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Chapter 1

Introduction

The purpose of this thesis is to study the Morita equivalence of idempotent rings using various algebraic constructions. Our goal is to find as many connections as possible between Morita equivalence and the considered constructions. The category of firm bimodules over two idempotent rings and especially monomorphisms in this category will be of special interest.

1.1 Background

The notion of a Morita equivalence for rings with identity first arose in 1958 from the seminal paper [36] by Kiiti Morita. He described when the module categories of two rings with identity are equivalent. Later this situation became known as Morita equivalence of the underlying rings. The resulting Morita theory has proven to be very useful in the development of the theory of rings with identity. First steps for extending Morita equivalence to non-unital rings were made by Abrams in 1983 with [1], who considered rings with local units. Further developments in extending Morita theory to a more wider class of rings were made by Komatsu in 1986 with [20] for s-unital rings and Ánh and Márki in 1987 with [7] for rings with local units. Later in 1991 García and Simón developed Morita theory for idempotent rings in [14].

One especially useful tool – which is widely used in this thesis – for studying Morita equivalence is the notion of a Morita context. Morita contexts were first introduced by Bass in 1962 in [8], who called them preequivalence datas. They were extensively used by Amitsur in 1971 in [3] and Müller in 1972 in [37] and have become increasingly popular for studying Morita equivalence ever since. Morita equivalence is defined using the equivalence of certain module categories. This makes it obvious that it is an equivalence relation on the class of rings, but on the other hand equivalence functors are hard to work with, especially if we wish to understand the structure of Morita equivalent rings. Morita contexts are helpful here, because they are much more concrete objects. Essentially, they consist of two bimodules and two bimodule homomorphisms.

Over the years, Morita theory has also been developed for many different algebraic structures, e.g. monoids (by Banaschewski and Knauer), semigroups (by Talwar in 1995 with [43]), quantales, C*-algebras etc. Morita equivalence of semigroups will be of particular interest in this thesis, because we will introduce several notions used to study Morita equivalence of semigroups into the ring case. In particular enlargements borrowed from Lawson's article [29] and strict local isomorphisms borrowed from Márki's and Steinfeld's paper [35]. Although the Morita theory of factorizable semigroups and idempotent rings are similar in some aspects, there exist some considerable differences. For instance if two monoids are Morita equivalent, then either of them is an enlargement of the other, but two Morita equivalent rings with identity need not be isomorphic to their joint enlargement. Also we will show that the only idempotent ring Morita equivalent to $\{0\}$ is $\{0\}$ itself. This is a considerable difference from the Morita equivalence of semigroups, because there are many infinitely semigroups Morita equivalent to the one-element semigroup.

Finally we will thoroughly study the category of firm bimodules over idempotent rings. The term "firm module" was first used by Quillen in 1996 in [39]. Although, a similar notion for modules over unital algebras was already introduced by Taylor in 1982 in [45] under the name regular modules. Categories of firm modules and their applications in Morita theory have been extensively studied by Marín in his master's thesis [33] and [34] in 1998 and later with García and González-Férez in articles [12], [13], [17] and [18].

1.2 Overview of the thesis

This thesis is divided into six chapters. The first chapter is the introduction, where we give a short historical overview of developments in Morita theory. Subsequently the summary of the thesis is presented.

In Chapter 2 we will give the preliminaries, which are necessary for understanding the material of this thesis. We will try to keep the text rather self-contained. First we will introduce some notions from category theory, which will be used in what follows. Namely we define adjoint functors and several kinds of monomorphisms. Next we present the basics of ring and module theory and after that introduce bimodules. Finally we will introduce Morita theory by defining and describing Morita equivalence for idempotent rings and Morita contexts.

In Chapter 3 we will define Rees matrix rings and tensor product rings for arbitrary rings. We will use both of these concepts to study Morita equivalence of idempotent rings. It turns out that every idempotent Rees matrix ring is Morita equivalent to its ground ring (Theorem 3.8). We define pseudo-surjective mappings. We see that every tensor product ring over an idempotent ring R, which is defined by a pseudo-surjective (R, R)-bilinear map, is Morita equivalent to R (Theorem 3.16). Then we define strict local isomorphisms of rings, inspired by a similar notion in semigroup theory introduced by Márki and Steinfeld. We show that if two rings are Morita equivalent, then any pseudo-surjectively defined tensor product ring over one of those rings is strictly locally isomorphic to the other one (Corollary 3.24). Finally we prove a result connecting the constructions of Rees matrix rings and tensor product rings (Theorem 3.40). We will also study the rings of adjoint endomorphisms of modules. This approach is a generalization of the ideas used by Ánh in [5]. We use adjoint endomorphisms to describe Morita equivalence of s-unital rings (Theorem 3.39). This section is based on [48].

In Chapter 4 we will define enlargements of rings, which is again a notion borrowed from semigroup theory. First we prove some simple properties of enlargements and then give two natural constructions that produce enlargements. We will show that enlargements – namely the existence of a joint enlargements – can be used to describe Morita equivalence of idempotent rings (Theorem 4.13). For instance this description allows us to easily conclude that the only ring Morita equivalent to $\{0\}$ is $\{0\}$ itself (Corollary 4.15). Furthermore, we will show that for any two Morita equivalent idempotent rings there exists a Morita context between those rings, where the bimodules are induced by their joint enlargement (Corollary 4.21). Finally we show that a joint enlargement of certain particular rings is lurking behind the strong Morita equivalence of semigroups (Theorem 4.25). This section is based on [27].

In Chapter 5 we will study unitary ideals of Morita equivalent idempotent rings. First we show that the set of all unitary ideals of an idempotent ring actually forms a unital quantale (Proposition 5.3). In particular we will prove that that the quantales of unitary ideals of Morita equivalent idempotent rings are isomorphic (Theorem 5.8). Next we will briefly consider socles and annihilators in connection to Morita equivalence. Finally, we will prove that if two idempotent rings are Morita equivalent, then their quotients, by the ideals that correspond to each other, are also Morita equivalent (Theorem 5.16). Essentially we will give a way of factorizing Morita contexts by ideals. This section is based on [49].

In Chapter 6 we will study the category of firm bimodules over two idempotent rings. First we will have a lengthy detour concerning the subcategories of firm, closed and torsion-free bimodules over idempotent rings. Due to the size of this section it is divided into subsections. After introducing the categories of firm and closed bimodules over some idempotent rings, we will show explicitly that these categories are equivalent to each other and also to the category of unitary torsion-free bimodules over the same rings (Theorem 6.14). Moreover, the category of closed bimodules over some idempotent rings is an essential localization of the category of all bimodules over those rings (Theorem 6.16). Next we will describe monomorphisms in the category of all bimodules and in the category of all unitary bimodules over two rings. There we will also give an example of a non-injective monomorphism in the category of unitary bimodules over some particular rings, proving that this category is not balanced (Example 6.23). Finally we will describe monomorphisms in the category of firm bimodules (Theorem 6.25) and show that the lattice of unitary sub-bimodules of a given firm bimodule is isomorphic to the lattice of categorical subobjects of this bimodule (Theorem 6.29). This chapter is a generalization of [47].

Chapter 2

Preliminaries

In this chapter we will introduce notions that are needed in this thesis. First we will dwell into category theory. Then we will introduce several special kinds of rings and modules, which will be important later. In order to consider Morita theory we will then study bimodules, especially the categories of firm and closed bimodules. Finally we will introduce the basics of Morita theory.

2.1 Some notions from category theory

In this thesis we will assume some prior knowledge of category theory. Still there are some notions which will be defined in this section, but first we must introduce some notation. Let $\mathcal{A}$ be a category. If A is an object of $\mathcal{A}$, then we will simply write $A \in \mathcal{A}$, and $\mathrm{Mor}_{\mathcal{A}}(A, B)$, where $A, B \in \mathcal{A}$, will denote the set of all morphisms of $\mathcal{A}$ from A to B. Also, $\mathrm{Mor}(\mathcal{A})$ wil denote the class of all morphisms in $\mathcal{A}$. Functors will usually be denoted by bold capital latin letters. Next we will define the notions of adjoint functors and the equivalence of categories.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ and $\mathbf{G}\colon\ \mathcal{B} \to \mathcal{A}$ functors. The functor $\mathbf{F}$ is called a **left adjoint** of $\mathbf{G}$ or, equivalently, G is called a **right adjoint** of $\mathbf{F}$ with the notation $\mathbf{F} \dashv \mathbf{G}$ or $\mathbf{G} \vdash \mathbf{F}$, if there exist two natural transformations $\varepsilon\colon\ \mathbf{F} \circ \mathbf{G} \to \mathrm{id}_{\mathcal{B}}$ and $\eta\colon\ \mathrm{id}_{\mathcal{A}} \to \mathbf{G} \circ \mathbf{F}$ such that for any objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ the so called *triangle identities* hold:

$$\mathrm{id}_{\mathbf{F}(A)} = \varepsilon_{\mathbf{F}(A)} \circ \mathbf{F}(\eta_A), \tag{2.1}$$

$$\mathrm{id}_{\mathbf{G}(B)} = \mathbf{G}(\varepsilon_B) \circ \eta_{\mathbf{G}(B)}. \tag{2.2}$$

Such η is called the **unit** and ε the **counit** of the adjunction $\mathbf{F} \dashv \mathbf{G}$. Adjoint functors can be composed in the following sence: if $\mathbf{F} \dashv \mathbf{G}$ and $\mathbf{F}' \dashv \mathbf{G}'$ are adjunctions, then there exists an adjunction

$$\mathbf{F} \circ \mathbf{F}' \dashv \mathbf{G}' \circ \mathbf{G}. \tag{2.3}$$

One of the most important properties of adjoint functors is that every functor that has a left (right) adjoint preserves limits (colimits) (Proposition 18.9 in [2]).

A functor $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ is called an **equivalence functor** (between categories $\mathcal{A}$ and $\mathcal{B}$) if there exists a functor $\mathbf{G}\colon\ \mathcal{B} \to \mathcal{A}$ and two natural isomorphisms $\varepsilon\colon\ \mathbf{F} \circ \mathbf{G} \to \mathrm{id}_{\mathcal{B}}$ and $\eta\colon\ \mathrm{id}_{\mathcal{A}} \to \mathbf{G} \circ \mathbf{F}$. In that case the functor $\mathbf{G}$ is also an equivalence functor between $\mathcal{A}$ and $\mathcal{B}$. If there exists an equivalence functor $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$, then we say that the categories $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** and write $\mathcal{A} \approx \mathcal{B}$. If $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ and $\mathbf{G}\colon\ \mathcal{B} \to \mathcal{A}$ are equivalence functors, then clearly $\mathbf{F}$ is a left and right adjoint of $\mathbf{G}$ and vice versa.

Next we must recall the notion of a monomorphism and its special cases regular and extremal monomorphisms. These will play an important role in what follows.

Definition 2.1. Let $\mathcal{A}$ be a category. A morphism $f\colon\ A \to B$ in $\mathcal{A}$ is called a **monomorphism**, if it is left cancellable, i.e., for every pair on morphisms $g, h\colon\ C \to A$ in $\mathcal{A}$, the following property holds:

$$f \circ g = f \circ h \quad\implies\quad g = h.$$

The dual notion of a monomorphism, i.e. a right cancellable morphism, is called an **epimorphism**. A morphism that is both a monomorphism and an epimorphism is called a **bimorphism**.

If $\mathcal{A}$ is a construct ([2, Definition 5.1]), then all injective (surjective) morphisms in $\mathcal{A}$ are monomorphisms (epimorphisms) in $\mathcal{A}$ (Corollary 7.38 in [2]).

Definition 2.2. A morphism $f\colon\ A \to B$ is called a **regular monomorphism**, if it is an equalizer of some morphisms $g, h\colon\ B \to C$.

It is easy to check that a regular monomorphism is indeed a monomorphism.

Definition 2.3. A monomorphism f is called an **extremal monomorphism** if $f = g \circ e$, where e is an epimorphism, implies that e is an isomorphism.

Also, we recall a very well known property of regular and extremal monomorphisms.

Lemma 2.4 ([2, Corollary 7.63]). *Every regular monomorphism is extremal.*

Finally we will need some notions concerning reflective and coreflective subcategories. The following definitions are taken from [9].

Definition 2.5. Let $\mathcal{A}$ be a category. A full subcategory $\mathcal{B} \subseteq \mathcal{A}$ is called **(co)reflective** if its inclusion functor $\mathbf{J}\colon\ \mathcal{B} \to \mathcal{A}$ has a left (right) adjoint $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$. The functor $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ is called a **(co)reflector**.

Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}$ with the reflector $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$. Due to the adjunction $\mathbf{F} \dashv \mathbf{J}$ there exist two natural transformations $\varepsilon\colon\ \mathbf{F} \circ \mathbf{J} \to \mathrm{id}_{\mathcal{B}}$ and $\eta\colon\ \mathrm{id}_{\mathcal{A}} \to \mathbf{J} \circ \mathbf{F}$ such that for every object $B \in \mathcal{B}$ we have

$$\mathrm{id}_B = \varepsilon_B \circ \eta_B.$$

On the other hand, using the naturality of ε and condition (2.1) (as shown on Figure 2.1), we calculate

$$\eta_B \circ \varepsilon_B = \mathbf{J}(\eta_B) \circ \varepsilon_B = \varepsilon_{\mathbf{F}(B)} \circ \mathbf{F}(\eta_B) = \mathrm{id}_{\mathbf{F}(B)}\,.$$

$$\begin{tikzcd} B \arrow[d, "\eta_B"'] & \mathbf{F}(B) \arrow[r, "\varepsilon_B"] \arrow[d, "\mathbf{F}(\eta_B)"'] & B \arrow[d, "\eta_B"] \\ \mathbf{F}(B) & \mathbf{F}(\mathbf{F}(B)) \arrow[r, "\varepsilon_{\mathbf{F}(B)}"'] & \mathbf{F}(B) \end{tikzcd}$$

Figure 2.1

Therefore we have shown that $\varepsilon_B\colon\ \mathbf{F}(B) \to B$ is an isomorphism. In conclusion, we have that if $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ is some reflector then the counit of the adjunction $\mathbf{F} \dashv \mathbf{J}$ is a natural isomorphism and its inverse is the unit restricted to the objects of $\mathcal{B}$. Dually, it can be shown that if $\mathcal{C}$ is a coreflective subcategory of $\mathcal{A}$ with a coreflector $\mathbf{G}\colon\ \mathcal{A} \to \mathcal{C}$, then the unit of the adjunction $\mathbf{J} \dashv \mathbf{G}$ is a natural isomorphism.

Definition 2.6 (Definition 3.5.6 in [9]). A reflective subcategory $\mathcal{B} \subseteq \mathcal{A}$ is called an **essential localization** of $\mathcal{A}$ if its reflector $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ has a left adjoint.

Next we will prove a lemma about essential localizations, which will prove to be useful in the following sections. This lemma was first published in [47].

Lemma 2.7. *Let $\mathcal{A}$ be a category and $\mathcal{B}$ an essential localization of $\mathcal{A}$. If monomorphisms and regular monomorphisms coincide in the category $\mathcal{A}$, then they also coincide in $\mathcal{B}$.*

Proof. Let $\mathcal{A}$ be a category where monomorphisms and regular monomorphisms coincide and $\mathcal{B} \subseteq \mathcal{A}$ its essential localization with a reflector $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ and let $\rho\colon\ \mathrm{id}_{\mathcal{A}} \to \mathbf{F}$ be the unit of the adjunction $\mathbf{F} \dashv \mathbf{J}$, where $\mathbf{J}\colon\ \mathcal{B} \to \mathcal{A}$ is the inclusion functor.

Let $f\colon\ B \to C$ be a monomorphism on $\mathcal{B}$ and $g, h \in \mathrm{Mor}_{\mathcal{A}}(A.B)$ such that $f \circ g = f \circ h$. Since $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}$, we may consider the morphisms $\mathbf{F}(g), \mathbf{F}(h)\colon\ \mathbf{F}(A) \to \mathbf{F}(B)$. Since f is a monomorphism in $\mathcal{B}$ we know that $\rho_B^{-1} \circ \mathbf{F}(g) = \rho_B^{-1} \circ \mathbf{F}(h)$. Now, since ρ is natural we obtain that

$$g = (\rho_B^{-1} \circ \mathbf{F}(g)) \circ \rho_A = (\rho_B^{-1} \circ \mathbf{F}(h)) \circ \rho_A = h,$$

which implies that $f\colon\ B \to C$ is also a monomorphism in $\mathcal{A}$.

By assumption, we know that f is a regular monomorphism in $\mathcal{A}$, which means that there exist morphisms $u, v \in \mathrm{Mor}_{\mathcal{A}}(C, D)$ such that f is an equalizer of u and v. Since $\mathcal{B}$ is an essential localization, the reflection functor $\mathbf{F}\colon\ \mathcal{A} \to \mathcal{B}$ has a left adjoint. Thus $\mathbf{F}$ is a right adjoint functor and by Proposition 18.6 in [2] it preserves equalizers. So, the morphism $\mathbf{F}(f)\colon\ \mathbf{F}(B) \to \mathbf{F}(C)$ is an equalizer of morphisms $\mathbf{F}(u), \mathbf{F}(v)\colon\ \mathbf{F}(C) \to \mathbf{F}(D)$ in $\mathcal{B}$ (as shown on Figure 2.2).

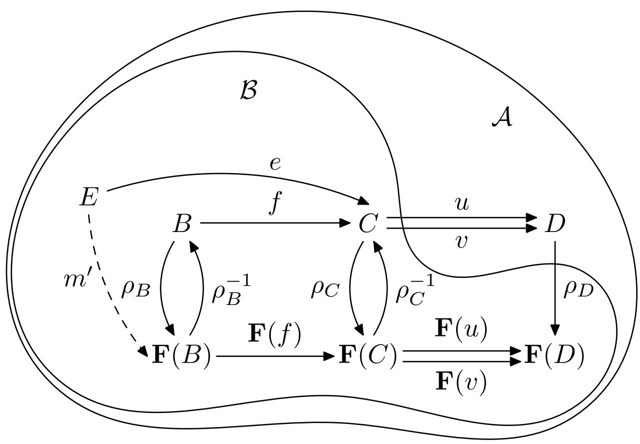

Figure 2.2

Due to ρ being a natural transformation, the equality

$$\rho_C \circ f = \mathbf{F}(f) \circ \rho_B$$

holds. This equality implies that $\mathbf{F}(u) \circ \rho_C \circ f = \mathbf{F}(v) \circ \rho_C \circ f$.

Let $e\colon\ E \to C$ be a morphism in $\mathcal{B}$ such that $\mathbf{F}(u) \circ \rho_C \circ e = \mathbf{F}(v) \circ \rho_C \circ e$. Since $\mathbf{F}(f)$ is the equalizer of $\mathbf{F}(u)$ and $\mathbf{F}(v)$, there exists a unique morphism $m'\colon\ E \to \mathbf{F}(B)$ in $\mathcal{B}$ such that $\mathbf{F}(f) \circ m' = \rho_C \circ e$. Morphisms ρ_B and ρ_C are isomorphisms in $\mathcal{B}$. Denoting $m := \rho_B^{-1} \circ m' : E \to B$ we have

$$f \circ m = f \circ \rho_B^{-1} \circ m' = \rho_C^{-1} \circ \mathbf{F}(f) \circ m' = \rho_C^{-1} \circ r_C \circ e = e.$$

Uniqueness of m follows from the fact that f is a monomorphism in $\mathcal{B}$. Thus we have shown that f is the equalizer of the morphisms $\mathbf{F}(u) \circ \rho_C, \mathbf{F}(v) \circ \rho_C\colon\ C \to \mathbf{F}(D)$ in $\mathcal{B}$. ■

Now we have all the necessary notions from category theory and we may move on to algebraic notions.

2.2 Rings and modules

In this thesis we will mostly consider associative but not necessarily having an identity element nor commutative rings, i.e. an abelian group $(R; +)$ will be called a **ring** if it is equipped with a mapping $R \times R \to R$, $(a, b) \mapsto ab$, called multiplication, which satisfies the condition $(ab)c = a(bc)$ for every $a, b, c \in R$ and addition and multiplication are connected by the distributivity conditions:

$$(a + b)c = ac + bc \qquad \text{and} \qquad c(a + b) = ca + cb,$$

for every $a, b, c \in R$.

We will need to consider modules over rings. Let R be a ring, denote by Mod_R the category whose objects are all right R-modules and morphisms are the homomorphisms of right R-modules; similarly ${}_R\mathsf{Mod}$ will be the category containing all left R-modules. Analogously, for all subsequent categories of modules, the position of the ground ring as an index will indicate either left or right modules. Let M and N be right R-modules. We will denote the set of all right R-module homomorphisms from M to N by the symbol $\mathrm{Hom}_R(M, N)$ and analogously the set of all left R-module homomorphisms by the symbol ${}_R\mathrm{Hom}(M, N)$, i.e.

$$\mathrm{Hom}_R(M, N) := \mathrm{Mor}_{\mathsf{Mod}_R}(M, N),$$
$${}_R\mathrm{Hom}(M, N) := \mathrm{Mor}_{{}_R\mathsf{Mod}}(M, N).$$

The set $\mathrm{Hom}_R(M, N)$ can actually be turned into a right R-module by defining addition and scalar multiplication as folows

$$(f + g)(x) := f(x) + g(x),$$

$$(fr)(x) := f(rx), \tag{2.4}$$

for every $f, g \in \mathrm{Hom}_R(M, N)$ and $r, x \in R$. The set ${}_R\mathrm{Hom}(M, N)$ can analogously be viewed as a left R-module by defining scalar multiplication as

$$(rf)(x) := f(xr), \tag{2.5}$$

for every $f \in \mathrm{Hom}_R(M, N)$ and $r, x \in R$.

If M_R is a right R-module, $A \subseteq M$ and $S \subseteq R$, then we denote

$$AS := \left\{ \sum_{k=1}^{k^*} a_k s_k \,\middle|\, k^* \in \mathbb{N};\ a_1, \ldots, a_{k^*} \in A;\ s_1, \ldots, s_{k^*} \in S \right\} \subseteq M.$$

For left modules (and later for bimodules) we will use a similar notation.

Next we will define several special kinds of modules and rings. We will formulate the definitions for right modules. Dually one can define such notions for left modules. All of these notions give rise to similar notions for rings, which will be defined by considering a ring R as an R-module R_R.

Definition 2.8. A right R-module M_R is called **unitary**, if $MR = R$, i.e. for every element $m \in M$ there exist elements $r_1, \ldots, r_{k^*} \in R$ and $m_1, \ldots, m_{k^*} \in M$ such that $m = m_1 r_1 + \ldots + m_{k^*} r_{k^*}$. The category of all unitary right R-modules is denoted by UMod_R.

It is easy to see that if R has an identity element 1, then M_R is unitary if and only if $m1 = m$ for every $m \in M$.

Definition 2.9. A ring R is called **idempotent** if the R-module R_R is unitary.

Idempotent rings are of central importance in this thesis. Clearly every ring with an identity element is idempotent.

We assume the familiriarity with the notion of tensor product of modules (see, for example, paragraph 12.1 in [51]), which will be used extencively in this thesis. Still, we will formulate the notion of a balanced mapping, because of its importance later.

Definition 2.10. Let R be a ring, M_R and ${}_RN$ R-modules and A an abelian group. A mapping $\alpha \colon M \times N \to A$ is called **R-balanced**, if, for every $r \in R$, $m, m' \in M$ and $n, n' \in N$, we have

1. $\alpha(m + m', n) = \alpha(m, n) + \alpha(m', n)$;
2. $\alpha(m, n + n') = \alpha(m, n) + \alpha(m, n')$;
3. $\alpha(mr, n) = \alpha(m, rn)$.

We will also formulate the universal property of the tensor product as the following proposition.

Proposition 2.11 (Universal property of the tensor product). *Let R be a ring, M_R and $_RN$ R-modules and A an abelian group. For every R-balanced map $\alpha\colon M \times N \to A$ there exists a unique homomorphism of abelian groups $\gamma\colon M \otimes_R N \to A$ such that $\gamma \circ \otimes = \alpha$ (Figure 2.3).*

$$\begin{tikzcd} & M \times N \arrow[dl, "\otimes"'] \arrow[dr, "\alpha"] & \\ M \otimes_R N \arrow[rr, "\gamma"'] & & A \end{tikzcd}$$

Figure 2.3

Next we will define firm modules.

Definition 2.12. A right R-module M_R is called **firm**, if the canonical homomorphism

$$\nu_M\colon\ M \otimes_R R \to M, \quad \sum_{k=1}^{k^*} m_k \otimes r_k \mapsto \sum_{k=1}^{k^*} m_k r_k \tag{2.6}$$

is bijective. The category of all firm right R-modules is denoted by FMod_R.

Definition 2.13. A ring R is called **firm**, if the R-module R_R is firm.

Clearly every firm module is also unitary. Namely, M_R is unitary if and only if ν_M is surjective. The converse is not always true. Hence, every firm ring is idempotent. Also every ring with identity is firm. Canepeel and Grandjean published the following example of a unitary but non-firm module in 1998.

Example 2.14 (Unitary non-firm module; Example 1.2 in [15]). Let $R := \mathbb{Z}_2 \oplus \mathbb{Z}$. Consider R as a ring with the usual componentwise addition and multiplication defined by

$$(\overline{z_1}, a_1)(\overline{z_2}, a_2) = (a_1\overline{z_2}, a_1 a_2).$$

The ring R is firm, because it has a left identity $(\overline{0}, 1)$.

Fix $c = (\overline{0}, 2) \in R$. The principal ideal

$$cR = \{(\overline{0}, 2b) \mid b \in \mathbb{Z}\} \cong 2\mathbb{Z}$$

is unitary, but not firm as a right R-module. For unitarity consider an element $(\overline{0}, 2b) \in cR$, then $(\overline{0}, 2b) = (\overline{0}, 2)(\overline{0}, b) = c(\overline{0}, b)$, where $(\overline{0}, b) \in R$ and $(\overline{0}, 2) \in cR$. Now, consider the element $(\overline{0}, 2) \otimes (\overline{1}, 0) \in cR \otimes_R R$. Obviously

$$\nu_{cR}((\overline{0}, 2) \otimes (\overline{1}, 0)) = (\overline{0}, 2)(\overline{1}, 0) = (\overline{0}, 0).$$

On the other hand there exists a well-defined right $\mathbb{Z}$-module homomorphism

$$cR \otimes_R R \to \mathbb{Z}_2, \quad (\overline{0}, 2b) \otimes (\overline{z}, a) \mapsto b\overline{z},$$

which maps $(\overline{0}, 2) \otimes (\overline{1}, 0) \mapsto \overline{1} \neq \overline{0}$. This proves that $(\overline{0}, 2) \otimes (\overline{1}, 0) \neq 0$ in $cR \otimes_R R$, because there exists a homomorphism of abelian groups that does not take $(\overline{0}, 2) \otimes (\overline{1}, 0)$ to zero. Hence cR is not firm, because ν_{cR} is not injective. □

González-Férez and Marín have also proved that there exist unitary but non-firm modules in [17] (Corollary 21).

Next we will give an example of an idempotent but non-firm ring, which was found by Ülo Reimaa.

Example 2.15 (Idempotent non-firm ring). Consider the following two semigroups $S = \{z, a, b, e\}$ and $B = \{0, 1, 2, 3, 4\}$ given by their Cayley tables:

S	z	a	b	e
z	z	z	z	z
a	z	z	z	z
b	z	z	z	b
e	z	a	z	e

and

B	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	2	1
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	3	4

.

Note that S is a non-firm semigroup, meaning that the S-acts $S \otimes_S S$ and S are not isomorphic (see Example 2.3 in [23]). Consider the mapping $\psi \colon S \times S \to B$ given by the following table:

ψ	z	a	b	e
z	0	0	0	0
a	0	0	0	0
b	0	2	0	1
e	0	3	0	4

.

It is easy to check that ψ is S-balanced, meaning that $\psi(ss', s'') = \psi(s, s's'')$ for every $s, s', s'' \in S$. Recall the notion of a semigroup ring (paragraph 5.3 in [51]) and consider the semigroup rings

$$\mathbb{Z}_2[S] = \{k_1 z + k_2 a + k_3 b + k_4 e \mid k_1, k_2, k_3, k_4 \in \mathbb{Z}_2\}$$

and $\mathbb{Z}_2[B]$. Clearly, the mapping induced by ψ,

$$\psi' \colon \ \mathbb{Z}_2[S] \times \mathbb{Z}_2[S] \to \mathbb{Z}_2[B],$$

is $\mathbb{Z}_2[S]$-balanced. By the universal property of the tensor product (Proposition 2.11), there exists a well-defined homomorphism of abelian groups

$$\overline{\psi'} \colon \ \mathbb{Z}_2[S] \otimes_{\mathbb{Z}_2[S]} \mathbb{Z}_2[S] \to \mathbb{Z}_2[B].$$

Note that $\overline{\psi'}(b \otimes a) = 2 \neq 0 = \overline{\psi'}(b \otimes b)$, which proves that $b \otimes a \neq b \otimes b$ in $\mathbb{Z}_2[S] \otimes_{\mathbb{Z}_2[S]} \mathbb{Z}_2[S]$. On the other hand

$$\nu_{\mathbb{Z}_2[S]}(b \otimes a) = ba = z = bb = \nu_{\mathbb{Z}_2[S]}(b \otimes b).$$

This proves that the mapping $\nu_{\mathbb{Z}_2[S]}$ is not injective and hence the ring $\mathbb{Z}_2[S]$ is not firm. It can be checked that $\mathbb{Z}_2[S]$ is idempotent. $\square$

Next we will define the notion of a torison-free module.

Definition 2.16. A right R-module M_R is called **torsion-free** if

$$\mathbf{t}_R(M) := \{m \in M \mid mR = \{0\}\} = \{0\}.$$

The category of all torsion-free right R-modules is denoted by TfMod_R.

The category of all unitary and torsion-free right R-modules is denoted by UTfMod_R.

Definition 2.17. A right R-module M_R is called **closed**, if the canonical homomorphism

$$\lambda_M \colon \ M \to \mathrm{Hom}_R(R, M), \quad (\lambda_M(m))(r) = mr$$

is bijective. The category of all closed right R-modules is denoted by CMod_R.

Clearly every closed module is also torsion-free. Namely, λ_M is injective if and only if M_R is torsion-free, because $\mathrm{Ker}(\lambda_M) = \mathbf{t}_R(M)$. The terms "firm module" and "closed module" were used by Quillen in [39]. Actually, firm modules appeared under the name "regular module" already in [45] by Taylor. Marín and González-Férez have studied the categories FMod_R and CMod_R and their properties extensively in [34], [17] and [18].

We will need the following theorem proven by Marín, which claims that the categories FMod_R, CMod_R and UTfMod_R are equivalent categories if R is idempotent.

Theorem 2.18 (Proposition 2.7 in [34]). *Let R be an idempotent ring. There exist equivalence functors*

$$\begin{aligned} \_R\colon\ & \mathsf{CMod}_R \to \mathsf{UTfMod}_R,\\ \mathrm{Hom}_R(R,\_)\colon\ & \mathsf{UTfMod}_R \to \mathsf{CMod}_R,\\ \_/\mathbf{t}_R(\_)\colon\ & \mathsf{FMod}_R \to \mathsf{UTfMod}_R,\\ \_\otimes_R R\colon\ & \mathsf{UTfMod}_R \to \mathsf{FMod}_R. \end{aligned}$$

These equivalences are realized by natural isomorphisms defined as follows

$$\begin{aligned} \lambda_C^{-1}\circ \mathrm{Hom}_R(R,\iota_C) = \lambda_C^{-1}\circ\iota_C\circ\_\colon\ & \mathrm{Hom}_R(R,CR)\to C,\\ \lambda_N|_{NR} = \lambda_{NR}\colon\ & N \to \mathrm{Hom}_R(R,N)R,\\ (\_/\mathbf{t}(\_))(\nu_N) = [\nu_N]\colon\ & (N\otimes_R R)/\mathbf{t}_R(N\otimes_R R)\to N,\\ ([\_]\otimes \mathrm{id}_R)\circ \nu_A^{-1}\colon\ & A \to A/\mathbf{t}_R(A)\otimes_R R, \end{aligned}$$

where $C \in \mathsf{CMod}_R$, $N \in \mathsf{UTfMod}_R$, $A \in \mathsf{FMod}_R$ and $\iota_C\colon\ CR \to C$ is the inclusion.

$$\begin{tikzcd} \mathsf{CMod}_R \arrow[r, bend left, "\_R"] \arrow[r, phantom, "\approx"] & \mathsf{UTfMod}_R \arrow[l, bend left, "\mathrm{Hom}_R(R,\_)"] \arrow[r, bend left, "\_\otimes_R R"] \arrow[r, phantom, "\approx"] & \mathsf{FMod}_R \arrow[l, bend left, "\_/\mathbf{t}_R(\_)"] \end{tikzcd}$$

Figure 2.4

Next we will define a few more special rings. Let R be a ring. An element $e \in R$ is called **idempotent**, if $ee = e$.

Definition 2.19 (Definition 1 in [7]). A ring R is said to have **local units**, if for every finite subset $\{r_1,\dots,r_n\} \subseteq R$ there exists an idempotent element $e \in R$ such that

$$r_1 = er_1 = r_1e, \quad \dots, \quad r_n = er_n = r_ne.$$

Every ring with local units is firm. We will also need the following weaker form of a ring with local units. A ring R is said to have **left local units** if for every subset $\{r_1,\dots,r_n\} \subseteq R$ there exists an idempotent $e \in R$ such that $r_1 = er_1, \dots, r_n = er_n$. A ring with **right local units** is defined dually. Here, the idempotent e is called a (**left**, **right**) **local unit** for the set $\{r_1,\dots,r_n\}$. Obviously, every ring with an identity element, is also a ring with (left, right) local units. In that case, the identity element 1 is the (left, right) local unit for any subset of R.

Now we will introduce the notion of s-unital rings.

Definition 2.20 ([46]). A ring R is called **left (right) s-unital** if for every $r \in R$ there exists an element $v \in R$ such that

$$r = vr \qquad (r = rv).$$

A ring R is called **s-unital** if it is both left and right s-unital, i.e. for every $r \in R$ there exist elements $u, v \in R$ such that $r = vr = ru$.

For example every ring with local units, including every von Neumann regular ring (see paragraph 3.1 in [51]), is s-unital. We will need the following result about s-unital rings, that was proved by Tominaga in [46].

Theorem 2.21 (Theorem 1 in [46]). *A ring R is left s-unital if and only if for every finite subset $\{r_1, \dots, r_n\} \subseteq R$ there exists $v \in R$ such that*

$$r_1 = vr_1, \quad \dots, \quad r_n = vr_n.$$

Next, we will prove that every left (or right) s-unital ring is firm.

Lemma 2.22. *Every left s-unital ring is firm and hence also idempotent.*

Proof. Let R be a left s-unital ring. Consider the homomorphism

$$\nu_R\colon\ R \otimes_R R \to R, \quad \sum_{k=1}^{k^*} r_k \otimes r'_k \mapsto \sum_{k=1}^{k^*} r_k r'_k$$

The homomorphism ν_R is surjective, because every $r \in R$ can be expressed $r = vr$ for some $v \in R$ and hence $r = vr = \nu_R(v \otimes r)$.

Next let $\sum_{k=1}^{k^*} r_k \otimes r'_k \in \mathrm{Ker}(\nu_R)$, then $\sum_{k=1}^{k^*} r_k r'_k = 0$. By Theorem 2.21, there exists an element $v \in R$ such that $r_k = vr_k$ for any $k \in \{1, \dots, k^*\}$. Now

$$\sum_{k=1}^{k^*} r_k \otimes r'_k = \sum_{k=1}^{k^*} vr_k \otimes r'_k = v \otimes \left(\sum_{k=1}^{k^*} r_k r'_k\right) = v \otimes 0 = 0.$$

Hence $\mathrm{Ker}(\nu_R) = \{0\}$, which proves that ν_R is injective. In conclusion, ν_R is an isomorphism, which proves that R is firm. Every firm ring is idempotent. ■

Lastly, we must recall the notion of an ideal of a ring. Let R be a ring. A subset $I \subseteq R$ is called a **right (left) ideal** of R if it is a subgroup of $(R;+)$ and $IR \subseteq I$ $(RI \subseteq I)$. Obvously, every right (left) ideal of R may be considered as a right (left) R-module. A subset $I \subseteq R$ is called an **ideal** of R if it is both a left and a right ideal of R. We will write $I \trianglelefteq R$ if I is an ideal of R and the symbol $\mathrm{Id}(R)$ will denote the set of all ideals of R. The set $\mathrm{Id}(R)$ is a complete lattice with respect to the inclusion relation. In $\mathrm{Id}(R)$ joins are sums and meets are intersections.

2.3 Bimodules

Let R and S be rings. A left S-module M, which is also a right R-module, is called an **(S, R)-bimodule**, if the condition

$$(sm)r = s(mr)$$

holds for every $s \in S$, $r \in R$ and $m \in M$. In such a case we write ${}_SM_R$ to indicate that M is an (S, R)-bimodule. A subset $A \subseteq M$ is called a **sub-bimodule** of an (S, R)-bimodule M if A is a submodule of both the left S-module ${}_SM$ and the right R-module M_R. The set of all sub-bimodules of an (S, R)-bimodule M is denoted by $\mathrm{Sub}(M)$.

The category of all (S, R)-bimodules is denoted by ${}_S\mathsf{Mod}_R$, morphisms in this category are mappings, which are both homomorphisms of left S-modules and also homomorphisms of right R-modules. For any $M, N \in {}_S\mathsf{Mod}_R$, denote

$${}_S\mathrm{Hom}_R(M, N) := \mathrm{Mor}_{{}_S\mathsf{Mod}_R}(M, N).$$

The set ${}_S\mathrm{Hom}_R(M, N)$ can be viewed as an (S, R)-bimodule by defining addition componentwise, right R-multiplication with (2.4) and left S-multiplication analogously.

Let $M \in {}_S\mathsf{Mod}_R$. Notice that the right R-module $\mathrm{Hom}_R(R, M)$ of right R-module homomorphisms can be viewed as an (S, R)-bimodule, by defining an S-multiplication for every $f \in \mathrm{Hom}_R(R, M)$ as follows

$$(sf)(r) := sf(r), \tag{2.7}$$

for any $s \in S$ and $r \in R$. The left S-module ${}_S\mathrm{Hom}(S, M)$ can analogously be viewed as an (S, R)-bimodule, i.e. addition in ${}_S\mathrm{Hom}(S, M)$ is defined componentwise and S-, R-multiplications are defined as follows

$$(sf)(s') = f(s's), \tag{2.8}$$

$$(fr)(s') = f(s')r, \tag{2.9}$$

for every $f \in {}_S\mathrm{Hom}(S, M)$, $s, s' \in S$ and $r \in R$.

Definition 2.23. An (S, R)-bimodule ${}_SM_R$ is called **unitary**, if ${}_SM$ is a unitary left S-module and M_R is a unitary right R-module. The category of all unitary (S, R)-bimodules is denoted by ${}_S\mathsf{UMod}_R$.

Firm, **torsion-free** and **closed** bimodules are defined completely analogously and their categories are denoted by ${}_S\mathsf{FMod}_R$, ${}_S\mathsf{TfMod}_R$ and ${}_S\mathsf{CMod}_R$, respectively. Also, we will adopt a convention of notation that if any of the abbrevations u, f, tf, c or utf is written to the left or right side of symbol Mod, then it denotes a category of bimodules whose objects have the respective properties as left or right modules. For example the category ${}_S\mathsf{utfModc}_R$ consists of all (S,R)-bimodules ${}_SM_R$ such that ${}_SM$ is a left unitary and torsion-free S-module and M_R is a right closed R-module. All of these categories are full subcategories of ${}_S\mathsf{Mod}_R$.

Now we will prove a simple, yet extremely useful description of unitary bimodules.

Lemma 2.24. *Let S and R be rings and ${}_SM_R$ an (S,R)-bimodule. A bimodule ${}_SM_R$ is unitary if and only if $SMR = M$.*

Proof. *Necessity.* Let ${}_SM_R$ be a unitary bimodule. Then $M = SM = SMR$.

Sufficiency. Let $M = SMR$ hold. Fix $m \in M$, then there exist elements $s_1, \ldots, s_{k^*} \in S$, $r_1, \ldots, r_{k^*}$ and $m_1, \ldots, m_{k^*} \in M$ such that $m = s_1m_1r_1 + \ldots + s_{k^*}m_{k^*}r_{k^*}$. Now

$$m = \sum_{k=1}^{k^*} s_k m_k r_k = \sum_{k=1}^{k^*} s_k(m_k r_k) \in SM,$$
$$m = \sum_{k=1}^{k^*} s_k m_k r_k = \sum_{k=1}^{k^*} (s_k m_k) r_k \in MR.$$

This proves the inclusions $M \subseteq SM$ and $M \subseteq MR$. The converse inclusions are obvious. ■

Let S and R be idempotent rings. Due to the previous lemma, we can construct a functor

$$\mathbf{U} = S\_R\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{UMod}_R, \quad M \mapsto SMR. \tag{2.10}$$

Indeed, for every $M \in {}_S\mathsf{Mod}_R$, we have $S(SMR)R = (SS)M(RR) = SMR$, meaning that $\mathbf{U}(M) \in {}_S\mathsf{UMod}_R$. The functor $\mathbf{U}$ maps morphisms to restrictions: $\mathbf{U}(f) = f|_{SMR}\colon\ SMR \to SNR$, for every $f \in \mathrm{Mor}_{{}_S\mathsf{Mod}_R}(M,N)$ with $M, N \in {}_S\mathsf{Mod}_R$. Clearly there exists a natural isomorphism $\mathbf{U} \cong \mathbf{U} \circ \mathbf{U}$, if we view $\mathbf{U}$ as an endofunctor of ${}_S\mathsf{Mod}_R$. It is easy to see that the functor $\mathbf{U}$ can be expressed as a composition

$$\mathbf{U} = (\_R) \circ (S\_)\colon \quad {}_S\mathsf{Mod}_R \to {}_S\mathsf{uMod}_R \to {}_S\mathsf{UMod}_R.$$

Let the symbol $\mathrm{USub}(M)$ denote the set of all unitary sub-bimodules of an (S,R)-bimodule ${}_SM_R$. The set $\mathrm{USub}(M)$ is a nonempty poset with respect to the inclusion relation. The following proposition shows that $\mathrm{USub}(M)$ is even a lattice with some good properties.

Proposition 2.25. *If ${}_SM_R$ is an (S, R)-bimodule, then* $\mathrm{USub}(M)$ *is a complete lattice. If R and S are idempotent rings, then this lattice is modular.*

PROOF. Let $M \in {}_S\mathsf{Mod}_R$ for some rings S and R. It is easy to see that the sum of any set of unitary sub-bimodules of M is a unitary sub-bimodule. Hence $\mathrm{USub}(M)$ is a complete lattice with

$$\bigvee_{k\in K} B_k := \sum_{k\in K} B_k,$$

for any set K with $B_k \in \mathrm{USub}(M)$, $k \in K$. The least element of $\mathrm{USub}(M)$ is $\{0\}$.

Now assume that the rings S and R are idempotent. Then the meet of an arbitrary subset $\{B_k \mid k \in K\} \subseteq \mathrm{USub}(M)$ is calculated as follows:

$$\bigwedge_{k\in K} B_k := S\left(\bigcap_{k\in K} B_k\right)R.$$

Let $A, B, C \in \mathrm{USub}(M)$ be such that $A \subseteq C$. Then $(A+B)\cap C = A+B\cap C$, because the lattice of all sub-bimodules $\mathrm{Sub}(M)$ is modular. Hence

$$\begin{aligned}(A \vee B) \wedge C &= R((A + B) \cap C)S = R(A + (B \cap C))S = RAS + R(B \cap C)S\\ &= A + R(B \cap C)S = A \vee (B \wedge C),\end{aligned}$$

which means that the complete lattice $\mathrm{USub}(M)$ is modular. ■

2.4 Morita theory

In this section we will introduce Morita contexts and show how they can be used to study Morita equivalence for idempotent rings.

Definition 2.26. A six-tuple $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, where R and S are rings and ${}_RP_S$, ${}_SQ_R$ are bimodules, is called a **Morita context**, if

$$\theta\colon\ {}_R(P \otimes_S Q)_R \to {}_RR_R, \quad \phi\colon\ {}_S(Q \otimes_R P)_S \to {}_SS_S$$

are bimodule homomorphisms such that

$$\theta(p \otimes q)p' = p\phi(q \otimes p'), \tag{2.11}$$

$$q\theta(p \otimes q') = \phi(q \otimes p)q' \tag{2.12}$$

for every $p, p' \in P$ and $q, q' \in Q$.

We say that a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is **unitary**, if the bimodules ${}_RP_S$ and ${}_SQ_R$ are unitary; and **surjective** (**bijective**), if the homomorphisms θ and ϕ are surjective (bijective). We will say that two rings S and R *are connected* by a Morita context, if there exists a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$.

Next we will prove one useful little proposition that first appeared in [27], which claims that unitary surjective Morita contexts only connect idempotent rings.

Proposition 2.27. *If $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is a unitary surjective Morita context, then the rings S and R are idempotent.*

Proof. Let $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ be an unitary surjective Morita context. Take $r \in R$. Using the surjectivity of θ we can find an element $\sum_{h=1}^{h^*} p_h \otimes q_h \in P \otimes Q$ such that $r = \theta(\sum_{h=1}^{h^*} p_h \otimes q_h)$. Since ${}_RP$ is unitary, for every $h \in \{1, \ldots, h^*\}$, there exist a natural number k^*, elements $r_{h1}, \ldots, r_{hk^*} \in S$ and $p_{h1}, \ldots, p_{hk^*} \in P$ such that $p_h = r_{h1}p_{h1} + \ldots + r_{hk^*}p_{hk^*}$ (if necessary, we add some zero summands to get the equal length of sums for all h's). Now

$$\begin{aligned} r = \theta\left(\sum_{h=1}^{h^*} p_h \otimes q_h\right) &= \sum_{h=1}^{h^*} \theta\left(p_h \otimes q_h\right) = \sum_{h=1}^{h^*} \theta\left(\sum_{k=1}^{k^*} r_{hk}p_{hk} \otimes q_h\right) \\ &= \sum_{h=1}^{h^*}\sum_{k=1}^{k^*} \theta\left(r_{hk}p_{hk} \otimes q_h\right) = \sum_{h=1}^{h^*}\sum_{k=1}^{k^*} r_{hk}\theta\left(p_{hk} \otimes q_h\right) \in RR\,. \end{aligned}$$

This proves that R is an idempotent ring. The proof that S is idempotent is analogous. ■

In [14] (after Corollary 2.9) García and Simón defined two idempotent rings S and R to be **Morita equivalent** if the categories UTfMod_R and UTfMod_S are equivalent categories. A somewhat similar idea for generalizing Morita equivalence for non-unital rings was also used by Nobusawa in [38] already in 1984. We will denote Morita equivalence of rings S and R by $S \approx_{\mathrm{ME}} R$. Due to Theorem 2.18 we could equivalently claim that two idempotent rings S and R are Morita equivalent if the categories CMod_R and CMod_S or categories FMod_R and FMod_S are equivalent. From these definitions it is easy to see that Morita equivalence is an equivalence relation on the class of all idempotent rings. The categories of CMod_R and CMod_S were also used by García and Marín to extend Morita theory to arbitrary rings in [13].

Propositions 2.3 and 2.6 in [14] give us a way to characterise Morita equivalence of idempotent rings in terms of unitary surjective Morita contexts. This characterization is given as the following theorem.

Theorem 2.28. *Two idempotent rings are Morita equivalent if and only if they are connected by a unitary surjective Morita context.*

It turns out that from each Morita context a new ring arises in a natural way.

Definition 2.29. Let $\Gamma = (R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ be a Morita context. Then the **Morita ring** $\overline{\Gamma}$ of the context Γ is defined as the matrix set

$$\overline{\Gamma} = \left\{ \begin{bmatrix} r & p \\ q & s \end{bmatrix} \middle| r \in R, s \in S, p \in P, q \in Q \right\}$$

with componentwise addition and with the multiplication

$$\begin{bmatrix} r & p \\ q & s \end{bmatrix} \begin{bmatrix} r' & p' \\ q' & s' \end{bmatrix} = \begin{bmatrix} rr' + \theta(p \otimes q') & rp' + ps' \\ qr' + sq' & \phi(q \otimes p') + ss' \end{bmatrix}. \tag{2.13}$$

It is easy to see that, if idempotent rings S and R are Morita equivalent, then their corresponding Morita ring is idempotent.

Note that the subsets

$$\overline{R} := \left\{ \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} \middle| r \in R \right\} \subseteq \overline{\Gamma}, \tag{2.14}$$

$$\overline{S} := \left\{ \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} \middle| s \in S \right\} \subseteq \overline{\Gamma} \tag{2.15}$$

are subrings of $\overline{\Gamma}$ that are isomorphic to R and S, respectively. This gives us a way of considering $\overline{\Gamma}$ as an (R, S)- or (S, R)-bimodule, by defining for any $r' \in R$, $s' \in S$ and $\left[\begin{smallmatrix} r & p \\ q & s \end{smallmatrix}\right] \in \overline{\Gamma}$ lefthand multiplications by

$$r' \begin{bmatrix} r & p \\ q & s \end{bmatrix} := \begin{bmatrix} r' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r & p \\ q & s \end{bmatrix} = \begin{bmatrix} r'r & r'p \\ 0 & 0 \end{bmatrix}, \tag{2.16}$$

$$s' \begin{bmatrix} r & p \\ q & s \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ 0 & s' \end{bmatrix} \begin{bmatrix} r & p \\ q & s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ s'q & s's \end{bmatrix} \tag{2.17}$$

and analogously on the righthand side. With these module structures in mind we can easily see that the mappings

$$P \to \overline{\Gamma}, \quad p \mapsto \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad Q \to \overline{\Gamma}, \quad q \mapsto \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}$$

are injective bimodule homomorphisms.

In conclusion we have seen that the Morita ring $\overline{\Gamma}$ has isomorphic copies of R, S, P and Q as corresponding substructures.

Chapter 3

Rees matrix rings and tensor product rings

In this chapter we will define Rees matrix rings and tensor product rings over arbitrary rings. We will show that both of these concepts can be used to study Morita equivalence of rings. We will also study the rings of adjoint endomorphisms of modules. In the last two sections we will study the connections between Rees matrix rings and tensor product rings and finally we will describe Morita equivalence of firm rings in terms of tensor product rings. This chapter is based on [48].

3.1 Rees matrix rings

Rees matrix rings over a ring with identity were introduced in [7] (Definition 2.1) by Ánh and Márki. We will use a similar definition for an arbitrary associative ring R. Firstly we shall define finite-dimensional Rees matrix rings. Let R be a ring, m, n some natural numbers and $M \in \mathrm{Mat}_{n,m}(R)$ a fixed matrix. Consider the ring

$$\mathcal{M} = \mathcal{M}(R; m, n; M) = (\mathrm{Mat}_{m,n}(R), +, *),$$

where addition $+$ is the usual componentwise addition of matrices and multiplication $*$ is defined as follows:

$$X * Y := XMY, \qquad X, Y \in \mathrm{Mat}_{m,n}(R).$$

Such a ring $\mathcal{M}$ is called a (finite-dimensional) **Rees matrix ring** over R. We will also use a more general definition of Rees matrix rings.

Definition 3.1 (Definition 2.1 in [6]). Let Λ and Ξ be non-empty sets and $M\colon\ \Xi \times \Lambda \to R$ a mapping. Consider the set $\mathcal{M} = \mathcal{M}(R;\Lambda,\Xi;M)$ of mappings $\Lambda \times \Xi \to R$ having only a finite number of non-zero values – these correspond to $\Lambda \times \Xi$ matrices over R with a finite number of non-zero entries. In $\mathcal{M}(R;\Lambda,\Xi;M)$ we define addition as the usual point-wise addition and multiplication $*$ with

$$X * Y = XMY,$$

where the multiplication on the right-hand side means the usual multiplication of matrices. With these operations $\mathcal{M} = \mathcal{M}(R;\Lambda,\Xi;M)$ is a ring, called a **Rees matrix ring**.

Elements of a Rees matrix ring $\mathcal{M}(R;\Lambda,\Xi;M)$ are called **matrices** and the mapping M is called a **sandwich matrix**. It is clear that, if we take $\Lambda = \{1,\dots,m\}$ and $\Xi = \{1,\dots,n\}$, then $\mathcal{M}(R;\Lambda,\Xi;M) = \mathcal{M}(R;m,n;M)$.

In this section we will give proofs with finite-dimensional Rees matrix rings, because they are easier to follow and more illustrative of the technique. These proofs can easily be generalized to arbitrary Rees matrix rings. This can be done by noticing that for every matrix $X \in \mathcal{M}(R;\Lambda,\Xi;M)$, there exists a minimal submatrix $\mu(X)$ such that every value outside of $\mu(X)$ is zero. This matrix $\mu(X)$ can be expressed as an element of $\mathcal{M}(R;m_X,n_X;M')$ for some numbers n_X and m_X and a submatrix M' of M. By adding zeros to $\mu(X)$ where necessary, we can also say that $\mu(X)$ is from a finite-dimensional Rees matrix ring $\mathcal{M}(R;m',n';M'')$ for every $m' \geq m_X$ and $n' \geq n_X$. Then whenever we have a finite collection of matrices $X_1,\dots,X_{k^*} \in \mathcal{M}(R;\Lambda,\Xi;M)$, we can do calculations in a finite-dimensional Rees matrix ring $\mathcal{M}(R;m,n;M')$ which is a subring of $\mathcal{M}(R;\Lambda,\Xi;M)$ and $\mu(X_k)$ is from $\mathcal{M}(R;m,n;M')$, for every $k \in \{1,\dots,k^*\}$. Such generalizations are given as corollaries. Firstly we will prove a proposition which describes idempotent Rees matrix rings.

Proposition 3.2. *A Rees matrix ring $\mathcal{M} = \mathcal{M}(R;m,n;M)$ is idempotent if and only if*

$$\mathrm{Mat}_{1,1}(R) = \mathrm{Mat}_{1,n}(R)M\,\mathrm{Mat}_{m,1}(R).$$

Proof. Let a Rees matrix ring $\mathcal{M} = \mathcal{M}(R;m,n;M)$ be idempotent. Then, for every $X \in \mathcal{M}$ there exist matrices $Y_1,Z_1,\dots,Y_{k^*},Z_{k^*} \in \mathcal{M}$ such that $X = Y_1 * Z_1 + \dots + Y_{k^*} * Z_{k^*}$. Therefore

$$X = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \dots & x_{mn} \end{bmatrix} = \sum_{k=1}^{k^*} Y_k * Z_k = \sum_{k=1}^{k^*} Y_k M Z_k$$

$$= \sum_{k=1}^{k*} \begin{bmatrix} y_{k11} & \cdots & y_{k1n} \\ \vdots & \ddots & \vdots \\ y_{km1} & \cdots & y_{kmn} \end{bmatrix} \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & \ddots & \vdots \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix} \begin{bmatrix} z_{k11} & \cdots & z_{k1n} \\ \vdots & \ddots & \vdots \\ z_{km1} & \cdots & z_{kmn} \end{bmatrix}$$

$$= \sum_{k=1}^{k^*} \begin{bmatrix} \sum_{h=1}^{n} y_{k1h}\mu_{h1} & \cdots & \sum_{h=1}^{n} y_{k1h}\mu_{hn} \\ \vdots & \ddots & \vdots \\ \sum_{h=1}^{n} y_{kmh}\mu_{h1} & \cdots & \sum_{h=1}^{n} y_{kmh}\mu_{hn} \end{bmatrix} \begin{bmatrix} z_{k11} & \cdots & z_{k1n} \\ \vdots & \ddots & \vdots \\ z_{km1} & \cdots & z_{kmn} \end{bmatrix} =$$

$$= \sum_{k=1}^{k^*} \begin{bmatrix} \sum_{j=1}^{m}\sum_{h=1}^{n} y_{k1h}\mu_{hj}z_{kj1} & \cdots & \sum_{j=1}^{m}\sum_{h=1}^{n} y_{k1h}\mu_{hj}z_{kjn} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{m}\sum_{h=1}^{n} y_{kmh}\mu_{hj}z_{kj1} & \cdots & \sum_{j=1}^{m}\sum_{h=1}^{n} y_{kmh}\mu_{hj}z_{kjn} \end{bmatrix}.$$

Now we see that for every $p \in \{1, \dots, m\}$ and $q \in \{1, \dots, n\}$, $[x_{pq}] \in \mathrm{Mat}_{1,1}(R)$ and

$$[x_{pq}] = \left[\sum_{k=1}^{k^*}\sum_{j=1}^{m}\sum_{h=1}^{n} y_{kph}\mu_{hj}z_{kjq}\right] = \sum_{k=1}^{k^*} \begin{bmatrix} y_{kp1} & \cdots & y_{kpn} \end{bmatrix} M \begin{bmatrix} z_{k1q} \\ \vdots \\ z_{kmq} \end{bmatrix}, \quad (3.1)$$

which implies that $[x_{pq}] \in \mathrm{Mat}_{1,n}(R)M\,\mathrm{Mat}_{m,1}(R)$. Since X was chosen arbitrarily, we have shown that $\mathrm{Mat}_{1,1}(R) = \mathrm{Mat}_{1,n}(R)M\,\mathrm{Mat}_{m,1}(R)$, which proves the necessity of our proposition. To prove the sufficiency one just has to retrace the previous steps in the opposite order. ■

Corollary 3.3. *A Rees matrix ring $\mathcal{M}(R; \Lambda, \Xi; M)$ is idempotent if and only if*

$$R = \Xi' M \Lambda',$$

where Ξ' is the set of mappings $\{1\} \times \Xi \to R$ with finite number of non-zero values and Λ' is the set of mappings $\Lambda \times \{1\} \to R$ with finite number of non-zero values and the set of mappings $\{1\} \times \{1\} \to R$ is identified with R.

From the decomposition (3.1), we deduce the following proposition.

Proposition 3.4. *If a Rees matrix ring $\mathcal{M}(R; \Lambda, \Xi; M)$ is idempotent, then the ring R is idempotent.*

Example 3.5 (Idempotent Rees matrix ring). If D is a division ring, then every Rees matrix ring over D is idempotent. Consider a Rees matrix ring $\mathcal{M}(D, m, n, M)$, where $M = [\mu_{hk}]_{h,k=1}^{n,m} \in \mathrm{Mat}_{n,m}(D)$ is not a zero matrix. If $\mu_{11} \neq 0$, then every one-element matrix $[d] \in \mathrm{Mat}_{1,1}(D)$ can be written as

$$[d] = \begin{bmatrix}\mu_{11}^{-1} & 0 & \dots & 0\end{bmatrix} \begin{bmatrix}\mu_{11} & \dots & \mu_{1m} \\ \vdots & \ddots & \vdots \\ \mu_{n1} & \dots & \mu_{nm}\end{bmatrix} \begin{bmatrix} d \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathrm{Mat}_{1,n}(D) M \,\mathrm{Mat}_{m,1}(D).$$

If $\mu_{11} = 0$, then there exists a $\mu_{hk} \neq 0$ for some h and k. The matrix $[d]$ can then be expressed analogously using μ_{hk}. Due to Proposition 3.2, the ring $\mathcal{M}(D; m, n; M)$ is idempotent. □

Next we will prove a little lemma, which will later become useful in several results.

Lemma 3.6. *For an idempotent ring R and $m, n \in \mathbb{N}$,*

$$\mathrm{Mat}_{m,n}(R) = \mathrm{Mat}_{m,1}(R)\,\mathrm{Mat}_{1,n}(R).$$

Proof. Clearly $\mathrm{Mat}_{m,1}(R)\,\mathrm{Mat}_{1,n}(R) \subseteq \mathrm{Mat}_{m,n}(R)$. Let $X = [x_{pq}]_{p,q=1}^{m,n} \in \mathrm{Mat}_{m,n}(R)$. Let $p \in \{1, \dots, m\}$ and $q \in \{1, \dots, n\}$ be arbitrary, then, due to R being idempotent, there exist elements $x_1, x_1', \dots, x_{k_{pq}}, x_{k_{pq}}' \in R$ such that $x_{pq} = x_1 x_1' + \dots + x_{k_{pq}} x_{k_{pq}}'$. Denote by $A_{pq}(r)$ the $m \times n$-matrix with the entry r at the position (p, q) and zeros elsewhere. Then

$$A_{pq}(x_{pq}) = \sum_{k=1}^{k_{pq}} A_{pq}(x_k x_k') = \sum_{k=1}^{k_{pq}} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_k \ \text{(}p\text{. line)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & \underset{(q.\ \text{column})}{x_k'} & 0 & \dots & 0 \end{bmatrix}. \tag{3.2}$$

Therefore every matrix $A_{pq}(x_{pq})$ can be expressed as an element of the set $\mathrm{Mat}_{m,1}(R)\,\mathrm{Mat}_{1,n}(R)$. Now, it follows that

$$X = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq}(x_{pq}) \in \mathrm{Mat}_{m,1}(R)\,\mathrm{Mat}_{1,n}(R).$$

Therefore $\mathrm{Mat}_{m,n}(R) = \mathrm{Mat}_{m,1}(R)\,\mathrm{Mat}_{1,n}(R)$. ■

Corollary 3.7. *Let $\mathcal{M}(R;\Lambda,\Xi;M)$ be a Rees matrix ring over an idempotent ring R. Then for every $f \in \mathcal{M}(R;\Lambda,\Xi;M)$, there exist $n \in \mathbb{N}$, $g_1,\ldots,g_n\colon \Lambda \to R$ and $h_1,\ldots,h_n\colon \Xi \to R$ such that for every $\lambda \in \Lambda$ and $\xi \in \Xi$*

$$f(\lambda,\xi) = \sum_{k=1}^{n} g_k(\lambda)h_k(\xi).$$

Now we are ready to prove the main theorem of this section. This theorem is the ring theoretic analogue of Proposition 2 in [22].

Theorem 3.8. *A ring R and a Rees matrix ring $\mathcal{M} = \mathcal{M}(R;m,n;M)$ are connected by a unitary surjective Morita context if and only if $\mathcal{M}$ is idempotent.*

Proof. *Necessity.* Let $\mathcal{M}$ and R be connected by a unitary surjective Morita context. Then by Proposition 2.27 the ring $\mathcal{M}$ is idempotent.

Sufficiency. Let the Rees matrix ring $\mathcal{M} = \mathcal{M}(R;m,n;M)$ be idempotent. Consider the left R-module ${}_R(\mathrm{Mat}_{1,n}(R))$ and the right R-module $(\mathrm{Mat}_{m,1}(R))_R$, where for every $r \in R$ the R-multiplications are defined as follows:

$$r\begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} := \begin{bmatrix} rx_1 & \ldots & rx_n \end{bmatrix} \in \mathrm{Mat}_{1,n}(R),$$
$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} r := \begin{bmatrix} y_1 r \\ \vdots \\ y_m r \end{bmatrix} \in \mathrm{Mat}_{m,1}(R).$$

Since $\mathcal{M}$ is idempotent, R is also idempotent by Proposition 3.4. Then, for arbitary $Y \in \mathrm{Mat}_{m,1}(R)$, we can write

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{k=1}^{k^*} y_{1k}y'_{1k} \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \sum_{k=1}^{k^*} \begin{bmatrix} y_{1k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} y'_{1k} + \begin{bmatrix} 0 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

where $y_1,\ldots,y_m,y_{11},y'_{11},\ldots,y_{1k^*},y'_{1k^*} \in R$ and $y_1 = y_{11}y'_{11} + \ldots + y_{1k^*}y'_{1k^*}$.

Continuing analogously, we can express every entry of Y as a sum of products of elements of R, and so the whole matrix Y as a sum of products of column-matrices and elements of R, which implies that the right R-module $(\mathrm{Mat}_{m,1}(R))_R$ is unitary. The left R-module ${}_R(\mathrm{Mat}_{1,n}(R))$ is analogously unitary.

We define a right and a left $\mathcal{M}$-multiplication for modules ${}_R(\mathrm{Mat}_{1,n}(R))$ and $(\mathrm{Mat}_{m,1}(R))_R$, respectively, as follows

$$X * Z := XMZ \in \mathrm{Mat}_{1,n}(R),$$
$$Z * Y := ZMY \in \mathrm{Mat}_{m,1}(R),$$

where $Z \in \mathcal{M}$, $X \in \mathrm{Mat}_{1,n}(R)$ and $Y \in \mathrm{Mat}_{m,1}(R)$. A straightforward verification shows that we have bimodules ${}_R(\mathrm{Mat}_{1,n}(R))_{\mathcal{M}}$ and ${}_{\mathcal{M}}(\mathrm{Mat}_{m,1}(R))_R$. Let $Y = [y_k]_{k=1}^m \in \mathrm{Mat}_{m,1}(R)$. By Proposition 3.2, there exist matrices $X_1 = [x_{1h}], \dots, X_{k^*} = [x_{k^*h}] \in \mathrm{Mat}_{1,n}(R)$ and $Y_1, \dots, Y_{k^*} \in \mathrm{Mat}_{m,1}(R)$ such that $y_1 = X_1 * Y_1 + \dots + X_{k^*} * Y_{k^*}$. Now

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{k^*} X_k * Y_k \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \sum_{k=1}^{k^*} \begin{bmatrix} x_{k1} & \dots & x_{kn} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} * Y_k + \begin{bmatrix} 0 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.$$

By continuing this process for every element $y_2, \dots, y_m$, we see that the module ${}_{\mathcal{M}}(\mathrm{Mat}_{m,1}(R))$ is unitary. Analogously, the module $(\mathrm{Mat}_{1,n}(R))_{\mathcal{M}}$ is also unitary. Therefore we have shown that ${}_R(\mathrm{Mat}_{1,n}(R))_{\mathcal{M}}$ and ${}_{\mathcal{M}}(\mathrm{Mat}_{m,1}(R))_R$ are unitary bimodules.

Define a mapping

$$\theta\colon\ {}_R(\mathrm{Mat}_{1,n}(R) \otimes_{\mathcal{M}} \mathrm{Mat}_{m,1}(R))_R \to {}_R R_R, \qquad \sum_{k=1}^{k^*} X_k \otimes Y_h \mapsto \sum_{k=1}^{k^*} X_k M Y_h.$$

Consider the mapping $\hat{\theta}\colon\ \mathrm{Mat}_{1,n}(R) \times \mathrm{Mat}_{m,1}(R) \to R$, $(X, Y) \mapsto XMY$. The mapping $\hat{\theta}$ clearly preserves addition and for every $Z \in \mathcal{M}$

$$\hat{\theta}(X * Z, Y) = (X * Z)MY = (XMZ)MY = XM(ZMY) = \hat{\theta}(X, Z * Y).$$

Therefore, the mapping $\hat{\theta}$ is $\mathcal{M}$-balanced. Due to the universal property of tensor product (see Proposition 2.11), the mapping θ is a well-defined homomorphism of abelian groups. For every $r \in R$ and $\sum_{k=1}^{k^*} X_k \otimes Y_k \in \mathrm{Mat}_{1,n}(R) \otimes \mathrm{Mat}_{m,1}(R)$, we have

$$\theta\left(r\left(\sum_{k=1}^{k^*} X_k \otimes Y_k\right)\right) = \theta\left(\sum_{k=1}^{k^*} (rX_k) \otimes Y_k\right) = r\sum_{k=1}^{k^*} X_k M Y_k = r\theta\left(\sum_{k=1}^{k^*} X_k \otimes Y_k\right).$$

Analogously $\theta((\sum_{k=1}^{k^*} X_k \otimes Y_k)r) = \theta(\sum_{k=1}^{k^*} X_k \otimes Y_k)r$, therefore θ is a homomorphism of bimodules. The homomorphism θ is surjective due to Proposition 3.2.

Now define a mapping

$$\phi\colon\ {}_{\mathcal{M}}(\mathrm{Mat}_{m,1}(R)\otimes_R \mathrm{Mat}_{1,n}(R))_{\mathcal{M}} \to {}_{\mathcal{M}}\mathcal{M}_{\mathcal{M}}, \qquad \sum_{k=1}^{k^*} Y_k\otimes X_k \mapsto \sum_{k=1}^{k^*} Y_kX_k.$$

Note that the multiplication of matrices distributes over addition and, for every $r\in R$, $Y\in \mathrm{Mat}_{m,1}(R)$ and $X\in \mathrm{Mat}_{1,n}(R)$, we have $(Yr)X = Y(rX)$, which implies that the mapping $\hat{\phi}\colon\ \mathrm{Mat}_{m,1}(R)\times \mathrm{Mat}_{1,n}(R)\to\mathcal{M}$, $(Y,X)\mapsto YX$ is R-balanced. Thereore ϕ is a well-defined homomorphism of abelian groups (see Proposition 2.11). For every $Z\in\mathcal{M}$ and $\sum_{k=1}^{k^*} Y_k\otimes X_k \in \mathrm{Mat}_{m,1}(R)\otimes_R \mathrm{Mat}_{1,n}(R)$ we have

$$\begin{aligned}\phi\left(Z*\left(\sum_{k=1}^{k^*} Y_k\otimes X_k\right)\right) &= \phi\left(\sum_{k=1}^{k^*}(Z*Y_k)\otimes X_k\right) = \sum_{k=1}^{k^*}(Z*Y_k)X_k\\ &= Z*\left(\sum_{k=1}^{k^*} Y_kX_k\right) = Z*\phi\left(\sum_{k=1}^{k^*} Y_k\otimes X_k\right).\end{aligned}$$

Analogously $\phi((\sum_{k=1}^{k^*} Y_k\otimes X_k)*Z) = \phi(\sum_{k=1}^{k^*} Y_k\otimes X_k)*Z$ and therefore ϕ is a homomorphism of bimodules. By Proposition 3.4 R is idempotent and Lemma 3.6 implies that ϕ is surjective.

Finally note that, for every $X,X'\in \mathrm{Mat}_{1,n}(R)$ and $Y,Y'\in\mathrm{Mat}_{m,1}(R)$, we have

$$\begin{aligned}\theta(X\otimes Y)X' &= (XMY)X' = XM(YX') = X*(YX') = X*\phi(Y\otimes X'),\\ Y'\theta(X\otimes Y) &= Y'(XMY) = (Y'X)MY = (Y'X)*Y = \phi(Y'\otimes X)*Y.\end{aligned}$$

In conclusion, we have shown that

$$(R,\mathcal{M},{}_R(\mathrm{Mat}_{1,n}(R))_{\mathcal{M}},{}_{\mathcal{M}}(\mathrm{Mat}_{m,1}(R))_R,\theta,\phi)$$

is a unitary surjective Morita context between rings R and $\mathcal{M}$. ■

Corollary 3.9. *A Rees matrix ring $\mathcal{M}(R;\Lambda,\Xi;M)$ and the ring R are connected by a unitary surjective Morita context if and only if $\mathcal{M}(R;\Lambda,\Xi;M)$ is idempotent.*

From the previous theorem and Theorem 2.28 we can deduce the following result.

Corollary 3.10. *If a Rees matrix ring $\mathcal{M}=\mathcal{M}(R;\Lambda,\Xi;M)$ is idempotent, then the rings R and $\mathcal{M}$ are Morita equivalent.*

We can also deduce the following classical result as a corollary.

Corollary 3.11. *If R is a ring with identity, then R and the Rees matrix ring $\mathcal{M}(R;n,n;I)=\mathrm{Mat}_n(R)$ are Morita equivalent, where I is the identity matrix.*

3.2 Tensor product rings

In this chapter we will consider tensor product rings. We show how to define a multiplication on the tensor product of modules over an arbitrary ring R using a certain bilinear mapping. This construction is analogous to that of Morita semigroups defined by Talwar in [44] (Section 6). For rings with an identity element and unitary modules this construction appears in [6] (Definition 2.2).

Let R be an arbitrary ring and ${}_RP$ and Q_R arbitrary R-modules. Also, let there be given an (R,R)-bilinear mapping

$$\langle\, ,\, \rangle\colon\ P \times Q \to R.$$

(R,R)-bilinearity here means that, for every $p, p' \in P$, $q, q' \in Q$ and $r \in R$,

$$\begin{aligned}
\langle p+p', q\rangle &= \langle p, q\rangle + \langle p', q\rangle,\\
\langle p, q+q'\rangle &= \langle p, q\rangle + \langle p, q'\rangle,\\
\langle rp, q\rangle &= r\langle p, q\rangle,\\
\langle p, qr\rangle &= \langle p, q\rangle r.
\end{aligned}$$

A pair of modules $({}_RP, Q_R)$ with an (R,R)-bilinear map $\langle\, ,\, \rangle\colon\ P \times Q \to R$ is often called a **pair over** R (e.g. Definition 2.2 in [6] and Definition 1 in [5]).

Define a multiplication $\star$ on the generators of the abelian group $Q \otimes_R P$ by

$$(q \otimes p) \star (q' \otimes p') := q \otimes \langle p, q'\rangle p' \tag{3.3}$$

and extend this definition to all elements of the tensor product $Q \otimes_R P$ via the distributivity property.

Note that, for every pair $(q,p) \in Q \times P$, we can define a mapping

$$f_{q,p}\colon\ Q \times P \to Q \otimes_R P,\ \ (q_1, p_1) \mapsto q_1 \otimes \langle p_1, q\rangle p.$$

The mappings $f_{q,p}$ are all R-balanced, because for every $q_1, q_2 \in Q$, $p_1, p_2 \in P$ and $r \in R$

$$\begin{aligned}
f_{q,p}(q_1+q_2, p_1) &= (q_1+q_2) \otimes \langle p_1, q\rangle p = q_1 \otimes \langle p_1, q\rangle p + q_2 \otimes \langle p_2, q\rangle p\\
&= f_{q,p}(q_1, p_1) + f_{q,p}(q_2, p_1),\\
f_{q,p}(q_1 r, p_1) &= (q_1 r) \otimes \langle p_1, q\rangle p = q_1 \otimes r\langle p_1, q\rangle p = q_1 \otimes \langle rp_1, q\rangle p = f_{q,p}(q_1, rp_1)
\end{aligned}$$

and analogously $f_{p,q}(q_1, p_1+p_2) = f_{q,p}(q_1, p_1) + f_{q,p}(q_1, p_2)$. Therefore there exist endomorphisms

$$\overline{f_{q,p}}\colon\ Q \otimes_R P \to Q \otimes_R P, \qquad \sum_{k=1}^{k^*} q_k \otimes p_k \mapsto \sum_{k=1}^{k^*} q_k \otimes \langle p_k, q\rangle p$$

of the abelian group $Q \otimes_R P$ (see Proposition 2.11).

Now define a mapping

$$\hat{\tau}\colon\ Q \times P \to \mathrm{End}(Q \otimes_R P), \qquad (q,p) \mapsto \overline{f_{q,p}}.$$

Here $\mathrm{End}(Q \otimes_R P)$ is an abelian group with respect to the pointwise addition of endomorphisms. Notice that, for every $q_1, q_2 \in Q$, $p_1, p_2 \in P$, $r \in R$ and $\sum_{k=1}^{k^*} \kappa_k \otimes \rho_k \in Q \otimes P$, we have

$$\begin{aligned}
(\hat{\tau}(q_1+q_2,p_1))\left(\sum_{k=1}^{k^*} \kappa_k \otimes \rho_k\right) &= \overline{f_{q_1+q_2,p_1}}\left(\sum_{k=1}^{k^*} \kappa_k \otimes \rho_k\right)\\
&= \sum_{k=1}^{k^*} \kappa_k \otimes \langle \rho_k, q_1+q_2\rangle p_1\\
&= \sum_{k=1}^{k^*} \kappa_k \otimes \langle \rho_k, q_1\rangle p_1 + \sum_{k=1}^{k^*} \kappa_k \otimes \langle \rho_k, q_2\rangle p_1\\
&= (\hat{\tau}(q_1,p_1) + \hat{\tau}(q_2,p_1))\left(\sum_{k=1}^{k^*} \kappa_k \otimes \rho_k\right),\\
(\hat{\tau}(q_1 r,p_1))\left(\sum_{k=1}^{k^*} \kappa_k \otimes \rho_k\right) &= \overline{f_{q_1 r,p_1}}\left(\sum_{k=1}^{k^*} \kappa_k \otimes \rho_k\right)\\
&= \sum_{k=1}^{k^*} \kappa_k \otimes \langle \rho_k, q_1 r\rangle p_1\\
&= \sum_{k=1}^{k^*} \kappa_k \otimes \langle \rho_k, q_1\rangle r p_1\\
&= (\hat{\tau}(q_1, r p_1))\left(\sum_{k=1}^{k^*} \kappa_k \otimes \rho_k\right)
\end{aligned}$$

and analogously $\hat{\tau}(q_1, p_1+p_2) = \hat{\tau}(q_1,p_1) + \hat{\tau}(q_1,p_2)$. Therefore $\hat{\tau}$ is R-balanced and hence, due to the universal property of the tensor product (Proposition 2.11), there exists a group homomorphism

$$\tau\colon\ Q \otimes P \to \mathrm{End}(Q \otimes P), \qquad q \otimes p \mapsto \overline{f_{q,p}}.$$

Now we can consider the well-defined mapping

$$\overline{\tau}\colon\ (Q \otimes P) \times (Q \otimes P) \to Q \otimes P, \qquad (x,y) \mapsto (\tau(x))(y).$$

We have, for every $q, q' \in Q$ and $p, p' \in P$,

$$\overline{\tau}(q \otimes p, q' \otimes p') = (\tau(p \otimes q))(q' \otimes p') = \overline{f_{q,p}}(q' \otimes p') = q \otimes \langle p, q' \rangle p'.$$

As we can see, the mapping $\overline{\tau}$ coincides with the multiplication $\star$ in definition (3.3), which means that the multiplication $\star$ is well-defined.

Finally notice that, for every $q_1 \otimes p_1, q_2 \otimes p_2, q_3 \otimes p_3 \in Q \otimes P$, we have

$$\begin{aligned}((q_1 \otimes p_1) \star (q_2 \otimes p_2)) \star (q_3 \otimes p_3) &= (q_1 \otimes \langle p_1, q_2 \rangle p_2) \star (q_3 \otimes p_3) \\ &= q_1 \otimes \langle \langle p_1, q_2 \rangle p_2, q_3 \rangle p_3 \\ &= q_1 \otimes \langle p_1, q_2 \rangle \langle p_2, q_3 \rangle p_3 \\ &= (q_1 \otimes p_1) \star (q_2 \otimes \langle p_2, q_3 \rangle p_3) \\ &= (q_1 \otimes p_1) \star ((q_2 \otimes p_2) \star (q_2 \otimes p_3)).\end{aligned}$$

This implies that the multiplication $\star$ is associative and therefore the abelian group $Q \otimes P$ with $\star$ is a ring.

Definition 3.12. Tensor product of modules $Q \otimes_R^\beta P$ with multiplication $\star$ defined in (3.3) is called a **tensor product ring** defined by an (R, R)-bilinear mapping $\beta = \langle\ ,\ \rangle$.

Often we will omit the mapping β from the tensor product symbol, i.e. we write $Q \otimes_R P := Q \otimes_R^\beta P$.

Next we will define the notion of a pseudo-surjective mapping. But first some notation, for any ring R and a set $A \subseteq R$, we will denote by $\langle A \rangle_{\mathrm{s}}$ the subgroup generated by A in the additive group $(R, +)$.

Definition 3.13. Let R be a ring and B a set. We call a mapping $f\colon B \to R$ **pseudo-surjective**, if $\langle \mathrm{Im} f \rangle_{\mathrm{s}} = R$, i.e. the additive subgroup of R generated by the set $\mathrm{Im} f$ is equal to R.

Clearly, every surjective mapping is also pseudo-surjective, but the converse is not always true. Next we will characterize pseudo-surjective bilinear mappings.

Lemma 3.14. *Let $\beta\colon\ {}_R P \times Q_R \to R$ be a bilinear mapping. Then $\langle \mathrm{Im}\beta \rangle_{\mathrm{s}}$ consists of all finite sums of the elements of* $\mathrm{Im}\beta$.

Proof. Let $\beta\colon\ {}_R P \times Q_R \to R$ be a bilinear mapping and $s \in \langle \mathrm{Im}\beta \rangle_{\mathrm{s}}$. Then there exist $k^* \in \mathbb{N}$, $p_1, \ldots, p_{k^*} \in P$ and $q_1, \ldots, q_{k^*} \in Q$ such that

$$s = \pm\beta(p_1, q_1) \pm \ldots \pm \beta(p_{k^*}, q_{k^*}).$$

Note that, for any $p \in P$ and $q \in Q$, we have

$$\beta(p,q) + \beta(-p,q) = \beta(p-p,q) = \beta(0,q) = \beta(0 \cdot 0, q) = 0\beta(0,q) = 0,$$

which proves that $-\beta(p,q) = \beta(-p,q)$. Therefore we can find elements $p'_1, \ldots, p'_{k^*} \in P$ such that $s = \beta(p'_1, q_1) + \ldots + \beta(p'_{k^*}, q_{k^*}) = \sum_{k=1}^{k^*} \beta(p'_k, q_k)$.■

If the mapping $\beta = \langle\, , \rangle$ is pseudo-surjective (surjective), then we say that the corresponding tensor product ring $Q \otimes_R^\beta P$ is **pseudo-surjectively (surjectively) defined**.

Proposition 3.15. *Let R be an idempotent ring and ${}_RP$, Q_R unitary R-modules. Then every pseudo-surjectively defined tensor product ring $Q \otimes_R P$ is idempotent.*

Proof. Let $\sum_{k=1}^{k^*} q_k \otimes p_k \in Q \otimes P$. Since the module ${}_RP$ is unitary, for every $k \in \{1, \ldots, k^*\}$ there exist elements $p_{k1}, \ldots, p_{kh^*} \in P$ and $r_{k1}, \ldots, r_{kh^*} \in R$ such that $p_k = r_{k1}p_{k1} + \ldots + r_{kh^*}p_{kh^*}$. Also, due to the pseudo-surjectivity of $\langle\, , \rangle$, for every $k \in \{1, \ldots, k^*\}$ and $h \in \{1, \ldots, h^*\}$, there exist elements $p_{kh1}, \ldots, p_{khj^*} \in P$ and $q_{kh1}, \ldots, q_{khj^*} \in Q$ such that $r_{kh} = \sum_{j=1}^{j^*} \langle p_{khj}, q_{khj} \rangle$. Therefore we have

$$\begin{aligned}
\sum_{k=1}^{k^*} q_k \otimes p_k &= \sum_{k=1}^{k^*} q_k \otimes \left(\sum_{h=1}^{h^*} r_{kh} p_{kh} \right) = \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} q_k \otimes r_{kh} p_{kh} \\
&= \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} q_k \otimes \left(\sum_{j=1}^{j^*} \langle p_{khj}, q_{khj} \rangle \right) p_{kh} \\
&= \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} q_k \otimes \langle p_{khj}, q_{khj} \rangle p_{kh} \\
&= \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} (q_k \otimes p_{khj}) \star (q_{khj} \otimes p_{kh}) \in (Q \otimes_R P) \star (Q \otimes_R P),
\end{aligned}$$

which implies that the ring $Q \otimes_R P$ is idempotent. ■

Next we will prove a result analogous to Theorem 5 in [44].

Theorem 3.16. *Let R be an idempotent ring, ${}_RP$ and Q_R unitary R-modules and $\langle\, , \rangle\colon\ P \times Q \to R$ a pseudo-surjective (R, R)-bilinear mapping. Then the tensor product ring $Q \otimes_R P$ defined by $\langle\, , \rangle$ is Morita equivalent to R.*

Proof. Define a right and a left $(Q \otimes_R P)$-multiplication on the R-modules $_RP$ and Q_R, respectively, as follows:

$$p\left(\sum_{k=1}^{k^*} q_k \otimes p_k\right) := \sum_{k=1}^{k^*} \langle p, q_k\rangle p_k, \tag{3.4}$$

$$\left(\sum_{k=1}^{k^*} q_k \otimes p_k\right) q := \sum_{k=1}^{k^*} q_k\langle p_k, q\rangle, \tag{3.5}$$

where $p \in P$, $q \in Q$ and $\sum_{k=1}^{k^*} q_k \otimes p_k \in Q \otimes_R P$. We will show that the multiplication (3.4) is well defined. Consider the mapping

$$Q \times P \to \mathrm{End}(_RP), \quad (q,p) \mapsto \langle \_\_, q\rangle p.$$

This mapping is clearly well defined and R-balanced, due to $\langle\, ,\, \rangle$ being (R,R)-bilinear. Now, by the universal property of the tensor product (Proposition 2.11), there exists a well-defined homomorphism of abelian groups

$$\tau\colon\ Q \otimes_R P \to \mathrm{End}(_RP), \quad \sum_{k=1}^{k^*} q_k \otimes p_k \mapsto \sum_{k=1}^{k^*} \langle \_\_, q_k\rangle p_k.$$

The multiplication (3.4) can now be expressed as $p\delta = \tau(\delta)(p)$, for every $p \in P$ and $\delta \in Q \otimes_R P$. Hence, the multiplication (3.4) is well defined. The multiplication (3.5) is analogously well defined. It is easy to check that we obtain bimodules $_RP_{Q\otimes P}$ and $_{Q\otimes P}Q_R$.

Let $p \in P$. Due to P being unitary, there exist $p_1, \dots, p_{k^*} \in P$ and $r_1, \dots, r_{k^*} \in R$ such that $p = r_1p_1 + \ldots + r_{k^*}p_{k^*}$. Also, for every $k \in \{1, \dots, k^*\}$, there exist $p_{k1}, \dots, p_{kh^*} \in P$ and $q_{k1}, \dots, q_{kh^*} \in Q$ such that $r_k = \sum_{h=1}^{h^*} \langle p_{kh}, q_{kh}\rangle$, because $\langle\, ,\, \rangle$ is pseudo-surjective. Now

$$p = \sum_{k=1}^{k^*} r_kp_k = \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} \langle p_{kh}, q_{kh}\rangle p_k = \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} p_{kh}(q_{kh} \otimes p_k) \in P(Q \otimes_R P),$$

which implies that $P_{Q\otimes P}$ is a unitary right module and therefore $_RP_{Q\otimes P}$ is a unitary bimodule. Analogously $_{Q\otimes P}Q_R$ is a unitary bimodule.

Define a mapping

$$\theta\colon\ _R(P \otimes_{Q\otimes R} Q)_R \to {_RR_R}, \qquad \sum_{h=1}^{h^*} p_h \otimes q_h \mapsto \sum_{h=1}^{h^*} \langle p_h, q_h\rangle.$$

Since $\langle\, ,\, \rangle$ is additive and, for every $p, p' \in P$ and $q, q' \in Q$,

$$\langle p(q' \otimes p'), q\rangle = \langle \langle p, q'\rangle p', q\rangle = \langle p, q'\rangle\langle p', q\rangle = \langle p, q'\langle p', q\rangle\rangle = \langle p, (q' \otimes p')q\rangle$$

we see that the mapping $\hat{\theta}\colon\ P \times Q \to R$, $(p,q) \mapsto \langle p,q\rangle$ is $(Q\otimes P)$-balanced. Now the universal property of the tensor product (see Proposition 2.11) implies that θ is a well-defined homomorphism of abelian groups. The bracket $\langle\,,\,\rangle$ being (R,R)-bilinear and pseudo-surjective implies that θ is a surjective homomorphism of bimodules.

Notice that, for every $p, p' \in P$ and $q, q' \in Q$, we have

$$\theta(p\otimes q)p' = \langle p,q\rangle p' = p(q\otimes p') = p\,\mathrm{id}(q\otimes p'),$$
$$q'\theta(p\otimes q) = q'\langle p,q\rangle = (q'\otimes p)q = \mathrm{id}(q'\otimes p)q.$$

In conclusion, we have shown that $(R, Q\otimes_R^{\hat{\theta}} P, P, Q, \theta, \mathrm{id}_{Q\otimes P})$ is a unitary surjective Morita context. By Proposition 3.15, the ring $Q\otimes_R P$ is idempotent and now, by Theorem 2.28, we conclude that the rings R and $Q\otimes_R P$ are Morita equivalent. ■

Let A be an abelian group, $P\otimes_S Q$ be an arbitrary tensor product of S-modules and $\psi\colon\ P\otimes_S Q \to A$ a homorphism of abelian groups. Denote $\hat{\psi} := \psi\circ\otimes$, i.e., for every $p\in P$ and $q \in Q$, we have

$$\hat{\psi}(p,q) = \psi(p\otimes q).$$

Then the mapping $\hat{\psi}\colon\ P\times Q \to A$ is clearly S-balanced. If ${}_RP_S$ and ${}_SQ_R$ are (R,S)- and (S,R)-bimodules, respectively, then $\hat{\psi}$ is also (R,R)-bilinear. If $\psi\colon P\otimes_R Q \to A$ is surjective, then $\hat{\psi}$ is pseudo-surjective, because in that case, for every $a\in A$ there exists $\sum_{k=1}^{k^*} p_k\otimes q_k \in P\otimes_R Q$ such that

$$a = \psi\left(\sum_{k=1}^{k^*} p_k\otimes q_k\right) = \sum_{k=1}^{k^*}\psi(p_k\otimes q_k) = \sum_{k=1}^{k^*}\hat{\psi}(p_k,q_k) \in \langle \mathrm{Im}\hat{\psi}\rangle_{\mathrm{s}}.$$

Next we give a simple corollary of Theorem 3.16, which is a ring-theoretic analogue of Proposition 4.7 in [26].

Corollary 3.17. *Let R be an idempotent ring. The rings R and $R\otimes_R^{\hat{\nu}} R$ are Morita equivalent with a corresponding surjective unitary Morita context $(R, R\otimes_R^{\hat{\nu}} R, R, R, \nu, \mathrm{id}_{R\otimes R})$, where $\nu\colon R\otimes_R R \to R$, $\sum_{k=1}^{k^*} r_k\otimes r'_k \mapsto \sum_{k=1}^{k^*} r_k r'_k$.*

If R is idempotent, then $R\otimes_R R$ is firm by Proposition 3.2 in [47]. Thus we can say that each idempotent ring is Morita equivalent to a firm ring.

Now we will prove an analogue of Proposition 4 in [44].

Proposition 3.18. *Let $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ be a unitary surjective Morita context connecting idempotent rings R and S, and let $Q\otimes_R^{\hat{\theta}} P$, $P\otimes_S^{\hat{\phi}} Q$ be tensor product rings defined by the mappings $\hat{\theta}$, $\hat{\phi}$, respectively. Then the rings R, S, $P\otimes_S^{\hat{\phi}} Q$ and $Q\otimes_R^{\hat{\theta}} P$ are all Morita equivalent.*

Proof. The definition of the tensor product implies that the mappings $\hat{\theta}\colon\ P \times Q \to R$ and $\hat{\phi}\colon\ Q \times P \to S$ are R- and S-balanced, respectively. Since θ and ϕ are surjective, the tensor product rings $P \otimes_S Q$ and $Q \otimes_R P$ are well- and pseudo-surjectively defined. By Theorem 3.16 we obtain Morita equivalences $R \approx_{\mathrm{ME}} Q \otimes_R P$ and $S \approx_{\mathrm{ME}} P \otimes_S Q$. By Theorem 2.28 and the transitivity of Morita equivalence we obtain the equivalences $R \approx_{\mathrm{ME}} S$ and $Q \otimes_R P \approx_{\mathrm{ME}} P \otimes_S Q$ (and also all other combinations). ■

In order to prove our next theorem, we must define locally injective homomorphisms and strict local isomorphisms of rings. Strict local isomorphisms for semigroups were first introduced by Márki and Steinfeld in [35].

Definition 3.19. We call a homomorphism $\tau\colon\ R \to S$ of rings **locally injective** if its restriction to any subring of the form aRb, where $a \in Ra$ and $b \in bR$, is injective.

A locally injective homomorphism of rings, which is also surjective, is called a **strict local isomorphism**.

Obviously, every injective ring homomorphism is also locally injective and every ring isomorphism is a strict local isomorphism. Later, in Example 3.22, we will see that there exist non-injective homomorphisms, which are locally injective.

We will give a description of locally injective ring homomorphisms $f\colon$ $S \to R$, where S is an s-unital ring.

Lemma 3.20. *Let S be a right s-unital ring and $f\colon S \to R$ a ring homomorphism. Then f is locally injective if and only if $f|_{Ss}$ is injective for every $s \in S$.*

Proof. *Necessity.* Let $f\colon\ S \to R$ be a locally injective ring homomorphism. Let $s \in S$ and consider the restriction $f|_{sS}$. Let $ss' \in \mathrm{Ker}(f|_{sS})$. Since S is right s-unital, there exists $u \in S$ such that $ss' = ss'u$. Now there also exists a $v \in S$ such that $u = uv \in uS$ and

$$ss' = ss'u \in \mathrm{Ker}(f|_{sSu}) = \{0\}.$$

(The last equality holds, because f is locally injective.) Hence $ss' = 0$, which proves that $f|_{sS}$ is injective.

Sufficiency. Let $s \in S$ and $f|_{sS}$ be injective. Note that for every $s' \in S$ we have

$$sSs' \subseteq sS,$$

which means that $f|_{sSs'}$ is a restriction of $f|_{sS}$ and therefore also injective. As can be seen, this implication actually does not assume anything from the ring S. ■

Next we will prove a useful proposition about locally injective homomorphisms and strict local isomorphisms of rings. Roughly speaking it says that strict local isomorphisms between rings are more or less the same thing as linear functionals.

Proposition 3.21. *Let R be a ring, M_R be an R-module and $f\colon\ M_R \to R_R$ a homomorphism of modules. If we define a multiplication on the abelian group M by*

$$m \bullet m' := mf(m'), \qquad (m, m' \in M), \tag{3.6}$$

then we obtain a ring and f is a locally injective homomorphism of rings. If S is a right s-unital ring then all strict local isomorphisms $S \to R$ can be obtained using this construction.

Proof. Let R be a ring, M_R an R-module and $f\colon\ M_R \to R_R$ a homomorphism of right R-modules. It is easy to see that M is a ring, where multiplication $\bullet$ is defined by (3.6), and that f is a homomorphism of rings.

Let $a = a' \bullet a \in M \bullet a$ and $b = b \bullet b' \in b \bullet M$. Also let $\rho = a \bullet \rho' \bullet b$ be such that, $f(\rho) = 0$. Then

$$\begin{aligned}\rho &= a \bullet \rho' \bullet b = (a' \bullet a) \bullet \rho' \bullet (b \bullet b') = a' \bullet (a \bullet \rho' \bullet b) \bullet b' = a' \bullet \rho \bullet b' \\ &= a' f(\rho \bullet b') = a' f(\rho f(b')) = a' f(\rho) f(b') = a' 0 f(b') = 0.\end{aligned}$$

Hence $\mathrm{Ker}(f|_{a\bullet M\bullet b}) = \{0\}$, which proves that f is locally injective.

To prove the second claim, we consider a strict local isomorphism $f\colon$ $S \to R$ where S is a right s-unital ring. We turn the abelian group $(S, +)$ into a right R-module by defining

$$s \cdot r := ss',$$

where $r \in R$, $s, s' \in S$ and $f(s') = r$ (using the surjectivity of f). We need to check if this is well defined. Suppose that also $f(s'') = r$. Then $f(ss') = f(s)f(s') = f(s)f(s'') = f(ss'')$. By Lemma 3.20, $f|_{sS}$ is injective, implying $ss' = ss''$, as required. We see that $f : S_R \to R_R$ is a module homomorphism by noticing that, for every $s, s' \in S$ and $r \in R$ such that $f(s') = r$, we have

$$f(s \cdot r) = f(ss') = f(s)f(s') = f(s)r.$$

If we now define a ring multiplication $\bullet$ on S using the module homomorphism f and the rule (3.6) then $\bullet$ coincides with the original multiplication of S, because $s \bullet s' = s \cdot f(s') = ss'$ for every $s, s' \in S$. ∎

The previous proposition gives us a way to construct many locally injective homomorphisms of rings. One such will be constructed in the following example.

Example 3.22 (Non-injective locally injective homomorphism). Let R be an s-unital ring (e.g. $\mathbb{Z}$). Consider the direct product $R \times R$ as a (right) R-module with componentwise addition and scalar multiplication. The mapping

$$f\colon\; R \times R \to R, \qquad (a, b) \mapsto a$$

is clearly a non-injective homomorphism of R-modules. We can turn $R \times R$ into a ring with multiplication defined as in (3.6). Now we see that f is a homomorphism of rings that is locally injective, but not injective. As f is surjective, it is also a strict local isomorphism. □

Now we are ready to prove a theorem which says that whenever R and S are arbitrary rings and $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ is a Morita context (not necessarily unitary or surjective), then there exist locally injective homomorphisms $P \otimes_S Q \to R$ and $Q \otimes_R P \to S$.

Theorem 3.23. *Let R and S be rings that are connected by a Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Consider the tensor product ring $P \otimes_S^{\hat{\phi}} Q$ defined by $\hat{\phi}$. Then $\theta\colon\; P \otimes_S^{\hat{\phi}} Q \to R$ is a locally injective homomorphism of rings.*

Proof. Let $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ be a Morita context. Notice that for every $\sum_{k=1}^{k^*} p_k \otimes q_k, \sum_{h=1}^{h^*} p'_h \otimes q'_h \in P \otimes_S Q$, we have

$$\begin{aligned}
\left(\sum_{k=1}^{k^*} p_k \otimes q_k\right) \star \left(\sum_{h=1}^{h^*} p'_h \otimes q'_h\right) &= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} p_k \otimes \hat{\phi}(q_k, p'_h)q'_h \\
&= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} p_k \otimes \phi(q_k \otimes p'_h)q'_h \\
&= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} p_k \otimes q_k\theta(p'_h \otimes q'_h) \\
&= \left(\sum_{k=1}^{k^*} p_k \otimes q_k\right) \theta \left(\sum_{h=1}^{h^*} p'_h \otimes q'_h\right).
\end{aligned}$$

Therefore, the multiplication $\star$ of the ring $P \otimes_S^{\hat{\phi}} Q$ is defined using the right R-module homomorphism $\theta : (P \otimes_S Q)_R \to R_R$. By Proposition 3.21, θ is a locally injective homomorphism of rings. ■

Corollary 3.24. *Let R and S be two Morita equivalent idempotent rings. Then there exist pseudo-surjectively defined tensor product rings $Q \otimes_R P$, $P \otimes_S Q$ and strict local isomorphisms $Q \otimes_R P \to S$ and $P \otimes_S Q \to R$.*

PROOF. Let R and S be idempotent rings such that $R \approx_{\mathrm{ME}} S$. Then, by Theorem 2.28, there exists a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. By Theorem 3.23, the mapping $\theta\colon\ P \otimes_S^{\hat{\phi}} Q \to R$ is a locally injective homomorphism of rings. Since θ is also surjective, θ is a strict local isomorphism. Analogously $\phi\colon\ Q \otimes_R^{\hat{\theta}} P \to S$ is a strict local isomorphism. ■

It turns out that if either of the mappings $P \otimes_S Q \to R$ or $Q \otimes_R P \to S$ is an isomorphism, then the converse of the previous corollary also holds.

Proposition 3.25. *Let R and S be idempotent rings. If R is isomorphic to some pseudo-surjectively defined tensor product ring $P \otimes_S Q$, where P_S and ${}_SQ$ are unitary modules, then the rings R and S are Morita equivalent.*

PROOF. Let R be isomorphic to some pseudo-surjectively defined tensor product ring $P \otimes_S Q$. The rings $P \otimes_S Q$ and S are Morita equivalent by Theorem 3.16. Since isomorphic rings are obviously Morita equivalent and Morita equivalence is transitive, we have that $R \approx_{\mathrm{ME}} S$. ■

3.3 Tensor product rings and adjoint endomorphisms

In this section we will explore the relationship between tensor product rings and rings of adjoint endomorphisms of modules.

Let ${}_RP$ and Q_R be R-modules and $\beta = \langle\, , \rangle\colon\ P \times Q \to R$ be an (R, R)-bilinear mapping. Adjoint endomorphisms of modules over a ring with local units were introduced in [5] (Definition 2).

Definition 3.26. Module endomorphisms $f \in \mathrm{End}({}_RP)$ and $g \in \mathrm{End}(Q_R)$ are called **adjoint** (with respect to $\beta = \langle\, , \rangle$) if, for every $p \in P$ and $q \in Q$, we have

$$\langle f(p), q\rangle = \langle p, g(q)\rangle.$$

We will denote the set of all pairs (f, g) of adjoint endomorphisms with respect to β by Ω^β. The set Ω^β is a subring of $((\mathrm{End}({}_RP))^{\mathrm{op}} \times \mathrm{End}(Q_R); +, \circ)$, where for every $f, f' \in \mathrm{End}({}_RP)$ and $g, g' \in \mathrm{End}(Q_R)$

$$(f, g) + (f', g') = (f + f', g + g'),$$

$$(f,g)\circ(f',g')=(f'\circ f,g\circ g')$$

and $(\mathrm{End}({}_RP))^{\mathrm{op}}$ denotes the opposite ring of $\mathrm{End}({}_RP)$.

Next we will introduce an important type of pairs of adjoint endomorphisms.

Lemma 3.27. *Let ${}_RP$ and Q_R be R-modules and $\beta=\langle\,,\,\rangle\colon\ P\times Q\to R$ an (R,R)-bilinear mapping. For any $k^*\in\mathbb{N}$, $p_1,\dots,p_{k^*}\in P$ and $q_1,\dots,q_{k^*}\in Q$, the mappings*

$$f:=\sum_{k=1}^{k^*}\langle\_\_,q_k\rangle p_k\colon\ {}_RP\to{}_RP \quad \text{and} \quad g:=\sum_{k=1}^{k^*}q_k\langle p_k,\_\_\rangle\colon\ Q_R\to Q_R \tag{3.7}$$

are adjoint endomorphisms.

Proof. Clearly the mappings f and g are endomorphisms of modules, due to $\beta=\langle\,,\,\rangle$ being (R,R)-bilinear. Note that, for every $p\in P$ and $q\in Q$, we have

$$\langle f(p),q\rangle=\left\langle\sum_{k=1}^{k^*}\langle p,q_k\rangle p_k,q\right\rangle=\sum_{k=1}^{k^*}\langle p,q_k\rangle\langle p_k,q\rangle=\left\langle p,\sum_{k=1}^{k^*}q_k\langle p_k,q\rangle\right\rangle=\langle p,g(q)\rangle,$$

which means that f and g are adjoint. Therefore $(f,g)\in\Omega^\beta$. ■

We will call the endomorphisms f and g from (3.7) **β-basic endomorphisms** of ${}_RP$ and Q_R, respectively.

Now we will study the subring of endomorfism pairs given by (3.7) more closely. Denote

$$\Sigma^\beta:=\left\{\sum_{k=1}^{k^*}(\langle\_\_,q_k\rangle p_k,q_k\langle p_k,\_\_\rangle)\in\Omega^\beta\,\middle|\,k^*\in\mathbb{N};\forall k\colon\ p_k\in P,q_k\in Q\right\}.$$

It can easily be seen from Lemma 3.27 that Σ^β is a subring of Ω^β. In fact Σ^β is the set of all pairs $(f,g)\in\Omega^\beta$ given by (3.7).

Theorem 3.28. *Let R be a ring. Then, for every (R,R)-bilinear mapping $\beta=\langle\,,\,\rangle\colon\ {}_RP\times Q_R\to R$, there exists a strict local isomorphism $Q\otimes_R^\beta P\to\Sigma^\beta$ of rings.*

Proof. Let R be a ring and $\beta=\langle\,,\,\rangle\colon\ {}_RP\times Q_R\to R$ an (R,R)-bilinear mapping. Define a mapping

$$\varphi\colon\ Q\otimes_R^\beta P\to\Sigma^\beta,\qquad \sum_{k=1}^{k^*}q_k\otimes p_k\mapsto\sum_{k=1}^{k^*}(\langle\_\_,q_k\rangle p_k,q_k\langle p_k,\_\_\rangle). \tag{3.8}$$

Consider the mapping $\hat{\varphi}\colon\ Q \times P \to \Sigma^\beta$, $(q,p) \mapsto (\langle \_, q\rangle p, q\langle p, \_\rangle)$. It is easy to see that $\hat{\varphi}$ is R-balanced, which means that, due to the universal property of tensor product (Proposition 2.11), φ is a well-defined homomorphism of abelian groups.

Let $\sum_{k=1}^{k^*} q_k \otimes p_k, \sum_{h=1}^{h^*} q'_h \otimes p'_h \in Q \otimes_R^\beta P$, then

$$\begin{aligned}
\varphi\left(\left(\sum_{k=1}^{k^*} q_k \otimes p_k\right) \star \left(\sum_{h=1}^{h^*} q'_h \otimes p'_h\right)\right) &= \varphi\left(\sum_{k=1}^{k^*}\sum_{h=1}^{h^*} q_k \otimes \langle p_k, q'_h\rangle p'_h\right) \\
&= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} (\langle _, q_k\rangle \langle p_k, q'_k\rangle p'_h, q_k\langle \langle p_k, q'_h\rangle p'_h, _\rangle) \\
&= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} (\langle \langle _, q_k\rangle p_k, q'_h\rangle p'_h, q_k\langle p_k, q'_h\langle p'_h, _\rangle\rangle) \\
&= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} (\langle _, q_k\rangle p_k, q_k\langle p_k, _\rangle) \circ (\langle _, q'_h\rangle p'_h, q'_h\langle p'_h, _\rangle) \\
&= \left(\sum_{k=1}^{k^*} \langle _, q_k\rangle p_k, q_k\langle p_k, _\rangle)\right) \circ \left(\sum_{h=1}^{h^*} (\langle _, q'_h\rangle p'_h, q'_h\langle p'_h, _\rangle)\right) \\
&= \varphi\left(\sum_{k=1}^{k^*} q_k \otimes p_k\right) \circ \varphi\left(\sum_{h=1}^{h^*} q'_h \otimes p'_h\right).
\end{aligned}$$

Therefore φ is a homomorphism of rings. Clearly, φ is surjective.

Let

$$\kappa = \sum_{k=1}^{k^*}\sum_{h=1}^{h^*}\sum_{j=1}^{j^*} (a_k \otimes b_k) \star (q_h \otimes p_h) \star (c_j \otimes d_j) \in \alpha * (Q \otimes_R^\beta P) \star \gamma,$$

where

$$\alpha = \sum_{k=1}^{k^*} a_k \otimes b_k = \sum_{x=1}^{x^*}\sum_{k=1}^{k^*} (a'_x \otimes b'_x) \star (a_k \otimes b_k) \in (Q \otimes_R^\beta P) \star \alpha$$

and $\gamma = \sum_j c_j \otimes d_j \in \gamma \star (Q \otimes_R^\beta P)$, be such that $\varphi(\kappa) = 0$. Then

$$\begin{aligned}
\varphi(\kappa) &= \varphi\left(\sum_{k=1}^{k^*}\sum_{h=1}^{h^*}\sum_{j=1}^{j^*} (a_k \otimes b_k) \star (q_h \otimes p_h) \star (c_j \otimes d_j)\right) \\
&= \varphi\left(\sum_{k=1}^{k^*}\sum_{h=1}^{h^*}\sum_{j=1}^{j^*} a_k \otimes \langle b_k, q_h\rangle \langle p_h, c_j\rangle d_j\right)
\end{aligned}$$

$$= \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} (\langle \_, a_k \rangle \langle b_k, q_h \rangle \langle p_h, c_j \rangle d_j, a_k \langle b_k, q_h \rangle \langle p_h, c_j \rangle \langle d_j, \_ \rangle) = 0.$$

Thus

$$\sum_{k=1}^{k^*} \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} \langle b'_x, a_k \rangle \langle b_k, q_h \rangle \langle p_h, c_j \rangle d_j = 0$$

for every $x \in \{1, \ldots, x^*\}$ and therefore

$$\begin{aligned}
\kappa &= \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} (a_k \otimes b_k) \star (q_h \otimes p_h) \star (c_j \otimes d_j) \\
&= \sum_{x=1}^{x^*} \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} (a'_x \otimes b'_x) \star (a_k \otimes b_k) \star (q_h \otimes p_h) \star (c_j \otimes d_j) \\
&= \sum_{x=1}^{x^*} \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} a'_x \otimes \langle b'_x, a_k \rangle \langle b_k, q_h \rangle \langle p_h, c_j \rangle d_j \\
&= \sum_{x=1}^{k^*} a'_x \otimes 0 = 0.
\end{aligned}$$

Hence $\operatorname{Ker}(\varphi|_{\alpha \star (Q \otimes P) \star \gamma}) = \{0\}$, which implies that φ is locally injective. In conclusion, we have proved that $\varphi \colon\ Q \otimes_R^\beta P \to \Sigma^\beta$ is a strict local isomorphism of rings. ∎

In order to strengthen the previous theorem, we must define the notion of a dual bilinear bracket.

Definition 3.29. An (R, R)-bilinear mapping $\langle\ ,\ \rangle\colon\ {}_R P \times Q_R \to {}_R R_R$ is said to be a **dual mapping**, if

(1) for every finite subset $Y \subseteq Q$, there exist $p_1, \ldots, p_{k^*} \in P$ and $q_1, \ldots, q_{k^*} \in Q$ such that for every $y \in Y$

$$y = \sum_{k=1}^{k^*} q_k \langle p_k, y \rangle;$$

(2) for every finite subset $X \subseteq P$, there exist $p_1, \ldots, p_{h^*} \in P$ and $q_1, \ldots, q_{h^*} \in Q$ such that for every $x \in X$

$$x = \sum_{h=1}^{h^*} \langle x, q_h \rangle p_h.$$

As can be seen, the previous definition could also be stated as follows: an (R,R)-bilinear mapping $\beta\colon {}_RP \times Q_R \to {}_RR_R$ is said to be dual if for any finite subset $Y \subseteq Q$, there exists a β-basic endomorphism of Q_R for which every $y \in Y$ is a fixed point; and for every finite subset $X \subseteq P$, there exists a β-basic endomorphism of ${}_RP$ for which every $x \in X$ is a fixed point.

It is easy to see that every locally projective pair (Definition 3 in [5]) is dual in the sense of the previous definition. Next we will give two examples, which show that dual mappings occur naturally in algebra.

Example 3.30 (Dual mapping I). Let V be a Euclidean space. It can be considered as a right or a left $\mathbb{R}$-module. The inner product of V is an $(\mathbb{R},\mathbb{R})$-bilinear mapping $\langle\, , \rangle\colon {}_{\mathbb{R}}V \times V_{\mathbb{R}} \to \mathbb{R}$. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis for V. Then

$$x = \sum_{h=1}^{n} \langle x, e_h\rangle e_h,$$

for every $x \in V$, thus (2) is satisfied for all subsets of V (not only finite). Similarly (1) is satisfied. Hence the inner product of any Euclidean space is a dual mapping. $\square$

We will give an example, which shows that two dual mappings arise naturally from a unitary surjective Morita context connecting s-unital rings.

Example 3.31 (Dual mapping II). Let R and S be s-unital rings that are connected by a unitary surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. We will show that

$$\hat{\theta} : {}_RP \times Q_R \to {}_RR_R, \quad (p,q) \mapsto \theta(p \otimes q)$$

is a dual mapping. (For $\hat{\phi}$ a similar proof works.)

Take a finite set $Y = \{y_1,\dots,y_n\} \subseteq Q$. Since ${}_SQ$ is unitary, every $y_k \in Y$ can be expressed as $y_k = \sum_{h=1}^{h^*} s_{kh}q_{kh}$, where $s_{kh} \in S$ and $q_{kh} \in Q$ for every $h \in \{1,\dots,h^*\}$. Due to left s-unitality, there exists $u \in S$ such that $s_{kh} = us_{kh}$ for every $k \in \{1,\dots,n\}$ and $h \in \{1,\dots,h^*\}$.

Since ϕ is surjective there exists $\sum_{j=1}^{j^*} q_j \otimes p_j \in Q \otimes_R P$ such that

$$u = \phi\left(\sum_{j=1}^{j^*} q_j \otimes p_j\right) = \sum_{j=1}^{j^*} \phi(q_j \otimes p_j).$$

Now, for every $k \in \{1,\dots,n\}$,

$$y_k = \sum_{h=1}^{h^*} s_{kh}q_{kh} = \sum_{h=1}^{h^*} us_{kh}q_{kh} = \sum_{h=1}^{h^*}\sum_{j=1}^{j^*} \phi(q_j \otimes p_j)s_{kh}q_{kh}$$

$$= \sum_{h=1}^{h^*} \sum_{j=1}^{j^*} q_j \theta(p_j \otimes s_{kh} q_{kh}) = \sum_{j=1}^{j^*} q_j \theta \left(p_j \otimes \sum_{h=1}^{h^*} s_{kh} q_{kh} \right)$$
$$= \sum_{j=1}^{j^*} q_j \theta \left(p_j \otimes y_k \right) = \sum_{j=1}^{j^*} q_j \hat{\theta}(p_j, y_k).$$

This proves condition (1) of the definition of duality. The proof of condition (2) is analogous using right s-unitality of S. □

Next we will prove that some dual mappings induce a Morita context.

Proposition 3.32. *Let R be a ring and $\beta = \langle \, , \rangle \colon {}_R P \times Q_R \to {}_R R_R$ a pseudo-surjective dual mapping. Then R is idempotent and the rings R and Σ^β are Morita equivalent.*

Proof. Let R be a ring and $\beta = \langle \, , \rangle \colon {}_R P \times Q_R \to {}_R R_R$ a dual mapping such that $\langle \operatorname{Im} \beta \rangle_{\mathrm{s}} = R$. To turn the abelian group P into a bimodule ${}_R P_{\Sigma^\beta}$ and the abelian group Q into a bimodule ${}_{\Sigma^\beta} Q_R$ we define

$$p(f, g) := f(p),$$
$$(f, g)q := g(q).$$

These multiplications are clearly well defined and turn P into a right Σ^β-module and Q into a left Σ^β-module. Let $r \in R$, $(f, g) \in \Sigma^\beta$ and $p \in P$. Then

$$(rp)(f, g) = f(rp) = rf(p) = r(p(f, g)).$$

Analogously we have $((f, g)q)r = (f, g)(qr)$ for any $q \in Q$. Hence ${}_R P_{\Sigma^\beta}$ and ${}_{\Sigma^\beta} Q_R$ are bimodules. Take $p \in P$. Then there exist $q_1, \ldots, q_{h^*} \in Q$ and $p_1, \ldots, p_{h^*} \in P$ such that $p = \sum_{h=1}^{h^*} \langle p, q_h \rangle p_h$, because of the duality of $\langle \, , \rangle$. Now note that $p \in RP$ and

$$p = \sum_{h=1}^{h^*} \langle p, q_h \rangle p_h = p \left(\sum_{h=1}^{h^*} (\langle \_\_, q_h \rangle p_h, q_h \langle p_h, \_\_ \rangle) \right) \in P\Sigma^\beta.$$

Hence ${}_R P_{\Sigma^\beta}$ is unitary. The bimodule ${}_{\Sigma^\beta} Q_R$ is analogously unitary.

We define

$$\theta \colon \; P \otimes_{\Sigma^\beta} Q \to R, \quad \sum_{k=1}^{k^*} p_k \otimes q_k \mapsto \sum_{k=1}^{k^*} \langle p_k, q_k \rangle,$$
$$\phi \colon \; Q \otimes_R P \to \Sigma^\beta, \quad \sum_{k=1}^{k^*} q_k \otimes p_k \mapsto \sum_{k=1}^{k^*} (\langle \_\_, q_k \rangle p_k, q_k \langle p_k, \_\_ \rangle).$$

Consider the mapping $\hat{\theta}\colon P \times Q \to R$, $(p,q) \mapsto \langle p,q\rangle$. Clearly $\hat{\theta}$ is additive in both of its arguments. Note that, for every $p \in P$, $q \in Q$ and $(f,g) \in \Sigma^\beta$, we have

$$\hat{\theta}(p(f,g),q) = \langle p(f,g),q\rangle = \langle f(p),q\rangle = \langle p,g(q)\rangle = \langle p,(f,g)q\rangle = \hat{\theta}(p,(f,g)q),$$

which proves that $\hat{\theta}$ is Σ^β-balanced. By the universal property of the tensor product (see Proposition 2.11), θ is a well-defined homomorphism of abelian groups. By the (R,R)-bilinearity of β, $\hat{\theta}$ is also a homomorphism of (R,R)-bimodules. The homomorphism θ is surjective, because $\langle \operatorname{Im}\beta\rangle_{\mathrm{s}} = R$.

The mapping ϕ is a well-defined homomorphism of bimodules due to Theorem 3.28. The homomorphism ϕ is clearly surjective.

Finally, note that, for every $p,p' \in P$ and $q,q' \in Q$, we have

$$\begin{aligned}\theta(p\otimes q)p' &= \langle p,q\rangle p' = p(\langle \_\_,q\rangle p', q\langle p',\_\_\rangle) = p\phi(q\otimes p'),\\ q'\theta(p\otimes q) &= q'\langle p,q\rangle = (\langle \_\_,q'\rangle p, q'\langle p,\_\_\rangle)q = \phi(q',p)q.\end{aligned}$$

In conclusion, $(R, \Sigma^\beta, {}_RP_{\Sigma^\beta}, {}_{\Sigma^\beta}Q_R, \theta, \phi)$ is a unitary surjective Morita context.

Since R and Σ^β are connected by a unitary surjective Morita context, we conclude that they are both idempotent by Proposition 2.27. Due to Theorem 2.28, we know that R and Σ^β are Morita equivalent rings. ∎

Note that for instance every surjective dual mapping $\beta\colon\ P \times Q \to R$ clearly satisfies $\langle \operatorname{Im}\beta\rangle_{\mathrm{s}} = R$, i.e. is pseudo-surjective, and therefore induces a unitary surjective Morita context.

Next we will show that Σ^β is isomorphic to a subring of $\operatorname{End}(Q_R)$. Similarly we could show that Σ^β is also isomorphic to an analogous subring of $(\operatorname{End}({}_RP))^{\mathrm{op}}$.

Proposition 3.33. *If R is a ring and $\beta = \langle\,,\,\rangle\colon\ {}_RP \times Q_R \to {}_RR_R$ is a dual mapping, then Σ^β is isomorphic to the subring*

$$\Pi^\beta := \left\{ \sum_{k=1}^{k^*} q_k\langle p_k,\_\_\rangle \,\middle|\, k^* \in \mathbb{N};\ \forall k\colon\ q_k \in Q, p_k \in P \right\} \tag{3.9}$$

of the endomorphism ring $\operatorname{End}(Q_R)$.

Proof. Let $\beta := \langle\,,\,\rangle\colon\ {}_RP \times Q_R \to {}_RR_R$ be a dual mapping. Define

$$\psi\colon\ \Sigma^\beta \to \operatorname{End}(Q_R), \qquad (f,g) \mapsto g.$$

Clearly ψ is a ring homomorphism, whose image is Π^β.

Let $(f,g) \in \Sigma^\beta$ be such that $g = 0$. Take an arbitrary $p \in P$. Since $\langle\, ,\rangle$ is dual, there exist $p_1,\dots,p_{k^*} \in P$ and $q_1,\dots,q_{k^*} \in Q$ such that $f(p) = \sum_{k=1}^{k^*}\langle f(p), q_k\rangle p_k$. Now

$$f(p) = \sum_{k=1}^{k^*}\langle f(p), q_k\rangle p_k = \sum_{k=1}^{k^*}\langle p, g(q_k)\rangle p_k = \sum_{k=1}^{k^*}\langle p, 0\rangle p_k = \sum_{k=1}^{k^*}\langle p, 0\rangle 0 p_k = 0.$$

Therefore $f = 0$, which implies that $\operatorname{Ker}\psi = \{0\}$. In conclusion, φ is an isomorphism of rings Σ^β and Π^β. ■

Corollary 3.34. *Let R be a ring and $\beta = \langle\, ,\rangle\colon {}_RP \times Q_R \to {}_RR_R$ a pseudo-surjective dual mapping. Then R is idempotent and the rings R and Π^β are Morita equivalent.*

The following result generalizes Proposition 2.2 in [5].

Proposition 3.35. *Let R be a ring. If $\langle\, ,\rangle\colon\ {}_RP \times Q_R \to {}_RR_R$ is a dual (R,R)-bilinear mapping, then the tensor product ring $Q \otimes_R P$ defined by $\langle\, ,\rangle$ is s-unital.*

Proof. Let $\beta = \langle\, ,\rangle\colon\ {}_RP \times Q_R \to {}_RR_R$ be a dual (R,R)-bilinear mapping. Fix an element $x = \sum_{k=1}^{k^*} q_k \otimes p_k \in Q \otimes_R^\beta P$. Consider the set $\{q_1,\dots,q_{k^*}\} \subseteq Q$. By the duality of $\langle\, ,\rangle$, there exist elements $p'_1,\dots,p'_{h^*} \in P$ and $q'_1,\dots,q'_{h^*} \in Q$ such that, for every $k \in \{1,\dots,k^*\}$, we have $q_k = \sum_{h=1}^{h^*} q'_h\langle p'_h, q_k\rangle$. Denote $a := \sum_{h=1}^{h^*} q'_h \otimes p'_h$. Now

$$\begin{aligned} x &= \sum_{k=1}^{k^*} q_k \otimes p_k = \sum_{k=1}^{k^*}\left(\sum_{h=1}^{h^*} q'_h\langle p'_h, q_k\rangle\right) \otimes p_k = \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} q'_h \otimes \langle p'_h, q_k\rangle p_k \\ &= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*}(q'_h \otimes p'_h) \star (q_k \otimes p_k) = \left(\sum_{h=1}^{h^*} q'_h \otimes p'_h\right) \star \left(\sum_{k=1}^{k^*} q_k \otimes p_k\right) = a \star x. \end{aligned}$$

Analogously we can construct an element $b \in Q \otimes_R^\beta P$ such that $x = x \star b$, which implies that $Q \otimes_R^\beta P$ is an s-unital ring. ■

Now we can prove a theorem which says that the subring Σ^β of the ring Ω^β of adjoint endomorphisms is isomorphic to a tensor product ring if the underlying bilinear bracket is dual.

Theorem 3.36. *Let R be a ring and $\beta = \langle\, ,\rangle\colon\ {}_RP \times Q_R \to {}_RR_R$ be a dual (R,R)-bilinear mapping. Then the tensor product ring $Q \otimes_R^\beta P$ is isomorphic to Σ^β and Π^β.*

Proof. Let $\beta = \langle\, , \,\rangle$ be a dual (R, R)-bilinear mapping. By Theorem 3.28 we know that the mapping $\varphi\colon\ Q \otimes^{\beta}_{R} P \to \Sigma^{\beta}$ defined by (3.8) is a strict local isomorphism. It suffices to prove that φ is injective.

Let $\sum_{k=1}^{k^*} q_k \otimes p_k \in \operatorname{Ker}(\varphi)$. Then $\sum_{k=1}^{k^*} q_k \langle p_k, \_\rangle = 0$. The ring $Q \otimes^{\beta}_{R} P$ is s-unital by Proposition 3.35, therefore applying Theorem 2.21 we can find an element $x = \sum_{j=1}^{j^*} \kappa_j \otimes \rho_j \in Q \otimes^{\beta}_{R} P$ such that for every $k \in \{1, \ldots, k^*\}$

$$(q_k \otimes p_k) \star x = q_k \otimes p_k.$$

Note that

$$\begin{aligned}
\sum_{k=1}^{k^*} q_k \otimes p_k &= \sum_{k=1}^{k^*} (q_k \otimes p_k) \star x = \sum_{k=1}^{k^*} \sum_{j=1}^{j^*} (q_k \otimes p_k) \star (\kappa_j \otimes \rho_j) \\
&= \sum_{k=1}^{k^*} \sum_{j=1}^{j^*} q_k \otimes \langle p_k, \kappa_j \rangle \rho_j = \sum_{k=1}^{k^*} \sum_{j=1}^{j^*} q_k \langle p_k, \kappa_j \rangle \otimes \rho_j \\
&= \sum_{j=1}^{j^*} \left(\sum_{k=1}^{k^*} q_k \langle p_k, \kappa_j \rangle \right) \otimes \rho_j = \sum_{j=1}^{j^*} 0 \otimes \rho_j = 0.
\end{aligned}$$

Hence $\operatorname{Ker}(\varphi) = \{0\}$. Therefore φ is injective, which means that φ is also an isomorphism. By Proposition 3.33 we have that $Q \otimes^{\beta}_{R} P$ is also isomorphic to Π^{β}. ■

3.4 Morita equivalence of firm rings

In this section we will prove a theorem that gives a necessary and sufficient condition for two firm rings to be Morita equivalent. We will need the following proposition about Morita contexts connecting two Morita equivalent firm rings.

Proposition 3.37. *If firm rings R and S are Morita equivalent, then they are connected by a bijective Morita context $(R, S, {}_RP_S, {}_SQ_R, \psi, \varphi)$, where P and Q are firm bimodules.*

Proof. Let R and S be firm rings and $R \approx_{\mathrm{ME}} S$. By Theorem 4.24 in [33], there exists a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \psi, \varphi)$, where ${}_RP_S$ and ${}_SQ_R$ are firm bimodules. The homomorphisms ψ and φ are bijective by Proposition 5.5 in [33]. ■

It should be remarked that we will take a closer look at firm bimodules in Section 6.1.1, which includes an explicit proof of the existence of Morita contexts with firm bimodules for Morita equivalent idempotent rings (see Proposition 6.4).

Now we are ready to prove the following description of Morita equivalent firm rings. This generalizes a part of Theorem 2.6 in [5] from rings with local units to firm rings.

Theorem 3.38. *Let R and S be firm rings. Then R and S are Morita equivalent if and only if R is isomorphic to a pseudo-surjectively defined tensor product ring $P \otimes_S Q$.*

PROOF. *Necessity.* Let $R \approx_{\mathrm{ME}} S$. By Proposition 3.37, there exists a bijective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then $P \otimes_S Q$ is a tensor product ring defined by $\hat{\phi}$ which is pseudo-surjective due to the bijectivity of ϕ. Also $\theta\colon\ P \otimes_S Q \to R$ is an isomorphism of rings by Theorem 3.23.

Sufficiency. Let R be isomorphic to a pseudo-surjectively defined tensor product ring $P \otimes_S Q$. By Theorem 3.16, the rings $P \otimes_S Q$ and S are Morita equivalent, which implies $R \approx_{\mathrm{ME}} S$. ■

Next we will prove that two s-unital rings R and S are Morita equivalent if and only if there exists a right R-module Q_R (with some additional properties) such that S is isomorphic to a subring of $\mathrm{End}(Q_R)$. This is a generalization of a well known result that two rings with identity R and S are Morita equivalent if and only if there exists a progenerator Q_R with $S \cong \mathrm{End}(Q_R)$ (Corollary 22.4 in [4]).

Theorem 3.39. *Two s-unital rings R and S are Morita equivalent if and only if there exist R-modules ${}_RP$, Q_R, a dual (R,R)-bilinear pseudo-surjective mapping $\beta = \langle\, ,\, \rangle\colon {}_RP \times Q_R \to {}_RR_R$ and $S \cong \Pi^\beta$ as rings.*

PROOF. *Necessity.* Let R and S be Morita equivalent s-unital rings. Since s-unital rings are firm, they are connected by a bijective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ (Proposition 3.37). From Example 3.31 we know that $\hat{\theta}\colon {}_RP \times Q_R \to {}_RR_R$ is a dual mapping. Due to Theorem 3.36 we have $Q \otimes_R^{\hat{\theta}} P \cong \Sigma^{\hat{\theta}}$. From Corollary 3.24 we obtain the isomorphism $S \cong Q \otimes_R^{\hat{\theta}} P \cong \Sigma^{\hat{\theta}}$, because ϕ is bijective. Also, $\Sigma^{\hat{\theta}} \cong \Pi^{\hat{\theta}}$ by Proposition 3.33. Finally, let $r \in R$. Since θ is surjective, there exists $\sum_{k=1}^{k^*} p_k \otimes q_k \in P \otimes_S Q$ such that $r = \theta(\sum_{k=1}^{k^*} p_k \otimes q_k)$. Now

$$r = \theta\left(\sum_{k=1}^{k^*} p_k \otimes q_k\right) = \sum_{k=1}^{k^*} \theta(p_k \otimes q_k) = \sum_{k=1}^{k^*} \hat{\theta}(p_k, q_k),$$

which proves that $\langle \mathrm{Im}\,\hat{\theta} \rangle_{\mathrm{s}} = R$.

Sufficiency. By Corollary 3.34, the rings R and $\Pi^\beta \cong S$ are Morita equivalent. ■

3.5 Connection between Rees matrix rings and tensor product rings

In this section we will prove a theorem, that sheds some light into how Rees matrix rings and tensor product rings are connected. Let R be a ring and Λ, Ξ some sets, denote by Λ' and Ξ' the sets of all mappings $\{1\} \times \Lambda \to R$ and $\Xi \times \{1\} \to R$ with a finite number of non-zero values, respectively. The sets Λ' and Ξ' are the infinite-dimensional analogues of sets $\mathrm{Mat}_{1,n}(R)$ and $\mathrm{Mat}_{m,1}(R)$, respectively, related to a Rees matrix ring $\mathcal{M}(R; m, n; M)$.

Theorem 3.40. *An idempotent Rees matrix ring $\mathcal{M}(R; \Lambda, \Xi; M)$ is a strict local isomorphic image of the tensor product ring $\Xi' \otimes_R \Lambda'$.*

Proof. Consider an idempotent Rees matrix ring $\mathcal{M} = \mathcal{M}(R; \Lambda, \Xi; M)$ and R-modules ${}_R\Lambda'$ and Ξ'_R with componentwise addition and R-multiplication. The ring R is idempotent by Proposition 3.4 and therefore the modules ${}_R\Lambda'$ and Ξ'_R are unitary (see the proof of Theorem 3.8).

Define a mapping

$$\langle\, , \rangle\colon\ {}_R(\Lambda' \times \Xi')_R \to {}_RR_R, \qquad \langle X, Y\rangle := X * Y = XMY.$$

The mapping $\langle\, , \rangle$ is clearly (R, R)-bilinear and, by Corollary 3.3, $\langle\, , \rangle$ is also surjective. Consider the tensor product ring $\Xi' \otimes_R \Lambda'$ surjectively defined by $\langle\, , \rangle$. Now define a mapping

$$\psi\colon\ \Xi' \otimes_R \Lambda' \to \mathcal{M}, \qquad \sum_{k=1}^{k^*} Y_k \otimes X_k \mapsto \sum_{k=1}^{k^*} Y_k X_k.$$

The mapping ψ is a well-defined homomorphism of abelian groups by the universal property of tensor product (see Proposition 2.11). The mapping ψ is surjective, because by Corollary 3.7, every $Z \in \mathcal{M}$ can be expressed as $Z = \sum_{k=1}^{k^*} Y_k X_k$, where $Y_k \in \Xi'$ and $X_k \in \Lambda'$ for every $k \in \{1, \ldots, k^*\}$.

Consider Λ' as a right $\mathcal{M}$-module similarly to the proof of Theorem 3.8, then the tensor product $(\Xi' \otimes_R \Lambda')_{\mathcal{M}}$ also becomes a right $\mathcal{M}$-module with multiplication

$$(Y \otimes X) * Z = Y \otimes (X * Z).$$

Let $\sum_{k=1}^{k^*} Y_k \otimes X_k \in \Xi' \otimes_R \Lambda'$ and $Z \in \mathcal{M}$, then

$$\psi\left(\left(\sum_{k=1}^{k^*} Y_k \otimes X_k\right) * Z\right) = \psi\left(\sum_{k=1}^{k^*} Y_k \otimes (X_k * Z)\right) = \sum_{k=1}^{k^*} Y_k (X_k * Z)$$

$$= \left(\sum_{k=1}^{k^*} Y_k X_k\right) * Z = \psi\left(\sum_{k=1}^{k^*} Y_k \otimes X_k\right) * Z.$$

Therefore ψ is also a homomorphism of right $\mathcal{M}$-modules.

Notice that, for arbitrary $\sum_{k=1}^{k^*} Y_k \otimes X_k, \sum_{h=1}^{h^*} Y'_h \otimes X'_h \in \Xi' \otimes_R \Lambda'$, we have

$$\begin{aligned}
\left(\sum_{k=1}^{k^*} Y_k \otimes X_k\right) \star \left(\sum_{h=1}^{h^*} Y'_h \otimes X'_h\right) &= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} Y_k \otimes \langle X_k, Y'_h\rangle X'_h \\
&= \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} Y_k \otimes (X_k * Y'_h) X'_h = \sum_{k=1}^{k^*} Y_k \otimes \left(X_k * \sum_{h=1}^{h^*} Y'_h X'_h\right) \\
&= \sum_{k=1}^{k^*} Y_k \otimes \left(X_k * \psi\left(\sum_{h=1}^{h^*} Y'_h \otimes X'_h\right)\right) \\
&= \left(\sum_{k=1}^{k^*} Y_k \otimes X_k\right) * \psi\left(\sum_{h=1}^{h^*} Y'_h \otimes X'_h\right),
\end{aligned}$$

which means that the multiplication $\star$ on $(\Xi' \otimes_R \Lambda')_{\mathcal{M}}$ coincides with the multiplication defined by the module homomorphism ψ. Now by Proposition 3.21, module homomorphism ψ is a locally injective homomorphism of rings.

Since ψ is surjective and locally injective homomorphism of rings, it is a strict local isomorphism of rings. ■

As a consequence of the previous theorem, we see that

$$(\Xi' \otimes_R \Lambda')/\operatorname{Ker}\psi \cong \mathcal{M}(R, \Lambda, \Xi, M),$$

that is, idempotent Rees matrix rings are quotients of tensor product rings.

Chapter 4

Enlargements of rings

In this chapter we will define the notion of an enlargement of a ring and use it to study Morita equivalence. The joint enlargement of two rings will prove to be especially effective. In particular, the existence of a joint enlargement of two idempotent rings turns out to be equivalent to those rings being Morita equivalent. This chapter is based on [27].

4.1 Definition and basic properties of enlargements

First we will define the enlargement of a ring. This definition is based on a similar notion for semigroups introduced by Lawson in [29].

Definition 4.1. We call a ring R an **enlargement** of its subring S if the conditions $R = RSR$ and $S = SRS$ hold. We also say that R is an **enlargement** of all rings isomorphic to such S.

We write $S \sqsubseteq R$ when R is an enlargement of its subring S. Next we will prove some simple properties of enlargements.

Proposition 4.2. *Let R and S be rings with $S \sqsubseteq R$. Then the following assertions hold.*

(1) The ring R is idempotent.
(2) If R is commutative then $R = S$.
(3) If S is an ideal of R, then $R = S$.
(4) If $S = \{0\}$, then $R = \{0\}$.

Proof. Let R and S be rings and $S \sqsubseteq R$.

1. Notice that
$$R = RSR = R(SR) \subseteq RR \subseteq R.$$
Hence $RR = R$ holds.
2. If R is a commutative ring then, due to (1), we have
$$R = RSR = RRS = RS = R(SRS) = SRRS = SRS = S.$$
3. If S is an ideal of R, then
$$R = RSR \subseteq S \subseteq R.$$
Therefore $R = S$.
4. This follows directly from (3). ■

The next proposition is the ring-theoretic analogue of Proposition 2 in [31]. In fact, there is no difference in the proof, but we will present it for the sake of completeness.

Proposition 4.3. *Let S, R and T be rings. The following assertions hold.*
(1) If $S \sqsubseteq R$ and $R \sqsubseteq T$, then $S \sqsubseteq T$.
(2) If $S \sqsubseteq R$ holds and $f\colon\ R \to T$ is a surjective ring homomorphism, then $f(S) \sqsubseteq T$.

Proof. Let S, R and T be rings and $S \sqsubseteq R$.

1. Let $R \sqsubseteq T$ hold, then obviously $S \subseteq T$. Notice that
$$TST \subseteq TRT = T = TRSRT = (TR)S(RT) \subseteq TST,$$
which implies that $TST = T$. Also notice that
$$S=SRS\subseteq STS=(SRS)T(SRS)=SR(STS)RS\subseteq SRTRS=SRS=S,$$
which implies that $S = STS$. In conclusion, we have shown that $S \sqsubseteq T$.
2. Let $f\colon\ R \to T$ be a surjective ring homomorphism. Then
$$T = f(R) = f(RSR) = f(R)f(S)f(R) = Tf(S)T,$$
$$f(S) = f(SRS) = f(S)f(R)f(S) = f(S)Tf(S).$$
Therefore $f(S) \sqsubseteq T$. ■

In the previous proposition we showed that the relation $\sqsubseteq$ is transitive. It is also antisymmetric. Clearly every idempotent ring is an enlargement of itself. This implies that the relation $\sqsubseteq$ is a partial order relation on the class of all idempotent rings.

Now we will take a look at enlargements of idempotent rings. Immediately from the definition of idempotent rings we have the following result, which says that in the case of idempotent rings it suffices to check only two inclusions instead of four.

Lemma 4.4. *A ring R is an enlargment of an idempotent ring S if and only if $R \subseteq RSR$ and $SRS \subseteq S$.*

Note that a subring S of a ring R, satisfying $SRS \subseteq S$, is called a **bi-ideal** of R (see [41], page 11).

Now we will prove another little property that simplifies finding enlargements of an idempotent ring. It is a ring-theoretic analogue of Proposition 3 in [31].

Proposition 4.5. *Let S be an idempotent subring of a ring R. If $S = SRS$ then $S \sqsubseteq RSR$.*

Proof. Let R be a ring and $S \subseteq R$ an idempotent subring with $S = SRS$. Denote $R' := RSR$; this is a subring of R. Then $S = SS = SSS \subseteq RSR = R'$. Therefore S is a subring of R'. Notice that

$$SR'S = S(RSR)S = SR(SRS) = SRS = S,$$
$$R'SR' = (RSR)S(RSR) = R(SRS)RSR = R(SRS)R = RSR = R'.$$

Hence $S \sqsubseteq RSR$. ■

Next we will give two series of examples of enlargements, which show that certain natural matrix constructions give rise to enlargements.

Example 4.6 (Enlargement of a ring I). A full matrix ring over an idempotent ring S is an enlargement of S.

Let $n \in \mathbb{N}$ and consider the full matrix ring $R := \mathrm{Mat}_n(S)$ over an idempotent ring S. We will prove that R is an enlargement of S using Lemma 4.4.

Let $A_{hk}(s)$ be an $(n \times n)$-matrix with entry s at the intersection of h-th row and k-th column, and zeroes elsewhere. Then

$$S' := \{A_{11}(s) \mid s \in S\}$$

is an idempotent subring of R which is isomorphic to the ring S. To prove the inclusion $R \subseteq RS'R$ it suffices to show that each $A_{hk}(s)$ belongs to $RS'R$.

Take $s \in S$. Since S is idempotent, we can write $s = \sum_{j=1}^{j^*} u_j s_j v_j$ for some $u_j, s_j, v_j \in S$. Hence

$$A_{hk}(s) = \sum_{j=1}^{j^*} A_{hk}(u_j s_j v_j) = \sum_{j=1}^{j^*} A_{h1}(u_j) \cdot A_{11}(s_j) \cdot A_{1k}(v_j) \in RS'R\,.$$

Also, we have the inclusion $S'RS' \subseteq S'$, because

$$A_{11}(s) \cdot A \cdot A_{11}(s') = A_{11}(sa_{11}s') \in S'.$$

for any $s, s' \in S$ and any matrix $A = [a_{hk}] \in R$. □

For the next example we will need the notion of a unital Rees matirix ring, whose definition is inspired from a similar notion for semigroups in [35]. We call a Rees matrix ring $\mathcal{M}(S;\Lambda,\Xi;M)$ **unital** if S is a ring with identity 1 and 1 is an entry of M.

Example 4.7 (Enlargement of a ring II). A unital Rees matrix ring over a ring S with identity is an enlargement of S.

Consider a unital Rees matrix ring $\mathcal{M} = \mathcal{M}(S;\Lambda,\Xi;M)$. For any $s \in S$ let $A_{uv}(s)$ denote the $(\Lambda \times \Xi)$-matrix over S such that $A_{uv}(s)(u,v) = s$ and $A_{uv}(s)(i,j) = 0$ for all other pairs $(i,j) \in \Lambda \times \Xi$. By our assumption, there exist $v_0 \in \Xi$ and $u_0 \in \Lambda$ such that $M(v_0,u_0) = 1 \in S$. Putting

$$S' := \{A_{u_0,v_0}(s) \mid s \in S\}$$

we obtain a subring of $\mathcal{M}$ which is isomorphic to S (with the isomorphism $\iota\colon\ S \to S'$, $s \mapsto A_{u_0,v_0}(s)$). Using Lemma 4.4 again, we will show that S' is an idempotent ring and $S' \sqsubseteq \mathcal{M}$.

To prove that S' is an idempotent ring we consider an arbitrary element $s \in S$. Then

$$A_{u_0,v_0}(s) = A_{u_0,v_0}(s)MA_{u_0,v_0}(1) = A_{u_0,v_0}(s) * A_{u_0,v_0}(1) \in S' * S',$$

and hence $S' = S' * S'$.

The inclusion $S' * \mathcal{M} * S' \subseteq S'$ holds because, for every $A \in \mathcal{M}$ and $s, s' \in S$, we have

$$A_{u_0,v_0}(s) * A * A_{u_0,v_0}(s') = A_{u_0,v_0}(s)MAMA_{u_0,v_0}(s')$$

and the last matrix product may have a nonzero entry only at the position (u_0,v_0).

Finally, to prove the inclusion $\mathcal{M} \subseteq \mathcal{M} * S' * \mathcal{M}$ we note that, by the definition of a Rees matrix ring, every element of $\mathcal{M}$ is a finite sum of matrices of type $A_{u,v}(s)$, and

$$\begin{aligned} A_{u,v}(s) &= A_{u,v_0}(s)MA_{u_0,v_0}(1)MA_{u_0,v}(1) \\ &= A_{u,v_0}(s) * A_{u_0,v_0}(1) * A_{u_0,v}(1) \in \mathcal{M} * S' * \mathcal{M}. \end{aligned}$$

In conclusion, we have shown that $S \sqsubseteq \mathcal{M}(S;\Lambda,\Xi;M)$. □

4.2 Enlargements and Morita equivalence

In this section we will show that enlargements of idempotent rings are very closely related to the Morita equivalence of these rings.

Proposition 4.8. *If R is an enlargement of an idempotent ring S then R and S are Morita equivalent.*

Proof. Let S be an idempotent ring and $S \sqsubseteq R$. Since isomorphic rings are Morita equivalent, it suffices to consider only the situation, where $S \subseteq R$. Consider the subring $SR \subseteq R$ as an (S,R)-bimodule and the subring $RS \subseteq R$ as an (R,S)-bimodule. From Proposition 2.27 we know that the ring R is idempotent. Therefore, the bimodules RS and SR are unitary.

Define the following mappings:

$$\theta\colon\ RS \otimes_S SR \to R, \quad \sum_{k=1}^{k^*} r_k s_k \otimes s'_k r'_k \mapsto \sum_{k=1}^{k^*} r_k s_k s'_k r'_k, \tag{4.1}$$

$$\phi\colon\ SR \otimes_R RS \to SRS = S, \quad \sum_{k=1}^{k^*} s_k r_k \otimes r'_k s'_k \mapsto \sum_{k=1}^{k^*} s_k r_k r'_k s'_k. \tag{4.2}$$

Note, that the mapping $\hat{\theta}\colon\ RS \times SR \to R$, $(rs, s'r') \mapsto rss'r'$ is S-balanced, and since S is an abelian group with respect to addition, we get from the universal property of tensor product (Proposition 2.11), that θ is a well-defined homomorphism of abelian groups. Analogously ϕ is well defined.

For every $r, r_1, r'_1, \ldots, r_{k^*}, r'_{k^*} \in R$ and $s_1, s'_1, \ldots, s_{k^*}, s'_{k^*} \in S$, we compute

$$\begin{aligned}\theta\left(r\left(\sum_{k=1}^{k^*} r_k s_k \otimes s'_k r'_k\right)\right) &= \theta\left(\sum_{k=1}^{k^*} r r_k s_k \otimes s'_k r'_k\right) = \sum_{k=1}^{k^*} r r_k s_k s'_k r'_k \\ &= r\sum_{k=1}^{k^*} r_k s_k s'_k r'_k = r\,\theta\left(\sum_{k=1}^{k^*} r_k s_k \otimes s'_k r'_k\right)\end{aligned}$$

and, analogously, $\theta((\sum_{k=1}^{k^*} r_k s_k \otimes s'_k r'_k)r) = \theta(\sum_{k=1}^{k^*} r_k s_k \otimes s'_k r'_k)r$. Therefore θ is a bimodule homomorphism.

Now, take an arbitrary element $r \in R$. Since $S \sqsubseteq R$ and S is idempotent, we have $R = RSR = R(SS)R = (RS)(SR)$. Hence, there exist elements $r_1, r'_1, \ldots, r_{k^*}, r'_{k^*} \in R$ and $s_1, s'_1, \ldots, s_{k^*}, s'_{k^*} \in S$ such that

$$r = \sum_{k=1}^{k^*} r_k s_k s'_k r'_k = \theta\left(\sum_{k=1}^{k^*} r_k s_k \otimes s'_k r'_k\right).$$

Thus, θ is surjective. Analogously, ϕ is a surjective bimodule homomorphism.

Finally, if $\rho, \rho' \in RS$ and $\sigma, \sigma' \in SR$, then

$$\theta(\rho \otimes \sigma)\rho' = (\rho\sigma)\rho' = \rho(\sigma\rho') = \rho\phi(\sigma \otimes \rho'),$$

$$\sigma'\theta(\rho \otimes \sigma) = \sigma'(\rho\sigma) = (\sigma'\rho)\sigma = \phi(\sigma' \otimes \rho)\sigma.$$

In conclusion, we have shown that $(R, S, RS, SR, \theta, \phi)$ is a unitary surjective Morita context connecting rings R and S. By Theorem 2.28, $R \approx_{\mathrm{ME}} S$.■

From the previous proposition and Examples 4.6 and 4.7 we obtain the following two corollaries. The first corollary is also a generalization of Corollary 3.11.

Corollary 4.9 (Cf. Corollary 22.6 in [4]). *A full matrix ring over an idempotent ring S is Morita equivalent to S.*

Corollary 4.10. *A unital Rees matrix ring over a ring S with identity is Morita equivalent to S.*

Now we will define the notion of a joint enlargement of rings and show that each unitary surjective Morita context gives rise to a joint enlargement.

Definition 4.11. Let S, R and T be rings. The ring T is called a **joint enlargement** of S and R if T is an enlargement of both S and R.

It turns out that if $S \approx_{\mathrm{ME}} R$, then the corresponding Morita ring is a joint enlargement of S and R.

Proposition 4.12. *If idempotent rings R and S are connected by a unitary surjective Morita context $\Gamma = (R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, then the Morita ring $\overline{\Gamma}$ is a joint enlargement of R and S.*

Proof. Let S and R be idempotent rings and $\Gamma = (R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ a unitary surjective Morita context. It is easy to see that

$$\overline{R} = \left\{ \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} \middle| r \in R \right\} \subseteq \overline{\Gamma}$$

is an idempotent subring of $\overline{\Gamma}$ that is isomorphic to R. We will prove the inclusions $\overline{\Gamma} \subseteq \overline{\Gamma}\,\overline{R}\,\overline{\Gamma}$ and $\overline{R}\,\overline{\Gamma}\,\overline{R} \subseteq \overline{R}$.

Every matrix $\left[\begin{smallmatrix} r & p \\ q & s \end{smallmatrix}\right] \in \overline{\Gamma}$ can be expressed as a sum

$$\begin{bmatrix} r & p \\ q & s \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}.$$

It suffices to show that the last four matrices belong to $\overline{\Gamma}\,\overline{R}\,\overline{\Gamma}$. For $\left[\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right]$ this is clear. Now consider $p \in P$. Since ${}_RP$ is unitary, we can find $p_1, \ldots, p_{k^*} \in P$ and $r_1, \ldots, r_{k^*} \in R$ such that $p = r_1p_1 + \ldots + r_{k^*}p_{k^*}$. Then we have

$$\begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} = \sum_{k=1}^{k^*} \begin{bmatrix} r_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & p_k \\ 0 & 0 \end{bmatrix} \in \overline{R}\,\overline{\Gamma} = \overline{R}\,\overline{R}\,\overline{\Gamma} \subseteq \overline{\Gamma}\,\overline{R}\,\overline{\Gamma}.$$

Analogously $\left[\begin{smallmatrix} 0 & 0 \\ q & 0 \end{smallmatrix}\right] \in \overline{\Gamma}\,\overline{R}\,\overline{\Gamma}$ for any $q \in Q$. If $s \in S$, then due to the surjectivity of ϕ, there exists $\sum_{k=1}^{k^*} q_k \otimes p_k \in Q \otimes_R P$ such that $s = \phi(\sum_{k=1}^{k^*} q_k \otimes p_k)$. Hence

$$\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} = \sum_{k=1}^{k^*} \begin{bmatrix} 0 & 0 \\ q_k & 0 \end{bmatrix} \begin{bmatrix} 0 & p_k \\ 0 & 0 \end{bmatrix} \in \overline{\Gamma}\left(\overline{\Gamma}\,\overline{R}\,\overline{\Gamma}\right) \subseteq \overline{\Gamma}\,\overline{R}\,\overline{\Gamma}.$$

And so we have proven the inclusion $\overline{\Gamma} \subseteq \overline{\Gamma}\,\overline{R}\,\overline{\Gamma}$.

Note that for any $r, r', r'' \in R$, $s \in S$, $q \in Q$ and $p \in P$, we have

$$\begin{bmatrix} r' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r & p \\ q & s \end{bmatrix} \begin{bmatrix} r'' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r'r & r'p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r'' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r'rr'' & 0 \\ 0 & 0 \end{bmatrix} \in \overline{R},$$

which implies the inclusion $\overline{R}\,\overline{\Gamma}\,\overline{R} \subseteq \overline{R}$. We have proven $\overline{R} \sqsubseteq \overline{\Gamma}$. The proof of $\overline{S} \sqsubseteq \overline{\Gamma}$ is analogous with $\overline{S} = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right] \mid s \in S\}$. ■

Now we are ready to prove the main theorem of this chapter.

Theorem 4.13. *Idempotent rings are Morita equivalent if and only if they have a joint enlargement.*

Proof. *Necessity.* If idempotent rings R and S are Morita equivalent then, by Theorem 2.28, they are connected by a unitary surjective Morita context Γ. By Proposition 4.12, the Morita ring $\overline{\Gamma}$ is their joint enlargement.

Sufficiency. If idempotent rings R and S have a joint enlargement T then, by Proposition 4.8, T is Morita equivalent to R and S. By transitivity of the Morita equivalence relation, the rings R and S are Morita equivalent. ■

Thus, two idempotent rings are Morita equivalent if and only if they can be embedded nicely in some ring T. This is a purely algebraic condition which does not refer to categories, and probably it is easier to use compared to the definition through equivalence functors.

We will draw some conclusions from Theorem 4.13.

Corollary 4.14. *Two rings with identity (two rings with local units, two s-unital rings) are Morita equivalent if and only if they have a joint enlargement which has identity (has local units, is s-unital).*

Proof. *Necessity.* Assume that two rings R and S are connected by a unitary surjective Morita context Γ. By Theorem 4.13, the rings R and S have a joint enlargement $\overline{\Gamma}$.

1. If R and S are rings with identity then the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is the identity element of their joint enlargement $\overline{\Gamma}$.

2. Let R and S be rings with local units. Fix a finite set
$$G := \left\{ \begin{bmatrix} r_1 & p_1 \\ q_1 & s_1 \end{bmatrix}, \dots, \begin{bmatrix} r_n & p_n \\ q_n & s_n \end{bmatrix} \right\} \subseteq \overline{\Gamma}.$$
For every $k \in \{1, \dots, n\}$ we can express $p_k = \sum_{h=1}^{h_k^*} r_{kh} p_{kh} s_{kh}$ and $q_k = \sum_{t=1}^{t_k^*} s'_{kt} q_{kt} r'_{kt}$, because P and Q are unitary. There exists a local unit $e \in R$ for the set $\{r_1, \dots, r_n, r_{11}, r_{12}, \dots, r_{nh_n^*}, r'_{11}, \dots, r'_{nt^*}\} \subseteq R$ and a local unit $d \in S$ for the set $\{s_1, \dots, s_n, s_{11}, s_{12}, \dots, s_{nh_n^*}, s'_{11} \dots, s'_{nt_n^*}\} \subseteq S$. Then the matrix $\left[\begin{smallmatrix} e & 0 \\ 0 & d \end{smallmatrix}\right]$ is a local unit for the set G.
3. Similar to part 2.

Sufficiency. This follows immediately from Theorem 4.13. ■

Corollary 4.15. *The only idempotent ring Morita equivalent to* $\{0\}$ *is* $\{0\}$ *itself.*

PROOF. Assume that T is an idempotent ring Morita equivalent to $\{0\}$. By Theorem 4.13 they have a joint enlargement S. Due to Proposition 4.2 (4), we have that $S = \{0\}$. Now it is clear that $T = \{0\}$ too. ■

The previous corollary shows one aspect how the Morita equivalence of rings differs from the Morita equivalence of semigroups. Namely, there exist many semigroups that are Morita equivalent to the one-element semigroup (Theorem 16 in [21]).

Next we consider some connections of enlargements and (sets of) idempotents. Let $\mathcal{E}(R)$ denote the set of all idempotent elements of a ring R. If $E \subseteq \mathcal{E}(R)$ is a nonempty set of idempotents then the set ERE is a subring of R.

Proposition 4.16. *Let R be a ring and let $\varnothing \neq E \subseteq \mathcal{E}(R)$. Then R is an enlargement of its subring ERE if and only if $R = RER$.*

PROOF. Let R be a ring and $\varnothing \neq E \subseteq \mathcal{E}(R)$.

Necessity. Let $ERE \sqsubseteq R$. Then we have
$$R = R(ERE)R = RE(RER) \subseteq RER \subseteq R.$$
Hence $R = RER$.

Sufficiency. Assume that $R = RER$. Then we have
$$(ERE)R(ERE) = (ERE)(RER)E = (ERE)RE = E(RER)E = ERE,$$
$$R(ERE)R = RE(RER) = RER = R.$$
Therefore, $ERE \sqsubseteq R$. ■

Corollary 4.17. *Let $e \in R$ be an idempotent element of a ring R. The condition $ReR = R$ holds if and only if R is an enlargement of its subring eRe.*

Corollary 4.18. *Let R be a ring and let $\varnothing \neq E \subseteq \mathcal{E}(R)$. If $R = RER$, then the rings R and ERE are Morita equivalent.*

Next we will give an example, where we calculate all of the subrings of $\mathrm{Mat}_2(\mathbb{Z}_2)$, to which it is an enlargement of.

Example 4.19 (Subrings and enlargements). Consider the ring $R := \mathrm{Mat}_2(\mathbb{Z}_2)$. By Corollary 4.9, the ring R is Morita equivalent to $\mathbb{Z}_2$. We computationally proved, that R has 27 proper subrings and 8 idempotents:

$$\mathcal{E}(R) = \left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} \right\}.$$

Every idempotent $e \in \mathcal{E}(R) \backslash \{ [\begin{smallmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{smallmatrix}], [\begin{smallmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{smallmatrix}] \}$ satisfies the condition $ReR = R$ and generates a subring of the form $eRe = \{ [\begin{smallmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{smallmatrix}], e \}$. By Corollary 4.17, R is an enlargement of all of these subrings.

Additionally there are 6 interesting four-element subrings:

$$\begin{gathered}
\left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix} \right\}, \\
\left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix} \right\}, \\
\left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} \right\}, \\
\left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix} \right\}, \\
\left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix} \right\}, \\
\left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix} \right\}.
\end{gathered}$$

All of the previous subrings are of the form $\{e, e'\}R\{e, e'\} = \{ [\begin{smallmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{smallmatrix}], e, e', e+e' \}$ for some idempotents $e, e' \in \mathcal{E}(R)$. Hence, by Proposition 4.16, R is also an enlargement of all of these subrings. In total there are 12 proper subrings of R, to which R is an enlargement (this has been proven computationally). Also, in this situation, if any of the aforementioned subrings is included in any other, then the bigger subring is an enlargement of the smaller one. This is collected onto Figure 4.1, where two subrings are connected by a line if the upper subring is an enlargement of the lower one.

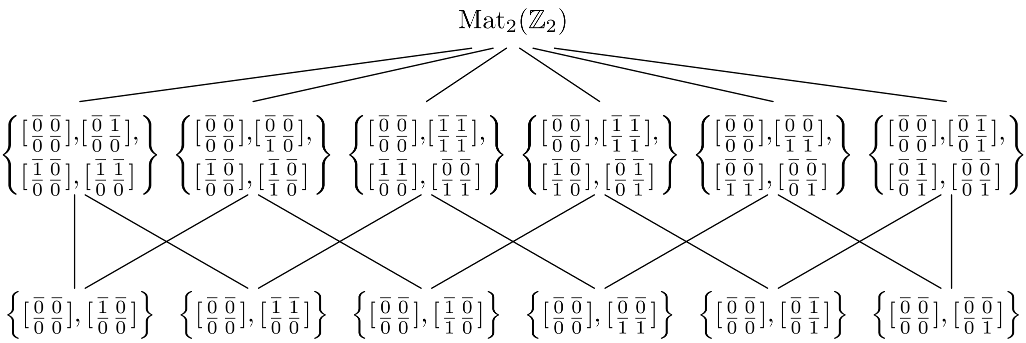

Figure 4.1

By Corollary 4.18, all of the rings in Figure 4.1 are Morita equivalent to each other (and to $\mathbb{Z}_2$). $\square$

4.3 Morita contexts come from enlargements

Let T be a joint enlargement of its subrings R and S. It is easy to see that it induces a Morita context with bimodules ${}_RP_S := RTS$ and ${}_SQ_R := STR$ and

$$\theta\colon\ RTS \otimes_S STR \to R, \quad \sum_{k=1}^{k^*} r_k t_k s_k \otimes s'_k t'_k r'_k \mapsto \sum_{k=1}^{k^*} r_k t_k s_k s'_k t'_k r'_k, \tag{4.3}$$

$$\phi\colon\ STR \otimes_R RTS \to S, \quad \sum_{k=1}^{k^*} s_k t_k r_k \otimes r'_k t'_k s'_k \mapsto \sum_{k=1}^{k^*} s_k t_l r_k r'_k t'_l s'_k. \tag{4.4}$$

We see that all the information about such a Morita context is encoded in a single ring T:

1. R and S are subrings (even bi-ideals) of T;
2. P and Q are subgroups of $(T, +)$;
3. the scalar multiplications of P and Q are defined using multiplication in T;
4. θ and ϕ are defined using the multiplication in T.

In our next theorem we will prove that any unitary Morita context between idempotent rings is isomorphic to a Morita context coming from a joint enlargement. But for that result we must first recall the notion of an isomorphism between Morita contexts, which appeared in [37] by Müller. We say that a Morita context $\Gamma = (R, S, P, Q, \theta, \phi)$ is **isomorphic** to a Morita context $\Gamma' = (R, S, P', Q', \theta', \phi')$, if there exists a pair (f, g), where

1. $f\colon\ P \to P'$ and $g\colon\ Q \to Q'$ are bimodule isomorphisms,
2. $\theta' \circ (f \otimes g) = \theta$ and $\phi' \circ (g \otimes f) = \phi$.

The pair (f, g) is also called an **isomorphism** between Morita contexts Γ and Γ'.

Theorem 4.20. *Every unitary Morita context Γ connecting idempotent rings R and S is isomorphic to the unitary Morita context $(R, S, R\overline{\Gamma}S, S\overline{\Gamma}R, \psi, \varphi)$.*

Proof. Let R and S be idempotent rings connected by a unitary Morita context $\Gamma = (R, S, P, Q, \theta, \phi)$.

The scalar multiplications of bimodules $R\overline{\Gamma}S$ and $S\overline{\Gamma}R$ are defined using the isomorphic copies of R and S in $\overline{\Gamma}$ (see (2.14) and (2.15)). Note that for any $r' \in R$, $s' \in S$ and $\left[\begin{smallmatrix} r & p \\ q & s \end{smallmatrix}\right] \in \overline{\Gamma}$, we have

$$r' \begin{bmatrix} r & p \\ q & s \end{bmatrix} s' = \begin{bmatrix} r' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r & p \\ q & s \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & s' \end{bmatrix} = \begin{bmatrix} r'r & r'p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & s' \end{bmatrix} = \begin{bmatrix} 0 & r'ps' \\ 0 & 0 \end{bmatrix}.$$

Hence we have

$$R\overline{\Gamma}S = \left\{ \sum_{k=1}^{k^*} \begin{bmatrix} 0 & r_k p_k s_k \\ 0 & 0 \end{bmatrix} \middle| \forall k\colon\ r_k \in R, p_k \in P, s_k \in S \right\} = \left\{ \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \middle| p \in P \right\},$$

where the last equality holds due to the unitarity of P. Analogously we have $S\overline{\Gamma}R = \{\left[\begin{smallmatrix} 0 & 0 \\ q & 0 \end{smallmatrix}\right] | q \in Q\}$.

Consider the Morita context $(R, S, R\overline{\Gamma}S, S\overline{\Gamma}R, \psi, \varphi)$, with $\psi = \iota_R \circ \psi'$ and $\varphi = \iota_S \circ \varphi'$, where

$$\psi'\colon R\overline{\Gamma}S \otimes_S S\overline{\Gamma}R \to \overline{R}, \quad \sum_{k=1}^{k^*} \begin{bmatrix} 0 & p_k \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ q_k & 0 \end{bmatrix} \mapsto \sum_{k=1}^{k^*} \begin{bmatrix} 0 & p_k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ q_k & 0 \end{bmatrix};$$

$\varphi'\colon S\overline{\Gamma}R \otimes_R R\overline{\Gamma}S \to \overline{S}$ is defined analogously; $\overline{R} = \{\left[\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right] \mid r \in R\}$; $\overline{S} = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right] \mid s \in S\}$; $\iota_R\colon \overline{R} \to R$, $\left[\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right] \mapsto r$ and $\iota_S\colon \overline{S} \to S$, $\left[\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right] \mapsto s$. The mappings ψ' and φ' are well-defined homomoprhisms, because the mappings in (4.1) and (4.2) are well-defined homomorphisms. Here $\overline{\Gamma}$ is considered as an (R, S)- and (S, R)-bimodule with multiplications defined in (2.16) and (2.17).

Define the mappings

$$f\colon\ P \to R\overline{\Gamma}S, \qquad p \mapsto \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix},$$

$$g\colon\ Q \to S\overline{\Gamma}R, \qquad q \mapsto \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}.$$

The mappings f and g are clearly bimodule isomorphisms. Now for any $p \in P$ and $q \in Q$ we have

$$\begin{aligned}(\psi \circ (f \otimes g))(p \otimes q) &= (\iota_R \circ \psi')\left(\begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}\right) = \iota_R\left(\begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}\right) \\ &= \iota_R\left(\begin{bmatrix} \theta(p \otimes q) & 0 \\ 0 & 0 \end{bmatrix}\right) = \theta(p \otimes q).\end{aligned}$$

Therefore we have $\psi \circ (f \otimes g) = \theta$ and analogously $\varphi \circ (g \otimes f) = \phi$, which proves that the Morita context Γ is isomorphic to $(R, S, R\overline{\Gamma}S, S\overline{\Gamma}R, \psi, \varphi)$. ■

Let R and S be idempotent rings with a joint enlargement T. We will call the Morita context $(R, S, RTS, STR, \theta, \phi)$, where θ and ϕ are defined as in (4.3) and (4.4), respectively, the **Morita context induced by** T. The previous theorem gives us a way to concretize the Morita context connecting Morita equivalent idempotent rings.

Corollary 4.21. *Two idempotent rings R and S are Morita equivalent if and only if they are connected by the Morita context induced by their joint enlargement T.*

4.4 Rings Morita equivalent to a ring with identity

In this section, we will give a necessary and sufficient condition, when a ring with left (or right) local units is Morita equivalent to a ring with identity. Our result will be a slight generalization of Proposition 3.5 in [7] by Ánh and Márki and Corollary 4.3 in [1] by Abrams. Also the following theorem is a special case of the Theorem in [11] by García. However, we use a different technique from all of them for proving it.

Theorem 4.22. *A ring R with left local units is Morita equivalent to a ring with identity if and only if there exists an idempotent $e \in R$ such that $R = ReR$. In that case R is Morita equivalent to its subring eRe.*

Proof. *Necessity.* Let a ring R with left local units be Morita equivalent to a ring S which has an identity 1. Then, by Theorem 2.28, there exist unitary bimodules ${}_RP_S$ and ${}_SQ_R$ and surjective homomorphisms $\theta\colon\ {}_R(P \otimes_S Q)_R \to {}_RR_R$ and $\phi\colon\ {}_S(Q \otimes_R P)_S \to {}_SS_S$, which satisfy conditions (2.11) and (2.12).

Since ϕ is surjective, there exist $q'_1, \ldots, q'_{h^*} \in Q$ and $p'_1, \ldots, p'_{h^*} \in P$ such that

$$\phi\left(\sum_{h=1}^{h^*} q'_h \otimes p'_h\right) = 1 \in S.$$

As ${}_RP$ is unitary, for every $h \in \{1, \ldots, h^*\}$, there exist elements $r_{h1}, \ldots, r_{hk^*} \in R$ and $p_{h1}, \ldots, p_{hk^*} \in P$ such that $p'_h = r_{h1}p_{h1} + \ldots + r_{hk^*}p_{hk^*}$. Consider the finite set $U := \{r_{hk} \mid h \in \{1, \ldots, h^*\}, k \in \{1, \ldots, k^*\}\}$. Since R has left local units, we can find an idempotent element $e \in R$ such that $r_{hk} = er_{hk}$ for every $r_{hk} \in U$. Now, for every $h \in \{1, \ldots, h^*\}$, we have

$$ep'_h = e\left(\sum_{k=1}^{k^*} r_{hk}p_{hk}\right) = \sum_{k=1}^{k^*} er_{hk}p_{hk} = \sum_{k=1}^{k^*} r_{hk}p_{hk} = p'_h.$$

Let $r \in R$. Due to the surjectivity of θ, there exist $p_1, \ldots, p_{j^*} \in P$ and $q_1, \ldots, q_{j^*} \in Q$ such that

$$r = \theta\left(\sum_{j=1}^{j^*} p_j \otimes q_j\right) = \sum_{j=1}^{j^*} \theta(p_j \otimes q_j).$$

Take any summand $\theta(p_j \otimes q_j)$ from the last sum. Then we have

$$\begin{aligned}
\theta(p_j \otimes q_j) = \theta(p_j \otimes 1q_j) &= \theta\left(p_j \otimes \phi\left(\sum_{h=1}^{h^*} q'_h \otimes p'_h\right) q_j\right) \\
&= \sum_{h=1}^{h^*} \theta(p_j \otimes \phi(q'_h \otimes p'_h)q_j) = \sum_{h=1}^{h^*} \theta(p_j \otimes q'_h\theta(p'_h \otimes q_j)) \\
&= \sum_{h=1}^{h^*} \theta(p_j \otimes q'_h)\theta(p'_h \otimes q_j) = \sum_{h=1}^{h^*} \theta(p_j \otimes q'_h)\theta(ep'_h \otimes q_j) \\
&= \sum_{h=1}^{h^*} \theta(p_j \otimes q'_h)e\theta(p'_h \otimes q_j) \in ReR.
\end{aligned}$$

It follows that $r \in ReR$. Since the inclusion $ReR \subseteq R$ is obvious, we conclude that $R = ReR$.

Sufficiency. If a ring R has left local units, then it is also idempotent. Let $e \in R$ be an idempotent element such that $R = ReR$. Then, due to Corollary 4.17, we have $eRe \sqsubseteq R$, where eRe is a ring with identity e.

By Proposition 4.8, we know that the rings R and eRe are Morita equivalent. ■

From Corollary 4.10 we obtain another result about the class of rings Morita equivalent to some ring S with identity.

Proposition 4.23. *Let $S \neq \{0\}$ be a ring with identity. There exists a ring with any cardinality, which is larger than the cardinality of S, that is Morita equivalent to S.*

Proof. Let $S \neq \{0\}$ be a ring with identity. We can construct a unital Rees matrix ring $\mathcal{M} = \mathcal{M}(S; \Lambda, \Xi; M)$ with any cardinality larger than the cardinality of S by choosing suitable sets Λ and Ξ. By Corollary 4.10, we have $S \approx_{\mathrm{ME}} \mathcal{M}$. ■

It should be noted here, that a unital Rees matrix ring over a ring with identity need not be a ring with identity itself.

Finally we will write a few words about the Morita equivalence of two finite rings with identity element. Let R and S be rings with identity. If $R \approx_{\mathrm{ME}} S$ and R is finite, then S is also finite. Indeed, by Corollary 22.7 in [4], there exists a natural number $n \in \mathbb{N}$ and an idempotent matrix $A \in \mathrm{Mat}_n(R)$ such that $S \cong A\,\mathrm{Mat}_n(R)A$. Since R is finite, $\mathrm{Mat}_n(R)$ is also finite for every $n \in \mathbb{N}$ and therefore S is finite. In conclusion, we see that finiteness is an invariant of Morita equivalence for rings with identity. But finiteness is not an invariant of Morita equivalence for idempotent rings, due to Proposition 4.23.

The following is a classical result about Morita equivalence of rings with identity.

Theorem 4.24 (Corollary 22.6 and Corollary 22.7 in [4]). *Let R and S be rings with identity. Then $R \approx_{\mathrm{ME}} S$ if and only if there exists a full matrix ring $T = \mathrm{Mat}_n(R)$ and an idempotent $A \in T$ such that $T = TAT$ and $S \cong ATA$.*

In the light of our results, we can recognize a joint enlargement here. Namely

- T is an enlargement of R by Example 4.6, and
- T is an enlargement of S by Corollary 4.17.

In general, neither R nor S need be isomorphic to T. This is in a sharp contrast with the monoid case. Namely, if A and B are monoids, then

$$A \sqsubseteq B \iff A \approx_{\mathrm{ME}} B \iff B \sqsubseteq A,$$

so each of the monoids is a joint enlargement. (This follows from Theorem 2.3 in [32].)

4.5 Enlargements of rings and Morita equivalence of semigroups

In this section we will prove a result that shows which if two semigroups are connected by a unitary surjective Morita context, then there exist natural rings which have a joint enlargement. First we must recall that if A and B are semigroups, then a six-tuple $(A, B, {}_AP_B, {}_BQ_A, \theta, \phi)$ is called a **Morita context** if P and Q are biacts and $\theta\colon\ P \otimes_B Q \to A$ and $\phi\colon\ Q \otimes_A P \to B$ are biact homomorphisms that satisfy conditions similar to (2.11) and (2.12). **Unitary** and **surjective** Morita contexts of semigroups are defined similarly to the case of rings (with unitary biacts in place of bimodules). If two semigroups A and B are connected by a unitary surjective Morita context, then they are called **strongly Morita equivalent** (Definition 7 in [44]). A semigroup S is called **factorizable** if $S = SS := \{ss' \mid s, s' \in S\}$. Strongly Morita equivalent semigroups must be factorizable.

It is natural to ask: do two factorizable strongly Morita equivalent semigroups have a joint enlargement? The answer to this question is not known. Lawson has proved (Theorem 1.1 in [30]) that a joint enlargement exists in the case of semigroups with local units. His construction is very different from the construction of a Morita ring of a context. It turns out that even if strongly Morita equivalent semigroups may not have a joint enlargement, they can be embedded into rings that have a joint enlargement.

The first part of the following theorem can be deduced from the theorem in [16], but we will write out all the necessary subsemigroups for the sake of completeness.

Theorem 4.25. *If semigroups A and B are strongly Morita equivalent, then there exists a ring T such that*

(1) A and B are isomorphic to some subsemigroups A' and B' of the multiplicative semigroup of T, respectively;

(2) T is a joint enlargement of rings $\langle A'\rangle$ and $\langle B'\rangle$, where $\langle S\rangle$ denotes the subring generated by the set S.

Proof. Let A and B be semigroups connected by a unitary surjective Morita context $(A, B, {}_AP_B, {}_BQ_A, \theta, \phi)$. Consider the ring

$$T := \left\{ \begin{bmatrix} x & f \\ g & y \end{bmatrix} \middle| x \in \mathbb{Z}[A],\ y \in \mathbb{Z}[B],\ f \in \mathbb{Z}^{(P)},\ g \in \mathbb{Z}^{(Q)} \right\},$$

where $\mathbb{Z}[A]$ and $\mathbb{Z}[B]$ are semigroup rings, $\mathbb{Z}^{(P)}$ and $\mathbb{Z}^{(Q)}$ are free abelian groups with bases P and Q respectively. Addition in T is defined componentwise and multiplication is defined analogously to the multiplication in a

Morita ring (2.13):

$$\begin{bmatrix} x_1 & f_1 \\ g_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 & f_2 \\ g_2 & y_2 \end{bmatrix} := \begin{bmatrix} x_1x_2 + \theta(f_1 \otimes g_2) & x_1f_2 + f_1y_2 \\ g_1x_2 + y_1g_2 & y_1y_2 + \phi(g_1 \otimes f_2) \end{bmatrix}.$$

Note that, for every $f \in \mathbb{Z}^{(P)}$ there exist $p_1, \ldots, p_{k^*} \in P$ and $z_1, \ldots, z_{k^*} \in \mathbb{Z}$ such that $f = \sum_{k=1}^{k^*} z_k p_k$. Analogously, for every $g \in \mathbb{Z}^{(Q)}$ can be expressed as $g = \sum_{h=1}^{h^*} z'_h q_h$, where $q_h \in Q$ and $z'_h \in \mathbb{Z}$ for every $h \in \{1, \ldots, h^*\}$. Since θ and ϕ are homomorphisms of abelian groups, we have

$$\theta(f \otimes g) = \sum_{k=1}^{k^*} \sum_{h=1}^{h^*} z_k z'_h \theta(p_k \otimes q_h) \quad \text{and} \quad \phi(g \otimes f) = \sum_{h=1}^{h^*} \sum_{k=1}^{k^*} z'_k z_h \phi(q_k \otimes p_h).$$

1. Consider the sets

$$A' := \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \middle| \, a \in A \right\} \subseteq T \quad \text{and} \quad B' := \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \middle| \, b \in B \right\} \subseteq T.$$

If $a_1, a_2 \in A$, then

$$\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ 0 & 0 \end{bmatrix} \in A'.$$

Therefore, A', and analogously B', is a subsemigroup of the multiplicative semigroup of T. Clearly $A \cong A'$ and $B \cong B'$.

2. Notice that the subring generated by the set $A' \subseteq T$ can be expressed as

$$\langle A' \rangle = \left\{ \sum_{k=1}^{k^*} z_k a_k \,\middle|\, k^* \in \mathbb{N}; \forall k\colon\ z_k \in \mathbb{Z}, a_k \in A' \right\} = \mathbb{Z}[A'] = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \middle| \, x \in \mathbb{Z}[A] \right\}.$$

The inclusion $T\langle A' \rangle T \subseteq T$ is obvious. Take an arbitrary matrix $\left[\begin{smallmatrix} x & f \\ g & y \end{smallmatrix}\right] \in T$ and express it as a sum

$$\begin{bmatrix} x & f \\ g & y \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}.$$

Since the Morita context $(A, B, {}_AP_B, {}_BQ_A, \theta, \phi)$ is unitary and surjective, the semigroup A is a factorizable semigroup (Lemma 7 in [21]). In turn $\mathbb{Z}[A]$ is an idempotent ring and there exist elements $x_1, x''_1, x''_1, \ldots, x_{k^*}, x'_{k^*}, x''_{k^*} \in \mathbb{Z}[A]$ such that $x = x_1x'_1x''_1 + \ldots + x_{k^*}x'_{k^*}x_{k^*}$. Now

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \sum_{k=1}^{k^*} \begin{bmatrix} x_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x'_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x''_k & 0 \\ 0 & 0 \end{bmatrix} \in \langle A' \rangle \langle A' \rangle \langle A' \rangle \subseteq T \langle A' \rangle T.$$

Biacts P and Q are unitary, hence $\mathbb{Z}^{(P)}$ and $\mathbb{Z}^{(Q)}$ are unitary bimodules and there exist elements $x_1, \ldots, x_{h^*} \in \mathbb{Z}[A]$ and $f_1, \ldots, f_{h^*} \in \mathbb{Z}^{(P)}$ such that $f = x_1 f_1 + \ldots + x_{h^*} f_{h^*}$. Then

$$\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} = \sum_{k=1}^{k^*} \begin{bmatrix} x_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & f_k \\ 0 & 0 \end{bmatrix} \in \langle A' \rangle T = (\langle A' \rangle \langle A' \rangle) T \subseteq T \langle A' \rangle T$$

and analogously $\left[\begin{smallmatrix} 0 & 0 \\ g & 0 \end{smallmatrix}\right] \in T\langle A' \rangle \subseteq T \langle A' \rangle T$. From the surjectivity of ϕ we know that there exists an element $\sum_{k=1}^{k^*} g_k \otimes f_k \in \mathbb{Z}^{(Q)} \otimes_{\mathbb{Z}[A]} \mathbb{Z}^{(P)}$ such that $y = \sum_{k=1}^{k^*} \phi(g_k \otimes f_k)$. Now

$$\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} = \sum_{k=1}^{k^*} \begin{bmatrix} 0 & 0 \\ g_k & 0 \end{bmatrix} \begin{bmatrix} 0 & f_k \\ 0 & 0 \end{bmatrix} \in (T\langle A' \rangle)(\langle A' \rangle T) = T \langle A' \rangle T.$$

Therefore we have shown that $T = T\langle A' \rangle T$.
Notice that since $\mathbb{Z}[A]$ is idempotent, we have $\langle A' \rangle = \langle A' \rangle \langle A' \rangle \langle A' \rangle \subseteq \langle A' \rangle T \langle A' \rangle$. For every $\xi_1, \xi_2, x \in \mathbb{Z}[A]$, $y \in \mathbb{Z}[B]$, $f \in \mathbb{Z}^{(P)}$ and $g \in \mathbb{Z}^{(Q)}$, we have

$$\begin{bmatrix} \xi_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & f \\ g & y \end{bmatrix} \begin{bmatrix} \xi_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \xi_1 x & \xi_1 f \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \xi_1 x \xi_2 & 0 \\ 0 & 0 \end{bmatrix} \in A' \subseteq \langle A' \rangle.$$

In conclusion we have shown that $\langle A' \rangle \sqsubseteq T$. The proof for $\langle B' \rangle \sqsubseteq T$ is analogous. ■

Due to Proposition 4.2 (1) we obtain that the ring T from the previous theorem is idempotent. Note that the ring $\langle A' \rangle$ from the previous theorem is isomorphic to the semigroup ring $\mathbb{Z}[A]$. This observation gives us the following corollary.

Corollary 4.26. *If semigroups A and B are strongly Morita equivalent, then the semigroup rings $\mathbb{Z}[A]$ and $\mathbb{Z}[B]$ are Morita equivalent.*

Chapter 5

Unitary ideals of rings

In this section we will study unitary ideals of rings. In particular we will prove that the set of unitary ideals of a ring forms a quantale and if two idempotent rings R and S are Morita equivalent, then their quantales of unitary ideals are isomorphic. Also we will show that the quotient rings by ideals that correspond to each other under that isomorphism are connected by a Morita context with surjective mappings. This section is based on the article [49].

5.1 Quantale of unitary ideals

Let R be a ring. A right (left) ideal I of R is called **unitary** if I is a unitary right (left) R-module, i.e., $IR = I$ ($RI = I$). An ideal $I \trianglelefteq R$ is called **unitary** if I is a unitary (R, R)-bimodule. By Lemma 2.24 we deduce that an ideal $I \trianglelefteq R$ is unitary if and only if $RIR = I$. The set of all unitary ideals of R will be denoted by $\mathrm{UId}(R)$. Unitary ideals of a ring are also studied in [10], where they are called *lower closed ideals* (Definition 3.1).

Next we will define the notion of a quantale. First recall that a poset L is called a **complete lattice** if every subset of L has both a meet and a join.

Definition 5.1 (Definition 2.1.1 in [40]). A complete lattice L is called a **quantale**, if it is equipped with an associative binary operation $*\colon\ L \times L \to L$, such that for every set K and for every $a, b_k \in L$, where $k \in K$, the following conditions hold

$$a * \left(\bigvee_{k \in K} b_k \right) = \bigvee_{k \in K} (a * b_k),$$

$$\left(\bigvee_{k\in K} b_k\right) * a = \bigvee_{k\in K} (b_k * a).$$

A quantale L is called **unital** if there exists an element $e \in L$ such that $a * e = e * a = a$ for every $a \in L$. The element e is called the **identity** element of the quantale A.

Let L and L' be quantales. A mapping $f\colon\ L \to L'$ is called an **isomorphism of quantales** if it is bijective, preserves arbitrary joins and

$$f(a_1 * a_2) = f(a_1) * f(a_2)$$

for every $a_1, a_2 \in L$. An **isomorphism of unital quantales** also has to preserve the identity element.

It is well known that the lattice $\mathrm{Id}(R)$ of all ideals of a ring R is a quantale (Example §2.6 in [40]). Now we will prove a proposition, which shows that the set of unitary ideals of a ring R naturally posesses the structure of a quantale.

Proposition 5.2. *Let R be a ring. The set $\mathrm{UId}(R)$ is a quantale.*

Proof. The poset $(\mathrm{UId}(R), \subseteq)$ is a complete lattice where, for every subset $U \subseteq \mathrm{UId}(R)$, we have

$$\bigvee U = \sum_{I\in U} I \quad \text{and} \quad \bigwedge U = \bigvee \left\{ V \in \mathrm{UId}(R) \;\middle|\; V \subseteq \bigcap_{I\in U} I \right\}.$$

By Proposition 3.2 in [10], any sum of unitary ideals is also a unitary ideal.

Define the operation $*\colon\ \mathrm{UId}(R) \times \mathrm{UId}(R) \to \mathrm{UId}(R)$ as $(I_1, I_2) \mapsto I_1 I_2$. If $J \in \mathrm{UId}(R)$ and $U \subseteq \mathrm{UId}(R)$ then

$$J * \left(\bigvee_{I\in U} I\right) = J\left(\sum_{I\in U} I\right) = \sum_{I\in U} JI = \bigvee_{I\in U} (J * I).$$

The other compatibility condition in the definition of a quantale holds analogously. ∎

We will see that if R is an idempotent ring, then the quantale $\mathrm{UId}(R)$ is even unital.

Proposition 5.3. *If R is an idempotent ring, then $\mathrm{UId}(R)$ is a unital quantale with identity element R.*

Proof. If R is an idempotent ring then, by Proposition 5.2, $\mathrm{UId}(R)$ is a quantale and R is a unitary ideal of itself. It is also clear from the definition of a unitary ideal that for every $I \in \mathrm{UId}(R)$ we have $RI = IR = I$, which means that R is an identity element of $\mathrm{UId}(R)$.

Meets are calculated here as follows:

$$\bigwedge U = R\left(\bigcap_{I \in U} I\right)R,$$

for any subset $U \subseteq \mathrm{UId}(R)$. ∎

From Proposition 2.25 we obtain that if R is an idempotent ring, then the quantale $\mathrm{UId}(R)$ is also a modular lattice.

Next we will give a description of unitary ideals generated by a subset of R, but first we need to give the definition.

Definition 5.4. Let R be a ring. It is said that an ideal $I \trianglelefteq R$ is **generated** by a subset $X \subseteq R$ if I is the smallest ideal that contains X. In that case we write $I = (X)_{\mathrm{g}}$. We say that an ideal $I \trianglelefteq R$ is **finitely generated** if it is generated by a finite set $X \subseteq R$.

One can give an explicit description of the ideal generated by X. According to [41] (page 5), the ideal $(X)_{\mathrm{g}}$ is

$$(X)_{\mathrm{g}} = \mathbb{Z}X + RX + XR + RXR. \tag{5.1}$$

Proposition 5.5. *Let R be a ring. If a unitary ideal $I \trianglelefteq R$ is generated by a set $X \subseteq R$, then $I = RXR$.*

Proof. Let $(X)_{\mathrm{g}} = I \in \mathrm{UId}(R)$. Then we have

$$\begin{aligned} I = RIR &= R(\mathbb{Z}X + RX + XR + RXR)R \\ &= \mathbb{Z}RXR + RRXR + RXRR + RRXRR \subseteq RXR. \end{aligned}$$

On the other hand, we see from the equality (5.1) that $RXR \subseteq I$. Therefore we have $I = RXR$. ∎

5.2 Unitary ideals and s-unital rings

In this section we will see that s-unital rings can be described in terms of unitary left and right ideals.

Proposition 5.6. *A ring R is right (left) s-unital if and only if all right (left) ideals of R are unitary.*

PROOF. *Necessity.* Let R be an s-unital ring. If I is a right ideal of R and $a \in I$, then $a = au$ for some $u \in R$. Hence $I = IR$.

Sufficiency. Let all right ideals of a ring R be unitary. Take an element $r \in R$. Since the right ideal $I = \mathbb{Z}r + rR$ is unitary, there exist elements $z_1, \ldots, z_{k^*} \in \mathbb{Z}$ and $r_1, u_1, \ldots, r_{k^*}, u_{k^*} \in R$ such that

$$
\begin{aligned}
r &= \sum_{k=1}^{k^*}(z_k r + r r_k)u_k = \sum_{k=1}^{k^*}(z_k r u_k + r r_k u_k) = \sum_{k=1}^{k^*}(r(z_k u_k) + r(r_k u_k)) \\
&= r \sum_{k=1}^{k^*}(z_k u_k + r_k u_k).
\end{aligned}
$$

The case for left s-unitality is completely analogous. ■

Corollary 5.7. *All ideals of an s-unital ring are unitary.*

5.3 Quantales of unitary ideals and Morita contexts

In this section we will study the quantales of unitary ideals of rings connected by a surjective but not necessarily unitary Morita context. It turns out that in that case these quantales are isomorphic. The following theorem is a ring theoretic analogue of Theorem 3.4 in [24]. It also generalizes Proposition 3.3 in [7] and Proposition 3.5 in [14].

Theorem 5.8. *Let R and S be rings. If R and S are connected by a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, then their quantales of unitary ideals* $\mathrm{UId}(R)$ *and* $\mathrm{UId}(S)$ *are isomorphic. This isomorphism takes finitely generated ideals to finitely generated ideals. If the rings R and S are idempotent, then the previous isomorphism is a morphism of unital quantales.*

PROOF. Let R and S be rings connected by a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$.

1. Note that, for every unitary ideal $J \in \mathrm{UId}(S)$, the set

$$
\theta(PJ \otimes_S Q) := \left\{ \theta\left(\sum_{k=1}^{k^*} p_k j_k \otimes q_k\right) \middle| \forall k\colon\ p_k \in P, j_k \in J, q_k \in Q \right\} \subseteq R
$$

is an ideal, because θ is an (R, R)-bimodule homomorphism (here the set $PJ \otimes_S Q$ is considered as a subset of the tensor product $P \otimes_S Q$). Additionally, we have

$$\begin{aligned}\theta(PJ \otimes_S Q) &= \theta(PSJS \otimes_S Q) = \theta(PSJ \otimes_S SQ)\\ &= \theta(P\mathrm{Im}(\phi)J \otimes_S \mathrm{Im}(\phi)Q) = \theta(\mathrm{Im}(\theta)PJ \otimes_S Q\mathrm{Im}(\theta))\\ &= \theta(RPJ \otimes_S QR) = R\theta(PJ \otimes_S Q)R.\end{aligned}$$

Therefore, the ideal $\theta(PJ \otimes Q)$ is unitary. Analogously, we can show that, for every $I \in \mathrm{UId}(R)$, the set $\phi(QI \otimes P)$ is a unitary ideal of S. This allows us to define the mappings

$$\Theta\colon\ \mathrm{UId}(S) \to \mathrm{UId}(R), \qquad \Theta(J) := \theta(PJ \otimes_S Q), \tag{5.2}$$

$$\Phi\colon\ \mathrm{UId}(R) \to \mathrm{UId}(S), \qquad \Phi(I) := \phi(QI \otimes_R P). \tag{5.3}$$

Let $J_1, J_2 \in \mathrm{UId}(S)$ be such that $J_1 \subseteq J_2$. Then we have the inclusion $\Theta(J_1) = \theta(PJ_1 \otimes_S Q) \subseteq \theta(PJ_2 \otimes_S Q) = \Theta(J_2)$, which means that the mapping Θ preserves order. Analogously, the mapping Φ also preserves order. If $J \in \mathrm{UId}(S)$, then

$$\begin{aligned}\Phi(\Theta(J)) &= \phi(Q\theta(PJ \otimes_S Q) \otimes_R P) = \phi(\phi(Q \otimes_R PJ)Q \otimes_R P)\\ &= \phi(Q \otimes_R P)J\phi(Q \otimes_R P) = SJS = J.\end{aligned}$$

Analogously, $\Theta(\Phi(I)) = I$ holds for every $I \in \mathrm{UId}(R)$, which means that the mappings Φ and Θ are inverses of each other. Hence, the mappings Φ and Θ are actually isomorphisms of posets. Consequently, Φ and Θ both preserve arbitrary joins.

If $J_1, J_2 \in \mathrm{UId}(S)$, then

$$\begin{aligned}\Theta(J_1)\Theta(J_2) &= \theta(PJ_1 \otimes_S Q)\theta(PJ_2 \otimes_S Q) = \theta(PJ_1 \otimes_S Q\theta(PJ_2 \otimes_S Q))\\ &= \theta(PJ_1 \otimes_S \phi(Q \otimes_R PJ_2)Q) = \theta(PJ_1 \otimes_S \phi(Q \otimes_R P)J_2Q)\\ &= \theta(PJ_1 \otimes_S SJ_2Q) = \theta(PJ_1 \otimes_S J_2Q)\\ &= \theta(P(J_1J_2) \otimes_S Q) = \Theta(J_1J_2).\end{aligned}$$

Analogously, we can show that, for every $I_1, I_2 \in \mathrm{UId}(R)$, the equality $\Phi(I_1)\Phi(I_2) = \Phi(I_1I_2)$ holds. Hence, Θ and Φ are isomorphisms of quantales.

2. Let $J \in \mathrm{UId}(S)$ be a finitely generated ideal. Then there exists a finite set $X = \{x_1, \dots, x_n\} \subseteq J$ such that $J = SXS$ (see Proposition 5.5). Fix an index $k \in \{1, \dots, n\}$. Then the element x_k can be written as

$$x_k = \sum_{h=1}^{h^*} s_{kh}x_{kh}s'_{kh},$$

where $s_{k1}, s'_{k1}, \ldots, s_{kh^*}, s'_{kh^*} \in S$ and $x_{k1}, \ldots, x_{kh^*} \in X$. Using the surjectivity of ϕ, we can also express x_k as follows:

$$x_k = \sum_{t=1}^{t^*} \phi(q_t \otimes p_t)\xi_t \phi(q'_t \otimes p'_t),$$

where $q_1, q'_1, \ldots, q_{t^*}, q'_{t^*} \in Q$, $p_1, p'_1, \ldots, p_{t^*}, p'_{t^*} \in P$ and $\xi_1, \ldots, \xi_{t^*} \in X$. Now, for every $p \in P$ and $q \in Q$, we have

$$\begin{aligned}
\theta(px_k \otimes q) &= \theta\left(p \sum_{t=1}^{t^*} \phi(q_t \otimes p_t)\xi_t \phi(q'_t \otimes p'_t) \otimes q\right) \\
&= \sum_{t=1}^{t^*} \theta(p\phi(q_t \otimes p_t)\xi_t \otimes \phi(q'_t \otimes p'_t)q) \\
&= \sum_{t=1}^{t^*} \theta(\theta(p \otimes q_t)p_t\xi_t \otimes q'_t\theta(p'_t \otimes q)) \\
&= \sum_{t=1}^{t^*} \theta(p \otimes q_t)\theta(p_t\xi_t \otimes q'_t)\theta(p'_t \otimes q) \in RYR,
\end{aligned}$$

where

$$Y := \{\theta(p_t\xi_t \otimes q'_t) \mid t \in \{1, \ldots, t^*\}\} \subseteq R.$$

Clearly, Y is a finite set. Note that

$$\begin{aligned}
\Theta(J) = \theta(PJ \otimes_S Q) &= \left\{\theta\left(\sum_{u=1}^{u^*} p_u j_u \otimes q_u\right) \,\middle|\, \forall u\colon\ p_u \in P, q_u \in Q, j_u \in J\right\} \\
&= \left\{\theta\left(\sum_{u=1}^{u^*} p_u \left(\sum_{h=1}^{h^*} s_{hu}x_{hu}s'_{hu}\right) \otimes q_u\right) \,\middle|\, \forall u, h\colon \begin{array}{ll} p_u \in P, & q_u \in Q, \\ x_{hu} \in X, & s_{hu}, s'_{hu} \in S \end{array}\right\} \\
&= \left\{\sum_{u=1}^{u^*} \sum_{h=1}^{h^*} \theta((p_u s_{hu})x_{hu} \otimes (s'_{hu}q_u)) \,\middle|\, \forall u, h\colon \begin{array}{ll} p_u \in P, & q_u \in Q, \\ x_{hu} \in X, & s_{hu}, s'_{hu} \in S \end{array}\right\} \\
&\subseteq RYR.
\end{aligned}$$

On the other hand, $Y \subseteq \Theta(J)$. Since $\Theta(J)$ is an ideal of R which contains Y,

$$(Y)_{\mathrm{g}} \subseteq \Theta(J) \subseteq RYR \subseteq (Y)_{\mathrm{g}},$$

which implies $\Theta(J) = (Y)_{\mathrm{g}}$. Hence, $\Theta(J)$ is a finitely generated ideal.

3. Let the rings R and S be idempotent. Then, by Proposition 5.3, the quantales $\mathrm{UId}(R)$ and $\mathrm{UId}(S)$ are unital quantales with identity elements R and S, respectively. Since sup-lattice isomorphisms preserve the largest elements, $\Theta(S) = R$ and $\Phi(R) = S$. ■

Remark. In Proposition 3.5 in the article [14], it has been shown that if idempotent rings R and S are connected by a unitary surjective Morita context, then the lattices $\mathrm{UId}(R)$ and $\mathrm{UId}(S)$ are isomorphic. We have proved that, additionally, they are isomorphic as quantales, that these isomorphisms behave well with respect to finitely generated ideals, and showed that assumig idempotence of rings and unitariness of bimodules in the Morita context is not necessary.

Theorem 5.8 implies that the isomorphisms Θ and Φ preserve all properties of unitary ideals that are defined using multiplication of ideals, inclusion relation, joins or meets. For example, if I is a semiprime element in the quantale $\mathrm{UId}(R)$ ([40, Definition 3.2.5]), then $\Phi(I)$ is semiprime in $\mathrm{UId}(S)$. An analogous statement holds for prime elements ([40, Definition 3.2.8]). In [42], the radical of a complete lattice is defined as the meet of all coatoms. Thus Φ takes the radical of the lattice $\mathrm{UId}(R)$ to the radical of $\mathrm{UId}(S)$.

Corollary 5.9. *If R is an idempotent ring and n a natural number, then $\mathrm{UId}(R)$ and $\mathrm{UId}(\mathrm{Mat}_n(R))$ are isomorphic quantales.*

Proof. Let R be an idempotent ring. By Corollary 4.9, $R \approx_{\mathrm{ME}} \mathrm{Mat}_n(R)$. The ring $\mathrm{Mat}_n(R)$ is idempotent by Example 4.6 and Proposition 4.2 (1). Then, by Theorem 2.28, the rings R and $\mathrm{Mat}_n(R)$ are connected by a unitary surjective Morita context. Now the claim follows from Theorem 5.8. ■

Corollary 5.10. *If R is an s-unital ring and n a natural number, then $\mathrm{Id}(R)$ and $\mathrm{Id}(\mathrm{Mat}_n(R))$ are isomorphic quantales.*

Proof. Let R be a s-unital ring. Using Theorem 2.21, we see that for every matrix $A \in \mathrm{Mat}_n(R)$ there exists an element $u \in R$ such that

$$A = \begin{bmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \dots & r_{nn} \end{bmatrix} = \begin{bmatrix} ur_{11} & \dots & ur_{1n} \\ \vdots & \ddots & \vdots \\ ur_{n1} & \dots & ur_{nn} \end{bmatrix} = \begin{bmatrix} u & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & u \end{bmatrix} \begin{bmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \dots & r_{nn} \end{bmatrix}.$$

Hence $\mathrm{Mat}_n(R)$ is left s-unital. Analogoualy we see that $\mathrm{Mat}_n(R)$ is right s-unital. The claim follows from Corollary 5.9 and Corollary 5.7. ■

In [10] Buys and Kyuno showed that the lattices $\mathrm{UId}(R)$ and $\mathrm{UId}(S)$ are also isomorphic to the lattices $\mathrm{USub}(P)$ and $\mathrm{USub}(Q)$, where P and Q come from the Morita context. The last two lattices will be studied with more detail in Section 6.4.

Theorem 5.11 (Theorem 3.3 in [10]). *Let the rings R and S be connected by a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. Then the following lattices are isomorphic:*
(1) $\mathrm{UId}(R)$, *(2)* $\mathrm{UId}(S)$, *(3)* $\mathrm{USub}({}_RP_S)$, *(4)* $\mathrm{USub}({}_SQ_R)$.

The isomorphisms in the previous theorem are obtained using the following mappings:

$$\begin{aligned}
\Psi\colon\ & \mathrm{UId}(R) \to \mathrm{USub}(P), && \Psi(I) := IP,\\
\Omega\colon\ & \mathrm{USub}(P) \to \mathrm{UId}(R), && \Omega(A) := \theta(A \otimes_S Q);\\
\Psi'\colon\ & \mathrm{UId}(R) \to \mathrm{USub}(Q), && \Psi'(I) := QI,\\
\Omega'\colon\ & \mathrm{USub}(Q) \to \mathrm{UId}(R), && \Omega'(B) := \theta(P \otimes_S B).
\end{aligned}$$

Remark. Among other things, Theorem 5.11 implies that if R and S are s-unital rings then

$$R \text{ is uniform} \iff S \text{ is uniform} \iff {}_RP_S \text{ is uniform} \iff {}_SQ_R \text{ is uniform},$$

where **uniformity** means that the intersection of every two non-zero ideals (sub-bimodules) is non-zero (see Paragraph 19.9 in [51]). An analogous claim holds for the dual notion – **hollowness** (see Paragraph 41.3 in [51]). In particular, uniformity and hollowness are Morita invariants on the class of s-unital rings.

In [10] (Definition 4.1), the **two sided socle** of a ring R is defined as

$$\mathrm{Soc}(R) := \sum\{I \mid I \text{ is a minimal ideal of } R\}.$$

Minimal ideals of R are precisely the atoms of the lattice $\mathrm{Id}(R)$ and $\mathrm{Soc}(R)$ is the join of all atoms of the complete lattice $\mathrm{Id}(R)$.

Definition 5.12. We define the **unitary two-sided socle** of a ring R as

$$\begin{aligned}
\mathrm{USoc}(R) &:= \sum\{I \in \mathrm{UId}(R) \mid I = \{0\} \text{ or } I \text{ is an atom of the lattice } \mathrm{UId}(R)\}\\
&= \bigvee\{I \in \mathrm{UId}(R) \mid I = \{0\} \text{ or } I \text{ is an atom of the lattice } \mathrm{UId}(R)\},
\end{aligned}$$

where the join is calculated in the lattice $\mathrm{UId}(R)$ (see also Section 2 in [42]).

If rings R and S are connected by a surjective Morita context, then by Theorem 5.8 there exists a sup-lattice isomorphism $\Theta\colon\ \mathrm{UId}(S) \to \mathrm{UId}(R)$ and it follows that

$$\Theta(\mathrm{USoc}(S)) = \mathrm{USoc}(R).$$

If the ring R (and analogously S) satisfies the condition

$$\forall r \in R\colon \quad (RrR = \{0\} \implies r = 0), \tag{5.4}$$

then every minimal ideal of R is unitary (Proposition 3.5 in [10]), hence $\mathrm{USoc}(R) = \mathrm{Soc}(R)$ and we may write

$$\Theta(\mathrm{Soc}(S)) = \mathrm{Soc}(R).$$

Definition 5.13 (Definition 4.5 in [10]). A ring R is called **completely reducible** if $\mathrm{Soc}(R) = R$.

If R is an idempotent ring, then the fact that the ring R is completely reducible means that the largest element R is the join of all atoms in the lattice $\mathrm{UId}(R)$.

Proposition 5.14. *Let R and S be s-unital rings. If R and S are connected by a surjective Morita context, then R is completely reducible if and only if S is completely reducible.*

Proof. Let R and S be s-unital rings connected by a surjective Morita context. Assume that S is completely reducible. Since R is s-unital, it satisfies (5.4). Indeed take $r \in R$ such that $RrR = \{0\}$. Due to the s-unitality we can find $u \in R$ such that $r = ru$ and also $v \in R$ such that $r = ru = v(ru) = vru \in RrR$. By our assumption we then have $r = 0$.

Hence $\mathrm{Soc}(R) = \mathrm{USoc}(R)$ and $\mathrm{Soc}(S) = \mathrm{USoc}(S)$. Due to Theorem 5.8, we have a sup-lattice isomorphism $\Theta\colon\ \mathrm{UId}(S) \to \mathrm{UId}(R)$. Now

$$\mathrm{Soc}(R) = \Theta(\mathrm{Soc}(S)) = \Theta(S) = R,$$

yielding that R is completely reducible. The other direction is similar. ■

5.4 Ideals and Morita contexts

In this section we will prove some results that will show how Morita contexts relate to the ideals of its underlying rings.

Recall that the **annihilator** of a right R-module M_R is defined as:

$$\mathrm{Ann}_R(M) := \{r \in R \mid Mr = 0\}.$$

It is easy to see that, for any right R-module M_R, its annihilator $\mathrm{Ann}_R(M)$ is an ideal of R. A right R-module M_R is called **faithful** if $\mathrm{Ann}_R(M) = \{0\}$.

Now we will prove a result which generalizes Proposition 18.47 in [28].

Proposition 5.15. *Let R and S be s-unital rings. If R and S are connected by a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ then there exists an isomorphism $\Phi\colon\ \mathrm{Id}(R) \to \mathrm{Id}(S)$. Moreover, for every right R-module M_R,*

- *$\Phi(\mathrm{Ann}_R(M)) = \mathrm{Ann}_S(M \otimes_R P)$;*
- *M_R is faithful if and only if the right S-module $M \otimes_R P$ is faithful.*

Proof. Let R and S be s-unital rings connected by a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$. By Corollary 5.7, we have $\mathrm{Id}(R) = \mathrm{UId}(R)$ and $\mathrm{Id}(S) = \mathrm{UId}(S)$. Due to Theorem 5.8, $\mathrm{Id}(R) \cong \mathrm{Id}(S)$ as quantales, where the isomorphism $\Phi\colon\ \mathrm{Id}(R) \to \mathrm{Id}(S)$ is defined as in (5.3). Note that

$$\begin{aligned}(M \otimes_R P)\Phi(\mathrm{Ann}_R(M)) &= (M \otimes_R P)\phi(Q\,\mathrm{Ann}_R(M) \otimes_R P)\\ &= M \otimes_R \theta(P \otimes_S Q)\,\mathrm{Ann}_R(M)P\\ &= M \otimes_R R\,\mathrm{Ann}_R(M)P = MR\,\mathrm{Ann}_R(M) \otimes_R P\\ &\subseteq M\,\mathrm{Ann}_R(M) \otimes_R P = 0 \otimes_R P = \{0\}.\end{aligned}$$

Therefore, we have $\Phi(\mathrm{Ann}_R(M)) \subseteq \mathrm{Ann}_S(M \otimes_R P)$. Analogously, we can show that $\Theta(\mathrm{Ann}_S(M \otimes_R P)) \subseteq \mathrm{Ann}_R(M \otimes_R P \otimes_S Q)$, where $\Theta\colon\ \mathrm{Id}(S) \to \mathrm{Id}(R)$ is an isomorphism defined as in (5.2).

Now, take $r \in \mathrm{Ann}_R(M \otimes_R P \otimes_S Q) \subseteq R$. Since R is s-unital, there exists an element $v \in R$ such that $r = vr$ and, due to the surjectivity of θ, there exist elements $p_1, \ldots, p_{k^*} \in P$ and $q_1, \ldots, q_{k^*} \in Q$ such that $v = \sum_{k=1}^{k^*} \theta(p_k \otimes q_k)$. Note that, for any $m \in M$, we have

$$\begin{aligned}mr = mvr = \nu_M(m \otimes v)r = \sum_{k=1}^{k^*} \nu_M(m \otimes \theta(p_k \otimes q_k))r\\ = \sum_{k=1}^{k^*} \nu_M((\mathrm{id}_M \otimes \theta)(m \otimes p_k \otimes q_k))r = \sum_{k=1}^{k^*} \nu_M((\mathrm{id}_M \otimes \theta)((m \otimes p_k \otimes q_k)r))\\ = \sum_{k=1}^{k^*} \nu_M((\mathrm{id}_M \otimes \theta)(0)) = 0,\end{aligned}$$

where $\nu_M\colon\ M \to R$ is a homomorphism defined as in (2.6). Hence, $r \in \mathrm{Ann}_R(M)$. Now we have proved the inclusions

$$\Theta(\mathrm{Ann}_S(M \otimes_R P)) \subseteq \mathrm{Ann}_R(M \otimes_R P \otimes_S Q) \subseteq \mathrm{Ann}_R(M).$$

Applying the poset isomorphism Φ to the previous sequence of inclusions we obtain

$$\mathrm{Ann}_S(M \otimes_R P) = \Phi(\Theta(\mathrm{Ann}_S(M \otimes_R P))) \subseteq \Phi(\mathrm{Ann}_R(M)).$$

In conclusion, we have shown that $\Phi(\mathrm{Ann}_R(M)) = \mathrm{Ann}_S(M \otimes_R P)$.

If M_R is faithful, then $\{0\} = \mathrm{Ann}_R(M) = \Theta(\mathrm{Ann}_S(M \otimes_R P))$, which implies that $\mathrm{Ann}_S(M \otimes_R P) = \{0\}$, because Θ is an isomorphism. ■

Next we will prove a theorem about finding quotients of Morita contexts, it is a generalization of Corollary 18.49 in [28]. It will imply that if R and S are Morita equivalent idempotent rings then every quotient ring of R is Morita equivalent to a certain quotient ring of S.

Theorem 5.16. *Let R and S be rings and $\Gamma = (R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$ a Morita context. Then, for every ideal $I \in \mathrm{Id}(R)$, the quotient rings R/I and $S/\Phi(I)$ are connected by a Morita context*

$$\Gamma_I = (R/I, S/\Phi(I), P/\Psi(I), Q/\Psi'(I), \zeta, \eta),$$

where

$$\begin{aligned} \Phi\colon\ & \mathrm{Id}(R) \to \mathrm{Id}(S), && \Phi(I) := \phi(QI \otimes P), \\ \Psi\colon\ & \mathrm{Id}(R) \to \mathrm{Sub}(P), && \Psi(I) := IP, \\ \Psi'\colon\ & \mathrm{Id}(R) \to \mathrm{Sub}(Q), && \Psi'(I) := QI. \end{aligned}$$

Moreover,

- *if Γ is surjective, then Γ_I is also surjective;*
- *if Γ is unitary, then Γ_I is also unitary.*

Proof. Let $I \in \mathrm{Id}(R)$. We must show that the abelian group $P/\Psi(I)$ is an $(R/I, S/\Phi(I))$-bimodule. Consider the mappings

$$R/I \times P/\Psi(I) \to P/\Psi(I), \qquad ([r],[p]) \mapsto [rp], \tag{5.5}$$
$$P/\Psi(I) \times S/\Phi(I) \to P/\Psi(I), \qquad ([p],[s]) \mapsto [ps]. \tag{5.6}$$

Let $p_1, p_2 \in P$ and $s_1, s_2 \in S$ be such that $[p_1]_{\Psi(I)} = [p_2]_{\Psi(I)}$ and $[s_1]_{\Phi(I)} = [s_2]_{\Phi(I)}$. Then we have $p_1 - p_2 \in \Psi(I) = IP$ and $s_1 - s_2 \in \Phi(I) = \phi(QI \otimes_R P)$. Note that

$$\begin{gathered} p_1 s_1 - p_2 s_1 = (p_1 - p_2)s_1 \in IPS \subseteq IP, \\ p_2 s_1 - p_2 s_2 = p_2(s_1 - s_2) \in P\phi(QI \otimes_R P) = \theta(P \otimes_S Q)IP \subseteq RIP \subseteq IP, \end{gathered}$$

which implies that

$$[p_1 s_1]_{\Psi(I)} = [p_2 s_1]_{\Psi(I)} = [p_2 s_2]_{\Psi(I)}.$$

Therefore the mapping (5.6) is well defined. Analogously, the mapping (5.5) is also well defined. Now it is easy to see that $P/\Psi(I)$ is an $(R/I, S/\Phi(I))$-bimodule with the mappings (5.5) and (5.6).

Analogously, the abelian group $Q/\Psi'(I)$ is an $(S/\Phi(I), R/I)$-bimodule.

Define the mappings ζ and η as follows:

$$\zeta\colon\ P/\Psi(I) \otimes_{S/\Phi(I)} Q/\Psi'(I) \to R/I, \quad \sum_{k=1}^{k^*}[p_k]\otimes[q_k] \mapsto \sum_{k=1}^{k^*}[\theta(p_k\otimes q_k)]_I,$$
$$\eta\colon\ Q/\Psi'(I) \otimes_{R/I} P/\Psi(I) \to S/\Phi(I), \quad \sum_{k=1}^{k^*}[q_k]\otimes[p_k] \mapsto \sum_{k=1}^{k^*}[\phi(q_k\otimes p_k)]_{\Phi(I)}.$$

To show that these mappings are well defined, we consider the mappings

$$\hat{\zeta}\colon\ P/\Psi(I) \times Q/\Psi'(I) \to R/I, \qquad ([p]_{\Psi(I)}, [q]_{\Psi'(I)}) \mapsto [\theta(p\otimes q)]_I,$$
$$\hat{\eta}\colon\ Q/\Psi'(I) \times P/\Psi(I) \to S/\Phi(I), \qquad ([q]_{\Psi'(I)}, [p]_{\Psi(I)}) \mapsto [\phi(q\otimes p)]_{\Phi(I)}.$$

Let $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ be such that $[p_1]_{\Psi(I)} = [p_2]_{\Psi(I)}$ and $[q_1]_{\Psi'(I)} = [q_2]_{\Psi'(I)}$. Then $p_1 - p_2 \in \Psi(I) = IP$ and $q_1 - q_2 \in \Psi'(I) = QI$, therefore there exist elements $\lambda_1, \ldots, \lambda_{k^*} \in P$, $\kappa_1, \ldots, \kappa_{h^*} \in Q$ and $\iota_1, \iota'_1, \ldots, \iota_{k^*}, \iota'_{h^*} \in I$ such that $p_1 - p_2 = \iota_1\lambda_1 + \ldots + \iota_{k^*}\lambda_{k^*}$ and $q_1 - q_2 = \kappa_1\iota'_1 + \ldots + \kappa_{h^*}\iota'_{h^*}$. Now

$$\hat{\zeta}([p_1],[q_1]) - \hat{\zeta}([p_2],[q_1]) = [\theta((p_1-p_2)\otimes q_1)]_I = \left[\sum_{k=1}^{k^*}\iota_k\theta(\lambda_k\otimes q_1)\right]_I = [0]_I,$$
$$\hat{\zeta}([p_2],[q_1]) - \hat{\zeta}([p_2],[q_2]) = [\theta(p_2\otimes(q_1-q_2))]_I = \left[\sum_{h=1}^{h^*}\theta(p_2\otimes\kappa_h)\iota'_h\right]_I = [0]_I.$$

Therefore we have

$$\hat{\zeta}([p_1]_{\Psi(I)}, [q_1]_{\Psi'(I)}) = \hat{\zeta}([p_2]_{\Psi(I)}, [q_1]_{\Psi'(I)}) = \hat{\zeta}([p_2]_{\Psi(I)}, [q_2]_{\Psi'(I)}),$$

which gives us that the mapping $\hat{\zeta}$ is well defined. Since $\hat{\zeta}$ is also $S/\Phi(I)$-balanced, due to the universal property of tensor product (see Proposition 2.11), the mapping ζ is a well-defined homomorphism of abelian groups. Analogously, the mappings $\hat{\eta}$ and η are well defined. Also, ζ and η are bimodule homomorphisms, because θ and ϕ are bimodule homomorphisms.

Now, for every $p, p' \in P$ and $q, q' \in Q$, we have

$$\zeta([p] \otimes [q])[p'] = [\theta(p \otimes q)][p'] = [\theta(p \otimes q)p'] = [p\phi(q \otimes p')] = [p]\eta([q] \otimes [p']),$$
$$[q']\zeta([p] \otimes [q]) = [q'][\theta(p \otimes q)] = [q'\theta(p \otimes q)] = [\phi(q' \otimes p)q] = \eta([q'] \otimes [p])[q].$$

In conclusion, we have shown that $(R/I, S/\Phi(I), P/\Psi(I), Q/\Psi'(I), \zeta, \eta)$ is a Morita context.

If θ and ψ are surjective, then ζ and η are also surjective. If P and Q are unitary, then their quotient bimodules are unitary too. ■

Corollary 5.17. *If two idempotent rings R and S are Morita equivalent, then, for every ideal $I \in \mathrm{Id}(R)$, the quotient rings R/I and $S/\Phi(I)$ are also Morita equivalent.*

Chapter 6

Monomorphisms and unitary sub-bimodules of firm bimodules

In this chapter we will study monomorphisms in the categories ${}_S\mathsf{UMod}_R$ and ${}_S\mathsf{FMod}_R$, for idempotent rings S and R. First we will study the categories ${}_S\mathsf{FMod}_R$ and ${}_S\mathsf{CMod}_R$ thoroughly. Then we show that the bimodule categories ${}_S\mathsf{FMod}_R$, ${}_S\mathsf{CMod}_R$ and ${}_S\mathsf{UTfMod}_R$ are equivalent and, moreover, that the category ${}_S\mathsf{CMod}_R$ is an essential localization of ${}_S\mathsf{Mod}_R$. Later we will use these results to show that, for a firm (S, R)-bimodule ${}_SM_R$, the lattice of unitary sub-bimodules $\mathrm{USub}(M)$ is isomorphic to the lattice of subobjects of M in the category ${}_S\mathsf{FMod}_R$. This chapter is a generalization of article [47] to the case of bimodules.

6.1 Subcategories of the category of all bimodules

In this section we will study the bimodule subcategories ${}_S\mathsf{UMod}_R$, ${}_S\mathsf{FMod}_R$, ${}_S\mathsf{CMod}_R$ and ${}_S\mathsf{UTfMod}_R$. As a main result we will prove that ${}_S\mathsf{FMod}_R$, ${}_S\mathsf{CMod}_R$ and ${}_S\mathsf{UTfMod}_R$ are all equivalent categories if S and R are idempotent rings. Finally we will show that if S and R are idempotent rings, then ${}_S\mathsf{CMod}_R$ is an essential localization of ${}_S\mathsf{Mod}_R$. The equivalence of the categories of right modules FMod_R, UTfMod_R and CMod_R was proved by Marín (Theorem 2.45 in [33]). Since this sections is somewhat of a detour from the rest of the thesis, but quite lengthy, it is divided in subsections.

6.1.1 The coreflective subcategory of firm bimodules

Firstly we will study the category of firm bimodules ${}_S\mathsf{FMod}_R$. We will show that it is a coreflective subcategory of ${}_S\mathsf{Mod}_R$. But first we will characterize firm bimodules in general.

Proposition 6.1. *Let S and R be rings and ${}_SM_R$ a firm bimodule. Then there exists an (S,R)-bimodule isomorphism*

$$\mu_M\colon\ S\otimes_S M\otimes_R R\to M,\quad \sum_{k=1}^{k^*} s_k\otimes m_k\otimes r_k\mapsto \sum_{k=1}^{k^*} s_k m_k r_k. \tag{6.1}$$

The familiy of morphisms $\mu=(\mu_M)_{M\in {}_S\mathsf{Mod}_R}$ is a natural transformation from the functor $S\otimes_S \_\_ \otimes_R R\colon\ {}_S\mathsf{Mod}_R\to {}_S\mathsf{Mod}_R$ to $\mathrm{id}_{{}_S\mathsf{Mod}_R}$.

Proof. Note that, for any $N\in {}_S\mathsf{Mod}_R$, the mapping $\mu_N\colon\ S\otimes_S N\otimes_R R\to N$ is a morphisms in ${}_S\mathsf{Mod}_R$. Fix $N_1,N_2\in {}_S\mathsf{Mod}_R$ and $f\in \mathrm{Mor}_{{}_S\mathsf{Mod}_R}(N_1,N_2)$ (as shown on Figure 6.1).

$$\begin{tikzcd}
S\otimes N_1\otimes R \arrow[r, "\mu_{N_1}"] \arrow[d, "\mathrm{id}_S\otimes f\otimes \mathrm{id}_R"'] & N_1 \arrow[d, "f"] \\
S\otimes N_2\otimes R \arrow[r, "\mu_{N_2}"'] & N_2
\end{tikzcd}$$

Figure 6.1

If $s\in S$, $r\in R$ and $a\in N_1$, then

$$\begin{aligned}(f\circ\mu_{N_1})(s\otimes a\otimes r) &= f(sar)=sf(a)r=\mu_{N_2}(s\otimes f(a)\otimes r)\\ &= (\mu_{N_2}\circ(\mathrm{id}_S\otimes f\otimes \mathrm{id}_R))(s\otimes a\otimes r).\end{aligned}$$

Therefore $\mu\colon\ S\otimes_S \_\_ \otimes_R R\to \mathrm{id}_{{}_S\mathsf{Mod}_R}$ is a natural transformation.

Let $M\in {}_S\mathsf{FMod}_R$. From the definition of firm bimodules, we know that there exist two canonical isomorphisms $\nu_{{}_SM}\colon\ S\otimes_S M\to M$ and $\nu_{M_R}\colon$ $M\otimes_R R\to M$. The mapping $\mathrm{id}_S\otimes\nu_{M_R}$ is an isomorphism, because the tensor product of isomorphisms is also an isomorphism (Property 12.3 (3) in [51]). For any $s\in S$, $r\in R$ and $m\in M$, we have

$$(\nu_{{}_SM}\circ(\mathrm{id}_S\otimes\nu_{M_R}))(s\otimes m\otimes r)=\nu_{{}_SM}(s\otimes mr)=smr. \tag{6.2}$$

Denote $\mu_M:=\nu_{{}_SM}\circ(\mathrm{id}_S\otimes\nu_{M_R})$. By extending (6.2) from its generators to the whole $S\otimes_S M\otimes_R R$, we have obtained the needed isomorphism μ_M. ■

Note that if $M \in {}_S\mathsf{Mod}_R$ is unitary, then μ_M is a surjective (S, R)-bimodule homomorphism. In the next proposition we will construct a functor $\mathbf{P}\colon {}_S\mathsf{Mod}_R \to {}_S\mathsf{FMod}_R$.

Proposition 6.2. *Let S and R be idempotent rings. For any $M \in {}_S\mathsf{Mod}_R$, the bimodule $S \otimes_S SMR \otimes_R R$ is firm and there exists a functor*

$$\begin{aligned}\mathbf{P} := S \otimes_S S\_R \otimes_R R\colon\ {}_S\mathsf{Mod}_R &\to {}_S\mathsf{FMod}_R,\\ M &\mapsto S \otimes_S SMR \otimes_R R,\\ f &\mapsto \mathrm{id}_S \otimes f|_{SMR} \otimes \mathrm{id}_R\,.\end{aligned}$$

Proof. Let S and R be idempotent rings and ${}_SM_R$ an (S, R)-bimodule. Denote $N := S \otimes_S SM$, clearly N is a right R-module. By Proposition 2.38 in [33], the module $NR \otimes_R R$ is a firm right R-module. Analogously, the module $S \otimes_S S(MR \otimes_R R) = S \otimes_S SMR \otimes_R R$ is a firm left S-module. In conclusion, the module $S \otimes_S SMR \otimes_R R$ is a firm (S, R)-bimodule.

Let $f \in \mathrm{Mor}_{{}_S\mathsf{Mod}_R}(A, B)$ and $g \in \mathrm{Mor}_{{}_S\mathsf{Mod}_R}(B, C)$, for some $A, B, C \in {}_S\mathsf{Mod}_R$. Then

$$\begin{aligned}\mathbf{P}(g \circ f) &= \mathrm{id}_S \otimes (g \circ f)|_{SAR} \otimes \mathrm{id}_R = (\mathrm{id}_S \circ \mathrm{id}_S) \otimes (g \circ f)|_{SAR} \otimes (\mathrm{id}_R \circ \mathrm{id}_R)\\ &= (\mathrm{id}_S \otimes g|_{SBR} \otimes \mathrm{id}_R) \circ (\mathrm{id}_S \otimes f|_{SAR} \otimes \mathrm{id}_R) = \mathbf{P}(g) \circ \mathbf{P}(f).\end{aligned}$$

Here we used the equality $(g \circ f)|_{SAR} = g|_{SBR} \circ f|_{SAR}$, which holds because, for every $\sum_{k=1}^{k^*} s_k a_k r_k \in SAR$, we have

$$f\left(\sum_{k=1}^{k^*} s_k a_k r_k\right) = \sum_{k=1}^{k^*} s_k f(a_k) r_k \in SBR.$$

Also $\mathbf{P}(\mathrm{id}_A) = \mathrm{id}_{S\otimes A\otimes R}$. Therefore $\mathbf{P}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{FMod}_R$ is a functor. ∎

We see, that the functor $\mathbf{P}$ can be expressed as the composition

$$\mathbf{P} = (S \otimes_S \_ \otimes_R R) \circ \mathbf{U}\colon\quad {}_S\mathsf{Mod}_R \to {}_S\mathsf{UMod}_R \to {}_S\mathsf{FMod}_R,$$

where $\mathbf{U} = S\_R$ is the functor defined in (2.10). It is also easy to see that there exists a natural isomorphism $\mathbf{P} \circ \mathbf{P} \cong \mathbf{P}$, if we consider $\mathbf{P}$ as an endofunctor of ${}_S\mathsf{Mod}_R$.

Analogously to Proposition 6.1, it can be shown that

$$\mu^{-1} = (\mu_A^{-1})_{A \in {}_S\mathsf{FMod}_R}\colon\ \mathrm{id}_{{}_S\mathsf{FMod}_R} \to S \otimes_S \_ \otimes_R R$$

is also a natural transformation.

Now we will prove that the functor $\mathbf{P}$ is a coreflector of ${}_S\mathsf{FMod}_R$.

Theorem 6.3. *Let S and R be idempotent rings. The category ${}_S\mathsf{FMod}_R$ is a coreflective subcategory of ${}_S\mathsf{Mod}_R$ with coreflector $\mathbf{P}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{FMod}_R$.*

Proof. Let S and R be idempotent rings. We will show that there exists an adjunction

$$\mathbf{J}_{\mathrm{F}} \dashv S \otimes_S S\_R \otimes_R R = \mathbf{P}, \tag{6.3}$$

where $\mathbf{J}_{\mathrm{F}}\colon\ {}_S\mathsf{FMod}_R \to {}_S\mathsf{Mod}_R$ is the inclusion functor. From Proposition 6.1 and the remark before this theorem, we know that $\mu\colon\ \mathbf{J}_{\mathrm{F}} \circ \mathbf{P} \to \mathrm{id}_{{}_S\mathsf{Mod}_R}$ and $\mu^{-1}\colon\ \mathrm{id}_{{}_S\mathsf{FMod}_R} \to \mathbf{P} \circ \mathbf{J}_{\mathrm{F}}$ are natural transformations. We will show that μ is the counit and μ^{-1} is the unit of the adjunction (6.3).

For any firm bimodule $A \in {}_S\mathsf{FMod}_R$, we have

$$\mu_{\mathbf{J}_{\mathrm{F}}(A)} \circ \mathbf{J}_{\mathrm{F}}(\mu_A^{-1}) = \mu_A \circ \mu_A^{-1} = \mathrm{id}_A = \mathrm{id}_{\mathbf{J}_{\mathrm{F}}(A)},$$

which proves the triangle identity (2.1).

Let $M \in {}_S\mathsf{Mod}_R$, then $\mathbf{P}(M) = S \otimes_S SMR \otimes_R R \in {}_S\mathsf{FMod}_R$. Fix a generator $s' \otimes m \otimes r' \in \mathbf{P}(M)$. Then, there exist elements $s_1, \ldots, s_{k^*} \in S$ and $r_1, \ldots, r_{h^*} \in R$ such that $m = s_1 m_1 r_1 + \ldots + s_{k^*} m_{k^*} r_{k^*}$. Now

$$\begin{aligned}
&\Big(\mathbf{P}(\mu_M) \circ \mu^{-1}_{\mathbf{P}(M)}\Big)(s' \otimes m \otimes r') \\
&\qquad = \big((\mathrm{id}_S \otimes \mu_M \otimes \mathrm{id}_R) \circ \mu^{-1}_{S \otimes SMR \otimes R}\big)\left(s' \otimes \left(\sum_{k=1}^{k^*} s_k m_k r_k\right) \otimes r'\right) \\
&\qquad = (\mathrm{id}_S \otimes \mu_M \otimes \mathrm{id}_R)\left(\sum_{k=1}^{k^*} s' \otimes s_k \otimes m_k \otimes r_k \otimes r'\right) \\
&\qquad = \sum_{k=1}^{k^*} s' \otimes s_k m_k r_k \otimes r' \\
&\qquad = s' \otimes \left(\sum_{k=1}^{k^*} s_k m_k r_k\right) \otimes r' \\
&\qquad = s' \otimes m \otimes r' = \mathrm{id}_{\mathbf{P}(M)}(s' \otimes m \otimes r'),
\end{aligned}$$

which proves the second triangle identity (2.2) $\mathbf{P}(\mu_M) \circ \mu^{-1}_{\mathbf{P}(M)} = \mathrm{id}_{\mathbf{P}(M)}$. Thus we have the adjunction (6.3). ∎

Note that, using the functor $\mathbf{P}$, we can construct a Morita context with firm bimodules between idempotent Morita equivalent rings. The following proposition can be deduced from Proposition 4.13 and Theorem 4.24 in [33], but we will give a direct proof inspired by Theorem 4.11 in [26].

Proposition 6.4. *Two idempotent rings S and R are Morita equivalent if and only if there exists a surjective Morita context $(R, S, {}_RP_S, {}_SQ_R, \theta, \phi)$, where ${}_RP_S$ and ${}_SQ_R$ are firm bimodules.*

Proof. *Necessity.* Let S and R be idempotent rings and $S \approx_{\mathrm{ME}} R$. By Theorem 2.28, there exists a unitary surjective Morita context $(S, R, {}_RP'_S, {}_SQ'_R, \psi, \varphi)$. Consider the bimodules

$$\begin{aligned} P &:= \mathbf{P}(P') = R \otimes_R RP'S \otimes_S S = R \otimes_R P' \otimes_S S, \\ Q &:= \mathbf{P}(Q') = S \otimes_S SQ'R \otimes_R R = S \otimes_S Q' \otimes_R R. \end{aligned}$$

(The last equalities hold, because P' and Q' are unitary). The bimodules P and Q are firm due to Proposition 6.2. The homomorphisms $\mu_{P'}\colon\ P \to P'$ and $\mu_{Q'}\colon\ Q \to Q'$ defined as in (6.1) are surjective, because P' and Q' are unitary. Define the compositions

$$\begin{aligned} \theta &:= \psi \circ (\mu_{P'} \otimes \mu_{Q'})\colon \quad P \otimes_S Q \to P' \otimes_S Q' \to R, \\ \phi &:= \varphi \circ (\mu_{Q'} \otimes \mu_{P'})\colon \quad Q \otimes_R P \to Q' \otimes_R P' \to S. \end{aligned}$$

The mappings θ and ϕ are surjective (R, R)-bimodule and (S, S)-bimodule homomorphisms, respectively, because they are defined as composites of two surjective bimodule homomorphisms (the tensor product of surjective homomorphisms is also a surjective homomorphism).

Note that, for every $p, p' \in P$, $q, q' \in Q$, $s, s', s'' \in S$ and $r, r', r'' \in R$, we have

$$\begin{aligned} &\theta((r \otimes p \otimes s) \otimes (s'' \otimes q \otimes r''))(r' \otimes p' \otimes s') = \psi(rps \otimes s''qr'')(r' \otimes p' \otimes s') \\ &\quad = r\psi(ps \otimes s''qr'')r' \otimes p' \otimes s' = r \otimes \psi(ps \otimes s''qr'')r'p' \otimes s' \\ &\quad = r \otimes ps\varphi(s''qr'' \otimes r'p') \otimes s' = r \otimes p \otimes s\varphi(s''qr'' \otimes r'p')s' \\ &\quad = (r \otimes p \otimes s)\varphi(s''qr'' \otimes r'p's') = (r \otimes p \otimes s)\phi((s'' \otimes q \otimes r'') \otimes (r' \otimes p' \otimes s')). \end{aligned}$$

The condition (2.12) is analogous. In conclusion, we have shown that the six-tuple $(R, S, P, Q, \theta, \phi)$ is a surjective Morita context with firm bimodules.

Sufficiency. Due to Theorem 2.28 and the fact that that firm bimodules are also unitary. ■

Remark 6.5. If S and R are firm rings, then a bimodule $M \in {}_S\mathsf{Mod}_R$ is firm if and only if $M \cong S \otimes M \otimes R$ by the isomorphism μ_M from (6.1). Necessity of this claim follows from Proposition 6.1. For sufficiency assume that μ_M is an isomorphism and notice that

$$S \otimes M \cong S \otimes (S \otimes M \otimes R) = (S \otimes S) \otimes M \otimes R \cong S \otimes M \otimes R \cong M.$$

Explicitly, the homomorphism $\nu_{SM}\colon S \otimes_S M \to M$, $s \otimes m \mapsto sm$ can be expressed as

$$\nu_{SM} = \mu_M \circ (\nu_S \otimes \mathrm{id}_M \otimes \mathrm{id}_R) \circ (\mathrm{id}_S \otimes \mu_M^{-1}),$$

which means that ν_{SM} is an isomorphism, because it is a composite of isomorphisms. The isomorphism $M \otimes R \cong M$ is analogous, which proves that M is firm.

6.1.2 The reflective subcategory of closed bimodules

Now we will study the category of closed bimodules ${}_S\mathsf{CMod}_R$. First we must introduce some notation. Let S and R be rings and $M \in {}_S\mathsf{Mod}_R$. Denote

$$\begin{aligned}
\mathbf{t}_R(M) &:= \{m \in M \mid mR = \{0\}\},\\
{}_S\mathbf{t}(M) &:= \{m \in M \mid Sm = \{0\}\},\\
\mathbf{t}(M) &:= \{m \in M \mid SmR = \{0\}\}\\
&= \{m \in M \mid \forall s \in S\, \forall r \in R\colon\ smr = 0\}.
\end{aligned} \tag{6.4}$$

The sets $\mathbf{t}_R(M)$, ${}_S\mathbf{t}(M)$ and $\mathbf{t}(M)$ are clearly sub-bimodules of M and also

$$\mathbf{t}_R(M) + {}_S\mathbf{t}(M) \subseteq \mathbf{t}(M). \tag{6.5}$$

Remark. Equation (6.4) deserves a bit more explanation. Let $m \in M$ be such that $SmR = \{0\}$. The set SmR includes all sums of the form $s_1mr_1 + \ldots + s_kmr_k$, where $k \in \mathbb{N}$, therefore it also includes "sums" with $k = 1$. Conversly, if $m \in M$ is such that $smr = 0$ for every $r \in R$ and $s \in S$, then clearly $SmR = \{0\}$.

Next we will prove one useful lemma about closed bimodules.

Lemma 6.6. *Let S and R be rings and C a closed (S,R)-bimodule. Then*

$$\mathbf{t}_R(C) = {}_S\mathbf{t}(C) = \mathbf{t}(C) = \{0\}.$$

Proof. Let $C \in {}_S\mathsf{CMod}_R$. Then, clearly $\mathbf{t}_R(C) = {}_S\mathbf{t}(C) = \{0\}$, because ${}_SC$ and C_R are both torsion-free modules.

Let $c \in \mathbf{t}(C)$. Then, for every $s \in S$ and $r \in R$,

$$0 = scr = s(cr).$$

Hence $cr \in {}_S\mathbf{t}(C) = \{0\}$, which means that $cr = 0$ for every $r \in R$. Therefore $c \in \mathbf{t}_R(C) = \{0\}$, which proves that $\mathbf{t}(C) = \{0\}$. ■

From the proof of the previous lemma and the inclusion (6.5), we obtain the following corollary.

Corollary 6.7. *A bimodule* $M \in {}_S\mathsf{Mod}_R$ *is torsion-free if and only if* $\mathbf{t}(M) = \{0\}$.

Now we can construct a functor, for any idempotent rings S and R,

$$\mathbf{T}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{TfMod}_R, \quad M \mapsto M/\mathbf{t}(M). \tag{6.6}$$

To see this, we show that if $M \in {}_S\mathsf{Mod}_R$, then $M/\mathbf{t}(M)$ is torsion-free. For every $m \in M$, we denote

$$[m] := m + \mathbf{t}(M) \in M/\mathbf{t}(M).$$

Thus $[m] = [m']$ if and only if $m - m' \in \mathbf{t}(M)$. Fix $[m] \in \mathbf{t}(\mathbf{T}(M)) = \mathbf{t}(M/\mathbf{t}(M))$. Then $[0] = s[m]r = [smr]$ and hence $smr \in \mathbf{t}(M)$ for every $r \in R$ and $s \in S$. Now let $s' \in S$ and $r' \in R$. Idempotency of S and R implies the existence of $s_1, s'_1, \ldots, s_{k^*}, s_{k^*} \in S$ and $r_1, r'_1, \ldots, r_{h^*}, r'_{h^*} \in R$ such that $s' = s_1 s'_1 + \ldots + s_{k^*} s'_{k^*}$ and $r' = r'_1 r_1 + \ldots + r'_{h^*} r_{h^*}$. Since $s'_k m r'_k \in \mathbf{t}(M)$, we have

$$s'mr' = \left(\sum_{k=1}^{k^*} s_k s'_k\right) m \left(\sum_{h=1}^{h^*} r'_h r_h\right) = \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} s_k (s'_k m r'_h) r_h = \sum_{k=1}^{k^*}\sum_{h=1}^{h^*} s_k 0 r_h = 0,$$

which implies $m \in \mathbf{t}(M)$ and therefore $[m] = [0]$. Hence we have shown (using Corollary 6.7) that $\mathbf{T}(M) \in {}_S\mathsf{TfMod}_R$ for every $M \in {}_S\mathsf{Mod}_R$.

Let $M, N \in {}_S\mathsf{Mod}_R$. For every $f \in {}_S\mathrm{Hom}_R(M, N)$, denote

$$\mathbf{T}(f) =: [f]\colon\ \mathbf{T}(M) \to \mathbf{T}(N), \quad [m] \mapsto [f(m)]. \tag{6.7}$$

We will show that $[f]$ is well defined. Let $[m_1], [m_2] \in \mathbf{T}(M) = M/\mathbf{t}(M)$ such that $[m_1] = [m_2]$. Then $m_1 - m_2 \in \mathbf{t}(M)$. If $s \in S$ and $r \in R$, then

$$sf(m_1 - m_2)r = f(s(m_1 - m_2)r) = f(0) = 0.$$

Hence $f(m_1 - m_2) = f(m_1) - f(m_2) \in \mathbf{t}(N)$, which implies that

$$[f]([m_1]) = [f(m_1)] = [f(m_2)] = [f]([m_2]).$$

Therefore $[f]$ is well defined. It is straightforward to check that $[f]$ is a homomorphism of (S, R)-bimodules. Also notice that

$$\mathbf{T}(g \circ f)([m]) = [g \circ f]([m]) = [g(f(m))] = ([g] \circ [f])([m]) = (\mathbf{T}(g) \circ \mathbf{T}(f))([m]),$$

$$\mathbf{T}(\mathrm{id}_M)([m]) = [\mathrm{id}_M]([m]) = [\mathrm{id}_M(m)] = [m] = \mathrm{id}_{\mathbf{T}(M)}([m]),$$

for every $f \in {}_S\mathrm{Hom}_R(M, N)$, $g \in {}_S\mathrm{Hom}_R(N, A)$ and $m \in M$. In conclusion, $\mathbf{T}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{TfMod}_R$ is a well-defined functor.

Next we will prove two lemmas that help understand the functor $\mathbf{T}$ a little better.

Lemma 6.8. *There exists a natural isomorphism $\mathbf{T} \circ \mathbf{T} \cong \mathbf{T}$, where $\mathbf{T}$ is considered as an endofunctor of ${}_S\mathsf{Mod}_R$.*

Proof. Let S, R be idempotent rings and $M \in {}_S\mathsf{Mod}_R$. By definition (6.6), $\mathbf{T}(M) = M/\mathbf{t}(M)$ is torsion-free. By Corollary 6.7, $\mathbf{t}(\mathbf{T}(M)) = \{[0]\}$. Now, clearly there exists a natural isomorphism $(\mathbf{T} \circ \mathbf{T})(M) = \mathbf{T}(M)/\mathbf{t}(\mathbf{T}(M)) \cong \mathbf{T}(M)$. ■

Lemma 6.9. *Let S and R be rings and ${}_SM_R$ a bimodule. The following isomorphism holds in ${}_S\mathsf{Mod}_R$:*

$$(M/\mathbf{t}_R(M))/{}_S\mathbf{t}(M/\mathbf{t}_R(M)) \cong M/\mathbf{t}(M). \tag{6.8}$$

Proof. Consider the sets

$$\begin{aligned} {}_S\mathbf{t}(M/\mathbf{t}_R(M)) &= \{m + \mathbf{t}_R(M) \mid S(m + \mathbf{t}_R(M)) = Sm + \mathbf{t}_R(M) = \mathbf{t}_R(M)\}, \\ \mathbf{t}(M)/\mathbf{t}_R(M) &= \{m + \mathbf{t}_R(M) \mid m \in \mathbf{t}(M)\}. \end{aligned}$$

Let $[m] = m + \mathbf{t}_R(M) \in {}_S\mathbf{t}(M/\mathbf{t}_R(M))$. If $m \in \mathbf{t}_R(M)$, then $[m] = \mathbf{t}_R(M) = [0] \in \mathbf{t}(M)/\mathbf{t}_R(M)$. Now assume, that $m \notin \mathbf{t}_R(M)$. Due to the definition of ${}_S\mathbf{t}(M/\mathbf{t}_R(M))$, we have

$$Sm + \mathbf{t}_R(M) = \mathbf{t}_R(M).$$

Fix $s \in S$, then either $sm = 0$ or $sm \in \mathbf{t}_R(M)$ must hold. In either case we see that $smR = \{0\}$. This implies that $m \in \mathbf{t}(M)$ and hence $[m] \in \mathbf{t}(M)/\mathbf{t}_R(M)$, which proves the inclusion ${}_S\mathbf{t}(M/\mathbf{t}_R(M)) \subseteq \mathbf{t}(M)/\mathbf{t}_R(M)$. The converse is obvious, therefore ${}_S\mathbf{t}(M/\mathbf{t}_R(M)) = \mathbf{t}(M)/\mathbf{t}_R(M)$.

Notice that $\mathbf{t}_R(M) \subseteq \mathbf{t}(M)$. By Corollary 4.3.3 in [19], we have the isomorphism of (S, R)-bimodules

$$(M/\mathbf{t}_R(M))/{}_S\mathbf{t}(M/\mathbf{t}_R(M)) = (M/\mathbf{t}_R(M))/(\mathbf{t}(M)/\mathbf{t}_R(M))) \cong M/\mathbf{t}(M),$$

which proves formula (6.8). ■

From the previous proof we can easily deduce that the following analogue of formula (6.8) holds:

$$(M/{}_S\mathbf{t}(M))/\mathbf{t}_R(M/{}_S\mathbf{t}(M)) \cong M/\mathbf{t}(M).$$

Hence we see that the functor $\mathbf{T}$ can be expressed as a composition

$$\mathbf{T} \cong (\_/\mathbf{t}_R(\_)) \circ (\_/{}_S\mathbf{t}(\_))\colon \quad {}_S\mathsf{Mod}_R \to {}_S\mathsf{tfMod}_R \to {}_S\mathsf{TfMod}_R \tag{6.9}$$

or equivalently $\mathbf{T} \cong (\_/{}_S\mathbf{t}(\_)) \circ (\_/\mathbf{t}_R(\_))$.

With the next proposition we will construct a functor from ${}_S\mathsf{Mod}_R$ to ${}_S\mathsf{CMod}_R$, where S and R are idempotent rings.

Proposition 6.10. *Let S and R be idempotent rings and ${}_SN_R$ a torsion-free (S,R)-bimodule. The set ${}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R,N))$ is a closed (S,R)-bimodule.*

Proof. Let $N \in {}_S\mathsf{TfMod}_R$. By Theorem 2.27 in [33], the (S,R)-bimodule $K := \mathrm{Hom}_R(R,N)$ is right closed, which means that the mapping

$$\lambda_K\colon K_R \to \mathrm{Hom}_R(R,K), \quad (\lambda_K(f))(r) = fr$$

is an isomorphism of right R-modules. Take $f \in \mathbf{t}(K)$. Then $\{0\} = SfR$, which means that, for every $r, r' \in R$ and $s \in S$,

$$sf(r')r = sf(r'r) = (sfr')(r) = 0.$$

Hence $\mathrm{Im}(f) \subseteq \mathbf{t}(N) = \{0\}$, which gives us

$$\mathbf{t}(K) = \mathbf{t}(\mathrm{Hom}_R(R,N)) = \{r \mapsto 0\}.$$

Therefore K is also torsion-free.

By the left sided dual of Theorem 2.27 in [33], the (S,R)-bimodule $H := {}_S\mathrm{Hom}(S,K) = {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R,N))$ is left closed. Consider the homomorphism

$$\lambda_H\colon H_R \mapsto \mathrm{Hom}_R(R,H), \quad \lambda_H(f)(r) = fr.$$

We will show that λ_H is an isomorphism of right R-modules. Consider the diagram on Figure 6.2,

$$\begin{tikzcd}
\mathrm{Hom}_R(R, {}_S\mathrm{Hom}(S,K)) \arrow[rr, bend left, "\varphi"] & & {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R,K)) \arrow[ll, bend left, "\psi"] \\
& {}_S\mathrm{Hom}(S,K) \arrow[ul, "\lambda_H"] \arrow[ur, "{}_S\mathrm{Hom}(S,\lambda_K) = \lambda_K \circ _"'] &
\end{tikzcd}$$

Figure 6.2

where

$$\begin{aligned} \varphi\colon &\quad f \mapsto (s \mapsto (r \mapsto f(r)(s))),\\ \psi\colon &\quad g \mapsto (r \mapsto (s \mapsto g(s)(r))). \end{aligned}$$

It is easy to see that the mappings φ and ψ are well defined. Note that, for every $f \in \mathrm{Hom}_R(R, H)$, $r \in R$ and $s \in S$,

$$(\psi \circ \varphi)(f)(r)(s) = \psi(\varphi(f))(r)(s) = \varphi(f)(s)(r) = f(r)(s).$$

Hence $\psi \circ \varphi = \mathrm{id}_{\mathrm{Hom}_R(R,H)}$ and analogously $\varphi \circ \psi = \mathrm{id}_{{}_S\mathrm{Hom}(S,H)}$, which means that ψ is bijective. For every $h \in H$, $s \in S$ and $r \in R$, we have

$$\begin{aligned} \lambda_H(h)(r)(s) = (hr)(s) \underset{(*)}{=} h(s)r = \lambda_K(h(s))(r) = (\lambda_K \circ h)(s)(r)\\ = (\lambda_K \circ \_\_)(h)(s)(r) = \psi((\lambda_K \circ \_\_)(h))(r)(s)\\ = (\psi \circ (\lambda_K \circ \_\_))(h)(r)(s). \end{aligned}$$

The equality $(*)$ holds due to the R-multiplication in H, defined as in (2.8). The homomorphism $\lambda_K \circ \_\_$ is an isomorphism, because λ_K is an isomorphism. We have obtained that λ_H is bijective because it can be expressed as a composition of two bijections $\lambda_H = \psi \circ (\lambda_K \circ \_\_)$. In conclusin, we have shown that H is closed. ■

Let S and R be idempotent rings. We can now construct a functor $\mathbf{K}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{CMod}_R$ as the composition

$$\mathbf{K} = {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, \_\_)) \circ \_\_/\mathbf{t}(\_\_)\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{TfMod}_R \to {}_S\mathsf{CMod}_R. \tag{6.10}$$

Clearly ${}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, \_\_)) = {}_S\mathrm{Hom}(S, \_\_) \circ \mathrm{Hom}_R(R, \_\_)$ is a functor and the functor $\mathbf{T} = \_\_/\mathbf{t}(\_\_)$ was introduced in (6.6).

From this composition we see that for any $f \in {}_S\mathrm{Hom}_R(M, N)$, with $M, N \in {}_S\mathsf{Mod}_R$,

$$\begin{aligned} \mathbf{K}(f)\colon\ {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M/\mathbf{t}(M))) \to {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, N/\mathbf{t}(N))),\\ g \mapsto [f] \circ g. \end{aligned}$$

Here we used the definition of $\mathbf{T}(f)$ given in (6.7) and how hom-functors map morphisms (Example 3.20 (4) in [2]).

The next corollary collects all the information we have on functor $\mathbf{K}$ so far.

Corollary 6.11. *Let S and R be idempotent rings. There exists a functor* $\mathbf{K}\colon {}_S\mathsf{Mod}_R \to {}_S\mathsf{CMod}_R$ *such that*

$$\mathbf{K}(M) = {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M/\mathbf{t}(M))),$$
$$\mathbf{K}(f)\colon\ \mathbf{K}(N_1) \to \mathbf{K}(N_2), \quad g \mapsto [f] \circ g,$$

where $f\colon N_1 \to N_2$, for some $N_1, N_2 \in {}_S\mathsf{Mod}_R$.

It is easy to see that there exists an natural isomorphism $\mathbf{K} \circ \mathbf{K} \cong \mathbf{K}$, if we consider $\mathbf{K}$ as an endofunctor of ${}_S\mathsf{Mod}_R$. Next we will prove that there exists a natural transformation from $\mathrm{id}_{{}_S\mathsf{Mod}_R}$ to $\mathbf{K}$ (again considered as an endofunctor of ${}_S\mathsf{Mod}_R$), even if S and R are arbitrary rings.

Proposition 6.12. *Let S and R be rings and ${}_SC_R$ a closed bimodule. Then there exists an (S, R)-bimodule isomorphism*

$$\gamma_C\colon\ C \to {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, C)), \quad c \mapsto (s \mapsto (r \mapsto scr)).$$

Moreover, the family $\gamma = (\gamma_M)_{M \in {}_S\mathsf{Mod}_R}$ of (S, R)-bimodule morphisms, where

$$\gamma_M\colon\ M \to {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M/\mathbf{t}(M))), \quad m \mapsto (s \mapsto (r \mapsto s[m]r)),$$

is a natural transformation from $\mathrm{id}_{{}_S\mathsf{Mod}_R}$ to ${}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, \_/\mathbf{t}(\_)))\colon {}_S\mathsf{Mod}_R \to {}_S\mathsf{Mod}_R$.

In order to not have to fix the elements of S and R all the time, denote $\gamma_M(m) := \underset{S}{\_}[m]\underset{R}{\_}$, for every $m \in M$.

Proof. Let $M \in {}_S\mathsf{Mod}_R$. First note that $\gamma_M(m)$ is a left S-module homomorphism and $\gamma_M(m)(s)$ is a right R-module homomorphism for any $m, m' \in M$ and $s \in S$. Hence, indeed $\mathrm{Im}(\gamma_M) \subseteq {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M/\mathbf{t}(M)))$. Fix $m \in M$, $r \in R$ and $s \in S$. Now

$$\begin{aligned}
\gamma_M(m + m') &= \underset{S}{_}[m + m']\underset{R}{_} = \underset{S}{_}([m]\underset{R}{_} + [m']\underset{R}{_}) = \underset{S}{_}[m]\underset{R}{_} + \underset{S}{_}[m']\underset{R}{_} \\
&= \gamma_M(m) + \gamma_M(m'), \\
\gamma_M(sm) &= \underset{S}{_}[sm]\underset{R}{_} = (\underset{S}{_}s)[m]\underset{R}{_} = s\gamma_M(m)
\end{aligned}$$

and analogously $\gamma_M(mr) = \gamma_M(m)r$, which implies that γ_M is a bimodule homomorphism.

Fix $M, N \in {}_S\mathsf{Mod}_R$ and $f \in \mathrm{Mor}_{{}_S\mathsf{Mod}_R}(M, N)$ (as shown on Figure 6.3).

$$\begin{tikzcd} M \arrow[r, "\gamma_M"] \arrow[d, "f"'] & {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M/\mathbf{t}(M))) \arrow[d, "{[f] \circ \_}"] \\ N \arrow[r, "\gamma_N"'] & {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, N/\mathbf{t}(N))) \end{tikzcd}$$

Figure 6.3

Fix an element $m \in M$. Then

$$\begin{aligned}(\gamma_N \circ f)(m) = \gamma_N(f(m)) &= \underset{S}{\_}[f(m)]\underset{R}{\_} = \underset{S}{\_}f(m)\underset{R}{\_} + \mathbf{t}(M) \\ &= f(\underset{S}{\_}m\underset{R}{\_}) + \mathbf{t}(M) = [\underset{S}{\_}f(m)\underset{R}{\_}] = (([f] \circ \_) \circ \gamma_M)(m).\end{aligned}$$

Hence, $\gamma\colon\ \mathrm{id}_{{}_S\mathsf{Mod}_R} \to {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, \_/\mathbf{t}(\_)))$ is a natural transformation.

Let $C \in {}_S\mathsf{CMod}_R$. Firstly notice that $C/\mathbf{t}(C) = C$ and therefore the two definitions of γ_C coincide. As a component of γ, γ_C is indeed an (S, R)-bimodule homomorphism. Notice that for every $c \in C$, $s \in S$ and $r \in R$

$$\begin{aligned}\gamma_C(c)(s)(r) &= scr = \lambda_C(sc)(r) = \lambda_C(\rho_C(c)(s))(r) = (\lambda_C \circ \rho_C(c))(s)(r) \\ &= ((\lambda_C \circ \_) \circ \rho_C)(c)(s)(r).\end{aligned}$$

$$\begin{tikzcd} C \arrow[rr, "\gamma_C"] \arrow[dr, "\rho_C"'] & & {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, C)) \\ & {}_S\mathrm{Hom}(S, C) \arrow[ur, "\lambda_C \circ \_"'] & \end{tikzcd}$$

Figure 6.4

Hence γ_C can be expressed as a composition of two bijections $\lambda_C \circ \_$ and ρ_C, which implies that γ_C is also a bijection. ■

It also follows that

$$\gamma^{-1} = (\gamma_C^{-1})_{C \in {}_S\mathsf{CMod}_R}\colon\ {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, \_)) \to \mathrm{id}_{{}_S\mathsf{CMod}_R}$$

is a natural isomorphism.

Next we will prove that the functor $\mathbf{K}$ turns out to be a reflector functor of ${}_S\mathsf{CMod}_R$.

Theorem 6.13. *Let S and R be idempotent rings. The category ${}_S\mathsf{CMod}_R$ is a reflective subcategory of ${}_S\mathsf{Mod}_R$ with reflector $\mathbf{K}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{CMod}_R$.*

Proof. Let S and R be idempotent rings. We will show that there exists an adjunction

$$\mathbf{K} = {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, \_/\mathbf{t}(\_))) \dashv \mathbf{J}_{\mathrm{C}}, \tag{6.11}$$

where $\mathbf{J}_{\mathrm{C}}\colon\ {}_S\mathsf{CMod}_R \to {}_S\mathsf{Mod}_R$ is the inclusion functor. From Proposition 6.12 and the paragraph after that, we know that $\gamma\colon\ \mathrm{id}_{{}_S\mathsf{Mod}_R} \to \mathbf{K} \circ \mathbf{J}_{\mathrm{C}}$ and $\gamma^{-1}\colon\ \mathbf{J}_{\mathrm{C}} \circ \mathbf{K} \to \mathrm{id}_{{}_S\mathsf{CMod}_R}$ are natural transformations. We will show that γ is the unit and γ^{-1} is the counit of adjunction (6.11).

For any closed bimodule $C \in {}_S\mathsf{CMod}_R$ we have

$$\mathbf{J}_{\mathrm{C}}(\gamma_C^{-1}) \circ \gamma_{\mathbf{J}_{\mathrm{C}}(C)} = \gamma_C^{-1} \circ \gamma_C = \mathrm{id}_C = \mathrm{id}_{\mathbf{J}_{\mathrm{C}}(C)},$$

which proves the triangle identity (2.2).

Let $M \in {}_S\mathsf{Mod}_R$. Then $\mathbf{K}(M) = {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M/\mathbf{t}(M))) \in {}_S\mathsf{CMod}_R$. Fixing $g \in \mathbf{K}(M)$, we have

$$\begin{aligned}\left(\gamma^{-1}_{\mathbf{K}(M)} \circ \mathbf{K}(\gamma_M)\right)(g) &= \gamma^{-1}_{\mathbf{K}(M)} \circ ([\gamma_M] \circ g) = \gamma^{-1}_{\mathbf{K}(M)} \circ \underset{S}{\_}\, g \underset{R}{\_} = \gamma^{-1}_{\mathbf{K}(M)}(\underset{S}{\_}\, g \underset{R}{\_}) \\ &= g = \mathrm{id}_{\mathbf{K}(M)}(g).\end{aligned}$$

Hence we have shown that the triangle identity (2.1) $\gamma^{-1}_{\mathbf{K}(M)} \circ \mathbf{K}(\gamma_M) = \mathrm{id}_{\mathbf{K}(M)}$ holds. ■

6.1.3 Equivalence of subcategories

Now we are ready to prove one of the main theorems of this section.

Theorem 6.14. *Let S and R be idempotent rings. The categories ${}_S\mathsf{FMod}_R$, ${}_S\mathsf{UTfMod}_R$ and ${}_S\mathsf{CMod}_R$ are equivalent categories.*

Proof. Consider the functors given on the diagram below (Figure 6.5).

$$\begin{tikzcd}
{}_S\mathsf{CMod}_R \arrow[r, bend left, "S_"] \arrow[r, phantom, "\mathbf{I}"] & {}_S\mathsf{utfModc}_R \arrow[l, bend left, "{}_S\mathrm{Hom}(S{,}_)"] \arrow[r, bend left, "_R"] \arrow[r, phantom, "\mathbf{II}"] & {}_S\mathsf{UTfMod}_R \arrow[l, bend left, "\mathrm{Hom}_R(R{,}_)"] \arrow[r, bend left, "_\otimes_R R"] \arrow[r, phantom, "\mathbf{III}"] & {}_S\mathsf{utfModf}_R \arrow[l, bend left, "_/\mathbf{t}_R(_)"] \arrow[r, bend left, "S\otimes_S_"] \arrow[r, phantom, "\mathbf{IV}"] & {}_S\mathsf{FMod}_R \arrow[l, bend left, "_/{}_S\mathbf{t}(_)"]
\end{tikzcd}$$

Figure 6.5

We will prove that in subdiagrams **II** and **III** we have equivalence functors. For subdiagrams **I** and **IV** the proof is similar and we will omit it. Notice that the functors

$$\_R\colon\ {}_S\mathsf{utfModc}_R \to {}_S\mathsf{UTfMod}_R,$$

$$\_/\mathbf{t}_R(\_)\colon\ {}_S\mathsf{utfModf}_R \to {}_S\mathsf{UTfMod}_R,$$
$$\_\otimes_R R\colon\ {}_S\mathsf{UTfMod}_R \to {}_S\mathsf{utfModf}_R$$

are all well defined. We will next show that we have a functor

$$\mathbf{F} := \mathrm{Hom}_R(R,\_)\colon\ {}_S\mathsf{UTfMod}_R \to {}_S\mathsf{utfModc}_R.$$

Due to Proposition 2.29 in [33] we have that

$$\mathbf{F}(M) = \mathrm{Hom}_R(R,M) = \mathrm{Hom}_R(R, M/\mathbf{t}_R(M)) \in {}_S\mathsf{utfModc}_R$$

for every $M \in {}_S\mathsf{UTfMod}_R$, because $\mathbf{t}_R(M) = \{0\}$ (see Corollary 6.7). If $f \in \mathrm{Mor}_{{}_S\mathsf{UTfMod}_R}(M,N)$, $s \in S$ and $g \in \mathrm{Hom}_R(R,M)$, then

$$(\mathbf{F}(f))(sg) = (f\circ \_)(sg) = f\circ sg = s(f\circ g) = s((f\circ\_)(g)) = s(\mathbf{F}(f)(g)),$$

which implies that $\mathbf{F}(f)\colon\ \mathrm{Hom}_R(R,M) \to \mathrm{Hom}_R(R,N)$ is an (S,R)-bimodule homomorphism. Thus $\mathrm{Hom}_R(R,\_)\colon\ {}_S\mathsf{UTfMod}_R \to {}_S\mathsf{utfModc}_R$ is a well-defined functor.

Fix bimodules $C \in {}_S\mathsf{utfModc}_R$, $N \in {}_S\mathsf{UTfMod}_R$ and $A \in {}_S\mathsf{FMod}_R$ and define

$$\begin{array}{ll}
\alpha_C := \lambda_C^{-1}\circ \mathrm{Hom}_R(R,\iota_C) = \lambda_C^{-1}\circ\iota_C\circ_\colon & \mathrm{Hom}_R(R,CR)\to C,\\
\beta_N := \lambda_N|_{NR} = \lambda_{NR}\colon & N \to \mathrm{Hom}_R(R,N)R,\\
\delta_N := (_/\mathbf{t}_R(_))(\mu_N) = [\mu_N]\colon & (N\otimes_R R)/\mathbf{t}_R(N\otimes_R R)\to N,\\
\epsilon_A := ([_]\otimes \mathrm{id}_R)\circ\mu_A^{-1}\colon & A \to A/\mathbf{t}_R(A)\otimes_R R,
\end{array}$$

where $\iota_C\colon\ CR \to C$ is the inclusion. From Theorem 2.18, we know that the mappings α_C, β_N, δ_N and ϵ_A are bijective homomorphisms of right R-modules and the corresponding families of mappings α, β, γ and ϵ are natural transformations. More precisely

$$\begin{array}{rl}
\alpha\colon & \mathrm{Hom}_R(R,_)\circ_R \to \mathrm{id}_{{}_S\mathsf{utfModc}_R},\\
\beta\colon & \mathrm{id}_{{}_S\mathsf{UTfMod}_R}\to _R\circ\mathrm{Hom}_R(R,_),\\
\delta\colon & (_/\mathbf{t}_R(_))\circ(_\otimes_R R)\to \mathrm{id}_{{}_S\mathsf{UTfMod}_R},\\
\epsilon\colon & \mathrm{id}_{{}_S\mathsf{FMod}_R}\to(_\otimes_R R)\circ(_/\mathbf{t}_R(_)).
\end{array}$$

We will prove that α_C, β_N, δ_N and ϵ_A are also homomorphisms of left S-modules. If $s\in S$, $c\underset{R}{\_} \in \mathrm{Hom}_R(R,CR)$, $n \in N$, $[n'\otimes r'] \in (N\otimes_R R)/\mathbf{t}_R(N\otimes_R R)$ and $ar \in A \in {}_S\mathsf{UMod}_R$, then

$$\alpha_C(s c\underset{R}{\_}) = \lambda_C^{-1}\circ\iota_C\circ(sc\underset{R}{\_}) = \lambda_C^{-1}(sc\underset{R}{\_}) = sc = s\lambda_C^{-1}(c\underset{R}{\_}) = s\alpha_C(c\underset{R}{\_}),$$

$$
\begin{aligned}
\beta_N(sn) &= \lambda_{NR}(sn) = sn\underset{R}{\_\_} = s(n\underset{R}{\_\_}) = s\beta_N(n),\\
\delta_N(s[n' \otimes r']) &= [\mu_N]([sn' \otimes r']) = [(sn')r'] \underset{(**)}{=} [s(n'r')] = s\delta_N([n' \otimes r']),\\
\epsilon_A(sar) &= ([\_\_] \otimes \mathrm{id}_R)(sa \otimes r) = [sa] \otimes r = s([a] \otimes r) = s\epsilon_A(ar).
\end{aligned}
$$

Equation $(**)$ holds because N is a bimodule. Hence, the mappings α_C, β_N, δ_N and ϵ_A are also (S,R)-bimodule isomorphisms. Therefore, the functors on subdiagrams **II** and **III** are equivalences.

Now using the transitivity of category equivalence, we obtain the equivalences ${}_S\mathsf{CMod}_R \approx {}_S\mathsf{UTfMod}_R \approx {}_S\mathsf{FMod}_R$. ■

From the previous proof we also see that the restriction functors $\mathbf{P}|_{{}_S\mathsf{CMod}_R}$, $\mathbf{P}|_{{}_S\mathsf{UTfMod}_R}$, $\mathbf{K}|_{{}_S\mathsf{FMod}_R}$ and $\mathbf{K}|_{{}_S\mathsf{UTfMod}_R}$ are equivalence functors. Because functors $\mathbf{P}$ and $\mathbf{K}$ can be expressed as the following compositions

$$
\begin{aligned}
\mathbf{P} &= (S \otimes_S \_\_) \circ (\_\_ \otimes_R R) \circ (\_\_R) \circ (S\_\_),\\
\mathbf{K} &= ({}_S\mathrm{Hom}(S,\_\_)) \circ (\mathrm{Hom}_R(R,\_\_)) \circ (\_\_/\mathbf{t}_R(\_\_)) \circ (\_\_/{}_S\mathbf{t}(\_\_)).
\end{aligned}
$$

6.1.4 An essential localization

We will need to consider one more functor, which we define as a composition of functor $\mathbf{U}$ defined in (2.10) and $\mathbf{T}$ defined in (6.6):

$$\mathbf{Q} := \mathbf{T} \circ \mathbf{U}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{UTfMod}_R, \quad M \mapsto (SMR)/\mathsf{t}(SMR).$$

(Here we will use the same symbols for functors and their restrictions.)

Now we will prove that $\mathbf{Q} = \mathbf{T} \circ \mathbf{U} \cong \mathbf{U} \circ \mathbf{T}$, i.e. the diagram on Figure 6.6 commutes (up to isomorphism).

$$
\begin{tikzcd}
{}_S\mathsf{Mod}_R \arrow[r, "\mathbf{U}"] \arrow[d, "\mathbf{T}"'] \arrow[dr, "\mathbf{Q}"] & {}_S\mathsf{UMod}_R \arrow[d, "\mathbf{T}"] \\
{}_S\mathsf{TfMod}_R \arrow[r, "\mathbf{U}"'] & {}_S\mathsf{UTfMod}_R
\end{tikzcd}
$$

Figure 6.6

Lemma 6.15. *The functor* $\mathbf{T} \circ \mathbf{U}$ *is naturally isomorphic to* $\mathbf{U} \circ \mathbf{T}$.

Proof. Fix a bimodule $M \in {}_S\mathsf{Mod}_R$. Clearly $\mathbf{t}(SMR) \subseteq \mathbf{t}(M) = \operatorname{Ker}(\kappa)$, where $\kappa\colon M \to M/\mathbf{t}(M)$ is the canonical surjection. Note, that for any $\sum_{k=1}^{k^*} s_k m_k r_k \in SMR$, we have

$$\kappa\left(\sum_{k=1}^{k^*} s_k m_k r_k\right) = \left[\sum_{k=1}^{k^*} s_k m_k r_k\right] = \sum_{k=1}^{k^*} s_k [m_k] r_k \in S\left(\frac{M}{\mathbf{t}(M)}\right)R.$$

Hence $\operatorname{Im}(\kappa|_{SMR}) \subseteq S(M/\mathbf{t}(M))R$. By the Fundamental Theorem of Homomorphisms (see Paragraph 6.5 in [51]) we have the following commutative diagram.

$$\begin{tikzcd}
SMR \arrow[rr, "\kappa|_{SMR}"] \arrow[dr, "\kappa'"'] & & S\left(\frac{M}{\mathbf{t}(M)}\right)R \\
& \frac{SMR}{\mathbf{t}(SMR)} \arrow[ur, "\alpha_M"'] &
\end{tikzcd}$$

Figure 6.7

We explicitly write out the (S,R)-bimodule homomorphism α_M:

$$\alpha_M\colon \quad (\mathbf{T}\circ\mathbf{U})(M) = \frac{SMR}{\mathbf{t}(SMR)} \to S\left(\frac{M}{\mathbf{t}(M)}\right)R = (\mathbf{U}\circ\mathbf{T})(M),$$
$$\sum_{k=1}^{k^*} s_k m_k r_k + \mathbf{t}(SMR) \mapsto \sum_{k=1}^{k^*} s_k (m_k + \mathbf{t}(M)) r_k.$$

Clearly α_M is surjective. Also, α_M is injective, because $\mathbf{t}(SMR) = \operatorname{Ker}(\kappa|_{SMR})$. Hence α_M is an isomorphism in ${}_S\mathsf{UTfMod}_R$.

Next, we will show that $\alpha = (\alpha_M)_{M\in {}_S\mathsf{Mod}_R}$ is natural. Note that, for any $[\sum_{k=1}^{k^*} s_k m_k r_k]_{\mathbf{t}(SMR)} \in SMR/\mathbf{t}(SMR)$, we have

$$\begin{aligned}
((\mathbf{U}\circ\mathbf{T})(f)\circ\alpha_M)\left(\left[\sum_{k=1}^{k^*} s_k m_k r_k\right]_{\mathbf{t}(SMR)}\right) &= [f]\big|_{S(\frac{M}{\mathbf{t}(M)})R}\left(\sum_{k=1}^{k^*} s_k [m_k]_{\mathbf{t}(M)} r_k\right) \\
&= \left[\sum_{k=1}^{k^*} s_k f(m_k) r_k\right]_{\mathbf{t}(N)} = \sum_{k=1}^{k^*} s_k [f(m_k)]_{\mathbf{t}(N)} r_k,
\end{aligned}$$

$$(\alpha_N\circ(\mathbf{T}\circ\mathbf{U})(f))\left(\left[\sum_{k=1}^{k^*} s_k m_k r_k\right]_{\mathbf{t}(SMR)}\right) = \alpha_N\left([f|_{SMR}]\left(\left[\sum_{k=1}^{k^*} s_k m_k r_k\right]_{\mathbf{t}(SMR)}\right)\right)$$

$$= \alpha_N \left(\left[\sum_{k=1}^{k^*} s_k f(m_k) r_k \right]_{\mathbf{t}(SNR)} \right) = \sum_{k=1}^{k^*} s_k [f(m_k)]_{\mathbf{t}(N)} r_k.$$

Hence $(\mathbf{U} \circ \mathbf{T})(f) \circ \alpha_M = \alpha_N \circ (\mathbf{T} \circ \mathbf{U})(f)$, which proves that α is natural. In conclusion, we have shown that $\alpha = (\alpha_M)_{M \in {}_S\mathsf{Mod}_R} \colon \mathbf{T} \circ \mathbf{U} \to \mathbf{U} \circ \mathbf{T}$ is a natural isomorphism. ∎

Introduce the following inclusion functors

$$\begin{aligned} \mathbf{J}_{\mathrm{C}} &\colon\ {}_S\mathsf{CMod}_R \to {}_S\mathsf{Mod}_R, \\ \mathbf{J}_{\mathrm{F}} &\colon\ {}_S\mathsf{FMod}_R \to {}_S\mathsf{Mod}_R, \\ \mathbf{J}_{\mathrm{Q}} &\colon\ {}_S\mathsf{UTfMod}_R \to {}_S\mathsf{Mod}_R. \end{aligned}$$

Now we will collect all the information we have proven so far to one diagram (Figure 6.8). We will denote the restrictions $\mathbf{K}' := \mathbf{K}|_{{}_S\mathsf{UTfMod}_R}$ and $\mathbf{P}' := \mathbf{P}|_{{}_S\mathsf{UTfMod}_R}$.

$$\begin{tikzcd}
 & {}_S\mathsf{CMod}_R \arrow[dl, bend left=15, "\mathbf{J}_{\mathrm{C}}"] \arrow[dr, bend left=15, "\mathbf{U}' = S_R"] & \\
{}_S\mathsf{Mod}_R \arrow[ur, bend left=15, "\mathbf{K} = {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, _/\mathbf{t}(_)))"] \arrow[rr, bend left=10, "\mathbf{Q}"] \arrow[dr, bend right=15, "\mathbf{P} = S \otimes_S S_R \otimes_R R"'] \arrow[r, phantom, "\dashv" near start, shift left=20] \arrow[r, phantom, "\vdash" near start, shift right=20] & & {}_S\mathsf{UTfMod}_R \arrow[ul, bend left=15, "\mathbf{K}'"] \arrow[ll, bend left=10, "\mathbf{J}_{\mathrm{Q}}"] \arrow[dl, bend left=15, "\mathbf{P}'"] \arrow[l, phantom, "\approx" near start, shift right=20] \arrow[l, phantom, "\approx" near start, shift left=20] \\
 & {}_S\mathsf{FMod}_R \arrow[ul, bend right=15, "\mathbf{J}_{\mathrm{F}}"'] \arrow[ur, bend right=15, "\mathbf{T}' = _/\mathbf{t}(_)"'] &
\end{tikzcd}$$

Figure 6.8

In the next theorem we will construct a left adjoint to the functor $\mathbf{K}$. This gives us the second main result of this section. The one sided analogue of the following theorem was proves by Marín in [33] (Proposition 3.17).

Theorem 6.16. *Let S and R be idempotent rings. The subcategory ${}_S\mathsf{CMod}_R$ is an essential localization of ${}_S\mathsf{Mod}_R$.*

Proof. Let S and R be idempotent rings. We will prove that the functor

$$\mathbf{K} = {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, \_/\mathbf{t}(\_)))\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{CMod}_R$$

has a left adjoint.

From Theorem 6.3 we have the adjunction $\mathbf{P} \vdash \mathbf{J}_{\mathrm{F}}$. Due to the equivalences proven in Theorem 6.14, we have the following adjunctions (see Figure 6.8)

$$\mathbf{K}' = \mathbf{K} \circ \mathbf{J}_{\mathrm{Q}} \quad \vdash \quad \mathbf{U}' = \mathbf{Q} \circ \mathbf{J}_{\mathrm{C}}, \tag{6.12}$$

$$\mathbf{T}' = \mathbf{Q} \circ \mathbf{J}_{\mathrm{F}} \quad \vdash \quad \mathbf{P}' = \mathbf{P} \circ \mathbf{J}_{\mathrm{Q}}. \tag{6.13}$$

The next part of the proof is divided into two parts.

1. We will first prove that $\mathbf{Q} \cong \mathbf{Q} \circ \mathbf{J}_{\mathrm{F}} \circ \mathbf{P}$. Fix $M \in {}_S\mathsf{Mod}_R$. Without loss of generality assume that bimodule M is unitary, because otherwise if $\mathbf{Q}|_{{}_S\mathsf{UMod}_R} \cong \mathbf{Q} \circ \mathbf{J}_{\mathrm{F}} \circ \mathbf{P}|_{{}_S\mathsf{UMod}_R}$ holds, then

$$\mathbf{Q} \circ \mathbf{J}_{\mathrm{F}} \circ \mathbf{P} = (\mathbf{Q} \circ \mathbf{J}_{\mathrm{F}} \circ \mathbf{P}|_{{}_S\mathsf{UMod}_R}) \circ \mathbf{U} \cong \mathbf{Q}|_{{}_S\mathsf{UMod}_R} \circ \mathbf{U} = \mathbf{T} \circ \mathbf{U} = \mathbf{Q}.$$

If ${}_SM_R$ is unitary, then $\mathbf{P}(M) = S \otimes_S SMR \otimes_R R = S \otimes_S M \otimes_R R$. Define a mapping

$$\eta_M \colon \ \mathbf{Q}(\mathbf{J}_{\mathrm{F}}(\mathbf{P}(M))) = \frac{S \otimes_S M \otimes_R R}{\mathbf{t}(S \otimes_S M \otimes_R R)} \to \frac{SMR}{\mathbf{t}(SMR)} = \mathbf{Q}(M),$$
$$\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k + \mathbf{t}(S \otimes_S M \otimes_R R) \mapsto \sum_{k=1}^{k^*} s_k m_k r_k + \mathbf{t}(SMR).$$

First we will show that η_M is well defined. Take $\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k \in \mathbf{t}(S \otimes_S M \otimes_R R)$. Now, for any $s \in S$ and $r \in R$, we have

$$\begin{aligned} s\left(\sum_{k=1}^{k^*} s_k m_k r_k\right) r &= s\left(\mu_M\left(\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k\right)\right) r \\ &= \mu_M\left(s\left(\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k\right) r\right) = \mu_M(0) = 0. \end{aligned}$$

Hence $\sum_{k=1}^{k^*} s_k m_k r_k \in \mathbf{t}(SMR)$ and therefore

$$\mathbf{t}(S \otimes_S M \otimes_R R) \subseteq \mathrm{Ker}(\kappa \circ \mu_M),$$

where $\kappa \colon SMR \to SMR/\mathbf{t}(SMR)$ is the canonical surjection. Now using the Fundamental Theorem of Homomorphisms (see Paragraph 6.5 in [51]) we see that η_M is a well-defined (S, R)-bimodule homomorphism (Figure 6.9).

$$\begin{tikzcd}
S \otimes_S M \otimes_R R \arrow[r, "\mu_M"'] \arrow[rr, bend left, "\kappa \circ \mu_M"] \arrow[dr, "\kappa'"'] & SMR \arrow[r, "\kappa"'] & \dfrac{SMR}{\mathbf{t}(SMR)} \\
& \dfrac{S \otimes_S M \otimes_R R}{\mathbf{t}(S \otimes_S M \otimes_R R)} \arrow[ur, "\eta_M"'] &
\end{tikzcd}$$

Figure 6.9

The homomorphism η_M is clearly surjective. We will show that it is injective. Fix $[\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k] \in \operatorname{Ker}(\eta_M)$. Then $\sum_{k=1}^{k^*} s_k m_k r_k \in \mathbf{t}(SMR)$ meaning that

$$s\left(\sum_{k=1}^{k^*} s_k m_k r_k\right) r = 0$$

for any $s \in S$ and $r \in R$. Fix $s \in S$ and $r \in R$. Since S and R are idempotent, there exist $s'_1, s''_1, \ldots, s'_{t^*}, s''_{t^*} \in S$ and $r'_1, r''_1, \ldots, r'_{h^*}, r''_{h^*} \in R$ such that $s = \sum_{t=1}^{t^*} s'_t s''_t$ and $r = \sum_{h=1}^{h^*} r'_h r''_h$. Now

$$\begin{aligned}
s\left[\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k\right] r &= \left(\sum_{t=1}^{t^*} s'_t s''_t\right)\left[\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k\right]\left(\sum_{h=1}^{h^*} r'_h r''_h\right) \\
&= \left[\sum_{t=1}^{t^*}\sum_{h=1}^{h^*}\sum_{k=1}^{k^*} s'_t s''_t s_k \otimes m_k \otimes r_k r'_h r''_h\right] \\
&= \left[\sum_{t=1}^{t^*}\sum_{h=1}^{h^*} s'_t \otimes s''_t \left(\sum_{k=1}^{k^*} s_k m_k r_k\right) r'_h \otimes r''_h\right] \\
&= \left[\sum_{t=1}^{t^*}\sum_{h=1}^{h^*} s''_t \otimes 0 \otimes r''_h\right] = [0].
\end{aligned}$$

Therefore $[\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k] \in \mathbf{t}(\mathbf{Q}(\mathbf{J}_{\mathrm{F}}(\mathbf{P}(M)))) = \{[0]\}$. Hence η_M is also injective and therefore a bimodule isomorphism. Clearly $\eta = (\eta_M)_{M \in {}_S\mathsf{Mod}_R}$ is natural in M. From the isomorphism $\mathbf{Q} \cong (\mathbf{Q} \circ \mathbf{J}_{\mathrm{F}}) \circ \mathbf{P} = \mathbf{T}' \circ \mathbf{P}$, we obtain the adjunction

$$\mathbf{Q} \vdash \mathbf{J}_{\mathrm{F}} \circ \mathbf{P}', \tag{6.14}$$

by composing the adjunctions (6.13) and $\mathbf{P} \vdash \mathbf{J}_{\mathrm{F}}$ as described in (2.3).

2. Next we will prove that $\mathbf{K} = \mathbf{K} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q}$. Fix a bimodule $M \in {}_S\mathsf{Mod}_R$. We must show that bimodules $\mathbf{K}(M)$ and $\mathbf{K}(\mathbf{J}_{\mathrm{Q}}(\mathbf{Q}(M))) = \mathbf{K}(SMR)$ are isomorphic. Without loss of generality assume that bimodule M is torsion-free, because otherwise if $\mathbf{K}|_{{}_S\mathsf{TfMod}_R} = \mathbf{K}|_{{}_S\mathsf{TfMod}_R} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q}$ holds, then

$$\begin{aligned}\mathbf{K} = \mathbf{K}|_{{}_S\mathsf{TfMod}_R} \circ \mathbf{T} &= \mathbf{K}|_{{}_S\mathsf{TfMod}_R} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q} \circ \mathbf{T} \\ &\cong \mathbf{K}|_{{}_S\mathsf{TfMod}_R} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{U} \circ \mathbf{T} \circ \mathbf{T} \cong \mathbf{K}|_{{}_S\mathsf{TfMod}_R} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{U} \circ \mathbf{T} \\ &\cong \mathbf{K}|_{{}_S\mathsf{TfMod}_R} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q} = \mathbf{K} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q}\end{aligned}$$

also holds (see Lemma 6.15 and Lemma 6.8). Clearly the inclusion

$$\begin{aligned}\mathbf{K}(\mathbf{J}_{\mathrm{Q}}(\mathbf{Q}(M))) &= {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, SMR)) \\ &\subseteq {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M)) = \mathbf{K}(M)\end{aligned}$$

holds. Fix $f \in {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, M)) = \mathbf{K}(M)$ and $s \in S$, $r \in R$. Since S and R are idempotent, there exist elements $s_1, s_1', \ldots, s_{k^*}, s_{k^*}' \in S$ and $r_1, r_1', \ldots, r_{h^*}, r_{h^*}' \in R$ such that $s = s_1 s_1' + \ldots + s_{k^*} s_{k^*}'$ and $r = r_1 r_1' + \ldots + r_{h^*} r_{h^*}'$. We have

$$\begin{aligned}f(s)(r) &= f(s)\left(\sum_{h=1}^{h^*} r_h r_h'\right) \\ &= \sum_{h=1}^{h^*} f(s)(r_h) r_h' && (f(s) \text{ is a right } R\text{-homomorphism}) \\ &= \sum_{h=1}^{h^*} f\left(\sum_{k=1}^{k^*} s_k s_k'\right)(r_h) r_h' \\ &= \sum_{h=1}^{h^*}\left(\sum_{k=1}^{k^*} s_k f(s_k')\right)(r_h) r_h' && (f \text{ is a left } S\text{-homomorphism}) \\ &= \sum_{h=1}^{h^*}\sum_{k=1}^{k^*} s_k (f(s_k')(r_h)) r_h' && (\text{left } S\text{-multipl. in } \mathrm{Hom}_R(R, M)) \\ &\in SMR.\end{aligned}$$

Hence $\mathrm{Im}(f) \subseteq SMR$, which implies that $f \in {}_S\mathrm{Hom}(S, \mathrm{Hom}_R(R, SMR))$. It suffices to show that the functors $\mathbf{K}$ and $\mathbf{K} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q}$ coincide on all morphisms of ${}_S\mathsf{TfMod}_R$. Take $M, N \in {}_S\mathsf{TfMod}_R$ and $g\colon M \to N$ (then $[g] = g$).

$$(\mathbf{K} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q})(g) = \mathbf{K}(\mathbf{J}_{\mathrm{Q}}(\mathbf{Q}(g))) = \mathbf{K}(\mathbf{J}_{\mathrm{Q}}(\mathbf{T}(\mathbf{U}(g))))$$

$$= \mathbf{K}(\mathbf{J}_{\mathrm{Q}}(\mathbf{T}(g|_{SMR}))) = \mathbf{K}([g|_{SMR}]) = g|_{SMR} \circ \_\_,$$
$$\mathbf{K}(g) = g \circ \_\_.$$

Now $(\mathbf{K} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q})(g) = \mathbf{K}(g)$ holds because we have previously shown that for every $f \in \mathbf{K}(M)$ we have $\mathrm{Im}(f) \subseteq SMR$, which means that $g|_{SMR} \circ f = g \circ f$. Therefore we have shown that $\mathbf{K} = \mathbf{K} \circ \mathbf{J}_{\mathrm{Q}} \circ \mathbf{Q}$.

In conclusion we obtain the following composition of adjunctions

$$\mathbf{K} = (\mathbf{K} \circ \mathbf{J}_{\mathrm{Q}}) \circ \mathbf{Q} \quad \vdash \quad (\mathbf{J}_{\mathrm{F}} \circ \mathbf{P}') \circ (\mathbf{Q} \circ \mathbf{J}_{\mathrm{C}}) = (\mathbf{J}_{\mathrm{F}} \circ \mathbf{P}') \circ \mathbf{U}'$$

$$\begin{tikzcd}
{}_S\mathsf{CMod}_R \arrow[r, bend left, "\mathbf{Q} \circ \mathbf{J}_{\mathrm{C}}"] \arrow[r, phantom, "\perp"] & {}_S\mathsf{UTfMod}_R \arrow[l, bend left, "\mathbf{K} \circ \mathbf{J}_{\mathrm{Q}}"] \arrow[r, bend left, "\mathbf{J}_{\mathrm{F}} \circ \mathbf{P}'"] \arrow[r, phantom, "\perp"] & {}_S\mathsf{Mod}_R \arrow[l, bend left, "\mathbf{Q}"]
\end{tikzcd}$$

Figure 6.10

from adjunctions (6.12) and (6.14) using (2.3) (see Figure 6.10). Hence, the functor $(\mathbf{J}_{\mathrm{F}} \circ \mathbf{P}') \circ \mathbf{U}' \colon\ {}_S\mathsf{CMod}_R \to {}_S\mathsf{Mod}_R$ is a left adjoint of $\mathbf{K}$. ■

6.2 Monomorphisms of (unitary) bimodules

Let S and R be rings. In this section we will study monomorphisms in the category ${}_S\mathsf{Mod}_R$ and give a sufficient condition for a morphism to be a monomorphism in the category ${}_S\mathsf{UMod}_R$.

First we will introduce the notion of an Ab-category (Appendix A.4.1 in [50]) and prove a simple lemma about monomorphisms in in these.

Definition 6.17. A category $\mathcal{A}$ is called an **Ab-category** or a **pre-additive category**, if every morphism-set $\mathrm{Mor}_{\mathcal{A}}(B, C)$ has a structure of an abelian group in such a way that composition distributes over addition.

Lemma 6.18. *Let $\mathcal{A}$ be an Ab-category and $\mathcal{C} \subseteq \mathcal{A}$ its full subcategory. A morphism $f \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ is a monomorphism if and only if*

$$f \circ u = 0 \implies u = 0 \tag{6.15}$$

for every $u \in \mathrm{Mor}_{\mathcal{C}}(D, B)$.

Proof. *Necessity.* Let f be a monomorphism. For every morphism u, condition $f \circ u = 0 = f \circ 0$ implies $u = 0$.

Sufficiency. For every $u \in \mathrm{Mor}_{\mathcal{C}}(D,B)$ assume condition (6.15). Let u and v be morphisms such that $f \circ u = f \circ v$. Then

$$\begin{aligned} f \circ u = f \circ v &\implies f \circ u - f \circ v = 0 \implies f \circ (u - v) = 0 \\ &\implies u - v = 0 \implies u = v. \end{aligned}$$

Therefore f is a monomorphism in the category $\mathcal{C}$. ■

Clearly, the category ${}_S\mathsf{Mod}_R$ is an Ab-category for arbitrary rings S and R. Monomorphisms in ${}_S\mathsf{Mod}_R$ can be described as follows.

Proposition 6.19. *Let S and R be rings and f a morphism in ${}_S\mathsf{Mod}_R$. The following assertions are equivalent:*
(1) f is a monomorphism;
(2) f is an extremal monomorphism;
(3) f is a regular monomorphism;
(4) f is injective.

Proof. ((3) $\implies$ (2) $\implies$ (1)). These implications hold in every category (see Lemma 2.4).

((4) $\implies$ (1)). This holds by Corollary 7.38 in [2], because ${}_S\mathsf{Mod}_R$ is a construct [2, Definition 5.1 (2)].

((1) $\implies$ (4)). Let $f\colon\ M \to N$ be a monomorphism in ${}_S\mathsf{Mod}_R$. Consider the inclusion $\iota_{\mathrm{Ker}\, f}\colon\ \mathrm{Ker}\, f \to M$. Since $\mathrm{Ker}\, f$ is a sub-bimodule of M, $\iota_{\mathrm{Ker}\, f}$ is an (S,R)-bimodule homomorphism. Clearly $f \circ \iota_{\mathrm{Ker}\, f} = 0$. Since f is a monomorphism, using Lemma 6.18, we obtain $\iota_{\mathrm{Ker}\, f} = 0$. We get $\{0\} = \mathrm{Im}\, \iota_{\mathrm{Ker}\, f} = \mathrm{Ker}\, f$, which implies that f is injective.

((4) $\implies$ (3)). Let $f\colon\ {}_SM_R \to {}_SN_R$ be an injective homomorphism. Then f is a monomorphism in ${}_S\mathsf{Mod}_R$. Consider the quotient bimodule

$$C := (N \times N)/(\mathrm{Im} f \times \mathrm{Im} f) \in {}_S\mathsf{Mod}_R.$$

Define the mappings $g_1, g_2\colon\ N \to C$ as follows:

$$\begin{aligned} g_1(n) &:= [(n,0)], \\ g_2(n) &:= [(0,n)], \end{aligned}$$

for every $n \in N$. Note that g_1 and g_2 are (S,R)-bimodule homomorphisms.

Let $m \in M$. Observe that

$$(f(m),0) - (0,f(m)) = (f(m),-f(m)) = (f(m),f(-m)) \in \mathrm{Im} f \times \mathrm{Im} f,$$

therefore $(g_1 \circ f)(m) = [(f(m),0)] = [(0,f(m))] = (g_2 \circ f)(m)$. Hence $g_1 \circ f = g_2 \circ f$. Denote $\mathcal{N} := \{n \in N \mid g_1(n) = g_2(n)\}$. We will show that $\mathrm{Im} f = \mathcal{N}$.

($\subseteq$). If $n \in \mathrm{Im} f$, then there exists $m \in M$ such that $n = f(m)$. Hence

$$g_1(n) = g_1(f(m)) = (g_1 \circ f)(m) = (g_2 \circ f)(m) = g_2(f(m)) = g_2(n).$$

Therefore $n \in \mathcal{N}$.

($\supseteq$). If $n \in \mathcal{N}$, then $g_1(n) = g_2(n)$ and

$$\mathrm{Im} f \times \mathrm{Im} f = [(0,0)] = g_1(n) - g_2(n) = [(n,0)] - [(0,n)] = [(n,-n)].$$

Therefore $(n,-n) \in \mathrm{Im} f \times \mathrm{Im} f$, which implies that $n \in \mathrm{Im} f$.

In conclusion, we have shown that $\mathrm{Im} f = \mathcal{N}$, hence $\iota_{\mathrm{Im} f} \colon\ \mathrm{Im} f \to N$ is an equalizer of morphisms g_1 and g_2 (Figure 6.11).

$$\begin{tikzcd}
\mathrm{Im} f \arrow[rr, "\iota_{\mathrm{Im} f}"] & & N \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \\
& M \arrow[ul, dashed, "f'"] \arrow[ur, "f"'] & &
\end{tikzcd}$$

Figure 6.11

Therefore there exists a unique homomorphism $f' \colon\ M \to \mathrm{Im} f$ such that $\iota_{\mathrm{Im} f} \circ f' = f$. Since f is injective, f' must also be injective. For every $m \in M$ we have $f(m) = \iota_{\mathrm{Im} f}(f'(m)) = f'(m)$, hence f' is also surjective. In conclusion, f' is a bimodule isomorphism and, therefore, f is also an equalizer of g_1 and g_2, which means that f is a regular monomorphism. ■

Next we will turn our attention to monomorphisms in ${}_S\mathsf{UMod}_R$. First we will describe regular and extremal monomorphisms in ${}_S\mathsf{UMod}_R$.

Proposition 6.20. *Let S and R be rings and f a morphism in ${}_S\mathsf{UMod}_R$. The following assertions are equivalent:*

(1) f is a regular monomorphism;
(2) f is an extremal monomorphism;
(3) f is injective.

PROOF. ((1) $\implies$ (2)). By Lemma 2.4.

((2) $\implies$ (3)) Let $f \colon {}_SM_R \to {}_SN_R$ be an extremal monomorphism in ${}_S\mathsf{UMod}_R$. Consider the composition given on Figure 6.12.

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "\kappa"'] & & N \\
& M/\operatorname{Ker} f \arrow[ur, "h"'] &
\end{tikzcd}$$

Figure 6.12

Here $\kappa\colon M \to M/\operatorname{Ker} f$ is the canonical surjection and $M/\operatorname{Ker} f \in {}_S\mathsf{UMod}_R$. The mapping $h\colon\ M/\operatorname{Ker} f \to N$ is a well-defined injective (S,R)-bimodule homomorphism due to the Fundamental Theorem of Homomorphisms. Since f is extremal, κ is bijective. Now f is injective, because it can be expressed as the composition of a bijective and an injective homomorphism $f = h \circ \kappa$.

((3) $\implies$ (1)) This implication can be proved exactly as the implication (4) $\implies$ (3) in Proposition 6.19 by noticing that the category ${}_S\mathsf{UMod}_R$ is closed under taking direct squares and quotients. ■

Next we will prove a necessary condition for a morphism being a monomorphism in ${}_S\mathsf{UMod}_R$.

Proposition 6.21. *Let S and R be rings and $f \in \operatorname{Mor}_{{}_S\mathsf{UMod}_R}(M,N)$. If the condition $S(\operatorname{Ker} f)R = \{0\}$ holds, then f is a monomorphism.*

Proof. Let S and R be rings, $f\colon\ M \to N$ a morphism in ${}_S\mathsf{UMod}_R$ and assume that $S(\operatorname{Ker} f)R = \{0\}$ holds. Take $g \in \operatorname{Mor}_{{}_S\mathsf{UMod}_R}(A,M)$ such that $f \circ g = 0$ and $a \in A$. Since ${}_SA_R$ is unitay, there exist $s_1,\dots,s_{k^*} \in S$, $r_1,\dots,r_{k^*} \in R$ and $a_1,\dots,a_{k^*} \in A$ such that $a = s_1a_1r_1 + \ldots + s_{k^*}a_{k^*}r_{k^*}$. For every index $k \in \{1,\dots,k^*\}$ we have $f(g(a_k)) = 0$. Now, by assumption we obtain $\sum_{k=1}^{k^*} s_kg(a_k)r_k = 0$ and therefore

$$g(a) = g\left(\sum_{k=1}^{k^*} s_ka_kr_k\right) = \sum_{k=1}^{k^*} s_kg(a_k)r_k = 0.$$

Hence $g = 0$ and, by Lemma 6.18, f is a monomorphism. ■

Corollary 6.22. *Let S and R be rings and $M \in {}_S\mathsf{UMod}_R$. The canonical homomorphism $\mu_M\colon\ S \otimes_S M \otimes_R R \to M$ is a monomorphism in ${}_S\mathsf{UMod}_R$.*

Proof. Let S and R be rings and $M \in {}_S\mathsf{UMod}_R$. Notice that, due to M being unitary, using Lemma 2.24 we have

$$S \otimes_S M \otimes_R R = S \otimes_S (SMR) \otimes_R R = SS \otimes_S M \otimes_R RR = S(S \otimes_S M \otimes_R R)R,$$

which implies that $S \otimes_S M \otimes_R R \in {}_S\mathsf{UMod}_R$. Clearly, μ_M defined as in (6.1) is a morphism in ${}_S\mathsf{UMod}_R$.

Now, arbitrary $\alpha \in \operatorname{Ker}\mu_M$ can be expressed as $\alpha = \sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k$. We have

$$0 = \mu_M(\alpha) = \mu_M\left(\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k\right) = \sum_{k=1}^{k^*} s_km_kr_k.$$

For every $s \in S$ and $r \in R$, we have

$$\begin{aligned} s\,\alpha\, r &= s\left(\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k\right) r = \sum_{k=1}^{k^*} s s_k \otimes m_k \otimes r_k r = \sum_{k=1}^{k^*} s \otimes s_k m_k r_k \otimes r \\ &= s \otimes \left(\sum_{k=1}^{k^*} s_k m_k r_k\right) \otimes r = s \otimes 0 \otimes r = 0. \end{aligned}$$

Therefore $S\alpha R = \{0\}$ and, by Proposition 6.21, μ_M is a monomorphism in ${}_S\mathsf{UMod}_R$. ■

Thanks to the previous corollary we can give an example of a non-injective monomorphism in ${}_S\mathsf{UMod}_R$.

Example 6.23 (Non-injective monomorphism). From Example 2.14 we know that the module $M = (\overline{0}, 2)(\mathbb{Z}_2 \oplus \mathbb{Z})$ is a unitary non-firm right $(\mathbb{Z}_2 \oplus \mathbb{Z})$-module. Denote $R := \mathbb{Z}_2 \oplus \mathbb{Z}$. As any right module, M can be viewed as a $(\mathbb{Z}, R)$-bimodule with the usual left $\mathbb{Z}$-multiplication. The new module ${}_\mathbb{Z}M_R$ retains its properties of being unitary, yet non-firm, because M_R is still non-firm. Now, by Corollary 6.22, the morphism

$$\mu_M\colon\ \mathbb{Z} \otimes_\mathbb{Z} M \otimes_R R \to M, \qquad z \otimes m \otimes r \mapsto zmr$$

is a monomorphism in ${}_\mathbb{Z}\mathsf{UMod}_R$. Consider $1 \otimes (\overline{0}, 2) \otimes (\overline{1}, 2) \in \mathbb{Z} \otimes_\mathbb{Z} M \otimes_R R$. Note that

$$\mu_M(1 \otimes (\overline{0}, 2) \otimes (\overline{1}, 0)) = 1(\overline{0}, 2)(\overline{1}, 0) = (\overline{0}, 0),$$

but

$$f(1 \otimes (\overline{0}, 2) \otimes (\overline{1}, 0)) = 1 \cdot 1 \cdot \overline{1} = \overline{1},$$

where $f\colon \mathbb{Z} \otimes_\mathbb{Z} M \otimes_R R \to \mathbb{Z}_2$, $k \otimes (\overline{0}, 2b) \otimes (\overline{z}, a) \mapsto kb\overline{z}$ is a $(\mathbb{Z}, \mathbb{Z})$-bimodule homomorphism. This proves that $1 \otimes (\overline{0}, 2) \otimes (\overline{1}, 0) \neq 0 \in \mathbb{Z} \otimes_\mathbb{Z} M \otimes_R R$, because every homomorphism takes zero to zero. Hence μ_M is a non-injective monomorphism in ${}_\mathbb{Z}\mathsf{UMod}_R$. The morphism μ_M is surjective, because M is unitary, and therefore μ_M is an epimorphism and a bimorphism. □

From the previous example we deduce that there exist rings S and R such that the category ${}_S\mathsf{UMod}_R$ is not balanced, as it contains a bimorphism, which is not an isomorphism.

6.3 Monomorphisms of firm bimodules

In this section we will describe monomorphisms in the category ${}_S\mathsf{FMod}_R$, where S and R are idempotent rings. But first we will prove some useful properties of the functor $\mathbf{P}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{FMod}_R$ from Proposition 6.2.

Lemma 6.24. *Let S and R be idempotent rings. If $f\colon\ M \to N$ is a monomorphism in ${}_S\mathsf{Mod}_R$, then $\mathbf{P}(f) = \mathrm{id}_S \otimes f|_{SMR} \otimes \mathrm{id}_R$ is a monomorphism in ${}_S\mathsf{FMod}_R$. Moreover, $\mathbf{P}\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{FMod}_R$ preserves surjective morphisms.*

PROOF.
1. Let S and R be idempotent rings. By Theorem 6.3, the category ${}_S\mathsf{FMod}_R$ is a coreflective subcategory of ${}_S\mathsf{Mod}_R$ with a coreflector $\mathbf{P} = S \otimes_S S\_R \otimes_R R\colon\ {}_S\mathsf{Mod}_R \to {}_S\mathsf{FMod}_R$. Therefore $\mathbf{P}$ has a left adjoint, which is the inclusion functor $\mathbf{J}_{\mathrm{F}}$. Hence P preserves all limits and therefore also monomorphisms.
2. Let $f\colon\ M \to N$ be a surjective homomorphism on bimodules. Take an arbitrary $\alpha = \sum_{k=1}^{k^*} s_k \otimes n_k \otimes r_k \in \mathbf{P}(N) = S \otimes_S SNR \otimes_R R$. For every $k \in \{1, \ldots, k^*\}$, there exists $m_k \in M$ such that $n_k = f(m_k)$, due to the surjectivity of f. Now

$$\sum_{k=1}^{k^*} s_k \otimes n_k \otimes r_k = \sum_{k=1}^{k^*} s_k \otimes f(m_k) \otimes r_k = (\mathrm{id}_S \otimes f \otimes \mathrm{id}_R)\left(\sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k\right),$$

which implies that $\mathrm{id}_S \otimes f \otimes \mathrm{id}_R$ is surjective. Also, since $n_k \in SNR$, we obtain that $\mathbf{P}(f) = \mathrm{id}_S \otimes f|_{SMR} \otimes \mathrm{id}_R$ is surjective. ■

Now we can present our main theorem of this section. This theorem is inspired by an analogous theorem for semigroups and firm acts in [25] (Theorem 2.10).

Theorem 6.25. *Let S and R be idempotent rings and $f\colon\ M \to N$ a morphism in ${}_S\mathsf{FMod}_R$. The following assertions are equivalent:*

(1) f is a monomorphism;
(2) f is an extremal monomorphism;
(3) f is a regular monomorphism;
(4) $S(\operatorname{Ker} f)R = \{0\}$;
(5) $f = \mu_N \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R) \circ g$, where $A \in {}_S\mathsf{UMod}_R$, $a\colon\ A \to N$ is an injective homomorphism and $g\colon\ M \to S \otimes_S A \otimes_R R$ an isomorphism;
(6) $f = h \circ (\mathrm{id}_S \otimes b \otimes \mathrm{id}_R) \circ g$, where $A, B \in {}_S\mathsf{UMod}_R$, $b\colon\ A \to B$ is an injective homomorphism and $g\colon\ M \to S \otimes_S A \otimes_R R$, $h\colon\ S \otimes_S B \otimes_R R \to N$ are isomorphisms.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \\ S \otimes_S A \otimes_R R \arrow[r, "\mathrm{id}_S \otimes a \otimes \mathrm{id}_R"'] & S \otimes_S N \otimes_R R \arrow[u, "\mu_N"'] \end{tikzcd}$$

Figure 6.13: Condition (5).

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \\ S \otimes_S A \otimes_R R \arrow[r, "\mathrm{id}_S \otimes b \otimes \mathrm{id}_R"'] & S \otimes_S B \otimes_R R \arrow[u, "h"'] \end{tikzcd}$$

Figure 6.14: Condition (6).

Proof. ((3) $\implies$ (2) $\implies$ (1)). Holds in every category (Lemma 2.4).

((1) $\implies$ (3)). The category ${}_S\mathsf{CMod}_R$ is an essential localization of ${}_S\mathsf{Mod}_R$ by Theorem 6.16. Now, by Proposition 6.19 and Lemma 2.7, we obtain that monomorphisms and regular monomorphisms coincide in ${}_S\mathsf{CMod}_R$. By Theorem 6.14, we know that ${}_S\mathsf{CMod}_R$ and ${}_S\mathsf{FMod}_R$ are equivalent categories, therefore monomorphisms and regular monomorphisms also coincide in ${}_S\mathsf{FMod}_R$.

((1) $\implies$ (4)). Let f be a monomorphism. Consider the bimodule $S \otimes_S S(\operatorname{Ker} f)R \otimes_R R$, which is firm by Proposition 6.2, and the morphism

$$\mu_{S(\operatorname{Ker} f)R}\colon\ S \otimes_S S(\operatorname{Ker} f)R \otimes_R R \to M, \qquad \sum_{k=1}^{k^*} s_k \otimes m_k \otimes r_k \mapsto \sum_{k=1}^{k^*} s_k m_k r_k.$$

Clearly, $f \circ \mu_{S(\operatorname{Ker} f)R} = 0$ and hence, by Lemma 6.18, $\mu_{S(\operatorname{Ker} f)R} = 0$. On the other hand $\operatorname{Im}(\mu_{\operatorname{Ker} f}) = S(S(\operatorname{Ker} f)R)R = (SS)(\operatorname{Ker} f)(RR) = S(\operatorname{Ker} f)R$, which implies that $S(\operatorname{Ker} f)R = \{0\}$.

((4) $\implies$ (1)). This is proved precisely as in Proposition 6.21.

((2) $\implies$ (5)). Let f be an extremal monomorphism in ${}_S\mathsf{FMod}_R$. According to the Fundamental Homomorphism Theorem, there exist a bimodule ${}_SA_R = M/\operatorname{Ker} f$, a surjective homomorphism $e\colon\ M \to A$ and an injective homomorphism $a\colon\ A \to N$ in ${}_S\mathsf{Mod}_R$, such that $f = a \circ e$ (Figure 6.15).

$$\begin{tikzcd} & M \arrow[rr, "f"] \arrow[dr, "e"] & & N \\ S \otimes_S M \otimes_R R \arrow[ur, "\mu_M"] \arrow[dr, "\mathrm{id}_S \otimes e \otimes \mathrm{id}_R"'] & & A \arrow[ur, "a"] & \\ & S \otimes_S A \otimes_R R \arrow[ur, "\mu_A"'] \arrow[rr, "\mathrm{id}_S \otimes a \otimes \mathrm{id}_R"'] & & S \otimes_S N \otimes_R R \arrow[uu, "\mu_N"'] \end{tikzcd}$$

Figure 6.15

As a quotient of a unitary bimodule M, the bimodule $A = M/\operatorname{Ker} f$ is also unitary. Still, A need not be a firm (S, R)-bimodule. Using the naturality of μ (see Proposition 6.1), we have

$$\begin{aligned} f \circ \mu_M &= a \circ e \circ \mu_M = a \circ \mu_A \circ (\mathrm{id}_S \otimes e \otimes \mathrm{id}_R) \\ &= \mu_N \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R) \circ (\mathrm{id}_S \otimes e \otimes \mathrm{id}_R). \end{aligned}$$

Since M is firm, μ_M is bijective by Proposition 6.1, and therefore

$$f = \mu_N \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R) \circ ((\mathrm{id}_S \otimes e \otimes \mathrm{id}_R) \circ \mu_M^{-1}). \tag{6.16}$$

By Proposition 6.2, we have that $S \otimes_S A \otimes_R R$ is firm. Equality (6.16) is a factorization of monomorphism f in ${}_S\mathsf{FMod}_R$ into a composition of a morphism $\mu_N \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R)$ and an epimorphism $(\mathrm{id}_S \otimes e \otimes \mathrm{id}_R) \circ \mu_M^{-1}$. Indeed, since e is surjective, by Lemma 6.14, $\mathrm{id}_S \otimes e \otimes \mathrm{id}_R$ is also surjective, and hence $\mathrm{id}_S \otimes e \otimes \mathrm{id}_R$ is an epimorphism. Due to the assumption that f is extremal, we conclude that $g := (\mathrm{id}_S \otimes e \otimes \mathrm{id}_R) \circ \mu_M^{-1}$ is an isomorphism.

((5) $\implies$ (6)). This is obvious (take $B := N$ and $h := \mu_N$).

((6) $\implies$ (1)). Assume that $f = h \circ (\mathrm{id}_S \otimes b \otimes \mathrm{id}_R) \circ g$ for some unitary (S, R)-bimodules A and B, injective homomorphism $b\colon A \to B$ and isomorphisms $g\colon M \to S \otimes_S A \otimes_R R$ and $h\colon S \otimes_S B \otimes_R R \to N$ (Figure 6.14). Since the homomorphism b is injective, by Proposition 6.19, b is a regular monomorphism in ${}_S\mathsf{Mod}_R$. Now according to Lemma 6.24 the morphism $\mathrm{id}_S \otimes b \otimes \mathrm{id}_R$ is a monomorphism in ${}_S\mathsf{FMod}_R$. Since g and h are isomorphisms, f is also a monomorphism. ■

Next, we will prove a result, which can be used to construct non-injective monomorphisms in ${}_S\mathsf{FMod}_R$, where S and R are idempotent rings.

Proposition 6.26. *Let S and R be idempotent rings, let a bimodule ${}_SM_R$ be firm and ${}_SN_R$ be a unitary, but non-firm, sub-bimodule of ${}_SM_R$. Let $\iota_N\colon N \to M$ be the inclusion mapping. Then $\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R\colon S \otimes_S N \otimes_R R \to S \otimes_S M \otimes_R R$ is a non-injective regular monomorphism in ${}_S\mathsf{FMod}_R$.*

Proof. Using the naturality of μ, we may consider the following commutative square (Figure 6.16).

$$\begin{tikzcd} S \otimes_S N \otimes_R R \arrow[r, "\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R"] \arrow[d, "\mu_N"'] & S \otimes_S M \otimes_R R \arrow[d, "\mu_M"] \\ N \arrow[r, "\iota_N"] & M \end{tikzcd}$$

Figure 6.16

Here μ_M is bijective and μ_N is surjective, but not injective. Suppose, to the contrary, that $\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R$ is injective. Then $\mu_M \circ (\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R)$ is aslo injective. From the equality

$$\mu_M \circ (\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R) = \iota_N \circ \mu_N,$$

we deduce that μ_N is injective. This is a contradiction to the assumtion that N is not firm. Therefore $\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R$ is non-injective.

On the other hand, $\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R$ is a regular monomorphism in ${}_S\mathsf{FMod}_R$. Because ι_N is a regular monomorphism in the category ${}_S\mathsf{Mod}_R$ by Proposition 6.19 and by Lemma 6.24 $\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R$ is a monomorphism in ${}_S\mathsf{FMod}_R$ (here N is unitary, hence $\iota_N|_{SNR} = \iota_N$). By Theorem 6.25, every monomorphism in ${}_S\mathsf{FMod}_R$ is also a regular monomorphism in ${}_S\mathsf{FMod}_R$. ■

The previous proposition is meaningful, because by Example 2.14, there exists a firm bimodule, which has a sub-bimodule that is unitary, but not firm.

Finally we will prove a result about bimodules over xst-rings. Recall that a ring R is called a **right (left) xst-ring**, if every submodule of any unitary right (left) R-module is unitary (Definition 1 in [12]).

Proposition 6.27. *For idempotent rings S and R the following assertions are equivalent:*

(1) S is a left xst-ring and R a right xst-ring;
(2) ${}_S\mathsf{UMod}_R = {}_S\mathsf{FMod}_R$;
(3) monomorphisms in ${}_S\mathsf{UMod}_R$ are injective;
(4) μ_M is injective for all (S,R)-bimodules ${}_SM_R$.

Proof. Let S and R be idempotent rings.

((1) $\iff$ (2)). This equivalence follows from Proposition 9 in [17].

((2) $\implies$ (3)). Assume that ${}_S\mathsf{UMod}_R = {}_S\mathsf{FMod}_R$ holds. By Theorem 6.25 (5) we have that any monomorphism $f\colon M \to N$ in ${}_S\mathsf{UMod}_R$ is of the form $f = \mu_N \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R) \circ g$ for a unitary (S,R)-bimodule A, an injective homomorphism $a\colon\ A \to N$ and an isomorphism $g\colon\ M \to S \otimes_S A \otimes_R R$. By assumption, μ_A is an isomorphism. Using the naturality of μ, we get that $f = a \circ \mu_A \circ g$. Now we have expressed f as a composite of injective homomorphisms, therefore f itself is also injective.

((3) $\implies$ (4)). Assume monomorphisms in ${}_S\mathsf{UMod}_R$ to be injective. Let $M \in {}_S\mathsf{Mod}_R$. The (S,R)-bimodule $S \otimes_S M \otimes_R R$ is unitary, because $S(S \otimes_S M \otimes_R R)R = (SS) \otimes_S M \otimes_R (RR) = S \otimes_S M \otimes_R R$. Clearly, $SMR = \mathbf{U}(M)$ is also unitary.

Obviously $\mathrm{Im}\mu_M \subseteq SMR$. Consider the homomorphism

$$\mu_M|^{SMR}\colon\ S \otimes_S M \otimes_R R \to SMR, \quad \mu_M|^{SMR}(s \otimes m \otimes r) = \mu_M(s \otimes m \otimes r).$$

By the proof of Corollary 6.22, $\mu_M|^{SMR}$ is a monomorphism in ${}_S\mathsf{UMod}_R$. Then, by the assumption, $\mu_M|^{SMR}$ is injective, therefore μ_M is also injective.

((4) $\implies$ (2)). Assume that μ_M is injective for all (S, R)-bimodules ${}_SM_R$. It is clear, that if ${}_SN_R$ is unitary, then μ_N is surjective, therefore μ_N is an isomorphism and ${}_S\mathsf{UMod}_R = {}_S\mathsf{FMod}_R$. ■

6.4 Lattice of unitary sub-bimodules of a firm bimodule

In this section we will show that, for a fixed firm bimodule M, the lattice of unitary sub-bimodules $\mathrm{USub}(M)$ and the lattice of subobjects of M are isomorphic. First we must recall the notion of subobjects of an object A in some category $\mathcal{A}$ (see Definition 7.77 and Definition 7.79 in [2]).

Let $\mathcal{A}$ be a category and fix an object A of $\mathcal{A}$. Let $\mathrm{Iso}(\mathcal{A})$ denote the class of all isomorphisms in $\mathcal{A}$. Consider the following equivalence relation defined on the class of monomorphisms with codomain A in category $\mathcal{A}$:

$$f \sim g \quad :\Longleftrightarrow \quad \exists h \in \mathrm{Iso}(\mathcal{A})\colon \quad f = g \circ h.$$

$$\begin{tikzcd} & A & \\ B \arrow[ur, "f"] \arrow[rr, "h"'] & & C \arrow[ul, "g"'] \end{tikzcd}$$

Figure 6.17

Denote $[f] = [f]_\sim$ the equivalence class of a monomorphism f by the relation $\sim$. We denote

$$\mathrm{SUB}_{\mathcal{A}}(A) := \{[f]_\sim \mid f\colon\ B \to A \text{ is a monomorphism}\}.$$

Equivalence classes $[f] \in \mathrm{SUB}_{\mathcal{A}}(A)$ are called **subobjects** of A. The relation $\preceq$ defined by

$$[f] \preceq [g] \quad :\Longleftrightarrow \quad \exists m \in \mathrm{Mor}(\mathcal{A})\colon \quad f = g \circ m$$

is a partial order on the class $\mathrm{SUB}_{\mathcal{A}}(A)$.

In [18, Theorem 6] Marín and González-Férez showed that $\mathrm{SUB}_{\mathsf{FMod}_R}(M)$, where $M \in \mathsf{FMod}_R$, is a lattice, gave formulas for computing joins and meets for two subobjects and proved the following result.

Theorem 6.28 (Theorem 6 in [18]). *In the category of firm right modules over a ring, the lattices of subobjects are modular.*

For every $M \in {}_S\mathsf{FMod}_R$ we write

$$\mathcal{S}(M) := \mathrm{SUB}_{{}_S\mathsf{FMod}_R}(M).$$

The following theorem shows that if S and R are idempotent rings, then for every bimodule $M \in {}_S\mathsf{FMod}_R$, the lattices $\mathrm{USub}(M)$ and $\mathcal{S}(M)$ are isomorphic. It is a ring theoretic analogue of Theorem 4.2 in [25] for the case of bimodules.

Theorem 6.29. *Let ${}_SM_R$ be a firm (S,R)-bimodule over idempotent rings S and R. Then there exists an isomorphism of lattices*

$$\Psi\colon\ \mathrm{USub}(M) \to \mathcal{S}(M).$$

Proof. Let S and R be idempotent rings and $M \in {}_S\mathsf{FMod}_R$. We consider the mapping $\Psi\colon\ \mathrm{USub}(M) \to \mathcal{S}(M)$ defined by

$$\Psi(N) := [\mu_M \circ (\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R)], \tag{6.17}$$

for every ${}_SN_R \in \mathrm{USub}(M)$ and the inclusion $\iota_N\colon\ N \to M$.

$$S \otimes_S N \otimes_R R \xrightarrow{\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R} S \otimes_S M \otimes_R R \xrightarrow{\mu_M} M$$

Figure 6.18

The (S,R)-bimodules $S \otimes_S N \otimes_R R$ and $S \otimes_S M \otimes_R R$ are firm by Proposition 6.2 (the bimodules M and N are both unitary). The inclusion ι_N is obviously injective and by Proposition 6.19 a monomorphism in ${}_S\mathsf{Mod}_R$. By Lemma 6.24, $\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R$ is a monomorphism in ${}_S\mathsf{FMod}_R$. Since M is firm, the morphism μ_M is an isomorphism and $\mu_M \circ (\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R)$ is a monomorphism as a composite of a monomorphism and an isomorphism. Therefore, Ψ is well defined.

Let $N, O \in \mathrm{USub}(M)$. Assume that $N \subseteq O$ and consider the inclusion $\iota'_N\colon\ N \to O$ (illustrated on Figure 6.19). Then $\iota_N = \iota_O \circ \iota'_N$ and

$$\begin{aligned}\Psi(N) &= [\mu_M \circ (\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R)] = [\mu_M \circ (\mathrm{id}_S \otimes (\iota_O \circ \iota'_N) \otimes \mathrm{id}_R)]\\ &= [\mu_M \circ (\mathrm{id}_S \otimes \iota_O \otimes \mathrm{id}_R) \circ (\mathrm{id}_S \otimes \iota'_N \otimes \mathrm{id}_R)]\\ &\preceq [\mu_M \circ (\mathrm{id}_S \otimes \iota_O \otimes \mathrm{id}_R)] = \Psi(O).\end{aligned}$$

$$\begin{tikzcd}
S \otimes_S N \otimes_R R \arrow[r, "\mathrm{id}_S \otimes \iota'_N \otimes \mathrm{id}_R"'] \arrow[d, "\mu_N"'] \arrow[rr, bend left, "\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R"] & S \otimes_S O \otimes_R R \arrow[r, "\mathrm{id}_S \otimes \iota_O \otimes \mathrm{id}_R"'] \arrow[d, "\mu_O"] & S \otimes_S M \otimes_R R \arrow[d, "\mu_M"] \\
N \arrow[r, "\iota'_N"'] \arrow[rr, bend right, "\iota_N"'] & O \arrow[r, "\iota_O"'] & M
\end{tikzcd}$$

Figure 6.19

On the other hand, if we assume that $\Psi(N) \preceq \Psi(O)$, then

$$[\iota_N \circ \mu_N] = [\mu_M \circ (\mathrm{id}_S \otimes \iota_N \otimes \mathrm{id}_R)] \preceq [\mu_M \circ (\mathrm{id}_S \otimes \iota_O \otimes \mathrm{id}_R)] = [\iota_O \circ \mu_O]$$

by the naturality of μ. Hence, there exists a morphism $g\colon\ S \otimes_S N \otimes_R R \to S \otimes_S O \otimes_R R$ in ${}_S\mathsf{Mod}_R$ such that $\iota_N \circ \mu_N = \iota_O \circ \mu_O \circ g$. If $n \in N$ then, by the unitarity of N, we know that there exist $s_1, \ldots, s_{k^*} \in S$, $r_1, \ldots, r_{k^*} \in R$ and $n_1, \ldots, n_{k^*} \in N$ such that $n = s_1 n_1 r_1 + \ldots + s_{k^*} n_{k^*} r_{k^*}$. Consequently,

$$\begin{aligned}
n = \iota_N \left(\sum_{k=1}^{k^*} s_k n_k r_k \right) &= \iota_N \left(\mu_N \left(\sum_{k=1}^{k^*} s_k \otimes n_k \otimes r_k \right) \right) \\
&= \iota_O \left(\mu_O \left(g \left(\sum_{k=1}^{k^*} s_k \otimes n_k \otimes r_k \right) \right) \right) \in \operatorname{Im} \iota_O = O
\end{aligned}$$

and hence $N \subseteq O$. This proves that Ψ is an order-embedding.

Let us consider an equivalence class $[f] \in \mathcal{S}$, where $f\colon\ N \to M$ is a monomorphism in ${}_S\mathsf{FMod}_R$. By Theorem 6.25 (5), $f = \mu_M \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R) \circ g$ for a unitary (S, R)-bimodule A, an injective homomorphism $a\colon\ A \to M$ and an isomorphism $g\colon\ N \to S \otimes_S A \otimes_R R$.

We write a as a composition $a = a' \circ \iota_{a(A)}$, where $a(A) = \mathrm{Im} a$ is a unitary sub-bimodule of M and $a'\colon\ x \mapsto a(x)$ is an isomorphism (Figure 6.20).

$$\begin{tikzcd}
A \arrow[r, "a'"] & a(A) \arrow[r, "\iota_{a(A)}"] & M
\end{tikzcd}$$

Figure 6.20

Using the naturality of μ and that $(\mathrm{id}_S \otimes a' \otimes \mathrm{id}_R)$ and g are isomorphisms, we obtain the following equalities (illustrated on Figure 6.21)

$$\Psi(a(A)) = [\mu_M \circ (\mathrm{id}_S \otimes \iota_{a(A)} \otimes \mathrm{id}_R)] = [\iota_{a(A)} \circ \mu_{a(A)}]$$

$$\begin{aligned}
&= [\iota_{a(A)} \circ \mu_{a(A)} \circ (\mathrm{id}_S \otimes a' \otimes \mathrm{id}_R)] = [\iota_{a(A)} \circ a' \circ \mu_A] \\
&= [a \circ \mu_A] = [\mu_M \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R)] = \\
&= [\mu_M \circ (\mathrm{id}_S \otimes a \otimes \mathrm{id}_R) \circ g] = [f].
\end{aligned}$$

This proves the surjectivity of Ψ.

$$\begin{tikzcd}
N \arrow[r, "g"] & S \otimes_S A \otimes_R R \arrow[r, "\mathrm{id}_S \otimes a' \otimes \mathrm{id}_R"'] \arrow[d, "\mu_A"'] \arrow[rr, bend left, "\mathrm{id}_S \otimes a \otimes \mathrm{id}_R"] & S \otimes_S a(A) \otimes_R R \arrow[r, "\mathrm{id}_S \otimes \iota_{a(A)} \otimes \mathrm{id}_R"'] \arrow[d, "\mu_{a(A)}"] & S \otimes_S M \otimes_R R \arrow[d, "\mu_M"] \\
& A \arrow[r, "a'"'] \arrow[rr, bend right, "a"'] & a(A) \arrow[r, "\iota_{a(A)}"'] & M
\end{tikzcd}$$

Figure 6.21

We have shown that Ψ is a surjective order-embedding and hence an isomorphism of posets and lattices. ■

Corollary 6.30. *Let S and R be idempotent rings and $M \in {}_S\mathsf{FMod}_R$. The lattice $\mathcal{S}(M)$ is complete and modular.*

Proof. Let S and R be idempotent rings and $M \in {}_S\mathsf{FMod}_R$. By Proposition 2.25, we know that the lattice $\mathrm{USub}(M)$ is complete and modular. By Theorem 6.29, we have the lattice isomorphism $\mathrm{USub}(M) \cong \mathcal{S}(M)$. Therefore $\mathcal{S}(M)$ is also a complete and modular lattice. ■

Summary in Estonian

Idempotentsete ringide Morita ekvivalentsusest ja püsivate bimoodulite monomorfismidest

Selles dissertatsioonis on uuritud idempotentsete ringide Morita ekvivalentsi ning viimases peatükis on täpsemalt vaadeldud erinevat tüüpi bimoodulite kategooriaid. Bimoodulitel on oluline roll Morita teoorias, näiteks esinevad nad Morita kontekstide komponentidena. Ringi nimetatakse idempotentseks, kui iga tema element on esitatav mingite elementide korrutiste summana. Idempotentsed ringid on ühikelemendiga ringide üldistus.

Ilma ühikelemendita ringide Morita ekvivalentsuse defineerimiseks on üldiselt kolm erinevat loomulikku viisi: öelda, et ringid R ja S on Morita ekvivalentsed parajasti siis, kui ringide R ja S püsivate, kinniste või unitaarseteväändeta parempoolsete moodulite kategooriad on ekvivalentsed. Idempotentsete ringide klass on üks suuremaid ringide klasse, kus kõik need viisid omavahel kokku langevad. Lisaks on idempotentsete ringide Morita ekvivalentsi mugav kirjeldada Morita kontekstide abil. Nimelt kehtib tingimus, et idempotentsed ringid R ja S on Morita ekvivalentsed parajasti siis, kui ringide R ja S vahel leidub unitaarne ja sürjektiivne Morita kontekst. See kontekstidega kirjeldus leiab siinses dissertatsioonis rohket kasutust.

Käesoleva dissertatsiooni põhieesmärk on uurida mitmeid algebralisi konstruktsioone, mis on seotud idempotentsete ringide Morita ekvivalentsusega ning nende abil avada idempotentsete ringide Morita ekvivalentsuse mõistet. Lisaks on viimases peatükis erilise vaatluse all just püsivate bimoodulite kategooria ning monomorfismid selles kategoorias.

Antud väitekiri koosneb kuuest peatükist. Esimene peatükk on sissejuhatus, kus antakse lühike ülevaade Morita teooria ajaloost ning seejärel tutvustatakse väitekirja struktuuri.

Teises peatükis on toodud vajalikud eelteadmised, mida läheb vaja, et mõista seda väitekirja. Alustuseks on tutvustatud mõningaid mõisteid kategooriateooriast, nimelt kaasfunktoritega seotud mõisteid ja erinevat liiki monomorfisme. Seejärel on ära toodud vajalikud mõisted ringiteooriast ning moodulite teooriast. Eelteadmiste peatükis on pikemalt tutvustatud ka bimooduleid ning defineeritud erinevad bimoodulite kategooriad. Lõpetuseks on antud Morita teooria algteadmised, s.h. on defineeritud idempotentsete ringide Morita ekvivalentsus ja Morita kontekst ning esitatud Morita ekvivalentsuse kirjeldus kasutades Morita kontekste.

Kolmandas peatükis defineeritakse Reesi-maatriksringi ja tensorkorrutisringi mõisted suvaliste ringide jaoks. Mõlemat konstruktsiooni on edukalt kasutatud, et uurida Morita ekvivalentsust ning on tõestatud tulemus, mis seob omavahel Reesi-maatriksringid ja tensorkorrutisringid. Lisaks on siin peatükis vaadeldud kaas-endomorfismide ringe, millede abil on kirjeldatud s-unitaalsete ringide Morita ekvivalentsus. See peatükk põhineb artiklil [48].

Neljandas peatükis on defineeritud ringide laiendid ning tõestatud mitmeid ringide laiendite lihtsamaid omadusi. Antud peatüki põhiteoreemina on tõestatud, et idempotentsed ringid R ja S on Morita ekvivalentsed parajasti siis, kui leidub nende ringide ühine laiend. Lisaks on seal näidatud, et iga unitaarne ja sürjektiivne Morita kontekst idempotentsete ringide R ja S vahel on isomorfne unitaarse ja sürjektiivse Morita kontekstiga, mis on indutseeritud ringide R ja S ühise laiendi poolt. Lõpetuseks on näidatud, et poolrühmade Morita ekvivalentsus on seotud teatavate ringide ühise laiendiga. Neljas peatükk põhineb artiklil [27].

Viiendas peatükis uuritakse ringi unitaarsete ideaalide kvantaali. Seal on tõestatud, et kui idempotentsed ringid R ja S on Morita ekvivalentsed, siis on R ja S unitaarsete ideaalide kvantaalid isomorfsed. Siin peatükis on seejärel lühidalt uuritud Morita ekvivalentsete ringide sokleid ja nende ringide moodulite annihilaatoreid. Lisaks on tõestatud, et kui kaks ringi on seotud Morita kontekstiga, siis on nende ringide faktorringid vastavate ideaalide järgi samuti seotud sama liiki Morita kontekstiga. Viies peatükk põhineb artiklil [49].

Viimases ehk kuuendas peatükis uuritakse põhjalikult püsivate bimoodulite kategooriat üle mingite idempotentse ringide S ja R. Kõigepealt on siin näidatud, et püsivate, kinniste ja unitaarsete-väändeta (S, R)-bimoodulite kategooriad on tõepoolest ekvivalentsed. Seejärel on kirjeldatud monomorfismid püsivate (S, R)-bimoodulite kategoorias. Lõpetuseks on tõestatud, et mingi püsiva (S, R)-bimooduli M unitaarsete alam-bimoodulite võre on isomorfne bimooduli M (kategoorsete) alamobjektide võrega. Kuues peatükk on artikli [47] üldistus bimoodulite juhule.

Summary in Latin

De aequivalentia Moritae anellorum idempotentium et monomorfismo bimodulorum firmorum

In hac dissertatione, aequivalentia Moritae anellorum idempotentium tractata est et in capitulo ultimo genera ex variis categoriis bimodulorum tractata sunt. Bimoduli in theoria Moritae magni momenti sunt, exampli gratia, ii partes in contextibus Moritae sunt.

Meta principalis huius dissertationis est studium quarundam constructionum algebrae, quae aequivalentiae Moritae anellorum idempotentium adiunctae sunt. Praeterea, in capitulo ultimo, categoria bimodulorum firmorum et monomorphismi in ea observati sunt.

Haec dissertatio sex capitula habet. Primum capitulum introductio est.

In secundo capitulo scientia necessaria precursoria exposita est. Quaedam notiones theoriae categoriarum introductae sunt. Deinde notiones necessariae theoriae anellorum et modulorum relatae sunt. Postremo, scientia elementaria theoriae Moritae tractata est, i.a. aequivalentia Moritae anellorum idempotentium et contextus Moritae definiti sunt.

In tertio capitulo anellus matricis Reesi et anellus tensor-multiplicationis anellis arbitrariis definitus est. Utraque constructio feliciter usa est studendo aequivalentiae Moritae. Theorema, quod anellos matricis Reesi et anellos tensor-multiplicationis conciliat, demonstratum est. Deinde anelli endomorphismorum adiunctorum considerati sunt, per quos aequivalentia Moritae anellorum s-unitalium descripta est. Hoc capitulum scripturae [48] fundatum est.

In quarto capitulo extensiones anellorum definitae sunt et qualitates simpliciores nonnullorum anellorum demonstratae sunt. In hoc capitulo theorema, quod anelli idempotentes R et S aequivalentiam Moritae habent, si anelli exensionem communem habent, demonstratum est. Hoc capitulum scripturae [27] fundatum est.

In quinto capitulo quantale idealium unitarium anelli tractatum est. In hoc capitulo demonstratum est, ut si anelli idempotentes R et S aequivalentiam Moritae habent, quantalia idealium unitarium R et S isomorpha sunt. Hoc capitulum scipturae [49] fundatum est.

In ultimo et sexto capitulo categoria bimodulorum firmorum supra quosdam anellos idempotentes S et R tractata est. Monomorphismi in categoria (S, R)-bimodulorum firmorum descripti sunt. Postremo demonstratum est, ut reticulum sub-bimodulorum unitarium cuiusdam (S, R)-bimoduli firmi M isomorphum est cum reticulo sub-obiectorum bimoduli M. Capitulum sextum est praesentatio generalior scripturae [47].

Curriculum vitae

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Teaduslikud huvid: ringiteooria, kategooriateooria, loogika

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