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Singular points binomial method for pricing American fixed lookback options

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Abstract: An option is the opportunity to buy or sell an underlying asset with a fixed price at a given time in the future. One of the biggest difficulties in option theory is determining the correct value of an option. In this thesis, we discuss what European and American options are, we further move on to price American lookback options using the singular point method. We use singular points which are formed on the nodes of the tree and apply the binomial method to find price of the option which are represented as continuous piecewise linear functions. The reflection principle and combinatorics are used in pricing European lookback options. Under the reflection principle the emphasis is on finding the appropriate probability convenient to use in pricing the option under the method suggested by John Hull. **CERCS research specialisation:**P160 statistics, operations research, programming, acturial mathematics.
Key words: financial mathematics, options, lookback options, binomial model, singular points method.

Singulaarsete punktide meetod Ameerika tagasivaatavate optsioonide hindamiseks

Lühikokkuvõte: Töös vaadeldakse tagasivaatavate (lookback) optsioonide hindamist. Tagasivaatavate optsioonide korral sõltub optsiooniga seotud väljamakse alusvara maksimaalsest või minimaalsest hinnast optsiooni eluajal. Töös vaadeldakse binoommeetodil tuginevaid võimalusi nii Euroopa kui Ameerika tüüpi tagasivaatavate optsioonide hinna leidmiseks. Ameerika tüüpi tagasivaatavate optsioonide hinna leidmiseks tutvustatakse töös singulaarsete punktide meetodit, mis on tunduvalt efektiiivsem võrreldes tavalise binoommeetodiga.
CERCS teaduseriala: P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika.
Märksõnad: finantsmatemaatika, optsioonid, tagasivaatavad optsioonid, binoommeetod, singulaarsete punktide meetod.

Contents

Introduction This work briefly talks about option pricing , some basic terminologies used in option pricing which are covered in brief as it is not the main point of concern. It focuses on pricing lookback options using a modified algorithm of the binomial approach namely the singular point method. In this thesis, we discuss what European and American options are, which further move on to elaborate on using the binomial method which serves as a pivot in this thesis and contribute to this in use of the singular point method, to price American lookback options. This is a standard binomial technique for pricing by providing singular points at the nodes. Nothing changes much from the binomial method except the introduction of singular points. The above is employed since we cannot find analytical formula for pricing lookback option. Even though Babbs[1] gives an accurate and efficient solution to the problem for American and floating strike lookback by using a procedure of complexity of order $0(n^2)$ by using a change of numeraire, this cannot be applied in the case of fixed strike price.

Nevertheless other possibilities are also catered for. For example the reflection principle and combinatorics are considered as alternatives in pricing European lookback options. The main concern here is to find the probability that the maximum or the minimum of the stock is greater or lesser than some security price levels.

For American options we consider singular points method but also looks at some technical notes from Hull for pricing American options.

In summary this thesis is divided into three parts namely- chapter 1 which deals with options and finding option values, chapter 2 which talks about path- dependent options with emphasis on lookback, asian and barrier options and finally chapter 3 on the use of the singular point method in pricing the American lookback options in addition to numerical analysis of sample results.

1 Chapter 1. Options

1.1 Options, option value

An option is the right but not the obligation , to buy or sell a security such as a stock for an agreed upon price known as the strike price for some time in the future known as the **exercise time or expiry date**. The right to buy a security is a call option and the right to sell a security is a put option[5]. We have the European and American type of options.

The European option is only exercised at maturity, the final time T . The value of the option at the final time T can also be called the intrinsic value. Hence an *European call option* allows its *holder* the right (not obligation) to buy from the *writer* a prescribed asset for a prescribed price at a prescribed time in the future. Mathematically the payment of the call option is given by $max(S(T) - K, 0)$ and that of the put is $max(K - S(T), 0)$ with $S(T)$ being the price of the underlying security at final time T[3]. The figure 1.1 highlights the payoff of the European call and put options

An American option its like the European option except that it can be exercised at any time between the start date and expiry date by its holder.

An American call option gives its holder the right(but not the obligation) to purchase from *the writer* a prescribed asset for a prescribed price at any time between the start date and specific expiry date in the future. In the time interval $[0, T]$.

Whereas an American put option gives its holder the right(but not the obligation) to sell to *the writer* a prescribed asset for a prescribed price at any time between the start date and specific expiry date in the future. In the time interval $[0, T]$.

The price of an option today should replicate its future value at time T else there is an arbitrage opportunity and this should not exist on the market for a longer time before there is a movement in prices to eliminate them. Otherwise investors may take advantage and buy what is cheaper and sell at a higher price. Loosely we say "there is no such thing as "free lunch". Formally, opportunities to make instantaneous risk-free profit do not exist[2].

We also have the Put-Call parity. This is an argument between the relationship between the value of C of the European call and the value P of the European put option, with the same strike price K and expiry date T. We consider two portfolios

$$\pi_A : \text{ one call option plus } Ke^{-rT} \text{ and } \pi_B : \text{one put option plus one unit of the asset.}$$

At expiry, the portfolio π_A is worth $\max(S(T) - K, 0) + K$ which can be written as $\max(S(T), K)$. Portfolio π_B is worth on expiry $\max(K - S(T), 0) + S(T)$ which

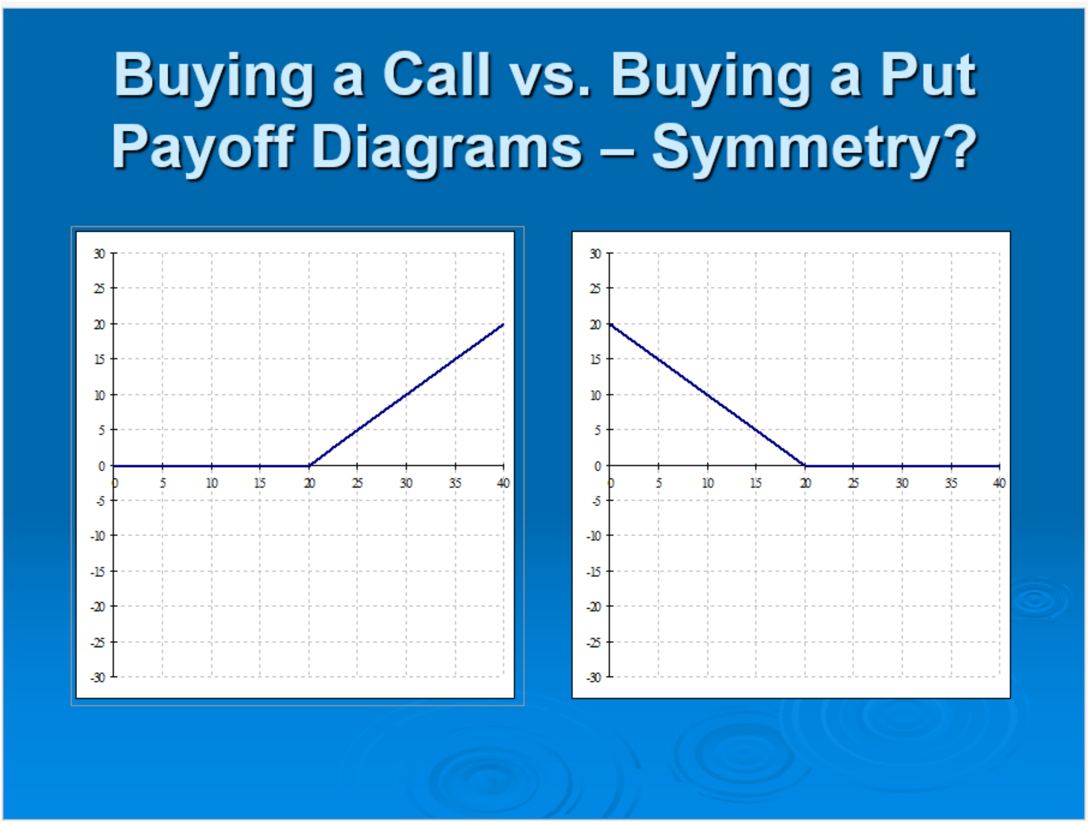

Figure 1.1: Payoff diagrams of European Call and Put Options

is also $\max(ST, K)$. If these two portfolios have the same value at expiry, we assume there is no arbitrage, then at $t = 0$ can say that the portfolios must have the same value at time 0, and

$$C + Ke^{-rT} = P + S(T) \tag{1.1}$$

This relationship which connects the call and put options is the *put-call parity*[2].

The argument behind (1.1) can be made more precise via the no arbitrage principle. If π_A is worth more than π_B at time 0 then it would be possible to sell π_A(that sell the call option and borrow the cash) and buy π_B (that is buy one put and one share). There is an instantaneous profit of $\pi_A - \pi_B$ (since we are sure that the payoff of π_B compensates for that of π_A at expiry). Such instantaneous profits violates the no arbitrage principle. A similar argument holds if π_B is worth more than π_A at time zero[2].

1.2 The Black-Scholes formula

We look at the Black-Scholes formula which is very important in option pricing. The evolution of stock is governed by

$$\Delta S = \mu \cdot S\Delta t + \sigma \cdot S \cdot \Delta W(t). \tag{1.2}$$

The ratio $\frac{\Delta S}{S}$ gives the return on the asset price and μ is the trend on the market and σ is the volatility. The quantity $\mu \cdot S\Delta t$ is known as the deterministic part and $\sigma \cdot S \cdot \Delta W(t)$ as the random part. The variable W is a stochastic variable (Brownian motion). The Variable $\Delta W(t)$ causes the uncertainty in the history of the stock price. The mean of W is 0 as intuitively wiggles up and down. Its variance over time T is still T. The higher the σ the higher the "jaggedness" of the path of the asset price.

A *Wiener process* also known as the *Brownian motion* is a particular type of *Markov* stochastic process . The behavior of the variable, W, which results from a Wiener process is ascertained by considering the changes in its value in small intervals of time. We consider a small interval of time of length Δt and let ΔW be the change in W during Δt. We have the basic properties as follows.

Property 1. The quantity ΔW must satisfy the equation

$$\Delta W = \epsilon\sqrt{\Delta t}$$

where ϵ is a random variable generated by the standardized normal distribution $N(0, 1)$.

Property 2. The values of ΔW for any two short intervals of time are independent. That means the intervals of time do not overlap.

The assumption that the value of the underlying security follows a geometric Brownian motion implies that

$$S(t) = S(0) \cdot exp(\mu - \sigma^2/2) \cdot t + \sigma \cdot \Delta W(t). \tag{1.3}$$

with all parameters as explained previously.

We make the following assumptions

- The stock price follows the log-normal distribution
- There are no taxes or transaction costs associated with hedging portfolio.
- Trading of the underlying asset takes place continuously.
- There are no dividends during the life of the option on the underlying assets. If dividends are known before hand this assumption can be dropped. They can be paid either at discrete intervals or continuously over the life of the option.
- There are no risk-less arbitrage opportunities. The absence of arbitrage opportunities means that all risk-free portfolios must earn the same return.
- The risk-free rate of interest, r, and the asset volatility σ are known functions of time over the life of the option.
- Short selling is permitted and the assets are divisible. We assume we can buy and sell any number(not necessarily an integer) of the underlying asset,and that we sell the assets we do not own.

If $\sigma = \sigma(t)$ and $r = r(t)$, then in partial differential sense $V = V(S(t), t)$, the price of the option with stock price S should satisfy the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$
$$S > 0 \text{ and } 0 \leq t < T.$$

This is known as the backward parabolic type if the final condition at time T is known and we solve backwards to find the price of the option else it is known as the forward parabolic type.

Without restrictions on the boundaries of the region which we solve for the value of the option price, a partial differential equation can have many different solutions. The domain of the unknown function V is a region in (S, t) -space. For a European call option the value at time T in the interval $[0, T]$ is of interest and T is noted as the exercise time of the option. The domain of finding the value of the option price is

$$\Omega = \{(S, t) \mid 0 \leq S < \infty \text{ and } 0 \leq t \leq T\}.$$

At time $t = T$ the value of the security will either exceed the strike price and the call option will be exercised with a payoff of $S(T) - K > 0$ or the security will attain a value less than the strike price, in this case the option expires unused and has no value. When $S(T) > K$ the call option is said to be **in the money**. The call option **out of the money** when $S(T) < K$. Thus the terminal value of the European call is

$$(S(T) - K)_+ = \max{(S(T) - K, 0)}.$$

where S(T) is the value of the underlying security at the exercise time T and K is the is the strike price of the option. Hence, if C represents a price of European call option, we can write the following equation as the final condition for the Black-Scholes PDE:

$$C(S, T) = (S(T) - K)_+.$$

At the boundary at $S = 0$, the call option is never exercised and has a value of zero. Thus we have one boundary condition namely

$$C(0, t) = 0.$$

Supposedly, if S is approaching infinity, it becomes likely that the call option will be exercised as $S \to \infty,$ it will exceed any finite value of K as K becomes less and less important. As $S \to \infty$, hence the value of the option becomes that of the asset price and we write

$$C(S, t) \approx S \text{ as } S \to \infty.$$

For a put option,with value $P(S, t)$, the final condition is the payoff

$$P(S, T) = (K - S(T))_+.$$

If S is zero the final payoff is known with certainty to be K. To determine $P(0, t)$ we have to calculate the present value of an amount K received at T. Assuming that interest rates are constant we find the boundary condition at $S = 0$ to be

$$P(0, t) = Ke^{-r(T-t)}.$$

More generally, for a time-dependent interest rate we have

$$P(0, t) = Ke^{-\int_t^T r(\tau)d\tau}.$$

As $S \to \infty$ the option is unlikely to be exercised and so (see [10])

$$P(S,t) \to 0 \text{ as } S \to \infty.$$

We assume C denotes the European call price, and P denotes the European put price. The BS formula follows:

$$C = S(0)N(d_1) - Ke^{-rT}N(d_2), P = Ke^{-rT}N(-d_2) - S(0)N(-d_1) \tag{1.4}$$

where

$$d_1 = \frac{\log(S(0)/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\log(S(0)/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}},$$

$N(x)$ - cumulative normal probability,
σ^2 - annualized variance of the continuously compounded return on the stock,
r - continuously compounded risk-free rate,
T - time to maturity .
Note:

$$S_0 = S(0)$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$

.

1.3 Binomial model

For American options and exotic options (lookback options) we cannot find analytical formula for the price of the option and we must use some numerical method. In general there are three methods for finding the price:

- Binomial or trinomial methods
- finite difference methods
- Monte-Carlo method

In this work we consider mainly the binomial method as the basic method in option pricing.

It serves as an alternative to partial differential equation solution to Black-Scholes equation. It was initially developed by Cox, Ross, and Rubeinstein [Cox et.

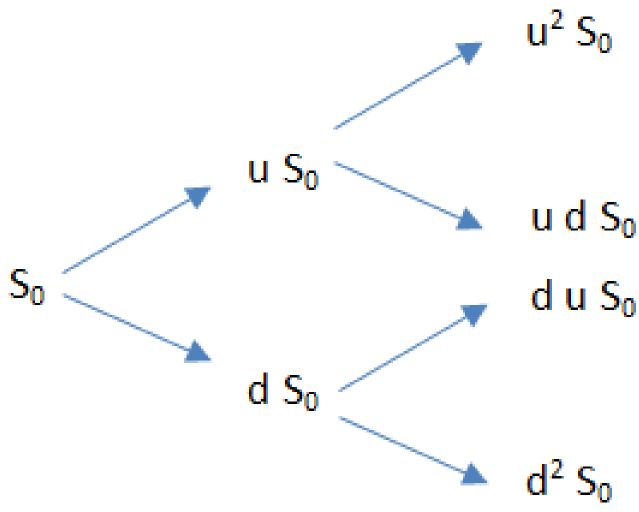

Figure 1.2: Binomial model

Al. (1979)] popularly known as **Cox-Rox-Rubeinstein model** which is again called the lattice model[6]. A graph of a double time step lattice model is shown in fig 1.0. It serves as an alternative model to the Blacks-Scholes equation in finding prices of option values. We make the following assumptions in the derivation

- K is the strike price of the call option
- T is the exercise time of the call option
- S_0, the initial price of the security
- r, the continuously compounded risk -free rate
- The price of the security follows a geometric Brownian motion with drift μ and σ as in (1.3)

Suppose the time interval $[0, T]$ is partitioned into n equal subintervals of length $\Delta t = \frac{T}{n}$. In the binomial model it is assumed that if the asset price is S at time-step $n\Delta t$, then it can either jump up to a higher value $uS, u > 1$ with probability p or down to a lower value $dS, d < 1$ with probability $1 - p$. Here we assume the the parameters $p, u,$ and d are constants and do not depend on time. Thus,

$$S_{t+1} = \begin{cases} uS_t \text{ with probability } p \\ dS_t \text{ with probability } 1 - p \end{cases} \tag{1.5}$$

We assume a risk-neutral world in which the underlying random walk for S is lognormally distributed. Then we can approximate this continuous random walk with a discrete random walk having the same mean and variance. Hence if the underlying security can take on only values uS_t and dS_t with probabilities p and $1-p$ respectively then after a time step of length Δt we have

$$puS_t + (1-p)dS_t = S_t e^{r\Delta t}. \tag{1.6}$$

And finally leads to

$$pu + (1-p)d = e^{r\Delta t}. \tag{1.7}$$

for all t. As

$$var(S_{t+1}) = \mathbb{E}(S_{t+1}^2) - (\mathbb{E}(S_{t+1}))^2,$$

then

$$var(S_{t+1}) = S_t^2(pu^2 + (1-p)d^2 - (pu + (1-p)d)^2).$$

Finally we have

$$pu^2 + (1-p)d^2 = e^{2r\Delta t} + \sigma^2 \Delta t. \tag{1.8}$$

By now we have written the probability p as a function of r and Δt and we have (1.8) which relates u and d to $r, \sigma,$ and Δt[8].

We can solve (1.7) as

$$p = \frac{ue^{r\Delta t} - 1}{u^2 - 1}$$

where

$$d = \frac{1}{u} \tag{1.9}$$

With (1.8) and (1.9) we can write

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{\sigma^2 \Delta t + e^{2r\Delta t} - d^2}{u^2 - d^2}$$

and hence

$$u + d = \frac{1}{d} + d = \frac{\sigma^2 \Delta t + e^{2r\Delta t} - d^2}{e^{r\Delta t} - d}$$

This is a quadratic expression for d, and to $O(\Delta t^{\frac{3}{2}})$ its solution is given by the expressions

$$p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \tag{1.10}$$

$$u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}}.$$

We may build up a lattice of possible asset prices. If at the current time $t = 0$ we know the asset price, S_0, then we divide the remaining life of the derivative security into n equal time-steps, $\Delta t = T/n$. At the first time step Δt there are two possible set prices, uS_0 and dS_0. At the second time-step, $2\Delta t$, there are three possible asset prices $u^2S_0, udS_0 = duS_0 = S_0$ and $d^2S_0 = \frac{S_0}{u^2}$.

In general, at the n-th time step $n\Delta t$ there are $n + 1$ possible values of the underlying asset price,

$$d^{n-j}u^jS_0 = u^{2j-n}S_0, j = 0, 1, ..., n.$$

Note that in figure 1.2 the lattice reconnects and lends itself to two lessons.The first is that the history of a particular asset price is lost, as there is clearly more than one path to any given point. Thus path dependent options cannot be valued using this reconnecting lattices. Secondly the total number of lattice points increases only quadratically with number of time-steps. This implies a number of time-steps can be taken.

Assuming that we we know the payoff function for our derivative security and that it depends only on the values of the underlying at expiry, this enables us to value it at expiry, time-step $T = N\Delta T$. If we consider a put, for example, we find that

$$P_{N,j} = \max(K - S_{N,j}, 0) \text{ and } j = 0, ...N,$$

where $P_{N,j}$ denotes the possible values of the put at the final time T and the j-th possible asset value $S_{N,j}$. For a call, we find that

$$C_{N,j} = \max(K - S_{N,j}, 0) \text{ for } j = 0, ...N$$

where $C_{N,j}$ denotes the possible values of the call at expiry. We can find the expected value of the option at the time-step prior to expiry, $(N-1)\Delta t$, and for possible asset price $S_{N-1,j}, j = 0, 1, ...N-1$, since we know that the probability of an asset priced at $S_{N-1,j}$ moving to $S_{N,j+1}$ during a time step is p and the probability of moving to $S_{N,j}$ is $(1-p)$. Using risk-neutral argument we can find the value of the security at each possible for time-step $(N - 1)$. Likewise this allows us to find the value of the security at time-step $(N - 2)$, and so on, back to time-step 0. This is the value of the security at the current time.

With European options we consider the following. Let $V_{n,j}$ be the value of the option at time-step $n\Delta t$ and asset price $S_{n,j}$(where $0 \leqslant j \leqslant n$). We calculate the expected value of the option at time step $j + 1$, given the asset price $S_{n,j}$, and discounting this for the riskless interest rate,

$$e^{r\cdot\Delta T}V_{n,j} = pV_{n+1,j+1} + (1-p)V_{n+1,j}.$$

This gives

$$V_{n,j} = e^{-r\cdot\Delta T}(pV_{n+1,j+1} + (1-p)V_{n+1,j}).$$

As we know the value of $V_{N,j}$, for $j = 0, ...N$ option value at final time $T = N\Delta T$, from the payoff function we can recursively determine the values $V_{n,j}$ for each $j = 0, ...n$ for $n < N$ to arrive at the current value of the option $V_{0,0}$. We do not require the asset prices $S_{n,j}$ during the evaluation of the option prices but $S_{N,j}$, when finding $V_{N,j}$. At each time-step we can discard the old $S_{n,j}$, as soon as we have found $S_{n+1,j}$. Once $V_{N,j}$ have been found, we can discard $S_{N,j}$ as well. This observation leads to extremely memory-efficient algorithm.

2 Chapter 2. Lookback options

According to **Desmond J.Higham (2005)**[2], the pay-off path dependent options depends on path of the underlying asset occuring within the time interval. Examples are Barrier options, Asian options and Lookback options.
The payoff of the lookback option depends on the maximum or minimum price of the underlying asset occurring over the life of the option. The option allows the holder to "look back" over time to determine the payoff. There exist two kinds of lookback options: with floating strike and with fixed strike. We assume K is the fixed strike price and M_T is the maximum value the underlying asset attains and m_T the minimum value in $[0, T]$. Hence the fixed lookback call and put can be priced in the following ways $(M_T - K)_+$ and $(K - m_T)_+$ respectively. Now we emphasise on the strike price K which attains the minimum value of the asset price, m_T or the maximum value, M_T giving way to a call or put respectively,hence we have $(S_T - m_T)_+$ and $(M_T - S_T)_+$ and this is known as the floating strike lookback.

Another type of the path dependent options is the Asian options. This is determined by average case behaviour. We take a look at this

An average price Asian call option has the pay-off at the expiry date T given by

$$\max\left(\frac{1}{T}\int_0^T S(\tau)d\tau - K, 0\right).$$

There is also the barrier option which has a payoff that switches on or off depending on whether the asset price crosses a pre-defined level. We have two types namely the *down -and- out call* option and *down -and- in call*. The former has a payoff that is zero if the asset crosses some predefined barriers $L < S_0$, $H > S_0$ at some time interval $[0, T]$. If the barrier is not crossed then the payoff becomes that of the European call, $\max\{S_T - K, 0\},$ if $L < S_t < H, t \in [0, T]$. Whilst the latter has a payoff zero unless the asset price crosses some predefined barrier $L < S_0$, $H > S_0$ at some time interval $[0, T]$. If the barrier is crossed then the payoff becomes that of the European call, $\max\{S_T - K, 0\},$ if $L < S_t < H, t \in [0, T]$

2.1 Price of the European Lookback option

Price of the European lookback option in the binomial model can be found using combinatorics and the reflection principle. First we consider the general idea about the reflection principle before considering how it can be used with the binomial method to price lookback options. Now we look at the reflection principle based on the binomial theorem for option pricing as suggested by Stanley R. Pliska

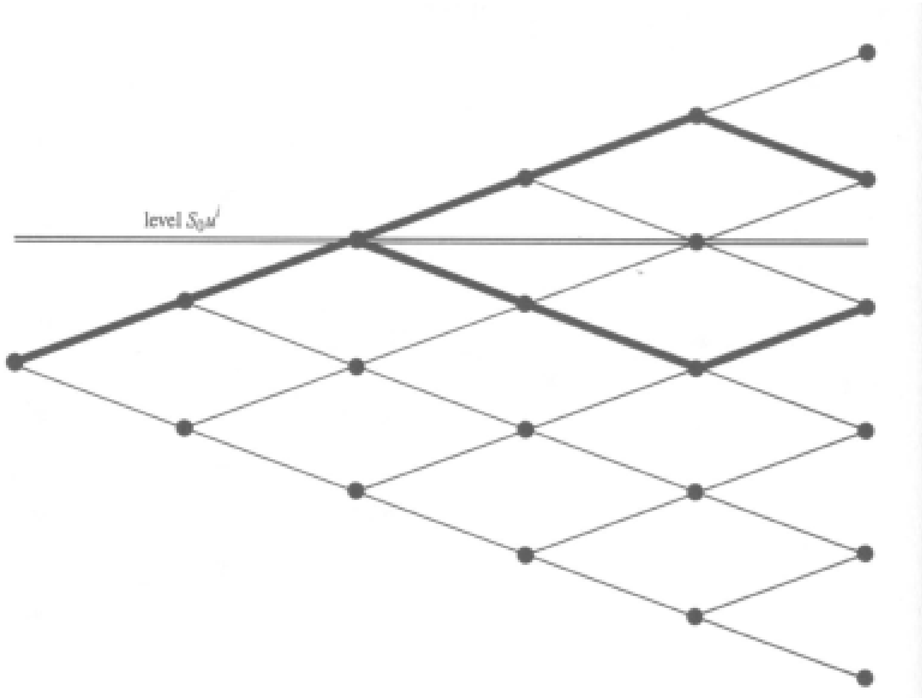

Figure 2.1: The reflection principle

(1979) [9]. We define the binomial security price model with T periods which features the the four parameters; $p, d, u,$ and S_0, where

$$0 \leq p \leq 1, 0 \leq d \leq 1 \leq u$$

and in fact $S_0 > 0$. At time t the price of the security is given by

$$S_t = S_0 u^{N_t} d^{t-N_t}, t = 1, 2...T.$$

, where N_t is the number of 'up' moves. The probability distribution of S_t is given by

$$P(S_t = S_0 u^n d^{t-n}) = \left(\begin{array}{c} t \\ n \end{array} \right) p^n (1-p)^{t-n} n = 0, 1, ...t$$

The binomial model can be used to compute the probability distribution for the maximum value achieved by the security process during the T periods. We derive this for the special case where $d = u^{-1}$ and this leads to the simplified version

$$S_t = S_0 u^n d^{2N_t - t}$$

.

$Y_T = \max\{S_t : t = 0, 1, ...T\}$, and this random variable takes the $T+1$ values $S_0, S_0u, ...S_0u^T$. We want to compute $P\{Y_T \geqslant S_0u^i$ for $i = 1, 2, ...T\}$. Fix i, we notice that $S_i \geqslant S_0u^i$ if and only if $2N_t - t \geqslant i$ so that $P\{Y_T \geqslant S_0u^i\}$ is the same as $P\{2N_t - t \geqslant i$ for some $t\}$. The latter is computed with the *reflection principle* as illustrated in figure 3.2. The main idea to find the first passage time $\tau_1 \equiv \min\{t : 2N_t - t = i\}$, where $\tau = \infty$ if $2N_t - t < i$ for all $t \leqslant T$, and take into account all the sample paths for which $\tau_1 \leqslant T$. It follows then that there are three mutually exclusive events. If i is one of the values $T, T-2, T-4, ...$, then it is possible to have the values $2N_T - T = i$,in this case $\tau_1 \leqslant T$. Secondly, it is likely to have $\tau_1 < T$ and $2N_T - T > i$. Thirdly $\tau_1 < T$ and $2N_T - T < i$. Thus

$$P\{Y_T \geqslant S_0u^i\} = P\{2N_t - t \geqslant i \text{ for some } t\} \tag{2.1}$$

$$P\{Y_T \geqslant S_0u^i\} = P\{\mathit{event}\ 1\ \} + P\{\mathit{event}\ 2\ \} + P\{\mathit{event}\ 3\ \} \tag{2.2}$$

The first probability is written as

$$P\{\mathit{event}\ 1\ \} = P\{N_T = (T+i)/2\} = \begin{pmatrix} T \\ \frac{T+i}{2} \end{pmatrix} p^{(T+i)/2}(1-p)^{(T-i)/2}$$

The above holds if $T + i$ is an even number else $P\{\text{event } 1\} = 0$. For the second probability, if $2N_T - T > i$, then definitely $\tau_i < T$, where $2N_T - T$ is the index of u at the final time T . Thus we have

$$P\{\mathit{event}\ 2\ \} = P\{N_T > (T+i)/2\} = \sum_{n=n^*}^{T} \begin{pmatrix} T \\ n \end{pmatrix} p^n(1-p)^{T-n}$$

where n^* is the smallest integer strictly greater than $(T + i)/2$. The sum is zero if $n^* > T$. To compute the third probability is a bit challenging therefore we use the reflection principle. Under the reflection principle, each sample path in event 2 is paired with a unique sample in event 3 as in figure 2.1. The sample paths coincide up to τ_i, and then each is the mirror image of the other across the level i. Hence the number of sample paths in the two events $2N_T - T > i$ and $2N_T - T < i$ are equal, although their probabilities are not equal unless $p = 1/2$.

To complete the computation of event 3, we consider an arbitrary sample path from event 2, and suppose it is such that $N_T = n(\leqslant n^*)$. This sample path occurs with probability $p^n(1-p)^{T-n}$ and there are $\begin{pmatrix} T \\ n \end{pmatrix}$ sample paths with $N_T = n$. Now looking at figure 2.1 it becomes apparent that "partner" of this sample path terminates with $N_T = T + i - n$, a symmetric distance below the level $(T + i)/2$. Hence N_T can be written as $N_T = T + i - (T + i)/2$. The probability of this "partner" sample path is $p^{T+i-n}(1-p)^{n-i}$. Since there are $\begin{pmatrix} T \\ n \end{pmatrix}$ sample paths in event 3 with $N_T = T + i - n$, it follows that

$$P\{\{event\ 3\ \cap \{N_T = T+i-n\}\} = \begin{pmatrix} T \\ n \end{pmatrix} p^{T+i-n}(1-p)^{n-i}$$

in which case

$$P\{event\ 3\ \} = \sum_{n=n^*}^{T} \begin{pmatrix} T \\ n \end{pmatrix} p^{T+i-n}(1-p)^{n-i}$$

Hence finally, we have

$$P\{Y_T \geqslant S_0 u^i\} = \begin{pmatrix} T \\ \frac{T+i}{2} \end{pmatrix} p^{(T+i)/2}(1-p)^{(T-i)/2} + \sum_{n=n^*}^{T} \begin{pmatrix} T \\ n \end{pmatrix} [p^n(1-p)^{T-n} + p^{T+i-n}(1-p)^{n-i}]$$

At this point we have in principle, the probability distribution for the maximum security price during T periods. Generally these formulas can be used for maximum security price during the first t periods when$t < T$. Since the event $\{Y \geqslant S_0 u^i\}$ is the same as the event $\{T \leqslant t\}$,we get the probability distribution for the first passage time to security price levels $S_0 u^i$. Same procedure can be used for the probability distributions of the minimum security price and the the first passage time to security price levels below S_0

There are exact formulas but if we use the binomial method, then we can find the price of the option using *combinatorics* (using reflection principle)

$$V(S,t) = e^{-r(T-t)} \sum_{i=0}^{n} P(Y_T = S_0 u^i) \max\{S_0 u^i - K\}.$$

At $t = 0$ we have

$$V(S,0) = e^{-r(T)} \sum_{i=0}^{n} P(Y_T = S_0 u^i) \max\{S_0 u^i - K\},$$

where $P\{Y_T = S_0 u^i\}$ is obtained from (2.3). We observe that $S_0 u^i$ as the maximum stock price attained and the above equation for the price of the option is the Fixed Strike European Call Lookback option.

A similar argument can be used to price the the Fixed strike European Put option by taking Y_T to be minimum security price attained and its first passage time levels below S_0.

2.2 Valuing American style lookback

Moreover number of researchers have suggested various approaches to valuing lookback options. **John Hull (1993)** provides one of such possibilities we can use to price an American -style lookback put. We consider a security with initial stock price 50, with volatility 0.04, risk-free interest rate is 0.1 and the time to expiry is 3 months. We assume that three steps are used to model the stock price movements.

When the option is exercised, there is a payoff equal to the excess of the maximum stock price over the current stock price. Therefore we define $G(t)$ as the maximum stock price achieved up to time t and we set

$$Y(t) = \frac{G(t)}{S(t)}.$$

We move on to use Cox,Ross, and Rubinstein tree for the stock price to produce a tree for Y. Initially, $Y = 1$ as $G = S$ as time $t = 0$. If there is an up movement by S during the first time step both G and S increase by the proportion u and the ratio Y is still 1. Mathematically we have

$$G(t+1) = \max\{S(t), S(t+1)\}$$

$$S(t+1) = uS(t)$$

hence

$$G(t+1) = uS(t)$$

and

$$Y(t+1) = \frac{G(t+1)}{S(t+1)}$$

$$Y(t+1) = 1.$$

Again if there is a down movement in the stock price S, G will stay the same and $Y(t+1) = \frac{1}{d} = u$. The rules for defining the geometry of the tree are

- When $Y = 1$ at time t, it is either u or 1 at time $t + \Delta t$.
- When $Y = u^m$ at time t for $m \geqslant 1$, it is either u^{m+1} or u^{m-1} at time $t + \Delta t$.

  An up movement in Y relates to a down movement in the stock price, and vice versa. The probability of an up movement in Y is $1 - p$, where as a down movement is p. We value the American lookback option in units of stock price rather in dollars. In dollars the payoff is

$$SY - S.$$

This is valid as SY is the maximum price attained up to time t. Algebraically SY produces G, the maximum price attained and is fixed.

Whilst in stock price units the payoff is

$$Y - 1.$$

We roll back through the tree in the usual way, valuing a derivative that provides this payoff except that we adjust for the differences in the stock price (i.e., the unit of measurement) at the nodes. If f_{ij} is the value of the lookback at the jth node at time $i\Delta t$ and Y_{ij} is the value of Y at this node, the rollback procedure gives

$$f_{ij} = \max(Y_{ij} - 1, e^{-r\Delta t}\left[(1-p)f_{i+1,j+1}d + pf_{i+1,j-1}u\right])$$

when $j \geqslant 1$[7].

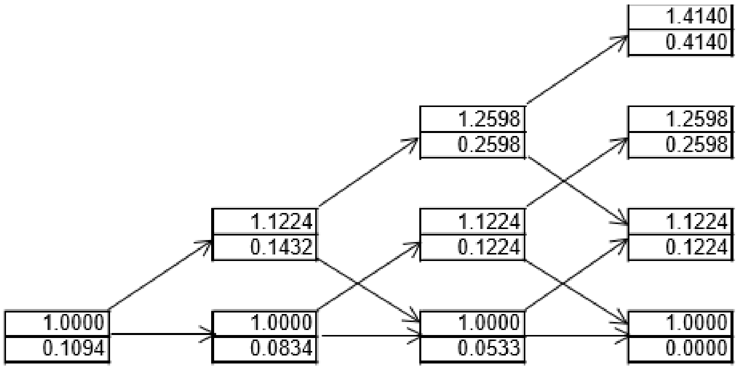

Figure 2.2: Procedure for valuing an American-style lookback options

we note that f_{ij} is to projected to $f_{i+1,j+1}$ by multiplying $f_{i+1,j+1}$ by d and to $f_{i+1,j-1}$ by multiplying $f_{i+1,j-1}$ by u

Similarly, when $j = 0$ the roll back procedure gives

$$f_{ij} = \max(Y_{ij} - 1, e^{-r\Delta t}\left[(1-p)f_{i+1,j+1}d + pf_{i+1,j}u\right]) .$$

As usual $f_{i,0}d$ and $f_{i,j}u$ are option price values formed at the edges of the binomial tree. At $j = 0$,Y_{ij} is the same for all i. This is because there is no change in S_0 whatsoever and hence the ratio of $G(t)$ to $S(t)$ does not change. The solution to example is shown the following diagram above.

3 Chapter 3. Singular points method for American lookback options

This chapter deals singular points method for American lookback options.
In the binomial (Cox-Ross-Rubinstein) model, the price at time $t = 0$ of the American lookback option is given by $V(0, S_0, S_0)$ where the functions $V(i, x, y)$ can be computed by the following backward dynamic programming equations:

$$V(n, x, y) = \psi(x, y)$$

$$V(i, x, y) = \max(\psi(x, y), V^c(i, x, y)$$

$$V^c(i, x, y) = e^{-r\Delta T}[pV(i+1, xu, \max(xu, y)) + (1-p)V(i+1, xd, y)],$$

where $\psi(x, y)$ is the payoff function and u, d, p are the parameters of the binomial model. The valuation of $V(0, S_0, S_0)$ requires a number of computations of order $O(n^3)$.
Here we consider a general framework for pricing European/American lookback options in a efficient way. The main idea of the singular points method is to give a continuous representation, at each node of the tree, of the option prices as a piecewise linear convex function of the path-dependent variable (maximum/minimum). These functions are characterized just by a set of points, which are called as "singular points". All such functions can be evaluated by backward induction in a straightforward way.

3.1 Piecewise linear convex functions

Definition 1. *Given a set of points :* $(x_1, y_1), ..., (x_n, y_n)$, *such that* $a = x_1 < x_2 < ...x_n = b$ and

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} < \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, i = 2, ...n-1, \tag{3.1}$$

let us consider the function f(x), x∈[a,b] obtained by interpolating the given points linearly. The points $(x_1, y_1), .., .(x_n, y_n)$*, which characterise the piecewise linear function* f *are called the singular points of* f*, while* $x_1, ...x_n$ *will be called singular values of* f*.*

Remark 1 *We considered only piecewise linear functions with strictly increasing slopes in the previous definition, hence the resulting function is convex.*

From henceforth only piecewise linear functions that are continuous and convex on the interval $[a, b]$ are taken into account. A set of singular points for these functions are found and they must satisfy (3.1).

Lemma 1 *Let f be a piecewise linear and convex function defined on the interval* $[a, b]$ *and let* $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ *be the set of its singular points. If we remove a point* (x_i, y_i) *with* $2 \le i \le n-1$*, from the set C, the resulting piecewise linear function* $\overline{f}$*, whose set of singular points is* $C \backslash \{(x_i, y_i)\}$*, is again convex in* $[a, b]$ *and we have*

$$f(x) \le \overline{f}(x), x \in [a, b]$$

.

Proof. The convexity of $\overline{f}$ follows from the fact that function $\overline{f}$ is the maximum between f and the function given by the straight line joining the points (x_{i-1}, y_{i-1}) and (x_{i+1}, y_{i+1}).

Remark 2. *It follows from Lemma 1 that every piecewise function* f *whose singular points are a subset of C (containing the first and the last singular points) is still convex and satisfies* $\overline{f}(x) > f$.

Lemma 2. *Let f be a piecewise linear and convex function defined on* $[a, b]$ *and let* $C = \{(x_i, y_i)\}, i = 1, ..., n$ *be the set of its singular points. We denote* $(\overline{x}, \overline{y})$*, the intersection between the straight lines joining* $(x_{i-1}, y_{i-1}), (x_i, y_i)$ *and the one joining* $(x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$*,* $2 \le i \le n-2$*. If we consider a new set of* $n-1$ *singular points* $\{(x_1, y_1), ..., (x_{i-1}, y_{i-1}), (\overline{x}, \overline{y}), (x_{i+2}, y_{i+2}), ..., (x_n, y_n)\}$*, the associated piecewise function* $\overline{f}$ *is convex on* $[a, b]$ *and* $f(x) \le \overline{f}(x), x \in [a, b]$.

Proof. The singular points of f satisfy the property of increasing slopes (3.1). The set of slopes of $\overline{f}$ are obtained by removing the slope of the line joining $(x_i, y_i), (x_{i+1}, y_{i+1})$, hence (3.1) is again satisfied and $\overline{f}$ is convex.

3.2 Pricing Lookback American options by singular points method

Now we describe the singular points method for fixed strike American lookback call option. The method consists in valuating the price of the option, at each node of the tree, for each possible choice of the maximum at that point. In the binomial model Let us denote by $N_{i,j}$ the node of the binomial tree whose underlying asset price is $S_{i,j} = S_0 u^{2j-i}, i = 0, ..., n, j = 0, ..., 1$. To each node $N_{i,j}$ we will associate a set of singular points, whose number is $L_{i,j}$. The singular point will be denoted by

$$(M_{i,j}^l, P_{i,j}^l), l = 1, 2, ..., L_{i,j}.$$

The singular values $M_{i,j}^l$ are called singular maximums and $P_{i,j}^l$ are called singular prices. At first we need to find the maximum and minimum values of the maximum $M_{n,j}^l$ underlying stock in the American case at the nodes $N_{n,j}, j = 0, 1, ..., n$. It follows that the maximum varies between a minimum value $M_{n,j}^{min}$ and a maximum value $M^{max}_{n,j}$, where

$$M^{min}_{n,j} = \max(S_{n,j}, S_0), M^{max}_{n,j} = S_0 u^j.$$

For each $M \in [M^{min}_{n,j}, M^{max}_{n,j}]$ the price of the option can be continuously defined by $V_{n,j}(M) = (M - K)_+$. The function $V_{n,j}(M)$ is a piecewise linear function satisfying Definition 1, whose singular points are valuable in a straightfowrard way. In fact:

- if $K \in (M^{min}_{n,j}, M^{max}_{n,j})$ then the price value function $V_{n,j}(M)$ is characterised by the 3 singular points $(M^l_{n,j}, P^l_{n,j}), l = 1, 2, 3$ (hence $L_{n,j} = 3$), where

$$M^1_{n,j} = M^{min}_{n,j}, \;\; P^1_{n,j} = 0; \tag{3.2}$$

$$M^2_{n,j} = K, \;\; P^2_{n,j} = 0; \tag{3.3}$$

$$M^3_{n,j} = M^{max}_{n,j}, \;\; P^3_{n,j} = M^{max}_{n,j} - K. \tag{3.4}$$

  Note: $P^1_{n,j} = 0$ is as a result of $\max(M^{min}_{i,j} - K, 0) = 0$ and $P^2_{n,j} = 0$ is obvious.

- If $K \notin (M^{min}_{n,j}, M^{max}_{n,j})$ then price value function $V_{n,j}(M)$ is characterised by the 2 singular points $(M^l_{n,j}, P^l_{n,j}), \;\; l = 1, 2$ (hence $L_{n,j} = 2$), where

$$M^1_{n,j} = M^{min}_{n,j}, \;\; P^1_{n,j} = (M^{min}_{n,j} - K)_+ \tag{3.5}$$

$$M^2_{n,j} = M^{max}_{n,j}, \;\; P^2_{n,j} = (M^{max}_{n,j} - K)_+. \tag{3.6}$$

- In the case $j = 0$ and $j = n$ the minimum and maximum of M coincide and $L_{n,j} = 1$.

Lemma 3. *At each node at maturity,the function $V_{n,j}(M)$ that provides the price of the option, is a piecewise linear function on the interval $(M^{min}_{n,j}, M^{min}_{n,j})$. Moreover,such function is convex on its domain.*

Consider now the step $i, 0 \leq i \leq n - 1$. At the node $N_{i,j}$ we can evaluate recursively the minimum and maximum value of the maximum M of the underlying by the relations

$$M^{min}_{i,j} = \max(M^{min}_{i+1,j+1}/u, S_0), \;\; M^{max}_{i,j} = M^{max}_{i+1,j}.$$

Lemma 4. *At each node $N_{i,j}$, $i = 0, ...n$, $j = 0, ..., i$, the function $V_{i,j}(M)$, which provides the price of the option as function of the maximum M, is piecewise linear and convex in the interval $[M^{min}_{i,j}, M^{max}_{i,j}]$.*

Proof. The claim is true at step $i = n$ (at maturity by Lemma 3). Consider the step $i = n - 1$. We extend the function $V_{i+1,j+1}(M)$ to the interval $[M^{min}_{i+1,j+1}/u, M^{max}_{i+1,j+1}]$ and we take $V_{i+1,j+1}(M) = V_{i+1,j+1}(M^{min}_{i+1,j+1})$ for $M \in [M^{min}_{i+1,j+1}/u, M^{min}_{i+1,j+1}]$. With such an extension the continuation value price function $V^c_{i,j}(M)$, becomes

$$V^c_{i,j}(M) = e^{-r\cdot\Delta T}[pV_{i+1,j+1}(M) + (1-p)V_{i+1,j}(M)]. \tag{3.7}$$

The price of an American lookback option can be obtained by computing only the singular points of the price function at each node. The structure of the tree in the lookback case gives us the opportunity to evaluate the singular points of $V_{i,j}$ in an efficient way. The procedure is elaborated in an ensuing Proposition 1 to be tackled soon. Hence we need to have some properties relating to the lookback case:

Lemma 5. *The price value function $V_{i,j}(M), M \in [M^{min}_{i,j}, M^{max}_{i,j}]$ has the following properties*:
a) if $K \in [M^{min}_{i,j}, M^{max}_{i,j}]$ then $V_{i,j}(M)$ is constant in $[M^{min}_{i,j}, K]$,
b) if $M \in [M^{min}_{i,j}, M^{max}_{i,j-1}]$ and $V_{i,j}(M) = M - K$ then $V_{i,j-1}(M) = M - K$,
c) if $M \in [M^{min}_{i+1,j+1}, M^{max}_{i,j}]$ and $v_{i+1,j+1}(M) = M - K$ then $v_{i,j}(M) = M - K$,
d) assume that $x_1 = S_0u^l$, $x_2 \in (S_0u^l, S_0u^{l+1})$, $x_3 = S_0u^{l+1}$ are singular values of $V_{i,j}$. If we delete the singular point $(x_2, V_{i,j}(x_2))$ then $V_{0,0}(S_0)$ does not change.

Proof. Properties (a) and (b) follows backward induction on the tree. Property (c) follows from (b) and we can conclude from (c) that, at the nodes $N_{i+1,j+1}$ and $N_{i,j}$ the same function passes through the singular points at those nodes.

The claim of property (d) follows by the fact that the value of the option at the nodes $N_{i,0}, N_{i,i}, i = 0, ..., n-1$, depends on the values assumed by $V_{i+1,j}$ at the nodes of the tree.

Again by Lemma $5(d)$, we deduce that every singular value which lies between consecutive nodal stock values and which are singular values as well, can be removed. This means the critical value $\overline{M}_{i,j}$ can be removed, during the backward iterations without affecting the price of the option if it lies between two consecutive nodal values.

In the ensuing proposition we shall get to know that the set of internal singular points of $v_{i,j}$ at each node can be reduced to a set of consecutive singular nodal values which are singular values of $v_{i,j}$ well as noted earlier, with the final addition of K. $\overline{M}_{i,j}$ lies between two consecutive nodal singular values and it can be ignored in the backward iteration by Lemma $5(d)$.

Proposition 1 Consider the price value function $V_{i,j}$ and denote by l_0 the smallest integer l such that $S_0u^l > max(K, M^{min}_{i,j})$. The set of singular values of $V_{i,j}$ can be reduced to : $M^{min}_{i,j}, M^{max}_{i,j}, K$ if $K \in (M^{min}_{i,j}, M^{max}_{i,j})$ and a set (eventually empty) of consecutive nodal stock values$S_0u^{l_0}, S_0u^{l_0+1}, ..., S_0u^{l_0+k}$ which are singular values of $V_{i+1,j+1}$ as well. Moreover if $M = S_0 u^{l_0+k} < \frac{M_{i,j}^{max}}{u}$, then $V_{i,j}(M) = M - K$.

Proof. Consider the case $i = n-1$. We take first $j \geq int[\frac{i}{2}]$ and $j < n-1$ (the case $j = n-1$ is trivial). At the node $N_{i,j}$, the singular values of $V_{i,j}^c$ are $M_{i,j}^{min}, M_{i,j}^{max}, K$ if $K \in [M_{i,j}^{min}, M_{i,j}^{max}]$ and eventually $uM_{i,j}^{min} = M_{i+1,j+1}^{min}$. By Lemma 5(a) $uM_{i,j}^{min}$ is a singular value of $V_{i,j}^c$ if and only if $uM_{i,j}^{min}$ is greater or equal than K.

Now we consider the function $V_{i,j}$. The singular values of $V_{i,j}$ are the same of $V_{i,j}^c$ and with possible addition of $\overline{M}_{i,j}$ (critical values) if it exists. If $\overline{M}_{i,j}$ exists then more importantly $uM_{i,j}^{min}$ is a singular value and $K < uM_{i,j}^{min}$. By Lemma 5(c) $V_{i,j}(uM_{i,j}^{min}) = uM_{i,j}^{min} - K$ since $V_{i+1,j+1}(uM_{i,j}^{min}) = uM_{i,j}^{min} - K$. We can then conclude that $\overline{M}_{i,j} \in [M_{i,j}^{min}, uM_{i,j}^{min}]$ and by Lemma $5(d)$ it can be removed. Hence the claim holds.

In the case $j < int\left[\frac{i}{2}\right]$ there are no singular values in $(K, M_{i,j}^{max)}$ so the claim is trivial.

Now we consider the general case $i < n-1$ and take $0 < j < n-1$ (the cases $j = 0$ and $j = n$ are trivial). Singular values of $V_{i+1,j+1}$ that belong to $[M_{i,j}^{min}, M_{i,j}^{max}]$ are singular values of $V_{i,j}^c$ as well. We can claim that $V_{i,j}^c$ has no internal singular values but possibly K, the strike price. If $M > \min(K, M_{i,j}^{min})$ is a singular value of $V_{i+1,j}$, then by induction it is a singular value of $V_{i+2,j+1}$, therefore it is a singular value of $V_{i+1,j+1}^c$. By Lemma $5(b)$ we can conclude that $V_{i+1,j+1}$ has it as a singular value as well. Hence we can say that the set of all singular values of $V_{i,j}^c$ is made of $M_{i,j}^{min}, M_{i,j}^{max}$ and eventually K and a sequence of consecutive nodal values $s_0 u_0^l, s_0 u^{l_0+1}, ..., s_0 u^{l_0+k}$ which are singular values of $V_{i+1,j+1}$.
Consider now $V_{i,j}$. If $V_{i,j}(M_{i,j}^{max}) \geq M_{i,j}^{max} - K$ then $V_{i,j} \equiv V_{i,j}^c$ and their singular points are the same. If $S_0 u^{l_0+k} < \frac{M_{i,j}^{max}}{u}$ then $S_0 u^{l_0+k+1}$ is not a singular value $V_{i+1,j+1}$ of by Proposition 1. If $V_{i+1,j+1}(S_0 u^{l+k}) = S_0 u^{l_0+k} - K$ then $V_{i,j}(S_0 u^{l_0+k}) = S_0 u^{l_0+k} - K$ by Lemma 6(c).

If we assume $V_{i,j}(M_{i,j}^{max}) < M_{i,j}^{max} - K$ and $V_{i,j}(M_{i,j}^{min)} \leq M_{i,j}^{min} - K$ then are no singular points $(M_{i,j}^{max}, M_{i,j}^{min})$ and the claim holds. If $V_{i,j}(M_{i,j}^{min}) > M_{i,j}^{min} - K$ then $\overline{M}_{i,j}$ exists. Let l_g be the largest index such that $S_0 u^{l_g} \in (K, M_{i,j}^{max})$ and $S_0 u^{l_g}$ is a singular value of $V_{i,j}^c$. If $S_0 u^{l_g} = \frac{M_{i,j}^{max}}{u}$ then the singular values of $V_{i,j}^c$ include all the nodal values from $S_0 u^{l_0}$ to $M_{i,j}^{max}$. By Lemma $5c$ $\overline{M}_{i,j} \leq S_0^{l_Q}$ We denote l_Q the smallest index such that $\overline{M}_{i,j} \leq S_0 u_Q^l$, we can then remove $S_0 u^{l_{Q+1}}, ..., S_0 u^{l_g}$ and the claim holds. We observe that $S_0 u^{l_g}$ might be smaller than $M_{i,j}^{max}/u$. If $S_0 u^{l_g} < \frac{M_{i,j}^{max}}{u}$ and by induction $V_{i+1,j+1}(S_0 u^{l_g}) = S_0 u^{l_g} - K$.Again $S_0 u^{l_{Q+1}}, ..., S_0 u^{l_g}$ can be removed and $V_{i,j}(S_0 u^{l_Q}) = S_0 u^{l_Q} - K$ proves the claim.
◇ **Note** It is evident that the singular values of $V_{i,j}$ and $V_{i+1,j+1}$ help the idea of

convexity to light and moreover $S_0 u^{l_Q}$ and $S_0 u^{l_g}$ are all greater than K by Lemma $5a$ as they are singular points and that these singular values form points on the increasing function $V_{i,j}$.

3.3 Sketch of the algorithm of the singular points method

Now we give the algorithm in order to obtain the exact binomial price for a fixed strike American lookback call option.

- Step n
  Compute the singular points at maturity by using formulas (3.2)-(3.6)

- Step $i, i = n-1, n-2, ..., 0$
  Compute $P^1_{i,0}, P^1_{i,i}$ using the formulas

$$P^1_{i,0} = \max\left(e^{-r\Delta T}(pP^1_{i+1,0} + (1-p)P^1_{i+1,1}), M^{max}_{i,0} - K\right),$$

$$P^1_{i,i} = \max\left(e^{-r\Delta T}(pP^1_{i+1,i+1} + (1-p)P^1_{i+1,i}), M^{max}_{i,i} - K\right).$$

  Note, that at nodes $N_{i,0}, N_{i,i}$ there is only a singular point and $M^{max} = M^{min}$. For each node $N_{i,j}, j = 1, ..., i-1$, compute the set of the singular points by the following steps:

  - Compute $V^c_{i,j}(M^{min}_{i,j}), V^c_{i,j}(M^{max}_{i,j})$.
  - If $V^c_{i,j}(M^{min}_{i,j}) \leq M^{min}_{i,j} - K$ then there are only 2 singular points: $(M^{min}_{i,j}, M^{min}_{i,j} - K)$, $(M^{max}_{i,j}, M^{max}_{i,j} - K)$, and the computation is concluded.
  - If $V^c_{i,j}(M^{min}_{i,j}) > M^{min}_{i,j} - K$ then insert singular points $(M^{min}_{i,j}, V_{i,j}(M^{min}_{i,j}))$, $(M^{max}_{i,j}, V_{i,j}(M^{max}_{i,j}))$.
  - If $K \in (M^{min}_{i,j}, M^{max}_{i,j})$ then insert singular point $(K, V_{i,j}(K))$.
  - For each singular value M of the node $N_{i+1,j+1}$ belonging to $(K, M^{max}_{i,j})$ add $(M, V^c_{i,j}(M))$. If $V^c_{i,j}(M^{max}_{i,j}) \geq M^{max}_{i,j} - K$ then $V^c_{i,j}$ and $V_{i,j}$ coincide and the computation is concluded.
  - Otherwise remove all singular points with singular value internal to $[M^{min}_{i,j}, M^{max}_{i,j}]$ and singular price given by early exercise, except from the one which has the smallest value.

The value $P^1_{0,0}$ is exactly the binomial price relative to the tree with n steps of fixed strike American lookback call option.

Table 1: Prices of the American lookback call option

n	$\sigma = 0.2$	$\sigma = 0.2$	$\sigma = 0.4$	$\sigma = 0.4$
	$K = 90$	$K = 110$	$K = 90$	$K = 110$
100	29.04519	12.31876	45.88504	28.78064
200	29.61321	12.78364	46.71973	29.63417
300	29.83982	12.95645	47.07974	29.86853
400	29.96725	13.0882	47.29169	30.09175
500	30.05073	13.13989	47.43499	30.23607
600	30.11082	13.20138	47.54004	30.33595
700	30.15633	13.22531	47.62165	30.371
800	30.19238	13.26299	47.68705	30.43735

3.4 Numerical examples

For pricing American lookback call option with singular points method we made the Matlab program (see Appendix). We use the following initial values: the initial value of the stock price is $S_0 = 100$, the maturity $T = 1$, the interest rate $r = 0.1$. We consider two choices for the volatility: $\sigma = 0.2$, $\sigma = 0.4$ and two choices for the strike price: $K = 90$ and $K = 110$. We consider different time steps n = 100; 200; 300; 400; 500; 600; 700; 800. The numerical results are in table 1.

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A Matlab code for singular point method

```
clc
clear
for s1=1:2
sigma=0.2*s1;
for s2=1:2
K=90+20*(s2-1);
for nn=1:8
%% Parameters of the binomial model
n=100*nn;
s0=100;
r=0.1;
T=1;
q=0.03;
delta_t=T/n;
u=exp(sigma*sqrt(delta_t));
d=1/u;
p=(exp(r*delta_t)-d)/(u-d);
P(1:n+1,1:100)=0;
M(1:n+1,1:100)=0;
P_new(1:n+1,1:100)=0;
M_new(1:n+1,1:100)=0;

%% Singular points at maturity (n)
for j=0:n
    j1=j+1;
    S(j1)=s0*u^(2*j-n);
    V(j1)=max(S(j1)-K,0);
    Mmin(j1)=max(s0,S(j1));
    Mmax(j1)=s0*u^j;
    if j==0 || j==n
        M(j1,1)=Mmin(j1);
        P(j1,1)=max(Mmin(j1)-K,0);
        L(j1)=1;
    else
        if K>Mmin(j1) && K<Mmax(j1)
            M(j1,1)=Mmin(j1);
```

```
            P(j1,1)=0;
            M(j1,2)=K;
            P(j1,2)=0;
            M(j1,3)=Mmax(j1);
            P(j1,3)=Mmin(j1)-K;
            L(j1)=3;
        else
            M(j1,1)=Mmin(j1);
            P(j1,1)=max(Mmin(j1)-K,0);
            M(j1,2)=Mmax(j1);
            P(j1,2)=max(Mmax(j1)-K,0);
            L(j1)=2;
        end
    end
end

%% Singular points j=0,1,...,n-1
for k=n-1:-1:0
    for j=0:1:k
        j1=j+1;
        S(j1)=s0*u^(2*j-k);
        Mmin(j1)=max(s0,S(j1));
        Mmax(j1)=s0*u^j;
        %% Singular points in case of j=0 or j=k
        if j==0 || j==k
          M_new(j1,1)=Mmin(j1);
          if j==0
              P_new(j1,1)=exp(-r*delta_t)*
(p*P(j1,1)+(1-p)*P(j1+1,1));
          else
              P_new(j1,1)=exp(-r*delta_t)*
(p*P(j1,1)+(1-p)*P(j1+1,L(j1+1)));
          end
           P_new(j1,1)=max(P_new(j1,1),Mmin(j1)-K);
           L_new(j1)=1;
        else
           %% singular points in case 1<=j<=k-1
            vcmin(j1)=Vc(Mmin(j1),P,M,L,j1,r,delta_t,p,u);
```

```
            vcmax(j1)=Vc(Mmax(j1),P,M,L,j1,r,delta_t,p,u);

            if vcmin(j1)<=Mmin(j1)-K
                M_new(j1,1)=Mmin(j1);
                P_new(j1,1)=Mmin(j1)-K;
                M_new(j1,2)=Mmax(j1);
                P_new(j1,2)=Mmax(j1)-K;
                L_new(j1)=2;
            else
                M_new(j1,1)=Mmin(j1);
                P_new(j1,1)=max(vcmin(j1),Mmin(j1)-K);
                L_new(j1)=1;
              if K>Mmin(j1) && K <Mmax(j1)
                 M_new(j1,2)=K;
                 vck(j1)=Vc(K,P,M,L,j1,r,delta_t,p,u);
                 P_new(j1,2)=max(vck(j1),K-K);
                 L_new(j1)=L_new(j1)+1;
               end
            for jt=1:L(j1+1)
                if M(j1+1,jt)>K && M(j1+1,jt)<Mmax(j1)
                   vck(j1)=Vc(M(j1+1,jt),P,M,L,j1,r,delta_t,p,u);
                   if vcmax(j1)>=Mmax(j1)-K &&
vck(j1)<=M(j1+1,jt)-K
                   else
                   M_new(j1,L_new(j1)+1)=M(j1+1,jt);
                   P_new(j1,L_new(j1)+1)=vck(j1);
                   L_new(j1)=L_new(j1)+1;
                   end
                end
            end
            M_new(j1,L_new(j1)+1)=Mmax(j1);
            P_new(j1,L_new(j1)+1)=max(vcmax(j1),Mmax(j1)-K);
            L_new(j1)=L_new(j1)+1;
            end

        end
    end
    for j=0:1:k
```

```
        j1=j+1;
        L(j1)=L_new(j1);
        for k1=1:L(j1)
        M(j1,k1)=M_new(j1,k1);
        P(j1,k1)=P_new(j1,k1);
        end
    end
end
hh=2*(s2-1)+s1;
Price(nn,hh)=P(1,1);
end
end
end
```

```
function VC = Vc(M1,P,M,L,j1,r,delta_t,p,u)
 if L(j1+1)==1
     V2=P(j1+1,1);
 else
     if M1<M(j1+1,1)  %%&& M1>=M(j1+1,1)/u
         V2=P(j1+1,1);
     else
     for jt=1:L(j1+1)-1
       if M1<=M(j1+1,jt+1) && M1>=M(j1+1,jt)
         V2=P(j1+1,jt)+(M1-M(j1+1,jt))*(P(j1+1,jt+1)-P(j1+1,jt))/
(M(j1+1,jt+1)-M(j1+1,jt));
       end
     end
     end
 end

  if L(j1)==1
      V1=P(j1,1);
  else
      for jt=1:L(j1)-1
         if M1<=M(j1,jt+1) && M1>=M(j1,jt)
             V1=P(j1,jt)+(M1-M(j1,jt))*(P(j1,jt+1)-P(j1,jt))/
(M(j1,jt+1)-M(j1,jt));
         end
      end
  end
  VC=exp(-r*delta_t)*(p*V2+(1-p)*V1);
```

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