)/k^j$, $j=0,\dots,d$, or $h^jf^{(j)}(0)/k^j$, $j=0,\dots,d_0$, are bounded, too, in the process $k\to\infty$.

Thus, (4.20) gives

$$||f||_{C^{d_0}[0,T]}\le const\ k^{d_0}. \tag{4.21}$$

On the other hand, due to $r_n=0$ for $n\ge 1$,

$$\alpha_{n+1}=(M+W)\alpha_n=...=(M+W)^n\alpha_1=\mu^n\alpha_1$$

and

$$||\alpha_{n+1}||=|\mu^n|||\alpha_1||\ge(1+\delta)^n||\alpha_1||$$

yields

$$||\alpha_{kN}|| \geq \frac{h}{k}(1+\delta)^{kN-1} \tag{4.22}$$

and (4.11) cannot be satisfied. The inequalities (4.21) and (4.22) mean also the exponential growth of approximate solution if we keep the norm of f bounded in C^{d_0}. The proof is complete.

The case where all eigenvalues of M are in the closed unit disk and there is some of them on the unit circle having different algebraic and geometric multiplicities can be treated as for VIE (see [24]). In fact, for VIDE the eigenvalue $\mu = 1$ has always different algebraic and geometric multiplicities. Thus, the collocation method is always at least weakly unstable. But this weak instability cannot be observed for low order splines (see next section for examples). In practice, the method is stable if and only if all eigenvalues of M are in the closed unit disk which we keep in view describing the examples.

4.3 Examples

Let us consider some special cases of d and m.

Case $d = 0$, $m \geq 1$.
We have

$$V = \left(\frac{1\ 0\ \dots\ 0}{C} \right), \qquad V_0 = \left(\frac{1\ 1\ \dots\ 1}{C} \right)$$

and $\det(V_0 - \mu V) = (1-\mu)^{m+1} \det C_0$ where C_0 is obtained from C omitting the first column. This means that the method is always stable.

Case $d = 1$, $m = 1$ (quadratic splines).
The equation $\det(V_0 - \mu V) = 0$ has solutions $\mu = 1$ and $\mu = 1 - 1/c_1$. The method is stable if and only if $1/2 \leq c_1 \leq 1$.

Case $d = 1$, $m = 2$ (Hermite cubic splines).
By proposition 4.3 $\mu = 1$ is a solution of $\det(V_0 - \mu V) = 0$ with geometric multiplicity 2 and with algebraic multiplicity at least 3. The other solution $\mu(c_1, c_2) = 1 - (c_1 + c_2 - 1)/c_1c_2$ shows that if $c_1 + c_2 < 1$ the method is unstable. Suppose $c_1 + c_2 > 1$. Then $1/2 < c_2 \leq 1$. As $\mu(c_1, 1) = 0$, only the possibility $1/2 < c_2 < 1$ needs some analysis. Then $1 - c_2 < c_1 < c_2$. As $\mu(1 - c_2, c_2) = 1$, $0 < \mu(c_2, c_2) < 1$ and $\mu(c_1, c_2)$ is strictly decreasing in c_1, we conclude that $0 \leq \mu(c_1, c_2) < 1$ for $c_1 + c_2 > 1$ which yields the stability. Clearly, the case $c_1 + c_2 = 1$ mean that $\mu = 1$ has algebraic multiplicity 4 and the method, being theoretically weakly unstable, is stable in practical calculations.

Case $d = 2$, $m = 1$ *(cubic splines).*
Here the geometric multiplicity of $\mu = 1$ as solution of $\det(V_0 - \mu V) = 0$ is 1 and its algebraic multiplicity is 2. We also get

$$c_1^2 \nu^2 - (2c_1 + 1)\nu + 2 = 0$$

with $\nu = 1 - \mu$. From $\nu = (1 + 2c_1 \pm \sqrt{1 + 4c_1(1 - c_1)})/2c_1^2$, we see that $\nu > 0$ and thus $\mu < 1$. For $c_1 = 1$, there are eigenvalues $\mu = 0$ and $\mu = -1$ corresponding to $\nu = 1$ and $\nu = 2$. The function $\phi(c_1) = (1 + 2c_1 + \sqrt{1 + 4c_1(1 - c_1)})/2c_1^2$ is decreasing ($\phi'(c_1) < 0$) and hence for $c_1 < 1$, we get $\nu > 2$ and $\mu < -1$. Thus, the method is stable if and only if $c_1 = 1$.

Chapter 5

STABILITY OF PIECEWISE POLYNOMIAL COLLOCATION METHOD FOR FIRST ORDER VIDE

In previous chapter we showed that, for general case of spline collocation method, the stability depends only on the collocation parameters. In this chapter we will show that, for case of piecewise polynomial collocation method (i.e. for $d = -1$), there is also dependence on the parameters of certain test equation.

5.1 Method in the case of test equation

Consider the test equation

$$y'(t) = \alpha y(t) + \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0, T], \tag{5.1}$$

where, in general, λ and $\alpha \neq 0$ may be any complex numbers.

Similarly to the Section 4.1, using the collocation conditions (2.3), applied to the test equation (5.1)

$$u'(t_{nj}) = \alpha u(t_{nj}) + \lambda \int_0^{t_{nj}} u(s)ds + f(t_{nj}),$$
$$j = 1, \ldots, m, \; n = 1, \ldots, N, \tag{5.2}$$

we get the equation in matrix form

$$(V - \alpha h V_1 - \lambda h^2 V_2)\alpha_{n+1} = (V - \alpha h V_1 - \lambda h^2 (V_2 - V_3))\alpha_n + h g_n, \tag{5.3}$$

with $m \times m$ matrices V, V_1, V_2, V_3 as follows:

$$V = \begin{pmatrix} 0 & 1 & 2c_1 & \dots & (m-1)c_1^{m-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2c_m & \dots & (m-1)c_m^{m-2} \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 1 & c_1 & c_1^2 & \dots & c_1^{m-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & c_m & c_m^2 & \dots & c_m^{m-1} \end{pmatrix}, \quad V_2 = \begin{pmatrix} c_1 & c_1^2/2 & \dots & c_1^m/m \\ \dots & \dots & \dots & \dots \\ c_m & c_m^2/2 & \dots & c_m^m/m \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 1 & 1/2 & \dots & 1/m \\ \dots & \dots & \dots & \dots \\ 1 & 1/2 & \dots & 1/m \end{pmatrix},$$

and the m dimensional vector

$$g_n = (f(t_{n+1,1}) - f(t_{n1}), \dots, f(t_{n+1,m}) - f(t_{nm})).$$

Thus $g_n = O(h)$ for $f \in C^1$.

Let us introduce the polynomials $P_k(\lambda, \alpha)$ and $Q_k(\lambda, \alpha)$ by the following recurrence relations

$$Q_k = P_{k-1} + \alpha Q_{k-1}, \tag{5.4}$$

$$P_k = \lambda Q_{k-1} \tag{5.5}$$

starting with $P_0 = 0$ and $Q_0 = 1$. Then we have, for example, $P_1 = \lambda$, $Q_1 = \alpha$, $P_2 = \lambda\alpha$, $Q_2 = \lambda + \alpha^2$ etc. Combining (5.4) and (5.5) we get also

$$Q_k = \alpha Q_{k-1} + \lambda Q_{k-2}. \tag{5.6}$$

Note that for all $k \geq 0$ we have $Q_k \neq 0$ or $Q_{k+1} \neq 0$ because the assumption $Q_{k+1} = 0$ and $Q_k = 0$ via (5.6) gives $Q_{k-1} = 0, \dots, Q_0 = 0$, which is not the case.

Denote by D_m Vandermonde's determinant formed by $c_1, \dots, c_m$, i.e., $D_m = \det V_1$.

Proposition 5.1 *We have*

$$\det(V - \alpha h V_1 - \lambda h^2 V_2) = (-1)^m Q_m D_m h^m$$
$$+ (-1)^m \lambda Q_{m-1}(c_1 + \dots + c_m) D_m h^{m+1}/m + O(h^{m+2}). \tag{5.7}$$

Proof. Writing the columns of the determinant as rows with representative element, we get

$$\det(V - \alpha h V_1 - \lambda h^2 V_2) = \begin{vmatrix} -\alpha h - \lambda h^2 c_i \\ 1 - \alpha h c_i - \lambda h^2 c_i^2/2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (m-1)c_i^{m-2} - \alpha h c_i^{m-1} - \lambda h^2 c_i^m/m \end{vmatrix}$$

$$= -Q_1 h \begin{vmatrix} 1 + (P_1/Q_1)hc_i \\ 1 - \alpha h c_i - \lambda h^2 c_i^2/2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (m-1)c_i^{m-2} - \alpha h c_i^{m-1} - \lambda h^2 c_i^m/m \end{vmatrix} = \ldots$$

$$= (-1)^m Q_m h^m \begin{vmatrix} 1 + (P_1/Q_1)hc_i \\ c_i + (P_2/Q_2)hc_i^2/2 \\ \cdots\cdots\cdots\cdots \\ c_i^{m-1} + (P_m/Q_m)hc_i^m/m \end{vmatrix},$$

which gives the representation (5.7), when $Q_m \neq 0, \ldots, Q_2 \neq 0$. In general case, take the sequences $\lambda_j \to \lambda, \alpha_j \to \alpha$ such that $Q_k(\lambda_j, \alpha_j) \neq 0$ for all j and k. Then from (5.7) for λ_j, α_j we get in limit process (5.7) for λ, α. The proof is complete.

Since $Q_m \neq 0$ or $Q_{m-1} \neq 0$, from (5.7) we get

Corollary 5.1 *The matrix $V - \alpha h V_1 - \lambda h^2 V_2$ is invertible for sufficiently small h.*

5.2 Stability of the method

Definition 5.1 *We say that the spline collocation method by piecewise polynomials is stable if, for any $f \in C^1[0,T]$, the approximate solution u of* (5.1) *remains bounded in $L_\infty(0,T)$ as $h \to 0$.*

Denote $W = V - \alpha h V_1 - \lambda h^2 V_2$, then the equation (5.3) takes the form

$$W\alpha_{n+1} = (W + \lambda h^2 V_3)\alpha_n + hg_n.$$

Therefore, the equation (5.3) may be written as follows

$$\alpha_{n+1} = (I + \lambda h^2 W^{-1} V_3)\alpha_n + hW^{-1} g_n. \tag{5.8}$$

Proposition 5.2 *Matrix $I + \lambda h^2 W^{-1} V_3$ has eigenvalue $\mu = 1$ with geometric multiplicity $m - 1$.*

Proof. It is clear that $\mathrm{Ker}(I + \lambda h^2 W^{-1} V_3 - \mu I)$=$\mathrm{Ker}(W + \lambda h^2 V_3 - \mu W)$. The geometric multiplicity of $\mu = 1$ is $\dim \mathrm{Ker} V_3$, but $\dim \mathrm{Ker} V_3 = m - \mathrm{rank} V_3$. As $\mathrm{rank} V_3 = 1$, we get the assertion.

Besides the eigenvalue $\mu = 1$ there is one more $\mu \in \mathrm{spec}(I + \lambda^2 W^{-1} V_3)$ which is equivalent to $\mu - 1 \in \mathrm{spec}(\lambda h^2 W^{-1} V_3)$. Thus, we have to find one additional solution of $\det(\lambda h^2 V_3 - \mu W) = 0$ having already 0 as solution of multiplicity $m - 1$ by Proposition 5.2.

Denote $A = \lambda h^2 V_3$ and $B = W$ with corresponding entries a_{ij} and b_{ij}. Taking into account

$$a_{11} = \ldots = a_{m1}, \ldots, a_{1m} = \ldots = a_{mm},$$

we get

$$\det(A - \mu B) = \begin{vmatrix} a_{11} - \mu b_{11} & \ldots & a_{1m} - \mu b_{1m} \\ a_{21} - \mu b_{21} & \ldots & a_{2m} - \mu b_{2m} \\ \ldots & \ldots & \ldots \\ a_{m1} - \mu b_{m1} & \ldots & a_{mm} - \mu b_{mm} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} - \mu b_{11} & \ldots & a_{1m} - \mu b_{1m} \\ \mu(b_{11} - b_{21}) & \ldots & \mu(b_{1m} - b_{2m}) \\ \ldots & \ldots & \ldots \\ \mu(b_{11} - b_{m1}) & \ldots & \mu(b_{1m} - b_{mm}) \end{vmatrix}$$

$$= \mu^{m-1} \begin{vmatrix} a_{11} - \mu b_{11} & \ldots & a_{1m} - \mu b_{1m} \\ b_{11} - b_{21} & \ldots & b_{1m} - b_{2m} \\ \ldots & \ldots & \ldots \\ b_{11} - b_{m1} & \ldots & b_{1m} - b_{mm} \end{vmatrix}. \tag{5.9}$$

Expansion by the first row gives us

$$\begin{aligned}\det(A-\mu B) = \mu^{m-1}&\left[(a_{11}-\mu b_{11})M_1 + (a_{12}-\mu b_{12})M_2 + \dots\right.\\ &\left.+(a_{1m}-\mu b_{1m})M_m\right]\\ &= \mu^{m-1}\left[a_{11}M_1 + a_{12}M_2 + \dots + a_{1m}M_m\right.\\ &\left.-\mu(b_{11}M_1 + b_{12}M_2 + \dots + b_{1m}M_m)\right].\end{aligned}$$

Thus, we have $\det(A-\mu B) = 0$ if

$$\mu = \frac{a_{11}M_1 + a_{12}M_2 + \ldots + a_{1m}M_m}{b_{11}M_1 + b_{12}M_2 + \ldots + b_{1m}M_m} \tag{5.10}$$

with some M_i obtained from the determinant in (5.9).

Lemma 5.1 *It holds*

$$(c_1 + c_2 + \dots + c_m)D_m = \begin{vmatrix} c_2 - c_1 & \dots & c_2^{m-2} - c_1^{m-2} & c_2^m - c_1^m \\ c_3 - c_1 & \dots & c_3^{m-2} - c_1^{m-2} & c_3^m - c_1^m \\ \dots & \dots & \dots & \dots \\ c_m - c_1 & \dots & c_m^{m-2} - c_1^{m-2} & c_m^m - c_1^m \end{vmatrix}.$$

This is a standard exercise result from Linear Algebra.

Lemma 5.2 *We have*

$$M_1 = Q_{m-1}h^{m-1}D_m + P_{m-1}h^m(c_1 + \ldots + c_m)D_m/m + O(h^{m+1}) \tag{5.11}$$

with $Q_{m-1} \neq 0$ *or* $P_{m-1} \neq 0$,

$$M_2 = -\lambda Q_{m-2}h^m D_m - \lambda P_{m-2}h^{m+1}(c_1 + \ldots + c_m)D_m/m + O(h^{m+2}) \tag{5.12}$$

with $Q_{m-2} \neq 0$ *or* $P_{m-2} \neq 0$,

$$M_3 = \lambda^2 h^{m+1} Q_{m-3} D_m/2 + O(h^{m+2}) \tag{5.13}$$

and

$$M_k = O(h^{m+2}), k \geq 4. \tag{5.14}$$

Proof. Let us start with matrix M_1. Writing the columns of the determinant as rows we get

$$M_1 = \begin{vmatrix} \alpha h(c_i - c_1) + \lambda h^2 \dfrac{c_i^2 - c_1^2}{2} \\ 2(c_1 - c_i) + \alpha h(c_i^2 - c_1^2) + \lambda h^2 \dfrac{c_i^3 - c_1^3}{3} \\ \dotfill \\ (m-1)(c_1^{m-2} - c_i^{m-2}) + \alpha h(c_i^{m-1} - c_1^{m-1}) + \lambda h^2 \dfrac{c_i^m - c_1^m}{m} \end{vmatrix}$$

$$= Q_1 h \begin{vmatrix} c_i - c_1 + \dfrac{P_1}{Q_1} h \dfrac{c_i^2 - c_1^2}{2} \\ \dfrac{Q_2}{P_1} h(c_i^2 - c_1^2) + \lambda h^2 \dfrac{c_i^3 - c_1^3}{3} \\ 3(c_1^2 - c_i^2) + \alpha h(c_i^3 - c_1^3) + \lambda h^2 \dfrac{c_i^4 - c_1^4}{4} \\ \dotfill \\ (m-1)(c_1^{m-2} - c_i^{m-2}) + \alpha h(c_i^{m-1} - c_1^{m-1}) + \lambda h^2 \dfrac{c_i^m - c_1^m}{m} \end{vmatrix}$$

$$= Q_{m-1} h^{m-1} \begin{vmatrix} c_i - c_1 + \dfrac{P_1}{Q_1} h \dfrac{c_i^2 - c_1^2}{2} \\ c_i^2 - c_1^2 + \dfrac{P_2}{Q_2} h \dfrac{c_i^3 - c_1^3}{3} \\ \dotfill \\ c_i^{m-1} - c_1^{m-1} + \dfrac{P_{m-1}}{Q_{m-1}} h \dfrac{c_i^m - c_1^m}{m} \end{vmatrix}.$$

The straightforwards calculations give an expansion of the last determinant as a sum of

$$\begin{vmatrix} c_i - c_1 \\ c_i^2 - c_1^2 \\ \dotfill \\ c_i^{m-1} - c_1^{m-1} \end{vmatrix}, \quad \begin{vmatrix} c_i - c_1 \\ c_i^2 - c_1^2 \\ \dotfill \\ c_i^{m-2} - c_1^{m-2} \\ \dfrac{P_{m-1}}{Q_{m-1}} h \dfrac{c_i^m - c_1^m}{m} \end{vmatrix}$$

and other terms of order $O(h^2)$. Now, basing on Lemma 5.1, we get (5.11). As in the proof of Proposition 5.1, this argument is correct if

$Q_{m-1} \neq 0, \ldots, Q_1 \neq 0$, but in general case the limit process will arrange the proof. In addition, $Q_{m-1} = 0$ and $P_{m-1} = 0$ yield by (5.4) that $Q_m = 0$ which is impossible as we have seen earlier. Thus, $Q_{m-1} \neq 0$ or $P_{m-1} \neq 0$. The other formulae (5.12) - (5.14) can be obtained by similar calculations.

Proposition 5.3 *For the solution* (5.10) *it holds*
1) if $Q_{m-1} \neq 0$ and $Q_m \neq 0$ then

$$\mu = -\lambda \frac{Q_{m-1}}{Q_m} h + O(h^2),$$

2) if $Q_{m-1} \neq 0$ and $Q_m = 0$ then

$$\mu = -\frac{m}{c_1 + \ldots + c_m} + O(h),$$

3) if $Q_{m-1} = 0$ and $Q_m \neq 0$ then

$$\mu = O(h^2).$$

Proof. The main term in the numerator of (5.10) is $\lambda h^{m+1} Q_{m-1} D_m$ for $Q_{m-1} \neq 0$. The denominator of (5.10) is

$$(-\alpha h - \lambda h^2 c_1)(Q_{m-1} D_m h^{m-1} + O(h^m))$$
$$+ (1 - \alpha h c_1 - \lambda h^2 \frac{c_1^2}{2})(-\lambda Q_{m-2} h^m D_m) + O(h^{m+1})) + O(h^{m+3}),$$

where we find, by (5.6), that the coefficient of h^m is $-Q_m D_m$. Therefore,

$$\mu = \frac{\lambda h^{m+1} Q_{m-1} D_m + O(h^{m+2})}{-Q_m D_m h^m + O(h^{m+1})} = -\lambda \frac{Q_{m-1}}{Q_m} h + O(h^2).$$

The third assertion follows immediately. If $Q_m = 0$, i.e., the coefficient of h^m in the denominator is zero, then the coefficient of h^{m+1} can be found as $-\lambda(c_1 + \ldots + c_m) D_m Q_{m-1}/m$ which yields the formula for μ in second case. The proof is complete.

It is natural to ask whether μ in (5.10) may have higher order in h than 2? In fact, more detailed calculations show that

$$M_1 = Q_{m-1} h^{m-1} D_m + \lambda Q_{m-2} h^m sym_1 D_m/m$$
$$+ \lambda^2 Q_{m-3} h^{m+1} sym_2 D_m/m(m-1) + \ldots$$
$$+ \lambda^{m-1} Q_0 h^{2m-2} sym_{m-1} D_m/m!,$$

$$M_2 = -\lambda Q_{m-2} h^m D_m - \lambda^2 Q_{m-3} h^{m+1} sym_1 D_m/m - \ldots$$
$$- \lambda^{m-1} Q_0 h^{2m-2} sym_{m-2} D_m/m!,$$

$$M_m = (-1)^{m-1} \lambda^{m-1} Q_0 h^{2m-2} D_m/(m-1)!,$$

where sym_i are standard symmetrical polynomials of $c_1, \ldots, c_m$ of order i, for example, $sym_1 = c_1 + \ldots + c_m$, $sym_2 = c_1c_2 + \ldots + c_{m-1}c_m$.

Proposition 5.4 *If $Q_m \neq 0$, $Q_{m-1} = 0$ and $c_1 + \ldots + c_m = m/2$, then*

$$\mu = \nu h^3 + O(h^4),\ \nu \neq 0,$$

and, for $k > 3$,

$$\mu = \nu h^k + O(h^{k+1}), \nu \neq 0,$$

is not possible.

Proof. The main term in the denominator of (5.10) is $-\lambda Q_{m-2} D_m h^m$ as $Q_{m-2} \neq 0$. In the numerator of (5.10) the coefficient of h^{m+2} is

$$\lambda^2 Q_{m-2} D_m \left(\frac{sym_1}{m} - \frac{1}{2} \right).$$

Therefore,

$$\mu = \lambda \left(\frac{sym_1}{m} - \frac{1}{2} \right) h^2 + O(h^3).$$

If the coefficient of h^{m+2} in the numerator of (5.10) is zero, i.e., $sym_1 = m/2$, we find that the coefficient of h^{m+3} is

$$\lambda^3 Q_{m-3} D_m \left(\frac{sym_2}{m(m-1)} - \frac{1}{2}\frac{sym_1}{m} + \frac{1}{6} \right).$$

Assuming that the coefficient of h^{m+3} is also zero, for $Q_{m-3} \neq 0$, we have

$$sym_2 = \frac{m(m-1)}{12}.$$

Now calculate

$$sym_1^2 = c_1^2 + ... + c_m^2 + 2sym_2$$
$$= c_1^2 + ... + c_m^2 + \frac{m(m-1)}{6},$$

from where

$$c_1^2 + ...c_m^2 = \frac{m^2 + 2m}{12}.$$

We get that $(m^2+2m)/12 \geq m/2$ for $m \geq 4$, i.e., $sym_2^2 \geq sym_1^2$. On the other side, always for $m \geq 2$ it holds $c_1 + ... + c_m > c_1^2 + ... + c_m^2$. Therefore,

$$sym_1 = \frac{m}{2}, \quad sym_2 = \frac{m(m-1)}{12}$$

cannot be valid together. Actually, $Q_{m-3} \neq 0$, because $Q_{m-2} \neq 0$ and $Q_{m-3} = 0$ yield, by (5.6), that $Q_{m-1} \neq 0$. This contradiction completes the proof.

Denote $M = I + \lambda h^2 W^{-1} V_3$.

Matrix M has eigenvalues with equal algebraic and geometric multiplicities. This implies that its Jordan form is diagonal matrix with $m-1$ entries 1 and one $1+\epsilon$ with $\epsilon = O(h^k), k = 0, \ldots, 3$. The Jordan representation $M = PJP^{-1}$ gives $M^n = PJ^nP^{-1}$ and at least for $k \geq 1$ the matrix J^n is bounded. We see that the boundedness of M^n depends also on behaviour of P and P^{-1} in process $h \to 0$.

Proposition 5.5 *Matrix $\lambda h^2 W^{-1} V_3$ or $W^{-1} V_3$ has the same eigenvectors as the matrix M.*

Proof. Let μ be an eigenvalue of $W^{-1}V_3$ and $v \neq 0$ a corresponding eigenvector. Then

$$W^{-1}V_3 v = \mu v \Leftrightarrow \lambda h^2 W^{-1} V_3 v = \lambda h^2 \mu v$$

$$\Leftrightarrow (I + \lambda h^2 W^{-1} V_3) v = (1 + \lambda h^2 \mu) v,$$

which gives the assertion.

The eigenvalues of $W^{-1}V_3$ could be chosen in such a way that they are the columns of P. Take them as an orthonormal system $p^1, \ldots, p^{m-1}$ corresponding to $0 \in \operatorname{spec}(W^{-1}V_3)$, which give $p^1, \ldots, p^{m-1} \in \operatorname{Ker} V_3$, and p^m of Euclidean norm 1 corresponding to $\epsilon \in \operatorname{spec}(W^{-1}V_3)$. Clearly P is bounded. The boundedness of P^{-1} can be guaranteed if $\mid \det P \mid \geq \delta$ for some $\delta > 0$. This takes place if we get $< p, q > \leq \sigma \|p\| \, \|q\|$ with $\sigma < 1$ for all $p \in \operatorname{Ker} V_3$ and all $q \in \operatorname{Ker}(\lambda h^2 V_3 - \mu W)$ which is equivalent to $< p, q > \leq \sigma \|p\| \, \|q\|$ for all $p \in (\operatorname{Ker} V_3)^\perp$ and all $q \in (\operatorname{Ker}(\lambda h^2 V_3 - \mu W))^\perp$. Here we may consider $p = (1, 1/2, \ldots, 1/m)$ because $\dim(\operatorname{Ker} V_3)^\perp = 1$ and $q = \sum_{1 \leq j \leq m-1} \lambda_j q^j$ with q^j (we write q^j here in column)

$$q^j = \begin{pmatrix} \lambda h^2 - \mu(-\alpha h - \lambda h^2 c_j) \\ \lambda h^2/2 - \mu(1 - \alpha h c_j - \lambda h^2 c_j^2/2) \\ \cdots \\ \lambda h^2/m - \mu((m-1)c_j^{m-2} - \alpha h c_j^{m-1} - \lambda h^2 c_j^m/m) \end{pmatrix}$$

as $q^1, \dots, q^{m-1}$ give a basis in $(\mathrm{Ker}\,(h^2 V_3 - \mu W))^\perp$ at least for small h. Let $\bar{q}^j = q^j - \lambda h^2 p$. Since $\det W \neq 0$, $\bar{q}^1, \dots, \bar{q}^{m-1}$ are linearly independent. Similarly, we get also the linear independence of p, $\bar{q}^1, \dots, \bar{q}^{m-1}$ for small h. Then

$$< p, q >= \Big(\sum_{j=1}^{m-1} \lambda_j \Big) \lambda h^2 < p, p > - \mu \Big\langle p, \sum_{j=1}^{m-1} \lambda_j \bar{q}^j \Big\rangle . \tag{5.15}$$

We may consider only the "worse" case, namely, when q is the projection of p onto $(\mathrm{Ker}\,(\lambda h^2 V_3 - \mu W))^\perp$. Then in the process $h \to 0$ the coefficients λ_j stabilize and

$$\Big\langle p, \sum_{j=1}^{m-1} \lambda_j \bar{q}^j \Big\rangle \approx \sigma_0 \|p\| \, \| \sum_{j=1}^{m-1} \lambda_j \bar{q}^j \|$$

for some fixed $\sigma_0 \in (-1, 1)$ due to the linear independence of p, $\bar{q}^1, \dots, \bar{q}^m$. In the cases $\mu \sim \nu h$, $\nu \neq 0$, and $\mu \sim const$ the last term in (5.15) is dominant and we get $< p, q > \leq \sigma \|p\| \, \|q\|$ with $\sigma < 1$ (actually, $\sigma \to \sigma_0$).

Note that the case $\mu \sim \nu h^k$, $\nu \neq 0$, $k \geq 2$ needs additional analysis but similar arguments lead us also to the boundedness of M^n.

Summing up the results of presented reasonings and Proposition 5.3 we have

Proposition 5.6 *The following holds*

1. *if $Q_m \neq 0$ then the method is stable,*
2. *if $Q_m = 0$ (and hence $Q_{m-1} \neq 0$) then for $c_1 + \ldots + c_m \geq m/2$ the method is stable, for $c_1 + \ldots + c_m < m/2$ unstable.*

For example, let $m = 3$. We have $Q_3 = 2\lambda\alpha + \alpha^3$. For $2\lambda\alpha + \alpha^3 \neq 0$ the method is stable and for $2\lambda\alpha + \alpha^3 = 0$ the stability region is $c_1 + c_2 + c_3 \geq 3/2$.

Chapter 6

STABILITY OF THE SPLINE COLLOCATION METHOD FOR SECOND ORDER VIDE

In this chapter we will investigate stability conditions for second order VIDE. We will show that there is connection between stability conditions for 1st order VIDE and 2nd one. The treatment is similar to those in Chapter 4.

6.1 Method in the case of test equation

Consider the test equation

$$y''(t) = \alpha y(t) + \beta y'(t) + \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0,T], \tag{6.1}$$

where, in general, α, β and λ may be any complex numbers. Similarly to Section 4.1, assume that the mesh sequence $\{\Delta_N\}$ is uniform, i.e., $h_n = h = T/N$ for all n. We will use the representation (4.2) on σ_n and smoothness conditions (4.3).

The collocation conditions (2.5), applied to the test equation (6.1), give

$$u''(t_{nj}) = \alpha y(t_{nj}) + \beta u'(t_{nj}) + \lambda \int_0^{t_{nj}} u(s)ds + f(t_{nj}),\ j = 1, \ldots, m,$$

$$n = 1, \ldots, N. \tag{6.2}$$

From (4.2) we get

$$u_n(t_{nj}) = \sum_{k=0}^{m+d} a_{nk} c_j^k,$$

$$u_n'(t_{nj}) = \frac{1}{h}\sum_{k=1}^{m+d} a_{nk}kc_j^{k-1}$$

and

$$u_n''(t_{nj}) = \frac{1}{h^2}\sum_{k=2}^{m+d} k(k-1)a_{nk}c_j^{k-2}.$$

Now the equation (6.2) becomes

$$\begin{aligned}
&\frac{1}{h^2}\sum_{k=0}^{m+d} k(k-1)a_{nk}kc_j^{k-2}\\
&= \alpha\sum_{k=0}^{m+d} a_{nk}c_j^k + \beta\frac{1}{h}\sum_{k=0}^{m+d} ka_{nk}c_j^{k-1} + \sum_{r=1}^{n-1}\lambda\int_{t_{r-1}}^{t_r} u_r(s)ds\\
&+ \lambda\int_{t_{n-1}}^{t_{nj}} u_n(s)ds + f(t_{nj})\\
&= \alpha\sum_{k=0}^{m+d} a_{nk}c_j^k + \beta\frac{1}{h}\sum_{k=0}^{m+d} ka_{nk}c_j^{k-1} + \sum_{r=1}^{n-1}\lambda h\int_0^1 \Big(\sum_{k=0}^{m+d} a_{rk}\tau^k\Big)d\tau\\
&+ \lambda h\int_0^{c_j} \Big(\sum_{k=0}^{m+d} a_{nk}\tau^k\Big)d\tau + f(t_{nj})\\
&= \alpha\sum_{k=0}^{m+d} a_{nk}c_j^k + \beta\frac{1}{h}\sum_{k=0}^{m+d} ka_{nk}c_j^{k-1} + \sum_{r=1}^{n-1}\lambda h\ \Big(\sum_{k=0}^{m+d} \frac{1}{k+1}a_{rk}\Big)\\
&+ \lambda h\sum_{k=0}^{m+d} a_{nk}\frac{c_j^{k+1}}{k+1} + f(t_{nj}). \qquad (6.3)
\end{aligned}$$

Using the notation $\alpha_n = (a_{n0}, \ldots, a_{n,m+d})$, we write (6.3) as follows:

$$\begin{aligned}
&\sum_{k=0}^{m+d} a_{nk}k(k-1)c_j^{k-2} - \alpha h^2\sum_{k=0}^{m+d} a_{nk}c_j^k - \beta h\sum_{k=0}^{m+d} a_{nk}kc_j^{k-1}\\
&- \lambda h^3\sum_{k=0}^{m+d} a_{nk}\frac{c_j^{k+1}}{k+1} = \lambda h^3\langle q, \sum_{r=1}^{n-1}\alpha_r\rangle + h^2 f(t_{nj}), \qquad (6.4)
\end{aligned}$$

where $q = (1, 1/2, \ldots, 1/(m+d+1))$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^{m+d+1}$. The difference of the equations (6.4) with n and $n+1$ yields

$$\begin{aligned}
&\sum_{k=0}^{m+d} a_{n+1,k} k(k-1) c_j^{k-2} - \beta h \sum_{k=0}^{m+d} a_{n+1,k} k c_j^{k-1} - \alpha h^2 \sum_{k=0}^{m+d} a_{n+1,k} c_j^k \\
&- \lambda h^3 \sum_{k=0}^{m+d} a_{n+1,k} \frac{c_j^{k+1}}{k+1} \\
&= \sum_{k=0}^{m+d} a_{nk} k(k-1) c_j^{k-2} - \beta h \sum_{k=0}^{m+d} a_{nk} k c_j^{k-1} - \alpha h^2 \sum_{k=0}^{m+d} a_{nk} c_j^k \\
&- \lambda h^3 \sum_{k=0}^{m+d} a_{nk} \frac{c_j^{k+1}}{k+1} + \lambda h^3 \langle q, \alpha_n \rangle + h^2 f(t_{n+1,j}) - h^2 f(t_{nj}), \\
&j = 1, \ldots, m, \; n = 1, \ldots, N-1. \qquad (6.5)
\end{aligned}$$

Now we may write together the equations (4.3) and (6.5) in the matrix form

$$\begin{aligned}
&(\widetilde{V} - \beta h \widetilde{V}_1 - \alpha h^2 \widetilde{V}_2 - \lambda h^3 \widetilde{V}_3) \alpha_{n+1} \\
&\qquad = (\widetilde{V}_0 - \beta h \widetilde{V}_1 - \alpha h^2 \widetilde{V}_2 - \lambda h^3 (\widetilde{V}_3 - \widetilde{V}_4)) \alpha_n + h^2 g_n, \\
&\qquad\qquad\qquad\qquad n = 1, \ldots, N-1, \quad (6.6)
\end{aligned}$$

with $(m+d+1) \times (m+d+1)$ matrices $\widetilde{V}$, $\widetilde{V}_0$, $\widetilde{V}_1$, $\widetilde{V}_2$, $\widetilde{V}_3$, $\widetilde{V}_4$ as follows:

$$\widetilde{V} = \begin{pmatrix} E \\ C \end{pmatrix}, \qquad \widetilde{V}_0 = \begin{pmatrix} A \\ C \end{pmatrix}, \qquad E = \begin{pmatrix} I & 0 \end{pmatrix}$$

I being the $(d+1) \times (d+1)$ identity matrix, 0 is the $(d+1) \times m$ zero matrix,

$$C = \begin{pmatrix} 0 & 0 & 2 & 6c_1 & \ldots & (m+d)(m+d-1)c_1^{m+d-2} \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 2 & 6c_m & \ldots & (m+d)(m+d-1)c_m^{m+d-2} \end{pmatrix},$$

A being a $(d+1) \times (m+d+1)$ matrix, defined in Section 3.1,

$$\widetilde{V}_1 = \begin{pmatrix} & & 0 & & \\ 0 & 1 & 2c_1 & \dots & (m+d)c_1^{m+d-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2c_m & \dots & (m+d)c_m^{m+d-1} \end{pmatrix},$$

$$\widetilde{V}_2 = \begin{pmatrix} & & 0 & & \\ 1 & c_1 & {c_1}^2 & \dots & c_1^{m+d} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & c_m & c_m^2 & \dots & c_m^{m+d} \end{pmatrix},$$

$$\widetilde{V}_3 = \begin{pmatrix} & & 0 & \\ c_1 & c_1^2/2 & \dots & c_1^{m+d+1}/(m+d+1) \\ \dots & \dots & \dots & \dots \\ c_m & c_m^2/2 & \dots & c_m^{m+d+1}/(m+d+1) \end{pmatrix},$$

$\widetilde{V}_4$ having the first $d+1$ rows equal to 0 and the last m rows the vector q, and, the $m+d+1$ dimensional vector

$$g_n = (0, \dots, 0, f(t_{n+1,1}) - f(t_{n1}), \dots, f(t_{n+1,m}) - f(t_{nm})).$$

Thus $g_n = O(h)$ for $f \in C^1$.

Proposition 6.1 *The matrix $\widetilde{V} - \beta h \widetilde{V}_1 - \alpha h^2 \widetilde{V}_2 - \lambda h^3 \widetilde{V}_3$ is invertible for sufficiently small h.*

Proof. If $d \geq 1$, we have

$$\det \widetilde{V} = \begin{vmatrix} (d+1)dc_1^{d-1} & \dots & (m+d)(m+d-1)c_1^{m+d-2} \\ (d+1)dc_2^{d-1} & \dots & (m+d)(m+d-1)c_2^{m+d-2} \\ \dots & \dots & \dots \\ (d+1)dc_m^{d-1} & \dots & (m+d)(m+d-1)c_m^{m+d-2} \end{vmatrix}$$

$$= (d+1)dc_1^{d-1} \dots (m+d)(m+d-1)c_m^{d-1} \cdot$$

$$\cdot \begin{vmatrix} 1 & c_1 & \dots & c_1^{m-1} \\ \dots & \dots & \dots & \dots \\ 1 & c_m^d & \dots & c_m^{m-1} \end{vmatrix} \neq 0,$$

and the matrix $\widetilde{V}$ is invertible. Such is also $\widetilde{V} - \beta h\widetilde{V}_1 - \alpha h^2\widetilde{V}_2 - \lambda h^3\widetilde{V}_3$ for small h. Although we have supposed, in general, that $d \geq 1$, let us remark that in cases $d = 0$ and $d = -1$ we may argue similarly to the proof in Chapter 5, and show that $\det(\widetilde{V} - \beta h\widetilde{V}_1 - \alpha h^2\widetilde{V}_2 - \lambda h^3\widetilde{V}_3) \neq 0$, for small h. Therefore, the equation (6.6) can be written in the form

$$\alpha_{n+1} = (\widetilde{V}^{-1}\widetilde{V}_0 + W)\alpha_n + r_n, \quad n = 1, \ldots, N-1, \tag{6.7}$$

where $W = O(h)$ and $r_n = O(h^3)$ for $f \in C^1$.

6.2 Stability of the method

We have seen that the spline collocation method (2.5) for the test equation (6.1) leads to the recursion (6.7). Denote $\widetilde{M} = \widetilde{V}^{-1}\widetilde{V}_0$.

We distinguish the method with initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \ldots, d$, and another method which uses $u_1(0) = y(0), u_1'(0) = y'(0)$ and additional collocation points $t_{0j} = t_0 + c_{0j}h$, $j = 1, \ldots, d-1$, with fixed $c_{0j} \in (0, 1] \setminus \{c_1, \ldots, c_m\}$ on the first interval σ_1.

Denote, in addition, $d_0 = \max\{d-2, 0\}$ for the method with initial values and $d_0 = 0$ for the method with additional initial collocation.

Definition 6.1 *We say that the spline collocation method is stable if for any $\alpha, \beta, \lambda \in \mathbb{C}$ and any $f \in C^{d_0}[0, T]$ the approximate solution u of* (6.1) *remains bounded in $C[0, T]$ in the process $h \to 0$.*

The principle of uniform boundedness allows us to establish

Proposition 6.2 *The spline collocation method is stable if and only if*

$$||u||_{C[0,T]} \leq const\, ||f||_{C^{d_0}[0,T]} \quad \forall f \in C^{d_0}[0, T], \tag{6.8}$$

where the constant may depend only on T, α, β, λ and on parameters c_j and c_{0j}.

Theorem 6.1 *For fixed c_j the eigenvalues of $\widetilde{M}$ for 2nd order VIDE in the case m and $d+2$ and eigenvalues of M for 1st order VIDE in the case m and $d+1$ coincide and have the same algebraic and geometric multiplicities, except $\mu = 1$ whose algebraic multiplicity for 2nd order VIDE is greater by one than for 1st order VIDE.*

Proof. The structure of the proof is similar to that of Theorem 4.1 in Chapter 4.

The eigenvalue problem for $\widetilde{M}$ is equivalent to the generalized eigenvalue problem for $\widetilde{V}_0$ and $\widetilde{V}$, i.e., $(\widetilde{M} - \mu I)v = 0$ for $v \neq 0$ if and only if

$(\widetilde{V}_0 - \mu\widetilde{V})v = 0$ and $(\widetilde{M} - \mu I)w = v$ takes place if and only if $(\widetilde{V}_0 - \mu\widetilde{V})w = \widetilde{V}v$. Denote $\nu = 1 - \mu$. Then for 2nd order VIDE with the parameters m and $d+2$ we have

$$
\widetilde{V}_0 - \mu\widetilde{V} =
$$

$$
= \left| \begin{array}{ccccccc}
\nu & 1 & 1 & 1 & \dots & \dots & 1 \\
0 & \nu & 2 & 3 & \dots & \dots & m+d+1 \\
0 & 0 & \nu & \binom{3}{2} & \dots & \dots & \binom{m+d+2}{2} \\
\multicolumn{7}{c}{\dotfill} \\
0 & \dots & \dots & \dots & \nu & \dots & \binom{m+d+2}{d+2} \\
0 & 0 & \nu\cdot 2 & \nu\cdot 6c_1 & \dots & \dots & \nu(m+d+2)(m+d+1)c_1^{m+d} \\
\multicolumn{7}{c}{\dotfill} \\
0 & 0 & \nu\cdot 2 & \nu\cdot 6c_m & \dots & \dots & \nu(m+d+2)(m+d+1)c_m^{m+d}
\end{array} \right| . \tag{6.9}
$$

Let $I_{i,p}$ be the diagonal matrix obtained from identity matrix, replacing the i-th diagonal element by the number p. Consider also the matrices V_0 and V defined in Chapter 4, with the parameters m and $d+1$. Now, using relation $\binom{p}{q}\frac{q}{p} = \binom{p-1}{q-1}$, we get from (6.9)

$$
\begin{aligned}
& I_{d+3,d+2}\dots I_{3,2}(\widetilde{V}_0 - \mu\widetilde{V})I_{3,1/2}\dots I_{d+m+3,1/(m+d+2)} \\
& = \begin{pmatrix} \nu & \overline{q} \\ 0 & V_0 - \mu V \end{pmatrix}
\end{aligned}
$$

or

$$
S(\widetilde{V}_0 - \mu\widetilde{V})S^{-1} = R\begin{pmatrix} \nu & \overline{q} \\ 0 & V_0 - \mu V \end{pmatrix}, \tag{6.10}
$$

where

$$
S = I_{d+3,d+2}\dots I_{3,2},
$$

$$
R = I_{d+m+3,d+m+2}\dots I_{d+4,d+3}
$$

and

$$
\overline{q} = \left(1, \frac{1}{2}, \dots, \frac{1}{m+d+2}\right).
$$

Now the equation (6.10) gives us

$$\det(\widetilde{V}_0 - \mu\widetilde{V}) = (d+3)\dots(d+m+2)\nu\det(V_0 - \mu V),$$

which permits to get the assertion about algebraic multiplicities of eigenvalues of $\widetilde{M}$ and M. Similarly to Propositions 4.3 or 3.1 we can prove that the eigenvalue $\mu = 1$ of $\widetilde{M}$ has geometric multiplicity m and similarly to proof of Theorem 4.1 that geometric multiplicities of $\mu \neq 1$ as an eigenvalue of $\widetilde{M}$ and M coincide. The proof is complete.

Proposition 6.3 *If $\widetilde{M}$ has an eigenvalue outside of the closed unit disk, then the spline collocation method is not stable with possible exponential growth of approximate solution.*

Proof. The structure of the proof is similar to that of Proposition 4.4 in Chapter 4 and we will deal only with main moments.

Consider an eigenvalue μ of $\widetilde{M} + W$ such that $|\mu| \geq 1 + \delta$ with some fixed $\delta > 0$ for any sufficiently small h. For $\alpha_1 \neq 0$, being an eigenvector of $\widetilde{M} + W$, we have here

$$(\widetilde{V} - \beta h\widetilde{V}_1 - \alpha h^2\widetilde{V}_2 - \lambda h^3\widetilde{V}_3)\alpha_1 = h^2 g_0, \tag{6.11}$$

where

$$g_0 = (\alpha_{10}, \dots, \alpha_{1d}, f(t_{11}), \dots, f(t_{1m}))$$

and

$$\alpha_{1j} = h^j \frac{y^{(j)}(0)}{j!},\ j = 0, \dots, d.$$

Because of

$$y''(0) = \alpha y(0) + \beta y'(0) + f(0),$$

$$y^{(j)}(0) = \alpha y^{(j-2)}(0) + \beta y^{(j-1)}(0) + \lambda y^{(j-3)}(0) + f^{(j-2)}(0), \quad j = 3, \dots, d, \tag{6.12}$$

the vector α_1 determines via (6.11) and (6.12) the values

$$f^{(j)}(0),\ j = 0, \dots, d-1,\ f(t_{11}), \dots, f(t_{1m}).$$

We take f on $[0, h]$ as the polynomial interpolating the values $f^{(j)}(0)$, $j = 0, \dots, d-2$, $f(t_{1j})$, $j = 1, \dots, m$, and $f^{(j)}(h) = 0$, $j = 0, \dots, d_0$ (if $c_m = 1$, then $f^{(j)}(h) = 0$, $j = 1, \dots, d_0$).

In the case of the method of additional knots let f be on $[0, h]$ the interpolating polynomial by the data $f(0)$, $f(t_{0j})$, $j = 0, \dots, d-1$, $f(t_{1j})$,

$j = 1, \ldots, m$, and $f^{(j)}(h) = 0$ (here $d_0 = 0$ and if $c_m = 1$, then $f(t_{1m}) = f(h)$ is already given and we drop the requirement $f(h) = 0$).

In both cases we ask f to be on $[nh, (n+1)h]$, $n \geq 1$, the interpolating polynomial by the values $f^{(j)}(nh) = 0$ and $f^{(j)}((n+1)h) = 0$, $j = 0, \ldots, d_0$ (if $c_m = 1$, then for $j = 1, \ldots, d_0$), and also $f(t_{n+1,j}) = f(t_{1j})$, $j = 1, \ldots, m$. This guarantees that $f \in C^{d_0}[0,T]$ and $r_n = 0$, $n \geq 1$.

The interpolant f can be represented on $[t_n, t_{n+1}]$ by the formula:

$$f(t) = f(t_n + \tau h) = \sum_{i=0}^{\kappa} \left(\sum_{l=0}^{k_i} h^{s_l} p_{il} f^{(s_l)}(\xi_l) \right) \prod_{r=0}^{i-1} (\tau - b_r) \tag{6.13}$$

with b_r being c_j or c_{0j}, ξ_l being t_{nj} or t_j, $0 \leq s_l \leq d_1$, $k_i \leq i$, constants p_{il} depending on c_j and c_{0j}.

In the case of initial conditions $\kappa = m + d + d_0 - 1$ ($\kappa = m + d + d_0 - 2$, if $c_m = 1$), in the case of additional knots $\kappa = m+d+1$ ($\kappa = m+d$, if $c_m = 1$) on the interval $[0,h]$ and $\kappa = m + 2d_0 + 1$ ($\kappa = m + 2d_0$ if $c_m = 1$) on the interval $[nh, (n+1)h]$, $n \geq 1$.

Replacing h by h/k, $k = 1, 2, \ldots$, and keeping $||\alpha_1|| = h^2/k^2$, we have $||g_0||_\infty$ bounded which means that $f(t_{1j})$, $j = 1, \ldots, m$, and $h^j y^{(j)}(0)/k^j$, $j = 0, \ldots, d$, or $h^j f^{(j)}(0)/k^j$, $j = 0, \ldots, d_0$, are bounded too in the process $k \to \infty$. Thus (6.13) gives

$$||f||_{C^{d_0}[0,T]} \leq const\, k^{d_0}. \tag{6.14}$$

On the other hand,

$$||\alpha_{n+1}|| \geq (1+\delta)^n ||\alpha_1||$$

yields

$$||\alpha_{kN}|| \geq \frac{h}{k}(1+\delta)^{kN-1} \tag{6.15}$$

and (6.8) cannot be satisfied. The inequalities (6.14) and (6.15) mean also the exponential growth of approximate solution if we keep the norm of f bounded in C^{d_0}. The proof is complete.

6.3 Examples

Let us consider some special cases of d and m.

Case $d = 1$, $m \geq 1$.
We have

$$\widetilde{V} = \left(\frac{1\,0\,\ldots\,0}{C} \right), \quad \widetilde{V} = \left(\frac{1\,1\,\ldots\,1}{C} \right)$$

and $\det(\widetilde{V}_0 - \mu \widetilde{V}) = (1-\mu)^{m+2} \det C_0$ where C_0 is obtained from C omitting first two columns. This means that the method is always stable.

Case $d = 2$, $m = 1$ *(cubic splines).*

The equation $\det(\widetilde{V}_0 - \mu\widetilde{V}) = 0$ besides $\mu = 1$ has the solution $\mu = 1 - 1/c_1$. The method is stable if and only if $1/2 \le c_1 \le 1$.

Case $d = 2$, $m = 2$.

Now the equation $\det(\widetilde{V}_0 - \mu\widetilde{V}) = 0$ has the root $\mu = 1$ with geometric multiplicity 2. Similarly to the case $d = 1$, $m = 2$ for 1st order VIDE (see Section 4.3) we get that from the solution $\mu(c_1, c_2) = 1 - (c_1 + c_2 + 1)/c_1c_2$ it follows that the method is stable if and only if $c_1 + c_2 \ge 1$.

Chapter 7

STABILITY OF THE SPLINE COLLOCATION METHOD WITH MULTIPLE NODES FOR FIRST ORDER VIDE

In this chapter we will analyze the stability of collocation method when, on each subinterval, there is only one collocation point with multiplicity m.

7.1 Method in the case of test equation

Consider the test equation

$$y'(t) = \alpha y(t) + \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0, T], \tag{7.1}$$

where, in general, λ and $\alpha \neq 0$ may be any complex numbers.

As in the previous chapters the smoothness conditions on uniform mesh (for any $u \in S_m^d(\Delta_N)$) give the equalities (4.3).

For given $c \in (0, 1]$ denote here $t_{nc} = t_{n-1} + ch$, $n = 1, \ldots, N$. From collocation conditions (2.3), applied to the test equation (7.1), we get

$$\begin{aligned}
&\sum_{k=0}^{m+d} a_{n+1,k} k c^{k-1} - \alpha h \sum_{k=0}^{m+d} a_{n+1,k} c^k - \lambda h^2 \sum_{k=0}^{m+d} a_{n+1,k} \frac{c^{k+1}}{k+1} \\
&= \sum_{k=0}^{m+d} a_{nk} k c^{k-1} - \alpha h \sum_{k=0}^{m+d} a_{nk} c^k - \lambda h^2 \sum_{k=0}^{m+d} a_{nk} \frac{c^{k+1}}{k+1} \\
&+ \lambda h^2 \sum_{k=0}^{m+d} \frac{1}{k+1} a_{nk} + h \left(f(t_{n+1,c}) - f(t_{nc}) \right), n = 1, \ldots, N-1. \qquad (7.2)
\end{aligned}$$

In addition to (7.2) we have $m-1$ equations

$$y^{(i)}(t) = \alpha y^{(i-1)}(t) + \lambda y^{(i-2)}(t) + f^{(i-1)}(t), \quad i = 2, ..., m,$$

which at collocation points can be written as follows

$$u_n^{(i)}(t_{nc}) = \alpha u_n^{(i-1)}(t_{nc}) + \lambda u_n^{(i-2)}(t_{nc}) + f^{(i-1)}(t_{nc}), \quad i = 2, ..., m. \quad (7.3)$$

Now using relations

$$u_n(t_{nc}) = \sum_{k=0}^{m+d} a_{nk} c^k,$$

$$u_n'(t_{nc}) = \frac{1}{h} \sum_{k=1}^{m+d} k a_{nk} c^{k-1},$$

and

$$u_n^{(i)}(t_{nc}) = \frac{1}{h^i} \sum_{k=i}^{m+d} \frac{k!}{(k-i)!} a_{nk} c^{k-i}, \quad i = 2, \ldots, m, \quad (7.4)$$

the equations (7.3) become

$$\frac{1}{h^i} \sum_{k=i}^{m+d} \frac{k!}{(k-i)!} a_{nk} c^{k-i} = \frac{\alpha}{h^{i-1}} \sum_{k=i-1}^{m+d} \frac{k!}{(k-i+1)!} a_{nk} c^{k-i+1}$$

$$+ \frac{\lambda}{h^{i-2}} \sum_{k=i-2}^{m+d} \frac{k!}{(k-i+2)!} a_{nk} c^{k-i+2} + f^{(i-1)}(t_{nc}), \quad i = 2, .., m,$$

or, in the form

$$\sum_{k=i}^{m+d} \frac{k!}{(k-i)!} a_{nk} c^{k-i} = \alpha h \sum_{k=i-1}^{m+d} \frac{k!}{(k-i+1)!} a_{nk} c^{k-i+1}$$

$$+ \lambda h^2 \sum_{k=i-2}^{m+d} \frac{k!}{(k-i+2)!} a_{nk} c^{k-i+2} + h^i f^{(i-1)}(t_{nc}), \quad i = 2, ..., m. \quad (7.5)$$

Remark 7.1 Even though in general $m > 0$, $d \geq -1$ and $c \in (0, 1]$ can be any numbers, if $c = 1$ and $m > d$ we should use one-sided derivates in (7.4). Therefore, it is natural to assume for $c = 1$ that $m \leq d$.

The difference of the equations (7.5) with n and $n+1$ yields

$$\sum_{k=i}^{m+d} \frac{k!}{(k-i)!} a_{n+1,k} c^{k-i} - \alpha h \sum_{k=i-1}^{m+d} \frac{k!}{(k-i+1)!} a_{n+1,k} c^{k-i+1}$$

$$- \lambda h^2 \sum_{k=i-2}^{m+d} \frac{k!}{(k-i+2)!} a_{n+1,k} c^{k-i+2}$$

$$= \sum_{k=i}^{m+d} \frac{k!}{(k-i)!} a_{nk} c^{k-i} - \alpha h \sum_{k=i-1}^{m+d} \frac{k!}{(k-i+1)!} a_{nk} c^{k-i+1}$$

$$- \lambda h^2 \sum_{k=i-2}^{m+d} \frac{k!}{(k-i+2)!} a_{nk} c^{k-i+2}$$

$$+ h^i \left(f^{(i-1)}(t_{n+1,c}) - f^{(i-1)}(t_{nc}) \right), \quad i = 2, ..., m. \tag{7.6}$$

Now we may write together the equations (4.3), (7.2) and (7.6) in matrix form

$$(V - \alpha h V_1 - \lambda h^2 V_2)\alpha_{n+1}$$
$$= (V_0 - \alpha h V_1 - \lambda h^2 (V_2 - V_3))\alpha_n + g_n,$$
$$n = 1, \dots, N-1, \tag{7.7}$$

with $(m+d+1) \times (m+d+1)$ matrices V, V_0, V_1, V_2, V_3 as follows:

$$V = \begin{pmatrix} I \;|\; 0 \\ \hline C \end{pmatrix}, \qquad V_0 = \begin{pmatrix} A \\ C \end{pmatrix},$$

I being the $(d+1) \times (d+1)$ identity matrix, 0 is the $(d+1) \times m$ zero matrix,

$$C = \begin{pmatrix} 0 & 1 & 2c & 3c^2 & 4c^3 & \dots & \dots & \dots & (m+d)c^{m+d-1} \\ 0 & 0 & 2! & 3!c & \frac{4!}{2!}c^2 & \dots & \dots & \dots & \frac{(m+d)!}{(m+d-2)!}c^{m+d-2} \\ 0 & 0 & 0 & 3! & 4!c & \dots & \dots & \dots & \frac{(m+d)!}{(m+d-3)!}c^{m+d-3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & m! & \dots & \frac{(m+d)!}{d!}c^d \end{pmatrix},$$

A being a $(d+1) \times (m+d+1)$ matrix, defined in Section 3.1,

$$V_1 = \begin{pmatrix} & & & & 0 & & & \\ 1 & c & c^2 & c^3 & \dots & \dots & \dots & c^{m+d} \\ 0 & 1 & 2c & 3c^2 & \dots & \dots & \dots & (m+d)c^{m+d-1} \\ 0 & 0 & 2! & 3!c & \dots & \dots & \dots & \dfrac{(m+d)!}{(m+d-2)!}c^{m+d-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & (m-1)! & \dots & \dfrac{(m+d)!}{(d+1)!}c^{d+1} \end{pmatrix},$$

$$V_2 = \begin{pmatrix} & & & & 0 & & & \\ c & \dfrac{c^2}{2} & \dfrac{c^3}{3} & \dfrac{c^4}{4} & \dots & \dots & \dots & \dfrac{c^{m+d+1}}{m+d+1} \\ 1 & c & c^2 & c^3 & \dots & \dots & \dots & c^{m+d} \\ 0 & 1 & 2c & 3c^2 & \dots & \dots & \dots & (m+d)c^{m+d-1} \\ 0 & 0 & 2! & 3!c & \dots & \dots & \dots & \dfrac{(m+d)!}{(m+d-2)!}c^{m+d-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & (m-2)! & \dots & \dfrac{(m+d)!}{(d+2)!}c^{d+2} \end{pmatrix},$$

$$V_3 = \begin{pmatrix} & & 0 & & \\ 1 & 1/2 & 1/3 & \dots & 1/(m+d+1) \\ & & 0 & & \end{pmatrix},$$

and the $m+d+1$ dimensional vector

$$g_n = (0, \dots, 0, h(f(t_{n+1,c}) - f(t_{nc})), \dots, h^m(f(t_{n+1,c}) - f(t_{nc}))) .$$

Alternative to the representation (7.7) is to write (7.5) for $n+1$

$$\sum_{k=i}^{m+d} \frac{k!}{(k-i)!} a_{n+1,k} c^{k-i} = \alpha h \sum_{k=i-1}^{m+d} \frac{k!}{(k-i+1)!} a_{n+1,k} c^{k-i+1}$$

$$+\lambda h^2 \sum_{k=i-2}^{m+d} \frac{k!}{(k-i+2)!} a_{n+1,k} c^{k-i+2} + h^i f^{(i-1)}(t_{n+1,c}),$$

$$i = 2, ..., m. \qquad (7.8)$$

Now equations (4.3), (7.2) and (7.1) give us

$$(V - \alpha h V_1 - \lambda h^2 V_2)\alpha_{n+1}$$
$$= (\widetilde{V}_0 - \alpha h \widetilde{V}_1 - \lambda h^2(\widetilde{V}_2 - V_3))\alpha_n + \widetilde{g}_n,$$
$$n = 1, \ldots, N-1, \quad (7.9)$$

where $\widetilde{V}_0$, $\widetilde{V}_1$, $\widetilde{V}_2$ are $(m+d+1) \times (m+d+1)$ matrices as follows:

$$\widetilde{V}_0 = \begin{pmatrix} & & & A & \\ 0 & 1 & 2c & \ldots & (m+d)c^{m+d-1} \\ & & & 0 & \end{pmatrix},$$

$$\widetilde{V}_1 = \begin{pmatrix} & & & 0 & \\ 1 & c & c^2 & \ldots & c^{m+d} \\ & & & 0 & \end{pmatrix},$$

$$\widetilde{V}_2 = \begin{pmatrix} & & & & 0 \\ c & c^2/2 & c^3/3 & \ldots & c^{m+d+1}/(m+d+1) \\ & & & & 0 \end{pmatrix},$$

and, finally, the $m+d+1$ dimensional vector

$$\widetilde{g}_n = \left(0, ..., 0, h(f(t_{n+1,c}) - f(t_{nc})), h^2 f'(t_{n+1,c}), ..., h^m f^{(m-1)}(t_{n+1,c})\right).$$

Proposition 7.1 *The matrix $V - \alpha h V_1 - \lambda h^2 V_2$ is invertible for sufficiently small h.*

Proof. We have

$$\det V = \begin{vmatrix} (d+1)c^d & \dots & (m+d)c^{m+d-1} \\ \dfrac{(d+1)!}{(d-1)!}c^{d-1} & \dots & \dfrac{(m+d)!}{(m+d-2)!}c^{m+d-2} \\ \dots & \dots & \dots \\ \dfrac{(d+1)!}{(d-(m-1))!}c^{d-(m-1)} & \dots & \dfrac{(m+d)!}{d!}c^d \end{vmatrix}$$

$$= c^{md} \begin{vmatrix} (d+1) & \dots & (m+d) \\ \dfrac{(d+1)!}{(d-1)!} & \dots & \dfrac{(m+d)!}{(m+d-2)!} \\ \dots & \dots & \dots \\ \dfrac{(d+1)!}{(d-(m-1))!} & \dots & \dfrac{(m+d)!}{d!} \end{vmatrix}$$

$$= c^{md} \begin{vmatrix} (d+1) & \dots & (m+d) \\ (d+1)d & \dots & (m+d)(m+d-1) \\ \dots & \dots & \dots \\ (d+1)\dots(d-(m-2)) & \dots & (m+d)\dots(d+1) \end{vmatrix}.$$

Transform the last determinant in the following way. Adding 1st row to 2nd one we get the squares in the 2nd row. But before, adding 2nd row twice to 3rd row and then, adding the obtained squares in the 2nd row to the new 3rd one, we will have cubes in 3rd row. This process could be extended also to get the powers in each row, thus, we get

$$\det V = c^{md} \begin{vmatrix} d+1 & d+2 & \dots & m+d \\ (d+1)^2 & (d+2)^2 & \dots & (m+d)^2 \\ \dots & \dots & \dots & \dots \\ (d+1)^m & (d+2)^m & \dots & (m+d)^m \end{vmatrix}$$

$$= c^{md}(d+1)\dots(m+d)V(d+1,\dots,m+d),$$

where here and in the sequel $V(x_1,\dots,x_n)$ denotes Vandermonde's determinant formed by the numbers $x_1,\dots,x_n$. So, the matrix V is invertible. Such is also $V - \alpha h V_1 - \lambda h^2 V_2$ for small h.

7.2 Stability of the method

We have proved that for sufficiently small h the matrix $V - \alpha h V_1 - \lambda h^2 V_2$ is invertible. Therefore the equation (7.7) can be written as

$$\alpha_{n+1} = (V^{-1}V_0 + W)\alpha_n + r_n, \quad n = 1, \ldots, N-1, \tag{7.10}$$

with $W = O(h)$ and $r_n = O(h^2)$. Note that the equation (7.9) could be treated as we will do with the equation (7.7) and we could get the same results.

As in the previous sections we define stability as the boundedness of approximate solutions in uniform norm when the number of knots increases. It means that we need to valuate the roots of equation $\det(V_0 - \mu V) = 0$. Denote $\nu = 1 - \mu$. Based on results from Chapter 4, we already have the next result.

Proposition 7.2 *For $m = 1$*
1. If $d = 1$, then the method is stable if and only if $1/2 \le c \le 1$;
2. If $d = 2$, then the method is stable if and only if $c = 1$;
3. If $d \ge 3$, then the method is unstable for all $c \in (0, 1]$.

In the following we assume that $m \ge 2$ and $c = 1$. Recall that we assume $m \le d$. Then

$$\det(V_0 - \mu V) =$$

$$= \begin{vmatrix} \nu & 1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ 0 & \nu & 2 & 3 & \ldots & d & d+1 & \ldots & m+d \\ 0 & 0 & \nu & \binom{3}{2} & \ldots & \binom{d}{2} & \binom{d+1}{2} & \ldots & \binom{m+d}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \nu & \binom{d+1}{d} & \ldots & \binom{m+d}{d} \\ & & & & & \nu C & & & \end{vmatrix}.$$

First, we expand the determinant by the first column. Then, writing ν outside of the determinant we continue the transformation of $\det(V_0 - \mu V)$

with

$$2!\ldots m!\,\nu^{m+1}\cdot$$

$$\cdot\begin{vmatrix} \nu & 2 & 3 & \ldots & d & d+1 & \ldots & m+d \\ 0 & \nu & \binom{3}{2} & \ldots & \binom{d}{2} & \binom{d+1}{2} & \ldots & \binom{m+d}{2} \\ \multicolumn{8}{c}{\dotfill} \\ \ldots & \ldots & \ldots & \ldots & \nu & \binom{d+1}{d} & \ldots & \binom{m+d}{d} \\ 1 & 2 & 3 & \ldots & d & d+1 & \ldots & m+d \\ 0 & 1 & \binom{3}{2} & \ldots & \binom{d}{2} & \binom{d+1}{2} & \ldots & \binom{m+d}{2} \\ \multicolumn{8}{c}{\dotfill} \\ \ldots & \ldots & \ldots & 1 & \ldots & \binom{d+1}{d} & \ldots & \binom{m+d}{d} \end{vmatrix}$$

$$= 2!\ldots m!\,\nu^{m+1}(\nu-1)^m\cdot$$

$$\cdot\begin{vmatrix} \nu & \binom{m+2}{m+1} & \ldots & \binom{d}{m+1} & \binom{d+1}{m+1} & \ldots & \binom{m+d}{m+1} \\ 0 & \nu & \ldots & \binom{d}{m+2} & \binom{d+1}{m+2} & \ldots & \binom{m+d}{m+2} \\ \multicolumn{7}{c}{\dotfill} \\ 0 & \ldots & \ldots & \nu & \binom{d+1}{d} & \ldots & \binom{m+d}{d} \\ m+1 & m+2 & \ldots & d & d+1 & \ldots & m+d \\ \binom{m+1}{2} & \binom{m+2}{2} & \ldots & \binom{d}{2} & \binom{d+1}{2} & \ldots & \binom{m+d}{2} \\ \multicolumn{7}{c}{\dotfill} \\ \binom{m+1}{m} & \binom{m+2}{m} & \ldots & \binom{d}{m} & \binom{d+1}{m} & \ldots & \binom{m+d}{m} \end{vmatrix}$$

$$= (-1)^{m(d+2)}2!\ldots m!\,\mu^m\nu^{m+1}\det\phi_{d,m}(\nu),$$

where

$$
\phi_{d,m}(\nu) = \begin{pmatrix}
m+1 & \dots & d & d+1 & \dots & m+d \\
\binom{m+1}{2} & \dots & \binom{d}{2} & \binom{d+1}{2} & \dots & \binom{m+d}{2} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\binom{m+1}{m} & \dots & \binom{d}{m} & \binom{d+1}{m} & \dots & \binom{m+d}{m} \\
\nu & \dots & \binom{d}{m+1} & \binom{d+1}{m+1} & \dots & \binom{m+d}{m+1} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & \nu & \binom{d+1}{d} & \dots & \binom{m+d}{d}
\end{pmatrix}. \tag{7.11}
$$

Denoting $k = d - m$, write (7.11) as

$$
\phi_{m+k,m}(\nu) = a_{k,m}\nu^k + b_{k,m}\nu^{k-1} + \ldots + c_{k,m}. \tag{7.12}
$$

The transformations indicated in the proof of Proposition 7.1 allow to find the coefficient $a_{k,m}$ from (7.11) as

$$
\begin{aligned}
a_{k,m} &= (-1)^{mk} \begin{vmatrix}
d+1 & d+2 & \dots & m+d \\
\binom{d+1}{2} & \binom{d+2}{2} & \dots & \binom{m+d}{2} \\
\dots & \dots & \dots & \dots \\
\binom{d+1}{m} & \binom{d+2}{m} & \dots & \binom{m+d}{m}
\end{vmatrix} \\
&= (-1)^{mk} \frac{1}{2! \ldots m!} \begin{vmatrix}
d+1 & d+2 & \dots & m+d \\
(d+1)^2 & (d+2)^2 & \dots & (m+d)^2 \\
\dots & \dots & \dots & \dots \\
(d+1)^m & (d+2)^m & \dots & (m+d)^m
\end{vmatrix} \\
&= (-1)^{mk} \frac{(d+1)\ldots(m+d)}{2! \ldots m!} V(d+1, \ldots, m+d) \\
&= (-1)^{mk} \frac{(d+1)\ldots(m+d)}{2! \ldots m!} (m-1)! \ldots 2! \\
&= (-1)^{mk} \binom{m+d}{m} = (-1)^{mk} \binom{m+d}{d}.
\end{aligned} \tag{7.13}
$$

Let us now discuss about the different choices of parameter m and d. First, assume that $d = m$. From (7.11) we have

$$\phi_{m,m}(\nu) = \begin{vmatrix} m+1 & m+2 & \dots & 2m \\ \binom{m+1}{2} & \binom{m+2}{2} & \dots & \binom{2m}{2} \\ \dots & \dots & \dots & \dots \\ \binom{m+1}{m} & \binom{m+2}{m} & \dots & \binom{2m}{m} \end{vmatrix}$$

$$= \frac{(m+1)\dots 2m}{2!\dots m!} V(m+1, \dots, 2m) \neq 0.$$

We see that the solutions of the equation $\det(V_0 - \mu V) = 0$ are only $\nu = 0$ (i.e. $\mu = 1$) and $\mu = 0$. We have proved the following

Theorem 7.1 *For $d = m$, the collocation method is stable.*

Let us now look at the case $d = m + 1$. Here

$$\phi_{m+1,m}(\nu) = \begin{vmatrix} m+1 & m+2 & \dots & 2m+1 \\ \binom{m+1}{2} & \binom{m+2}{2} & \dots & \binom{2m+1}{2} \\ \dots & \dots & \dots & \dots \\ \binom{m+1}{m} & \binom{m+2}{m} & \dots & \binom{2m+1}{m} \\ \nu & \binom{m+2}{m+1} & \dots & \binom{2m+1}{m+1} \end{vmatrix}$$

$$= a_{1,m}\nu + b_{1,m},$$

with

$$a_{1,m} = (-1)^m \binom{2m+1}{m}.$$

Taking $\nu = 1 = \binom{m+1}{m+1}$ we get

$$\phi_{m+1,m}(1) = \frac{(m+1)\dots(2m+1)}{2!\dots(m+1)!} V(m+1, \dots, 2m+1)$$

$$= \binom{2m+1}{m+1} = (-1)^m a_{1,m}.$$

On the other hand,

$$\phi_{m+1,m}(1) = a_{1,m} + b_{1,m},$$

thus,

$$b_{1,m} = (-1)^m a_{1,m} - a_{1,m} = ((-1)^m - 1)\, a_{1,m}.$$

Hence, the polynomial $\phi_{m+1,m}(\nu)$ has the root $\nu = 1 - (-1)^m$. This means that $\det(V_0 - \mu V) = 0$ has the corresponding root $\mu = 1$ or $\mu = -1$. We have proved

Theorem 7.2 *For $d = m + 1$, the collocation method is stable*

In the next case we need an auxiliary result. Suppose $q \le p$. Let us consider

$$W_{p,q,k}(\nu) = a\nu + b \tag{7.14}$$

$$= \begin{vmatrix} p & p+1 & \dots & p+k & p+k+2 & \dots & p+q \\ \binom{p}{2} & \binom{p+1}{2} & \dots & \binom{p+k}{2} & \binom{p+k+2}{2} & \dots & \binom{p+q}{2} \\ \hdotsfor{7} \\ \binom{p}{q-1} & \binom{p+1}{q-1} & \dots & \binom{p+k}{q-1} & \binom{p+k+2}{q-1} & \dots & \binom{p+q}{q-1} \\ \nu & \binom{p+1}{q} & \dots & \binom{p+k}{q} & \binom{p+k+2}{q} & \dots & \binom{p+q}{q} \end{vmatrix}.$$

Then

$$a = (-1)^{q-1} \begin{vmatrix} p+1 & \dots & p+k & p+k+2 & \dots & p+q \\ \binom{p+1}{2} & \dots & \binom{p+k}{2} & \binom{p+k+2}{2} & \dots & \binom{p+q}{2} \\ \hdotsfor{6} \\ \binom{p+1}{q-1} & \dots & \binom{p+k}{q-1} & \binom{p+k+2}{q-1} & \dots & \binom{p+q}{q-1} \end{vmatrix}$$

$$= (-1)^{q-1} \frac{(p+1)\dots(p+k)(p+k+2)\dots(p+q)}{2!\dots(q-1)!} \cdot$$

$$\cdot V(p+1, \dots, p+k, p+k+2, \dots, p+q).$$

Similarly

$$a\binom{p}{q} + b = \frac{p\dots(p+k)(p+k+2)\dots(p+q)}{2!\dots q!} \cdot$$
$$\cdot V(p, \dots, p+k, p+k+2, \dots, p+q).$$

Note that

$$V(p,\dots,p+k,p+k+2,\dots,p+q)$$
$$= \frac{q!}{k+1} V(p+1,\dots,p+k,p+k+2,\dots,p+q).$$

Thus,

$$\begin{aligned} W_{p,q,k}(0) = b &= a\binom{p}{q} + b - a\binom{p}{q} \\ &= \frac{(p+1)\dots(p+k)(p+k+2)\dots(p+q)}{2!\dots q!} \cdot \\ &\cdot V(p+1,\dots,p+k,p+k+2,\dots,p+q) \cdot \\ &\cdot \left[\frac{p}{k+1} - (-1)^{q-1}\binom{p}{q}\right]. \end{aligned} \tag{7.15}$$

Now for the case $d = m+2$ we have

$$\phi_{m+2,m}(\nu) = a_{2,m}\nu^2 + b_{2,m}\nu + c_{2,m}.$$

We will calculate explicitly the coefficients $a_{2,m}$, $b_{2,m}$ and $c_{2,m}$. Actually, the coefficient $a_{2,m}$ is already found in (7.13) as

$$a_{2,m} = \binom{2m+2}{m+2}.$$

From the representations (7.11) and (7.12) we find

$$c_{2,m} = \begin{vmatrix} m+1 & m+2 & m+3 & \dots & 2m+2 \\ \binom{m+1}{2} & \binom{m+2}{2} & \binom{m+3}{2} & \dots & \binom{2m+2}{2} \\ \dots & \dots & \dots & \dots & \dots \\ \binom{m+1}{m} & \binom{m+2}{m} & \binom{m+3}{m} & \dots & \binom{2m+2}{m} \\ 0 & \binom{m+2}{m+1} & \binom{m+3}{m+1} & \dots & \binom{2m+2}{m+1} \\ 0 & 0 & \binom{m+3}{m+2} & \dots & \binom{2m+2}{m+2} \end{vmatrix}. \tag{7.16}$$

Intending to develop the determinant (7.16) by last row, we see that the coefficient of the entry $\binom{2m+2}{m+2}$ will be

$$b_{1,m} = (1 - (-1)^m)\binom{2m+1}{m}.$$

The other minors occurring in the development are of kind $W_{p,q,k}(0)$, where $W_{p,q,k}$ is determined in (7.14), and they could be evaluated by the formula (7.15). The calculations, for m even, give

$$\begin{aligned}
c_{2,m} &= \binom{2m+2}{m+2}\cdot 0 - \binom{2m+1}{m+2}\frac{1}{m}\frac{(2m+2)!}{(m-1)!\,(m+1)!\,(2m+1)} \\
&+ \binom{2m}{m+2}\frac{2}{m-1}\frac{(2m+2)!}{(m-2)!\,(m+1)!\,(2m)\cdot 2!} \\
&- \binom{2m-1}{m+2}\frac{3}{m-2}\frac{(2m+2)!}{(m-3)!\,(m+1)!\,(2m-1)\cdot 3!} + \dots \\
&- \binom{m+3}{m+2}\frac{m-1}{2}\frac{(2m+2)!}{1!\,(m+1)!\,(m+3)(m-1)!} \\
&= \frac{(2m+2)!}{(m+2)!}\left[-\binom{2m}{m+1}\frac{1}{m!\,0!} + \binom{2m-1}{m+1}\frac{1}{(m-1)!\,1!}\right. \\
&\left. - \binom{2m-2}{m+1}\frac{1}{(m-2)!\,2!} + \dots - \binom{m+3}{m+1}\frac{1}{2!\,(m-2)!}\right] \qquad (7.17)
\end{aligned}$$

and, for m odd, we have

$$\begin{aligned}
c_{2,m} &= \frac{(2m+2)!}{(m+1)!}\left[\binom{2m+2}{m+2}\frac{1}{(m+1)!\,0!} - \binom{2m+1}{m+2}\frac{1}{m!\,1!}\right. \\
&\left. + \binom{2m}{m+2}\frac{1}{(m-1)!\,2!} - \dots + \binom{m+3}{m+2}\frac{1}{2!\,(m-1)!}\right]. \qquad (7.18)
\end{aligned}$$

Denote by p_m the expression in brackets in (7.18). From (7.17) and (7.18) we get, for m odd,

$$c_{2,m} = \frac{(2m+2)!}{(m+1)!}p_m$$

and, for $m+1$ even (then m is odd),

$$c_{2,m+1} = \frac{(2(m+1)+2)!}{(m+1+2)!}(-p_m) = -\frac{(2m+4)!}{(m+3)!}p_m.$$

Denote $q_m = c_{2,m}/a_{2,m}$ and recalling that $a_{2,m} = \binom{2m+2}{m+2}$, we have for m odd

$$q_m = \frac{(2m+2)!}{(m+1)!}\,p_m\frac{(m+2)!\,m!}{(2m+2)!} = (m+2)m!\,p_m.$$

Similarly, for $m+1$ even,

$$q_{m+1} = -(m+1)!\,p_m = -\frac{m+1}{m+2}\,q_m.$$

We have proved the following

Proposition 7.3 *For m odd (then $m+1$ is even), it holds*

$$q_{m+1} = -\frac{m+1}{m+2}\,q_m.$$

Next, for m even, we calculate q_m. Using (7.17) we find

$$\begin{aligned} q_m = \frac{c_{2,m}}{a_{2,m}} = \frac{1}{(m+1)!}\Bigg[&-\binom{m}{0}\frac{(2m)!}{(m-1)!} + \binom{m}{1}\frac{(2m-1)!}{(m-2)!} \\ &-\binom{m}{2}\frac{(2m-2)!}{(m-3)!} + \ldots - \binom{m}{m-2}\frac{(m+2)!}{1!}\Bigg]. \end{aligned} \tag{7.19}$$

To give an explicit value to the right hand side of (7.19) we need following results.

Lemma 7.1 *It holds*

$$\frac{d^m}{dx^m}\left(\frac{1}{x}-1\right)^m\Bigg|_{x=1} = (-1)^m m!\,. \tag{7.20}$$

Proof. For $m=1$ we have

$$\left(\frac{1}{x}-1\right)'\Bigg|_{x=1} = -\frac{1}{x^2}\Bigg|_{x=1} = -1 = (-1)^1\cdot 1!\,,$$

which gives us the basis of induction. Assume that the formula (7.20) holds for $m-1$. We will show that then it holds for m. Now, using the Leibniz formula

$$(uv)^{(m)} = u^{(m)}v + \binom{m}{1}u^{(m-1)}v' + \binom{m}{2}u^{(m-2)}v'' + \ldots + uv^{(m)},$$

we have

$$\frac{d^m}{dx^m}\left(\frac{1}{x}-1\right)^m\bigg|_{x=1} = \frac{d^{m-1}}{dx^{m-1}}\left[m\left(\frac{1}{x}-1\right)^{m-1}\left(-\frac{1}{x^2}\right)\right]\bigg|_{x=1}$$

$$= m\left(\left[\frac{d^{m-1}}{dx^{m-1}}\left(\frac{1}{x}-1\right)^{m-1}\right]\left(-\frac{1}{x^2}\right)\right.$$

$$\left. + \binom{m-1}{1}\left[\frac{d^{m-2}}{dx^{m-2}}\left(\frac{1}{x}-1\right)^{m-1}\right]\frac{d}{dx}\left(-\frac{1}{x^2}\right)+\dots\right)\bigg|_{x=1}.$$

Except the first term, the derivatives from $1/x-1$ will contain positive powers of it and, thus, give at $x=1$ zero terms. Therefore, we get

$$\frac{d^m}{dx^m}\left(\frac{1}{x}-1\right)^m\bigg|_{x=1} = m\left[\frac{d^{m-1}}{dx^{m-1}}\left(\frac{1}{x}-1\right)^{m-1}\right]\left(-\frac{1}{x^2}\right)\bigg|_{x=1}$$

$$= m(-1)^{m-1}(m-1)!(-1) = (-1)^m m!\,,$$

which completes the proof.

Lemma 7.2 *It holds*

$$\frac{d^{m+1}}{dx^{m+1}}\left(\frac{1}{x}-1\right)^m\bigg|_{x=1} = (-1)^{m+1}m(m+1)!\,. \tag{7.21}$$

Proof. For $m=1$ we have

$$\left(\frac{1}{x}-1\right)''\bigg|_{x=1} = \frac{2}{x^3}\bigg|_{x=1} = 2 = (-1)^2\cdot 1\cdot 2!$$

as a basis of induction. Again, by Leibniz formula, we find

$$\frac{d^{m+1}}{dx^{m+1}}\left(\frac{1}{x}-1\right)^m\bigg|_{x=1} = m\left(\left[\frac{d^m}{dx^m}\left(\frac{1}{x}-1\right)^{m-1}\right]\left(-\frac{1}{x^2}\right)\right.$$

$$\left. + \binom{m}{1}\frac{d^{m-1}}{dx^{m-1}}\left(\frac{1}{x}-1\right)^{m-1}\frac{2}{x^3}+\dots\right)\bigg|_{x=1}.$$

Now using (7.21) in the first term and Lemma 7.1 in the second one (other terms are zero at $x = 1$), we get

$$\frac{d^{m+1}}{dx^{m+1}}\left(\frac{1}{x}-1\right)^m\bigg|_{x=1} = m((-1)^m(m-1)m!(-1)$$

$$+ m(-1)^{m-1}(m-1)!\cdot 2)$$

$$= (-1)^{m+1}m(m+1)!\,.$$

The proof is complete.

Let us calculate, for m even, the left hand side of (7.21) otherwise:

$$\frac{d^{m+1}}{dx^{m+1}}\left(\frac{1}{x}-1\right)^m\bigg|_{x=1}$$

$$= \frac{d^{m+1}}{dx^{m+1}}\left[\binom{m}{0}\left(\frac{1}{x}\right)^m - \binom{m}{1}\left(\frac{1}{x}\right)^{m-1} + \ldots - \binom{m}{m-1}\frac{1}{x} + \binom{m}{m}\right]\bigg|_{x=1}$$

$$= -\binom{m}{0}\frac{(2m)!}{(m-1)!} + \binom{m}{1}\frac{(2m-1)!}{(m-2)!} - \cdots + \binom{m}{m-1}\frac{(m+1)!}{0!}.$$

Taking into account the last result, Lemma 7.2 and (7.19), for m even, we get

$$q_m = \frac{1}{(m+1)!}\left[\frac{d^{m+1}}{dx^{m+1}}\left(\frac{1}{x}-1\right)^m\bigg|_{x=1} - \binom{m}{m-1}(m+1)!\right]$$

$$= \frac{1}{(m+1)!}\left[(-1)^{m+1}m(m+1)! - \frac{m!}{(m-1)!}(m+1)!\right]$$

$$= \frac{1}{(m+1)!}\left(-m(m+1)! - m(m+1)!\right) = -2m.$$

Now, for m odd (then $m+1$ is even), by Proposition 7.3 we have

$$q_m = -\frac{m+2}{m+1}\,q_{m+1} = -\frac{m+2}{m+1}(-2(m+1)) = 2(m+2).$$

In consequence, we have proved the following

Proposition 7.4 *For m even, $q_m = -2m$ and, for m odd, $q_m = 2(m+2)$.*

Clearly, for $m \geq 3$, $|q_m| > 4$ and the collocation method is unstable. If $m = 2$, then $q_m = -4$, i.e., $\nu_1\nu_2 = -4$, ν_1 and ν_2 being the roots of the polynomial $\phi_{4,2}(\nu)$. Therefore, it is not possible to have $\nu_1 = \nu_2 = 2$ and at least one of the solution of the equation $\det(V_0 - \mu V) = 0$ is located outside of the unit circle. Thus, we have proved the following

Theorem 7.3 *For $d = m + 2$, the collocation method is unstable.*

Although the knowledge of $q_m = c_{2,m}/a_{2,m}$ has allowed to establish the instability of the method for $d = m+2$, we may find explicitly the roots of $\phi_{m+2,m}(\nu)$. These roots characterize quantitatively the unstable behaviour of the method. We have already $a_{2,m} = \binom{2m+2}{m+2}$ and $c_{2,m} = q_m a_{2,m}$. The coefficient $b_{2,m}$ can be found as $b_{2,m} = \phi_{m+2,m}(1) - a_{2,m} - c_{2,m}$. Developing the determinant

$$\phi_{m+2,m}(1) = \begin{vmatrix} m+1 & m+2 & \dots & 2m+2 \\ \binom{m+1}{2} & \binom{m+2}{2} & \dots & \binom{2m+2}{2} \\ \dots & \dots & \dots & \dots \\ \binom{m+1}{m+1} & \binom{m+2}{m+1} & \dots & \binom{2m+2}{m+1} \\ 0 & \binom{m+2}{m+2} & \dots & \binom{2m+2}{m+2} \end{vmatrix}$$

by the last row and using the technics indicated in the proof of Proposition 7.1, we obtain

$$\phi_{m+2,m}(1) = \frac{(2m+2)!}{m!\,(m+2)!}\left[\binom{2m+1}{m} - \binom{m}{1}\binom{2m}{m} + \binom{m}{2}\binom{2m-1}{m} + \ldots + (-1)^m\binom{m}{m}\binom{m+1}{m}\right]. \tag{7.22}$$

Calculate the following derivative, using before the binomial expansion:

$$\frac{d^m}{dx^m}\left(\left(\frac{1}{x}-1\right)^m \frac{1}{x^2}\right)\Bigg|_{x=1} = (-1)^m m!\left[\binom{m}{0}\frac{(2m+1)!}{(m+1)!\,m!} - \binom{m}{1}\frac{(2m)!}{m!\,m!} + \binom{m}{2}\frac{(2m-1)!}{(m-1)!\,m!} - \ldots + (-1)^m\binom{m}{m}\frac{(m+1)!}{1!\,m!}\right]. \tag{7.23}$$

On the other hand,

$$\frac{d^m}{dx^m}\left(\left(\frac{1}{x}-1\right)^m \frac{1}{x^2}\right)\Bigg|_{x=1} = (-1)^m m!\,. \tag{7.24}$$

Taking into account (7.22), (7.23) and (7.24), we obtain

$$\phi_{m+2,m}(1) = \frac{(2m+2)!}{m!\,(m+2)!}(-1)^m \frac{1}{m!}\left[\frac{d^m}{dx^m}\left(\left(\frac{1}{x}-1\right)^m \frac{1}{x^2}\right)\right]\Big|_{x=1}$$

$$= \frac{(2m+2)!}{m!\,(m+2)!} = a_{2,m}. \tag{7.25}$$

Therefore,

$$b_{2,m} = \phi_{m+2,m}(1) - a_{2,m} - c_{2,m} = -a_{2,m}q_m$$

and the roots of the polynomial $\phi_{m+2,m}(\nu) = a_{2,m}(\nu^2 - q_m\nu + q_m)$ are

$$\nu = (q_m \pm \sqrt{q_m(q_m-4)})/2.$$

Thus, for m even, we get the roots

$$\nu_{1,m} = -m + \sqrt{m(m+2)},$$

$$\nu_{2,m} = -m - \sqrt{m(m+2)}$$

and, for m odd,

$$\nu_{1,m} = m + 2 + \sqrt{m(m+2)},$$

$$\nu_{2,m} = m + 2 - \sqrt{m(m+2)}.$$

The elementary analysis of the asymptotics implies

Proposition 7.5 *In the case $d = m + 2$ it holds*
for m even, $\nu_{1,m} \to 1$ and $\nu_{2,m} \to -\infty$ as $m \to \infty$;
for m odd, $\nu_{1,m} \to \infty$ and $\nu_{2,m} \to 1$ as $m \to \infty$.

Let us now consider the general case $d = m + k$. As we have seen, the polynomial $\phi_{d,m}(\nu) = \psi_{m,k}(\nu) = a_{k,m}\nu^k + \ldots + c_{k,m}$ has the main coefficient

$$a_{k,m} = (-1)^{mk}\binom{m+d}{m}.$$

Denote here the maximal root by modulus of $\psi_{m,k}(\nu)$ by ν_m, k. We have already proved that, for m even, $\nu_{m,2} \to -\infty$ and, for m odd, $\nu_{m,2} \to \infty$ as $m \to \infty$. We state the following

Conjecture For all $k \geq 2$ it holds
for m even, $\nu_{m,k} \to -\infty$ as $m \to \infty$;
for m odd, $\nu_{m,k} \to \infty$ as $m \to \infty$.

This assertion could be proved, e.g., taking into account the behaviour of $\psi_{m,k}(\nu)$ as $\nu \to \infty$ or $\nu \to -\infty$ and showing that, for k even, $\psi_{m,k+1}(\nu_{m,k}) > 0$, and, for k odd, $\psi_{m,k+1}(\nu_{m,k}) < 0$. In the following table we present some numerical results about the value of $\psi_{m,3}(\nu_{m,2})$ supporting the conjecture:

m	2	3	4
$\nu_{m,2}$	-4.828	8.873	-8.899
$\psi_{m,3}(\nu_{m,2})$	$9.456 \cdot 10^3$	$1.364 \cdot 10^5$	$1.395 \cdot 10^6$

m	5	6	7
$\nu_{m,2}$	12.916	-12.928	16.937
$\psi_{m,3}(\nu_{m,2})$	$1.172 \cdot 10^7$	$8.650 \cdot 10^7$	$5.852 \cdot 10^8$

However, the validity of the conjecture yields the instability of the collocation method for all $k = d - m \geq 2$. This would be in complete accordance with the results by H. N. Mülthei about the convergence of step-by-step collocation for the Cauchy problem of ordinary differential equations (see Section 1.1).

Another way to prove the instability for $k \geq 2$ is to show that $c_{k,m}/a_{k,m} > 2^k$. But this would not characterize quantitatively the unstable behaviour of the method as well as the conjecture.

Chapter 8

NUMERICAL TESTS

8.1 First order VIDE

Consider the 1st order Volterra integro-differential equation

$$y'(t) = y(t) + \int_0^t y(s)ds - (\cos t - 3\sin t - e^t)/2 \tag{8.1}$$

with $y(0) = 1$. This equation has the exact solution $y(t) = (\sin t + \cos t + e^t)/2$. As an approximate value of $||u||_\infty$ we actually calculate

$$\max_{1\le n\le N} \max_{0\le k\le 10} \left|u_n(t_{n-1} + kh/10)\right|.$$

The results are presented in following tables.

Case $d = 0$, $m = 1$ (linear splines)

N	4	16	64	256	4096
$c_1 = 1.0$	2.105018	2.059782	2.052299	2.050586	2.050062
$c_1 = 0.5$	2.049933	2.050022	2.050027	2.050028	2.050028

Case $d = 0$, $m = 2$

N	4	16	64	256	4096
$c_1 = 0.7$, $c_2 = 1.0$	2.042611	2.049641	2.050004	2.050026	2.050028
$c_1 = 0.4$, $c_2 = 0.6$	2.047681	2.049882	2.050018	2.050027	2.050028

Case $d = 1$, $m = 1$ *(quadratic splines)*

N	4	16	64	256	4096
$c_1 = 1.0$	2.055503	2.050359	2.050048	2.050029	2.050028
$c_1 = 0.5$	2.047524	2.049863	2.050017	2.050027	2.050028
$c_1 = 0.4$	2.047418	2.049880	8.962233	$2.69 \cdot 10^{32}$	$1.83 \cdot 10^{165}$

Case $d = 1$, $m = 2$ *(Hermite cubic splines)*

N	4	16	64	256
$c_1 = 0.5$, $c_2 = 1.0$	2.050006	2.050027	2.050028	2.050028
$c_1 = 0.3$, $c_2 = 0.7$	2.049615	2.050001	2.050026	2.050027
$c_1 = 0.2$, $c_2 = 0.5$	2.043332	$3.21 \cdot 10^2$	$9.21 \cdot 10^{28}$	$1.39 \cdot 10^{142}$

Case $d = 2$, $m = 1$ *(cubic splines)*

N	4	16	64	256
$c_1 = 1.0$	2.050148	2.050028	2.050028	2.050028
$c_1 = 0.9$	2.049806	2.049999	5.773942	$1.60 \cdot 10^{29}$
$c_1 = 0.5$	2.054945	$3.30 \cdot 10^4$	$7.30 \cdot 10^{38}$	$2.77 \cdot 10^{183}$

For piecewise polynomial splines we look at the equation

$$y'(t) = \alpha y(t) + \lambda \int_0^t y(s)ds - (\cos t - 3\sin t - e^t)/2 \tag{8.2}$$

with $y(0) = 1$, but with different choices of parameters α and λ. The results are presented in following tables.

Case $m = 2$

$\alpha = 1$, $\lambda = 1$

N	4	16	64	256	1024
$c_1 = 0.1$, $c_2 = 0.2$	1.81554	1.77374	1.76551	1.76302	1.76254
$c_1 = 0.5$, $c_2 = 1.0$	1.78039	1.76705	1.76353	1.76264	1.76242

$\alpha = 1,\ \lambda = -1$

N	4	16	64	256	1024
$c_1 = 0.2$ $c_2 = 0.5$	5.01	12.76	$1.27 \cdot 10^{15}$	$1.32 \cdot 10^{66}$	$9.82 \cdot 10^{271}$
$c_1 = 0.3$ $c_2 = 0.7$	3.71489	3.59568	3.56853	3.56193	3.56029
$c_1 = 0.5$ $c_2 = 1.0$	3. 48369	3.52181	3.55030	3.55739	3.55916

Case $m = 3$

$\alpha = 1,\ \lambda = 1$

N	4	16	64	256	1024
$c_1 = 0.1$ $c_2 = 0.2$ $c_3 = 0.3$	1.64625	1.66144	1.66684	1.66827	1.66863
$c_1 = 0.2$ $c_2 = 0.5$ $c_3 = 0.9$	1.65331	1.66493	1.66779	1.66851	1.66869

$\alpha = 2,\ \lambda = -2$

N	4	16	64	256
$c_1 = 0.1$ $c_2 = 0.2$ $c_3 = 0.3$	58.07	$9.08 \cdot 10^{8}$	$7.11 \cdot 10^{37}$	$2.79 \cdot 10^{153}$
$c_1 = 0.2$ $c_2 = 0.5$ $c_3 = 0.8$	3.82646	3.78772	3.77323	3.76931
$c_1 = 0.3$ $c_2 = 0.6$ $c_3 = 0.9$	3.02214	2.56511	2.56229	2.56247

We can see different dependence of the stability on the cases $Q_m = 0$ and $Q_m \neq 0$ (depending on the choice of α and λ), as well as on different choices of c_i.

8.2 Second order VIDE

We consider the 2nd order integro-differential equation

$$y''(t) = y(t) + y'(t) + \int_0^t y(s)ds - \sin(t) - \cos(t) - e^t \tag{8.3}$$

with $y'(0) = 1, y'(0) = 1$ on the interval $[0, 1]$. This equation has the exact solution $y(t) = (\sin t + \cos t + e^t)/2$ (which was also the solution of (8.1)). The results are presented in following tables.

Case $d = 1$, $m = 1$ (quadratic splines)

N	4	16	64	256	4096
$c_1 = 0.5$	2.053593	2.050242	2.050041	2.050028	2.050028
$c_1 = 1.0$	2.112955	2.060136	2.052332	2.050591	2.050062

Case $d = 1$, $m = 2$ (Hermite cubic splines)

N	4	16	64	256	4096
$c_1 = 0.4$ $c_2 = 0.6$	2.047625	2.049880	2.050018	2.050027	2.050028
$c_1 = 0.7$ $c_2 = 1.0$	2.042264	2.049630	2.050004	2.050026	2.050028

Case $d = 2$, $m = 1$ (cubic splines)

N	4	16	64	256	512
$c_1 = 0.4$	2.047252	2.049817	61.720406	$1.60 \cdot 10^{33}$	$1.20 \cdot 10^{77}$
$c_1 = 0.5$	2.047590	2.049861	2.050017	2.050027	2.050027
$c_1 = 1.0$	2.055555	2.050364	2.050048	2.050028	2.050028

Case $d = 2$, $m = 2$

N	4	64	256	512
$c_1 = 0.2$ $c_2 = 0.5$	2.049254	$7.65 \cdot 10^{26}$	$2.89 \cdot 10^{139}$	$1.21 \cdot 10^{292}$
$c_1 = 0.3$ $c_2 = 0.7$	2.049935	2.050027	2.050028	2.050028
$c_1 = 0.5$ $c_2 = 1.0$	2.050015	2.050028	2.050028	2.050028

8.3 Collocation with multiple nodes for first order VIDE

We explore the equation

$$y'(t) = y(t) + \int_0^t y(s)ds - (\cos t - 3\sin t - e^t)/2 \tag{8.4}$$

with $y(0) = 1$. The results are presented in following tables.

Case $d = 1$, $m = 1$

N	4	16	64	256
$c = 0.2$	2.051823	$1.86 \cdot 10^{2}$	$5.86 \cdot 10^{28}$	$9.07 \cdot 10^{141}$
$c = 0.7$	2.049380	2.049953	2.050022	2.050027
$c = 1.0$	2.055503	2.050359	2.050048	2.050029

Case $d = 2$, $m = 1$

N	4	16	64	256
$c = 0.5$	2.054945	$3.30 \cdot 10^{4}$	$7.30 \cdot 10^{38}$	$2.77 \cdot 10^{183}$
$c = 0.9$	2.049805	2.049999	5.572743	$1.61 \cdot 10^{29}$
$c = 1.0$	2.050148	2.050028	2.050028	2.050028

Case $d = 2$, $m = 2$

N	4	16	64
$c = 0.2$	60.572511	$1.34 \cdot 10^{27}$	$3.59 \cdot 10^{137}$
$c = 0.5$	2.047744	$1.73 \cdot 10^{6}$	$1.02 \cdot 10^{51}$
$c = 1.0$	2.050016	2.050027	2.050028

Case $d = 3$, $m = 1$

N	4	16	64
$c = 0.2$	73.516030	$4.85 \cdot 10^{26}$	$1.30 \cdot 10^{135}$
$c = 0.5$	2.084259	$4.27 \cdot 10^{11}$	$1.03 \cdot 10^{73}$
$c = 1.0$	2.049860	2.267400	$2.94 \cdot 10^{24}$

Case $d = 3$, $m = 2$

N	4	16	64
$c = 0.2$	5.304158	$1.81 \cdot 10^{36}$	$7.00 \cdot 10^{195}$
$c = 0.5$	2.049929	$1.95 \cdot 10^{12}$	$2.11 \cdot 10^{93}$
$c = 1.0$	2.050027	2.050027	2.050027

Case $d = 4$, $m = 1$

N	4	16	64
$c = 0.2$	11.640978	$1.72 \cdot 10^{34}$	$2.12 \cdot 10^{180}$
$c = 0.5$	2.049464	$9.87 \cdot 10^{14}$	$1.20 \cdot 10^{100}$
$c = 1.0$	2.050028	61.283346	$2.38 \cdot 10^{46}$

From these numerical examples we can observe a good conformity in the preceding sections and corresponding results given in this section.

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KOKKUVÕTE
Splain-kollokatsioonimeetodi stabiilsus Volterra integro-diferentsiaalvõrrandi korral

Integraalvõrrandite teooria uurimine on tunduvalt intensiivistunud viimasel paarikümnel aastal. Võrrandite rakendusi võib leida erinevates eluvaldkondades: meditsiinis, bioloogias, majanduses. Praktikas esinevad integraalvõrrandid lahendatakse enamasti ligikaudselt ehk kasutades erinevaid diskretiseerimismeetodeid. Erinevad diskretiseerimismeetodid on aga praktikas rakendatavad vaid juhul, kui need on stabiilsed. Käesolevas dissertatsioonis on vaatluse all sammhaaval rakendatava splain-kollokatsioonimeetodi stabiilsus Volterra integro-diferentsiaalvõrrandi korral.

Me ütleme, et splain-kollokatsioonimeetod on stabiilne, kui teatava testvõrrandi ligikaudne lahend on tõkestatud protsessis, kus ühtlase võrgu sõlmede arv kasvab.

Käesolevas doktoritöös on vaatluse all nii esimest kui ka teist järku Volterra integro-diferentsiaalvõrrandid. Selgub, et kasutades splain-kollokatsioonimeetodit, tekib üleminekul ühest osalõigust teise teatav üleminekumaatriks ning stabiilsuse tingimused on leitavad vastava üleminekumaatriksi omaväärtuste abil.

Töö esimeses peatükis antakse lühike ülevaade integraalvõrrandite teooria ajaloost. Näidatud on, kuidas saab esimest järku Volterra integro-diferentsiaalvõrrandi lahendamist taandada Volterra integraalvõrrandi lahendamisele ja teist järku integro-diferentsiaalvõrrandi lahendamist esimest järku võrrandi lahendamisele.

Selgub, et teatud konstantse tuumaga testvõrrand, mida kasutatakse stabiilsuse uurimisel, teisendub taandamisel mittekonstantse tuumaga võrrandiks. Seega tulemused, mis on saadud integraalvõrrandite korral, ei ole otseselt rakendatavad integro-diferentsiaalvõrranditele.

Teises peatükis on kirjeldatud kasutatavat splain-kollokatsioonimeetodit nii esimest kui ka teist järku integro-diferentsiaalvõrrandite korral.

Kolmas peatükk annab esmalt lühikese ülevaate tulemustest, mis on saadud Volterra integraalvõrrandi stabiilsuse uurimisel. Teise punktina on toodud vajaminevad tulemused lineaaralgebrast.

Neljandas peatükis on vaadeldud kollokatsioonimeetodi stabiilsust, kus kasutatavad splainid on vähemalt pidevad. Näidatud on stabiilsustingimuste vaheline seos esimest järku integro-diferentsiaalvõrrandi ja integraalvõrrandi korral. Mõningatel juhtudel on saadud täpsed tulemused, mis näitavad stabiilsuse sõltuvust kollokatsiooniparameetritest.

Viiendas peatükis on vaadeldud kollokatsioonimeetodi stabiilsust esimest järku integro-diferentsiaalvõrrandi korral, kus kasutatavad splainid on tükiti polünomiaalsed. Sellisel juhul sõltub meetodi stabiilsus ka testvõrrandi parameetritest.

Kuues peatükk käsitleb kollokatsioonimeetodi stabiilsust teist järku võrrandi korral. Näidatud on stabiilsustingimuste vaheline seos esimest ja teist järku integro-diferentsiaalvõrrandite korral.

Seitmendas peatükis uuritakse meetodi stabiilsust kordsete kollokatsioonisõlmede korral. Lähemalt on vaadeldud juhtu, kui meil on vaid üks kollokatsiooniparameeter, mille kordsus on m.

Töö kaheksandas peatükis on toodud rida numbrilisi eksperimente, millest selgub, et numbrilised tulemused on täielikus kooskõlas teoreetiliste tulemustega.

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