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HELLE HALLIK

Rational spline histopolation

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Faculty of Mathematics and Computer Science, University of Tartu, Tartu, Estonia

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Supervisor:

Cand. Sc., Assoc. Prof. Peeter Oja
University of Tartu
Tartu, Estonia

Opponents:

Dr. math., Assoc. Prof. Natalia Budkina
Riga Technical University
Riga, Latvia

Cand. Sc. Prof. Jaan Janno
Tallinn University of Technology
Tallinn, Estonia

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Chapter 1

Introduction

In statistics and other fields there is often the need to restore data or transform data to the form where further analysis can be made.

Spline functions are a fundamental and prevalent ingredient in the endeavors of scientists and engineers. The design of curves and surfaces plays an important role not only in the construction of different products such as car bodies, ship hulls, airplane fuselages and wings, propellers blades, etc., but also in the description of geological, physical and even medical phenomena. In the majority of these applications, it is important to construct curves and surfaces that have certain shape properties.

Piecewise polynomials were used in the approximation theory already in the early 1900s but the terminology *spline function* was first introduced by Schoenberg in 1946 [51]. Schoenberg states that he started to use this terminology by the connection of piecewise polynomials with a certain mechanical device called a spline – a thin rod of some elastic material equipped with a groove and a set of weights with attached arms designed to fit into the groove. It appears that the use of piecewise polynomial functions offers significant advantages – it is simpler and more powerful (see, e.g., [8]). Piecewise polynomial functions are more adaptable to special problems as well. Up until 1960 the theory of spline functions had a rather modest development and prior to the mid-1960s there were only a few papers which dealt with the problem of how well classes of smooth functions can be approximated by piecewise polynomials or splines.

In the early 1900s there was also quite extensive development of interpolation using piecewise polynomials. Interpolation is the simplest way to reconstruct a function according to discrete data. The intensive development of the theory of interpolating splines began in the early 1960s. It led to achieving error bounds. Some of the early contributors are Ahlberg and Nilson [1], Birkhoff and de Boor [7], Ahlberg, Nilson, and Walsh [2, 3], Atkinson [6], Schoenberg [52, 53, 54].

The works by Stechkin and Subbotin [60], Zavyalov, Kvasov and Miroshnicenko [62] and Schumaker [58] present a comprehensive treatment of the theory and numerical analysis of polynomial spline functions (see also [12, 28]). The book [60] is intended as a supplement and complement to the book [4]. Thus, much space is given to detailed analysis of parabolic spline interpolation. Splines of higher degree than cubic appear only with uniformly spaced knots. In his work the author of [58] states that his original intention was to cover both the theory and application of spline functions. This book covers the main algebraic, analytic and approximation-theoretic properties of various spaces of splines. The detailed study of approximation of functions, numerical differentiation and integration, and solution of boundary value problems for ordinary differential equations is given in [62].

Interpolation and histopolation problems are connected in the sense that the derivative of the interpolant is a solution of the corresponding histopolation problem (as done in [23]) and, vice versa, the integral of the histopolant is a solution of the corresponding interpolation problem, see, e.g., [55]. In case of the interpolation problem we have a given mesh $a = x_0 < x_1 < \ldots < x_n = b$ and function values f_i, $i = 0, \ldots, n$, corresponding to points x_i. We need to construct function $S : [a, b] \to \mathbb{R}$ such that interpolation conditions

$$S(x_i) = f_i, \quad i = 0, \ldots, n,$$

apply.

In histopolation problem we have a mesh $a = x_0 < x_1 < \ldots < x_n = b$ and given numbers z_i, $i = 1, \ldots, n$, which correspond to the mean values of data on subintervals $[x_{i-1}, x_i]$. We need to construct function $T : [a, b] \to \mathbb{R}$ in such a way that histopolation conditions

$$\int_{x_{i-1}}^{x_i} T(x)dx = z_i(x_i - x_{i-1}), \quad i = 1, \ldots, n,$$

hold.

If we have histopolation problem we can fix $f_0 \in \mathbb{R}$ and calculate $f_i = f_{i-1} + z_i(x_i - x_{i-1})$, $i = 1, \ldots, n$. Finding interpolant S corresponding to this data we get that S' is solution of the primary given histopolation problem. In other way, if we have a given interpolation problem and we calculate $z_i = (f_i - f_{i-1})/(x_i - x_{i-1})$, $i = 1, \ldots, n$, then we can find the histopolant T. It turns out that the function

$$S(x) = f_0 + \int_a^x T(s)ds$$

is the solution of the interpolation problem. Therefore, if one problem is thoroughly studied then the result of another one could be immediately derived. Such an approach is adequate in the case of polynomial spline interpolants and histopolants.

In practical cases, it is often important to preserve geometrical properties of data: positivity (nonnegativity), monotonicity, convexity. It is the classical knowledge that polynomial splines do not preserve geometric properties like positivity, monotonicity, convexity of the function to approximate. A good example is the function $f(x) = 1/x^2$, $x \in [-2, -0.2]$, in [34] with appropriate knots and boundary values and it is valid also as an example in the case of different problems. The same numerical example is used in [25] which studies interpolation with quadratic/linear rational splines.

There are two traditional ways to preserve these properties, both are using suitable choice of free parameters. The first approach of them uses additional knots [32, 33, 35, 36, 38, 57]. The second strategy uses higher degree polynomials with less smoothness [44, 49, 50]. In book [31] and in article [49] general information and references about preserving geometrical properties are presented. Author of the article [47] also studies the shape-preservation of histopolation. On the other hand, linear/linear rational spline of class C^1 is monotone by itself, quadratic/linear of class C^2 convex (or concave).

The first development of nonlinear spline spaces with the rational functions and generalizations of them was carried out by Schaback [45] and by Werner [64]. A very general space of rational splines was also defined by Schumaker [56]. A generalization of the results of [45] is presented in [9]. Since then various classes of rational splines have been studied. In article [63] is presented an interpolating rational spline with which solving nonlinear equations can be avoided. For example, in [14] a class of rational C^2 quadratic/quadratic and in [15] a class of C^1 cubic/cubic splines for interpolation are considered. In [30] algorithms for interpolation by rational splines containing, as a special case, parabolic splines and piecewise-linear interpolation are discussed. A class of rational C^2 cubic/quadratic splines is studied in [20]. The accuracy $O(h^3)$ or $O(h^4)$ is achieved. These splines may have some advantages over rational linear/linear and quadratic/linear splines because of their possibly large choice of coefficients but low degree rational splines are simpler and more convenient to use. They do not lose the accuracy, too. For a smooth function f and interpolating linear/linear rational spline S it is known that $\| S - f \|_\infty = O(h^3)$, see, e.g., [24, 39]. For consistent data, the linear/linear rational spline interpolant of class C^1 always exists and is unique [39, 41]. In [24], the expansions on subintervals via the derivatives of the smooth function to interpolate could be found. They give the superconvergence of the spline values and its derivatives in certain points. In interpolation the linear/linear rational splines of class C^1 have the same accuracy as the classical quadratic splines and none of them have an advantage in comparison with real errors [24].

The problem of shape-preserving interpolation has been considered by several authors [13, 14, 15, 19, 20, 43, 48]. Firstly they kept in mind monotonicity and convexity. For example, in [10], cubic interpolant is used to preserve local convexity or concavity of data with necessary and sufficient conditions for second derivative of spline. A construction of convex histogram is studied in [61]. A review with 164 references of shape preserving approximation methods and algorithms univariate functions or discrete data is given in [27]. Main ideas about the methods of solving the nonlinear system of equations could be found in [42].

The theory of adaptive interpolation is developed, e.g., in [46] with cubic polynomial and quadratic/linear rational splines and in [41] for any data with quadratic polynomial and linear/linear splines.

In case of linear/linear rational splines, the number of parameters can be minimal. This idea is used in [16, 39, 40, 41, 46]. With linear/linear rational splines we can solve the histopolation problem, where histogram heights are making a sudden rise [16] (the primal data of this example is given in [5]). In [16] the convergence of classical Newton method (see [29]) is used for calculating the convergence rate.

Linear/linear rational splines are versatilely studied at interpolation [41], at histopolation [16, 17, 18, 21], at solving differential equations [26, 37]. Quadratic/linear rational splines are studied at interpolation [45, 46]. But there are no studies about interpolation with splines which derivative is quadratic/linear rational spline. Thus, the histopolation with quadratic/linear rational splines of class C^2 is an independent problem and article [22] is an attempt to provide some answers to basic questions in the field. Existence and uniqueness of the polynomial spline interpolant in general cases, i.e., at arbitrary placement and multiplicity of interpolation and spline knots, at arbitrary degree of polynomial pieces, is characterized by Schoenberg-Whitney theorem [58], the interpolant exists for arbitrary interpolation values. In the particular case of cubic and quadratic polynomial spline interpolation as the beginning of this kind of researches, we refer to [4]. The existence of linear/linear spline interpolant also takes place for arbitrary strictly monotone values [41], quadratic/linear spline interpolant exists for arbitrary strictly convex data [46]. The same is known for linear/linear histopolant for any strictly monotone histogram[16, 18]. It occurs that the situation is completely different at histopolation with quadratic/linear rational splines.

Among other works more closely related to this dissertation about histopolation we mention [44, 50, 47, 49, 11]. General information about shape-preservation could be found in [59] and [31].

Let us mention that in general, the convergence of interpolating splines is better studied than that of histopolating ones [31, 59].

In the following we provide a brief overview of the dissertation by chapters. The present work consists of seven chapters.

In Chapter 1 we have already given an overview of histopolation problems and a short review of main books and papers on spline histopolation with rational splines.

In Chapter 2 we refer to article [16]. For given monotone data we propose the construction of an histopolating linear/linear rational spline of class C^1. The uniqueness and existence of this spline is proved. The method is implemented via the representation with histogram heights and first derivatives of the spline. The use of Newton's method and ordinary iterations are discussed.

In Chapter 3 we refer to article [17]. The convergence rate of histopolation on arbitrary nonuniform mesh with linear/linear rational splines of class C^1 is studied. Established convergence rate depends on Lipschitz smoothness class of the function to histopolate.

In Chapter 4 we present the algorithms for constructing histopolating splines consisting of linear/linear rational or quadratic polynomial pieces. A unique comonotone histospline of such kind exists for any histogram with weak alternation of data. In general, without weak alternation of data, a modified comonotone spline histopolation strategy should be used. The method is implemented via the representation with histogram heights and knot values of first derivatives of the spline. The results of Chapter 4 are published in [18].

In Chapter 5 we study the convergence rate of histopolation on an interval with combined splines of class C^1 having linear/linear rational or quadratic polynomial pieces. The function to histopolate may have a finite number of derivative zeros and the established convergence rate depends mainly on the behaviour of the derivative near its zeros. The results of Chapter 5 are published in [21].

Chapter 6 is devoted to the quadratic/linear rational spline histopolation problem. These splines keep the sign of its second derivative on the whole interval and therefore the given histogram should be strictly convex or strictly concave. The grid points of the histogram and a suitable number of the spline knots between them are supposed to be placed arbitrarily. The uniqueness of such an histopolant is established. It is shown that the histopolant may not exist but some sufficient conditions for the existence are given. The results of Chapter 6 are presented in a submitted article [22].

Chapter 7 provides numerical results which completely support the theoretical analysis.

Chapter 2

Monotonicity preserving rational spline histopolation

This chapter consists mainly of the results from the paper [16]. The notation will be used later when presenting other results. We find that the inclusion of the material from [16] to the thesis makes it much more self-contained and considerably facilitates reading. Here we expand several reasonings from [16].

2.1 The histopolation problem

Let x_i be given points in an interval $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$ and let z_i, $i = 1, \ldots, n$, be given real numbers. We want to construct a C^1 smooth function S on $[a, b]$ of the form

$$S(x) = \frac{a_i + b_i(x - x_{i-1})}{1 + d_i(x - x_{i-1})} \tag{2.1}$$

with $1 + d_i(x - x_{i-1}) > 0$ for $x \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$, (i.e., a linear/linear rational spline) satisfying the histopolation (area–matching) conditions

$$\int\limits_{x_{i-1}}^{x_i} S(x)dx = z_i(x_i - x_{i-1}), \quad i = 1, \ldots, n. \tag{2.2}$$

In addition, we impose the boundary conditions

$$S'(x_0) = \alpha, \quad S'(x_n) = \beta \tag{2.3}$$

or

$$S(x_0) = \alpha, \quad S(x_n) = \beta \tag{2.4}$$

for given α and β. However, one condition from (2.3) and another from (2.4) at different endpoints x_0 and x_n may be used.

Observe at once that on $[x_{i-1}, x_i]$, we have

$$S'(x) = \frac{b_i - a_i d_i}{(1 + d_i(x - x_{i-1}))^2}\,, \tag{2.5}$$

which means that S being in $C^1[a,b]$ is strictly increasing or strictly decreasing or constant on $[a,b]$. This, in turn, implies that for the existence of the solution of (2.2) with (2.3) or (2.4), it is necessary that

$$z_1 < \ldots < z_n \quad \text{or} \quad z_1 > \ldots > z_n \quad \text{or} \quad z_1 = \ldots = z_n\,, \tag{2.6}$$

and the boundary data has to be consistent with z_i, for example, in the first case $S'(x_0) = \alpha > 0$ and $S'(x_n) = \beta > 0$ or $S(x_0) = \alpha < z_1$ and $S(x_n) = \beta > z_n$.

2.2 Uniqueness of the histopolant

Theorem 2.1. *There are no two different linear/linear rational splines of class C^1 satisfying histopolation conditions (2.2) and boundary conditions (2.3) or (2.4).*

Proof. Let S_1 and S_2 be linear/linear rational functions on $[x_{i-1}, x_i]$, then

$$S_1(x) = \frac{a_{1i} + b_{1i}(x - x_{i-1})}{1 + d_{1i}(x - x_{i-1})}$$

and

$$S_2(x) = \frac{a_{2i} + b_{2i}(x - x_{i-1})}{1 + d_{2i}(x - x_{i-1})},$$

if $x \in [x_{i-1}, x_i]$. Denoting $c_{1i} = b_{1i} - a_{1i}d_{1i}$ and $c_{2i} = b_{2i} - a_{2i}d_{2i}$ and using (2.5), we get

$$g'(x) = \frac{c_{1i}}{(1 + d_{1i}(x - x_{i-1}))^2} - \frac{c_{2i}}{(1 + d_{2i}(x - x_{i-1}))^2}.$$

Equation $g'(x) = 0$ is equivalent to

$$\frac{c_{1i}}{(1 + d_{1i}(x - x_{i-1}))^2} = \frac{c_{2i}}{(1 + d_{2i}(x - x_{i-1}))^2}.$$

Considering the sign of the nominator in (2.1) ($1 + d_{1i}(x - x_{i-1}) > 0$ and $1 + d_{2i}(x - x_{i-1}) > 0$), we can see that c_{1i} and c_{2i} have the same sign. If $c_{1i} = c_{2i} = 0$ then $g'(x) = 0$ everywhere on $[x_{i-1}, x_i]$. If $c_{1i}c_{2i} > 0$ then $g'(x) = 0$ is equivalent to

$$\frac{1 + d_{1i}(x - x_{i-1})}{1 + d_{2i}(x - x_{i-1})} = \left(\frac{c_{1i}}{c_{2i}}\right)^{1/2}$$

or

$$1 + d_{1i}(x - x_{i-1}) = \left(\frac{c_{1i}}{c_{2i}}\right)^{1/2} (1 + d_{2i}(x - x_{i-1})).$$

Last linear equation is satisfied only in one point on $[x_{i-1}, x_i]$ or everywhere or nowhere. Therefore, if there is no subinterval with $g'(x) = 0$ everywhere, then the function g' can have at most n zeros on $[x_0, x_n]$.

Suppose S_1 and S_2 satisfy (2.2) with the same z_i and the same boundary conditions. Then

$$\int_{x_{i-1}}^{x_i} g(x)dx = 0\,, \quad i = 1, \dots, n\,, \tag{2.7}$$

which implies that there are $\xi_i \in (x_{i-1}, x_i)$ so that $g(\xi_i) = 0$.

First assume that there is no interval $[x_{i-1}, x_i]$ where $g'(x) = 0$ everywhere. From conditions (2.4) we get $g(x_0) = 0$ and $g(x_n) = 0$. This means that the function g has at least $n + 2$ different zeros on $[x_0, x_n]$. Then by Rolle's theorem the function g' has at least $n+1$ different zeros in (x_0, x_n). This is in contradiction with the fact that the function g' has no more than n zeros. From other boundary conditions (2.3) we get $g'(x_0) = 0$ and $g'(x_n) = 0$. Additionally, the function g' has $n - 1$ different zeros between points ξ_i, $i = 1, \dots, n$. Therefore, we get at least $n+1$ zeros for the function g', which is a contradiction. In any case, from equation (2.2) with boundary conditions (2.3) or (2.4), we get that the function g' has at least $n + 1$ zeros on $[x_0, x_n]$, which is a contradiction.

If there is an interval $[x_{i-1}, x_i]$, where $g'(x) = 0$ everywhere, we can apply the same discussion on maximal sequence of adjacent intervals, where the function g' is not zero everywhere. If one of adjacent intervals endpoint is x_i, $i = 1, \dots, n-1$, then we can use boundary condition $g'(x_i) = 0$.

Finally, if $g'(x) = 0$ in $[x_0, x_n]$, then g is constant and any of conditions (2.7) implies that $g(x) = 0$ everywhere. This completes the proof. □

2.3 Representation of the histopolant

Denoting $c_i = b_i - a_i d_i$, $i = 1, \dots, n$, then we can write the derivative (2.5) of the function (2.1) on $[x_{i-1}, x_i]$ as

$$S'(x) = \frac{c_i}{(1 + d_i(x - x_{i-1}))^2}. \tag{2.8}$$

Let us denote $S'(x_i) = m_i$, $i = 0, \dots, n$, and $h_i = x_i - x_{i-1}$, $i = 1, \dots, n$. From the formula (2.8) we get

$$c_i = m_{i-1} \quad \text{and} \quad \frac{c_i}{(1 + d_i h_i)^2} = m_i\,. \tag{2.9}$$

In Section 2.2 it was already stated the strict increase or decrease or constancy of S and in terms of m_i these properties may be expressed that either $m_i > 0$ or $m_i < 0$ or $m_i = 0$ for any $i = 0, \ldots, n$.

If $m_i \neq 0$, we get from (2.9)

$$\frac{m_{i-1}}{m_i} = (1 + h_i d_i)^2,$$

from what

$$d_i = \frac{1}{h_i}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right). \tag{2.10}$$

Let $d_i \neq 0$ (then $m_{i-1} \neq m_i$). Then

$$S(x) = \frac{b_i}{d_i} - \frac{c_i}{d_i(1 + d_i(x - x_{i-1}))}. \tag{2.11}$$

Since

$$\begin{aligned}\int_{x_{i-1}}^{x_i} S(x)dx &= \int_{x_{i-1}}^{x_i}\left[\frac{b_i}{d_i} - \frac{c_i}{d_i(1 + d_i(x - x_{i-1}))}\right]dx \\ &= \frac{b_i}{d_i}h_i - \frac{c_i}{d_i^2}\log(1 + h_i d_i),\end{aligned}$$

then from histopolation conditions (2.2) we get

$$\frac{b_i}{d_i} - \frac{c_i}{h_i d_i^2}\log(1 + h_i d_i) = z_i. \tag{2.12}$$

From equation (2.12) we have

$$\frac{b_i}{d_i} = z_i + \frac{c_i}{h_i d_i^2}\log(1 + h_i d_i).$$

Thus, rational spline (2.11) represents in form

$$S(x) = z_i + \frac{m_{i-1}}{h_i d_i^2}\log(1 + h_i d_i) - \frac{m_{i-1}}{d_i(1 + d_i(x - x_{i-1}))}, \tag{2.13}$$

or, taking into account the equation (2.10), as

$$\begin{aligned}S(x) = z_i &+ h_i \frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)^2}\log\left(\frac{m_{i-1}}{m_i}\right)^{1/2} \\ &- h_i \frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\left(1 + t\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)},\end{aligned} \tag{2.14}$$

$x \in [x_{i-1}, x_i], \quad i = 1, \ldots, n, \quad t = (x - x_{i-1})/h_i.$

In the case $d_i = 0$, however, $m_{i-1} = m_i$ and $b_i = c_i = m_i$. The condition (2.2) gives $a_i = z_i - h_i m_i/2$ and we obtain for $x \in [x_{i-1}, x_i]$ the representation

$$S(x) = z_i + m_i\left(x - \left(x_{i-1} + \frac{h_i}{2}\right)\right). \tag{2.15}$$

Clearly, for $m_i = 0$ we have $S(x) = z_i$, $x \in [a, b]$, which is also consistent with (2.15).

2.4 Continuity conditions

The representation of S in terms of m_i as in (2.14) and (2.15) ensures the continuity of S'. In this section we will express the continuity of S in the knots $x_1, \ldots, x_{n-1}$ by the corresponding equations.

We restrict ourselves to the case $m_i > 0$ for all i, the case $m_i < 0$ for all i may be treated similarly. Let $d_i \neq 0$. Since $m_{i-1} = m_i(1 + h_i d_i)^2$ we can write because of (2.13) that

$$\begin{aligned} S(x_i - 0) &= z_i + h_i m_i \frac{(1 + h_i d_i)^2 \log(1 + h_i d_i) - h_i d_i (1 + h_i d_i)}{h_i^2 d_i^2} \\ &= z_i + h_i m_i \frac{(1 + h_i d_i)^2 (\log(1 + h_i d_i) - 1) + 1 + h_i d_i}{h_i^2 d_i^2} \\ &= z_i + h_i m_i \frac{\frac{m_{i-1}}{m_i}\left(\log\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right) + \left(\frac{m_{i-1}}{m_i}\right)^{1/2}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)^2}. \end{aligned} \tag{2.16}$$

Also from (2.13) we get

$$\begin{aligned} S(x_{i-1} + 0) &= z_i + h_i m_{i-1} \frac{\log(1 + h_i d_i) - h_i d_i}{h_i^2 d_i^2} \\ &= z_i - h_i m_{i-1} \frac{(1 + h_i d_i)^{-2} \left(\log(1 + h_i d_i)^{-1} - 1\right) + (1 + h_i d_i)^{-1}}{((1 + h_i d_i)^{-1} - 1)^2} \end{aligned}$$

$$= \quad z_i - h_i m_{i-1} \frac{\dfrac{m_i}{m_{i-1}}\left(\log\left(\dfrac{m_i}{m_{i-1}}\right)^{1/2} - 1\right) + \left(\dfrac{m_i}{m_{i-1}}\right)^{1/2}}{\left(\left(\dfrac{m_i}{m_{i-1}}\right)^{1/2} - 1\right)^2}. \qquad (2.17)$$

From (2.15) it follows (in the case $d_i = 0$)

$$S(x_i - 0) = z_i + \frac{h_i}{2} m_i, \quad S(x_{i-1} + 0) = z_i - \frac{h_i}{2} m_i.$$

Let us introduce the function

$$\varphi(x) = \begin{cases} \dfrac{x^2(\log x - 1) + x}{(x-1)^2} & \text{for} \quad x > 0\,,\ x \neq 1\,, \\ \dfrac{1}{2} \quad \text{for} \quad x = 1\,. \end{cases}$$

Lemma 2.1. *It holds*

1) $\varphi(x) > 0$, $\varphi'(x) > 0$ *and* $\varphi''(x) < 0$ *for* $x > 0$,

2) $\lim\limits_{x\to 1} \varphi(x) = \dfrac{1}{2}$, $\lim\limits_{x\to 1} \varphi'(x) = \dfrac{1}{3}$,

3) $\lim\limits_{x\to 0+} \dfrac{\varphi(x)}{x} = 1$, $\lim\limits_{x\to\infty} \dfrac{\varphi(x)}{\log x} = 1$,

4) $\dfrac{1}{2} < \dfrac{\varphi(x)}{x} < 1$ *for* $0 < x < 1$,

5) $\varphi(x^{1/2}) \leqslant \log x$ *for* $x \geqslant 1.84$.

Proof. 1) Assume that $x > 0$ and consider the expression $(x^2(\log x - 1) + x)/(x-1)^2$. Here the limit of the numerator is

$$\lim_{x\to 0_+} (x^2(\log x - 1) + x) = \lim_{x\to 0_+} \frac{\log x - 1}{\dfrac{1}{x^2}} = \lim_{x\to 0_+} \frac{\dfrac{1}{x}}{-\dfrac{2}{x^3}} = \lim_{x\to 0_+} \frac{-x^2}{2} = 0.$$

Therefore, $\lim_{x\to 0_+} \varphi(x) = 0$.

Next, if $x > 0$, $x \neq 1$, we find

$$\varphi'(x) = \frac{(2x(\log x - 1) + x^2 \cdot \dfrac{1}{x} + 1)(x-1)^2}{(x-1)^4} - \frac{(x^2(\log x - 1) + x) \cdot 2(x-1)}{(x-1)^4}$$
$$= \frac{x^2 - 2x\log x - 1}{(x-1)^3}.$$

If $x > 1$, then $(x-1)^3 > 0$. Let $f(x) = x^2 - 2x\log x - 1$. It is clear that $f(1) = 0$. Now $f'(x) = 2x - 2\log x - 2 = 2(x - \log x - 1)$. We see that $f'(1) = 0$. In addition

$f''(x) = 2(1 - 1/x) > 0$ if $x > 1$, thus $f'(x) > 0$ if $x > 1$ which gives us that $f(x) > 0$ if $x > 1$. In consequence $\varphi'(x) > 0$ if $x > 1$. We find

$$\lim_{x \to 1_+} \varphi'(x) = \lim_{x \to 1_+} \frac{x^2 - 2x \log x - 1}{(x-1)^3} = \lim_{x \to 1_+} \frac{\frac{2}{x^2}}{6} = \frac{1}{3} > 0.$$

Let us now consider the domain $0 < x < 1$ and investigate the sign of $\varphi'(x)$. Then $(x-1)^3 < 0$. For the function $f(x) = x^2 - 2x \log x - 1$ we have $\lim_{x \to 0_+} f(x) = -1$ and also $\lim_{x \to 0_+} f'(x) = \infty$. Furthermore $f''(x) = 2(1 - 1/x) < 0$ if $0 < x < 1$, e.g., f' is strictly decreasing in $(0, 1)$. Because of $f'(1) = 0$, this means that $f'(x) > 0$ in $(0, 1)$. This gives us that f is strictly increasing in $(0, 1)$. Therefore, $f(x) < 0$ if $0 < x < 1$. Hence, $\varphi'(x) > 0$ always if $0 < x < 1$. In addition, we get $\lim_{x \to 1_-} \varphi(x) = 1/3$. In conclusion we have proved that $\varphi'(x) > 0$ if $x > 0$ which means that φ is strictly increasing if $x > 0$. Due to the property $\lim_{x \to 0_+} \varphi(x) = 0$, we have shown that $\varphi(x) > 0$ if $x > 0$.

Similar calculations allow to establish the negativity of φ''.

2) We get

$$\begin{aligned}
\lim_{\substack{x \to 1 \\ x \neq 1}} \varphi(x) &= \lim_{\substack{x \to 1 \\ x \neq 1}} = \frac{2x(\log x - 1) + x^2 \cdot \frac{1}{x} + 1}{2(x-1)} \\
&= \lim_{\substack{x \to 1 \\ x \neq 1}} \frac{2x \log x - x + 1}{2(x-1)} \\
&= \lim_{\substack{x \to 1 \\ x \neq 1}} \frac{2 \log x + 2x \cdot \frac{1}{x} - 1}{2} \\
&= \lim_{\substack{x \to 1 \\ x \neq 1}} \frac{2 \log x + 1}{2} = \frac{1}{2}.
\end{aligned}$$

Thus,

$$\lim_{x \to 1} \varphi(x) = \frac{1}{2}.$$

The limit of φ' at the point 1 was already found in part 1) of the proof.

3) We have

$$\lim_{x \to 0_+} \frac{\varphi(x)}{x} = \lim_{x \to 0_+} \frac{x(\log x - 1) + 1}{(x-1)^2} = 1$$

because $\lim_{x \to 0_+} x \log x = 0$.

We already know that $\varphi(x) > 0$ for $x > 0$. Actually

$$\lim_{x\to\infty} \frac{\varphi(x)}{\log x} = \lim_{x\to\infty} \frac{x^2(\log x - 1)}{(x-1)^2 \log x} + \lim_{x\to\infty} \frac{x}{(x-1)^2 \log x}.$$

Second addend is approaching to zero in process $x \to \infty$. Hence

$$\lim_{x\to\infty} \frac{\varphi(x)}{\log x} = \lim_{x\to\infty} \frac{x^2}{(x-1)^2} \cdot \lim_{x\to\infty} \frac{\log x - 1}{\log x}$$

where both factors are equal to one. We get that

$$\lim_{x\to\infty} \frac{\varphi(x)}{\log x} = 1.$$

4) Denote $\psi(x) = \varphi(x)/x = (x(\log x - 1) + 1)/(x-1)^2$. Then

$$\begin{aligned} \psi'(x) &= \frac{((\log x - 1) + x \cdot \frac{1}{x})(x-1)^2 - (x(\log x - 1) + 1)2(x-1)}{(x-1)^4} \\ &= \frac{-(x+1)\log x + 2(x-1)}{(x-1)^3}. \end{aligned}$$

Let us investigate the sign of $\psi'(x)$ if $x \in (0, 1)$.

We use the notation $\chi(x) = -(x+1)\log x + 2(x-1)$ for numerator of the last fraction. Then $\lim_{x\to 0_+} \chi(x) = \infty$. It is clear that $\chi(1) = 0$. We also get that $\chi''(x) = -1/x + 1/x^2 = (-x+1)/x^2 > 0$ if $0 < x < 1$. Thus, χ' is increasing if $0 < x < 1$. Considering the equality $\chi'(1) = 0$ we get $\chi'(x) < 0$ if $0 < x < 1$ which means that χ is decreasing if $0 < x < 1$. From $\chi(1) = 0$ we get that $\chi(x) > 0$ if $0 < x < 1$. With that we have shown that $\psi'(x) = \chi(x)/(x-1)^3 < 0$ if $0 < x < 1$. Therefore, the function $\varphi(x)/x$ is decreasing if $0 < x < 1$. From properties 3) and 2) follows $\lim_{x\to 0_+} \varphi(x)/x = 1$ and $\lim_{x\to 1} \varphi(x)/x = 1/2$, thus, the property 4) holds.

5) From property 3) we get that if x is great enough then $\varphi(x)/\log x \leqslant 2$ or $\varphi(x) \leqslant 2\log x$. Let us find maximal solution x^* of equation $\varphi(x) = 2\log x$. Note that because of $\varphi(1) = 1/2$ and $\log 1 = 0$ we get $\lim_{x\to 1_+} \varphi(x)/\log x = \infty$ and x^* exists. If $x \geqslant x^*$ then $\varphi(x)/\log x \leqslant 2$. Actually $x^* < \sqrt{1.84}$, therefore, if $x \geqslant \sqrt{1.84}$ then $x \geqslant x^*$ and $\varphi(x)/\log x \leqslant 2$.

With this Lemma 2.1 is proved. $\square$

Using the function φ we can write equalities (2.16) and (2.17) as

$$S(x_i - 0) = z_i + h_i m_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) \tag{2.18}$$

and

$$S(x_{i-1}+0) = z_i - h_i m_{i-1}\varphi\left(\left(\frac{m_i}{m_{i-1}}\right)^{1/2}\right). \tag{2.19}$$

From continuity conditions of spline S, i.e., $S(x_i - 0) = S(x_i + 0)$, $i = 1, \dots, n-1$, we get

$$m_i\left(h_i\varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + h_{i+1}\varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)\right) = \delta_i, \tag{2.20}$$

where $\delta_i = z_{i+1} - z_i$. Assuming $m_i > 0$ the spline S is strictly monotone and therefore we have to assume that $\delta_i > 0$.

If $d_i = 0$ and $d_{i+1} = 0$ then from equalities

$$S(x_i - 0) = z_i + \frac{h_i}{2}m_i \tag{2.21}$$

and

$$S(x_{i-1}+0) = z_i - \frac{h_i}{2}m_i$$

we get

$$m_i(h_i + h_{i+1}) = 2\delta_i. \tag{2.22}$$

If $d_i = 0$ and $d_{i+1} \neq 0$ (case $d_i \neq 0$, $d_{i+1} = 0$ is analogical), then from (2.19) and (2.21), we get

$$m_i\left(\frac{h_i}{2} + h_{i+1}\varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)\right) = \delta_i. \tag{2.23}$$

Note that (2.22) and (2.23) are special cases of (2.20). The condition $d_i = 0$ is the same as $m_{i-1} = m_i$, remember that $\varphi(1) = 1/2$.

Boundary conditions (2.3) fix the values $m_0 = \alpha$ and $m_n = \beta$. But boundary conditions (2.4) do not depend on values of d_0 and d_n and could be written as

$$\begin{aligned} z_1 - h_1 m_0\varphi\left(\left(\frac{m_1}{m_0}\right)^{1/2}\right) &= \alpha\,, \\ z_n + h_n m_n\varphi\left(\left(\frac{m_{n-1}}{m_n}\right)^{1/2}\right) &= \beta. \end{aligned} \tag{2.24}$$

Notice that the equations (2.24) may be considered as special cases of (2.20) with $i = 0$, $i = n$ and $h_0 = 0$, $z_0 = \alpha$, $h_{n+1} = 0$, $z_{n+1} = \beta$.

2.5 Existence of histosplines

The main result of this chapter is the existence of linear/linear rational spline for any strictly monotone data z_i and consistent boundary condition values α, β.

Theorem 2.2. *For any z_i with the property (2.6) and consistent boundary values α, β, there is a linear/linear rational spline of class C^1 satisfying (2.2) and (2.3) or (2.2) and (2.4).*

Proof. Suppose there is $\delta_i = 0$. Then, by (2.20) (or by any of the equations (2.22), (2.23) or (2.24) as special cases of (2.20)) $m_i = 0$ for the solution S of (2.2) with (2.3) or (2.4). But this yields constancy of S. Therefore, we may assume $\delta_i > 0$, $i = 1, \ldots, n-1$, with $\alpha > 0$, $\beta > 0$ in (2.3) and $\alpha < z_1$, $\beta > z_n$ in (2.4).

We write all the equations of type (2.20)–(2.23) with two additional ones obtained from the boundary conditions in the form $m_i = \varphi_i(m)$ with $m = (m_0, \ldots, m_n)$ and we show that there is an interval $[c, M]$ such that all functions map as $\varphi_i : [c, M]^{n+1} \to [c, M]$ and φ_i are continuous. Then by the Bohl-Brouwer fixed point theorem the system $m_i = \varphi_i(m)$, $i = 0, \ldots, n$, has a solution.

The equation (2.20) is in fact

$$m_i = \varphi_i(m) = \frac{\delta_i}{h_i \varphi\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big) + h_{i+1}\varphi\Big(\Big(\frac{m_{i+1}}{m_i}\Big)^{1/2}\Big)} \,. \tag{2.25}$$

Suppose m_{i-1}, m_i, $m_{i+1} \in [c, M]$ where $0 < c < M$. Then, as φ is strictly increasing, we have

$$\varphi_i(m) \leqslant \frac{\delta_i}{(h_i + h_{i+1})\varphi\Big(\Big(\frac{c}{M}\Big)^{1/2}\Big)} \,.$$

By Lemma 2.1, 4), $\varphi((c/M)^{1/2}) \geqslant (c/M)^{1/2}/2$. Continuing with the estimation we get

$$\varphi_i(m) \leqslant \frac{2\delta_i}{h_i + h_{i+1}} \left(\frac{M}{c}\right)^{1/2} \leqslant M$$

if

$$\frac{2\delta_i}{h_i + h_{i+1}} \leqslant (Mc)^{1/2} \,. \tag{2.26}$$

On the other hand, for sufficiently small c and large M (actually, it sufficies to take $M/c \geqslant 1.84$), it holds $\varphi((M/c)^{1/2}) \leqslant 2\log(M/c)^{1/2}$, and we get

$$\varphi_i(m) \geqslant \frac{\delta_i}{2(h_i + h_{i+1})\log\left(\frac{M}{c}\right)^{1/2}} \geqslant c$$

if

$$\frac{\delta_i}{h_i + h_{i+1}} \geqslant c \log \frac{M}{c}\,. \tag{2.27}$$

We have mentioned in the previous section that the equations (2.22) and (2.23) are special cases of (2.20), therefore, the estimates with them also lead to the inequalities (2.26) and (2.27). The same could be said about (2.24), but as for boundary conditions (2.3) we simply have to take care of the requirement $\alpha, \beta \in [c, M]$ in the choice of c and M.

Denote (we mean $h_0 = 0$, $h_{n+1} = 0$ as needed in (2.24))

$$A = \max_{0 \leqslant i \leqslant n} \frac{2\delta_i}{h_i + h_{i+1}}\,, \quad B = \min_{0 \leqslant i \leqslant n} \frac{\delta_i}{h_i + h_{i+1}}\,.$$

Keeping $A = (Mc)^{1/2}$ and decreasing c (accompaning the increase of M) we achieve $B \geqslant c\,\log(M/c)$. The proof is complete. □

2.6 Rate of convergence

Suppose we have additional knots ξ_i in intervals (x_{i-1}, x_i), $i = 1, \dots, n$, and consider the interpolation conditions

$$S(\xi_i) = f(\xi_i)\,, \quad i = 1, \dots, n\,, \tag{2.28}$$

for some function f defined on $[a, b]$. In [39] it is proved the following

Proposition 2.1. *Given strictly monotone f with $f'' \in \mathrm{Lip}\,1$, the linear/linear rational interpolating spline S of class C^1 satisfying (2.28) and (2.3) or (2.4) on uniform mesh (i.e. $x_i = a + ih$, $h = (b-a)/n$, $i = 0, \dots, n$) has the rate of convergence $O(h^3)$ in uniform norm.*

Remark 2.1. *The proof of the rate $O(h^3)$ is presented explicitly in [39] merely in the case of boundary conditions (2.4). As for (2.3), the argument is quite similar. The only modification is that, for example, the condition $S'(x_0) = f'_0 = f'(x_0)$ leads to the equation (with $S_i = S(x_i)$, $i = 0, 1$)*

$$h^2 f'_0(S_1 - f(\xi_1)) - h(f(\xi_1) - S_0)(S_1 - S_0) = 0$$

which replaces the first one in the system (3.1) in [39].

Consider a linear/linear rational spline S satisfying

$$\int\limits_{x_{i-1}}^{x_i} S(x)dx = \int\limits_{x_{i-1}}^{x_i} f(x)dx\,, \quad i = 1, \dots, n\,, \tag{2.29}$$

for a function f being at least continuous. By the mean value theorem there are $\xi_i \in (x_{i-1}, x_i)$ such that

$$\int_{x_{i-1}}^{x_i} (S(x) - f(x))dx = (S(\xi_i) - f(\xi_i))(x_i - x_{i-1}) = 0$$

which gives $S(\xi_i) = f(\xi_i)$. This means that the histopolant S is also an interpolant.

Consider the boundary conditions

$$S'(x_0) = f'(x_0)\,, \quad S'(x_n) = f'(x_n) \tag{2.30}$$

and

$$S(x_0) = f(x_0)\,, \quad S(x_n) = f(x_n)\,. \tag{2.31}$$

Based on Proposition 2.1 we have the following

Theorem 2.3. *For given strictly monotone function f with $f'' \in \mathrm{Lip}\,1$ (in particular, with $f \in C^3[a,b]$), the linear/linear rational histopolating spline S of class C^1 satisfying (2.29) on uniform mesh and boundary conditions (2.30) or (2.31) has the rate of convergence $O(h^3)$ in uniform norm.*

We will present a considerable improvement of this result in Chapter 3.

2.7 On actual construction of histopolants

The representation of the histopolant (2.14) or (2.15) requires the knowledge of parameters m_i. They are uniquely determined by the equations (2.20) for $i = 1, \ldots, n-1$ (including (2.22) and (2.23) as special cases) with additional two ones (2.24) (special cases of (2.20), too) obtained from boundary conditions (2.4) or simply $m_0 = \alpha$, $m_n = \beta$. This system could be written in the form

$$\begin{aligned} &\psi_0(m) \equiv m_0 - \alpha = 0\,, \\ &\psi_i(m) \equiv m_i\left(h_i\varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + h_{i+1}\varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)\right) - \delta_i = 0\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, n-1\,, \\ &\psi_n(m) \equiv m_n - \beta = 0 \end{aligned} \tag{2.32}$$

with obvious modifications if we use the equations (2.24). We will use the function $\Psi(m) = (\psi_0(m), \ldots, \psi_m(m))$ and $\Psi(m) = 0$ as a brief form of (2.32).

One way to find the solution of (2.32) is the Newton's method. A step of Newton's method means the solution of the linear system

$$\Psi'(m^k)m^{k+1} = \Psi'(m^k)m^k - \Psi(m^k) \tag{2.33}$$

with tridiagonal matrix. The entries of the matrix $\Psi\,'$ are (in i-th row)

$$\frac{\partial \psi_i}{\partial m_{i-1}} = \frac{1}{2} h_i \frac{1}{u_i} \varphi'(u_i)\,,$$

$$\frac{\partial \psi_i}{\partial m_i} = h_i \varphi(u_i) - \frac{1}{2} h_i u_i \varphi'(u_i) + h_{i+1} \varphi(v_i) - \frac{1}{2} h_{i+1} v_i \varphi'(v_i)\,,$$

$$\frac{\partial \psi_i}{\partial m_{i+1}} = \frac{1}{2} h_{i+1} \frac{1}{v_i} \varphi'(v_i)$$

with $u_i = (m_{i-1}/m_i)^{1/2}$ and $v_i = (m_{i+1}/m_i)^{1/2}$.

Proposition 2.2. *For any $u_i > 0$, $v_i > 0$ it holds $\partial\psi_i/\partial m_i > 0$.*

Proof. Consider the function $\chi(x) = \varphi(x) - x\varphi'(x)/2$. Then we can find $\chi'(x) = (\varphi'(x) - x\varphi''(x))/2$. From Lemma 2.1, 1), it follows $\chi'(x) > 0$ for $x > 0$. Since $\chi(0) = 0$ (more precisely, $\lim \chi(x) = 0$ as $x \to 0_+$), we get the assertion of the proposition. $\square$

It is clear that $\partial\psi_i/\partial m_{i-1} > 0$ and $\partial\psi_i/\partial m_{i+1} > 0$ (except some boundary cases (2.24) where these derivatives may be equal to zero). Therefore, the difference of domination in i-th row is

$$\begin{aligned} &\frac{\partial \psi_i}{\partial m_i} - \frac{\partial \psi_i}{\partial m_{i-1}} - \frac{\partial \psi_i}{\partial m_{i+1}} \\ &= h_i \left(\varphi(u_i) - \frac{1}{2}\left(u_i + \frac{1}{u_i}\right)\varphi'(u_i)\right) + h_{i+1}\left(\varphi(v_i) - \frac{1}{2}\left(v_i + \frac{1}{v_i}\right)\varphi'(v_i)\right). \end{aligned}$$

Let us introduce the function

$$\delta(x) = \varphi(x) - \frac{1}{2}\left(x + \frac{1}{x}\right)\varphi'(x)\,.$$

Then

$$\delta'(x) = \frac{1}{2}\left(1 + \frac{1}{x^2}\right)\varphi'(x) - \frac{1}{2}\left(x + \frac{1}{x}\right)\varphi''(x)$$

and, by Lemma 2.1, $\delta'(x) > 0$ for $x > 0$. The equation $\delta(x) = 0$ (or, equivalently, the equation $2x^4 \log x - 3x^4 + 4x^3 - 2x^2 + 2x \log x + 1 = 0$) has the solution $x^* \approx 0.734$. Consequently, we have the following:

Proposition 2.3. *The diagonal of the matrix in Newton's method is dominant if $m_{i-1}/m_i \geqslant (x^*)^2 \approx 0.54$, $m_{i+1}/m_i \geqslant (x^*)^2$ and at least one of these inequalities is strict.*

Let us remark that, histopolating a strictly monotone function in the process $n \to \infty$, we have m_{i-1}, m_i and m_{i+1} to be close each other and, thus we have the diagonal dominance.

In practical calculations with given data we do not know in advance which representation, (2.14) or (2.15), should be used. The solution of the system (2.22) itself determines which case we have to deal with because $d_i = 0$ is equivalent to $m_{i-1} = m_i$.

We will use the following (see, e.g. [29]) classical result about the convergence of Newton's method.

Proposition 2.4. *Suppose there is a ball $B(m^0, R)$ such that, with respect to some norm,*

$$\big\|\Psi'(x) - \Psi'(y)\big\| \leqslant L\|x-y\| \quad \forall\, x, y \in B(m^0, R)\,,$$

$$\big\|\big(\Psi'(m^0)\big)^{-1}\big\| \leqslant b_0\,, \tag{2.34}$$

$$\big\|\big(\Psi'(m^0)\big)^{-1}\Psi(m^0)\big\| \leqslant b_1 \tag{2.35}$$

with $b_0 b_1 L \leqslant 1/2$ and $R \geqslant (1-\sqrt{1-2b_0})b_1/b_0$. Then the method (2.33) converges to the solution $m^* \in B(m^0, R)$ of the system (2.32).

Let us also recall that due to Lipschitz continuity of Ψ' the convergence is quadratic, i.e. $\|m^{k+1} - m^*\| = O(\|m^k - m^*\|^2)$.

In our case, Lipschitz continuity of Ψ' is guaranteed by the smoothness of the function φ. The estimate (2.34) is satisfied in many natural situations, e.g. in the case described just after Proposition 2.3. Finally, if m^0 (or any m^k which could be considered as an initial value) is quite close to the solution then (2.35) is satisfied for some small b_1.

To be more precise, suppose we have to histopolate a function $f \in C^3[a,b]$ with given histogram heights

$$z_i = \frac{1}{h_i}\int\limits_{x_{i-1}}^{x_i} f(x)dx\,, \quad i = 1, \ldots, n\,.$$

Take the initial values

$$m_i = \frac{2(z_{i+1} - z_i)}{h_i + h_{i+1}}\,, \quad i = 1, \ldots, n-1\,. \tag{2.36}$$

The use of Taylor's formula gives that $m_i^0 = f'(x_i) + O(h)$ (here $h = \max h_i$) and even $m_i^0 = f'(x_i) + O(h^2)$ in the case of uniform mesh. More complicated but straightforward calculations lead to $m_i^* = f'(x_i) + O(h)$ and, consequently, $m_i^0 - m_i^* = O(h)$. Thus, for small h, Proposition 2.4 is applicable to ensure the convergence of Newton's method starting with indicated choice of initial values.

Let us add that, if possible, we should use the boundary conditions $m_0 = f'(x_0)$ and $m_n = f'(x_n)$. Having only given histogram we may form (at the left end of the interval $[a, b]$, the other end of $[a, b]$ is similar) the linear or quadratic interpolant, say $p(x)$, at the points $(x_{i-1}+x_i)/2$, $i = 1, 2$ for linear and $i = 1, 2, 3$ for quadratic case using the values z_1, z_2 and z_1, z_2, z_3 correspondingly and take $S(x_0) = p(x_0)$ or $m_0 = p'(x_0)$. Then we have $m_0 = f'(x_0) + O(h)$ for the linear interpolant and $m_0 = f'(x_0) + O(h^2)$ for the quadratic interpolant even on nonuniform mesh. In both cases we have $S(x_0) = f(x_0) + O(h^2)$. The precision $S(x_0) = f(x_0) + O(h^3)$ could be obtained by the quadratic histopolant using z_1, z_2, z_3.

However, the described choice of initial values (2.36) and m_0, m_n is natural having arbitrary monotone histogram.

Consider again the system

$$m_i = \varphi_i(m)\,, \quad i = 0, \ldots, n\,, \tag{2.37}$$

briefly, $m = \Phi(m)$ with $\Phi(m) = (\varphi_0(m), \ldots, \varphi_n(m))$, consisting of equations (2.25), $i = 1, \ldots, n-1$, and two ones obtained from (2.24) or simply the given values $m_0 = \alpha$ and $m_n = \beta$. Another natural method for solving (2.37) is ordinary iterations $m_i^k = \varphi_i(m^{k-1})$, $k = 1, 2, \ldots$. We will analyze the behaviour of iterations only for $i = 1, \ldots, n-1$ as for $i = 0$ and $i = n$ this is similar and even simpler. Represent the difference $\Delta_i^k = m_i^k - m_i^* = \varphi_i(m^{k-1}) - \varphi_i(m^*)$ by Lagrange formula taking the derivative at the point $\mu = \xi m^{k-1} + (1-\xi)m^*$, $\xi \in (0, 1)$, where, for briefness, we do not indicate the dependence of μ on k. We have

$$\frac{\Delta_i^k}{m_i^*} = \frac{1}{2}\alpha_i\left((\beta_i + \gamma_i)\frac{\Delta_i^{k-1}}{\mu_i} - \beta_i\frac{\Delta_{i-1}^{k-1}}{\mu_{i-1}} - \gamma_i\frac{\Delta_{i+1}^{k-1}}{\mu_{i+1}}\right)$$

where

$$\alpha_i = \frac{h_i\varphi\Big(\Big(\frac{m_{i-1}^*}{m_i^*}\Big)^{1/2}\Big) + h_{i+1}\varphi\Big(\Big(\frac{m_{i+1}^*}{m_i^*}\Big)^{1/2}\Big)}{h_i\varphi\Big(\Big(\frac{\mu_{i-1}}{\mu_i}\Big)^{1/2}\Big) + h_{i+1}\varphi\Big(\Big(\frac{\mu_{i+1}}{\mu_i}\Big)^{1/2}\Big)}\,,$$

$$\beta_i = \frac{h_i\Big(\frac{\mu_{i-1}}{\mu_i}\Big)^{1/2}\varphi'\Big(\Big(\frac{\mu_{i-1}}{\mu_i}\Big)^{1/2}\Big)}{h_i\varphi\Big(\Big(\frac{\mu_{i-1}}{\mu_i}\Big)^{1/2}\Big) + h_{i+1}\varphi\Big(\Big(\frac{\mu_{i+1}}{\mu_i}\Big)^{1/2}\Big)}\,,$$

$$\gamma_i = \frac{h_{i+1}\Big(\frac{\mu_{i+1}}{\mu_i}\Big)^{1/2}\varphi'\Big(\Big(\frac{\mu_{i+1}}{\mu_i}\Big)^{1/2}\Big)}{h_i\varphi\Big(\Big(\frac{\mu_{i-1}}{\mu_i}\Big)^{1/2}\Big) + h_{i+1}\varphi\Big(\Big(\frac{\mu_{i+1}}{\mu_i}\Big)^{1/2}\Big)}\,.$$

By Lemma 2.1, it holds $\beta_i + \gamma_i < 1 - \delta$ for some $\delta > 0$ depending on particular system (2.37). It may happen that $\alpha_i > 1$.

Suppose we are in a small neighborhood of m^*, i.e. $m^k \approx m^*$ and, thus, $\mu \approx m^*$. Then $\alpha_i \approx 1$ and relative errors $|\Delta_i^k|/m_i^*$ (consequently, absolute errors

$|\Delta_i^k|$, too) converge to zero geometrically with the quotient $\beta_i + \gamma_i$. Actually, as $\varphi(1) = 1/2$ and $\varphi'(1) = 1/3$, this quotient is $2/3$. Note that the worse case is $\Delta_i^{k-1}\Delta_{i-1}^{k-1} < 0$ and $\Delta_i^{k-1}\Delta_{i+1}^{k-1} < 0$ which always has been realized in practical calculations where, nevertheless, the actual quotient is rather $1/2$ than $2/3$.

The behaviour of iterations nearby the solution is almost as an exact geometric progression and this suggests the use of some acceleration method. For example, coordinatewise Aitken's transform could be recommended.

Chapter 3

Convergence rate of monotonicity preserving rational spline histopolation

In this chapter we consider the histopolation problem introduced in Section 2.1. The text originates from [17] and some following results are based on these considerations. The purpose is the same as in the previous chapter - to be more self-contained in the text of the thesis.

3.1 Estimates of first moments

Suppose that for given function f we calculate

$$z_i = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x)dx, \quad i = 1, \ldots, n.$$

Consider the boundary conditions $S(x_0) = f(x_0)$ or $S'(x_0) = f'(x_0)$ with their similar counterparts in x_n. We derive our convergence rate results basing on the estimates of m_i which will be established in this section.

Lemma 3.1. *Suppose $f' \in \operatorname{Lip} \alpha$ for some $\alpha \in (0, 1]$ and $f'(x) > 0$ for all $x \in [a, b]$. Then $m_i - f'(x_i) = O(h^\alpha)$.*

Proof. Take $K_i = [f'(x_i) - ch^\alpha, f'(x_i) + ch^\alpha]$ with a number $c > 0$ independent of h and which will be specified later. Showing that $\varphi_i : \prod_{i=0}^{n} K_i \to K_i$ for all i, we may use Bohl-Brouwer fixed point theorem and the uniqueness of the solution of the system $m_i = \varphi_i(m)$, $i = 0, \ldots, n$, to state that $m_i \in [f'(x_i) - ch^\alpha, f'(x_i) + ch^\alpha]$.

First, let us analyze the main case $i = 1, \dots, n-1$. Using in integrals of

$$\delta_i = \frac{1}{h_{i+1}} \int\limits_{x_i}^{x_{i+1}} f(x)dx - \frac{1}{h_i} \int\limits_{x_{i-1}}^{x_i} f(x)dx$$

the Taylor expansion $f(x) = f(x_i) + f'(x_i)(x - x_i) + R$, where it holds $|R| \leqslant (L/(1+\alpha))|x - x_i|^{1+\alpha}$ and L is the Lipschitz constant of f', we get

$$\delta_i = \frac{1}{2}(h_i + h_{i+1})f'(x_i) \pm \frac{L}{(1+\alpha)(2+\alpha)}\big(h_i^{1+\alpha} + h_{i+1}^{1+\alpha}\big) \tag{3.1}$$

(the compact writing $p = q \pm r$, as usual, denotes the two-sided inequality $q - r \leqslant p \leqslant q + r$).

Next, consider the expansion

$$\varphi\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big) = \varphi(1) + \varphi'(\xi_i)\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2} - 1\Big), \tag{3.2}$$

$$\xi_i \in \Big(1, \Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big).$$

Choose $m_i = f'(x_i) \pm ch^\alpha$, $i = 0, \dots, n$. Then we obtain

$$\frac{m_{i-1}}{m_i} - 1 = \frac{m_{i-1} - m_i}{m_i} = \pm \frac{2c + L}{f'(x_i) - ch^\alpha} h^\alpha. \tag{3.3}$$

Let us remark, in addition, that this yields $m_{i-1}/m_i \to 1$ as $h \to 0$. Using the Taylor expansion up to the second derivative for $(1+x)^{1/2}$ at 0, we obtain

$$\begin{aligned} \sqrt{x} - 1 &= \sqrt{1 + (x-1)} - 1 \\ &= \frac{x-1}{2} - \frac{(x-1)^2}{8(1+\xi)^{3/2}}, \quad \xi \in (0, x-1). \end{aligned}$$

This, applied in the case $x = m_{i-1}/m_i$ with the help of (3.3) leads to

$$\begin{aligned} \Big(\frac{m_{i-1}}{m_i}\Big)^{1/2} - 1 &= \frac{1}{2}\Big(\frac{m_{i-1}}{m_i} - 1\Big) + O(h^{1+\alpha}) \\ &= \pm\Big(\frac{c + \frac{L}{2}}{f'(x_i) - ch^\alpha} h^\alpha + O(h^{1+\alpha})\Big) \\ &= \pm\Big(\frac{c + \frac{L}{2}}{f'(x_i)} h^\alpha + O(h^{2\alpha})\Big). \end{aligned} \tag{3.4}$$

We may conclude that, in (3.2) and then in (2.25), it holds

$$\varphi'(\xi_i) = \frac{1}{3} + O(h^\alpha). \tag{3.5}$$

Analogous calculations could be done for the term $\varphi((m_{i+1}/m_i)^{1/2})$ in (2.25).

Taking in (2.25) into account (3.1) and (3.2), (3.4), (3.5) with their counterparts for m_{i+1}, we obtain

$$\begin{aligned}
\varphi_i(m) &= \frac{\frac{1}{2}(h_i+h_{i+1})f'(x_i) \pm \frac{L}{(1+\alpha)(2+\alpha)}(h_i^{1+\alpha}+h_{i+1}^{1+\alpha})}{\frac{1}{2}(h_i+h_{i+1}) \pm (h_i+h_{i+1})\Big(\frac{1}{3}+O(h^\alpha)\Big)\Big(\frac{c+\frac{L}{2}}{f'(x_i)}h^\alpha+O(h^{2\alpha})\Big)} \\
&= \frac{f'(x_i) \pm \frac{2L}{(1+\alpha)(2+\alpha)}h^\alpha}{1 \pm \Big(\frac{1}{3}+O(h^\alpha)\Big)\Big(\frac{2c+L}{f'(x_i))}h^\alpha+O(h^{2\alpha})\Big)} \\
&= \Big(f'(x_i) \pm \frac{2L}{(1+\alpha)(2+\alpha)}h^\alpha\Big)\Big(1 \pm \Big(\frac{\frac{2}{3}c+\frac{L}{3}}{f'(x_i)}h^\alpha+O(h^{2\alpha})\Big)\Big) \\
&= f'(x_i) \pm \Big(\Big(\frac{2}{3}c+\Big(\frac{1}{3}+\frac{2}{(1+\alpha)(2+\alpha)}\Big)L\Big)h^\alpha+Mh^{2\alpha}\Big) \qquad (3.6)
\end{aligned}$$

with certain $M > 0$ depending, however, on c, L and f. We have the inclusion $\varphi_i(m) \in K_i$ if

$$\frac{2}{3}c+\Big(\frac{1}{3}+\frac{2}{(1+\alpha)(2+\alpha)}\Big)L+Mh^\alpha \leqslant c,$$

which, in turn, takes place for sufficiently large c (e.g., in the case $\alpha = 1$, for $c > 2L$) and small h.

The boundary condition $S'(x_0) = f'(x_0)$ does not need any analysis and we deal briefly with $S(x_0) = f(x_0)$ leading to

$$m_0 = \varphi_0(m) = \frac{\delta_0}{h_1\varphi\big((\frac{m_1}{m_0})^{1/2}\big)}. \qquad (3.7)$$

Then it holds

$$\delta_0 = \frac{h_1}{2}f'(x_0) \pm \frac{L}{(1+\alpha)(2+\alpha)}h_1^{1+\alpha}$$

and with the help of the expansion

$$\varphi\Big(\Big(\frac{m_1}{m_0}\Big)^{1/2}\Big) = \varphi(1)+\varphi'(\xi_0)\Big(\Big(\frac{m_1}{m_0}\Big)^{1/2}-1\Big),$$

$$\xi_0 \in \Big(1, \Big(\frac{m_1}{m_0}\Big)^{1/2}\Big),$$

we get for φ_0 the same final form of two-sided estimate (3.6). This completes the proof. $\square$

Lemma 3.2. *Suppose $f'' \in \operatorname{Lip}\alpha$ for some $\alpha \in (0,1]$ and $f'(x) > 0$ for all $x \in [a,b]$. Then $m_i - f'(x_i) = O(h^{1+\alpha})$.*

Proof. Let us write the equations (2.25) in the form

$$F_i(m_{i-1},m_i,m_{i+1}) = h_i m_i \varphi\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big) + h_{i+1} m_i \varphi\Big(\Big(\frac{m_{i+1}}{m_i}\Big)^{1/2}\Big)$$
$$= \delta_i, \qquad i = 1,\dots,n-1, \tag{3.8}$$

at the same time introducing functions F_i. By Taylor expansion we establish

$$\delta_i = \frac{1}{2}(h_i + h_{i+1})f'(x_i) + \frac{1}{6}(h_{i+1}^2 - h_i^2)f''(x_i) + O(h_i^{2+\alpha} + h_{i+1}^{2+\alpha}). \tag{3.9}$$

At left hand side of (3.8) we use the Taylor expansion

$$F_i(m_{i-1},m_i,m_{i+1}) = F_i(m_i,m_i,m_i) + F_i'(m_i,m_i,m_i)\overline{h}_i + \frac{F_i''(\xi_\lambda)}{2!}\overline{h}_i^2$$

with $\overline{h}_i = (m_{i-1} - m_i, 0, m_{i+1} - m_i)$, some $\lambda \in (0,1)$ and $\xi_\lambda = (m_i,m_i,m_i) + \lambda\overline{h}_i$. Here we have at once $F_i(m_i,m_i,m_i) = (h_i + h_{i+1})m_i/2$. Concerning the term with F_i' we calculate

$$\frac{\partial F_i}{\partial m_{i-1}} = \frac{h_i}{2}\varphi'\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big)\Big(\frac{m_{i-1}}{m_i}\Big)^{-1/2}$$

which gives $\partial F_i/\partial m_{i-1}(m_i,m_i,m_i) = h_i/6$ and similarly we obtain the value $\partial F_i/\partial m_{i+1}(m_i,m_i,m_i) = h_{i+1}/6$. In F_i'' we actually need only

$$\frac{\partial^2 F_i}{\partial m_{i-1}^2} = \frac{h_i}{4}\frac{1}{m_{i-1}}\Big(\varphi''\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big) - \varphi'\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big)\Big(\frac{m_{i-1}}{m_i}\Big)^{-1/2}\Big)$$

and similar derivative $\partial^2 F_i/\partial m_{i+1}^2$. Observe that, by Lemma 3.1, it holds $m_i \in [c_1,c_2]$ with $c_1,c_2 > 0$ for sufficiently small values of h. This gives $\lambda m_{i-1} + (1-\lambda)m_i,\ \lambda m_{i+1} + (1-\lambda)m_i \in [c_1,c_2]$. After standard calculations we can conclude that

$$\frac{F_i''(\xi_\lambda)}{2}\overline{h}_i^2 = h_i\alpha_i(m_{i-1} - m_i)^2 + h_{i+1}\beta_i(m_{i+1} - m_i)^2$$

where α_i and β_i are bounded. Thus, the left hand side of (3.8) reduces to

$$\frac{1}{6}h_i m_{i-1} + \frac{1}{3}(h_i + h_{i+1})m_i + \frac{1}{6}h_{i+1}m_{i+1}$$
$$+ h_i\alpha_i(m_{i-1} - m_i)^2 + h_{i+1}\beta_i(m_{i+1} - m_i)^2. \tag{3.10}$$

In addition, using the formulae $f'(x_i) - h_i f''(x_i) = f'(x_{i-1}) + O(h_i^{1+\alpha})$ and $f'(x_i) + h_{i+1}f''(x_i) = f'(x_{i+1}) + O(h_{i+1}^{1+\alpha})$ let us write (3.9) as

$$\delta_i = \frac{1}{6}h_i f'(x_{i-1}) + \frac{1}{3}(h_i + h_{i+1})f'(x_i) + \frac{1}{6}h_{i+1}f'(x_{i+1})$$
$$+ O(h_i^{2+\alpha} + h_{i+1}^{2+\alpha}). \tag{3.11}$$

Now (3.10) and (3.11) permit to transform (3.8) to the form

$$\lambda_i(m_{i-1} - f'(x_{i-1})) + 2(m_i - f'(x_i)) + \mu_i(m_{i+1} - f'(x_{i+1}))$$
$$= -6\lambda_i\alpha_i(m_{i-1} - m_i)^2 - 6\mu_i\beta_i(m_{i+1} - m_i)^2 + O(h^{1+\alpha}) \quad (3.12)$$

with $\lambda_i = h_i/(h_i + h_{i+1})$ and $\mu_i = 1 - \lambda_i$.

In the case of boundary condition $S(x_0) = f(x_0)$ write (3.7) as

$$F_0(m_0, m_1) = h_1 m_0 \varphi\Big(\Big(\frac{m_1}{m_0}\Big)^{1/2}\Big) = \delta_0\,. \quad (3.13)$$

Here we use the expansions

$$\begin{aligned} \delta_0 &= \frac{1}{2}h_1 f'(x_0) + \frac{1}{6}h_1^2 f''(x_0) + O(h_1^{2+\alpha}) \\ &= \frac{1}{3}h_1 f'(x_0) + \frac{1}{6}h_1 f'(x_1) + O(h_1^{2+\alpha}) \end{aligned}$$

and

$$F_0(m_0, m_1) = F_0(m_0, m_0) + F_0'(m_0, m_0)\overline{h}_0 + \frac{F_0''(\xi_\lambda)}{2!}\overline{h}_0^2$$

with $\overline{h}_0 = (0, m_1 - m_0)$ and $\xi_\lambda = (m_0, m_0) + \lambda\overline{h}_0$. In the last formula we have $F_0(m_0, m_0) = h_1 m_0/2$, $\partial F_0/\partial m_1(m_0, m_0) = h_1/6$ and

$$\frac{F_0''(\xi_\lambda)}{2}\overline{h}_0^2 = h_1\alpha_0(m_1 - m_0)^2$$

where α_0 is bounded. The equation (3.13) takes the form

$$2(m_0 - f'(x_0)) + (m_1 - f'(x_1)) = -6\alpha_0(m_1 - m_0)^2 + O(h_1^{1+\alpha}). \quad (3.14)$$

Observe that the assumption $f'' \in \operatorname{Lip}\alpha$ guarantees $f' \in \operatorname{Lip} 1$ and then, by Lemma 3.1, $m_i - m_{i-1} = O(h)$ or $(m_i - m_{i-1})^2 = O(h^2)$ for all i.

Considering now the equations (3.12) and (3.14) with its analogue at x_n as a linear system with respect to $m_i - f'(x_i)$, $i = 0, \dots, n$, we find out that there is the diagonal dominance in rows. However, the condition $S'(x_0) = f'(x_0)$ gives the trivial equation $m_0 - f'(x_0) = 0$ which preserves the property of diagonal dominance. This yields $m_i - f'(x_i) = O(h^{1+\alpha})$ which completes the proof. $\square$

Remark 3.1. *Instead of exact boundary conditions $S(x_0) = f(x_0)$ and $S'(x_0) = f'(x_0)$ it may be used their perturbed versions $S(x_0) = f(x_0) + O(h_1^{1+\alpha})$ and $S'(x_0) = f'(x_0) + O(h_1^{\alpha})$ in Lemma 3.1, as well $S(x_0) = f(x_0) + O(h_1^{2+\alpha})$ and $S'(x_0) = f'(x_0) + O(h_1^{1+\alpha})$ in Lemma 3.2.*

3.2 Convergence estimates

In this section we establish the convergence rate of uniform norm $\|S - f\|_\infty = \max_{a \leqslant x \leqslant b} |S(x) - f(x)|$ for S being the linear/linear rational spline histopolant to a function f as was described in Sections 2.1 and 3.1. In addition, the convergence rate of $\|S' - f'\|_\infty$ is obtained.

Lemma 3.3. *In the assumptions of Lemma 3.1 (respectively, of Lemma 3.2) it holds $\|S' - f'\|_\infty = O(h^\alpha)$ (respectively, $\|S' - f'\|_\infty = O(h^{1+\alpha})$).*

Proof. Let us recall the representation (2.14) of histopolating spline

$$\begin{aligned} S(x) = z_i \quad &+ \quad h_i \frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)^2} \log\left(\frac{m_{i-1}}{m_i}\right)^{1/2} \\ &- \quad h_i \frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\left(1 + \frac{x - x_{i-1}}{h_i}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)} \end{aligned}$$

for $x \in [x_{i-1}, x_i]$. This gives

$$S'(x) = \frac{m_{i-1}}{\left(1 + \frac{x - x_{i-1}}{h_i}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)\right)^2}.$$

First, let f satisfy the assumptions of Lemma 3.1. We have found in its proof that

$$A = \left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1 = \frac{1}{2}\left(\frac{m_{i-1}}{m_i} - 1\right) + O(h^{1+\alpha})$$

and we know that $A = O(h^\alpha)$. Thus, we have

$$\begin{aligned} S'(x) &= \frac{m_{i-1}}{1 + 2\frac{x - x_{i-1}}{h_i} A + O(h^{2\alpha})} \\ &= m_{i-1}\left(1 - 2\frac{x - x_{i-1}}{h_i} A + O(h^{2\alpha})\right) \\ &= m_{i-1} - \frac{x - x_{i-1}}{h_i} \frac{m_{i-1}}{m_i}(m_{i-1} - m_i) + O(h^{2\alpha}). \end{aligned} \tag{3.15}$$

Using here the replacements $m_{i-1} = f'(x_{i-1}) + O(h^\alpha)$ and $m_{i-1} - m_i = f'(x_{i-1}) + O(h^\alpha) - (f'(x_i) + O(h^\alpha)) = O(h^\alpha)$ together with $m_{i-1}, m_i \in [c_1, c_2]$ for some $c_1, c_2 > 0$, we obtain

$$S'(x) = f'(x_{i-1}) + O(h^\alpha), \quad x \in [x_{i-1}, x_i]. \tag{3.16}$$

Obviously, $f'(x) = f'(x_{i-1}) + O(h^\alpha)$, $x \in [x_{i-1}, x_i]$, and this with (3.16) gives one of the assertions of Lemma 3.3.

Secondly, consider the case of f satisfying the assumptions of Lemma 3.2. Now use in (3.15) the replacements

$$m_{i-1} = f'(x_{i-1}) + O(h^{1+\alpha}),$$

$$\frac{m_{i-1}}{m_i} = \frac{f'(x_{i-1}) + O(h^{1+\alpha})}{f'(x_i) + O(h^{1+\alpha})} = \frac{f'(x_i) + O(h)}{f'(x_i) + O(h^{1+\alpha})} = 1 + O(h),$$

$$m_{i-1} - m_i = f'(x_{i-1}) + O(h^{1+\alpha}) - (f'(x_i) + O(h^{1+\alpha})) = -h_i f''(x_{i-1}) + O(h^{1+\alpha}).$$

Observe also that, at this time, $A = O(h)$ and the rest term in (3.15) is $O(h^2)$. Then we have

$$S'(x) = f'(x_{i-1}) + (x - x_{i-1}) f''(x_{i-1}) + O(h^{1+\alpha}), \quad x \in [x_{i-1}, x_i].$$

This with the Taylor expansion $f'(x) = f'(x_{i-1}) + (x - x_{i-1}) f''(x_{i-1}) + O(h^{1+\alpha})$, $x \in [x_{i-1}, x_i]$, implies the other assertion of Lemma 3.3. The proof is complete. $\square$

We summarize the estimates of values in lemmas in the following:

Theorem 3.1. *Suppose $f'(x) > 0$ for all $x \in [a, b]$ and $f' \in \operatorname{Lip}\alpha$ for some $\alpha \in (0, 1]$. Then the histopolating spline S satisfies $\|S - f\|_\infty = O(h^{1+\alpha})$. If, in addition, $f'' \in \operatorname{Lip}\alpha$, $\alpha \in (0, 1]$, then $\|S - f\|_\infty = O(h^{2+\alpha})$.*

Proof. The histopolation condition

$$\frac{1}{h_i} \int_{x_{i-1}}^{x_i} S(x) dx = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x) dx$$

is equivalent to $\int_{x_{i-1}}^{x_i} (S(x) - f(x)) dx = 0$ which implies the existence of $\xi_i \in (x_{i-1}, x_i)$ such that $S(\xi_i) = f(\xi_i)$. Therefore, it holds $S(x) - f(x) = \int_{\xi_i}^{x} (S'(s) - f'(s)) ds$. Assuming $\|S' - f'\|_\infty \leqslant M h^\beta$ for some $M > 0$, we have for $x \in [x_{i-1}, x_i]$

$$\begin{aligned} |S(x) - f(x)| &\leqslant \left| \int_{\xi_i}^{x} \big|S'(s) - f'(s)\big| ds \right| \\ &\leqslant M h^\beta |x - \xi_i| \leqslant M h^{\beta+1}. \end{aligned}$$

Basing now on Lemma 3.3 we get the assertion of Theorem 3.1. $\square$

Chapter 4

Comonotone shape-preserving histopolation

While in Chapter 2 we looked at the case of strictly monotone data, in this chapter we consider any kind of data given as histogram to histopolate. Our purpose is to preserve the shape of the data as much as possible.

The main results of this chapter are published in [18].

4.1 Histopolation problem

Let $[a, b]$, $a, b \in \mathbb{R}$, $a < b$, and we choose points $x_i, i = 0, \ldots, n$, such that

$$a \leqslant x_0 < x_1 < \ldots < x_n \leqslant b, \quad n \in \mathbb{N}.$$

And let z_i, $i = 1, \ldots, n$, be given real numbers. We are interested in the construction of a C^1 function S on $[a, b]$ that, in every subinterval $[x_{i-1}, x_i]$, $i = 1, \ldots, n$, is either quadratic polynomial or a rational function of the form (2.1) with $1 + d_i(x - x_{i-1}) > 0$ satisfying the histopolation conditions (2.2).

In addition to (2.2), we impose two boundary conditions (2.3) or (2.4). Or we can use a combination of these conditions, for example $S'(x_0) = \alpha$ and $S(x_n) = \beta$.

4.2 Uniqueness of the histopolant

In the previous section we considered the histopolation problem with the given mesh x_i, $i = 0, \ldots, n$, and given numbers z_i, $i = 1, \ldots, n$. Let it be set when the spline is linear/linear rational function of a form (2.1) or quadratic polynomial on subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots, n$.

Theorem 4.1. *There are no two different splines being equally linear/linear rational function or quadratic polynomial on particular intervals and satisfying the same histopolating conditions (2.2) and boundary conditions (2.3) or (2.4).*

Proof. Suppose S_1 and S_2 are two different histopolating splines with the same z_i, the same boundary conditions and the same choice of kinds for particular intervals. Denote $g = S_1 - S_2$ and analyze zeros of the function g' on interval $[x_{i-1}, x_i]$. If S_1 and S_2 are both quadratic polynomials on this interval then g is quadratic and g' is linear, therefore $g'(x) = 0$ everywhere or nowhere or only in one point on $[x_{i-1}, x_i]$.

If S_1 and S_2 are both linear/linear rational functions on $[x_{i-1}, x_i]$ then by proof of Theorem 2.1 we have that $g'(x) = 0$ everywhere or at most in one point on this interval.

Assume that S_1 and S_2 satisfy conditions (2.2) with the same numbers z_i and boundary conditions. As in proof of Theorem 2.1 we get that only the case which does not lead to a contradiction is $g'(x) = 0$ everywhere on $[x_0, x_n]$ and then condition (2.7) gives us that $g(x) = 0$ everywhere. This completes the proof. □

4.3 Representation of spline

In this section we see how to present spline which corresponds to the problem in Section 4.1, where spline S from class C^1 is on $[x_{i-1}, x_i]$ either quadratic polynomial or linear/linear rational function. Depending on spline S type we are naming intervals $[x_{i-1}, x_i]$ quadratic or rational. We also assume that we have numbers z_i and histopolation conditions (2.2) hold.

In this section we give representations of S for quadratic and rational intervals. For rational intervals we use representation (2.14) or (2.15) if $d_i = 0$.

On a quadratic interval $[x_{i-1}, x_i]$ we use the representation

$$S(x) = a_0 + a_1(x - x_{i-1}) + a_2(x - x_{i-1})^2.$$

From $S'(x) = a_1 + 2a_2(x - x_{i-1})$ we get

$$\begin{aligned} S'(x_{i-1}) &= a_1 = m_{i-1}, \\ S'(x_i) &= a_1 + 2a_2 h_i = m_i. \end{aligned}$$

These last equations give us

$$a_2 = \frac{1}{2h_i}(m_i - m_{i-1}).$$

From histopolation conditions (2.2) we can find

$$a_0 = z_i - \frac{1}{6}h_i(m_i + 2m_{i-1}).$$

Thus the spline S represents on quadratic interval $[x_{i-1}, x_i]$ as

$$S(x) = z_i + \frac{h_i}{6}((-2 + 6t - 3t^2)m_{i-1} + (-1 + 3t^2)m_i), \tag{4.1}$$

where $x = x_{i-1} + th_i$.

4.4 Continuity conditions

In this section we give the basic equations depending on continuity conditions.

If we denote $m_i = S'(x_i)$, $i = 0, \ldots, n$, and the spline S is found on subinterval $[x_{i-1}, x_i]$ presented by parameters m_{i-1} and m_i for every i then by this we mean that $m_i = S'(x_i - 0)$ and $m_i = S'(x_i + 0)$, $i = 1, \ldots, n-1$, which means that the derivative S' is continuous in points $x_1, \ldots, x_{n-1}$.

Let there be intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ both rational. We consider the case where $m_i > 0$ for every i (the case $m_i < 0$ is analogical). Continuity of S is analyzed in Section 2.4 and we can write the continuity of S in point x_i, $i = 1, \ldots, n-1$, as equation (2.25).

On a quadratic interval $[x_{i-1}, x_i]$ we get from (4.1) that

$$S(x_i - 0) = z_i + \frac{h_i}{6}(m_{i-1} + 2m_i), \tag{4.2}$$

$$S(x_{i-1} + 0) = z_i + \frac{h_i}{6}(-2m_{i-1} - m_i). \tag{4.3}$$

If the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ are quadratic then for the continuity of S in point x_i, $i = 1, \ldots, n-1$, it has to hold the equality $S(x_i - 0) = S(x_i + 0)$. From that condition we get

$$z_{i+1} + \frac{h_{i+1}}{6}(-2m_i - m_{i+1}) = z_i + \frac{h_i}{6}(m_{i-1} + 2m_i)$$

or, denoting $\delta_i = z_{i+1} - z_i$,

$$\frac{h_i}{6}m_{i-1} + \frac{h_i + h_{i+1}}{3}m_i + \frac{h_{i+1}}{6}m_{i+1} = \delta_i, \tag{4.4}$$

what we can write as

$$m_i = \varphi_i(m) = \frac{6\delta_i - h_i m_{i-1} - h_{i+1}m_{i+1}}{2(h_i + h_{i+1})}. \tag{4.5}$$

Let the interval $[x_{i-1}, x_i]$ be rational and the interval $[x_i, x_{i+1}]$ quadratic. From (2.18) and (4.3) we get continuity condition

$$z_i + h_i m_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) = z_{i+1} + \frac{h_{i+1}}{6}(-2m_i - m_{i+1})$$

or

$$h_i m_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) - \frac{h_{i+1}}{6}(-2m_i - m_{i+1}) = \delta_i, \tag{4.6}$$

and we may write it as

$$m_i = \varphi_i(m) = \frac{\delta_i - \frac{h_{i+1}}{6} m_{i+1}}{h_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + \frac{h_{i+1}}{3}}. \tag{4.7}$$

In symmetrical case where the interval $[x_{i-1}, x_i]$ is quadratic and the interval $[x_i, x_{i+1}]$ rational we use (4.2) and (2.19) and get the continuity of S in point x_i, $i = 1, \ldots, n-1$, as

$$z_i + \frac{h_i}{6}(m_{i-1} + 2m_i) = z_{i+1} - h_{i+1} m_i \varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)$$

or

$$\frac{h_i}{6}(m_{i-1} + 2m_i) + h_{i+1} m_i \varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right) = \delta_i,$$

and

$$m_i = \varphi_i(m) = \frac{\delta_i - \frac{h_i}{6} m_{i-1}}{\frac{h_i}{3} + h_{i+1} \varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)}. \tag{4.8}$$

For actual construction of the spline we have to solve a system of $n+1$ equations (generally nonlinear) with $n+1$ unknowns $m_0, \ldots, m_n$. Equations that form this system are of type (2.25), (4.5), (4.7) or (4.8) depending on the interval types. In addition, we use boundary conditions. Histopolation conditions (2.2) are already counted in these equations. If the parameters $m_0, \ldots, m_n$ are found then, the spline is represented on the rational interval as (2.14) and on the quadratic interval as (4.1).

It is clear that an algorithm of finding described combined splines consists in determination of kinds, linear/linear rational or quadratic, for any interval and then the solution of a nonlinear system.

4.5 A priori estimates

In this section we continue the groundwork for researching the existence of histopolating spline.

In the following we give some a priori estimates for functions φ_i on closed intervals, which are defined by equations (2.25), (4.5), (4.7) and (4.8). These estimates will be used later for proving the existence of histopolating spline.

Lemma 4.1. *Let $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ be rational intervals where $\delta_i > 0$. Assume that $K_j = [r_j, R]$, $r_j > 0$, $j = i-1, i, i+1$. If $m_j \in K_j$, $j = i-1, i, i+1$, then $\varphi_i(m) \in K_i$ if next inequalities are satisfied:*

$$2\delta_i \leqslant (h_i r_{i-1}^{1/2} + h_{i+1} r_{i+1}^{1/2}) R^{1/2}$$

and

$$\frac{\delta_i}{h_i + h_{i+1}} \geqslant r_i \log \frac{R}{r_i}$$

with $R/r_i \geqslant 1.84$.

Proof. Let the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ be rational and $\delta_i > 0$. Then

$$\varphi_i(m) = \frac{\delta_i}{h_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + h_{i+1} \varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)}.$$

Assume that $K_j = [r_j, R]$, $r_j > 0$, and $m_j \in K_j$, $j = i-1, i, i+1$.

From the upper estimate of $\varphi_i(m)$ we get

$$\begin{aligned} \varphi_i(m) &= \frac{\delta_i}{h_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + h_{i+1} \varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)} \\ &\leqslant \frac{\delta_i}{h_i \varphi\left(\left(\frac{r_{i-1}}{R}\right)^{1/2}\right) + h_{i+1} \varphi\left(\left(\frac{r_{i+1}}{R}\right)^{1/2}\right)}. \end{aligned}$$

Using the property 4) for function φ from Lemma 2.1 we get that

$$\varphi\left(\left(\frac{r_{i-1}}{R}\right)^{1/2}\right) > \frac{1}{2}\left(\frac{r_{i-1}}{R}\right)^{1/2}$$

and

$$\varphi\left(\left(\frac{r_{i+1}}{R}\right)^{1/2}\right) > \frac{1}{2}\left(\frac{r_{i+1}}{R}\right)^{1/2}.$$

With the help of these estimates

$$\varphi_i(m) \leqslant \frac{2\delta_i}{h_i \left(\frac{r_{i-1}}{R}\right)^{1/2} + h_{i+1}\left(\frac{r_{i+1}}{R}\right)^{1/2}} \leqslant R$$

if

$$2\delta_i \leqslant \left(h_i r_{i-1}^{1/2} + h_{i+1} r_{i+1}^{1/2}\right) R^{1/2}.$$

Estimating $\varphi_i(m)$ from below we get

$$\varphi_i(m) = \frac{\delta_i}{h_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + h_{i+1} \varphi\left(\left(\frac{m_{i+1}}{m_i}\right)^{1/2}\right)}$$

$$\geqslant \quad \frac{\delta_i}{h_i\varphi\left(\left(\frac{R}{r_i}\right)^{1/2}\right)+h_{i+1}\varphi\left(\left(\frac{R}{r_i}\right)^{1/2}\right)}.$$

From the property 5) of Lemma 2.1 we get $\varphi((R/r_i)^{1/2}) \leqslant \log(R/r_i)$ if $R/r_i \geqslant 1.84$. Then

$$\varphi_i(m) \geqslant \frac{\delta_i}{(h_i+h_{i+1})\log\frac{R}{r_i}} \geqslant r_i$$

if

$$\frac{\delta_i}{h_i+h_{i+1}} \geqslant r_i \log\frac{R}{r_i}.$$

We have proved the lemma. □

Lemma 4.2. *Let the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ be quadratic with $\delta_i \geqslant 0$. Suppose $K_{i-1} = [-R, -\varepsilon_{i-1}]$, $K_i = [\varepsilon_i, R]$, $K_{i+1} = [-R, -\varepsilon_{i+1}]$ and $\varepsilon_j \geqslant 0$, $j = i-1, i, i+1$. If $m_j \in K_j$, $j = i-1, i, i+1$, then $\varphi_i(m) \in K_i$ on assumption that*

$$6\delta_i \leqslant (h_i + h_{i+1})R \tag{4.9}$$

and

$$6\delta_i + h_i\varepsilon_{i-1} + h_{i+1}\varepsilon_{i+1} \geqslant 2(h_i + h_{i+1})\varepsilon_i, \tag{4.10}$$

in special case of $\delta_i = 0$ we may take (4.10) in form

$$\varepsilon_{i+1} = 2\left(1+\frac{h_i}{h_{i+1}}\right)\varepsilon_i - \frac{h_i}{h_{i+1}}\varepsilon_{i-1}.$$

Proof. Let $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ be quadratic intervals with $\delta_i \geqslant 0$. Then

$$\varphi_i(m) = \frac{6\delta_i - h_i m_{i-1} - h_{i+1}m_{i+1}}{2(h_i+h_{i+1})}.$$

Suppose that $K_{i-1} = [-R, -\varepsilon_{i-1}]$, $K_i = [\varepsilon_i, R]$, $K_{i+1} = [-R, -\varepsilon_{i+1}]$ and $\varepsilon_j \geqslant 0$, $m_j \in K_j$, $j = i-1, i, i+1$.

First let us estimate $\varphi_i(m)$ from above. We get

$$\begin{aligned}\varphi_i(m) \quad &= \quad \frac{6\delta_i - h_i m_{i-1} - h_{i+1}m_{i+1}}{2(h_i+h_{i+1})}\\ &\leqslant \quad \frac{6\delta_i + R(h_i+h_{i+1})}{2(h_i+h_{i+1})} \leqslant R,\end{aligned}$$

if

$$6\delta_i + R(h_i+h_{i+1}) \leqslant 2R(h_i+h_{i+1})$$

or

$$6\delta_i \leqslant R(h_i+h_{i+1}).$$

Estimating the value $\varphi_i(m)$ from below we get

$$\begin{aligned}\varphi_i(m) &= \frac{6\delta_i - h_i m_{i-1} - h_{i+1} m_{i+1}}{2(h_i + h_{i+1})} \\ &\geqslant \frac{6\delta_i + h_i \varepsilon_{i-1} + h_{i+1}\varepsilon_{i+1}}{2(h_i + h_{i+1})} \geqslant \varepsilon_i,\end{aligned}$$

if

$$6\delta_i + h_i\varepsilon_{i-1} + h_{i+1}\varepsilon_{i+1} \geqslant 2(h_i + h_{i+1})\varepsilon_i.$$

If $\delta_i = 0$ then the equation

$$h_i\varepsilon_{i-1} + h_{i+1}\varepsilon_{i+1} = 2(h_i + h_{i+1})\varepsilon_i$$

or

$$\varepsilon_{i+1} = 2\left(1 + \frac{h_i}{h_{i+1}}\right)\varepsilon_i - \frac{h_i}{h_{i+1}}\varepsilon_{i-1}$$

gives us (4.10). Lemma is proved. □

Lemma 4.3. *Let $[x_{i-1}, x_i]$ be rational and $[x_i, x_{i+1}]$ quadratic intervals with $\delta_i \geqslant 0$. Suppose that $K_{i-1} = [\varepsilon_{i-1}, R]$, $K_i = [\varepsilon_i, R]$, $K_{i+1} = [-R, -\varepsilon_{i+1}]$, with $\varepsilon_{i-1} > 0$, $\varepsilon_i > 0$ and $\varepsilon_{i+1} \geqslant 0$. If $m_j \in K_j$, $j = i-1, i, i+1$, then $\varphi_i(m) \in K_i$ on the assumptions*

$$\delta_i \leqslant \frac{h_i}{2}(\varepsilon_{i-1}R)^{1/2} + \frac{h_{i+1}}{6}R$$

and

$$\delta_i + \frac{h_{i+1}}{6}\varepsilon_{i+1} \geqslant h_i\varepsilon_i \log\frac{R}{\varepsilon_i} + \frac{h_{i+1}}{3}\varepsilon_i.$$

In symmetrical case, i.e., for $[x_{i-1}, x_i]$ being quadratic and $[x_i, x_{i+1}]$ rational with $\delta_i \geqslant 0$, we take $K_{i-1} = [-R, \varepsilon_{i-1}]$, $K_i = [\varepsilon_i, R]$, $K_{i+1} = [\varepsilon_{i+1}, R]$, with $\varepsilon_{i-1} \geqslant 0$, $\varepsilon_i > 0$, $\varepsilon_{i+1} > 0$ and assume that

$$\delta_i \leqslant \frac{h_i}{6}R + \frac{h_{i+1}}{2}(\varepsilon_{i+1}R)^{1/2}$$

and

$$\delta_i + \frac{h_i}{6}\varepsilon_{i-1} \geqslant \frac{h_i}{3}\varepsilon_i + h_{i+1}\varepsilon_i \log\frac{R}{\varepsilon_i}.$$

In both cases we suppose that $R/\varepsilon_i \geqslant 1.84$.

Proof. Let $[x_{i-1}, x_i]$ be rational and $[x_i, x_{i+1}]$ quadratic interval with $\delta_i \geqslant 0$. Then

$$\varphi_i(m) = \frac{\delta_i - \frac{h_{i+1}}{6}m_{i+1}}{h_i\varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + \frac{h_{i+1}}{3}}.$$

Suppose that $K_{i-1} = [\varepsilon_{i-1}, R]$, $K_i = [\varepsilon_i, R]$, $K_{i+1} = [-R, -\varepsilon_{i+1}]$, with $\varepsilon_{i-1} > 0$, $\varepsilon_i > 0$, $\varepsilon_{i+1} \geqslant 0$ and $m_j \in K_j$, $j = i-1, i, i+1$.

Again we estimate the value $\varphi_i(m)$ from above and get

$$\varphi_i(m) = \frac{\delta_i - \frac{h_{i+1}}{6} m_{i+1}}{h_i \varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + \frac{h_{i+1}}{3}} \leqslant \frac{\delta_i + \frac{h_{i+1}}{6} R}{h_i \varphi\left(\left(\frac{\varepsilon_{i-1}}{R}\right)^{1/2}\right) + \frac{h_{i+1}}{3}}.$$

Using the property 4) from Lemma 2.1 we get $\varphi((\varepsilon_{i-1}/R)^{1/2}) > (\varepsilon_{i-1}/R)^{1/2}/2$. Further estimating gives us

$$\varphi_i(m) \leqslant \frac{\delta_i + \dfrac{h_{i+1}}{6} R}{\dfrac{h_i}{2}\left(\dfrac{\varepsilon_{i-1}}{R}\right)^{1/2} + \dfrac{h_{i+1}}{3}} \leqslant R$$

if

$$\delta_i + \frac{h_{i+1}}{6} R \leqslant \frac{h_i}{2}\left(\frac{\varepsilon_{i-1}}{R}\right)^{1/2} R + \frac{h_{i+1}}{3} R$$

or

$$\delta_i \leqslant \frac{h_i}{2}\left(\varepsilon_{i-1} R\right)^{1/2} + \frac{h_{i+1}}{6} R.$$

Now we estimate the value $\varphi_i(m)$ from below. We get

$$\varphi_i(m) = \frac{\delta_i - \dfrac{h_{i+1}}{6} m_{i+1}}{h_i \varphi\left(\left(\dfrac{m_{i-1}}{m_i}\right)^{1/2}\right) + \dfrac{h_{i+1}}{3}} \geqslant \frac{\delta_i + \dfrac{h_{i+1}}{6} \varepsilon_{i+1}}{h_i \varphi\left(\left(\dfrac{R}{\varepsilon_i}\right)^{1/2}\right) + \dfrac{h_{i+1}}{3}}.$$

Using the property 5) from Lemma 2.1 we obtain $\varphi((R/\varepsilon_i)^{1/2}) \leqslant \log(R/\varepsilon_i)$ if $R/\varepsilon_i \geqslant 1.84$. Continuing with the estimation of $\varphi_i(m)$ we get

$$\varphi_i(m) \geqslant \frac{\delta_i + \dfrac{h_{i+1}}{6} \varepsilon_{i+1}}{h_i \log\left(\dfrac{R}{\varepsilon_i}\right)^{1/2} + \dfrac{h_{i+1}}{3}} \geqslant \varepsilon_i,$$

if

$$\delta_i + \frac{h_{i+1}}{6} \varepsilon_{i+1} \geqslant h_i \varepsilon_i \log \frac{R}{\varepsilon_i} + \frac{h_{i+1}}{3} \varepsilon_i.$$

In symmetrical case the proofs of estimates are similar.

This completes the proof. □

Note that, for opposite signs of δ_i, obvious modifications should be made in Lemmas 4.1–4.3.

4.6 Comonotone shape-preserving strategy

In this section we prove the existence of histospline in the case of weak alternation of data.

Denote

$$\delta_i = z_{i+1} - z_i, \quad i = 1, \dots, n-1,$$

$$\delta_0 = \alpha, \quad \delta_n = \beta \quad \text{(for boundary conditions (2.3))},$$

$$\delta_0 = z_1 - \alpha, \quad \delta_n = \beta - z_n \quad \text{(for boundary conditions (2.4))}.$$

As comonotone shape-preseving strategy, we determine a subinterval $[x_{i-1}, x_i]$ to be rational if $\delta_{i-1}\delta_i > 0$ and quadratic otherwise.

A quadratic section, i.e., a maximal sequence of adjacent quadratic intervals $[x_i, x_{i+1}], \dots, [x_{i+k-1}, x_{i+k}]$, has a weak alternation of data if the interval $[x_{i-1}, x_i]$ is rational, intervals $[x_i, x_{i+1}], \dots, [x_{i+k-1}, x_{i+k}]$ quadratic, interval $[x_{i+k}, x_{i+k+1}]$ rational and

$$\delta_i > 0, \delta_{i+1} \leqslant 0, \delta_{i+2} \geqslant 0, \dots, (-1)^k \delta_{i+k-1} \leqslant 0, (-1)^k \delta_{i+k} > 0$$

or

$$\delta_i < 0, \delta_{i+1} \geqslant 0, \delta_{i+2} \leqslant 0, \dots, (-1)^k \delta_{i+k-1} \geqslant 0, (-1)^k \delta_{i+k} < 0.$$

Theorem 4.2. *If a weak alternation of data takes place on quadratic sections then a comonotone shape-preserving histospline exists and it is strictly monotone on rational intervals.*

Proof. First, we write all equations of type (2.25),(4.5), (4.7), (4.8) with two additional ones which we get from boundary conditions in the form $m_i = \varphi_i(m)$. We find a compact convex set $K = \prod_{i=0}^{n} K_i \subset \mathbb{R}^{n+1}$ where intervals K_i are closed, such that $\varphi_i : K \to K_i$ and the functions φ_i are continuous. Then by the Bohl–Brouwer fixed point theorem the system $m_i = \varphi_i(m)$, $i = 0, \dots, n$, has a solution.

For every rational interval $[x_{i-1}, x_i]$, if $\delta_{i-1} > 0$, $\delta_i > 0$ choose $K_{i-1} = K_i = [r, R]$ and in case $\delta_{i-1} < 0$, $\delta_i < 0$ let $K_{i-1} = K_i = [-R, -r]$ with $0 < r < R$. We assume that between rational intervals $[x_{i-1}, x_i]$ and $[x_{i+k}, x_{i+k+1}]$ there are quadratic intervals $[x_i, x_{i+1}], \dots, [x_{i+k-1}, x_{i+k}]$ with the weak alternation of data. If $K_i = [r, R]$ then we choose $K_{i+1} = [-R, 0]$, $K_{i+2} = [0, R], \dots$. But if $K_i = [-R, -r]$ then we take $K_{i+1} = [0, R]$, $K_{i+2} = [-R, 0], \dots$. Choose $m \in K$ or $m_i \in K_i$, $i = 0, \dots, n$. We have the next three cases to analyze.

First, for rational intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ with $\delta_i > 0$, by Lemma 4.1 with $r_j = r$, $j = i-1, i, i+1$, we have $\varphi_i(m) \in K_i$ if R/r is large enough,

$$2\delta_i \leqslant (h_i + h_{i+1})(rR)^{1/2} \tag{4.11}$$

and

$$\frac{\delta_i}{h_i + h_{i+1}} \geqslant r \log \frac{R}{r}. \tag{4.12}$$

Secondly, suppose the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ are quadratic. If, for example, $\delta_j \geqslant 0$, then $K_j = [0, R]$ and $K_{j-1} \subset [-R, 0]$, $K_{j+1} \subset [-R, 0]$. Lemma 4.2 implies $\varphi_j(m) \in K_j$ if

$$6\delta_j \leqslant (h_j + h_{j+1})R. \tag{4.13}$$

Thirdly, for example, let the interval $[x_{i-1}, x_i]$ be rational and the interval $[x_i, x_{i+1}]$ quadratic. Let $K_{i-1} = K_i = [r, R]$ and, therefore, $\delta_i > 0$. Then $K_{i+1} = [-R, 0]$ or $K_{i+1} = [-R, -r]$ (if $[x_{i+1}, x_{i+2}]$ is a rational interval). Lemma 4.3 gives us that $\varphi_i(m) \in K_i$ if

$$\delta_i \leqslant \frac{h_{i+1}}{6} R \tag{4.14}$$

and

$$\delta_i \geqslant h_i r \log \frac{R}{r} + \frac{h_{i+1}}{3} r \tag{4.15}$$

with sufficiently large R/r.

Now we choose R such that all conditions (4.13) and (4.14) are satisfied. Notice that the increase of R does not spoil them. Then, decreasing the value of r and increasing R, if needed, we may satisfy (4.11), (4.12), their counterparts for other pairs of neighboring rational intervals and all inequalities (4.15).

With that we have proved the existence of the comonotone shape-preserving histospline. We have also shown that at the endpoints of rational intervals first derivatives of the spline are different from zero, hence, we have also the last assertion of the theorem. The proof is complete. □

Remark 4.1. *If all $\delta_i \neq 0$, $i = 0, \ldots, n$, then a weak alternation of data takes place on quadratic sections.*

4.7 Modified comonotone shape-preserving strategy

In this section we treat the data without weak alternation on some quadratic sections. For this it is necessary that there is an interior knot with $\delta_i = 0$, i.e., there are two neighbouring histogram heights with the same height. The strategy around quadratic sections with alternating data is the same as indicated in Section 4.6. We will show how to proceed with quadratic sections without alternating data.

Particular case 4.1 Let us consider a histogram with $\delta_{i-2} > 0$, $\delta_{i-1} > 0$, $\delta_i = 0$, $\delta_{i+1} < 0$, $\delta_{i+2} < 0$. We choose the intervals $[x_{i-2}, x_{i-1}]$ and $[x_{i+1}, x_{i+2}]$ to be rational. At first, we analyze the comonotone shape-preserving strategy in the choice of intermediate intervals.

Example 4.1 By comonotone shape-preserving strategy let the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ be rational. Write the equations (4.7), (4.5) and (4.8) for knots x_{i-1}, x_i and x_{i+1} in the following form:

$$\frac{h_i}{6}(2m_{i-1} + m_i) + h_{i-1}m_{i-1}\varphi\left(\left(\frac{m_{i-2}}{m_{i-1}}\right)^{1/2}\right) = \delta_{i-1}, \tag{4.16}$$

$$h_i m_{i-1} + 2(h_i + h_{i+1})m_i + h_{i+1}m_{i+1} = 0, \tag{4.17}$$

$$\frac{h_{i+1}}{6}(m_i + 2m_{i+1}) + h_{i+2}m_{i+1}\varphi\left(\left(\frac{m_{i+2}}{m_{i+1}}\right)^{1/2}\right) = \delta_{i+1}. \tag{4.18}$$

Retaining the notation of general context, consider x_{i-2} and x_{i+2} as boundary points and let $m_{i-2} = S'(x_{i-2}) > 0$ and $m_{i+2} = S'(x_{i+2}) < 0$ be fixed. Fix also h_j, $j = i-1, \dots, i+2$, and δ_{i+1}. Assume that $\delta_{i-1} \to \infty$. We look for the solution of system (4.16) – (4.18) as $m_{i-1} > 0$ and $m_{i+1} < 0$. First, (4.16) implies that $m_{i-1} \to \infty$ or $m_i \to \infty$. The condition $m_i \to \infty$ and equation (4.18) together give us that $m_{i+1} \to -\infty$. Taking into account $\varphi(x)/x \to 1$ if $x \to 0_+$ we get from (4.18) that $m_i/m_{i+1} \to -2$ and therefore $m_{i+1} = -m_i/2 + o(m_i)$. Now we have (4.17) in the form

$$h_i m_{i-1} + (2h_i + \frac{3}{2}h_{i+1})m_i + o(m_i) = 0,$$

but on the other hand, it must be $m_{i-1} > 0$ and $m_i \to \infty$, which is a contradiction. The possibility $m_i \to -\infty$ does not match with (4.8) and $m_{i+1} < 0$. If m_i remains bounded then the equation (4.18) gives us the boundedness of m_{i+1} but this with $m_{i-1} \to \infty$ is contradicting to (4.17).

With this example we have shown that, generally, it may happen that a comonotone shape-preserving histopolating spline does not exist. However, more detailed analysis show that such a choice of interval types may be successful for some data.

This example forces us to renounce the comonotone shape-preserving strategy and try some other approach. Let the intervals $[x_{i-2}, x_{i-1}]$ and $[x_{i+1}, x_{i+2}]$ be rational like in the example. We now choose the interval $[x_{i-1}, x_i]$ to be rational, too. Then $[x_i, x_{i+1}]$ must be quadratic. Let $K_{i-2} = K_{i-1} = [r, R]$, $K_{i+1} = K_{i+2} = [-R, -r]$ and $K_i = [\varepsilon, R]$ with $\varepsilon > 0$. We assume that $m_j \in K_j$ for every j. By Lemma 4.1 we have $\varphi_{i-1}(m) \in K_{i-1}$ if

$$2\delta_{i-1} \leqslant h_{i-1}(rR)^{1/2} \tag{4.19}$$

and

$$\frac{\delta_{i-1}}{h_{i-1} + h_i} \geqslant \log\frac{R}{r}. \tag{4.20}$$

Lemma 4.3 with $\delta_i = 0$ gives us $\varphi_i(m) \in K_i$ if

$$\frac{h_{i+1}}{6}r \geqslant h_i\varepsilon\log\frac{R}{\varepsilon} + \frac{h_{i+1}}{3}\varepsilon. \tag{4.21}$$

Finally, symmetrical part of Lemma 4.3 with inequalities (4.9) and (4.10) gives us $\varphi_{i+1}(m) \in K_{i+1}$ if

$$|\delta_{i+1}| \leqslant \frac{h_{i+1}}{6} R + \frac{h_{i+2}}{2} (rR)^{1/2} \tag{4.22}$$

and

$$|\delta_{i+1}| \geqslant \frac{h_{i+1}}{3} r + h_{i+2} r \log \frac{R}{r}. \tag{4.23}$$

In conclusion, we choose r and R so that they satisfy the conditions given in the proof of Theorem 4.2. Then decreasing, if needed, the value of r and increasing R we could satisfy inequalities (4.19), (4.20), (4.22) and (4.23). Finally, we take ε such that condition (4.21) holds. Then by the Bohl-Brouwer fixed point the theorem histopolating spline exists.

With this we have described the strategy for Particular case 4.1 and showed that this modified comonotone strategy could be used.

Particular case 4.2 Assume that a quadratic section of the given histogram does not have a weak alternation of data with $\delta_{i-2} > 0$, $\delta_{i-1} > 0$, $\delta_i = 0, \ldots, \delta_{i+k} = 0$, $(-1)^k \delta_{i+k+1} < 0$, $(-1)^k \delta_{i+k+2} < 0$. If $k = 0$ then we have Particular case 4.1. From Particular case 4.1 we know that there may be no comonotone shape-preserving histopolating spline. Suppose now that $k \geqslant 1$. The reasoning of Example 4.1 works as well for $k \geqslant 1$, i.e., taking $\delta_{i-1} \to 0$ and other parameters fixed, we get a contradiction in comonotone shape-preserving strategy.

Let us describe the modified comonotone shape-preserving strategy at choosing the type of subintervals. The intervals $[x_{i-2}, x_{i-1}]$ and $[x_{i+k+1}, x_{i+k+2}]$ are chosen to be rational in any case. As in Particular case 4.1 we choose the interval $[x_{i-1}, x_i]$ to be rational and the intervals $[x_i, x_{i+1}], \ldots, [x_{i+k}, x_{i+k+1}]$ to be quadratic. Now we prove that such choice guarantees the existence of the histopolating spline.

Let $K_{i-2} = K_{i-1} = [r, R]$, $K_j = [\varepsilon_j, R]$, $j = i, i+2, \ldots,$ $K_j = [-R, -\varepsilon_j]$, $j = i+1, i+3, \ldots,$ $K_{i+k+1} = K_{i+k+2} = [r, R]$ if k is odd and $K_{i+k+1} = K_{i+k+2} = [-R, -r]$ if k is even with $\varepsilon_j > 0$ for all j. We show that, under certain conditions, we can get $\varphi_j(m) \in K_j$, $j = i-1, \ldots, i+k+1$ if $m_j \in K_j$ for all j. If conditions (4.19) and (4.20) hold we get $\varphi_{i-1}(m) \in K_{i-1}$. If

$$\frac{h_{i+1}}{6} \varepsilon_{i+1} = h_i \varepsilon_i \log \frac{R}{\varepsilon_i} + \frac{h_{i+1}}{3} \varepsilon_i \tag{4.24}$$

then $\varphi_i(m) \in K_i$. By Lemma 4.2 $\varphi_j(m) \in K_j$, $j = i+1, \ldots, i+k-1$, if

$$\varepsilon_{j+1} = 2\left(1 + \frac{h_j}{h_{j+1}}\right) \varepsilon_j - \frac{h_j}{h_{j+1}} \varepsilon_{j-1} \tag{4.25}$$

and $\varphi_{i+k}(m) \in K_{i+k}$ if

$$r \geqslant 2\left(1 + \frac{h_{i+k}}{h_{i+k+1}}\right) \varepsilon_{i+k} - \frac{h_{i+k}}{h_{i+k+1}} \varepsilon_{i+k-1}. \tag{4.26}$$

By Lemma 4.3, $\varphi_{i+k+1}(m) \in K_{i+k+1}$ if

$$|\delta_{i+k+1}| \leqslant \frac{h_{i+k+1}}{6}R + \frac{h_{i+k+2}}{2}(rR)^{1/2} \tag{4.27}$$

and

$$|\delta_{i+k+1}| \geqslant \frac{h_{i+k+1}}{3}r + h_{i+k+2}r\log\frac{R}{r}. \tag{4.28}$$

We choose r and R from inequalities (4.19), (4.20), (4.27), (4.28) and by conditions in other knots outside of this quadratic section. Starting with sufficiently small ε_i, determining ε_{i+1} from condition (4.24) and next values of ε_j by (4.25), we can satisfy the inequality (4.26).

Remark 4.2. *Previous argumentation works as well if, for some $j = i+1, \dots, i+k$, it occurs that $\delta_j > 0$ corresponding to $K_j = [\varepsilon_j, R]$ or $\delta_j < 0$ corresponding to $K_j = [-R, -\varepsilon_j]$. The suitable signs of δ_j do not spoil upper estimate of $|\varphi_j(m)|$ (value of R should be sufficiently large) and helps to improve a little bit the lower estimate without any changes in conditions (4.19), (4.20), (4.24) – (4.28).*

Remark 4.3. *Instead of choosing interval $[x_{i-1}, x_i]$ to be rational, we can choose $[x_{i+k}, x_{i+k+1}]$ to be rational if $\delta_j = 0$, $j = i, \dots, i+k$.*

General case. Let us now describe a modified comonotone shape-preserving strategy in the general case of a quadratic section without weak alternation of data $\delta_{i-2} > 0$, $\delta_{i-1} > 0$, $\delta_i \leqslant 0$,..., $(-1)^l\delta_{i+k} \geqslant 0$, $(-1)^l\delta_{i+k+1} < 0$, $(-1)^l\delta_{i+k+2} < 0$. The intervals $[x_{i-2}, x_{i-1}]$ and $[x_{i+k+1}, x_{i+k+2}]$ are rational. We choose the closed intervals K_{i-2}, K_{i-1}, K_{i+k+1} and K_{i+k+2} as in Particular cases. Consider that signs of $\delta_i \leqslant 0$, $\delta_{i+1} \geqslant 0, \dots$ are coincident with the weak alternation of data. For example, in Particular case 4.2 we have all $\delta_i, \dots, \delta_{i+k}$ with suitable signs and δ_{i+k+1} has the first unsuitable sign. We have seen in Particular case 4.2 that if we choose one additional rational interval, we can create a quadratic section with suitable signs for all δ_j. Following the same idea, for the first value δ_{i+p} of unsuitable sign (this means that $\delta_{i+p} \neq 0$ and $\delta_{i+p+1} = 0$), we take the interval $[x_{i+p-1}, x_{i+p}]$ to be rational and the intervals $[x_{i-1}, x_i], \dots, [x_{i+p-2}, x_{i+p-1}]$ to be quadratic. In this case, the form of K_j, $j = i, \dots, i+p$, is determined to be $K_j = [\varepsilon_j, R]$ or $K_j = [-R, -\varepsilon_j]$. Similarly to the Particular cases, we take $\varepsilon_{i+p-1} = \varepsilon_{i+p}$ and continue with the part $\delta_{i+p-1}, \dots, \delta_{i+k+2}$, where the intervals $[x_{i+p-1}, x_{i+p}]$ and $[x_{i+k+1}, x_{i+k+2}]$ are chosen to be rational. Remarks 4.2 and 4.3 allow to establish the inclusion $\varphi_j(m) \in K_j$ for all j. Existence of the solution follows again from the Bohl–Brouwer theorem.

Remark 4.4. *Instead of starting from the left end of quadratic section, we can also start using Remark 4.3 from the right end. In this case some subintervals are of different type than they would be if started from the left end.*

The main idea of the modified comonotone shape-preserving strategy is that in quadratic sections without weak alternation of data we choose one of the outermost intervals to be rational (instead quadratic). In this way we create a new quadratic section with the weak alternation of data.

Considering Theorem 4.1 we have proved:

Theorem 4.3. *For any data and boundary conditions there is a modified comonotone shape-preserving histopolating spline which is linear/linear rational function or quadratic polynomial on particular intervals. This spline is unique for determined choice of the kind of intervals.*

Consider again Particular case 4.2 and the presented modified comonotone shape-preserving strategy. This approach introduces a nonsymmetry in the construction of histopolant. With the given data we actually have $m_{i-1} > 0$, $m_i > 0$ and the equation (4.7) gives us $m_{i+1} < 0$ and also $|m_{i+1}| > 2m_i$. Using the equation (4.5) in the form

$$m_{j+1} = -\left(2 + \frac{3h_j}{3h_{j+1}}\right) m_j - \frac{h_j}{2h_{j+1}}(m_j + 2m_{j-1}),\ j = i+1, \ldots, i+k,$$

we get, by induction, that $(-1)^{j-i} m_j > 0$, $|m_j| > 2|m_{j-1}|$, $j = i+1, \ldots, i+k+1$. Because of that undesirable amplifying effect we try to find other approaches.

The second strategy may be, for sufficiently large k, to histopolate separately on $[x_0, x_i]$ and $[x_{i+k}, x_n]$ with the boundary conditions $S(x_i) = z_i$ and $S(x_{i+k}) = z_{i+k} = z_i$. We also choose $S(x) = z_i$ if $x \in [x_i, x_{i+k}]$. A weakness of this strategy is the possible lack of C^1 smoothness in the points x_i and x_{i+k}.

Note that it is possible to histopolate separately on intervals $[x_0, x_{i-1}]$ and $[x_{i+k+1}, x_n]$ with boundary conditions $S(x_{i-1}) = z_i$ and $S(x_{i+k+1}) = z_{i+k}$. If we take $S(x) = z_i$, $x \in [x_{i-1}, x_{i+k+1}]$ then it is somehow unnatural and is not C^1 smooth in the points x_{i-1} and x_{i+k+1}.

The third opportunity is to histopolate separately on $[x_0, x_i]$ and $[x_{i+k}, x_n]$ like in the second case. But here we choose the function S to be cubic polynomial on intervals $[x_{i-1}, x_i]$ and $[x_{i+k}, x_{i+k+1}]$ with boundary conditions $S(x_i) = z_i$, $S'(x_i) = 0$ and $S(x_{i+k}) = z_{i+k}$, $S'(x_{i+k}) = 0$ keeping $S(x) = z_i$, $x \in [x_i, x_{i+k}]$. In this case the function S preserves the C^1 smoothness.

These last two possibilities for strategies do not guarantee the constancy of the function S on $[x_{i-1}, x_{i+k+1}]$. Having such an objective we may take $S(x) = z_i$, $x \in [x_{i-1}, x_{i+k+1}]$ and on intervals $[x_{i-2}, x_{i-1}]$ and $[x_{i+k+1}, x_{i+k+2}]$ let S be cubic polynomial, with boundary conditions in x_{i-1} and x_{i+k+1} for S to be with C^1 smoothness. Then, for $x \in [x_{i-2}, x_{i-1}]$, we can write the spline in the form

$$S(x) = z_{i-1} + \delta_{i-1}(-1 + 6t^2 - 4t^3) + \frac{h_{i-1}}{6} m_{i-2}(-1 + 6t - 9t^2 + 4t^3)$$

with $x = x_{i-2} + th_{i-1}$ and

$$S'(x) = \frac{12}{h_{i-1}}\delta_{i-1}(t - t^2) + m_{i-2}(1 - 3t + 2t^2).$$

Assume $\delta_{i-1} > 0$. A direct analysis gives us that $S'(x) \geqslant 0$, $x \in [x_{i-2}, x_{i-1}]$, if and only if $0 \leqslant m_{i-2} \leqslant 12\delta_{i-1}/h_{i-1}$. In the opposite case, when this method gives us a nonincreasing cubic polynomial, some tension function between linear and cubic polynomial could be used on $[x_{i-2}, x_{i-1}]$ (see, e.g., [31]). In case of $\delta_i - 1 < 0$ obvious modifications should be done. On the interval $[x_{i+k+1}, x_{i+k+2}]$ the analysis is similar.

Chapter 5

Convergence of comonotone histopolating splines

The main purpose of this chapter is to find out the convergence rate of histopolating combined splines consisting of linear/linear rational or quadratic polynomial pieces if the function to histopolate is not strictly monotone. The results of this chapter are published in [21].

5.1 Analysis of basic equations

Here we consider the histopolation problem given in Section 4.1. We will follow the notation and notions of Chapter 4.

Suppose that the given function f has a finite number of points c_i in $[a,b]$ such that $f'(c_i) = 0$. Then the technics of Chapter 3 cannot be applied directly. At each point c_i we may allow the Taylor expansion

$$\begin{aligned} f(x) = f(c_i) + f'(c_i)(x-c_i) + \ldots + &\frac{f^{(k_i-1)}(c_i)}{(k_i-1)!}(x-c_i)^{k_i-1} \\ + &\frac{f^{(k_i)}(c_i)}{k_i!}(x-c_i)^{k_i} + o((x-c_i)^{k_i}) \end{aligned}$$

with $f'(c_i) = 0, \ldots, f^{(k_i-1)}(c_i) = 0, f^{(k_i)}(c_i) \neq 0$. Generalizing the situation, we carry out our analysis for the differentiable functions f such that there are $\lim_{x\to c_i,\, x>c_i} f'(x)/|x-c_i|^{\alpha_1} = \gamma_1 \neq 0$ and $\lim_{x\to c_i,\, x<c_i} f'(x)/|x-c_i|^{\alpha_2} = \gamma_2 \neq 0$ with some constants α_1, α_2 being positive and γ_1, γ_2. Our reasoning is adequate in the small neighbourhood of points c_i. For the regions far enough from c_i the analysis of Chapter 3 could be applied.

Let us indicate some general observations.

Supposing that $f' \in \operatorname{Lip}\alpha$, $\quad 0 < \alpha \leqslant 1$, i.e., $|f'(x) - f'(y)| \leqslant L|x-y|^\alpha$ for some L, we have in the Taylor expansion $f(x) = f(a) + f'(a)(x-a) + R$ that $|R| \leqslant (L/(1+\alpha))|x-a|^{1+\alpha}$. Then

$$\delta_i = \frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}} f(x)dx - \frac{1}{h_i}\int_{x_{i-1}}^{x_i} f(x)dx = \frac{1}{2}(h_i + h_{i+1})f'(x_i) + R_i$$

with $|R_i| \leqslant (L/((1+\alpha)(2+\alpha)))(h_i^{1+\alpha} + h_{i+1}^{1+\alpha})$.

At histopolation the estimate $\|S' - f'\|_\infty = O(h^\alpha)$ always yields $\|S - f\|_\infty = O(h^{1+\alpha})$, see Chapter 3. We also discuss in Section 5.2 how the estimate $|m_i - f_i'| = O(h^\alpha)$ gives $\|S' - f'\|_\infty = O(h^\alpha)$.

We will perform the reasoning in particular cases and later see the ways of generalization. For simplicity of presentation, take the uniform partition of $[a,b]$, i.e., $x_i = a + ih$, $\quad i = 0, \dots, n$, $h = (b-a)/n$. In the case of sufficiently smooth function f we have

$$\frac{1}{h}\delta_i = f_i' + \frac{h^2}{12}f_i''' + O(h^3),$$

here and in the sequel we mean $f_i' = f'(x_i)$ with similar significance for other functions. Observe that outside of certain (small) neighbourhood of each c_i the reasoning of Chapter 3 is applicable and the estimate $|m_i - f_i'| = O(h^2)$ for smooth functions holds (or $|m_i - f_i'| = O(h^\alpha)$, $0 < \alpha < 2$, in the case of lower smoothness, see Chapter 3). Thus, the study only in the neighbourhoods of c_i is needed. Nevertheless, we give a complete analysis independent of Chapter 3.

Let us start with the function $f(x) = x^2 \operatorname{sgn} x$, $x \in [-1,1]$.

For n even, we have $x_i = 0, i = n/2$, and elementary calculations give

$$\delta_i = \frac{2}{3}h^2, \quad \delta_{i+k} = \delta_{i-k} = 2kh^2, \quad k = 1, \dots, \frac{n}{2} - 1.$$

As boundary conditions (2.3) in form $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$ are consistent we choose all subintervals to be rational. For n odd, the point 0 is the midpoint of the interval $[x_i, x_{i+1}]$ with $i = (n-1)/2$. Then

$$\delta_i = \delta_{i+1} = \frac{13}{12}h^2, \quad \delta_{i+k} = (2k-1)h^2, \quad k = 2, \dots, (n-1)/2,$$

and

$$\delta_{i-1} = \delta_{i+k+1} = (2k+1)h^2, \quad k = 1, \dots, (n-3)/2.$$

Here also all subintervals are rational. We will focus our attention mainly on the case of n even.

The equations of the type (2.20) corresponding to the points x_j are now

$$m_j\Big(\varphi\Big(\Big(\frac{m_{j-1}}{m_j}\Big)^{1/2}\Big) + \varphi\Big(\Big(\frac{m_{j+1}}{m_j}\Big)^{1/2}\Big)\Big) = \frac{1}{h}\delta_j = f_j' \tag{5.1}$$

or

$$m_j = \frac{f'_j}{\varphi\Big(\Big(\frac{m_{j-1}}{m_j}\Big)^{1/2}\Big) + \varphi\Big(\Big(\frac{m_{j+1}}{m_j}\Big)^{1/2}\Big)}$$

and thus

$$m_j - f'_j = f'_j \frac{1 - \varphi\Big(\Big(\dfrac{m_{j-1}}{m_j}\Big)^{1/2}\Big) - \varphi\Big(\Big(\dfrac{m_{j+1}}{m_j}\Big)^{1/2}\Big)}{\varphi\Big(\Big(\dfrac{m_{j-1}}{m_j}\Big)^{1/2}\Big) + \varphi\Big(\Big(\dfrac{m_{j+1}}{m_j}\Big)^{1/2}\Big)}. \tag{5.2}$$

Denote $\mu_j = m_{j-1}/m_j, j = 1, \ldots, i$, and $\mu_j = m_j/m_{j-1}, j = i+1, \ldots, n$. Our aim is to estimate the values μ_j and then use them in (5.2). Let us consider the situation at right hand side from $x_i = 0$ (by symmetry, $\mu_{i+k} = \mu_{i-k+1}$ for all k), then the equations (5.1) for $j = i + k$ and $j + 1$ give

$$\mu_{j+1} = \frac{\delta_{j+1}}{\delta_j} \frac{\varphi(\mu_j^{-1/2}) + \varphi(\mu_{j+1}^{1/2})}{\varphi(\mu_{j+1}^{-1/2}) + \varphi(\mu_{j+2}^{1/2})} \tag{5.3}$$

with $\delta_{j+1}/\delta_j = 1 + 1/k$. Denote $\mu = (\mu_1, \ldots, \mu_n)$ and introduce the functions

$$\Phi_j(\mu) = \frac{\varphi(\mu_{j-1}^{-1/2}) + \varphi(\mu_j^{1/2})}{\varphi(\mu_j^{-1/2}) + \varphi(\mu_{j+1}^{1/2})}$$

and

$$\Psi_j(\mu) = \frac{\delta_j}{\delta_{j-1}} \Phi_j(\mu).$$

Then the equations (5.3) are $\mu_j = \Psi_j(\mu)$.

We have seen in Section 3.2 that the estimate $|m_i - f'_i| = O(h^\alpha)$ yields $\|S' - f'\|_\infty = O(h^\alpha)$ and this in turn, gives $\|S - f\|_\infty = O(h^{1+\alpha})$. While the last consequence is universal at histopolation, the first one was given in the case of linear/linear rational splines for smooth function having strictly positive derivative. We will return to these estimates later in more general case. We need the following technical

Lemma 5.1. *Suppose* $\mu_{j+l} \in [1 + 1/(k + l - 1) - \delta, 1 + 1/(k + l - 1) + \delta]$, $l = 0, 1, 2,$ *with sufficiently small* $\delta > 0$. *Then for some positive constants* c_1, c_2 *it holds*

$$\Phi_{j+1}(\mu) \in \Big[1 - \frac{c_1}{k^3} - \frac{2}{3}\delta - c_2\delta^2, 1 + \frac{c_1}{k^3} + \frac{2}{3}\delta + c_2\delta^2\Big].$$

Proof. Take $\bar{\mu}_{j+l} = 1 + 1/(k + l - 1), l = 0, 1, 2,$ and estimate then $\Phi_{j+1}(\bar{\mu})$. The use of the Taylor expansion

$$\varphi(\bar{\mu}_{j+1}^{1/2}) = \varphi\Big(\Big(1 + \frac{1}{k}\Big)^{1/2}\Big) = \varphi(1) + \varphi'(1)\Big(\Big(1 + \frac{1}{k}\Big)^{1/2} - 1\Big)$$

$$+\frac{\varphi''(1)}{2}\Big(\Big(1+\frac{1}{k}\Big)^{1/2}-1\Big)^2+\frac{\varphi'''(\xi)}{6}\Big(\Big(1+\frac{1}{k}\Big)^{1/2}-1\Big)^3$$

and inside that

$$x^{1/2}-1=\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+O((x-1)^3) \tag{5.4}$$

gives

$$\varphi\Big(\bar\mu_{j+1}^{1/2}\Big)=\frac{1}{2}+\frac{1}{6k}-\frac{1}{16k^2}+O\Big(\frac{1}{k^3}\Big).$$

We get also

$$\varphi\Big(\bar\mu_{j+1}^{-1/2}\Big)=\varphi\Big(\Big(1-\frac{1}{k+1}\Big)^{1/2}\Big)=\frac{1}{2}-\frac{1}{6(k+1)}-\frac{1}{16(k+1)^2}+O\Big(\frac{1}{k^3}\Big).$$

Then

$$\varphi\Big(\bar\mu_{j}^{-1/2}\Big)+\varphi\Big(\bar\mu_{j+1}^{1/2}\Big)=1-\frac{1}{8k^2}+O\Big(\frac{1}{k^3}\Big)$$

and

$$\varphi\Big(\bar\mu_{j+1}^{-1/2}\Big)+\varphi\Big(\bar\mu_{j+2}^{1/2}\Big)=1-\frac{1}{8(k+1)^2}+O\Big(\frac{1}{k^3}\Big).$$

Now

$$\Phi_{j+1}(\bar\mu)=\Big(1-\frac{1}{8k^2}+O\Big(\frac{1}{k^3}\Big)\Big)\Big(1+\frac{1}{8(k+1)^2}+O\Big(\frac{1}{k^3}\Big)\Big)=1+O\Big(\frac{1}{k^3}\Big).$$

If the components of μ are such that $\mu_{j+l}\in[\bar\mu_{j+l}-\delta,\bar\mu_{j+l}+\delta], l=0,1,2,$ we take into account $\varphi'(1)=1/3$, the boundedness of φ'' and the Taylor expansion (5.4) to arrive at the inclusion of $\Phi_{j+1}(\mu)$ stated in the assertion. $\square$

As a consequence, in the assumption of Lemma 5.1, we get the inclusion

$$\Psi_{j+1}(\mu)\in[1+\frac{1}{k}-\frac{c_1}{k^3}-\frac{2}{3}\delta-c_2\delta^2, 1+\frac{1}{k}+\frac{c_1}{k^3}+\frac{2}{3}\delta+c_2\delta^2] \tag{5.5}$$

where the constants c_1, c_2 may be different compared to those of Lemma 5.1.

Take in Lemma 5.1 $\delta=c_0/k, c_0>0$, then $\Psi_{j+1}(\mu)\in[1+1/k-\delta, 1+1/k+\delta]$ if $c_1/k^2+c_2c_0^2/k\leqslant c_0/3$ for c_1, c_2 in (5.5). This holds for some $c_0>0$ and for $k\geqslant k_0$ with some fixed k_0 taken after the choice of c_0. Basing on the proof of Lemma 5.1 we get for the solution of $\mu_{j+1}=\Psi_{j+1}(\mu)$ the estimate

$$\frac{1-\varphi(\mu_j^{-1/2})-\varphi(\mu_{j+1}^{1/2})}{\varphi(\mu_j^{-1/2})+\varphi(\mu_{j+1}^{1/2})}=O\Big(\frac{1}{k^2}\Big)+O(\delta)$$

and, taking into account $f'_j=2kh$,

$$m_j-f'_j=2kh(O\Big(\frac{1}{k^2}\Big)+O(\delta))=O(h). \tag{5.6}$$

It remains to study the behaviour of $m_j - f_j'$ for $j = i, \ldots, i + k_0 - 1$. Because of $f_j' = 2kh, j = i+k$, it is sufficient to establish the boundedness of μ_j from below and from above by positive constants (not depending on h) for finite number of indicies k. The symmetry considerations allow to assert that $m_{i-1} = m_{i+1}$ (still for n even and $i = n/2$) and then $\mu_i = \mu_{i+1}$. The equation (5.3) is in this case

$$\mu_i = 3\frac{\varphi(\mu_i^{-1/2}) + \varphi(\mu_i^{1/2})}{\varphi(\mu_i^{-1/2}) + \varphi(\mu_{i+2}^{1/2})}. \tag{5.7}$$

First, assume that $\mu_i \to \infty$. Then $\varphi(\mu_i^{1/2}) \sim (\log \mu_i)/2$, $\mu_i^{-1} \to 0$ and $\varphi(\mu_i^{-1/2}) \sim \mu_i^{-1/2} \to 0$ (here and in the sequel $\sim$ means that the quotient of these terms converges to some positive constant, mainly to 1). In the equation (5.7) in the form

$$\mu_i(\varphi(\mu_i^{-1/2}) + \varphi(\mu_{i+2}^{1/2})) = 3(\varphi(\mu_i^{-1/2}) + \varphi(\mu_i^{1/2})) \tag{5.8}$$

the right hand side of (5.8) behaves as $3(\log \mu_i)/2$ but the left hand side increases at least as $\mu_i^{1/2}$ which gives a contradiction. Secondly, let $\mu_i \to 0$. Then $\varphi(\mu_i^{1/2}) \to 0$, $\mu_i^{-1/2} \to \infty$ and the right hand side of (5.8) is of order $3(\log \mu_i^{-1})/2$ while in the left hand side this order has to have the term $\mu_i \varphi(\mu_{i+2}^{1/2})$. This yields that $\mu_{i+2} \to \infty$ and $\varphi(\mu_i^{-1/2})/\varphi(\mu_{i+2}^{1/2}) \sim 2\mu_i/3 \to 0$. The particular case of (5.3) for $j = i + 1$ gives

$$\mu_{i+2}\Big(\varphi(\mu_{i+2}^{-1/2}) + \varphi(\mu_{i+3}^{1/2})\Big) = 2\Big(\varphi(\mu_i^{-1/2}) + \varphi(\mu_{i+2}^{1/2})\Big)$$

where at right the main term is of order $\log \mu_{i+2}$ but at left we have at least $\mu_{i+2}^{1/2}$ which is impossible.

This completes the proof of boundedness of μ_i (and μ_{i+1}). It remains to carry out the induction step which differs from the just presented reasoning only by details and we omit it.

We have proved the estimate $m_i - f_i' = O(h)$ for the investigated function $f(x) = x^2 \operatorname{sgn} x, x \in [-1, 1]$.

Let us consider now briefly some other particular cases.

The function $f(x) = x^3, x \in [-1, 1]$, generates $\delta_j > 0$ for all j and any partition. On uniform partition, for n even and $i = n/2$ we get $\delta_i/h = h^2/2$ and $\delta_{i+k}/h = h^2(6k^2 + 1)/2,\ k \geqslant 1$, and then

$$\frac{\delta_{i+k}}{\delta_{i+k-1}} = 1 + \frac{2}{k} + O\Big(\frac{1}{k^2}\Big). \tag{5.9}$$

For n odd and $i = (n-1)/2$ we have $x_i = -h/2$, $x_{i+1} = h/2$, $\delta_i/h = \delta_{i+1}/h = 5h^2/4$, $\delta_{i+k+1}/h = (3(2k+1)^2/4 + 1/2)h^2$ and still (5.9) holds. The assertion of Lemma 5.1 about the inclusion of the value $\Phi_{j+1}(\mu)$ takes places if we choose

$\mu_{j+l} \in [1+2/(k+l-1)-\delta, 1+2/(k+l-1)+\delta]$, $\quad l=0,1,2$, we only have to take $\bar{\mu}_{j+l} = 1+2/(k+l-1)$ in the proof. Then

$$\varphi(\bar{\mu}_j^{-1/2}) + \varphi(\bar{\mu}_{j+1}^{1/2}) = 1 - \frac{1}{6k^2} + O\Big(\frac{1}{k^2}\Big)$$

and other changes in the proof of Lemma 5.1 are obvious. Taking into account (5.9) we establish the inclusion $\Psi_{j+1}(\mu) \in [1+2/k-\delta, 1+2/k+\delta]$ if the inequality

$$\Big(1+\frac{c_1}{k^3}+\frac{2}{3}\delta+c_2\delta^2\Big)\Big(1+\frac{2}{k}+\frac{c_3}{k^2}\Big) \leqslant 1+\frac{2}{k}+\delta$$

holds (c_3 reflects (5.9)). This is achieved with $\delta = c_0/k^2$ where c_0 is sufficiently great and $k \geqslant k_0$ for some fixed k_0. Now $\delta_j/h = 3k^2h^2 + h^2 f_j'''/12$ and instead of (5.2) we have

$$\begin{aligned} m_j - f_j' =& f_j' \frac{1-\varphi\Big(\Big(\dfrac{m_{j-1}}{m_j}\Big)^{1/2}\Big) - \varphi\Big(\Big(\dfrac{m_{j+1}}{m_j}\Big)^{1/2}\Big)}{\varphi\Big(\Big(\dfrac{m_{j-1}}{m_j}\Big)^{1/2}\Big) + \varphi\Big(\Big(\dfrac{m_{j+1}}{m_j}\Big)^{1/2}\Big)} \\ &+ \frac{\dfrac{h^2}{12} f_j'''}{\varphi\Big(\Big(\dfrac{m_{j-1}}{m_j}\Big)^{1/2}\Big) + \varphi\Big(\Big(\dfrac{m_{j+1}}{m_j}\Big)^{1/2}\Big)}, \end{aligned}$$

thus (5.6) is replaced by

$$m_j - f_j' = 3k^2h^2\Big(O\Big(\frac{1}{k^2}\Big) + O(\delta)\Big) + O(h^2) = O(h^2). \tag{5.10}$$

The boundedness of μ_j for finite number of j could be obtained exactly as presented above and, consequently, we have $m_j - f_j' = O(h^2)$ for the function $f(x) = x^3, x \in [-1,1]$.

Next, consider the function $f(x) = x^2, x \in [-1,1]$, and uniform partition on $[-1,1]$. For n even, we have $x_i = 0, i = n/2$, and $z_i = z_{i+1}$, thus $\delta_i = 0$. We have also $\delta_j < 0, j = 0,\dots,i-1$, and $\delta_j > 0, j = i+1,\dots,n$. By the comonotone strategy it should be chosen $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ quadratic, all other particular intervals rational. But there is no weak alternation of data in this section of quadratic intervals and, by the modified strategy, we choose $[x_{i-1}, x_i]$ to be rational, too. For n odd, the point 0 is the midpoint of the interval $[x_i, x_{i+1}]$ with $i = (n-1)/2$. Then $\delta_j < 0$, $j = 0,\dots,i$, and $\delta_j > 0$, $j = i+1,\dots,n$. The comonotone strategy makes the subinterval $[x_i, x_{i+1}]$ to be quadratic and all others rational. Here we have the weak alternation of data. It is clear that only a finite number of basic equations corresponding to the neighbourhood of $[x_i, x_{i+1}]$ need the study. The equations of the types (2.20), (4.4), (4.6) corresponding to the points x_{i-1}, x_i, x_{i+1} are now

$$m_{i-1}\Big(\varphi\Big(\Big(\frac{m_{i-2}}{m_{i-1}}\Big)^{1/2}\Big)+\varphi\Big(\Big(\frac{m_i}{m_{i-1}}\Big)^{1/2}\Big)\Big)=\frac{1}{h}\delta_{i-1}=f'_{i-1}, \tag{5.11}$$

$$m_i\varphi\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big)+\frac{1}{3}m_i+\frac{1}{6}m_{i+1}=\frac{1}{h}\delta_i=f'_i, \tag{5.12}$$

$$\frac{1}{6}m_i+\frac{1}{3}m_{i+1}+m_{i+1}\varphi\Big(\Big(\frac{m_{i+2}}{m_{i+1}}\Big)^{1/2}\Big)=\frac{1}{h}\delta_{i+1}=f'_{i+1}. \tag{5.13}$$

Actually, we have to estimate $m_j - f'_j$ for $j = i-1, i, i+1$, for $j < i-1$ and $j > i+1$ the reasoning about the function $f(x) = x^2 \operatorname{sgn} x$ is valid provided we prove the boundedness of $\mu_i = m_{i-1}/m_i$ and $\mu_{i+2} = m_{i+2}/m_{i+1}$. We see that, for n even, in (5.11) $f'_{i-1} = -2h$, in (5.12) $f'_i = 0$, in (5.13) $f'_{i+1} = 2h$, for n odd, $f'_{i-1} = -3h$, $f_i = -h$, $f_{i+1} = h$.

Suppose $m_{i-1}/m_i \to \infty$. First, consider the case $c_1 \leqslant m_i/h \leqslant c_2$ with some $c_1, c_2 < 0$. Dividing (5.12) by h, we see that $m_{i+1}/h \to \infty$. Then (5.13) divided by h gives a contradiction. Secondly, let $m_i/h \to -\infty$. Dividing (5.12) by m_i we conclude that $m_{i+1}/m_i \to -\infty$. Then (5.13) divided by m_i leads to a contradiction. Thirdly, let $m_i/h \to 0$, however $m_i < 0$. Then $h/m_i \to -\infty$ and $|h/m_i| \to \infty$. Consider the case n even (then $f'_i = 0$). Dividing (5.12) by m_i gives $m_{i+1}/m_i \to -\infty$, we have also $\varphi((m_{i-1}/m_i)^{1/2}) \sim |m_{i+1}/m_i|/6$ or

$$\frac{m_{i-1}}{m_i} \sim e^{\frac{1}{3}\left|\frac{m_{i+1}}{m_i}\right|}. \tag{5.14}$$

Dividing (5.11) by m_i we get

$$\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}+\frac{m_{i-1}}{m_i}\varphi\Big(\Big(\frac{m_{i-2}}{m_{i-1}}\Big)^{1/2}\Big)\sim 2\Big|\frac{h}{m_i}\Big|$$

and according to (5.14)

$$e^{\frac{1}{6}\left|\frac{m_{i+1}}{m_i}\right|}+e^{\frac{1}{3}\left|\frac{m_{i+1}}{m_i}\right|}\varphi\Big(\Big(\frac{m_{i-2}}{m_{i-1}}\Big)^{1/2}\Big)\sim 2\Big|\frac{h}{m_i}\Big|. \tag{5.15}$$

Dividing (5.13) by m_i gives

$$\frac{1}{3}\Big|\frac{m_{i+1}}{m_i}\Big|+\Big|\frac{m_{i+1}}{m_i}\Big|\varphi\Big(\Big(\frac{m_{i+2}}{m_{i+1}}\Big)^{1/2}\Big)\sim 2\Big|\frac{h}{m_i}\Big|.$$

The boundedness of m_{i+2}/m_{i+1} means that $|m_{i+1}/m_i|$ and $2|h/m_i|$ have the same order and the left hand side of (5.15) has also the order of $|m_{i+1}/m_i|$ which is impossible. It remains to consider the case $m_{i+2}/m_{i+1} \to \infty$. Dividing the counterpart of (5.11) corresponding to x_{i+2} by m_{i+1} gives

$$\frac{m_{i+2}}{m_{i+1}}\Big(\varphi\Big(\Big(\frac{m_{i+1}}{m_{i+2}}\Big)^{1/2}\Big)+\varphi\Big(\Big(\frac{m_{i+3}}{m_{i+2}}\Big)^{1/2}\Big)\Big) \sim \frac{h}{m_{i+1}} \tag{5.16}$$

which means also that $h/m_{i+1} \to \infty$ because $m_{i+1}/m_{i+2} \to 0$. Then (5.13) divided by m_{i+1} implies that h/m_{i+1} has the order of $\log(m_{i+2}/m_{i+1})$ which is in contradiction with (5.16). Consider now the case of n odd. Dividing (5.12) by m_i we get

$$\varphi\Big(\Big(\frac{m_{i-1}}{m_i}\Big)^{1/2}\Big)+\frac{1}{3}=\frac{1}{6}\Big|\frac{m_{i+1}}{m_i}\Big|+\Big|\frac{h}{m_i}\Big|.$$

The possibility $\varphi((m_{i-1}/m_i)^{1/2}) \sim |m_{i+1}/m_i|$ (the quotient is bounded from above and below by positive constants) could be treated as in the case of n even. If $\varphi((m_{i-1}/m_i)^{1/2}) \sim |h/m_i|$ then dividing (5.11) by m_i we get

$$\frac{m_{i-1}}{m_i}\varphi\Big(\Big(\frac{m_{i-2}}{m_{i-1}}\Big)^{1/2}\Big)+\frac{m_{i-1}}{m_i}\varphi\Big(\Big(\frac{m_i}{m_{i-1}}\Big)^{1/2}\Big)=3\Big|\frac{h}{m_i}\Big|.$$

Here the right hand side is of order $\log(m_{i-1}/m_i)$ while the left hand side is of order at least $(m_{i-1}/m_i)^{1/2}$ which is a contradiction. This completes the proof of the boundedness of m_{i-1}/m_i from above.

Suppose next that $m_{i-1}/m_i \to 0$, then $m_i/m_{i-1} \to \infty$. Dividing (5.11) by m_{i-1} we get

$$\varphi\Big(\Big(\frac{m_{i-2}}{m_{i-1}}\Big)^{1/2}\Big)+\varphi\Big(\Big(\frac{m_i}{m_{i-1}}\Big)^{1/2}\Big)=l\Big|\frac{h}{m_{i-1}}\Big|$$

with $l = 2$ or $l = 3$ depending on n to be even or odd. This means that $|h/m_{i-1}| \to \infty$. If $\varphi\Big(\Big(m_{i-2}/m_{i-1}\Big)^{1/2}\Big)$ is of order $|h/m_{i-1}|$ then

$$\frac{m_i}{m_{i-1}} \sim e^{k\left|\frac{h}{m_{i-1}}\right|} \tag{5.17}$$

with some constant k (here also, $\sim$ means that the quotient is two-sided bounded). Dividing (5.12) by m_{i-1} we see that

$$\frac{m_{i+1}}{m_{i-1}}/\frac{m_i}{m_{i-1}} \to -2. \tag{5.18}$$

Dividing (5.13) by m_{i-1} gives

$$\frac{1}{6}\frac{m_i}{m_{i-1}}+\Big(\frac{1}{3}+\varphi\Big(\Big(\frac{m_{i+2}}{m_{i+1}}\Big)^{1/2}\Big)\Big)\frac{m_{i+2}}{m_{i-1}}=-l\Big|\frac{h}{m_{i-1}}\Big|$$

with $l = 1$ or $l = 2$ and this is a contradiction due to (5.18) and (5.17). If $\varphi((m_{i-2}/m_{i-1})^{1/2})$ is of order $|h/m_{i-1}|$ then

$$\frac{m_{i-2}}{m_{i-1}} \sim e^{k\left|\frac{h}{m_{i-1}}\right|}. \tag{5.19}$$

Now the counterpart of (5.11) corresponding to the point x_{i-2} divided by m_{i-1} gives

$$\frac{m_{i-2}}{m_{i-1}}\varphi\Big(\Big(\frac{m_{i-3}}{m_{i-2}}\Big)^{1/2}\Big)+\Big(\frac{m_{i-2}}{m_{i-1}}\Big)^{1/2}\sim\Big|\frac{h}{m_{i-1}}\Big|$$

which is a contradiction due to (5.19) and this ends the proof of boundedness of m_{i-1}/m_i from below by a positive constant.

As any point x_j is an endpoint of a rational interval and the quotient of two values m_j in any rational interval is bounded from above and below by positive constants, we obtain the estimate $m_j - f'_j = O(h)$ for all j like in the case of function $f(x) = x^2 \operatorname{sgn} x$.

The function $f(x) = |x|^3$, $x \in [-1, 1]$, could be treated joining the arguments from treatments of two previous functions. Here we have the estimate $m_j - f'_j = O(h^2)$.

Ending this section let us indicate the framework at establishing the estimates. First, we prove the boundedness of μ_j for finite number of indices $j = i, \ldots, i + k_0$ (or for $j < i$), with suitable fixed k_0 and then use (5.6) or (5.10). For $j > i + k_0$ (or $j < i - k_0$) we use Bohl – Brouwer fixed point principle as it was done by us several times [16, 18, 17] basing on inclusions (5.5) or its analogue in the analysis of function $f(x) = x^3$.

However, we need a boundary value $\mu_{j+1} = \mu_{i+k_0}$ but, e.g., we take it from the estimates $m_j = f'_j + \alpha_1$, $m_{j+1} - f'_{j+1} = \alpha_2$, $|\alpha_1| \leqslant ch$, $|\alpha_2| \leqslant ch$, c =const, then $\mu_{j+1} = m_{j+1}/m_j = 1 + 1/k_0 + \delta$, $|\delta| \leqslant c_0/k_0$. Such an estimate is valid due to (5.6) but works as well if we use (5.10). At the other end we choose as boundary condition $\mu_{n+1} = 1 + 2/n$ or $\mu_{n+1} = 1 + 1/n$ for the cases $f(x) = x^2$ or $f(x) = x^3$, respectively.

5.2 Convergence results

In Section 5.1 we established the estimates $m_i - f'_i = O(h^\alpha)$, $0 < \alpha \leqslant 2$, depending on the function of f to histopolate. In the beginning we show how this implies the estimate $\|S' - f'\|_\infty = O(h^\alpha)$. We have already mentioned that this yields $\|S - f\|_\infty = O(h^{1+\alpha})$.

The representation (2.14) gives on a rational interval $[x_{i-1}, x_i]$

$$S'(x) = \frac{m_{i-1}}{\Big(1 + \dfrac{x - x_{i-1}}{h}\Big(\Big(\dfrac{m_{i-1}}{m_1}\Big)^{1/2} - 1\Big)\Big)^2} \tag{5.20}$$

and on a quadratic interval $[x_{i-1}, x_i]$ from (4.1) follows

$$S'(x) = (1 - t)m_{i-1} + tm_i, \tag{5.21}$$

actually, as the linear interpolation representation for derivative S'. On rational intervals we have using (5.20) and (5.1)

$$\begin{aligned} S'(x)-f'(x) &= S'(x)-f'_{j-1}+f'_{j-1}-f'(x) \\ &= \frac{\frac{1}{h}\delta_{i-1}}{\left(1+\frac{x-x_{i-1}}{h}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}-1\right)\right)^2\left(\varphi\left(\left(\frac{m_{i-2}}{m_{i-1}}\right)^{1/2}\right)+\varphi\left(\left(\frac{m_i}{m_{i-1}}\right)^{1/2}\right)\right)} \\ &\quad - f'_{i-1}+O(h). \end{aligned} \tag{5.22}$$

In the case of function $f(x)=x^2$ and $f(x)=x^2\operatorname{sgn}x$ it holds $\delta_{i-1}/h=f'_{i-1}$ and (5.22) gives $S'(x)-f'(x)=O(h)$ because $f'_{j+1}=2kh$, $j=n/2+k$ (or $j+1=(n-1)/2+k$), $j\geqslant n/2$ and $\mu_{j+1}=1+O(1/k)$. Near the central knot we only use the boudedness of μ_j. The same reasoning works as well in the case of functions $f(x)=x^3$ and $f(x)=|x|^3$ for $k\leqslant k_0$, k_0 fixed, because of $f'_{j+1}=3k^2h^2$ and $f'_{i-1}-f(x)=O(h^2)$ due to $f''(x)=O(h)$, $0\leqslant x\leqslant x_{n/2+k_0}$. Then we obtain $S'(x)-f'(x)=O(h^2)$. For $k>k_0$, use the expansion

$$\begin{aligned} S'(x)-f'(x) =& m_{i-1}-f'_{i-1}+(x-x_{i-1})(S''_{i-1}-f''_{i-1}) \\ &+\frac{(x-x_{i-1})^2}{2}(S'''-f''')(\xi). \end{aligned} \tag{5.23}$$

Let us show how to establish $S''_{i-1}-f''_{i-1}=O(h)$. From (5.20) follows

$$S''_{i-1}=-\frac{2m_{i-1}}{h}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}-1\right).$$

It was shown that ($i\geqslant n/2+k_0$) $\mu_i=m_i/m_{i-1}=1+2/k+O(1/k^2)$, then $(m_{i-1}/m_i)^{1/2}-1=-1/k+O(1/k^2)$(see, e.g., [17]), $f'_{i-1}=3k^2h^2$ and we get due to $m_{i-1}=f'_{i-1}+O(h^2)$

$$S''_{i-1}=\frac{-2(3k^2h^2+O(h^2))}{h}\left(-\frac{1}{k}+O\left(\frac{1}{k^2}\right)\right)=6kh+O(h).$$

Now, as $f''_{i-1}=6kh$, we get $S''_{i-1}-f''_{i-1}=O(h)$. In addition, (5.20) gives

$$S'''(x)=\frac{6m_{i-1}\frac{1}{h^2}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}-1\right)^2}{\left(1+\frac{x-x_{i-1}}{h}\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}-1\right)\right)^4}$$

and using again $(m_{i-1}/m_i)^{1/2}-1=-1/k+O(1/k^2)$, $m_{i-1}=3k^2h^2+O(h^2)$, we obtain the uniform boundedness of S''' and from (5.23) $S'(x)-f'(x)=O(h^2)$ for $k>k_0$, too. On the quadratic interval (5.21) gives

$$S'(x)-f'(x)=(1-t)(m_{i-1}-f'_{i-1})+t(m_i-f'_i)+(1-t)f'_{i-1}+tf'_i-f'(x)$$

and from this the required order follows if we take into account the order of linear interpolation on dependence on the smoothness of f.

It is quite clear how all presented reasonings could be generalized to the case considered at the beginning of Section 5.1. Let us indicate some important moments. If in the point c_i the function f' does not change the sign, we should argue as in the case of $f(x) = x^2 \operatorname{sgn} x$ or $f(x) = x^3$. Then in (5.3) $\delta_{j+1}/\delta_j = 1 + l/k + O(1/k^2)$ with some constant l which can be fractional. Such a fractionality of constants appears also in other formulae but all presented steps of the proofs are valid. If f' changes the sign in the point c_i, we follow the proof of the cases of $f(x) = x^2$ or $f(x) = |x|^3$ and again the fractionality phenomina should be taken into account. We have proved the following:

Theorem 5.1. *Suppose that a function f has a finite number of points c_i in $[a, b]$ such that $f'(c_i) = 0$ and*

$$\lim_{\substack{x \to c_i \\ x < c_i}} \frac{f'(x)}{|x - c_i|^{\alpha_{i1}}} = \gamma_{i1} \neq 0, \quad \lim_{\substack{x \to c_i \\ x > c_i}} \frac{f'(x)}{|x - c_i|^{\alpha_{i2}}} = \gamma_{i2} \neq 0.$$

Let $\alpha = \min_i \min\{\alpha_{i1}, \alpha_{i2}, 2\}$. We also assume that $f' \in Lip\ \alpha$ if $0 < \alpha \leqslant 1$ or $f'' \in Lip\ (\alpha - 1)$ if $1 < \alpha \leqslant 2$. Then the combined histopolating spline S which is constructed by comonotone or modified comonotone strategy, has in the uniform norm on $[a, b]$ the convergence rate $\|S - f\|_\infty = O(h^{1+\alpha})$ together with $\|S' - f'\|_\infty = O(h^\alpha)$.

Remark 5.1. *We formulated and proved the results in the case of uniform mesh. It is quite evident that our arguments work also in the case of a mesh with $0 < q_1 \leqslant h_j/h_i \leqslant q_2$ for $|i - j| = 1$ where q_1, q_2 are constants. The reason here is that far enough from the points c_i the results hold by Section 3.2, but in finite number of intervals around the points c_i such nonuniformity can change only the constants, not the order of all given estimates. However, around the points c_i where the theoretical rate of convergence is lower than in regions with strict uniform monotonicity, we can use the mesh with smaller step as the idea of adaptive meshes. This compensates such a lower rate.*

Chapter 6

Convexity preserving rational spline histopolation

6.1 The histopolation problem

Let x_i be given points in an interval $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$ and let z_i, $i = 1, \ldots, n$, be given real numbers corresponding to subintervals $[x_{i-1}, x_i]$, as histopolation data in the form of a given histogram. It is natural to suppose that $n \geqslant 2$. Now we choose spline knots ξ_i, $i = 1, \ldots, n$, such that $\xi_1 = x_0, x_1 < \xi_2 < x_2, \ldots, x_{i-1} < \xi_i < x_i, \ldots, x_{n-2} < \xi_{n-1} < x_{n-1}, \xi_n = x_n$. We want to construct a C^2 smooth function S on $[a, b]$ in the form

$$S(x) = a_i + b_i(x - x_i) + \frac{c_i}{1 + d_i(x - x_i)} \tag{6.1}$$

with $1+d_i(x-x_i) > 0$ for $x \in [\xi_i, \xi_{i+1}]$, (i.e., a quadratic/linear rational spline) satisfying the histopolation (area–matching) conditions (2.2). In addition, we impose the boundary conditions

$$S''(x_0) = \alpha, \quad S''(x_n) = \beta \tag{6.2}$$

or

$$S'(x_0) = \alpha, \quad S'(x_n) = \beta \tag{6.3}$$

or

$$S(x_0) = \alpha, \quad S(x_n) = \beta \tag{6.4}$$

for given α and β. However, for example, one condition from (6.2) and another from (6.3) at different endpoints x_0 and x_n may be used or some other combination of boundary conditions may be used.

A direct calculation from (6.1) shows that the second derivative

$$S''(x) = \frac{2c_i d_i^2}{(1 + d_i(x - x_i))^3} \tag{6.5}$$

preserves the sign on each particular interval $[\xi_i, \xi_{i+1}]$ and, consequently, on the whole interval $[a, b]$ and because of that S is strictly convex or strictly concave or constant.

6.2 Uniqueness of the histopolant

Theorem 6.1. *There are no two different splines of class C^2 being quadratic/linear rational function on intervals and satisfying the same histopolating conditions (2.2) and boundary conditions (6.2) or (6.3) or (6.4).*

Proof. Let there be two different quadratic/linear splines of class C^2 S_1 and S_2 which satisfy the same histopolating conditions and boundary conditions. Then

$$\int_{x_{i-1}}^{x_i} (S_1(x) - S_2(x))dx = 0, \quad i = 1, \ldots, n,$$

implies that there are $\eta_i \in (x_{i-1}, x_i)$ such that $S_1(\eta_i) = S_2(\eta_i)$, $i = 1, \ldots, n$.

Denote $g = S_1 - S_2$. We have the following possibilities:

1) If we use boundary conditions (6.4), the function g has $n+2$ different zeros $a, \eta_1, \ldots, \eta_n, b$ in $[a, b]$. Then the function g' has $n+1$ and g'' has n different zeros in (a, b).

2) If we use boundary conditions (6.3) and from the knowledge that the function g has n different zeros $\eta_1, \ldots, \eta_n$ in (a, b) we get that the function g' has $n-1$ zeros in (a, b) and 2 additional zeros from boundary conditions. Therefore, the function g'' has n zeros in (a, b).

3) From boundary conditions (6.2) and from the fact that the function g'' has $n-2$ different zeros in (a, b) we get altogether that g'' has n zeros in $[a, b]$.

Thus, we have at least one interval $[\xi_i, \xi_{i+1}]$ where there are two different zeros of g''. From (6.5) we get

$$g''(x) = \frac{2c_{1i}d_{1i}^2}{(1 + d_{1i}(x - x_i))^3} - \frac{2c_{2i}d_{2i}^2}{(1 + d_{2i}(x - x_i))^3}.$$

If, e.g., $c_{1i}d_{1i}^2 = 0$ then $S_1''(x) = 0$ for all $x \in [\xi_i, \xi_{i+1}]$ and $g''(x) = 0$ for some $x \in [\xi_i, \xi_{i+1}]$ gives also that $S_2''(x) = 0$ for all $x \in [\xi_i, \xi_{i+1}]$ and, thus, $g''(x) = 0$ for all $x \in [\xi_i, \xi_{i+1}]$. Consider next the case $c_{1i}d_{1i}^2 \neq 0$ and $c_{2i}d_{2i}^2 \neq 0$. Then $g''(x) = 0$ if and only if

$$\frac{c_{1i}d_{1i}^2}{(1 + d_{1i}(x - x_i))^3} = \frac{c_{2i}d_{2i}^2}{(1 + d_{2i}(x - x_i))^3}$$

or

$$\Big(\frac{1 + d_{1i}(x - x_i)}{1 + d_{2i}(x - x_i)}\Big)^3 = \frac{c_{1i}d_{1i}^2}{c_{2i}d_{2i}^2}$$

or

$$\frac{1 + d_{1i}(x - x_i)}{1 + d_{2i}(x - x_i)} = \Big(\frac{c_{1i}d_{1i}^2}{c_{2i}d_{2i}^2}\Big)^{1/3}. \tag{6.6}$$

Equation (6.6) could be true only for every $x \in [\xi_i, \xi_{i+1}]$ or only in one point. At least on interval $[\xi_i, \xi_{i+1}]$ the function g'' has two different zeros, therefore (6.6) applies for every $x \in [\xi_i, \xi_{i+1}]$. Then in endpoints ξ_i and ξ_{i+1} the functions S_1'' and S_2'' are equal. Now we repeat the discussion in intervals $[a, \xi_i]$ and $[\xi_{i+1}, b]$. Finally we get that $g''(x) = 0$ for every $x \in [a, b]$. This means that g is at most first degree polynomial. As $n \geqslant 2$ the histopolation condition $\int_{x_{i-1}}^{x_i} g(x)dx = 0$ gives us that $g(x) = 0$ for every $x \in [a, b]$ which completes the proof. □

6.3 Representation of the histopolant

In this section we show what kind of representation we will use for S. We assume that S satisfies (2.2) and also we require the smoothness C^2 for S. Let us denote $M_i = S''(\xi_i)$, $i = 1, \ldots, n$, and $\varepsilon_i = x_i - \xi_i$, $\eta_i = \xi_{i+1} - x_i$, particularly, $\varepsilon_1 = h_1$, $\delta_1 = h_1 + \eta_1$, $\eta_{n-1} = h_n$, $\delta_{n-1} = \varepsilon_{n-1} + h_n$ and $\delta_i = \varepsilon_i + \eta_i = \xi_{i+1} - \xi_i$, $i = 1, \ldots, n-1$, also $h_i = \eta_{i-1} + \varepsilon_i = x_i - x_{i-1}$, $i = 1, \ldots, n$.

We mentioned that S'' preserves the sign on the whole interval $[a, b]$. Therefore, we consider the case $M_i > 0$ and $d_i \neq 0$, $i = 1, \ldots, n$. Later we give a separate remark about the case $M_i = 0$ and, more generally, about $d_i = 0$.

From (6.1) we can calculate

$$S'(x) = b_i + \frac{-c_i d_i}{(1 + d_i(x - x_i))^2}, \quad x \in [\xi_i, \xi_{i+1}],$$

and $S''(x)$ is presented as (6.5). Then

$$M_i = S''(\xi_i) = \frac{2c_i d_i^2}{(1 - d_i\varepsilon_i)^3}$$

and

$$M_{i+1} = S''(\xi_{i+1}) = \frac{2c_i d_i^2}{(1 + d_i\eta_i)^3}. \tag{6.7}$$

From these we get

$$\Big(\frac{M_{i+1}}{M_i}\Big)^{1/3} = \frac{1 - d_i\varepsilon_i}{1 + d_i\eta_i}$$

or

$$d_i = \frac{M_i^{1/3} - M_{i+1}^{1/3}}{\varepsilon_i M_i^{1/3} + \eta_i M_{i+1}^{1/3}} \tag{6.8}$$

or

$$d_i = \frac{1 - \mu_i}{\varepsilon_i + \eta_i\mu_i}, \quad \mu_i = \Big(\frac{M_{i+1}}{M_i}\Big)^{1/3}.$$

Denote

$$\lambda_i = \int_{\xi_i}^{x_i} S(x)dx, \quad \rho_i = \int_{x_i}^{\xi_{i+1}} S(x)dx, \quad i = 1, \dots, n-1.$$

From histopolation conditions (2.2) we get

$$\rho_{i-1} + \lambda_i = h_i z_i, \quad i = 1, \dots, n,$$

particularly, $\lambda_1 = h_1 z_1$ and $\rho_{n-1} = h_n z_n$ with $\rho_0 = 0$, $\lambda_n = 0$ if needed. From (6.1) follows

$$\lambda_i = a_i \varepsilon_i - \frac{b_i}{2}\varepsilon_i^2 - \frac{c_i}{d_i}\log(1 - d_i\varepsilon_i) \tag{6.9}$$

and

$$\rho_i = a_i \eta_i + \frac{b_i}{2}\eta_i^2 + \frac{c_i}{d_i}\log(1 + d_i\eta_i). \tag{6.10}$$

Now we additionally denote $\alpha_i = 1 - d_i\varepsilon_i$ and $\beta_i = 1 + d_i\eta_i$, $i = 1, \dots, n$, then $\alpha_i = \mu_i\beta_i$. We see that $\alpha_i = \mu_i\delta_i/(\varepsilon_i + \eta_i\mu_i) > 0$ and $\beta_i = \delta_i/(\varepsilon_i + \eta_i\mu_i) > 0$ because $\mu_i > 0$. By symmetry consideration we use also $\nu_i = (M_{i-1}/M_i)^{1/3}$, then, e.g., $d_i = (\nu_{i+1} - 1)/(\varepsilon_i\nu_{i+1} + \eta_i)$.

From equations (6.9) and (6.10) we can eliminate a_i

$$\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} = \frac{b_i}{2}(\eta_i + \varepsilon_i) + \frac{c_i}{d_i}(\frac{1}{\eta_i}\log\beta_i + \frac{1}{\varepsilon_i}\log\alpha_i). \tag{6.11}$$

From (6.7) we get

$$\frac{c_i}{d_i} = \frac{(1 + d_i\eta_i)^3 M_{i+1}}{2d_i^3}$$

and replacing $1 + d_i\eta_i = \beta_i$ gives us for $\gamma_i = c_i/d_i$

$$\gamma_i = \frac{\beta_i^3}{2d_i^3}M_{i+1} = \frac{\delta_i^3}{2(1-\mu_i)^3}M_{i+1} = \frac{\delta_i^3 M_i M_{i+1}}{2\left(M_i^{1/3} - M_{i+1}^{1/3}\right)^3}. \tag{6.12}$$

From (6.11) we can calculate

$$b_i = \frac{2}{\delta_i}\Big(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\Big(\frac{1}{\eta_i}\log\beta_i + \frac{1}{\varepsilon_i}\log\alpha_i\Big)\Big) \tag{6.13}$$

and then from (6.9)

$$a_i = \frac{1}{\delta_i}\Big(\frac{\eta_i}{\varepsilon_i}\lambda_i + \frac{\varepsilon_i}{\eta_i}\rho_i + \gamma_i\Big(\frac{\eta_i}{\varepsilon_i}\log\alpha_i - \frac{\varepsilon_i}{\eta_i}\log\beta_i\Big)\Big). \tag{6.14}$$

So using (6.8), (6.12), (6.13) and (6.14) we can represent the spline S via M_i, M_{i+1}, λ_i, ρ_i and this gives the representation of the histopolant (we keep here d_i given in (6.8)) if $x \in [\xi_i, \xi_{i+1}]$ as

$$\begin{aligned} S(x) =& \frac{1}{\delta_i}\Big(\frac{\eta_i}{\varepsilon_i}\lambda_i + \frac{\varepsilon_i}{\eta_i}\rho_i + \gamma_i\Big(\frac{\eta_i}{\varepsilon_i}\log\alpha_i - \frac{\varepsilon_i}{\eta_i}\log\beta_i\Big)\Big) \\ &+ \frac{2}{\delta_i}\Big(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\Big(\frac{1}{\eta_i}\log\beta_i + \frac{1}{\varepsilon_i}\log\alpha_i\Big)\Big)(x - x_i) + \frac{\gamma_i d_i}{1 + d_i(x - x_i)}. \end{aligned} \tag{6.15}$$

6.4 Continuity conditions

With the representation of S in terms M_i we ensure the continuity of S''. We also need the continuity of S and S' in knots ξ_i, $i = 2, \dots, n-1$, this means we want

$$S(\xi_i - 0) = S(\xi_i + 0), \quad i = 2, \dots, n-1, \tag{6.16}$$

and

$$S'(\xi_i - 0) = S'(\xi_i + 0), \quad i = 2, \dots, n-1. \tag{6.17}$$

From (6.15)

$$S(\xi_i + 0) = \frac{1}{\delta_i}\Big(\Big(2 + \frac{\eta_i}{\varepsilon_i}\Big)\lambda_i - \frac{\varepsilon_i}{\eta_i}\rho_i + \gamma_i\Big(\Big(2 + \frac{\eta_i}{\varepsilon_i}\log\alpha_i + \frac{\varepsilon_i}{\eta_i}\log\beta_i\Big)\Big) + \frac{\gamma_i d_i}{\alpha_i}$$

and

$$S(\xi_{i+1} - 0) = \frac{1}{\delta_i}\Big(-\frac{\eta_i}{\varepsilon_i}\lambda_i + \Big(2 + \frac{\varepsilon_i}{\eta_i}\Big)\rho_i + \gamma_i\Big(-\frac{\eta_i}{\varepsilon_i}\log\alpha_i - \Big(2 + \frac{\varepsilon_i}{\eta_i}\Big)\log\beta_i\Big)\Big) + \frac{\gamma_i d_i}{\beta_i}.$$

Because of (6.16) we get

$$\begin{aligned}
&-\frac{1}{\delta_{i-1}}\frac{\eta_{i-1}}{\varepsilon_{i-1}}\lambda_{i-1} + \frac{1}{\delta_{i-1}}\Big(2 + \frac{\varepsilon_{i-1}}{\eta_{i-1}}\Big)\rho_{i-1} - \frac{1}{\delta_i}\Big(2 + \frac{\eta_i}{\varepsilon_i}\Big)\lambda_i + \frac{\varepsilon_i}{\delta_i\eta_i}\rho_i \\
=&\frac{1}{\delta_{i-1}}\gamma_{i-1}\Big(\frac{\eta_{i-1}}{\varepsilon_{i-1}}\log\alpha_{i-1} + \Big(2 + \frac{\varepsilon_{i-1}}{\eta_{i-1}}\Big)\log\beta_{i-1}\Big) \qquad (6.18)\\
&+\frac{1}{\delta_i}\gamma_i\Big(\Big(2 + \frac{\eta_i}{\varepsilon_i}\Big)\log\alpha_i + \frac{\varepsilon_i}{\eta_i}\log\beta_i\Big) - \frac{\gamma_{i-1}d_{i-1}}{\beta_{i-1}} + \frac{\gamma_i d_i}{\alpha_i}, \quad i = 2, \dots, n-1.
\end{aligned}$$

The derivative of (6.15) is

$$S'(x) = \frac{2}{\delta_i}\Big(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\Big(\frac{1}{\eta_i}\log\beta_i + \frac{1}{\varepsilon_i}\log\alpha_i\Big)\Big) - \frac{\gamma_i d_i}{(1 + d_i(x - x_i))^2}, \quad x \in [\xi_i, \xi_{i+1}].$$

From that

$$S'(\xi_i + 0) = \frac{2}{\delta_i}\Big(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\Big(\frac{1}{\eta_i}\log\beta_i + \frac{1}{\varepsilon_i}\log\alpha_i\Big)\Big) - \frac{\gamma_i d_i^2}{\alpha_i^2}$$

and

$$S'(\xi_{i+1} - 0) = \frac{2}{\delta_i}\Big(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i} - \gamma_i\Big(\frac{1}{\eta_i}\log\beta_i + \frac{1}{\varepsilon_i}\log\alpha_i\Big)\Big) - \frac{\gamma_i d_i^2}{\beta_i^2}.$$

We require (6.17), so we get the equations

$$\begin{aligned}
&-\frac{1}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1} + \frac{1}{\delta_{i-1}\eta_{i-1}}\rho_{i-1} + \frac{1}{\delta_i\varepsilon_i}\lambda_i - \frac{1}{\delta_i\eta_i}\rho_i \\
=&\frac{1}{\delta_{i-1}}\gamma_{i-1}\Big(\frac{1}{\varepsilon_{i-1}}\log\alpha_{i-1} + \frac{1}{\eta_{i-1}}\log\beta_{i-1}\Big) \qquad (6.19)\\
&-\frac{1}{\delta_i}\gamma_i\Big(\frac{1}{\varepsilon_i}\log\alpha_i + \frac{1}{\eta_i}\log\beta_i\Big) + \frac{\gamma_{i-1}d_{i-1}^2}{2\beta_{i-1}^2} - \frac{\gamma_i d_i^2}{2\alpha_i^2}, \quad i = 2, \dots, n-1.
\end{aligned}$$

6.5 Basic equations

Let us write the histopolation conditions in the form

$$\rho_{i-1} + \lambda_i - h_i z_i = 0, \quad i = 1, \ldots, n. \tag{6.20}$$

We refer to the just introduced equations (6.18)–(6.20) also as $(6.18, i)$–$(6.20, i)$.

Consider now together the equations $(6.18, i-1)$, $(6.20, i-1)$, $(6.19, i-1)$, $(6.18, i)$, $(6.20, i)$, $(6.19, i)$, $(6.18, i+1)$, $(6.20, i+1)$, $(6.19, i+1)$. Although, in general, the ordering here is not important, in the presented form we have a system of 9 linear equations with respect to 8 unknowns λ_{i-2}, ρ_{i-2}, λ_{i-1}, ρ_{i-1}, λ_i, ρ_i, λ_{i+1}, ρ_{i+1} and the entries of the matrix are placed more compactly. Thus, there is a nontrivial linear combination of these equations where all coefficients of λ_j, ρ_j, $j = i-2, \ldots, i+1$, become equal to zero. We can indicate this combination with the coefficients as follows:

$$\begin{array}{ll}
(6.18, i-1) & \dfrac{h_i + h_{i+1}}{h_{i-1}} \eta_{i-2} \\[2ex]
(6.20, i-1) & -\dfrac{h_i + h_{i+1}}{h_{i-1}} \\[2ex]
(6.19, i-1) & -\dfrac{h_i + h_{i+1}}{h_{i-1}} \eta_{i-2}^2 \\[2ex]
(6.18, i) & \dfrac{(h_i + h_{i+1})\varepsilon_i - \eta_{i-1}(h_{i-1} + h_i)}{h_i} \\[2ex]
(6.20, i) & \dfrac{h_{i-1} + 2h_i + h_{i+1}}{h_i} \\[2ex]
(6.19, i) & -\dfrac{\eta_{i-1}^2(h_{i-1} + h_i) - (h_{i-1}h_i + \eta_{i-1}(h_i + \varepsilon_i))(h_i + h_{i+1})}{h_i} \\[2ex]
(6.18, i+1) & -\dfrac{h_{i-1} + h_i}{h_{i+1}} \varepsilon_{i+1} \\[2ex]
(6.20, i+1) & -\dfrac{h_{i-1} + h_i}{h_{i+1}} \\[2ex]
(6.19, i+1) & -\dfrac{h_{i-1} + h_i}{h_{i+1}} \varepsilon_{i+1}^2.
\end{array}$$

At the left hand side of this combination it remains (we introduce here the notation of D_i, too)

$$D_i = (h_i + h_{i+1})z_{i-1} - (h_{i-1} + 2h_i + h_{i+1})z_i + (h_{i-1} + h_i)z_{i+1}, \quad i = 2, \ldots, n-1.$$

We suppose in the sequel that $D_i > 0$ for all i, which is, in fact, the strict convexity of the given histogram.

The right hand side of this combination contains as unknowns M_{i-2}, M_{i-1}, M_i, M_{i+1}, M_{i+2}. This combination in the form

$$\Phi_i(M_{i-2}, M_{i-1}, M_i, M_{i+1}, M_{i+2}) = D_i \tag{6.21}$$

will be called basic equation. Our following purpose is to study the structure of the function Φ_i.

Only $(6.18, i-1)$ and $(6.19, i-1)$ contain γ_{i-2} and in (6.21) we obtain from them the term with the coefficient of γ_{i-2}

$$\frac{h_i + h_{i+1}}{h_{i-1}}\Big(\log \beta_{i-2} - \frac{\eta_{i-2}d_{i-2}}{\beta_{i-2}} - \frac{1}{2}\Big(\frac{\eta_{i-2}d_{i-2}}{\beta_{i-2}}\Big)^2\Big).$$

Denoting $\sigma_{i-2} = \varepsilon_{i-2}/\eta_{i-2}$ (recall that $\nu_{i-1} = (M_{i-2}/M_{i-1})^{1/3}$), we calculate

$$\gamma_{i-2} = \frac{\delta_{i-2}^3 M_{i-2}}{2(\nu_{i-1}-1)^3},$$

$$\beta_{i-2} = \frac{(\sigma_{i-2}+1)\nu_{i-1}}{\sigma_{i-2}\nu_{i-1}+1},$$

$$\frac{\eta_{i-2}d_{i-2}}{\beta_{i-2}} = \frac{\nu_{i-1}-1}{(\sigma_{i-2}\nu_{i-1}+1)\nu_{i-1}},$$

and obtain the term

$$\frac{h_i + h_{i+1}}{h_{i-1}}\frac{\delta_{i-2}^3}{2}\frac{M_{i-2}}{(\nu_{i-1}-1)^3}\Big(\log\frac{(\sigma_{i-2}+1)\nu_{i-1}}{\sigma_{i-2}\nu_{i-1}+1} - \frac{\nu_{i-1}-1}{(\sigma_{i-2}+1)\nu_{i-1}} - \frac{1}{2}\Big(\frac{\nu_{i-1}-1}{(\sigma_{i-2}+1)\nu_{i-1}}\Big)^2\Big).$$

Here appears the function of the argument ν_{i-1} (denote $\sigma = \sigma_{i-2}+1$)

$$\varphi_A(x) = \frac{1}{(x-1)^3}\Big(\log\frac{\sigma x}{(\sigma-1)x+1} - \frac{x-1}{\sigma x} - \frac{1}{2}\Big(\frac{x-1}{\sigma x}\Big)^2\Big) \tag{6.22}$$

which deserves our special attention. Similar reasoning with $(6.18, i+1)$ and $(6.19, i+1)$ containing γ_{i+1} gives the term

$$\frac{h_{i-1} + h_i}{h_{i+1}}\frac{\delta_{i+1}^3}{2}M_{i+2}\varphi_A(\mu_{i+1})$$

where now $\mu_{i+1} = (M_{i+2}/M_{i+1})^{1/3}$, $\sigma = \eta_{i+1}/\varepsilon_{i+1} + 1$. The value γ_{i-1} is present in $(6.18, i-1)$, $(6.19, i-1)$, $(6.18, i)$ and $(6.19, i)$. This part is represented in the basic equation (6.21) as the summand consisting of γ_{i-1} with the multiplier

$$\frac{h_i + h_{i+1}}{h_{i-1}}\Big(-\log\alpha_{i-1} + \frac{\eta_{i-2}d_{i-1}}{\alpha_{i-1}} + \frac{1}{2}\Big(\frac{\eta_{i-2}d_{i-1}}{\alpha_{i-1}}\Big)^2\Big)$$

$$-\frac{h_{i-1}+2h_i+h_{i+1}}{h_i}\Big(\log\beta_{i-1}-\frac{\eta_{i-1}d_{i-1}}{\beta_{i-1}}-\frac{1}{2}\Big(\frac{\eta_{i-1}d_{i-1}}{\beta_{i-1}}\Big)^2\Big)$$

$$-(h_i+h_{i+1})\frac{d_{i-1}}{\beta_{i-1}}-(h_i+h_{i+1})(2\eta_{i-1}+h_{i-1})\frac{d_{i-1}^2}{2\beta_{i-1}^2}.$$

Similar term could be written with γ_i and they will generate in (6.23) the summand containing the function φ_B.

The study in general case is technically quite complicated and we restrict ourselves in what follows mainly to the particular case of uniform mesh and corresponding uniform replacement of spline knots. Nevertheless, sometimes we add some results in general case.

6.6 Basic equations on uniform mesh

Let us consider the mesh where $h_i = h$, $i = 1,\dots,n$, with spline (interior) knots $\xi_i = (x_{i-1}+x_i)/2$, $i = 2,\dots,n-1$. Then $\varepsilon_1 = h$, $\varepsilon_i = h/2$, $i = 2,\dots,n-1$, $\eta_i = h/2$, $i = 2,\dots,n-1$, $\eta_{n-1} = h$, $\delta_1 = \delta_{n-1} = (3/2)h$, $\delta_i = h$, $i = 2,\dots,n-2$. We have $\varepsilon_i/\eta_i = 1$ and $\sigma = 2$. The basic equation (6.21) takes the form

$$M_{i-2}\varphi_A\Big(\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{1/3}\Big)+M_{i-1}\varphi_B\Big(\Big(\frac{M_{i-1}}{M_i}\Big)^{1/3}\Big)+M_{i+1}\varphi_B\Big(\Big(\frac{M_{i+1}}{M_i}\Big)^{1/3}\Big)$$
$$+M_{i+2}\varphi_A\Big(\Big(\frac{M_{i+2}}{M_{i+1}}\Big)^{1/3}\Big)=D_i \quad (6.23)$$

where

$$\varphi_A(x)=\frac{1}{(x-1)^3}\Big(\log\frac{2x}{x+1}-\frac{1}{2}\frac{x-1}{x}-\frac{1}{8}\Big(\frac{x-1}{x}\Big)^2\Big), \qquad (6.24)$$

$$\begin{aligned}\varphi_B(x) &= \frac{1}{(x-1)^3}\Big(\log\frac{x+1}{2}-2\log\frac{2x}{x+1}+\frac{1}{2}(x-1)+\frac{1}{8}(x-1)^2-\frac{3}{4}\Big(\frac{x-1}{x}\Big)^2\Big)\\ &= \frac{1}{(x-1)^3}\Big(\log\frac{2x}{x+1}-\frac{1}{2}\frac{x-1}{x}+\frac{1}{8}\frac{(x-1)^2(x^2-6)}{x^2}\Big) \qquad (6.25)\end{aligned}$$

with rescaled values $D_i = (2z_{i-1}-4z_i+2z_{i+1})/h^2$, $i = 2,\dots,n-1$.

Le us collect together the main properties of the functions (6.24) and (6.25) in the following

Lemma 6.1. *It holds*

1)$\lim_{x\to1}\varphi_A(x) = 1/24$, $\lim_{x\to1}\varphi_B(x) = 23/24$,

2)$\varphi_A(x) > 0$, $\varphi_A'(x) < 0$, $\varphi_A''(x) > 0$, *for* $x > 0$, *the same properties has* φ_B, *3)* $\varphi_A(x) + x\varphi'_A(x)/3 > 0$, $\varphi_B(x) + x\varphi'_B(x)/3 > 0$ *for* $x > 0$,

4) $\varphi_B(x) \geqslant 3/(4x^2)$ *for* $x \leqslant 1$, $\varphi_B(x) \leqslant 0.393$ *for* $x \geqslant 3$.

The proof is a quite standard calculus, similar to that of Lemma 2.1, but sometimes long and consists of several steps.

To compare the general case (6.22) the calculations give there $\varphi_A(1) = 1/(3\sigma^3)$, $\varphi'_A(1) = -(4\sigma - 1)/(4\sigma^4)$, $\varphi''_A(1) = 2(10\sigma^2 - 5\sigma + 1)/(5\sigma^5)$.

6.7 Boundary equations

Near the boundary of the interval $[a, b]$ appear some differences compared to the equation (6.21). We analyze only the vicinity of the endpoint a, the changes for b are obvious. The 7 equations $(6.20, 1)$, $(6.18, 2)$, $(6.19, 2)$, $(6.20, 2)$, $(6.18, 3)$, $(6.19, 3)$, $(6.20, 3)$, are linear with respect to 6 unknowns λ_1, ρ_1, λ_2, ρ_2, λ_3, ρ_3 and we eliminate them by a nontrivial linear combination. Actually, the same coefficients as in general case are convenient here if we consider them for $i = 2$. The only difference is the absence of $(6.18, 1)$ and $(6.19, 1)$. Instead of (6.21) we have

$$\Phi_2(M_1, M_2, M_3, M_4) = D_2. \tag{6.26}$$

Continue now with the uniform partition of knots. Then the equation (6.23) is replaced by

$$M_1\varphi_C\Big(\Big(\frac{M_1}{M_2}\Big)^{1/3}\Big) + M_3\varphi_B\Big(\Big(\frac{M_3}{M_2}\Big)^{1/3}\Big) + M_4\varphi_A\Big(\Big(\frac{M_4}{M_3}\Big)^{1/3}\Big) = D_2 \tag{6.27}$$

where

$$\varphi_C(x) = -\frac{27}{8(x-1)^3}\Big(\log\frac{27x^2}{(2x+1)^3} + \frac{1}{3}\Big(\frac{x-1}{x}\Big)^2\Big),$$

however, with rescaled D_2.

The assertions 2), 3) of Lemma 6.1 take place as well for φ_C and we have $\lim_{x\to 1}\varphi_c(x) = 1$.

Let us analyze now the boundary conditions.

Conditions (6.3) fix the values M_1 and M_n. Then we have for $n-2$ unknowns $M_2, \ldots, M_{n-1}$ the equal quantity of equations, namely (6.26) with its counterpart containing D_{n-1} and (6.21) for $i = 3, \ldots, n-2$.

At conditions (6.4) and (6.5) M_1 and M_n are also unknowns. Thus, we need two more equations to determine all the values M_i.

Consider first $S(a) = \alpha$. We calculated already $S(\xi_1 + 0)$ and we have

$$\frac{\delta_1 + h_1}{\delta_1\varepsilon_1}(\lambda_1 + \gamma_1\log\alpha_1) + \frac{h_1}{\delta_1\eta_1}(-\rho_1 + \gamma_1\log\beta_1) + \frac{\gamma_1 d_1}{\alpha_1} = \alpha. \tag{6.28}$$

Together with this we take $(6.20, 1)$ (which actually gives the value λ_1), $(6.18, 2)$, $(6.19, 2)$, $(6.20, 2)$. These 5 equations are linear with respect to 4 parameters λ_1, ρ_1, λ_2, ρ_2. A nontrivial combination gives us the equation

$$\begin{aligned}&(h_1+h_2)\alpha-(2h_1+h_2)z_1+h_1z_2\\ =&\frac{2h_1+h_2}{h_1}\gamma_1\log\alpha_1+\frac{h_1}{h_2}\gamma_1\log\beta_1-\frac{h_1}{h_2}\gamma_2\log\alpha_2\\ &+(h_1+h_2)\frac{\gamma_1d_1}{\alpha_1}-\frac{\varepsilon_2h_1}{h_2}\Big(-\frac{\gamma_1d_1}{\beta_1}+\frac{\gamma_2d_2}{\alpha_2}+\varepsilon_2\frac{\gamma_1d_1^2}{2\beta_1^2}-\varepsilon_2\frac{\gamma_2d_2^2}{2\alpha_2^2}\Big)\end{aligned}$$

containing unknowns M_1, M_2, M_3.

In the case of uniform mesh this equation takes the form

$$M_2\varphi_D\Big(\Big(\frac{M_2}{M_1}\Big)^{1/3}\Big)+M_3\varphi_E\Big(\Big(\frac{M_3}{M_2}\Big)^{1/3}\Big)=D_1 \tag{6.29}$$

where

$$\varphi_D(x)=\frac{27}{16}\frac{1}{(1-x)^3}\log\frac{81x^3}{(2+x)^4}+\frac{3}{32}\frac{x^2+5x+24}{x(1-x)^2},$$

$$\varphi_E(x)=-\frac{1}{2(1-x)^3}\log\frac{2x}{1+x}+\frac{1-5x}{16x^2(1-x)^2},$$

$$D_1=\frac{1}{h^2}(2\alpha-3z_1+z_2).$$

Now we have $\lim_{x\to1}\varphi_D(x)=1/48$ and $\lim_{x\to1}\varphi_E(x)=31/48$.

Next consider the condition $S'(a)=\alpha$. Then instead of (6.28) we get

$$-\frac{2}{\delta_1\varepsilon_1}(\lambda_1+\gamma_1\log\alpha_1)+\frac{2}{\delta_1\eta_1}(-\rho_1+\gamma_1\log\beta_1)-\frac{\gamma_1d_1^2}{\alpha_1^2}=\alpha$$

and this with $(6.20, 1)$, $(6.18, 2)$, $(6.19, 2)$, $(6.20, 2)$ admits a nontrivial combination

$$\begin{aligned}z_2-z_1-(h_1+h_2)\frac{\alpha}{2}=&\frac{1}{\varepsilon_1}\gamma_1\log\alpha_1+\frac{1}{h_2}\gamma_1\log\beta_1-\frac{1}{h_2}\gamma_2\log\alpha_2\\ &+(h_1+h_2)\frac{\gamma_1d_1^2}{2\alpha_1^2}+\frac{\varepsilon_2}{h_2}\Big(\frac{\gamma_1d_1}{\beta_1}-\varepsilon_2\frac{\gamma_1d_1^2}{2\beta_1^2}-\frac{\gamma_2d_2}{\alpha_2}+\varepsilon_2\frac{\gamma_2d_2^2}{2\alpha_2^2}\Big).\end{aligned}$$

Note that the consistency of $S(a)=\alpha$ and histogram means here that $(h_1+h_2)\alpha-(2h_1+h_2)z_1+h_1z_2>0$ and for $S'(a)=\alpha$ that $2(z_2-z_1)-(h_1+h_2)\alpha>0$. However, for $S''(a)=\alpha$ the consistency is the requirement $\alpha>0$.

In the case of uniform mesh this equation is

$$M_2\varphi_F\Big(\Big(\frac{M_2}{M_1}\Big)^{1/3}\Big)+M_3\varphi_E\Big(\Big(\frac{M_3}{M_2}\Big)^{1/3}\Big)=D_1 \tag{6.30}$$

where at this time

$$\varphi_F(x) = \frac{27}{16}\frac{1}{(1-x)^3}\log\frac{9x}{(2+x)^2} + \frac{3}{32}\frac{x^3+5x^2-8x+8}{x^2(1-x)^2},$$

$$D_1 = \frac{1}{h^2}(z_2 - z_1 - h\alpha).$$

We calculate $\lim_{x\to 1}\varphi_F(x) = 47/48$. The assertions 2), 3) of Lemma 6.1 take place for φ_D, φ_E and φ_F.

6.8 Existence of the histopolant

The quadratic/linear rational histopolant could be constructed if the values λ_i, ρ_i and M_i are known. These parameters satisfy the equations (6.18), (6.19), (6.20) and therefore the basic equations. Thus, for the existence of the histopolant, it is necessary and sufficient that the system of basic (nonlinear) equations and then a linear system determining λ_i, ρ_i have a solution.

Suppose the system of basic equations has the solution, it is unique due to the uniqueness of the histopolant. Let us derive an appropriate system to determine the values λ_i, ρ_i.

Consider the equations (6.18) and (6.19). The combination (6.18) + ε_i(6.19) eliminates ρ_i, the combination (6.18) $-$ η_{i-1}(6.19) eliminates λ_{i-1} and we obtain the equations

$$-\frac{h_i}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1} + \frac{\delta_{i-1}+h_i}{\delta_{i-1}\eta_{i-1}}\rho_{i-1} - \frac{1}{\varepsilon_i}\lambda_i = p_i, \tag{6.31}$$

$$\frac{1}{\eta_{i-1}}\rho_{i-1} - \frac{h_i+\delta_i}{\delta_i\varepsilon_i}\lambda_i + \frac{h_i}{\delta_i\eta_i}\rho_i = q_i \tag{6.32}$$

with the known right hand sides p_i and q_i. We take the equations $(6.32, 2)$, $(6.31, 3), \ldots, (6.31, n-1)$ as interior equations of a three-diagonal system for unknowns in ordering ρ_1, λ_2, $\rho_2, \ldots, \rho_{n-2}$, λ_{n-1}. We get the first equation in this system as $(6.18, 2) + \varepsilon_2(6.19, 2)$ of the form

$$\frac{\delta_1+h_2}{\delta_1\eta_1}\rho_1 - \frac{1}{\varepsilon_2}\lambda_2 = p_2 \tag{6.33}$$

and the last one as $(6.18, n-1) - \eta_{n-2}(6.19, n-1)$ of the form

$$\frac{1}{\eta_{n-2}}\rho_{n-2} - \frac{h_{n-1}+\delta_{n-1}}{\delta_{n-1}\varepsilon_{n-1}}\lambda_{n-1} = q_{n-1} \tag{6.34}$$

where p_2 and q_{n-1} contain the known values λ_1 and ρ_{n-1}, respectively.

Proposition 6.1. *The system (6.31)–(6.34) has unique solution.*

Proof. We use the standard elimination to show that the matrix of the system is regular. With the help of (6.33) eliminate ρ_1 in (6.32, 2), then the coefficient of λ_2 will be $h_2(\delta_1+\delta_2+h_2)/(\varepsilon_2\delta_2(\delta_1+h_2)) \neq 0$ and the coefficient of ρ_2 does not change. With the help of this equation eliminate then in (6.31, 3) λ_2, the coefficient of ρ_2 will be $(\delta_1+\delta_2+h_2+h_3)/(\eta_2(\delta_1+\delta_2+h_2)) \neq 0$. Continuing by the induction, we have the coefficients of λ_i (diagonal entries of transformed triangular matrix)

$$\frac{h_i}{\varepsilon_i\delta_i}\frac{\sum\limits_{k=1}^{i}\delta_k+\sum\limits_{k=2}^{i}h_k}{\sum\limits_{k=1}^{i-1}\delta_k+\sum\limits_{k=2}^{i}h_k} \neq 0$$

and of ρ_i

$$\frac{1}{\eta_i}\frac{\sum\limits_{k=1}^{i}\delta_k+\sum\limits_{k=2}^{i+1}h_k}{\sum\limits_{k=1}^{i}\delta_k+\sum\limits_{k=2}^{i}h_k} \neq 0$$

which completes the proof. □

Let us turn to the analysis of basic equations in uniform mesh case. Denote $M=(M_1,\dots,M_n)$, then the equation (6.23) is, in fact, $\Phi_i(M)=D_i$. We calculate

$$\begin{aligned}
\frac{\partial\Phi_i}{\partial M_{i-2}} =&\varphi_A\Big(\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{1/3}\Big)+\frac{1}{3}\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{1/3}\varphi_A'\Big(\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{1/3}\Big),\\
\frac{\partial\Phi_i}{\partial M_{i-2}} =&\varphi_B\Big(\Big(\frac{M_{i-1}}{M_{i}}\Big)^{1/3}\Big)+\frac{1}{3}\Big(\frac{M_{i-1}}{M_{i}}\Big)^{1/3}\varphi_B'\Big(\Big(\frac{M_{i-1}}{M_{i}}\Big)^{1/3}\Big)\\
&-\frac{1}{3}\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{4/3}\varphi_A'\Big(\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{1/3}\Big),\\
\frac{\partial\Phi_i}{\partial M_{i}} =&-\frac{1}{3}\Big(\frac{M_{i-1}}{M_{i}}\Big)^{4/3}\varphi_B'\Big(\Big(\frac{M_{i-1}}{M_{i}}\Big)^{1/3}\Big)-\frac{1}{3}\Big(\frac{M_{i+1}}{M_{i}}\Big)^{4/3}\varphi_B'\Big(\Big(\frac{M_{i+1}}{M_{i}}\Big)^{1/3}\Big),\\
\frac{\partial\Phi_i}{\partial M_{i+1}} =&\varphi_B\Big(\Big(\frac{M_{i+1}}{M_{i}}\Big)^{1/3}\Big)+\frac{1}{3}\Big(\frac{M_{i+1}}{M_{i}}\Big)^{1/3}\varphi_B'\Big(\Big(\frac{M_{i+1}}{M_{i}}\Big)^{1/3}\Big)\\
&-\frac{1}{3}\Big(\frac{M_{i+2}}{M_{i+1}}\Big)^{4/3}\varphi_A'\Big(\Big(\frac{M_{i+2}}{M_{i+1}}\Big)^{1/3}\Big),\\
\frac{\partial\Phi_i}{\partial M_{i+2}} =&\varphi_A\Big(\Big(\frac{M_{i+2}}{M_{i+1}}\Big)^{1/3}\Big)+\frac{1}{3}\Big(\frac{M_{i+2}}{M_{i+1}}\Big)^{1/3}\varphi_A'\Big(\Big(\frac{M_{i+2}}{M_{i+1}}\Big)^{1/3}\Big).
\end{aligned}$$

Basing on Lemma 6.1 we conclude that $(\partial\Phi_i/\partial M_j)(M)>0$, $j=i-2,\dots,i+2$. Similar assertions hold for (6.27), (6.29) and (6.30). This means that all functions Φ_i are increasing by each argument. The whole system for boundary conditions (6.3) is $\Phi(M)=D$ with $\Phi(M)=(\Phi_2(M),\dots,\Phi_{n-1}(M))$ and $D=(D_2,\dots,D_{n-1})$.

For (6.4) and (6.5) Φ and D contain two more components. Write $\Phi(M) = D$ in equivalent form $M = \Psi(M)$ introducing $\Psi(M) = M + \gamma(\Phi(M) - D)$, $\gamma \neq 0$. We intend to use Bohl-Brouwer fixed point theorem and try to find numbers r and R such that $R > r > 0$ and $M \in [r, R]^n$ implies $\Psi(M) \in [r, R]^n$. More generally, look for r_i and R_i, $R_i > r_i > 0$, implying Ψ mapping the set $K = \prod_{i=1}^n [r_i, R_i]$ into itself.

Observe that, for $M_i = R_i$, $\Psi_i(M) = R_i + \gamma(\Phi_i(M) - D_i) > R_i$ if $\gamma > 0$ and $\Phi_i(M) > D_i$. The last inequality is very natural to hold. Similarly, for $M_i = r_i$, $\Psi_i(M) = r_i + \gamma(\Phi_i(M) - D_i) < r_i$ if $\gamma > 0$ and $\Phi_i(M) < D_i$. Consequently, the natural choice is $\gamma < 0$ and the actual value will be chosen later.

We analyze equations (6.23) in more details, the reasoning for near-boundary is very similar. Taking $M \in K$, it holds $\Psi_i(M) = M_i + \gamma(\Phi_i(M) - D_i) \leqslant R_i$ if and only if

$$\max_{M \in K}(M_i + \gamma(\Phi_i(M) - D_i)) \leqslant R_i. \tag{6.35}$$

As $\partial\Phi_i/\partial M_j > 0$, $j = i-2, \ldots, i+2$, it holds $\partial\Psi_i/\partial M_i > 0$ for sufficiently small $|\gamma|$ and $\partial\Psi_i/\partial M_j < 0$, $j = i-2, i-1, i+1, i+2$. Thus, (6.35) is equivalent to $\Psi_i(r_{i-2}, r_{i-1}, R_i, r_{i+1}, r_{i+2}) \leqslant R_i$ or

$$\begin{aligned} r_{i-2}\varphi_A\Big(\Big(\frac{r_{i-2}}{r_{i-1}}\Big)^{1/3}\Big) + r_{i-1}\varphi_B\Big(\Big(\frac{r_{i-1}}{R_i}\Big)^{1/3}\Big) &+ r_{i+1}\varphi_B\Big(\Big(\frac{r_{i+1}}{R_i}\Big)^{1/3}\Big) \\ &+ r_{i+2}\varphi_A\Big(\Big(\frac{r_{i+2}}{r_{i+1}}\Big)^{1/3}\Big) \geqslant D_i. \end{aligned} \tag{6.36}$$

Similar calculations give that, for all $M \in K$, the inequality $\Psi_i(M) \geqslant r_i$ is satisfied if and only if

$$\begin{aligned} R_{i-2}\varphi_A\Big(\Big(\frac{R_{i-2}}{R_{i-1}}\Big)^{1/3}\Big) + R_{i-1}\varphi_B\Big(\Big(\frac{R_{i-1}}{r_i}\Big)^{1/3}\Big) &+ R_{i+1}\varphi_B\Big(\Big(\frac{R_{i+1}}{r_i}\Big)^{1/3}\Big) \\ &+ R_{i+2}\varphi_A\Big(\Big(\frac{R_{i+2}}{R_{i+1}}\Big)^{1/3}\Big) \leqslant D_i. \end{aligned} \tag{6.37}$$

It could be checked that it holds $\varphi_B(x) \geqslant 3/(4x^2)$ for $x \leqslant 1$ (actually, $\lim_{x\to 0_+} x^2\varphi_B(x) = 3/4$) and $x\varphi_B(x) \leqslant 2/5$ for $x \geqslant 3$. Basing on the just indicated behaviour of φ_B and $\varphi_A(1) = 1/24$ we see that the inequality (6.36) is satisfied if

$$\frac{1}{24}(\min\{r_{i-2}, r_{i-1}\} + \min\{r_{i+1}, r_{i+2}\}) + \frac{3}{4}(r_{i-1}^{1/3} + r_{i+1}^{1/3})R_i^{2/3} \geqslant D_i \tag{6.38}$$

and (6.37) holds if, e.g., $R_{i-1}/r_i \geqslant 3$, $R_{i+1}/r_i \geqslant 3$ and

$$\frac{1}{24}(\max\{R_{i-2}, R_{i-1}\} + \max\{R_{i+1}, R_{i+2}\}) + \frac{2}{5}r_i^{1/3}(R_{i-1}^{2/3} + R_{i+1}^{2/3}) \leqslant D_i. \tag{6.39}$$

Taking $r_i = r$ and $R_i = R$ for all i the inequality (6.38) turns to

$$\frac{1}{12}r + \frac{3}{2}r^{1/3}R^{2/3} \geqslant D_i \tag{6.40}$$

and (6.39) to

$$\frac{1}{12}R + \frac{4}{5}r^{1/3}R^{2/3} \leqslant D_i. \tag{6.41}$$

Analogously we establish the estimates for (6.27), (6.29) and (6.30).

The given analysis suggests that the system of basic equations may not have the solution. This is confirmed by

Proposition 6.2. *There is strictly convex histogram on uniform mesh with corresponding spline knots where quadratic/linear spline histopolant does not exist.*

Proof. Consider the basic equations $(6.23, i-1)$ and $(6.23, i)$. Suppose that the solution exists for arbitrary values $D_{i-1} > 0$, $D_i > 0$. Take, e.g., $D_{i-1} \to 0$, $D_i \to \infty$. Then at left hand side of $(6.23, i-1)$ all summands converge to zero. At the left hand side of $(6.23, i)$ at least one summand goes to infinity (by appropriate subsequence).

1) Let $M_{i-2}\varphi_A((M_{i-2}/M_{i-1})^{1/3}) \to \infty$.

If $M_{i-2} \leqslant$ const then $\varphi_A((M_{i-2}/M_{i-1})^{1/3}) \to \infty$ and $M_{i-2}/M_{i-1} \to 0$. It holds $\varphi_A((M_{i-2}/M_{i-1})^{1/3}) \sim (M_{i-2}/M_{i-1})^{-2/3}$ (the sign $\sim$ between terms means here that the quotient of these terms converges to some positive constant) and $M_{i-2}\varphi_A((M_{i-2}/M_{i-1})^{1/3}) \sim M_{i-2}^{1/3}M_{i-1}^{2/3} \to \infty$ which yields $M_{i-1} \to \infty$. Now in $(6.23, i-1)$ $M_{i-2}\varphi_B((M_{i-2}/M_{i-1})^{1/3}) \sim M_{i-2}^{1/3}M_{i-1}^{2/3} \to \infty$ which is a contradiction.

If $M_{i-2} \to \infty$ then $M_{i-1} \leqslant$ const yields $M_{i-2}\varphi_A((M_{i-2}/M_{i-1})^{1/3}) \sim M_{i-1} \to \infty$, contradiction. It remains $M_{i-1} \to \infty$. Then in $(6.23, i-1)$

$$M_{i-2}\varphi_B\Big(\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{1/3}\Big) \geqslant M_{i-2}\frac{\text{const}}{\Big(\frac{M_{i-2}}{M_{i-1}}\Big)^{1/3}} \sim M_{i-2}^{2/3}M_{i-1}^{1/3} \to \infty,$$

which is again contradiction.

2) Suppose $M_{i-1}\varphi_B((M_{i-1}/M_i)^{1/3}) \to \infty$.

If $M_{i-1} \leqslant$ const then $\varphi_B((M_{i-1}/M_i)^{1/3}) \to \infty$ and $M_{i-1}/M_i \to 0$ with $\varphi_B((M_{i-1}/M_i)^{1/3}) \sim (M_{i-1}/M_i)^{-2/3}$. But in this case $M_{i-1}\varphi_B((M_{i-1}/M_i)^{1/3}) \sim M_{i-1}^{1/3}M_i^{2/3} \to \infty$. In the equation $(6.23, i-1)$, due to $M_i/M_{i-1} \to \infty$, we get $M_i\varphi_B((M_i/M_{i-1})^{1/3}) \sim M_{i-1}^{1/3}M_i^{2/3} \to \infty$, contradiction.

If $M_{i-1} \to \infty$ then $M_i \leqslant$ const implies the convergence $M_{i-1}/M_i \to \infty$ and $M_{i-1}\varphi_B((M_{i-1}/M_i)^{1/3}) \sim M_{i-1}^{2/3}M_i^{1/3} \to \infty$. In $(6.23, i-1)$, due to $M_i/M_{i-1} \to 0$, $M_i\varphi_B((M_i/M_{i-1})^{1/3}) \sim M_{i-1}^{2/3}M_i^{1/3} \to \infty$ which is impossible. If $M_i \to \infty$ then in $(6.23, i-1)$ we have

$$M_i\varphi_B\Big(\Big(\frac{M_i}{M_{i-1}}\Big)^{1/3}\Big) \geqslant M_i\frac{\text{const}}{\Big(\frac{M_i}{M_{i-1}}\Big)^{1/3}} \sim M_{i-1}^{1/3}M_i^{2/3} \to \infty,$$

which is contradiction.

Clearly, two other summands in $(6.23, i)$ could be treated in the same way. Note that the structure of the proof could be used as well in the case of D_{i-1} bounded and $D_i \to \infty$. The proof is complete. $\square$

Despite of quite restrictive sufficient conditions of existence (6.36) and (6.37) they are satisfied if we histopolate a strictly convex function $f \in C^2[a,b]$ ($f''(x) > 0$ for all $x \in [a,b]$) and calculate the histogram heights $z_i = h_i^{-1} \int_{x_{i-1}}^{x_i} f(x)dx$, $i = 1, \ldots, n$. Then in general case (equation (6.21)) the Taylor expansion allows to get

$$D_i = \frac{1}{2}(h_{i-1} + h_i)(h_i + h_{i+1})h_i f''(\xi_i) + o(\bar{h}_i^3)$$

where $\bar{h}_i = \max\{h_{i-1}, h_i, h_{i+1}\}$. In uniform case (equation (6.23)) rescaled values are $D_i = 2f''(\xi_i) + o(1)$. Sufficiently small h assures the quotients D_i/D_{i-1} to be approximately equal to 1 and (6.36), (6.37) or (6.38), (6.39) are satisfied with reasonable gaps between r_i and R_i.

Remark 6.1. *The representation (6.1) does not include quadratic polynomial pieces of S. With this the case of $S''(x) = \text{const} \neq 0$, $x \in [\xi_i, \xi_{i+1}]$, is excluded. The simplest way here is to allow additionally on some intervals the representation*

$$S(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2, \quad x \in [\xi_i, \xi_{i+1}].$$

Then we obtain $c_i = M_i/2$ and

$$\begin{aligned} b_i &= \frac{2(\rho_i\varepsilon_i - \lambda_i\eta_i)}{\varepsilon_i\eta_i\delta_i} + \frac{M_i}{3}(\varepsilon_i - \eta_i), \\ a_i &= \frac{1}{\delta_i}\Big(\frac{\eta_i}{\varepsilon_i}\lambda_i + \frac{\varepsilon_i}{\eta_i}\rho_i\Big) - \frac{M_i}{6}\varepsilon_i\eta_i \end{aligned}$$

representing S via unknowns λ_i, ρ_i, M_i. In equations (6.18) and (6.19) the left hand sides remain unchanged which gives the validity of all results concerning the determination of the values λ_i, ρ_i. However, basic equations contain this situation as limit case with the values of functions $\varphi_A, \varphi_B, \ldots, \varphi_F$ in the point 1, corresponding to the equality of neighbouring values M_i.

Chapter 7

Numerical tests

7.1 Monotonicity preserving spline histopolation

First, we histopolated the function $f(x) = \sin x$ on the interval $[0, 1]$ with uniform mesh using (2.29) and end conditions $S'(0) = f'(0)$, $S'(1) = f'(1)$. The system (2.32) was solved by Newton's method with starting values for m_i as $\cos x_i + 0.1$ and iterations were stopped at $\|m^k - m^{k-1}\|_\infty < 10^{-7}$. Actually, 4–5 steps were needed. We calculated the error $\|S - f\|_\infty$ approximately as $\max_{0 \leqslant i \leqslant 10n} |(S - f)(i/(10n))|$. The results are presented in Table 7.1 and confirm the rate of convergence $O(h^3)$.

Tabel 7.1. Errors of histopolation of the function $f(x) = \sin x$ on the interval $[0, 1]$

n	4	8	16	32	64
$\|S - f\|_\infty$	$8.17 \cdot 10^{-4}$	$1.07 \cdot 10^{-4}$	$1.36 \cdot 10^{-5}$	$1.70 \cdot 10^{-6}$	$2.13 \cdot 10^{-7}$

In the second example taken from [47] we histopolated with uniform mesh on the interval $[0, 7]$ the values z_i as 5.78, 3.51, 2.11, 1.27, 0.75, 0.49, 0.29, using end conditions $S'(0) = -2.6$, $S'(7) = -0.19$. Newton's method for solving the system (2.32) was started from values $m_1 = -2.5$, $m_2 = -2.2$, $m_3 = -2$, $m_4 = -1.7$, $m_5 = -1.5$, $m_6 = -1.2$. The histopolant is shown in Figure 7.1 and is practically coinciding with that of [47].

In the next example, we used two different meshes for the same histogram heights z_i written in Table 7.2.

Tabel 7.2. Histogram heights and meshes used in Figs. 7.2 and 7.3.

x_i	0		2		2.5		3		4		5
	0		2.4		2.5		2.6		4		5
z_i		1.25		2.5		2.6		3		11	

In both cases, as boundary conditions we took $S'(0) = 1$, $S'(5) = 5$ and starting values were $m_1 = 1.1$, $m_2 = 1.2$, $m_3 = 1.3$, $m_4 = 1.4$. The histopolants are represented in Figures 7.2 and 7.3.

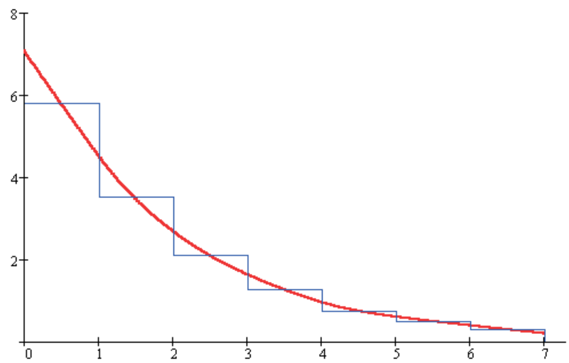

Fig. 7.1. Linear/linear rational spline histopolant for the data taken from [47].

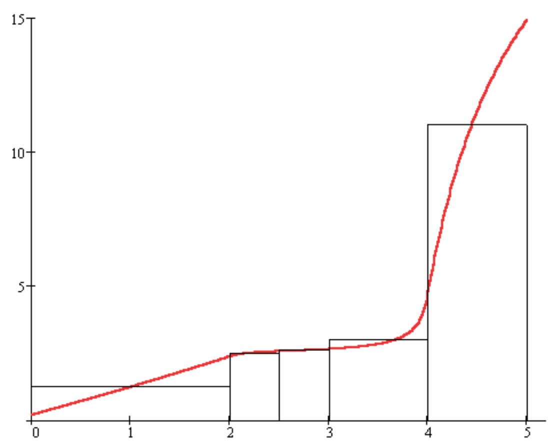

Fig. 7.2. Histogram and linear/linear rational histopolant corresponding to the first mesh in Table 7.2.

We considered an example inspired by classical Akima's test data [5] for interpolation. On uniform mesh $x_i = i$, $i = 0, \ldots, 5$, we took histogram heights $z_1 = 10$, $z_2 = 10.1$, $z_3 = 100$, $z_4 = 100.1$, $z_5 = 101$. The heights z_i were interpolated linearly at points 1/2 and 3/2 to get $\alpha = S(x_0)$ and similarly $\beta = S(x_5)$. Fixed $\delta_1 = z_1 - \alpha$ and $\delta_5 = \beta - z_5$ were then used in ordinary iterations with initial values calculated by (2.36) and also with $m_0^0 = m_1^0$ and $m_5^0 = m_4^0$. Ordinary iterations converged to $m_0 = 0.371$, $m_1 = 0.0116$, $m_2 = 1.83 \cdot 10^5$, $m_3 = 0.0106$, $m_4 = 2.62$, $m_5 = 0.555$.

In the third example we also took the mesh $x_i = i$, $i = 0, \dots, 5$, and the histogram heights $z_1 = 100$, $z_2 = 200$, $z_3 = 300$, $z_4 = 400$, $z_5 = 500$. The initial values were taken as $m_0 = 100$ (fixed), $m_1^0 = 1$, $m_2^0 = 101$, $m_3^0 = 1$, $m_4^0 = 101$, $m_5 = 100$ (fixed). The matrix $\Phi'(m^0)$ has the (maximal by modulus) eigenvalue 7.66 which is the spectral radius of $\Phi'(m^0)$. Thus, Φ could not be contractive in the neighborhood of m^0 with respect to some norm in $\mathbf{R}^{n+1}$.

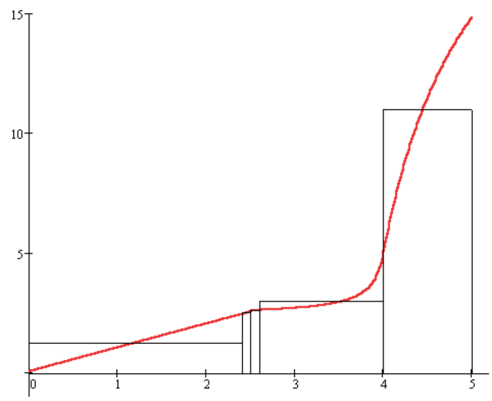

Fig. 7.3. Histogram and linear/linear rational histopolant corresponding to the second mesh in Table 7.2.

All considered examples were tested with ordinary iterations and Seidel's method. Linear convergence with a quotient between 1/2 and 2/3 was observed. This rate determines the number of steps depending directly on required precision. Aitken's transform accelerated considerably the convergence but Newton's method nearby the solution being easy to implement, could be recommended.

7.2 Comonotone shape-preserving histopolation

For spline S representation with equations (2.14) and (4.1) on rational and quadratic intervals we need parameters m_i. We can find them from the system which contains equations of type (2.25), (4.5), (4.7) or (4.8) with corresponding boundary conditions. To solve this system we can successfully use Newton's method [16, 18].

Now we look at some examples. In the first example we have a weak alternation of the data. We take the mesh $x_0 = 0$, $x_1 = 1$, $x_2 = 1.9$, $x_3 = 2.8$, $x_4 = 4$, $x_5 = 4.9$, $x_6 = 6.2$, $x_7 = 7.5$ and histogram heights are $z_1 = 2$, $z_2 = 3$, $z_3 = z_4 = z_5 = 9$, $z_6 = 5$, $z_7 = 2$. We use boundary conditions $S'(0) = 1$ and $S'(7.5) = -2$. In this case the intervals $[x_2, x_3]$, $[x_3, x_4]$ and $[x_4, x_5]$ are quadratic, all remaining intervals are rational. Figure 7.4 shows plots of corresponding histospline.

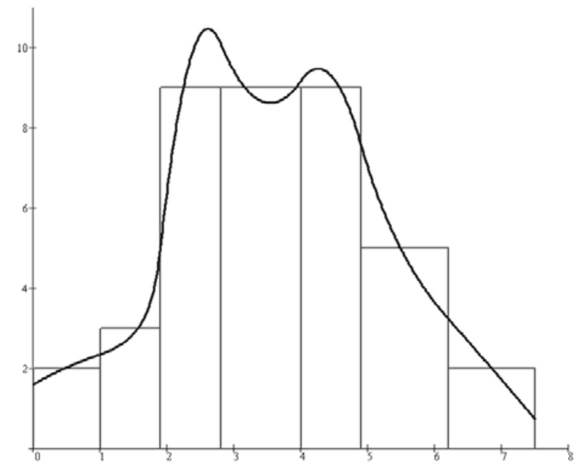

Fig. 7.4. Histogram and comonotone shape-preserving histospline.

Next we present four numerical examples without weak alternation of data.

In the second example we have the mesh $x_0 = 0$, $x_1 = 1$, $x_2 = 1.9$, $x_3 = 2.8$, $x_4 = 4$, $x_5 = 4.9$, $x_6 = 6.2$ and histogram heights $z_1 = 2$, $z_2 = 3$, $z_3 = z_4 = 7$, $z_5 = 6$, $z_6 = 4$ with boundary conditions $S'(0) = 1$ and $S'(6.2) = -1$. We use the modified comonotone strategy to find the corresponding shape-preserving spline and we choose $[x_2, x_3]$ to be rational. Then $[x_3, x_4]$ is quadratic interval and all others are rational intervals. Corresponding spline is on Figure 7.5.

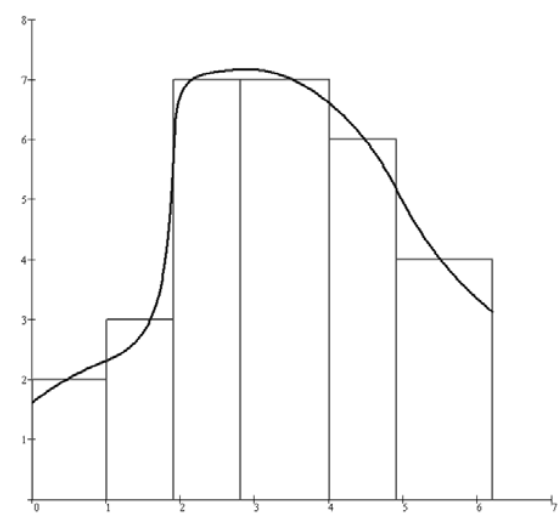

Fig. 7.5. Histogram and modified comonotone shape-preserving histospline.

For the third example we take the mesh $x_0 = 0$, $x_1 = 1$, $x_2 = 1.9$, $x_3 = 2.8$, $x_4 = 4$, $x_5 = 4.9$, $x_6 = 6.2$, $x_7 = 7.5$ and histogram heights are $z_1 = 2$, $z_2 = 3$, $z_3 = z_4 = z_5 = 7$, $z_6 = 8$, $z_7 = 10$ and boundary conditions $S'(0) = 1$ and $S'(7.5) = 12$. We choose $[x_3, x_4]$ and $[x_4, x_5]$ to be quadratic and remaining intervals to be rational (interval $[x_2, x_3]$ is chosen to be rational because of modified comonotone shape-preserving strategy). This spline is presented in Figure 7.6. Figure 7.6 shows the growth of absolute values of m_i on the quadratic section as predicted by the theory.

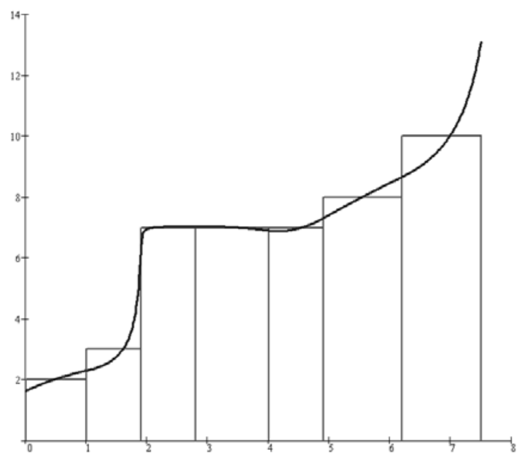

Fig. 7.6. Histogram and modified comonotone shape-preserving histospline.

In the fourth example we have mesh $x_0 = 0$, $x_1 = 1$, $x_2 = 1.9$, $x_3 = 2.8$, $x_4 = 4$, $x_5 = 4.9$, $x_6 = 6.2$, $x_7 = 7.5$, $x_8 = 8.5$ and histogram heights $z_1 = 2$, $z_2 = 6$, $z_3 = z_4 = z_5 = z_6 = 7$, $z_7 = 5$, $z_8 = 2$. We use boundary conditions $S'(0) = 1$ and $S'(8.5) = -1$. Histospline which corresponds to this data is shown in Figure 7.7.

Finally, for the mesh $x_0 = 0$, $x_1 = 1$, $x_2 = 1.9$, $x_3 = 2.8$, $x_4 = 4$, $x_5 = 4.9$, $x_6 = 6.2$ we chose histogram heights $z_1 = 1$, $z_2 = 4.7$, $z_3 = z_4 = 7$, $z_5 = 6.75$, $z_6 = 3$. This histospline is plotted in Figure 7.8. Spline uses constant function on intervals $[x_2, x_3]$ and $[x_3, x_4]$, linear/linear rational function on $[x_0, x_1]$ and $[x_5, x_6]$ and cubic polynomial on $[x_1, x_2]$ and $[x_4, x_5]$. Boundary conditions are $S'(0) = 1$ and $S'(6.2) = -1$ with appropriate C^1 smoothness in points x_2 and x_4. Calculated values $m_1 = 2.801$ and $m_5 = -7.549$ correspond to monotone and nonmonotone shape of cubic polynomials (because of condition $0 \leqslant m_{i-2} \leqslant 12\delta_{i-1}/h_{i-1}$).

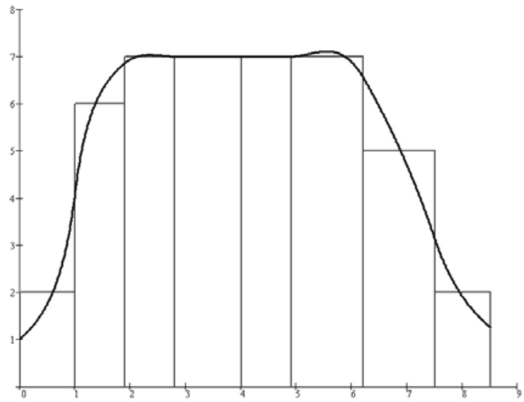

Fig. 7.7. Histogram and modified comonotone shape-preserving histospline.

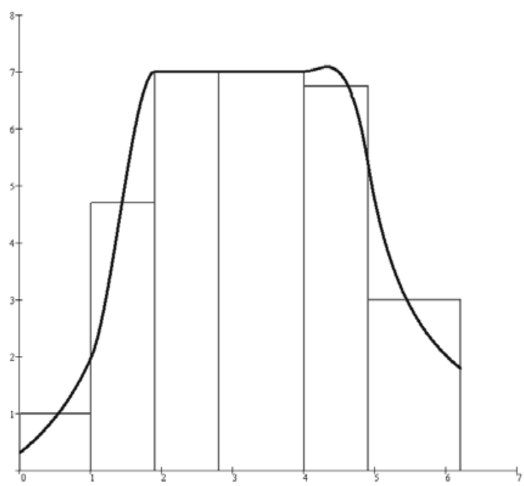

Fig. 7.8. Histogram and C^1 smooth histospline with constant part.

7.3 Convexity preserving histopolation

We histopolated the functions $f(x) = e^x$ and $f(x) = x^4$ for $x \in [-2, 2]$ on uniform mesh. Histogram heights were calculated by $z_i = h_i^{-1} \int_{x_{i-1}}^{x_i} f(x)dx$, $i = 1, \ldots, n$. Boundary conditions $S''(a) = f''(a)$ and $S''(b) = f''(b)$ were used. Numerical results are presented in Tables 7.3 and 7.4, histopolants in Figures 7.9 and 7.10.

Table 7.3. Histogram heights and spline parameters for $f(x) = e^x$, $n = 8$

x_i	-2		-1.5		-1		-0.5		0		0.5		1		1.5		2
ξ_i	-2			-1.25		-0.75		-0.25		0.25		0.75		1.25			2
z_i		0.176		0.289		0.477		0.787		1.297		2.139		3.527		5.815	
λ_i		0.088		0.081		0.134		0.221		0.365		0.601		0.992		0	
ρ_i	0		0.063		0.105		0.172		0.284		0.469		0.772		2.907		
M_i	0.135			0.29		0.475		0.783		1.297		2.117		3.551			7.389

Table 7.4. Histogram heights and spline parameters for $f(x) = x^4$, $n = 8$

x_i	-2		-1.5		-1		-0.5		0		0.5		1		1.5		2
ξ_i	-2			-1.25		-0.75		-0.25		0.25		0.75		1.25			2
z_i		9.762		2.638		0.388		0.013		0.013		0.388		2.638		9.762	
λ_i		4.881		0.408		0.047		-0.001		0.011		0.151		0.912		0	
ρ_i	0		0.912		0.151		0.0112		-0.001		0.047		0.408		4.881		
M_i	48			19.98		7.944		0.782		0.781		7.946		19.98			48

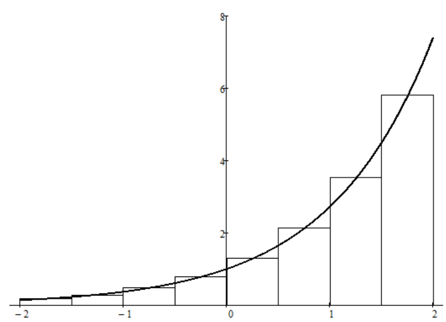

Fig. 7.9. Histogram and shape-preserving histospline corresponding to the data in Table 7.3.

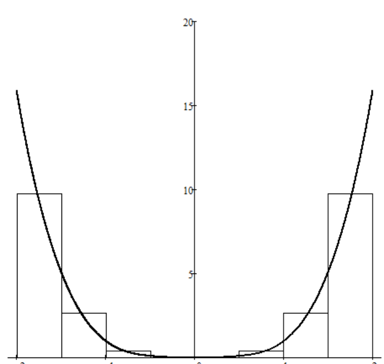

Fig. 7.10. Histogram and shape-preserving histospline corresponding to the data in Table 7.4.

We tested the solution of basic equations on uniform mesh where in equations (6.23) the right hand sides were taken alternately $D_i = c$, $D_{i+1} = c^{-1}$, $D_{i+2} = c$ etc. Boundary values were $M_1 = M_n = 1$ and $n = 8$. The Mathcad package solved the system for $c = 1.8$ but not for $c = 1.9$. This is in consistence with the estimates (6.40) and (6.41) confirming that they are quite adequate.

7.4 Convergence rate of monotonicity preseving spline histopolation

We histopolated on the interval $[0, 1]$ the function $f(x) = \sin x$ to confirm the highest theoretical rate $O(h^3)$ and also the piecewise quadratic function

$$f(x) = \begin{cases} -\dfrac{x^2}{2} + x & \text{for} \quad 0 \leqslant x \leqslant \dfrac{1}{2}, \\ \dfrac{x^2}{2} + \dfrac{1}{4} & \text{for} \quad \dfrac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

having $f' \in \operatorname{Lip} 1$. However, the last function is such that $f'' \in \operatorname{Lip} \alpha$ does not hold for no one $\alpha \in (0, 1]$. Thus, here the rate $O(h^2)$ coincides with those predicted by Theorem 3.1.

The mesh was nonuniform of the following form. Taking $h = 1/n$, central knots were calculated as

$$x_{\frac{n}{2}} = \frac{1+h}{2}, \quad x_{\frac{n}{2}-1} = x_{\frac{n}{2}} - h,$$

$$x_{\frac{n}{2}-2} = x_{\frac{n}{2}-1} - h, \quad x_{\frac{n}{2}+1} = x_{\frac{n}{2}} + \frac{h}{10}.$$

Another ones were spaced uniformly on rest parts of the interval, i.e.,

$$x_i = i \frac{x_{\frac{n}{2}-2}}{\frac{n}{2} - 2}, \quad i = 1, \ldots, \frac{n}{2} - 3,$$

$$x_{\frac{n}{2}+1+i} = x_{\frac{n}{2}+1} + i \frac{1 - x_{\frac{n}{2}+1}}{\frac{n}{2} - 1}, \quad i = 1, \ldots, \frac{n}{2} - 2.$$

We used the boundary conditions (2.3) with $\alpha = f'(x_0)$ and $\beta = f'(x_n)$.

The "tridiagonal" nonlinear system to determine the values of m_i consisting of equations (2.25) was solved by Newton's method.

The errors $\|S - f\|_\infty$ were calculated approximately on tenfold refined grid as

$$\varepsilon_n = \max_{1 \leqslant i \leqslant n} \max_{0 \leqslant k \leqslant 10} |(S - f)(x_{i-1} + kh_i/10)|.$$

Results of numerical tests are presented in Tables 7.5 and 7.6.

Table 7.5. Numerical results for $f(x) = \sin x$.

n	8	16	32	64	128
ε_n	$1.15 \cdot 10^{-4}$	$1.46 \cdot 10^{-5}$	$1.84 \cdot 10^{-6}$	$2.30 \cdot 10^{-7}$	$2.87 \cdot 10^{-8}$
$\varepsilon_{n/2}/\varepsilon_n$		7.874	7.961	7.988	7.996

Table 7.6. Numerical results for piecewise quadratic function.

n	8	16	32	64	128
ε_n	$7.39 \cdot 10^{-4}$	$1.79 \cdot 10^{-4}$	$4.41 \cdot 10^{-5}$	$1.09 \cdot 10^{-5}$	$2.71 \cdot 10^{-6}$
$\varepsilon_{2/n}/\varepsilon_n$		4.117	4.071	4.040	4.021

7.5 Convergence rate of comonotone shape-preseving spline histopolation

We histopolated the functions $f(x) = x^2 \operatorname{sgn} x, f(x) = x^3, f(x) = x^2$ and $f(x) = |x|^3$ on the interval $[-1, 1]$. For the first and the third functions we obtain the convergence rate $O(h^2)$ and for others the rate $O(h^3)$. In our tests we used the uniform mesh, for the third function we also used a nonuniform mesh, where central knots were calculated as $x_{(n+1)/2} = h/2$, $x_{(n+1)/2-1} = -h/2$, $x_{(n+1)/2-2} = -3h/2$, $x_{(n+1)/2+1} = 3h/5$. Other knots were spaced uniformly on rest parts of the interval, i.e., $x_i = x_0 + i(x_{(n+1)/2-2} - x_0)/((n+1)/2-2), \quad i = 1, \dots, (n+1)/2-3$ and $x_{(n+1)/2+1+i} = x_n - (n-1)(x_n - x_{(n+1)/2+1})/((n+1)/2-2), \quad i = 1, \dots, n-1$. For the function $f(x) = x^2$ we made tests for both, n odd and n even. Other functions were tested only for n odd or for only n even and on uniform mesh. Selection of the subinterval was made by comonotone shape-preserving strategy in case of n odd and by modified comonotone shape-preserving strategy in case of n even.

We used the boundary conditions $\alpha = f'(x_0)$ and $\beta = f'(x_n)$. The approximates to the errors $\|S - f\|_\infty$ were calculated on ten times refined grid as

$$\varepsilon_n = \max_{1 \leqslant i \leqslant n} \max_{1 \leqslant k \leqslant 10} |S(x_{i-1} + \frac{kh}{10}) - f(x_{i-1} + \frac{kh}{10})|.$$

Nonlinear system of $m_i, i = 0, \dots, n$, was solved by Newton's method. Results of numerical test are presented in the next tables.

Table 7.7. Numerical results for $f(x) = x^2,\ x \in [-1, 1]$, uniform mesh, n odd

n	5	15	45	135	405
ε_n	7.34×10^{-3}	8.39×10^{-4}	9.33×10^{-5}	1.04×10^{-5}	1.15×10^{-6}
$\varepsilon_{n/3}/\varepsilon_n$		8.741958	9.000082	9.000000	9.000000

Table 7.8. Numerical results for $f(x) = x^2, \quad x \in [-1, 1]$, uniform mesh, n even

n	8	16	32	64	128
ε_n	7.88×10^{-3}	1.97×10^{-3}	4.93×10^{-4}	1.23×10^{-4}	3.08×10^{-5}
$\varepsilon_{n/2}/\varepsilon_n$		3.998068	3.999997	4.000000	4.000000

Table 7.9 Numerical results for $f(x) = x^2, \quad x \in [-1, 1]$, nonuniform mesh, n odd

n	5	15	45	135	405
ε_n	3.95×10^{-2}	9.44×10^{-4}	1.05×10^{-4}	1.16×10^{-5}	1.29×10^{-6}
$\varepsilon_{n/3}/\varepsilon_n$		41.790849	9.012392	9.003344	9.001046

Table 7.10 Numerical results for $f(x) = x^3, \quad x \in [-1, 1]$, uniform mesh, n even

n	8	16	32	64	128
ε_n	3.70×10^{-3}	5.23×10^{-4}	6.81×10^{-5}	8.66×10^{-6}	1.09×10^{-6}
$\varepsilon_{n/2}/\varepsilon_n$		7.086933	7.681549	7.856177	7.931600

Table 7.11 Numerical results for $f(x) = |x^3|, \quad x \in [-1, 1]$, uniform mesh, n odd

n	5	15	45	135	405
ε_n	1.27×10^{-2}	6.32×10^{-4}	2.47×10^{-5}	9.31×10^{-7}	3.47×10^{-8}
$\varepsilon_{n/3}/\varepsilon_n$		20.130665	25.566869	26.557522	26.858469

Table 7.12 Numerical results for $f(x) = x^2 \operatorname{sgn} x, \quad x \in [-1, 1]$, uniform mesh, n even

n	8	16	32	64	128
ε_n	4.39×10^{-3}	1.10×10^{-3}	2.75×10^{-4}	6.86×10^{-5}	1.72×10^{-5}
$\varepsilon_{n/2}/\varepsilon_n$		3.998591	3.999998	4.000000	4.000000

These numerical results are completely in concordance with theoretical ones.

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Sisukokkuvõte

Ratsionaalsplainidega histopoleerimine

Funktsioonide taastamisel mõõtmistulemuste või katseandmete põhjal on praktikas esinevatel juhtudel tihti oluline, et säilitataks algandmete geomeetrilisi omadusi: positiivsust (mittenegatiivsust), monotoonsust, kumerust. On üldiselt teada, et siledad polünomiaalsed interpoleerivad splainid ei säilita lähteandmete geomeetrilisi omadusi. Klassikaline näide selle kohta on funktsioon $f(x) = 1/x^2$, $x \in [-2, -0.2]$, ja tema interpoleerimine kuupsplainidega [34]. See funktsioon on ka näide sellest, kuidas andmete positiivsus, monotoonsus ja kumerus ei säili ruutsplainidega interpoleerimisel [31], ja seotuse tõttu histopoleerimisülesandega ei säilitata geomeetrilisi omadusi ka histopoleerimisel.

Kui meil on ülesandeks konstrueerida monotoonne histopolant, siis üks võimalus on moodustada ekvivalentne interpolatsiooniülesanne ja selle lahendi S tuletis S' võtta histopolandiks T. Küllaldase sileduse korral on T monotoonselt kasvav parajasti siis, kui $T'(x) \geqslant 0$, $x \in [a, b]$, mis on samaväärne sellega, et $S''(x) \geqslant 0$, $x \in [a, b]$, ehk S on kumer. Seega saab monotoonse histopolandi leidmise ülesande taandada kumera interpolandi leidmise ülesandele. Kumeraks interpolandiks sobib näiteks ruut/lineaar ratsionaalfunktsioonidest koosnev klassi C^2 kuuluv splain [46]. Kuid ruut/lineaar ratsionaalfunktsiooni tuletis ei pruugi olla lineaar/lineaar ratsionaalfunktsioon, seepärast ei saada selliselt käesolevas doktoritöös kirjeldatud meetodit. Lisaks sellele ei ole ruut/lineaar ratsionaalfunktsioonidest koosnevate splainidega interpolatsiooniülesannet kergem lahendada kui lineaar/lineaar splainidega histopoleerimisülesannet. Sarnast arutluskäiku võime läbi viia ka kumeruse säilitamiseks.

Käesolev doktoritöö koosneb seitsmest peatükist. Esimeses peatükis antakse lühike ülevaade tööst, ratsionaalsplainide ajaloost ning tutvustatakse varasemaid töid uuritava valdkonna kohta.

Teine ja kolmas peatükk baseeruvad artiklitel [16] ja [17]. Need peatükid on lisatud, et töö moodustaks iseseisva terviku. Teises peatükis on artiklis kirjutatule lisaks tõestatud mõned artiklis esitatud väited ja laiendatud mõningate võrrandite tuletamist. Selles peatükis tutvustatakse monotoonsust säilitavat lineaar/lineaar ratsionaalsplainidega histopoleerimist. Lineaar/lineaar ratsionaalsplaini esitus on antud esimeste momentide $m_i = S'(x_i)$ ja histogrammi kõrguste z_i kaudu. Töös on tõestatud selliselt esitatud ratsionaalsplaini ühesus ja olemasolu. Peatüki lõpus antakse esmane hinnang meetodi koonduvuskiirusele ning kolmandas peatükis on seda hinnangut täpsustatud. Saadud koonduvuskiirus oleneb histopoleeritava funktsiooni või selle tuletise Lipschitzi tingimuse täidetusest.

Neljandas peatükis on uuritud monotoonsuse säilitamist adaptiivsel juhul. Peatükis esitatud meetod kasutab splaini konstrueerimiseks osalõikudel kas lineaar/lineaar ratsionaalfunktsioone või ruutpolünoome, vastavalt valitud osalõigu tüübile. Splaini esitus on antud kasutades histogrammi kõrguseid ja splaini S esimesi tuletisi splaini sõlmedes. Selline adaptiivne splain eksisteerib ja on ühene igasuguste andmete korral, millel on olemas nõrga alterneerimise omadus. Kui andmetel pole nõrga laterneerimise omadust, kasutatakse modifitseeritud kaasmonotoonset strateegiat osalõikude liigi määramiseks.

Viies peatükk on pühendatud neljandas peatükis kirjeldatud strateegia koonduvuskiiruse leidmisele. Eeldatakse, et histopoleeritava funktsiooni tuletisel on lõplik arv nullkohti. Meetodi koonduvuskiirus sõltub peamiselt selle funktsiooni tuletise käitumisest nullkohtade ümbruses.

Kuuendas peatükis uuritakse ruut/lineaar ratsionaalsplainidega histopoleerimist. Uuritakse juhtu, kus etteantud histogramm on rangelt kumer või rangelt nõgus. Ruut/lineaar ratsionaalsplaini esitus on antud histogrammi kõrguste z_i, osalõigu pikkuste h_i, splaini teiste tuletiste $M_i = S''(\xi_i)$ ja osaintegraalide λ_i ja ρ_i kaudu, mis rahuldavad võrdust $\rho_{i-1} + \lambda_i - h_i z_i = 0$. Selliselt esitatud splain on ühene. Samuti on peatükis näidatud, et selline histopoleeriv splain ei eksisteeri igasuguste andmete korral. Samas on peatükis leitud tingimused algandmetele, mille korral selline splain eksisteerib.

Seitsmendast peatükist leiab teoreetiliste tulemuste numbrilised testid. Saadud arvulised tulemused on kooskõlas doktoritöös toodud teoreetiliste tulemustega.

Neljanda peatüki tulemused on avaldatud artiklis [18] ja viienda peatüki tulemused artiklis [21]. Kuuenda peatüki tulemused [22] on publitseerimiseks valmis ja vormistatud preprindina. Dissertatsioonis esitatud tulemusi on tutvustatud neljal rahvusvahelisel teaduskonverentsil ja vastavates konverentsiteesides.

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Curriculum Vitae

Helle Hallik

Born : July 13, 1982, Viljandi, Estonia

Citizenship : Estonian

Address : Institute of Mathematics, J. Liivi 2, 50409 Tartu, Estonia

Phone : +372 56 48 59 82

E-mail : helle.hallik@ut.ee

Education

1988 - 1997 Retla Primary School

1997 - 2000 Türi Gymnasium

2000 - 2004 Faculty of Mathematics, University of Tartu,
Baccalaureus Scientiarum in mathematics

2004 - 2006 Faculty of Mathematics, University of Tartu,
Institute of Applied Mathematics,
Magister Scientiarum in mathematics

2006 - 2015 Faculty of Mathematics, University of Tartu,
PhD student at the Institute of Applied Mathematics

Professional employment

2003-2004 Tartu Emajõe School, teacher of mathematics

2004-2005 Otepää Gymnasium, teacher of mathematics

Scientific Work

The main field of interest is shape-preserving rational spline histopolation. Results have been published in three papers and presented at the following four conferences:

"The 10th Mathematical Modelling and Analysis" in Trakai, Lithuania (2005),

"The Second Finnish-Estonian Mathematical Colloquium" in Tampere, Finland (2006),

"The 11th Mathematical Modelling and Analysis" in Jurmala, Latvia (2006),

"The 20th Mathematical Modelling and Analysis" in Sigulda, Latvia (2015).

Elulookirjeldus

Helle Hallik

Sünniaeg ja -koht : 13. juuli 1982, Viljandi, Eesti

Kodakondsus : Eesti

Aadress: TÜ matemaatika instituut, J.Liivi 2, 50409, Tartu, Eesti

Telefon : +372 56 48 59 82

E-kiri : helle.hallik@ut.ee

Haridus

1988 - 1997 Retla Põhikool

1997 - 2000 Türi Gümnaasium

2000 - 2004 Tartu Ülikooli matemaatika-informaatikateaduskond,
baccalaureus scientiarum matemaatika erialal,
gümnaasiumi matemaatikaõpetaja kutse

2004 - 2006 Tartu Ülikooli matemaatika-informaatikateaduskond,
magister scientiarum matemaatika erialal

2006 - 2015 Tartu Ülikooli matemaatika-informaatikateaduskond,
doktoriõpe matemaatika instituudis

Teenistuskäik

2003-2004 Tartu Emajõe Kool, matemaatikaõpetaja

2004-2005 Otepää Gümnaasium, matemaatikaõpetaja

Teadustegevus

Peamine uurimisvaldkond on splainidega lähendusmeetodid. Tulemused on ilmunud kahes teadusartiklis ja üks artikkel on esitatud publitseerimiseks. Esinenud neljal teaduskonverentsil:

“The 10th Mathematical Modelling and Analysis” Trakais (2005),

“The Second Finnish-Estonian Mathematical Colloquium” Tamperes (2006),

“The 11th Mathematical Modelling and Analysis” Jurmalas (2006),

“The 20th Mathematical Modelling and Analysis” Siguldas (2015).

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