TARTU RIIKLIK ÜLIKOOL

INTEGRAALIDE TABELID

1985

TARTU RIIKLIK ÜLIKOOL
Matemaatilise analüüsi kateeder

INTEGRAALIDE TABELID

TARTU 1985

Kinnitatud matemaatikateaduskonna nõukogus
27. septembril 1984. a.

Koostaja M. Tõnnov

Käesolevatesse integraalide tabelitesse on koondatud tähtsamad määramatud integraalid ja väike valik kõige enam vajaminevaid päratuid integraale. Tabelite kasutamisel tuleb arvestada järgmisi märkusi:

1. integreerimiskonstant on ära jäetud, välja arvatud juhul, kui integraali võib esitada erinevate avaldistena erinevate suvaliste konstantidega;

2. kõikides valemites, kus algfunktsiooni avaldises esineb $\ln f(x)$, tuleb teda mõista, kui $\ln |f(x)|$ (absoluutväärtuse märgid on kirjutuse lihtsustuseks ära jäetud);

3. juhul, kui algfunktsioon on esitatud astmereana, siis see tähendab, et teda ei saa esitada lõpliku arvu elementaarfunktsioonide kaudu.

1. Ratsionaalfunktsioonide integraalid

1.1. Integraalid, mis sisaldavad avaldist $ax+b$.

Tähistus: $X = ax + b$.

1) $\int X^n\,dx = \frac{1}{a(n+1)} X^{n+1} \quad (n \neq -1)$.

2) $\int \frac{dx}{X} = \frac{1}{a} \ln X$.

3) $\int xX^n\,dx = \frac{1}{a^2(n+2)} X^{n+2} - \frac{b}{a^2(n+1)} X^{n+1} \quad (n \neq -1,\ -2)$.

4) $\int \frac{x\,dx}{X} = \frac{x}{a} - \frac{b}{a^2} \ln X$.

5) $\int \frac{x\,dx}{X^2} = -\frac{b}{a^2 X} + \frac{1}{a^2} \ln X$.

6) $\int \frac{x\,dx}{X^3} = \frac{1}{a^2}\left(-\frac{1}{X} + \frac{b}{2X^2}\right)$.

7) $\int \frac{x\,dx}{X^n} = \frac{1}{a^2}\left(\frac{-1}{(n-2)X^{n-2}} + \frac{b}{(n-1)X^{n-1}}\right) \quad (n \neq 1,\ 2)$.

8) $\int x^n X^m\,dx = \frac{1}{a^{n+1}} \int (X-b)^n X^m dX \quad (n \geq 0,\ m < n)$.

9) $\int \frac{x^2\,dx}{X} = \frac{1}{a^3}\left(\frac{1}{2}X^2 - 2bX + b^2 \ln X\right)$.

10) $\int \frac{x^2\,dx}{X^2} = \frac{1}{a^3}\left(X - 2b \ln X - \frac{b^2}{X}\right)$.

11) $\int \frac{x^2\,dx}{X^3} = \frac{1}{a^3}\left(\ln X + \frac{2b}{X} - \frac{b^2}{2X^2}\right)$.

12) $\int \frac{x^2\,dx}{X^n} = \frac{1}{a^3}\left[\frac{-1}{(n-3)X^{n-3}} + \frac{2b}{(n-2)X^{n-2}} - \frac{b^2}{(n-1)X^{n-1}}\right] \quad (n \neq 1,\ 2,\ 3)$.

13) $\int \frac{x^3\,dx}{X} = \frac{1}{a^4}\left(\frac{X^3}{3} - \frac{3bX^2}{2} + 3b^2X - b^3 \ln X\right)$.

14) $\int \frac{x^3\,dx}{X^2} = \frac{1}{a^4}\left(\frac{X^2}{2} - 3bX + 3b^2 \ln X + \frac{b^3}{X}\right)$.

15) $\int \frac{x^3\,dx}{X^3} = \frac{1}{a^4}\left(X - 3b \ln X - \frac{3b^2}{X} + \frac{b^3}{2X^2}\right)$.

16) $\int \frac{x^3\,dx}{X^4} = \frac{1}{a^4}\left(\ln X + \frac{3b}{X} - \frac{3b^2}{2X^2} + \frac{b^3}{3X^3}\right)$.

17) $\int \frac{x^3\,dx}{X^n} = \frac{1}{a^4}\left[\frac{-1}{(n-4)X^{n-4}} + \frac{3b}{(n-3)X^{n-3}} - \frac{3b^2}{(n-2)X^{n-2}} + \right.$

$\left. + \frac{b^3}{(n-1)X^{n-1}}\right]$ $(n \neq 1, 2, 3, 4)$.

18) $\int \frac{dx}{xX} = -\frac{1}{b}\ln\frac{X}{x}$. 19) $\int \frac{dx}{xX^2} = \frac{1}{b^2}\left(\ln\frac{X}{x} + \frac{ax}{X}\right)$.

20) $\int \frac{dx}{xX^3} = -\frac{1}{b^3}\left(\ln\frac{X}{x} + \frac{2ax}{X} - \frac{a^2x^2}{2X^2}\right)$.

21) $\int \frac{dx}{xX^n} = -\frac{1}{b^n}\left[\ln\frac{X}{x} - \sum_{i=1}^{n-1} C_{n-1}^i \frac{(-a)^i x^i}{iX^i}\right]$ $(n \geqslant 2)$.

22) $\int \frac{dx}{x^2X} = -\frac{1}{bx} + \frac{a}{b^2}\ln\frac{X}{x}$. 23) $\int \frac{dx}{x^2X^2} = -a\left[\frac{1}{b^2X} + \frac{1}{ab^2x} - \frac{2}{b^3}\ln\frac{X}{x}\right]$.

24) $\int \frac{dx}{x^2X^3} = -a\left[\frac{1}{2b^2X^2} + \frac{2}{b^3X} + \frac{1}{ab^3x} - \frac{3}{b^4}\ln\frac{X}{x}\right]$.

25) $\int \frac{dx}{x^2X^n} = -\frac{1}{b^{n+1}}\left[-\sum_{i=2}^{n} C_n^i \frac{(-a)^i x^{i-1}}{(i-1)X^{i-1}} + \frac{X}{x} - na\ln\frac{X}{x}\right]$ $(n \geqslant 2)$.

26) $\int \frac{dx}{x^3X} = -\frac{1}{b^3}\left[a^2\ln\frac{X}{x} - \frac{2aX}{x} + \frac{X^2}{2x^2}\right]$.

27) $\int \frac{dx}{x^3X^2} = -\frac{1}{b^4}\left[3a^2\ln\frac{X}{x} + \frac{a^3x}{X} + \frac{X^2}{2x^2} - \frac{3aX}{x}\right]$.

28) $\int \frac{dx}{x^3X^3} = -\frac{1}{b^5}\left[6a^2\ln\frac{X}{x} + \frac{4a^3x}{X} - \frac{a^4x^2}{2X^2} + \frac{X^2}{2x^2} - \frac{4aX}{x}\right]$.

29) $\int \frac{dx}{x^3X^n} = -\frac{1}{b^{n+2}}\left[-\sum_{i=3}^{n+1} C_{n+1}^i \frac{(-a)^i x^{i-2}}{(i-2)X^{i-2}} + \frac{a^2X^2}{2x^2} - \frac{(n+1)aX}{x} + \right.$

$\left. + \frac{n(n+1)a^2}{2}\ln\frac{X}{x}\right]$ $(n \geqslant 3)$.

30) $\int \frac{dx}{x^mX^n} = -\frac{1}{b^{m+n-1}}\sum_{i=0}^{m+n-2} C_{m+n-2}^i \frac{X^{m-i-1}(-a)^i}{(m-i-1)x^{m-i-1}}$

[kui mingi liikme nimetaja Σ märgi all on null, tuleb see liige asendada avaldisega $C_{m+n-2}^{m-1}(-a)^{m-1}\ln\frac{X}{x}$].

Tähistus: $\Delta = bf - ag$.

31) $\int \frac{ax+b}{fx+g}dx = \frac{ax}{f} + \frac{\Delta}{f^2}\ln(fx+g)$.

32) $\int \frac{dx}{(ax+b)(fx+g)} = \frac{1}{\Delta}\ln\frac{fx+g}{ax+b}$ $(\Delta \neq 0)$.

33) $\int \frac{x\,dx}{(ax+b)(fx+g)} = \frac{1}{\Delta}\left[\frac{b}{a}\ln(ax+b) - \frac{g}{f}\ln(fx+g)\right]$ $(\Delta \neq 0)$.

34) $\int \frac{dx}{(ax+b)^2(fx+g)} = \frac{1}{\Delta}\left(\frac{1}{ax+b} + \frac{f}{\Delta}\ln\frac{fx+g}{ax+b}\right)$ $(\Delta \neq 0)$.

35) $\int \frac{x\,dx}{(a+x)(b+x)^2} = \frac{b}{(a-b)(b+x)} - \frac{a}{(a-b)^2}\ln\frac{a+x}{b+x}$ $(a \neq b)$.

36) $\int \frac{x^2\,dx}{(a+x)(b+x)^2} = \frac{b^2}{(b-a)(b+x)} + \frac{a^2}{(b-a)^2}\ln(a+x) +$

$$+ \frac{b^2-2ab}{(b-a)^2}\ln(b+x) \quad (a \neq b).$$

37) $\int \frac{dx}{(a+x)^2(b+x)^2} = \frac{-1}{(a-b)^2}\left(\frac{1}{a+x} + \frac{1}{b+x}\right) + \frac{2}{(a-b)^3}\ln\frac{a+x}{b+x}$ $(a \neq b)$.

38) $\int \frac{x\,dx}{(a+x)^2(b+x)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{a+x} + \frac{b}{b+x}\right) + \frac{a+b}{(a-b)^3}\ln\frac{a+x}{b+x}$ $(a \neq b)$.

39) $\int \frac{x^2\,dx}{(a+x)^2(b+x)^2} = \frac{-1}{(a-b)^2}\left(\frac{a^2}{a+x} + \frac{b^2}{b+x}\right) + \frac{2ab}{(a-b)^3}\ln\frac{a+x}{b+x}$ $(a \neq b)$.

1.2. Integraalid, mis sisaldavad avaldist ax^2+bx+c.

Tähistused: $X = ax^2+bx+c$, $\Delta = 4ac - b^2$.

40) $\int \frac{dx}{X} = \begin{cases} \frac{2}{\sqrt{\Delta}}\operatorname{arctg}\frac{2ax+b}{\sqrt{\Delta}} & (\Delta > 0), \\ -\frac{2}{\sqrt{-\Delta}}\operatorname{Arth}\frac{2ax+b}{\sqrt{-\Delta}} = \frac{1}{\sqrt{-\Delta}}\ln\frac{2ax+b-\sqrt{-\Delta}}{2ax+b+\sqrt{-\Delta}} & (\Delta < 0). \end{cases}$

41) $\int \frac{dx}{X^2} = \frac{2ax+b}{\Delta X} + \frac{2a}{\Delta}\int \frac{dx}{X}$ (№ 40).

42) $\int \frac{dx}{X^3} = \frac{2ax+b}{\Delta}\left(\frac{1}{2X^2} + \frac{3a}{\Delta X}\right) + \frac{6a^2}{\Delta^2}\int \frac{dx}{X}$ (№ 40).

43) $$\int \frac{dx}{X^n} = \frac{2ax+b}{(n-1)\,\Delta X^{n-1}} + \frac{(2n-3)\,2a}{(n-1)\,\Delta} \int \frac{dx}{X^{n-1}}.$$

44) $$\int \frac{x\,dx}{X} = \frac{1}{2a} \ln X - \frac{b}{2a} \int \frac{dx}{X} \quad (\text{№ } 40).$$

45) $$\int \frac{x\,dx}{X^2} = -\frac{bx+2c}{\Delta X} - \frac{b}{\Delta} \int \frac{dx}{X} \quad (\text{№ } 40).$$

46) $$\int \frac{x\,dx}{X^n} = -\frac{bx+2c}{(n-1)\,\Delta X^{n-1}} - \frac{b\,(2n-3)}{(n-1)\,\Delta} \int \frac{dx}{X^{n-1}}.$$

47) $$\int \frac{x^2\,dx}{X} = \frac{x}{a} - \frac{b}{2a^2} \ln X + \frac{b^2-2ac}{2a^2} \int \frac{dx}{X} \quad (\text{№ } 40).$$

48) $$\int \frac{x^2\,dx}{X^2} = \frac{(b^2-2ac)\,x+bc}{a\,\Delta X} + \frac{2c}{\Delta} \int \frac{dx}{X} \quad (\text{№ } 40).$$

49) $$\int \frac{x^2\,dx}{X^n} = \frac{-x}{(2n-3)\,aX^{n-1}} + \frac{c}{(2n-3)\,a} \int \frac{dx}{X^n} - \frac{(n-2)\,b}{(2n-3)\,a} \int \frac{x\,dx}{X^n} \quad (\text{№№ } 43, 46).$$

50) $$\int \frac{x^m\,dx}{X^n} = -\frac{x^{m-1}}{(2n-m-1)\,aX^{n-1}} + \frac{(m-1)\,c}{(2n-m-1)\,a} \int \frac{x^{m-2}\,dx}{X^n} -$$

$$- \frac{(n-m)\,b}{(2n-m-1)\,a} \int \frac{x^{m-1}\,dx}{X^n} \quad (m \neq 2n-1).$$

51) $$\int \frac{x^{2n-1}\,dx}{X^n} = \frac{1}{a} \int \frac{x^{2n-3}\,dx}{X^{n-1}} - \frac{c}{a} \int \frac{x^{2n-3}\,dx}{X^n} - \frac{b}{a} \int \frac{x^{2n-2}\,dx}{X^n}$$

52) $$\int \frac{dx}{xX} = \frac{1}{2c} \ln \frac{x^2}{X} - \frac{b}{2c} \int \frac{dx}{X} \quad (\text{№ } 40).$$

53) $$\int \frac{dx}{xX^n} = \frac{1}{2c\,(n-1)\,X^{n-1}} - \frac{b}{2c} \int \frac{dx}{X^n} + \frac{1}{c} \int \frac{dx}{xX^{n-1}}$$

54) $$\int \frac{dx}{x^2X} = \frac{b}{2c^2} \ln \frac{X}{x^2} - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right) \int \frac{dx}{X} \quad (\text{№ } 40).$$

55) $$\int \frac{dx}{x^mX^n} = -\frac{1}{(m-1)\,cx^{m-1}X^{n-1}} - \frac{(2n+m-3)\,a}{(m-1)\,c} \int \frac{dx}{x^{m-2}X^n} -$$

$$- \frac{(n+m-2)\,b}{(m-1)\,c} \int \frac{dx}{x^{m-1}X^n} \quad (m > 1).$$

56) $$\int \frac{dx}{(fx+g)\,X} = \frac{1}{2\,(cf^2-gbf+g^2a)} \left[f \ln \frac{(fx+g)^2}{X} \right] +$$

$$+ \frac{2ga-bf}{2\,(cf^2-gbf+g^2a)} \int \frac{dx}{X} \quad (\text{№ } 40).$$

1.3. Integraalid, mis sisaldavad avaldist $a^2 \pm x^2$.

Tähistused: $\boxed{X = a^2 \pm x^2,}$

$$Y = \begin{cases} \operatorname{arctg} \dfrac{x}{a} & \text{märgi } + \text{ korral,} \\ \operatorname{Arth} \dfrac{x}{a} = \dfrac{1}{2} \ln \dfrac{a+x}{a-x} & \text{märgi } - \text{ korral, kui } |x| < a, \\ \operatorname{Arcth} \dfrac{x}{a} = \dfrac{1}{2} \ln \dfrac{x+a}{x-a} & \text{märgi } - \text{ korral, kui } |x| > a. \end{cases}$$

Kui avaldise ees on märk $\pm$ või $\mp$, siis ülemise märgiga võrduses $X = a^2 + x^2$ ja alumise märgiga võrduses $X = a^2 - x^2$.

57) $\int \dfrac{dx}{X} = \dfrac{1}{a} Y.$ 58) $\int \dfrac{dx}{X^2} = \dfrac{x}{2a^2 X} + \dfrac{1}{2a^2} Y.$

59) $\int \dfrac{dx}{X^3} = \dfrac{x}{4a^2 X^2} + \dfrac{3x}{8a^4 X} + \dfrac{3}{8a^5} Y.$

60) $\int \dfrac{dx}{X^{n+1}} = \dfrac{x}{2na^2 X^n} + \dfrac{2n-1}{2na^2} \int \dfrac{dx}{X^n}.$

61) $\int \dfrac{x\,dx}{X} = \pm \dfrac{1}{2} \ln X.$ 62) $\int \dfrac{x\,dx}{X^2} = \mp \dfrac{1}{2X}.$

63) $\int \dfrac{x\,dx}{X^3} = \mp \dfrac{1}{4X^2}.$ 64) $\int \dfrac{x\,dx}{X^{n+1}} = \mp \dfrac{1}{2nX^n} \quad (n \neq 0).$

65) $\int \dfrac{x^2\,dx}{X} = \pm x \mp a Y.$ 66) $\int \dfrac{x^2\,dx}{X^2} = \mp \dfrac{x}{2X} \pm \dfrac{1}{2a} Y.$

67) $\int \dfrac{x^2\,dx}{X^3} = \mp \dfrac{x}{4X^2} \pm \dfrac{x}{8a^2 X} \pm \dfrac{1}{8a^3} Y.$

68) $\int \dfrac{x^2\,dx}{X^{n+1}} = \mp \dfrac{x}{2nX^n} \pm \dfrac{1}{2n} \int \dfrac{dx}{X^n} \quad (n \neq 0).$

69) $\int \dfrac{x^3\,dx}{X} = \pm \dfrac{x^2}{2} - \dfrac{a^2}{2} \ln X.$ 70) $\int \dfrac{x^3\,dx}{X^2} = \dfrac{a^2}{2X} + \dfrac{1}{2} \ln X.$

71) $\int \dfrac{x^3\,dx}{X^3} = -\dfrac{1}{2X} + \dfrac{a^2}{4X^2}.$ 72) $\int \dfrac{x^3\,dx}{X^{n+1}} = -\dfrac{1}{2(n-1)X^{n-1}} + \dfrac{a^2}{2nX^n} \quad (n > 1).$

73) $\int \dfrac{dx}{xX} = \dfrac{1}{2a^2} \ln \dfrac{x^2}{X}.$ 74) $\int \dfrac{dx}{xX^2} = \dfrac{1}{2a^2 X} + \dfrac{1}{2a^4} \ln \dfrac{x^2}{X}.$

75) $\int \dfrac{dx}{xX^3} = \dfrac{1}{4a^2 X^2} + \dfrac{1}{2a^4 X} + \dfrac{1}{2a^6} \ln \dfrac{x^2}{X}.$ 76) $\int \dfrac{dx}{x^2 X} = -\dfrac{1}{a^2 x} \mp \dfrac{1}{a^3} Y.$

77) $\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} \mp \frac{x}{2a^4X} \mp \frac{3}{2a^5}Y.$

78) $\int \frac{dx}{x^2X^3} = -\frac{1}{a^6x} \mp \frac{x}{4a^4X^2} \mp \frac{7x}{8a^6X} \mp \frac{15}{8a^7}Y.$

79) $\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} \mp \frac{1}{2a^4}\ln\frac{x^2}{X}.$

80) $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} \mp \frac{1}{2a^4X} \mp \frac{1}{a^6}\ln\frac{x^2}{X}.$

81) $\int \frac{dx}{x^3X^3} = -\frac{1}{2a^6x^2} \mp \frac{1}{a^6X} \mp \frac{1}{4a^4X^2} \mp \frac{3}{2a^8}\ln\frac{x^2}{X}.$

82) $\int \frac{dx}{(b+cx)X} = \frac{1}{a^2c^2 \pm b^2}\left[c\ln(b+cx) - \frac{c}{2}\ln X \pm \frac{b}{2}Y\right].$

1.4. Integraalid, mis sisaldavad avaldist $a^3 \pm x^3$.

Tähistus: $X = a^3 \pm x^3$.

Kui avaldise ees on märk $\pm$ või $\mp$, siis ülemise märgiga võrduses $X = a^3 + x^3$ ja alumise märgiga võrduses $X = a^3 - x^3$.

83) $\int \frac{dx}{X} = \pm\frac{1}{6a^2}\ln\frac{(a \pm x)^2}{a^2 \mp ax + x^2} + \frac{1}{a^2\sqrt{3}}\operatorname{arctg}\frac{2x \mp a}{a\sqrt{3}}.$

84) $\int \frac{dx}{X^2} = \frac{x}{3a^3X} + \frac{2}{3a^3}\int\frac{dx}{X}$ (№ 83).

85) $\int \frac{x\,dx}{X} = \frac{1}{6a}\ln\frac{a^2 \mp ax + x^2}{(a \pm x)^2} \pm \frac{1}{a\sqrt{3}}\operatorname{arctg}\frac{2x \mp a}{a\sqrt{3}}.$

86) $\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^2X} + \frac{1}{3a^3}\int\frac{x\,dx}{X}$ (№ 85).

87) $\int \frac{x^2\,dx}{X} = \pm\frac{1}{3}\ln X.$ 88) $\int \frac{x^2\,dx}{X^2} = \mp\frac{1}{3X}.$

89) $\int \frac{x^3\,dx}{X} = \pm x \mp a^3\int\frac{dx}{X}$ (№ 83). 90) $\int \frac{x^3\,dx}{X^2} = \mp\frac{x}{3X} \pm \frac{1}{3}\int\frac{dx}{X}$ (№ 83).

91) $\int \frac{dx}{xX} = \frac{1}{3a^3}\ln\frac{x^3}{X}.$ 92) $\int \frac{dx}{xX^2} = \frac{1}{3a^3X} + \frac{1}{3a^6}\ln\frac{x^3}{X}.$

93) $\int \frac{dx}{x^2X} = -\frac{1}{a^3x} \mp \frac{1}{a^3}\int\frac{x\,dx}{X}$ (№ 85).

94) $\int \frac{dx}{x^2X^2} = -\frac{1}{a^6x} \mp \frac{x^2}{3a^6X} \mp \frac{4}{3a^6}\int \frac{x\,dx}{X}$ (№ 85).

95) $\int \frac{dx}{x^3X} = -\frac{1}{2a^3x^2} \mp \frac{1}{a^3}\int \frac{dx}{X}$ (№ 83).

96) $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^6x^2} \mp \frac{x}{3a^6X} \mp \frac{5}{3a^6}\int \frac{dx}{X}$ (№ 83).

1.5. Integraalid, mis sisaldavad avaldist $a^4 + x^4$.

97) $\int \frac{dx}{a^4+x^4} = \frac{1}{4a^3\sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3\sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2}$.

98) $\int \frac{x\,dx}{a^4+x^4} = \frac{1}{2a^2} \operatorname{arctg} \frac{x^2}{a^2}$.

99) $\int \frac{x^2\,dx}{a^4+x^4} = -\frac{1}{4a\sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a\sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2}$.

100) $\int \frac{x^3\,dx}{a^4+x^4} = \frac{1}{4} \ln (a^4 + x^4)$.

1.6. Integraalid, mis sisaldavad avaldist $a^4 - x^4$.

101) $\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3} \ln \frac{a+x}{a-x} + \frac{1}{2a^3} \operatorname{arctg} \frac{x}{a}$. 102) $\int \frac{x\,dx}{a^4 - x^4} = \frac{1}{4a^3} \ln \frac{a^2 + x^2}{a^2 - x^2}$.

103) $\int \frac{x^2\,dx}{a^4 - x^4} = \frac{1}{4a} \ln \frac{a+x}{a-x} - \frac{1}{2a} \operatorname{arctg} \frac{x}{a}$. 104) $\int \frac{x^3\,dx}{a^4 - x^4} = -\frac{1}{4} \ln (a^4 - x^4)$.

1.7. Osamurdudeks lahutamise valemeid.

105) $\frac{1}{(a+bx)(f+gx)} \equiv \frac{1}{fb - ag}\left(\frac{b}{a+bx} - \frac{g}{f+gx}\right)$.

106) $\frac{1}{(x+a)(x+b)(x+c)} \equiv \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$,

$A = \frac{1}{(b-a)(c-a)}$, $B = \frac{1}{(a-b)(c-b)}$, $C = \frac{1}{(a-c)(b-c)}$.

107) $\frac{1}{(x+a)(x+b)(x+c)(x+d)} \equiv \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c} + \frac{D}{x+d}$,

$A = \frac{1}{(b-a)(c-a)(d-a)}$, $B = \frac{1}{(a-b)(c-b)(d-b)}$, ...

108) $\frac{1}{(a+bx^2)(f+gx^2)} \equiv \frac{1}{fb - ag}\left(\frac{b}{a+bx^2} - \frac{g}{f+gx^2}\right)$.

2. Irratsionaalsete funktsioonide integraalid

2.1. Integraalid, mis sisaldavad avaldisi: $\sqrt{x}$, $a^2 \pm b^2x$.

Tähistused:

$$X = a^2 \pm b^2x, \quad Y = \begin{cases} \operatorname{arctg} \dfrac{b\sqrt{x}}{a} & \text{märgi + korral,} \\ \dfrac{1}{2} \ln \dfrac{a + b\sqrt{x}}{a - b\sqrt{x}} & \text{märgi - korral.} \end{cases}$$

Kui avaldise ees on märk $\pm$ või $\mp$, siis ülemise märgiga võrduses $X = a^2 + b^2x$ ja alumise märgiga võrduses $X = a^2 - b^2x$.

109) $\displaystyle\int \frac{\sqrt{x}\,dx}{X} = \pm \frac{2\sqrt{x}}{b^2} \mp \frac{2a}{b^3} Y.$

110) $\displaystyle\int \frac{\sqrt{x^3}\,dx}{X} = \pm \frac{2\sqrt{x^3}}{3b^2} - \frac{2a^2\sqrt{x}}{b^4} + \frac{2a^3}{b^5} Y.$

111) $\displaystyle\int \frac{\sqrt{x}\,dx}{X^2} = \mp \frac{\sqrt{x}}{b^2X} \pm \frac{1}{ab^3} Y.$

112) $\displaystyle\int \frac{\sqrt{x^3}\,dx}{X^2} = \pm \frac{2\sqrt{x^3}}{b^2X} + \frac{3a^2\sqrt{x}}{b^4X} - \frac{3a}{b^5} Y.$

113) $\displaystyle\int \frac{dx}{X\sqrt{x}} = \frac{2}{ab} Y.$

114) $\displaystyle\int \frac{dx}{X\sqrt{x^3}} = -\frac{2}{a^2\sqrt{x}} \mp \frac{2b}{a^3} Y.$

115) $\displaystyle\int \frac{dx}{X^2\sqrt{x}} = \frac{\sqrt{x}}{a^2X} + \frac{1}{a^3b} Y.$

116) $\displaystyle\int \frac{dx}{X^2\sqrt{x^3}} = -\frac{2}{a^2X\sqrt{x}} \mp \frac{3b^2\sqrt{x}}{a^4X} \mp \frac{3b}{a^5} Y.$

2.2. Integraalid, mis sisaldavad avaldist $\sqrt{x}$.

117) $\displaystyle\int \frac{\sqrt{x}\,dx}{a^4 + x^2} = -\frac{1}{2a\sqrt{2}} \ln \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a\sqrt{2}} \operatorname{arctg} \frac{a\sqrt{2x}}{a^2 - x}.$

118) $\displaystyle\int \frac{dx}{(a^4 + x^2)\sqrt{x}} = \frac{1}{2a^3\sqrt{2}} \ln \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a^3\sqrt{2}} \operatorname{arctg} \frac{a\sqrt{2x}}{a^2 - x}.$

119) $\displaystyle\int \frac{\sqrt{x}\,dx}{a^4 - x^2} = \frac{1}{2a} \ln \frac{a + \sqrt{x}}{a - \sqrt{x}} - \frac{1}{a} \operatorname{arctg} \frac{\sqrt{x}}{a}.$

120) $\displaystyle\int \frac{dx}{(a^4 - x^2)\sqrt{x}} = \frac{1}{2a^3} \ln \frac{a + \sqrt{x}}{a - \sqrt{x}} + \frac{1}{a^3} \operatorname{arctg} \frac{\sqrt{x}}{a}.$

2.3. Integraalid, mis sisaldavad avaldist $\sqrt{ax+b}$.

Tähistus: $X = ax + b$.

121) $\int \sqrt{X}\,dx = \frac{b}{3a}\sqrt{X^3}$.

122) $\int x\sqrt{X}\,dx = \frac{2(3ax - 2b)\sqrt{X^3}}{15a^2}$.

123) $\int x^2\sqrt{X}\,dx = \frac{2(15a^2x^2 - 12abx + 8b^2)\sqrt{X^3}}{105a^3}$.

124) $\int \frac{dx}{\sqrt{X}} = \frac{2\sqrt{X}}{a}$.

125) $\int \frac{x\,dx}{\sqrt{X}} = \frac{2(ax - 2b)}{3a^2}\sqrt{X}$.

126) $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{2(3a^2x^2 - 4abx + 8b^2)\sqrt{X}}{15a^3}$.

127) $\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\frac{2}{\sqrt{b}}\operatorname{Arth}\sqrt{\frac{X}{b}} = \frac{1}{\sqrt{b}}\ln\frac{\sqrt{X}-\sqrt{b}}{\sqrt{X}+\sqrt{b}} & (b > 0), \\ \frac{2}{\sqrt{-b}}\operatorname{arctg}\sqrt{\frac{X}{-b}} & (b < 0). \end{cases}$

128) $\int \frac{\sqrt{X}}{x}dx = 2\sqrt{X} + b\int \frac{dx}{x\sqrt{X}}$ (№ 127).

129) $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{X}}$ (№ 127).

130) $\int \frac{\sqrt{X}}{x^2}dx = -\frac{\sqrt{X}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{X}}$ (№ 127).

131) $\int \frac{dx}{x^n\sqrt{X}} = -\frac{\sqrt{X}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{dx}{x^{n-1}\sqrt{X}}$.

132) $\int \sqrt{X^3}\,dx = \frac{2\sqrt{X^5}}{5a}$.

133) $\int x\sqrt{X^3}\,dx = \frac{2}{35a^2}(5\sqrt{X^7} - 7b\sqrt{X^5})$.

134) $\int x^2\sqrt{X^3}\,dx = \frac{2}{a^3}\left(\frac{\sqrt{X^9}}{9} - \frac{2b\sqrt{X^7}}{7} + \frac{b^2\sqrt{X^5}}{5}\right)$.

135) $\int \frac{\sqrt{X^3}}{x}dx = \frac{2\sqrt{X^3}}{3} + 2b\sqrt{X} + b^2\int \frac{dx}{x\sqrt{X}}$ (№ 127).

136) $\int \frac{x\,dx}{\sqrt{X^3}} = \frac{2}{a^2}\left(\sqrt{X} + \frac{b}{\sqrt{X}}\right)$.

137) $\int \frac{x^2\,dx}{\sqrt{X^3}} = \frac{2}{a^3}\left(\frac{\sqrt{X^3}}{3} - 2b\sqrt{X} - \frac{b^2}{\sqrt{X}}\right)$.

138) $\int \frac{dx}{x\sqrt{X^3}} = \frac{2}{b\sqrt{X}} + \frac{1}{b}\int \frac{dx}{x\sqrt{X}}$ (№ 127).

139) $\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{bx\sqrt{X}} - \frac{3a}{b^2\sqrt{X}} - \frac{3a}{2b^2}\int \frac{dx}{x\sqrt{X}}$ (№ 127).

140) $\int X^{\pm n/2}\,dx = \dfrac{2X^{(2\pm n)/2}}{a(2\pm n)}$. 141) $\int xX^{\pm n/2}\,dx = \dfrac{2}{a^2}\left(\dfrac{X^{(4\pm n)/2}}{4\pm n} - \dfrac{bX^{(2\pm n)/2}}{2\pm n}\right)$.

142) $\int x^2X^{\pm n/2}\,dx = \dfrac{2}{a^3}\left(\dfrac{X^{(6\pm n)/2}}{6\pm n} - \dfrac{2bX^{(4\pm n)/2}}{4\pm n} + \dfrac{b^2X^{(2\pm n)/2}}{2\pm n}\right)$.

143) $\int \dfrac{X^{n/2}\,dx}{x} = \dfrac{2X^{n/2}}{n} + b\int \dfrac{X^{(n-2)/2}}{x}\,dx$.

144) $\int \dfrac{dx}{xX^{n/2}} = \dfrac{2}{(n-2)\,bX^{(n-2)/2}} + \dfrac{1}{b}\int \dfrac{dx}{xX^{(n-2)/2}}$.

145) $\int \dfrac{dx}{x^2X^{n/2}} = -\dfrac{1}{bxX^{(n-2)/2}} - \dfrac{na}{2b}\int \dfrac{dx}{xX^{n/2}}$.

2.4. Integraalid, mis sisaldavad avaldisi : $\sqrt{ax+b}$, $\sqrt{fx+g}$.

Tähistused: $X = ax + b$, $Y = fx + g$, $\Delta = bf - ag$.

146) $\int \dfrac{dx}{\sqrt{XY}} = \begin{cases} \dfrac{2}{\sqrt{-af}}\operatorname{arctg}\sqrt{-\dfrac{fX}{aY}} & (af<0), \\ \dfrac{2}{\sqrt{af}}\operatorname{Arth}\sqrt{\dfrac{fX}{aY}} = \dfrac{2}{\sqrt{af}}\ln\left(\sqrt{aY}+\sqrt{fX}\right) & (af>0). \end{cases}$

147) $\int \dfrac{x\,dx}{\sqrt{XY}} = \dfrac{\sqrt{XY}}{af} - \dfrac{ag+bf}{2af}\int \dfrac{dx}{\sqrt{XY}}$ (№ 146).

148) $\int \dfrac{dx}{\sqrt{X}\sqrt{Y^3}} = -\dfrac{2\sqrt{X}}{\Delta\sqrt{Y}}$.

149) $\int \dfrac{dx}{Y\sqrt{X}} = \begin{cases} \dfrac{2}{\sqrt{-\Delta f}}\operatorname{arctg}\dfrac{f\sqrt{X}}{\sqrt{-\Delta f}} & (\Delta f<0), \\ \dfrac{1}{\sqrt{\Delta f}}\ln\dfrac{f\sqrt{X}-\sqrt{\Delta f}}{f\sqrt{X}+\sqrt{\Delta f}} & (\Delta f>0). \end{cases}$

150) $\int \sqrt{XY}\,dx = \dfrac{\Delta+2aY}{4af}\sqrt{XY} - \dfrac{\Delta^2}{8af}\int \dfrac{dx}{\sqrt{XY}}$ (№ 146).

151) $\int \sqrt{\dfrac{Y}{X}}\,dx = \dfrac{1}{a}\sqrt{XY} - \dfrac{\Delta}{2a}\int \dfrac{dx}{\sqrt{XY}}$ (№ 146).

152) $\int \dfrac{\sqrt{X}\,dx}{Y} = \dfrac{2\sqrt{X}}{f} + \dfrac{\Delta}{f}\int \dfrac{dx}{Y\sqrt{X}}$ (№ 149).

153) $\int \frac{Y^n\,dx}{\sqrt{X}} = \frac{2}{(2n+1)\,a}\left(\sqrt{X}\,Y^n - n\Delta \int \frac{Y^{n-1}\,dx}{\sqrt{X}}\right)$.

154) $\int \frac{dx}{\sqrt{X}\,Y^n} = -\frac{1}{(n-1)\,\Delta}\left\{\frac{\sqrt{X}}{Y^{n-1}} + \left(n-\frac{3}{2}\right)a\int \frac{dx}{\sqrt{X}\,Y^{n-1}}\right\}$.

155) $\int \sqrt{X}\,Y^n\,dx = \frac{1}{(2n+3)\,f}\left(2\sqrt{X}\,Y^{n+1} + \Delta \int \frac{Y^n\,dx}{\sqrt{X}}\right)$ (№ 153).

156) $\int \frac{\sqrt{X}\,dx}{Y^n} = \frac{1}{(n-1)\,f}\left(-\frac{\sqrt{X}}{Y^{n-1}} + \frac{a}{2}\int \frac{dx}{\sqrt{Y}\,Y^{n-1}}\right)$.

2.5. Integraalid, mis sisaldavad avaldist $\sqrt{a^2-x^2}$.

Tähistus: $X = a^2 - x^2$.

157) $\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} + a^2 \arcsin \frac{x}{a}\right)$. 158) $\int x\sqrt{X}\,dx = -\frac{1}{3}\sqrt{X^3}$.

159) $\int x^2\sqrt{X}\,dx = -\frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left(x\sqrt{X} + a^2 \arcsin \frac{x}{a}\right)$.

160) $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} - a^2\frac{\sqrt{X^3}}{3}$. 161) $\int \frac{\sqrt{X}}{x}dx = \sqrt{X} - a \ln \frac{a+\sqrt{X}}{x}$.

162) $\int \frac{\sqrt{X}}{x^2}dx = -\frac{\sqrt{X}}{x} - \arcsin \frac{x}{a}$. 163) $\int \frac{\sqrt{X}}{x^3}dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\ln \frac{a+\sqrt{X}}{x}$.

164) $\int \frac{dx}{\sqrt{X}} = \arcsin \frac{x}{a}$. 165) $\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X}$.

166) $\int \frac{x^2\,dx}{\sqrt{X}} = -\frac{x}{2}\sqrt{X} + \frac{a^2}{2}\arcsin \frac{x}{a}$.

167) $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}$. 168) $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln \frac{a+\sqrt{X}}{x}$.

169) $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x}$. 170) $\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} - \frac{1}{2a^3}\ln \frac{a+\sqrt{X}}{x}$.

171) $\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\arcsin \frac{x}{a}\right)$.

172) $\int x\sqrt{X^3}\,dx = -\frac{1}{5}\sqrt{X^5}$.

173) $\int x^2\sqrt{X^3}\,dx = -\frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} + \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\arcsin \frac{x}{a}$.

174) $\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}$.

175) $\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3 \ln\frac{a+\sqrt{X}}{x}$.

176) $\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} - \frac{3x\sqrt{X}}{2} - \frac{3a^2}{2}\arcsin\frac{x}{a}$.

177) $\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} - \frac{3\sqrt{X}}{2} + \frac{3a}{2}\ln\frac{a+\sqrt{X}}{x}$.

178) $\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}$. 179) $\int \frac{x\,dx}{\sqrt{X^3}} = \frac{1}{\sqrt{X}}$. 180) $\int \frac{x^2\,dx}{\sqrt{X^3}} = \frac{x}{\sqrt{X}} - \arcsin\frac{x}{a}$.

181) $\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}$. 182) $\int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln\frac{a+\sqrt{X}}{x}$.

183) $\int \frac{dx}{x^2\sqrt{X^3}} = \frac{1}{a^4}\left(-\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$.

184) $\int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} + \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\ln\frac{a+\sqrt{X}}{x}$.

2.6. Integraalid, mis sisaldavad avaldist $\sqrt{x^2+a^2}$.

Tähistus: $X = x^2 + a^2$.

185) $\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} + a^2\,\mathrm{Arsh}\frac{x}{a}\right) + C = \frac{1}{2}\left[x\sqrt{X} + a^2\ln(x+\sqrt{X})\right] + C_1$.

186) $\int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}$.

187) $\int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left(x\sqrt{X} + a^2\,\mathrm{Arsh}\frac{x}{a}\right) + C =$

$= \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left[x\sqrt{X} + a^2\ln(x+\sqrt{X})\right] + C_1$.

188) $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} - \frac{a^2\sqrt{X^3}}{3}$.

189) $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln\frac{a+\sqrt{X}}{x}$.

190) $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \mathrm{Arsh}\frac{x}{a} + C = -\frac{\sqrt{X}}{x} + \ln(x+\sqrt{X}) + C_1$.

191) $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\ln\frac{a+\sqrt{X}}{x}$.

192) $\int \frac{dx}{\sqrt{X}} = \operatorname{Arsh} \frac{x}{a} + C = \ln (x + \sqrt{X}) + C_1.$ 193) $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}.$

194) $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2} \operatorname{Arsh} \frac{x}{a} + C = \frac{x}{2}\sqrt{X} - \frac{a^2}{2} \ln (x + \sqrt{X}) + C_1.$

195) $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}.$ 196) $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a} \ln \frac{a + \sqrt{X}}{x}.$

197) $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2 x}.$ 198) $\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3} \ln \frac{a + \sqrt{X}}{x}.$

199) $\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2} \operatorname{Arsh} \frac{x}{a}\right) + C =$

$= \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2} \ln (x + \sqrt{X})\right) + C_1.$

200) $\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}.$

201) $\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} - \frac{a^6}{16} \operatorname{Arsh} \frac{x}{a} + C =$

$= \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} - \frac{a^6}{16} \ln (x + \sqrt{X}) + C_1.$

202) $\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}.$

203) $\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3 \ln \frac{a + \sqrt{X}}{x}.$

204) $\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} + \frac{3x}{2}\sqrt{X} + \frac{3a^2}{2} \operatorname{Arsh} \frac{x}{a} + C =$

$= -\frac{\sqrt{X^3}}{x} + \frac{3x}{2}\sqrt{X} + \frac{3a^2}{2} \ln (x + \sqrt{X}) + C_1.$

205) $\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3a}{2} \ln \frac{a + \sqrt{X}}{x}.$

206) $\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}.$ 207) $\int \frac{x\,dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}.$

208) $\int \frac{x^2\,dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \operatorname{Arsh} \frac{x}{a} + C = -\frac{x}{\sqrt{X}} + \ln (x + \sqrt{X}) + C_1.$

209) $\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}.$ 210) $\int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3} \ln \frac{a + \sqrt{X}}{x}.$

211) $\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$.

212) $\int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^5}\ln\frac{a+\sqrt{X}}{x}$.

2.7. Integraalid, mis sisaldavad avaldist $\sqrt{x^2-a^2}$.

Tähistus: $X = x^2 - a^2$.

213) $\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} - a^2 \operatorname{Arch}\frac{x}{a}\right) + C = \frac{1}{2}\left[x\sqrt{X} - a^2\ln(x+\sqrt{X})\right] + C_1$.

214) $\int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}$.

215) $\int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left(x\sqrt{X} - a^2 \operatorname{Arch}\frac{x}{a}\right) + C =$

$= \frac{x}{4}\sqrt{X^3} + \frac{a^3}{8}\left[x\sqrt{X} - a^2\ln(x+\sqrt{X})\right] + C_1$.

216) $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} + \frac{a^2\sqrt{X^3}}{3}$.

217) $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\arccos\frac{a}{x}$.

218) $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \operatorname{Arch}\frac{x}{a} + C = -\frac{\sqrt{X}}{x} + \ln(x+\sqrt{X}) + C_1$.

219) $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\arccos\frac{a}{x}$.

220) $\int \frac{dx}{\sqrt{X}} = \operatorname{Arch}\frac{x}{a} + C = \ln(x+\sqrt{X}) + C_1$.

221) $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}$.

222) $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\operatorname{Arch}\frac{x}{2} + C = \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\ln(x+\sqrt{X}) + C_1$.

223) $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X}$.

224) $\int \frac{dx}{x\sqrt{X}} = \frac{1}{a}\arccos\frac{a}{x}$.

225) $\int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2x}$.

226) $\int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\arccos\frac{a}{x}$.

227) $\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} - \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\operatorname{Arch}\frac{x}{a}\right) + C =$

$= \frac{1}{4}\left(x\sqrt{X^3} - \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\ln(x+\sqrt{X})\right) + C_1$.

228) $\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}.$

229) $\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\operatorname{Arch}\frac{x}{a} + C =$

$= \frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\ln(x+\sqrt{X}) + C_1.$

230) $\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} + \frac{a^2\sqrt{X^5}}{5}.$

231) $\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X} + a^3\arccos\frac{a}{x}.$

232) $\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{2} + \frac{3x}{2}\sqrt{X} - \frac{3a^2}{2}\operatorname{Arch}\frac{x}{a} + C =$

$= -\frac{\sqrt{X^3}}{2} + \frac{3x}{2}\sqrt{X} - \frac{3a^2}{2}\ln(x+\sqrt{X}) + C_1.$

233) $\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3\sqrt{X}}{2} - \frac{3a}{2}\arccos\frac{a}{x}.$

234) $\int \frac{dx}{\sqrt{X^3}} = -\frac{x}{a^2\sqrt{X}}.$ 235) $\int \frac{x\,dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}.$

236) $\int \frac{x^2\,dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \operatorname{Arch}\frac{x}{a} + C = -\frac{x}{\sqrt{X}} + \ln(x+\sqrt{X}) + C_1.$

237) $\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} - \frac{a^2}{\sqrt{X}}.$ 238) $\int \frac{dx}{x\sqrt{X^3}} = -\frac{1}{a^2\sqrt{X}} - \frac{1}{a^2}\arccos\frac{a}{x}.$

239) $\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right).$

240) $\int \frac{dx}{x^3\sqrt{X^3}} = \frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\arccos\frac{a}{x}.$

2.8. Integraalid, mis sisaldavad avaldist $\sqrt{ax^2+bx+c}$

Tähistused: $X = ax^2+bx+c$, $\Delta = 4ac-b^2$, $k = \frac{4a}{\Delta}$.

241) $\int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}}\ln(2\sqrt{aX}+2ax+b)+C & (a>0), \\ \frac{1}{\sqrt{a}}\operatorname{Arsh}\frac{2ax+b}{\sqrt{\Delta}}+C_1 & (a>0,\ \Delta>0), \end{cases}$

241) $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}} \ln (2ax + b) & (a > 0,\ \Delta = 0), \\ -\frac{1}{\sqrt{-a}} \arcsin \frac{2ax + b}{\sqrt{-\Delta}} & (a < 0,\ \Delta < 0). \end{cases}$$

242) $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2ax + b)}{\Delta \sqrt{X}}.$$ 243) $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax + b)}{3\Delta\sqrt{X}} \left(\frac{1}{X} + 2k\right).$$

244) $$\int \frac{dx}{X^{(2n+1)/2}} = \frac{2(2ax + b)}{2(n-1)\Delta X^{(2n-1)/2}} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{(2n-1)/2}}.$$

245) $$\int \sqrt{X}\, dx = \frac{(2ax + b)\sqrt{X}}{4a} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}$$ (№ 241).

246) $$\int X\sqrt{X}\, dx = \frac{(2ax + b)\sqrt{X}}{8a} \left(X + \frac{3}{2k}\right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}$$ (№ 241).

247) $$\int X^2\sqrt{X}\, dx = \frac{(2ax + b)\sqrt{X}}{12a} \left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}$$ (№ 241).

248) $$\int X^{(2n+1)/2}\, dx = \frac{(2ax + b)X^{(2n+1)/2}}{4a(n+1)} + \frac{2n+1}{2k(n+1)} \int X^{(2n-1)/2}\, dx.$$

249) $$\int \frac{x\, dx}{\sqrt{X}} = \frac{\sqrt{X}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{X}}$$ (№ 241).

250) $$\int \frac{x\, dx}{X\sqrt{X}} = -\frac{2(bx + 2c)}{\Delta\sqrt{X}}.$$

251) $$\int \frac{x\, dx}{X^{(2n+1)/2}} = -\frac{1}{(2n-1)aX^{(2n-1)/2}} - \frac{b}{2a} \int \frac{dx}{X^{(2n+1)/2}}$$ (№ 244).

252) $$\int \frac{x^2\, dx}{\sqrt{X}} = \left(\frac{x}{2a} - \frac{3b}{4a^2}\right)\sqrt{X} + \frac{3b^2 - 4ac}{8a^2} \int \frac{dx}{\sqrt{X}}$$ (№ 241).

253) $$\int \frac{x^2\, dx}{X\sqrt{X}} = \frac{(2b^2 - 4ac)x + 2bc}{a\Delta\sqrt{X}} + \frac{1}{a} \int \frac{dx}{\sqrt{X}}$$ (№ 241).

254) $$\int x\sqrt{X}\, dx = \frac{X\sqrt{X}}{3a} - \frac{b(2ax + b)}{8a^2}\sqrt{X} - \frac{b}{4ak} \int \frac{dx}{\sqrt{X}}$$ (№ 241).

255) $$\int xX\sqrt{X}\, dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a} \int X\sqrt{X}\, dx$$ (№ 246).

256) $$\int xX^{(2n+1)/2}\, dx = \frac{X^{(2n+3)/2}}{(2n+3)a} - \frac{b}{2a} \int X^{(2n+1)/2}\, dx$$ (№ 248).

257) $$\int x^2\sqrt{X}\,dx=\left(x-\frac{5b}{6a}\right)\frac{X\sqrt{X}}{4a}+\frac{5b^2-4ac}{16a^2}\int\sqrt{X}\,dx \qquad (\text{№ } 245).$$

258) $$\int\frac{dx}{x\sqrt{X}}=\begin{cases}-\dfrac{1}{\sqrt{c}}\ln\left(\dfrac{2\sqrt{cX}}{x}+\dfrac{2c}{x}+b\right)+C & (c>0),\\[2ex] -\dfrac{1}{\sqrt{c}}\operatorname{Arsh}\dfrac{bx+2c}{x\sqrt{\Delta}}+C_1 & (c>0,\ \Delta>0),\\[2ex] -\dfrac{1}{\sqrt{c}}\ln\dfrac{bx+2c}{x} & (c>0,\ \Delta=0),\\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{x\sqrt{-\Delta}} & (c<0,\ \Delta<0).\end{cases}$$

259) $$\int\frac{dx}{x^2\sqrt{X}}=-\frac{\sqrt{X}}{cx}-\frac{b}{2c}\int\frac{dx}{x\sqrt{X}} \qquad (\text{№ } 258).$$

260) $$\int\frac{\sqrt{X}\,dx}{x}=\sqrt{X}+\frac{b}{2}\int\frac{dx}{\sqrt{X}}+c\int\frac{dx}{x\sqrt{X}} \qquad (\text{№№ } 241 \text{ и } 258).$$

261) $$\int\frac{\sqrt{X}\,dx}{x^2}=-\frac{\sqrt{X}}{x}+a\int\frac{dx}{\sqrt{X}}+\frac{b}{2}\int\frac{dx}{x\sqrt{X}} \qquad (\text{№№ } 241 \text{ и } 258).$$

262) $$\int\frac{X^{(2n+1)/2}}{x}dx=\frac{X^{(2n+1)/2}}{2n+1}+\frac{b}{2}\int X^{(2n-1)/2}\,dx+c\int\frac{X^{(2n-1)/2}}{x}dx \qquad (\text{№№ } 248, 260).$$

2.9. Integraalid, mis sisaldavad teisi irratsionaalseid avaldisi.

263) $$\int\frac{dx}{x\sqrt{ax^2+bx}}=-\frac{2}{bx}\sqrt{ax^2+bx}.$$

264) $$\int\frac{dx}{\sqrt{2ax-x^2}}=\arcsin\frac{x-a}{a}.$$

265) $$\int\frac{x\,dx}{\sqrt{2ax-x^2}}=-\sqrt{2ax-x^2}+a\arcsin\frac{x-a}{a}.$$

266) $$\int\sqrt{2ax-x^2}\,dx=\frac{x-a}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\arcsin\frac{x-a}{a}.$$

267) $$\int \frac{dx}{(ax^2+b)\sqrt{fx^2+g}} =$$

$$= \begin{cases} \dfrac{1}{\sqrt{b}\sqrt{ag-bf}} \operatorname{arctg} \dfrac{x\sqrt{ag-bf}}{\sqrt{b}\sqrt{fx^2+g}} & (ag-bf>0), \\ \dfrac{1}{2\sqrt{b}\sqrt{bf-ag}} \ln \dfrac{\sqrt{b}\sqrt{fx^2+g}+x\sqrt{bf-ag}}{\sqrt{b}\sqrt{fx^2+g}-x\sqrt{bf-ag}} & (ag-bf<0). \end{cases}$$

268) $$\int \sqrt[n]{ax+b}\,dx = \frac{n(ax+b)}{(n+1)a}\sqrt[n]{ax+b}.$$ 269) $$\int \frac{dx}{\sqrt[n]{ax+b}} = \frac{n(ax+b)}{(n-1)a}\cdot\frac{1}{\sqrt[n]{ax+b}}.$$

270) $$\int \frac{dx}{x\sqrt{x^n+a^2}} = -\frac{2}{na}\ln\frac{a+\sqrt{x^n+a^2}}{\sqrt{x^n}}.$$ 271) $$\int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{na}\operatorname{arccos}\frac{a}{\sqrt{x^n}}.$$

272) $$\int \frac{\sqrt{x}\,dx}{\sqrt{a^3-x^3}} = \frac{2}{3}\arcsin\sqrt{\left(\frac{x}{a}\right)^3}.$$

2.10. Integraal diferentsiaalbinoomist.

273) $$\int x^m(ax^n+b)^p\,dx =$$

$$= \frac{1}{m+np+1}\left[x^{m+1}(ax^n+b)^p + npb\int x^m(ax^n+b)^{p-1}\,dx\right],$$

$$= \frac{1}{bn(p+1)}\left[-x^{m+1}(ax^n+b)^{p+1} + (m+n+np+1)\int x^m(ax^n+b)^{p+1}\,dx\right],$$

$$= \frac{1}{(m+1)b}\left[x^{m+1}(ax^n+b)^{p+1} - a(m+n+np+1)\int x^{m+n}(ax^n+b)^p\,dx\right],$$

$$= \frac{1}{a(m+np+1)}\left[x^{m-n+1}(ax^n+b)^{p+1} - (m-n+1)b\int x^{m-n}(ax^n+b)^p\,dx\right].$$

3. Transtsendentsete funktsioonide integraalid

3.1. Siinusfunktsiooni sisaldavad integraalid.

274) $$\int \sin ax\,dx = -\frac{1}{a}\cos ax.$$ 275) $$\int \sin^2 ax\,dx = \frac{1}{2}x - \frac{1}{4a}\sin 2ax.$$

276) $$\int \sin^3 ax\,dx = -\frac{1}{a}\cos ax + \frac{1}{3a}\cos^3 ax.$$

277) $$\int \sin^4 ax\,dx = \frac{3}{8}x - \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax.$$

278) $\int \sin^n ax\,dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx \quad (n = 1, 2, \ldots)$.

279) $\int x \sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$.

280) $\int x^2 \sin ax\,dx = \frac{2x}{a^2}\sin ax - \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\cos ax$.

281) $\int x^3 \sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax - \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\cos ax$.

282) $\int x^n \sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx \quad (n > 0)$.

283) $\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} - \frac{(ax)^7}{7 \cdot 7!} + \ldots$ *).

284) $\int \frac{\sin ax}{x^2}\,dx = -\frac{\sin ax}{x} + a\int \frac{\cos ax\,dx}{x}$ (№ 322).

285) $\int \frac{\sin ax}{x^n}\,dx = -\frac{1}{n-1}\cdot\frac{\sin ax}{x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\,dx$ (№ 324).

286) $\int \frac{dx}{\sin ax} = \frac{1}{a}\ln \operatorname{tg}\frac{ax}{2}$.

287) $\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\operatorname{ctg} ax$.

288) $\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a \sin^2 ax} + \frac{1}{2a}\ln \operatorname{tg}\frac{ax}{2}$.

289) $\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)}\cdot\frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax} \quad (n > 1)$.

290) $\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left(ax + \frac{(ax)^3}{3 \cdot 3!} + \frac{7(ax)^5}{3 \cdot 5 \cdot 5!} + \frac{31(ax)^7}{3 \cdot 7 \cdot 7!} + \frac{127(ax)^9}{3 \cdot 5 \cdot 9!} + \ldots\right.$
$\left. \ldots + \frac{2(2^{2n-1} - 1)}{(2n+1)!} B_n (ax)^{2n+1} + \ldots\right)$. **)

291) $\int \frac{x\,dx}{\sin^2 ax} = -\frac{x}{a}\operatorname{ctg} ax + \frac{1}{a^2}\ln \sin ax$.

*) Määratud integraali $\int_0^x \frac{\sin t}{t}\,dt$ nimetatakse integraalseks siinuseks:

$$\operatorname{Si}(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \ldots .$$

**) B_n — Bernoulli arvud (vt. lk. 44).

292) $$\int \frac{x\,dx}{\sin^n ax} = -\frac{x\cos ax}{(n-1)\,a\sin^{n-1} ax} - \frac{1}{(n-1)(n-2)\,a^2\sin^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x\,dx}{\sin^{n-2} ax} \quad (n>2).$$

293) $$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\operatorname{tg}\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$ 294) $$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\operatorname{tg}\left(\frac{\pi}{4}+\frac{ax}{2}\right).$$

295) $$\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\operatorname{tg}\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\ln\cos\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

296) $$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\operatorname{ctg}\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

297) $$\int \frac{\sin ax\,dx}{1\pm\sin ax} = \pm x + \frac{1}{a}\operatorname{tg}\left(\frac{\pi}{4}\mp\frac{ax}{2}\right).$$

298) $$\int \frac{dx}{\sin ax\,(1\pm\sin ax)} = \frac{1}{a}\operatorname{tg}\left(\frac{\pi}{4}\mp\frac{ax}{2}\right) + \frac{1}{a}\ln\operatorname{tg}\frac{ax}{2}.$$

299) $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\operatorname{tg}\left(\frac{\pi}{4}-\frac{ax}{2}\right) - \frac{1}{6a}\operatorname{tg}^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

300) $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\operatorname{ctg}\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{1}{6a}\operatorname{ctg}^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

301) $$\int \frac{\sin ax\,dx}{(1+\sin ax)^2} = -\frac{1}{2a}\operatorname{tg}\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{1}{6a}\operatorname{tg}^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

302) $$\int \frac{\sin ax\,dx}{(1-\sin ax)^2} = -\frac{1}{2a}\operatorname{ctg}\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{1}{6a}\operatorname{ctg}^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

303) $$\int \frac{dx}{1+\sin^2 ax} = \frac{1}{2\sqrt{2}\,a}\arcsin\left(\frac{3\sin^2 ax-1}{\sin^2 ax+1}\right).$$

304) $$\int \frac{dx}{1-\sin^2 ax} = \int \frac{dx}{\cos^2 ax} = \frac{1}{a}\operatorname{tg} ax.$$

305) $$\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \quad (|a|\neq|b|).$$

306) $$\int \frac{dx}{b+c\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{b^2-c^2}}\operatorname{arctg}\dfrac{b\operatorname{tg}\frac{ax}{2}+c}{\sqrt{b^2-c^2}} & (b^2>c^2), \\ \dfrac{1}{a\sqrt{c^2-b^2}}\ln\dfrac{b\operatorname{tg}\frac{ax}{2}+c-\sqrt{c^2-b^2}}{b\operatorname{tg}\frac{ax}{2}+c+\sqrt{c^2-b^2}} & (b^2<c^2). \end{cases}$$

307) $\int \frac{\sin ax\,dx}{b + c\sin ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b + c\sin ax}$ (№ 306).

308) $\int \frac{dx}{\sin ax\,(b + c\sin ax)} = \frac{1}{ab}\ln \operatorname{tg}\frac{ax}{2} - \frac{c}{b}\int \frac{dx}{b + c\sin ax}$ (№ 306).

309) $\int \frac{dx}{(b + c\sin ax)^2} = \frac{c\cos ax}{a(b^2 - c^2)(b + c\sin ax)} + \frac{b}{b^2 - c^2}\int \frac{dx}{b + c\sin ax}$ (№ 306).

310) $\int \frac{\sin ax\,dx}{(b + c\sin ax)^2} = \frac{b\cos ax}{a(c^2 - b^2)(b + c\sin ax)} + \frac{c}{c^2 - b^2}\int \frac{dx}{b + c\sin ax}$ (№ 306).

311) $\int \frac{dx}{b^2 + c^2\sin^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}}\operatorname{arctg}\frac{\sqrt{b^2 + c^2}\operatorname{tg} ax}{b}$ $(b > 0)$.

312) $\int \frac{dx}{b^2 - c^2\sin^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}}\operatorname{arctg}\frac{\sqrt{b^2 - c^2}\operatorname{tg} ax}{b}$ $(b^2 > c^2,\ b > 0)$,

$= \frac{1}{2ab\sqrt{c^2 - b^2}}\ln\frac{\sqrt{c^2 - b^2}\operatorname{tg} ax + b}{\sqrt{c^2 - b^2}\operatorname{tg} ax - b}$ $(c^2 > b^2,\ b > 0)$.

3.2. Koosinusfunktsiooni sisaldavad integraalid.

313) $\int \cos ax\,dx = \frac{1}{a}\sin ax$. 314) $\int \cos^2 ax\,dx = \frac{1}{2}x + \frac{1}{4a}\sin 2ax$.

315) $\int \cos^3 ax\,dx = \frac{1}{a}\sin ax - \frac{1}{3a}\sin^3 ax$.

316) $\int \cos^4 ax\,dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax$.

317) $\int \cos^n ax\,dx = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$.

318) $\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$.

319) $\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$.

320) $\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$.

321) $\int x^n\cos ax\,dx = \frac{x^n\sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx$.

322) $\int \frac{\cos ax}{x}\,dx = \ln(ax) - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \ldots$ *).

323) $\int \frac{\cos ax}{x^2}\,dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax\,dx}{x}$ (№ 283).

324) $\int \frac{\cos ax}{x^n}\,dx = -\frac{\cos ax}{(n-1)\,x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax\,dx}{x^{n-1}}$ $(n \neq 1)$ (№ 285).

325) $\int \frac{dx}{\cos ax} = \frac{1}{a}\ln \operatorname{tg}\left(\frac{ax}{2} + \frac{\pi}{4}\right)$. 326) $\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\operatorname{tg} ax$.

327) $\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{ax}{2}\right)$.

328) $\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)}\frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax}$ $(n > 1)$.

329) $\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left(\frac{(ax)^2}{2} + \frac{(ax)^4}{4\cdot 2!} + \frac{5(ax)^6}{6\cdot 4!} + \frac{61(ax)^8}{8\cdot 6!} + \right.$

$\left. + \frac{1385(ax)^{10}}{10\cdot 8!} + \ldots + \frac{E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \ldots\right)$ **).

330) $\int \frac{x\,dx}{\cos^2 ax} = \frac{x}{a}\operatorname{tg} ax + \frac{1}{a^2}\ln\cos ax$.

331) $\int \frac{x\,dx}{\cos^n ax} = \frac{x\sin ax}{(n-1)\,a\cos^{n-1} ax} - \frac{1}{(n-1)(n-2)\,a^2\cos^{n-2} ax} +$

$+ \frac{n-2}{n-1}\int \frac{x\,dx}{\cos^{n-2} ax}$ $(n > 2)$.

332) $\int \frac{dx}{1+\cos ax} = \frac{1}{a}\operatorname{tg}\frac{ax}{2}$. 333) $\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\operatorname{ctg}\frac{ax}{2}$.

334) $\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\operatorname{tg}\frac{ax}{2} + \frac{2}{a^2}\ln\cos\frac{ax}{2}$.

---

*) Määratud integraali $\int_x^{+\infty} \frac{\cos t}{t}\,dt$ $(x > 0)$ nimetatakse integraalseks kosinuseks:

$$\operatorname{Ci}(x) = C + \ln x - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \ldots$$

kus C on Euleri konstant (vt. lk. 44).

**) E_n — Euleri arvud (vt. lk. 44).

335) $\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\operatorname{ctg}\frac{ax}{2} + \frac{2}{a^2}\ln\sin\frac{ax}{2}.$

336) $\int \frac{\cos ax\,dx}{1+\cos ax} = x - \frac{1}{a}\operatorname{tg}\frac{ax}{2}.$ 337) $\int \frac{\cos ax\,dx}{1-\cos ax} = -x - \frac{1}{a}\operatorname{ctg}\frac{ax}{2}.$

338) $\int \frac{dx}{\cos ax\,(1+\cos ax)} = \frac{1}{a}\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\operatorname{tg}\frac{ax}{2}.$

339) $\int \frac{dx}{\cos ax\,(1-\cos ax)} = \frac{1}{a}\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\operatorname{ctg}\frac{ax}{2}.$

340) $\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\operatorname{tg}\frac{ax}{2} + \frac{1}{6a}\operatorname{tg}^3\frac{ax}{2}.$

341) $\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\operatorname{ctg}\frac{ax}{2} - \frac{1}{6a}\operatorname{ctg}^3\frac{ax}{2}.$

342) $\int \frac{\cos ax\,dx}{(1+\cos ax)^2} = \frac{1}{2a}\operatorname{tg}\frac{ax}{2} - \frac{1}{6a}\operatorname{tg}^3\frac{ax}{2}.$

343) $\int \frac{\cos ax\,dx}{(1-\cos ax)^2} = \frac{1}{2a}\operatorname{ctg}\frac{ax}{2} - \frac{1}{6a}\operatorname{ctg}^3\frac{ax}{2}.$

344) $\int \frac{dx}{1+\cos^2 ax} = \frac{1}{2\sqrt{2}\,a}\arcsin\left(\frac{1-3\cos^2 ax}{1+\cos^2 ax}\right).$

345) $\int \frac{dx}{1-\cos^2 ax} = \int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\operatorname{ctg} ax.$

346) $\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad (|a| \neq |b|).$

347) $\int \frac{dx}{b+c\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{b^2-c^2}}\operatorname{arctg}\dfrac{(b-c)\operatorname{tg}\frac{ax}{2}}{\sqrt{b^2-c^2}} & (b^2 > c^2), \\ \dfrac{1}{a\sqrt{c^2-b^2}}\ln\dfrac{(c-b)\operatorname{tg}\frac{ax}{2} + \sqrt{c^2-b^2}}{(c-b)\operatorname{tg}\frac{ax}{2} - \sqrt{c^2-b^2}} & (b^2 < c^2). \end{cases}$

348) $\int \frac{\cos ax\,dx}{b+c\cos ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b+c\cos ax}$ (№ 347).

349) $\int \frac{dx}{\cos ax\,(b+c\cos ax)} = \frac{1}{ab}\ln\operatorname{tg}\left(\frac{ax}{2}+\frac{\pi}{4}\right) - \frac{c}{b}\int \frac{dx}{b+c\cos ax}$ (№ 347).

350) $\int \frac{dx}{(b+c\cos ax)^2} = \frac{c\sin ax}{a(c^2-b^2)(b+c\cos ax)} - \frac{b}{c^2-b^2}\int \frac{dx}{b+c\cos ax}$ (№ 347).

351) $$\int \frac{\cos ax\,dx}{(b+c\cos ax)^2} = \frac{b\sin ax}{a(b^2-c^2)(b+c\cos ax)} - \frac{c}{b^2-c^2}\int \frac{dx}{b+c\cos ax}$$ (№ 347).

352) $$\int \frac{dx}{b^2+c^2\cos^2 ax} = \frac{1}{ab\sqrt{b^2+c^2}}\operatorname{arctg}\frac{b\operatorname{tg} ax}{\sqrt{b^2+c^2}} \quad (b>0).$$

353) $$\int \frac{dx}{b^2-c^2\cos^2 ax} = \begin{cases} \dfrac{1}{ab\sqrt{b^2-c^2}}\operatorname{arctg}\dfrac{b\operatorname{tg} ax}{\sqrt{b^2-c^2}} & (b^2>c^2,\quad b>0), \\ \dfrac{1}{2ab\sqrt{c^2-b^2}}\ln\dfrac{b\operatorname{tg} ax-\sqrt{c^2-b^2}}{b\operatorname{tg} ax+\sqrt{c^2-b^2}} & (c^2>b^2,\quad b>0). \end{cases}$$

3.3. Siinus- ja koosinusfunktsiooni sisaldavad integraalid.

354) $$\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax.$$ 355) $$\int \sin^2 ax\cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}.$$

356) $$\int \sin^n ax\cos ax\,dx = \frac{1}{a(n+1)}\sin^{n+1} ax \quad (n\neq -1).$$

357) $$\int \sin ax\cos^n ax\,dx = -\frac{1}{a(n+1)}\cos^{n+1} ax \quad (n\neq -1).$$

358) $$\int \sin^n ax\cos^m ax\,dx = -\frac{\sin^{n-1} ax\cos^{m+1} ax}{a(n+m)} + \frac{n-1}{n+m}\int \sin^{n-2} ax\cos^m ax\,dx,$$

$$= \frac{\sin^{n+1} ax\cos^{m-1} ax}{a(n+m)} + \frac{m-1}{n+m}\int \sin^n ax\cos^{m-2} ax\,dx \quad (m,n>0).$$

359) $$\int \frac{dx}{\sin ax\cos ax} = \frac{1}{a}\ln\operatorname{tg} ax.$$

360) $$\int \frac{dx}{\sin^2 ax\cos ax} = \frac{1}{a}\left[\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{\sin ax}\right]$$

361) $$\int \frac{dx}{\sin ax\cos^2 ax} = \frac{1}{a}\left(\ln\operatorname{tg}\frac{ax}{2} - \frac{1}{\cos ax}\right).$$

362) $$\int \frac{dx}{\sin^3 ax\cos ax} = \frac{1}{a}\left(\ln\operatorname{tg} ax - \frac{1}{2\sin^2 ax}\right).$$

363) $$\int \frac{dx}{\sin ax\cos^3 ax} = \frac{1}{a}\left(\ln\operatorname{tg} ax + \frac{1}{2\cos^2 ax}\right).$$

364) $$\int \frac{dx}{\sin^2 ax\cos^2 ax} = -\frac{2}{a}\operatorname{ctg} 2ax.$$

365) $$\int \frac{dx}{\sin^2 ax\cos^3 ax} = \frac{1}{a}\left[\frac{\sin ax}{2\cos^2 ax} - \frac{1}{\sin ax} + \frac{3}{2}\ln\operatorname{tg}\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right].$$

366) $$\int \frac{dx}{\sin^3 ax \cos^2 ax} = \frac{1}{a}\left(\frac{1}{\cos ax} - \frac{\cos ax}{2\sin^2 ax} + \frac{3}{2}\ln \operatorname{tg}\frac{ax}{2}\right).$$

367) $$\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax} \quad (n \neq 1)$$

(№№ 361, 363).

368) $$\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax} \quad (n \neq 1)$$

(№№ 360 362).

369) $$\int \frac{dx}{\sin^n ax \cos^m ax} = -\frac{1}{a(n-1)} \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} +$$

$$+ \frac{n+m-2}{n-1} \int \frac{dx}{\sin^{n-2} ax \cos^m ax} \qquad (m > 0, n > 1),$$

$$= \frac{1}{a(m-1)} \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} +$$

$$+ \frac{n+m-2}{m-1} \int \frac{dx}{\sin^n ax \cos^{m-2} ax} \qquad (n > 0, m > 1).$$

370) $$\int \frac{\sin ax\, dx}{\cos^2 ax} = \frac{1}{a\cos ax}.$$ 371) $$\int \frac{\sin ax\, dx}{\cos^3 ax} = \frac{1}{2a\cos^2 ax} + C = \frac{1}{2a}\operatorname{tg}^2 ax + C_1.$$

372) $$\int \frac{\sin ax\, dx}{\cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax}.$$

373) $$\int \frac{\sin^2 ax\, dx}{\cos ax} = -\frac{1}{a}\sin ax + \frac{1}{a}\ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{ax}{2}\right).$$

374) $$\int \frac{\sin^4 ax\, dx}{\cos^3 ax} = \frac{1}{a}\left[\frac{\sin ax}{2\cos^2 ax} - \frac{1}{2}\ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right].$$

375) $$\int \frac{\sin^2 ax\, dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{n-1}\int \frac{dx}{\cos^{n-2} ax} \quad (n \neq 1)$$

(№№ 325, 326, 328).

376) $$\int \frac{\sin^3 ax\, dx}{\cos ax} = -\frac{1}{a}\left(\frac{\sin^2 ax}{2} + \ln \cos ax\right).$$

377) $$\int \frac{\sin^3 ax\, dx}{\cos^2 ax} = \frac{1}{a}\left(\cos ax + \frac{1}{\cos ax}\right).$$

378) $$\int \frac{\sin^3 ax\, dx}{\cos^n ax} = \frac{1}{a}\left[\frac{1}{(n-1)\cos^{n-1} ax} - \frac{1}{(n-3)\cos^{n-3} ax}\right] \quad (n \neq 1,\ n \neq 3).$$

379) $$\int \frac{\sin^n ax}{\cos ax}\,dx = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax\,dx}{\cos ax} \qquad (n \neq 1).$$

380) $$\int \frac{\sin^n ax}{\cos^m ax}\,dx = \frac{\sin^{n+1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\sin^n ax}{\cos^{m-2} ax}\,dx \qquad (m \neq 1),$$

$$= -\frac{\sin^{n-1} ax}{a(n-m)\cos^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\sin^{n-2} ax\,dx}{\cos^m ax} \qquad (m \neq n),$$

$$= \frac{\sin^{n-1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\sin^{n-1} ax\,dx}{\cos^{m-2} ax} \qquad (m \neq 1).$$

381) $$\int \frac{\cos ax\,dx}{\sin^2 ax} = -\frac{1}{a\sin ax}.$$

382) $$\int \frac{\cos ax\,dx}{\sin^3 ax} = -\frac{1}{2a\sin^2 ax} + C = -\frac{\operatorname{ctg}^2 ax}{2a} + C_1.$$

383) $$\int \frac{\cos ax\,dx}{\sin^n ax} = -\frac{1}{a(n-1)\sin^{n-1} ax}.$$

384) $$\int \frac{\cos^2 ax\,dx}{\sin ax} = \frac{1}{a}\left(\cos ax + \ln \operatorname{tg}\frac{ax}{2}\right).$$

385) $$\int \frac{\cos^2 ax\,dx}{\sin^3 ax} = -\frac{1}{2a}\left(\frac{\cos ax}{\sin^2 ax} - \ln \operatorname{tg}\frac{ax}{2}\right).$$

386) $$\int \frac{\cos^2 ax\,dx}{\sin^n ax} = -\frac{1}{(n-1)}\left(\frac{\cos ax}{a\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} dx}\right) \qquad (n \neq 1) \quad (\text{№ } 289).$$

387) $$\int \frac{\cos^3 ax\,dx}{\sin ax} = \frac{1}{a}\left(\frac{\cos^2 ax}{2} + \ln \sin ax\right).$$

388) $$\int \frac{\cos^3 ax\,dx}{\sin^2 ax} = -\frac{1}{a}\left(\sin ax + \frac{1}{\sin ax}\right).$$

389) $$\int \frac{\cos^3 ax\,dx}{\sin^n ax} = \frac{1}{a}\left[\frac{1}{(n-3)\sin^{n-3} ax} - \frac{1}{(n-1)\sin^{n-1} ax}\right] \qquad (n \neq 1,\ n \neq 3).$$

390) $$\int \frac{\cos^n ax}{\sin ax}\,dx = \frac{\cos^{n-1} ax}{a(n-1)} + \int \frac{\cos^{n-2} ax\,dx}{\sin ax} \qquad (n \neq 1).$$

391) $$\int \frac{\cos^n ax\,dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\cos^n ax\,dx}{\sin^{m-2} ax} \qquad (m \neq 1).$$

$$= \frac{\cos^{n-1} ax}{a(n-m)\sin^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cos^{n-2} ax\,dx}{\sin^m ax} \qquad (m \neq n),$$

$$= -\frac{\cos^{n-1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax\,dx}{\sin^{m-2} ax} \qquad (m \neq 1).$$

392) $$\int \frac{dx}{\sin ax\,(1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \ln \operatorname{tg} \frac{ax}{2}.$$

393) $$\int \frac{dx}{\cos ax\,(1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{ax}{2}\right).$$

394) $$\int \frac{\sin ax\,dx}{\cos ax\,(1 \pm \cos ax)} = \frac{1}{a} \ln \frac{1 \pm \cos ax}{\cos ax}.$$

395) $$\int \frac{\cos ax\,dx}{\sin ax\,(1 \pm \sin ax)} = -\frac{1}{a} \ln \frac{1 \pm \sin ax}{\sin ax}.$$

396) $$\int \frac{\sin ax\,dx}{\cos ax\,(1 \pm \sin ax)} = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a} \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{ax}{2}\right).$$

397) $$\int \frac{\cos ax\,dx}{\sin ax\,(1 \pm \cos ax)} = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a} \ln \operatorname{tg} \frac{ax}{2}.$$

398) $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a} \ln (\sin ax \pm \cos ax).$$

399) $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm \frac{x}{2} + \frac{1}{2a} \ln (\sin ax \pm \cos ax).$$

400) $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \ln \operatorname{tg}\left(\frac{ax}{2} \pm \frac{\pi}{8}\right).$$

401) $$\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm \frac{1}{a} \ln \left(1 \pm \operatorname{tg} \frac{ax}{2}\right).$$

402) $$\int \frac{dx}{b \sin ax + c \cos ax} = \frac{1}{a\sqrt{b^2 + c^2}} \ln \operatorname{tg} \frac{ax + \theta}{2},$$

$$\sin \theta = \frac{c}{\sqrt{b^2 + c^2}}, \quad \operatorname{tg} \theta = \frac{c}{b}.$$

403) $$\int \frac{\sin ax\,dx}{b + c \cos ax} = -\frac{1}{ac} \ln (b + c \cos ax).$$

404) $$\int \frac{\cos ax\,dx}{b + c \sin ax} = \frac{1}{ac} \ln (b + c \sin ax).$$

405) $$\int \frac{dx}{b + c \cos ax + f \sin ax} = \int \frac{d\left(x + \frac{\theta}{a}\right)}{b + \sqrt{c^2 + f^2} \sin (ax + \theta)},$$

$$\sin \theta = \frac{c}{\sqrt{c^2 + f^2}}, \quad \operatorname{tg} \theta = \frac{c}{f} \qquad (\text{№ } 306).$$

406) $$\int \frac{dx}{b^2\cos^2 ax + c^2\sin^2 ax} = \frac{1}{abc}\operatorname{arctg}\left(\frac{c}{b}\operatorname{tg} ax\right).$$

407) $$\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc}\ln\frac{c\operatorname{tg} ax + b}{c\operatorname{tg} ax - b}.$$

408) $$\int \sin ax\cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \qquad (a^2 \neq b^2).$$

tan

3.4. Tangensfunktsiooni sisaldavad integraalid.

409) $$\int \operatorname{tg} ax\,dx = -\frac{1}{a}\ln\cos ax.$$ 410) $$\int \operatorname{tg}^2 ax\,dx = \frac{\operatorname{tg} ax}{a} - x.$$

411) $$\int \operatorname{tg}^3 ax\,dx = \frac{1}{2a}\operatorname{tg}^2 ax + \frac{1}{a}\ln\cos ax.$$

412) $$\int \operatorname{tg}^n ax\,dx = \frac{1}{a(n-1)}\operatorname{tg}^{n-1} ax - \int \operatorname{tg}^{n-2} ax\,dx.$$

413) $$\int x\operatorname{tg} ax\,dx = \frac{ax^3}{3} + \frac{a^3x^5}{15} + \frac{2a^5x^7}{105} + \frac{17a^7x^9}{2835} + \ldots$$

$$\ldots + \frac{2^{2n}(2^{2n}-1)B_n a^{2n-1}x^{2n+1}}{(2n+1)!} + \ldots\ ^{*}).$$

414) $$\int \frac{\operatorname{tg} ax\,dx}{x} = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \frac{17(ax)^7}{2205} + \ldots$$

$$\ldots + \frac{2^{2n}(2^{2n}-1)B_n (ax)^{2n-1}}{(2n-1)(2n)!} + \ldots\ ^{*}).$$

415) $$\int \frac{\operatorname{tg}^n ax}{\cos^2 ax}\,dx = \frac{1}{a(n+1)}\operatorname{tg}^{n+1} ax \qquad (n \neq -1).$$

416) $$\int \frac{dx}{\operatorname{tg} ax \pm 1} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax).$$

417) $$\int \frac{\operatorname{tg} ax\,dx}{\operatorname{tg} ax \pm 1} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax).$$

---

*) B_n - Bernoulli arvud (vt. lk. 44).

3.5. Kootangensfunktsiooni sisaldavad integraalid.

418) $\int \operatorname{ctg} ax\,dx = \frac{1}{a} \ln \sin ax.$ 419) $\int \operatorname{ctg}^2 ax\,dx = -\frac{\operatorname{ctg} ax}{a} - x.$

420) $\int \operatorname{ctg}^3 ax\,dx = -\frac{1}{2a} \operatorname{ctg}^2 ax - \frac{1}{a} \ln \sin ax.$

421) $\int \operatorname{ctg}^n ax\,dx = -\frac{1}{a(n-1)} \operatorname{ctg}^{n-1} ax - \int \operatorname{ctg}^{n-2} ax\,dx \quad (n \neq 1).$

422) $\int x \operatorname{ctg} ax\,dx = \frac{x}{a} - \frac{ax^3}{9} - \frac{a^3x^5}{225} - \ldots - \frac{2^{2n}B_n a^{2n-1}x^{2n+1}}{(2n+1)!} - \ldots$ *).

423) $\int \frac{\operatorname{ctg} ax\,dx}{x} = -\frac{1}{ax} - \frac{ax}{3} - \frac{(ax)^3}{135} - \frac{2(ax)^5}{4725} - \ldots - \frac{2^{2n}B_n (ax)^{2n-1}}{(2n-1)(2n)!} - \ldots$ *).

424) $\int \frac{\operatorname{ctg}^n ax}{\sin^2 ax}\,dx = -\frac{1}{a(n+1)} \operatorname{ctg}^{n+1} ax \quad (n \neq -1).$

425) $\int \frac{dx}{1 \pm \operatorname{ctg} ax} = \int \frac{\operatorname{tg} ax\,dx}{\operatorname{tg} ax \pm 1}$ (№ 417).

3.6. Hüperboolsete funktsioonide integraalid.

426) $\int \operatorname{sh} ax\,dx = \frac{1}{a} \operatorname{ch} ax.$ 427) $\int \operatorname{ch} ax\,dx = \frac{1}{a} \operatorname{sh} ax.$

428) $\int \operatorname{sh}^2 ax\,dx = \frac{1}{2a} \operatorname{sh} ax \operatorname{ch} ax - \frac{1}{2} x.$ 429) $\int \operatorname{ch}^2 ax\,dx = \frac{1}{2a} \operatorname{sh} ax \operatorname{ch} ax + \frac{1}{2} x.$

430) $\int \operatorname{sh}^n ax\,dx = \frac{1}{an} \operatorname{sh}^{n-1} ax \operatorname{ch} ax - \frac{n-1}{n} \int \operatorname{sh}^{n-2} ax\,dx \qquad (n > 0),$

$= \frac{1}{a(n+1)} \operatorname{sh}^{n+1} ax \operatorname{ch} ax - \frac{n+2}{n+1} \int \operatorname{sh}^{n+2} ax\,dx \quad (n < 0;\ n \neq -1).$

431) $\int \operatorname{ch}^n ax\,dx = \frac{1}{an} \operatorname{sh} ax \operatorname{ch}^{n-1} ax + \frac{n-1}{n} \int \operatorname{ch}^{n-2} ax\,dx \qquad (n > 0),$

$= -\frac{1}{a(n+1)} \operatorname{sh} ax \operatorname{ch}^{n+1} ax + \frac{n+2}{n+1} \int \operatorname{ch}^{n+2} ax\,dx \quad (n < 0;\ n \neq -1).$

432) $\int \frac{dx}{\operatorname{sh} ax} = \frac{1}{a} \ln \operatorname{th} \frac{ax}{2}$ 433) $\int \frac{dx}{\operatorname{ch} ax} = \frac{2}{a} \operatorname{arctg} e^{ax}.$

---

*) B_n - Bernoulli arvud (vt. lk. 44).

434) $\int x \operatorname{sh} ax\, dx = \frac{1}{a} x \operatorname{ch} ax - \frac{1}{a^2} \operatorname{sh} ax.$

435) $\int x \operatorname{ch} ax\, dx = \frac{1}{a} x \operatorname{sh} ax - \frac{1}{a^2} \operatorname{ch} ax.$

436) $\int \operatorname{th} ax\, dx = \frac{1}{a} \ln \operatorname{ch} ax.$ 437) $\int \operatorname{cth} ax\, dx = \frac{1}{a} \ln \operatorname{sh} ax.$

438) $\int \operatorname{th}^2 ax\, dx = x - \frac{\operatorname{th} ax}{a}.$ 439) $\int \operatorname{cth}^2 ax\, dx = x - \frac{\operatorname{cth} ax}{a}$

440) $\int \operatorname{sh} ax \operatorname{sh} bx\, dx = \frac{1}{a^2 - b^2} (a \operatorname{sh} bx \operatorname{ch} ax - b \operatorname{ch} bx \operatorname{sh} ax).$

441) $\int \operatorname{ch} ax \operatorname{ch} bx\, dx = \frac{1}{a^2 - b^2} (a \operatorname{sh} ax \operatorname{ch} bx - b \operatorname{sh} bx \operatorname{ch} ax).$ $a^2 \neq b^2$

442) $\int \operatorname{ch} ax \operatorname{sh} bx\, dx = \frac{1}{a^2 - b^2} (a \operatorname{sh} bx \operatorname{sh} ax - b \operatorname{ch} bx \operatorname{ch} ax).$

443) $\int \operatorname{sh} ax \sin ax\, dx = \frac{1}{2a} (\operatorname{ch} ax \sin ax - \operatorname{sh} ax \cos ax).$

444) $\int \operatorname{ch} ax \cos ax\, dx = \frac{1}{2a} (\operatorname{sh} ax \cos ax + \operatorname{ch} ax \sin ax).$

445) $\int \operatorname{sh} ax \cos ax\, dx = \frac{1}{2a} (\operatorname{ch} ax \cos ax + \operatorname{sh} ax \sin ax).$

446) $\int \operatorname{ch} ax \sin ax\, dx = \frac{1}{2a} (\operatorname{sh} ax \sin ax - \operatorname{ch} ax \cos ax).$

3.7. Eksponentfunktsioonide integraalid.

447) $\int e^{ax}\, dx = \frac{1}{a} e^{ax}.$ 448) $\int x e^{ax}\, dx = \frac{e^{ax}}{a^2} (ax - 1).$

449) $\int x^2 e^{ax}\, dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right).$

450) $\int x^n e^{ax}\, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx.$

451) $\int \frac{e^{ax}}{x}\,dx = \ln x + \frac{ax}{1\cdot 1!} + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \ldots$ *).

452) $\int \frac{e^{ax}}{x^n}\,dx = \frac{1}{n-1}\left(-\frac{e^{ax}}{x^{n-1}} + a\int \frac{e^{ax}}{x^{n-1}}\,dx\right) \quad (n \neq 1).$

453) $\int \frac{dx}{1+e^{ax}} = \frac{1}{a}\ln\frac{e^{ax}}{1+e^{ax}}.$

454) $\int \frac{dx}{b+ce^{ax}} = \frac{x}{b} - \frac{1}{ab}\ln(b+ce^{ax}).$ 455) $\int \frac{e^{ax}\,dx}{b+ce^{ax}} = \frac{1}{ac}\ln(b+ce^{ax}).$

456) $\int \frac{dx}{be^{ax}+ce^{-ax}} = \frac{1}{a\sqrt{bc}}\operatorname{arctg}\left(e^{ax}\sqrt{\frac{b}{c}}\right) \quad (ac > 0),$

$$= \frac{1}{a\sqrt{-bc}}\ln\frac{c+e^{ax}\sqrt{-bc}}{c-e^{ax}\sqrt{-bc}} \quad (bc < 0).$$

457) $\int \frac{xe^{ax}\,dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$

458) $\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$ (№ 451).

459) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx).$

460) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$

461) $\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx$

(№№ 447–459).

462) $\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1}x}{a^2+n^2}(a\cos x + n\sin x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx$

(№№ 447–460).

---

*) Määratud integraali $\int_{-\infty}^{x} \frac{e^t}{t}\,dt$ $(x<0)$ nimetatakse integraalseks eksponentfunktsiooniks ja tähistatakse: $\mathrm{Ei}(x)$. Kui $x>0$, siis $\mathrm{Ei}(x)$ defineeritakse päratu integraali $\int_{-\infty}^{x} \frac{e^t}{t}\,dt = C + \ln|x| + \frac{x}{1\cdot 1!} + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \ldots + \frac{x^n}{n\cdot n!} + \ldots$ peaväärtusena, kus C on Euleri konstant (vt. lk. 44).

463) $$\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin bx - 2ab\cos bx].$$

464) $$\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos bx + 2ab\sin bx].$$

3.8. Logaritmfunktsioonide integraalid.

465) $\int \ln x\,dx = x\ln x - x.$ 466) $\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x.$

467) $\int (\ln x)^3\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x.$

468) $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \quad (n \neq -1).$

469) $$\int \frac{dx}{\ln x} = \ln\ln x + \ln x + \frac{(\ln x)^2}{2\cdot 2!} + \frac{(\ln x)^3}{3\cdot 3!} + \ldots\ ^{*}).$$

470) $$\int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1)(\ln x)^{n-1}} + \frac{1}{n-1}\int \frac{dx}{(\ln x)^{n-1}} \quad (n \neq 1)\ (\text{№ } 469).$$

471) $$\int x^m \ln x\,dx = x^{m+1}\left[\frac{\ln x}{m+1} - \frac{1}{(m+1)^2}\right] \quad (m \neq -1).$$

472) $$\int x^m (\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\,dx$$

$(m \neq -1,\ n \neq -1)$ (№ 470).

473) $$\int \frac{(\ln x)^n}{x}\,dx = \frac{(\ln x)^{n+1}}{n+1}.$$

474) $$\int \frac{\ln x}{x^m}\,dx = -\frac{\ln x}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2 x^{m-1}} \quad (m \neq 1).$$

---

*) Määratud integraali $\int_0^x \frac{dt}{\ln t}$ nimetatakse integraalseks logaritmiks ja tähistatakse $\mathrm{li}\,x$. Kui $x > 1$, siis integraal hajub, sel juhul $\mathrm{li}\,x$ defineeritakse päratu integraali peaväärtusena; $\mathrm{li}\,x = \mathrm{Ei}(\ln x)$.

475) $\int \frac{(\ln x)^n}{x^m}\,dx = -\frac{(\ln x)^n}{(m-1)\,x^{m-1}} + \frac{n}{m-1}\int \frac{(\ln x)^{n-1}}{x^m}\,dx \qquad (m \neq 1)$ (№ 474).

476) $\int \frac{x^m\,dx}{\ln x} = \int \frac{e^{-y}}{y}\,dy, \quad y = -(m+1)\ln x$ (№ 451).

477) $\int \frac{x^m\,dx}{(\ln x)^n} = -\frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\ln x)^{n-1}} \qquad (n \neq 1)$.

478) $\int \frac{dx}{x \ln x} = \ln \ln x$.

479) $\int \frac{dx}{x^n \ln x} = \ln \ln x - (n-1)\ln x + \frac{(n-1)^2(\ln x)^2}{2 \cdot 2!} - \frac{(n-1)^3(\ln x)^3}{3 \cdot 3!} + \ldots$

480) $\int \frac{dx}{x\,(\ln x)^n} = \frac{-1}{(n-1)(\ln x)^{n-1}} \qquad (n \neq 1)$.

481) $\int \frac{dx}{x^p (\ln x)^n} = \frac{-1}{x^{p-1}(n-1)(\ln x)^{n-1}} - \frac{p-1}{n-1}\int \frac{dx}{x^p (\ln x)^{n-1}} \qquad (n \neq 1)$.

482) $\int \ln \sin x\,dx = x \ln x - x - \frac{x^3}{18} - \frac{x^5}{900} - \ldots - \frac{2^{2n-1} B_n x^{2n+1}}{n\,(2n+1)!} - \ldots$ *).

483) $\int \ln \cos x\,dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{315} - \ldots - \frac{2^{2n-1}(2^{2n}-1)\,B_n}{n\,(2n+1)!}\,x^{2n+1} - \ldots$ *).

484) $\int \ln \operatorname{tg} x\,dx = x \ln x - x + \frac{x^3}{9} + \frac{7x^5}{450} + \ldots + \frac{2^{2n}(2^{2n-1}-1)\,B_n}{n\,(2n+1)!}\,x^{2n+1} + \ldots$ *).

485) $\int \sin \ln x\,dx = \frac{x}{2}(\sin \ln x - \cos \ln x)$.

486) $\int \cos \ln x\,dx = \frac{x}{2}(\sin \ln x + \cos \ln x)$.

487) $\int e^{ax} \ln x\,dx = \frac{1}{a}\,e^{ax} \ln x - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$ (№ 451).

3.9. Arkusfunktsioonide integraalid.

488) $\int \arcsin \frac{x}{a}\,dx = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}$.

489) $\int x \arcsin \frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin \frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2}$.

---

*) B_n - Bernoulli arvud (vt. lk. 44).

490) $$\int x^2 \arcsin\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}.$$

491) $$\int \frac{\arcsin\frac{x}{a}\,dx}{x} = \frac{x}{a} + \frac{1}{2\cdot 3\cdot 3}\frac{x^3}{a^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} + \dots$$

492) $$\int \frac{\arcsin\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}.$$

493) $$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}.$$

494) $$\int x\arccos\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2}.$$

495) $$\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}.$$

496) $$\int \frac{\arccos\frac{x}{a}\,dx}{x} = \frac{\pi}{2}\ln x - \frac{x}{a} - \frac{1}{2\cdot 3\cdot 3}\frac{x^3}{a^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} - \dots$$

497) $$\int \frac{\arccos\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\arccos\frac{x}{a} + \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}$$

498) $$\int \operatorname{arctg}\frac{x}{a}\,dx = x\operatorname{arctg}\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2)$$

499) $$\int x\operatorname{arctg}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\operatorname{arctg}\frac{x}{a} - \frac{ax}{2}$$

500) $$\int x^2\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arctg}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(a^2+x^2)$$

501) $$\int x^n\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arctg}\frac{x}{a} - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{a^2+x^2} \qquad (n\neq -1).$$

502) $$\int \frac{\operatorname{arctg}\frac{x}{a}\,dx}{x} = \frac{x}{a} - \frac{x^3}{3^2a^3} + \frac{x^5}{5^2a^5} - \frac{x^7}{7^2a^7} + \dots \qquad (|x|<|a|).$$

503) $$\int \frac{\operatorname{arctg}\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\operatorname{arctg}\frac{x}{a} - \frac{1}{2a}\ln\frac{a^2+x^2}{x^2}$$

504) $$\int \frac{\operatorname{arctg}\frac{x}{a}\,dx}{x^n} = -\frac{1}{(n-1)\,x^{n-1}}\operatorname{arctg}\frac{x}{a} + \frac{a}{n-1}\int \frac{dx}{x^{n-1}(a^2+x^2)} \qquad (n \neq 1).$$

505) $$\int \operatorname{arcctg}\frac{x}{a}\,dx = x\operatorname{arcctg}\frac{x}{a} + \frac{a}{2}\ln(a^2+x^2).$$

506) $$\int x\operatorname{arcctg}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\operatorname{arcctg}\frac{x}{a} + \frac{ax}{2}.$$

507) $$\int x^2\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arcctg}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(a^2+x^2).$$

508) $$\int x^n\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arcctg}\frac{x}{a} + \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{a^2+x^2} \qquad (n \neq -1).$$

509) $$\int \frac{\operatorname{arcctg}\frac{x}{a}\,dx}{x} = \frac{\pi}{2}\ln x - \frac{x}{a} + \frac{x^3}{3^2a^3} - \frac{x^5}{5^2a^5} + \frac{x^7}{7^2a^7} - \dots$$

510) $$\int \frac{\operatorname{arcctg}\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\operatorname{arcctg}\frac{x}{a} + \frac{1}{2a}\ln\frac{a^2+x^2}{x^2}.$$

511) $$\int \frac{\operatorname{arcctg}\frac{x}{a}\,dx}{x^n} = -\frac{1}{(n-1)\,x^{n-1}}\operatorname{arcctg}\frac{x}{a} - \frac{a}{n-1}\int \frac{dx}{x^{n-1}(a^2+x^2)} \qquad (n \neq 1).$$

3.10. Areafunktsioonide integraalid.

512) $$\int \operatorname{Arsh}\frac{x}{a}\,dx = x\operatorname{Arsh}\frac{x}{a} - \sqrt{x^2+a^2}.$$

513) $$\int \operatorname{Arch}\frac{x}{a}\,dx = x\operatorname{Arch}\frac{x}{a} - \sqrt{x^2-a^2}.$$

514) $$\int \operatorname{Arth}\frac{x}{a}\,dx = x\operatorname{Arth}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2).$$

515) $$\int \operatorname{Arcth}\frac{x}{a}\,dx = x\operatorname{Arcth}\frac{x}{a} + \frac{a}{2}\ln(x^2-a^2).$$

4. Päratud integraalid

4.1. Eksponentfunktsiooni sisaldavad integraalid.

1) $\int_0^{+\infty} x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$ *) $\quad (a>0,\ n>-1).$

2) $\int_0^{+\infty} x^n e^{-ax^2}\,dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2a^{(n+1)/2}}$ *) $\quad (a>0,\ n>-1).$

3) $\int_0^{+\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a} \quad (a>0).$ 4) $\int_0^{+\infty} x^2 e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{4a^3} \quad (a>0).$

5) $\int_0^{+\infty} e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a} e^{-b^2/4a^2} \quad (a>0).$

6) $\int_0^{+\infty} \frac{x\,dx}{e^x-1} = \frac{\pi^2}{6}.$ 7) $\int_0^{+\infty} \frac{x\,dx}{e^x+1} = \frac{\pi^2}{12}.$

8) $\int_0^{+\infty} \frac{e^{-ax}\sin x}{x}\,dx = \operatorname{arcctg} a = \operatorname{arctg}\frac{1}{a} \quad (a>0).$

9) $\int_0^{+\infty} e^{-x}\ln x\,dx = -C \approx -0{,}5772$ **).

10) $\int_0^{+\infty} e^{-x^2}\ln x\,dx = \frac{1}{4}\Gamma\left(\frac{1}{2}\right) = -\frac{\sqrt{\pi}}{4}(C+2\ln 2)$ **).

11) $\int_0^{+\infty} e^{-x^2}\ln^2 x\,dx = \frac{\sqrt{\pi}}{8}\left[(C+2\ln 2)^2 + \frac{\pi^2}{2}\right]$ **).

---

*) Γ - gammafunktsioon: $\Gamma(n+1) = n\Gamma(n)$, $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, kui n on naturaalarv, siis $\Gamma(n+1) = n!$ (vt. lk 44).

**) C - Euleri konstant (vt. lk. 44).

4.2. Trigonomeetrilisi funktsioone sisaldavad integraalid.

12) $\int_0^{\pi/2} \sin^{2\alpha+1} x \cos^{2\beta+1} x\, dx = \frac{1}{2} B(\alpha+1, \beta+1) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2)}$ *).

13) $\int_0^{+\infty} \frac{\sin ax}{x} dx = \begin{cases} \frac{\pi}{2}, & a > 0, \\ -\frac{\pi}{2}, & a < 0. \end{cases}$

14) $\int_0^{+\infty} \frac{\sin^p x}{x} dx = 2^{p-2} \frac{[\Gamma(p/2)]^2}{\Gamma(p)}$ **).

15) $\int_0^{+\infty} \frac{\sin ax}{x^s} dx = \frac{\pi a^{s-1}}{2\Gamma(s)\sin(s\pi/2)} \quad (0 < s < 2).$

16) $\int_0^{+\infty} \frac{\cos ax\, dx}{x} = \infty.$

17) $\int_0^{+\infty} \frac{\cos ax}{x^s} dx = \frac{\pi a^{s-1}}{2\Gamma(s)\cos(s\pi/2)} \quad (0 < s < 1).$

18) $\int_0^{+\infty} \frac{\operatorname{tg} ax\, dx}{x} = \begin{cases} \frac{\pi}{2}, & a > 0, \\ -\frac{\pi}{2}, & a < 0. \end{cases}$

19) $\int_0^{+\infty} \frac{\cos ax - \cos bx}{x} dx = \ln \frac{b}{a}.$

20) $\int_0^{+\infty} \frac{\sin x \cos ax}{x} dx = \begin{cases} \pi/2, & |a| < 1, \\ \pi/4, & |a| = 1, \\ 0, & |a| > 1. \end{cases}$

21) $\int_0^{+\infty} \frac{\sin x}{\sqrt{x}} dx = \int_0^{+\infty} \frac{\cos x}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}.$

22) $\int_0^{+\infty} \frac{x \sin ax}{b^2 + x^2} dx = \frac{\pi}{2} e^{-|ab|} \operatorname{sign} a.$

23) $\int_0^{+\infty} \frac{\cos ax}{1 + x^2} dx = \frac{\pi}{2} e^{-|a|}.$

24) $\int_0^{+\infty} \frac{\sin^2 ax}{x^2} dx = \frac{\pi}{2} |a|.$

---

*) $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ - beetafunktsioon, $\Gamma(x)$ - gammafunktsioon (vt. lk. 44).

**) p - paaritu lugeja ja nimetajaga ratsionaalarv, $\Gamma(x)$ - gammafunktsioon (vt. lk. 44).

25) $\int_{-\infty}^{+\infty} \sin(x^2)\,dx = \int_{-\infty}^{+\infty} \cos(x^2)\,dx = \sqrt{\frac{\pi}{2}}$.

26) $\int_0^{\pi/2} \frac{\sin x\,dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2k} \ln \frac{1+k}{1-k}$. $|k| < 1$.

27) $\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k} \arcsin k$, $|k| < 1$.

28) $\int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k^2}(K - E)$ *) $(|k| < 1)$.

29) $\int_0^{\pi/2} \frac{\cos^2 x\,dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k^2}[E - (1 - k^2)K]$ *) $(|k| < 1)$

30) $\int_0^{\pi} \frac{\cos ax\,dx}{1 - 2b\cos x + b^2} = \frac{\pi b^2}{1 - b^2}$ $(|b| < 1,\ a = 0, 1, 2, \ldots)$.

4.3. Logaritmfunktsioone sisaldavad integraalid.

31) $\int_0^1 \ln \ln x\,dx = -C$ **) 32) $\int_0^1 \frac{\ln x}{x-1}\,dx = \frac{\pi^2}{6}$. 33) $\int_0^1 \frac{\ln x}{x+1}\,dx = -\frac{\pi^2}{12}$.

34) $\int_0^1 \frac{\ln x}{x^2 - 1}\,dx = \frac{\pi^2}{8}$. 35) $\int_0^1 \frac{\ln(1+x)}{x^2+1}\,dx = \frac{\pi}{8}\ln 2$.

36) $\int_0^1 \frac{(1-x^\alpha)(1-x^\beta)}{(1-x)\ln x}\,dx = \ln \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}$ $(\alpha > -1,\ \beta > -1,\ \alpha + \beta > -1)$.

37) $\int_0^1 \frac{x^{a-1} - x^{-a}}{(1+x)\ln x}\,dx = \ln \operatorname{tg} \frac{a\pi}{2}$ $(0 < a < 1)$

---

*) $E = E\left(k, \frac{\pi}{2}\right)$, $K = F\left(k, \frac{\pi}{2}\right)$ – täielikud elliptilised integraalid (vt. lk. 43).

**) $C \approx 0.5772$ – Euleri konstant (vt. lk. 44).

38) $\int_0^1 \ln\left(\frac{1}{x}\right)^a dx = \Gamma(a+1)$ *) $\quad (-1 < a < +\infty)$.

39) $\int_0^{\pi/2} \ln \sin x\, dx = \int_0^{\pi/2} \ln \cos x\, dx = -\frac{\pi}{2} \ln 2$. 40) $\int_0^{\pi} x \ln \sin x\, dx = -\frac{\pi^2 \ln 2}{2}$.

41) $\int_0^{\pi/2} \sin x \ln \sin x\, dx = \ln 2 - 1$. 42) $\int_0^{+\infty} \frac{\sin x}{x} \ln x\, dx = -\frac{\pi}{2}(C + \ln a)$ **) $(a > 0)$.

43) $\int_0^{+\infty} \frac{\sin ax}{x} \ln^2 x\, dx = \frac{\pi}{2} C^2 + \frac{\pi^3}{24} + \pi C \ln a + \frac{\pi}{2} \ln^2 a \qquad (a > 0)$.

44) $\int_0^{\pi} \ln(a \pm b \cos x)\, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \qquad (a \geqslant b)$.

45) $\int_0^{\pi} \ln(a^2 - 2ab \cos x + b^2)\, dx = \begin{cases} 2\pi \ln a & (a \geqslant b > 0), \\ 2\pi \ln b & (b \geqslant a > 0). \end{cases}$

46) $\int_0^{\pi/2} \ln \operatorname{tg} x\, dx = 0$. 47) $\int_0^{\pi/4} \ln(1 + \operatorname{tg} x)\, dx = \frac{\pi}{8} \ln 2$.

4.4. Algebralisi funktsioone sisaldavad integraalid.

48) $\int_0^1 x^{\alpha} (1-x)^{\beta} dx = B(\alpha+1, \beta+1) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$ *).

49) $\int_0^{+\infty} \frac{dx}{(1+x)x^a} = \frac{\pi}{\sin a\pi} \quad (0 < a < 1)$. 50) $\int_0^{+\infty} \frac{dx}{(1-x)x^a} = -\pi \operatorname{ctg} a\pi \quad (0 < a < 1)$.

51) $\int_0^{+\infty} \frac{x^{a-1}}{1+x^b} dx = \frac{\pi}{b \sin \frac{a\pi}{b}} \quad (0 < a < b)$. 52) $\int_0^1 \frac{dx}{\sqrt{1-x^a}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{a}\right)}{a\Gamma\left(\frac{2+a}{2a}\right)}$ *).

53) $\int_0^1 \frac{dx}{1 + 2x \cos a + x^2} = \frac{a}{2 \sin a} \quad \left(0 < a < \frac{\pi}{2}\right)$.

54) $\int_0^{+\infty} \frac{dx}{1 + 2x \cos a + x^2} = \frac{a}{\sin a} \quad \left(0 < a < \frac{\pi}{2}\right)$.

---

*) $\Gamma(x)$ - gammafunktsioon, $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ - beetafunktsioon (vt. lk. 44).

**) $C \approx 0{,}5772$ - Euleri konstant (vt. lk. 44).

4.5. Esimest liiki elliptiline integraal.

$$F(k,\varphi)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}=\int_0^{\sin\varphi}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}},\quad k=\sin\alpha.$$

φ \ α	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
0°	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
10	0,1745	0,1746	0,1746	0,1748	0,1749	0,1751	0,1752	0,1753	0,1754	0,1754
20	0,3491	0,3493	0,3499	0,3508	0,3520	0,3533	0,3545	0,3555	0,3561	0,3564
30	0,5236	0,5243	0,5263	0,5294	0,5334	0,5379	0,5422	0,5459	0,5484	0,5493
40	0,6981	0,6997	0,7043	0,7116	0,7213	0,7323	0,7436	0,7535	0,7604	0,7629
50	0,8727	0,8756	0,8842	0,8982	0,9173	0,9401	0,9647	0,9876	1,0044	1,0107
60	1,0472	1,0519	1,0660	1,0896	1,1226	1,1643	1,2126	1,2619	1,3014	1,3170
70	1,2217	1,2286	1,2495	1,2853	1,3372	1,4068	1,4944	1,5959	1,6918	1,7354
80	1,3963	1,4056	1,4344	1,4846	1,5597	1,6660	1,8125	2,0119	2,2653	2,4362
90	1,5708	1,5828	1,6200	1,6858	1,7868	1,9356	2,1565	2,5046	3,1534	∞

4.6. Teist liiki elliptiline integraal.

$$E(k,\varphi)=\int_0^{\varphi}\sqrt{1-k^2\sin^2\psi}\,d\psi=\int_0^{\sin\varphi}\sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt,\quad k=\sin\alpha.$$

φ \ α	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
0°	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
10	0,1745	0,1745	0,1744	0,1743	0,1742	0,1740	0,1739	0,1738	0,1737	0,1736
20	0,3491	0,3489	0,3483	0,3473	0,3462	0,3450	0,3438	0,3429	0,3422	0,3420
30	0,5236	0,5229	0,5209	0,5179	0,5141	0,5100	0,5061	0,5029	0,5007	0,5000
40	0,6981	0,8966	0,6921	0,6851	0,6763	0,6667	0,6575	0,6497	0,6446	0,6428
50	0,8727	0,8698	0,8614	0,8483	0,8317	0,8134	0,7954	0,7801	0,7697	0,7660
60	1,0472	1,0426	1,0290	1,0076	0,9801	0,9493	0,9184	0,8914	0,8728	0,8660
70	1,2217	1,2149	1,1949	1,1632	1,1221	1,0750	1,0266	0,9830	0,9514	0,9397
80	1,3963	1,3870	1,3597	1,3161	1,2590	1,1926	1,1225	1,0565	1,0054	0,9848
90	1,5708	1,5589	1,5238	1,4675	1,3931	1,3055	1,2111	1,1184	1,0401	1,0000

4.7. Täielikud elliptilised integraalid.

$$K = F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = \int_0^1 \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - k^2 t^2}},$$

$$E = E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi = \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}}\, dt, \qquad k = \sin\alpha .$$

α	K	E	α	K	E	α	K	E
0	1,5708	1,5708	30	1,6858	1,4675	60	2,1565	1,2111
1	1,5709	1,5707	31	1,6941	1,4608	61	2,1842	1,2015
2	1,5713	1,5703	32	1,7028	1,4539	62	2,2132	1,1920
3	1,5719	1,5697	33	1,7119	1,4469	63	2,2435	1,1826
4	1,5727	1,5689	34	1,7214	1,4397	64	2,2754	1,1732
5	1,5738	1,5678	35	1,7312	1,4323	65	2,3088	1,1638
6	1,5751	1,5665	36	1,7415	1,4248	66	2,3439	1,1545
7	1,5767	1,5649	37	1,7522	1,4171	67	2,3809	1,1453
8	1,5785	1,5632	38	1,7633	1,4092	68	2,4198	1,1362
9	1,5805	1,5611	39	1,7748	1,4013	69	2,4610	1,1272
10	1,5828	1,5589	40	1,7868	1,3931	70	2,5046	1,1184
11	1,5854	1,5564	41	1,7992	1,3849	71	2,5507	1,1096
12	1,5882	1,5537	42	1,8122	1,3765	72	2,5998	1,1011
13	1,5913	1,5507	43	1,8256	1,3680	73	2,6521	1,0927
14	1,5946	1,5476	44	1,8396	1,3594	74	2,7081	1,0844
15	1,5981	1,5442	45	1,8541	1,3506	75	2,7681	1,0764
16	1,6020	1,5405	46	1,8691	1,3418	76	2,8327	1,0686
17	1,6061	1,5367	47	1,8848	1,3329	77	2,9026	1,0611
18	1,6105	1,5326	48	1,9011	1,3238	78	2,9786	1,0538
19	1,6151	1,5283	49	1,9180	1,3147	79	3,0617	1,0468
20	1,6200	1,5238	50	1,9356	1,3055	80	3,1534	1,0401
21	1,6252	1,5191	51	1,9539	1,2963	81	3,2553	1,0338
22	1,6307	1,5141	52	1,9729	1,2870	82	3,3699	1,0278
23	1,6365	1,5090	53	1,9927	1,2776	83	3,5004	1,0223
24	1,6426	1,5037	54	2,0133	1,2681	84	3,6519	1,0172
25	1,6490	1,4981	55	2,0347	1,2587	85	3,8317	1,0127
26	1,6557	1,4924	56	2,0571	1,2492	86	4,0528	1,0086
27	1,6627	1,4864	57	2,0804	1,2397	87	4,3387	1,0053
28	1,6701	1,4803	58	2,1047	1,2301	88	4,7427	1,0026
29	1,6777	1,4740	59	2,1300	1,2206	89	5,4349	1,0008
30	1,6858	1,4675	60	2,1565	1,2111	90	∞	1,0000

Lisa

1. Bernoulli arvud B_k.

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \ldots = \frac{\pi^{2k} 2^{2k-1}}{(2k)!} B_k$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{2k}} = 1 - \frac{1}{2^{2k}} + \frac{1}{3^{2k}} - \frac{1}{4^{2k}} + \ldots = \frac{\pi^{2k} (2^{2k-1} - 1)}{(2k)!} B_k$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = 1 + \frac{1}{3^{2k}} + \frac{1}{5^{2k}} + \frac{1}{7^{2k}} + \ldots = \frac{\pi^{2k} (2^{2k} - 1)}{2 \cdot (2k)!} B_k$$

k	1	2	3	4	5	6	7	8	9	10	11
B_k	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2630}$	$\frac{7}{6}$	$\frac{3617}{510}$	$\frac{43867}{798}$	$\frac{174611}{330}$	$\frac{854513}{138}$

2. Euleri arvud E_k.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)^{2k+1}} = 1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \frac{1}{7^{2k+1}} + \ldots = \frac{\pi^{2k+1}}{2^{2k+2} \cdot (2k)!} E_k$$

k	1	2	3	4	5	6	7
E_k	1	5	61	1385	50521	2702765	199360981

3. Euleri konstant.

$$C = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right), \quad C = 0{,}577\,215\,664\,901\,532\ldots$$

4. Euleri integraalid (beeta- ja gammafunktsioon).

$$B(x, y) = \int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt \quad (0 < x,\ 0 < y). \qquad \Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt \quad (x > 0).$$

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \qquad \Gamma(x+1) = x\Gamma(x). \qquad \Gamma(n+1) = n! \quad (n = 0, 1, 2, \ldots).$$

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin x\pi} \quad (0 < x < 1). \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

SISUKORD

1. Ratsionaalfunktsioonide integraalid .. 3
1.1. Integraalid, mis sisaldavad avaldist $ax+b$ 3
1.2. Integraalid, mis sisaldavad avaldist ax^2+bx+c 5
1.3. Integraalid, mis sisaldavad avaldist $a^2 \pm x^2$ 7
1.4. Integraalid, mis sisaldavad avaldist $a^3 \pm x^3$ 8
1.5. Integraalid, mis sisaldavad avaldist $a^4 + x^4$ 9
1.6. Integraalid, mis sisaldavad avaldist $a^4 - x^4$ 9
1.7. Osamurdudeks lahutamise valemeid ... 9
2. Irratsionaalsete funktsioonide integraalid 10
2.1. Integraalid, mis sisaldavad avaldisi $\sqrt{x}$, $a^2 \pm b^2x$ 10
2.2. Integraalid, mis sisaldavad avaldist $\sqrt{x}$ 10
2.3. Integraalid, mis sisaldavad avaldist $\sqrt{ax+b}$ 11
2.4. Integraalid, mis sisaldavad avaldisi $\sqrt{ax+b}$, $\sqrt{fx+g}$ 12
2.5. Integraalid, mis sisaldavad avaldist $\sqrt{a^2-x^2}$ 13
2.6. Integraalid, mis sisaldavad avaldist $\sqrt{x^2+a^2}$ 14
2.7. Integraalid, mis sisaldavad avaldist $\sqrt{x^2-a^2}$ 16

2.8. Integraalid, mis sisaldavad avaldist $\sqrt{ax^2+bx+c}$ 17
2.9. Integraalid, mis sisaldavad teisi irratsionaalseid avaldisi 19
2.10. Integraal diferentsiaalbinoomist .. 20
3. Transtsendentsete funktsioonide integraalid 20
3.1. Siinusfunktsiooni sisaldavad integraalid 20
3.2. Koosinusfunktsiooni sisaldavad integraalid 23
3.3. Siinus- ja koosinusfunktsiooni sisaldavad integraalid 26
3.4. Tangensfunktsiooni sisaldavad integraalid 30
3.5. Kootangensfunktsiooni sisaldavad integraalid 31
3.6. Hüperboolsete funktsioonide integraalid 31
3.7. Eksponentfunktsioonide integraalid . 32
3.8. Logaritmfunktsioonide integraalid .. 34
3.9. Arkusfunktsioonide integraalid 35
3.10 Areafunktsioonide integraalid 37
4. Päratud integraalid 38
4.1. Eksponentfunktsiooni sisaldavad integraalid 38
4.2. Trigonomeetrilisi funktsioone sisaldavad integraalid 39
4.3. Logaritmfunktsioone sisaldavad integraalid 40
4.4. Algebralisi funktsioone sisaldavad integraalid 41
4.5. Esimest liiki elliptiline integraal. 42
4.6. Teist liiki elliptiline integraal .. 42
4.7. Täielikud elliptilised integraalid . 43
Lisa 44

ТАБЛИЦЫ ИНТЕГРАЛОВ.
Составитель Маргус Т ы н н о в.
На эстонском языке.
Тартуский государственный университет.
ЭССР, 202400, г.Тарту, ул.Ülikooli, 18.
Paljundamisele antud 28.11. 1984.
Formaat 60x84/16.
Kirjutuspaber.
Masinakiri. Rotaprint.
Tingtrükipoognaid 2,79.
Arvestuspoognaid 2,4. Trükipoognaid 3,0.
Trükiarv 1200.
Tell. nr. 1162.
Hind 10 kop.
TRÜ trükikoda. ENSV, 202400 Tartu, Pälsoni t. 14.