DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
43

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
43

UNIFORM FACTORIZATION FORCOMPACT SUBSETS OF BANACH SPACES OF OPERATORS

KRISTEL MIKKOR

TARTU UNIVERSITY
PRESS

Faculty of Mathematics and Computer Science, University of Tartu, Tartu, Estonia

Dissertation is accepted for the commencement of the degree of Doctor of Philosophy (PhD) in mathematics on April 21, 2006, by the Council of the Faculty of Mathematics and Computer Science, University of Tartu.

Supervisor:

Cand. Sc., Professor Eve Oja,
University of Tartu,
Tartu, Estonia

Opponents:

Cand. Sc., Professor Anne Tali,
Tallinn University,
Tallinn, Estonia

PhD, Academy Research Fellow
Hans-Olav Johannes Tylli,
Academy of Finland,
Helsinki, Finland

Commencement will take place on June 20, 2006.

Publication of this dissertation is granted by the Institute of Pure Mathematics of the University of Tartu (research project TMTPM0085) and the Estonian Science Foundation grant GMTPM5704.

ISSN 1024–4212
ISBN 9949–11–313–X (trükis)
ISBN 9949–11–314–8 (PDF)

Tartu Ülikooli Kirjastus
www.tyk.ee
Tellimus nr. 266

Contents

Acknowledgements

I am indepted to many people who helped and supported me throughout the thesis writing period and during my studies in past ten years. Undoubtely, the most important are my supervisor Eve Oja and my family. Thank you, Eve, for all the ideas, advice, good teaching, encouragement, and continuous support you provided.

There are, of course, my other colleagues in the Faculty of Mathematics and Computer Science at Tartu University whose suggestions, ideas, remarks, and general support mathematically as well as personally I greatly appreciate, and who must be acknowledged. I especially mention Rainis Haller, Joosep Lippus, Annemai Mölder, and Märt Põldvere.

During the preparation of this thesis, I have also been helped by other mathematicians who have always given very helpful answers to all the questions I had and kindly sent their papers and other material I needed. Hereby, I thank Richard Aron and Eckart Plewnia.

Special thanks go also to all my colleagues from the field of geodesy and all my friends for the support.

List of original publications

1. K. Mikkor, *On polynomials that are weakly uniformly continuous on the unit ball of a Banach space*, Proc. Estonian Acad. Sci. Phys. Math. **55** (2006), 16-23.

2. K. Mikkor, E. Oja, *Uniform factorization for compact sets of weakly compact operators*, Studia Math. (to appear).

3. K. Mikkor, E. Oja, *Uniform factorization for compact sets of operators acting from a Banach space to its dual space*, Acta Comment. Univ. Tartuensis Math. **9** (2005), 87-93.

Chapter 1

Introduction

1.1 Background

Wojtaszczyk [W, page 285] writes: "The factorization theorems ... assert that the operator is actually better that it seems to be. This is useful in both ways; we get stronger information once we prove something weaker or conversely we show that "very" bad behaviour once we show a "moderately" bad one."

Let X and Y be Banach spaces over the same, either real or complex, field $\mathbb{K}$. We denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous linear operators from X to Y, and by $\mathcal{F}(X,Y)$, $\overline{\mathcal{F}}(X,Y)$, $\mathcal{K}(X,Y)$, and $\mathcal{W}(X,Y)$ its subspaces of finite rank, approximable, compact, and weakly compact operators. If $\mathcal{A}$ is $\mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{K}$, $\mathcal{W}$, or $\mathcal{L}$, then $\mathcal{A}_{w^*}(X^*,Y)$ denotes the subspace of $\mathcal{A}(X^*,Y)$ consisting of those operators which are weak*-weak continuous.

In 1971, Johnson [J] proved that, for any fixed p, every approximable operator factors through C_p.

In 1973, basing on Johnson's theorem, Figiel [F, Proposition 3.1] proved that every compact operator factors through some closed subspace of C_p.

Theorem 1.1.1 (Figiel-Johnson). *Let $1 \leq p \leq \infty$. Let X and Y be Banach spaces. If $S \in \mathcal{K}(X,Y)$, then there exist a closed subspace W of C_p and operators $u \in \mathcal{K}(X,W)$ and $v \in \mathcal{K}(W,Y)$ such that $S = v \circ u$.*

Randtke (see [R, Corollary 7]), Terzioğlu (see [Te, page 252]), and Dazord (see [Da, Proposition 5.12]) proved the following factorization results for compact operators between special spaces.

Theorem 1.1.2 (Randtke). *Let X be an $\mathcal{L}_1$-space and let Y be a Banach space. If $S \in \mathcal{K}(X,Y)$, then there exists operators $u \in \mathcal{K}(X,\ell_1)$ and $A \in \mathcal{K}(\ell_1,Y)$ such that $S = A \circ u$.*

Theorem 1.1.3 (Terzioğlu-Dazord). *Let X be an $\mathcal{L}_\infty$-space and let Y be a Banach space. If $S \in \mathcal{K}(X,Y)$, then there exists operators $u \in \mathcal{K}(X, c_0)$ and $A \in \mathcal{K}(c_0, Y))$ such that $S = A \circ u$.*

More than ten years later, in 1987, Graves and Ruess (see [GR2, Theorem 2.1]) extended Theorems 1.1.2 and 1.1.3 from single compact operators to compact subsets of compact operators as follows.

Theorem 1.1.4 (Graves-Ruess). *Let X be an $\mathcal{L}_1$-space (respectively, an $\mathcal{L}_\infty$-space) and let Y be a Banach space. Let $\mathcal{C}$ be a relatively compact subset of $\mathcal{K}(X,Y)$. Then there exist an operator $u \in \mathcal{K}(X, \ell_1)$ (respectively, $u \in \mathcal{K}(X, c_0)$) and a relatively compact subset $\{A_S : S \in \mathcal{C}\}$ of $\mathcal{K}(\ell_1, Y)$ (respectively, of $\mathcal{K}(c_0, Y)$) such that $S = A_S \circ u$ for all $S \in \mathcal{C}$.*

The uniform factorization of compact operators in a general setting was studied by Aron, Lindström, Ruess, and Ryan. In 1999, the following result was obtained (see [ALRR, Theorem 1]) where Z_{FJ} denotes a universal factorization space of Figiel [F] and Johnson [J] (for instance, $Z_{FJ} = (\sum_{W \subset C_p} W)_p$ where W runs through the closed subspaces of C_p for any fixed p; see Section 4.1).

Theorem 1.1.5 (Aron-Lindström-Ruess-Ryan). *Let X and Y be Banach spaces and let $\mathcal{C}$ be a relatively compact subset of $\mathcal{K}_{w^*}(X^*, Y)$. Then there exist operators $u \in \mathcal{K}_{w^*}(X^*, Z_{FJ})$ and $v \in \mathcal{K}(Z_{FJ}, Y)$, and a relatively compact subset $\{A_S : S \in \mathcal{C}\}$ of $\mathcal{K}(Z_{FJ}, Z_{FJ})$ such that $S = v \circ A_S \circ u$ for all $S \in \mathcal{C}$.*

Since, in the setting of Theorem 1.1.4, every single compact operator factors compactly through ℓ_1 (see Theorem 1.1.2) or, respectively, through c_0 (see Theorem 1.1.3), Theorem 1.1.5 easily implies Theorem 1.1.4 (this was observed in [ALRR, Corollary 4]).

Theorem 1.1.4 and Theorem 1.1.5, together with their proofs in [GR2] and [ALRR], do not give much information about mapping properties of the correspondence $S \to A_S$, $S \in \mathcal{C}$. For instance, one does not even have any estimate for $\operatorname{diam}\{A_S : S \in \mathcal{C}\}$.

1.2 Thesis

The main purpose of this thesis is to get quantitative strengthenings of Theorems 1.1.5 and 1.1.4. This will be done in Chapter 4 of the thesis (see Theorems 4.2.1 and 4.4.3). For this end, in Chapter 3, we shall apply a general unified approach, different from [GR2] and [ALRR], and, in our opinion, much easier, to obtain uniform factorization results for compact subsets of compact operators as well as of weakly compact operators. Our idea (see Lemmas 3.1.1 and 3.1.2 and Theorems 3.2.1, 3.2.2, and 3.2.3 in Chapter 3) consists in constructing a mapping $S \to A_S$ from a compact subset $\mathcal{C}$ of weakly compact operators that preserves compact operators, as well as finite rank operators. This mapping is Hölder continuous, being also bijective and having a 1-Lipschitz continuous inverse, and $\operatorname{diam}\{A_S : S \in \mathcal{C}\} = \operatorname{diam} \mathcal{C}$ whenever $0 \in \mathcal{C}$.

Our construction in Chapter 3 will be based on the isometric version of the famous Davis-Figiel-Johnson-Pełczyński factorization lemma [DFJP] due to Lima, Nygaard, and Oja [LNO] that is presented in Chapter 2 (see Lemma 2.2.1). For comparison, let us remark that the technical proof in [GR2] relies on Ruess's characterization [Ru] of relatively compact sets in $\mathcal{K}_{w^*}(X^*, Y)$, and uses Saphar's tensor products machinery [S]. The paper [ALRR] presents two different methods of proof: one being essentially based on Grothendieck's characterization [G] of relatively compact sets in the projective tensor product of Banach spaces, the other – on the Banach-Dieudonné theorem.

In Chapter 5, we prove a uniform factorization result that describes the factorization of compact sets of compact and weakly compact operators acting from X to X^* via Hölder continuous homeomorphisms having Lipschitz continuous inverses. In Chapter 6, results of Chapters 4 and 5 are applied to polynomials. We prove a factorization result (similar to the result in Chapter 5) for compact sets of 2-homogeneous polynomials and quantitative strengthenings of results on polynomials that are weakly uniformly continuous on the unit ball of a Banach space due to Aron, Lindström, Ruess, and Ryan, and to Toma.

Chapters 3 and 4 develop results from [MO1], Chapter 5 is based on [MO2], and main results of Chapter 6 are from [M1] and [MO2].

1.3 Notation

Our notation is rather standard (see, e.g., [LT]).

A Banach space X will always be regarded as a subspace of its bidual X^{**} under the canonical embedding. The closed unit ball of X is denoted by B_X. The closure of a set $A \subset X$ is denoted by $\overline{A}$. The linear span of A is denoted by $\operatorname{span} A$ and the closed convex hull by $\overline{\operatorname{conv}}A$. The circled hull of A is denoted by $\operatorname{circ}A$.

Let us recall that $T \in \mathcal{L}(X^*, Y)$ is weak*-weak continuous if and only if $\operatorname{ran} T^*$, the range of T^*, is contained in X. Recall also that $\mathcal{L}_{w^*}(X^*, Y) = \mathcal{W}_{w^*}(X^*, Y)$ (if $T \in \mathcal{L}(X^*, Y)$ is weak*-weak continuous, then $T(B_{X^*})$ is weakly compact because B_{X^*} is compact in the weak* topology).

Let X and Y be isomorphic Banach spaces. Recall that the Banach-Mazur distance $\mathrm{d_{BM}}(X, Y)$ between spaces X and Y is defined as

$$\mathrm{d_{BM}}(X, Y) := \inf\{\|T\|\|T^{-1}\| \ : T \textit{ is an isomorphism from } X \textit{ onto } Y\}.$$

The $\mathcal{L}_{p,\lambda}$- and $\mathcal{L}_p$-spaces were introduced by Lindenstrauss and Pełczyński [LP] in 1968. Let $1 \le p \le \infty$, $1 \le \lambda < \infty$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space if for every finite-dimensional subspace E of X there is a finite-dimensional subspace F of X containing E and such that

$$\mathrm{d_{BM}}(F, \ell_p^m) \le \lambda,$$

where $m = \dim F$, the dimension of F. A Banach space X is said to be an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some λ. For the basic properties of $\mathcal{L}_{p,\lambda}$-spaces and $\mathcal{L}_p$-spaces, the reader is referred to [LP] and [LR] or [JL, pages 57–60].

We use the symbol ℓ_∞ for the Banach space of null sequences, usually denoted by c_0.

Now we recall the definition of the infinite direct sum of Banach spaces in the sense of ℓ_p for $1 \le p \le \infty$ (see, e.g., [Day, pages 35–36] or [W, page 43]).

Let $(X_\alpha)_{\alpha \in \mathcal{A}}$ be a family of Banach spaces. Let $1 \le p < \infty$. We denote by

$$(\sum_\alpha X_\alpha)_p$$

the Banach space of all functions $f : \mathcal{A} \to \cup_{\alpha \in \mathcal{A}} X_\alpha$ such that $f(\alpha) \in X_\alpha$ whenever $\alpha \in \mathcal{A}$, for which the norm

$$\|f\|_p = (\sum_\alpha \|f(\alpha)\|^p)^{1/p} < \infty.$$

We denote by

$$(\sum_\alpha X_\alpha)_\infty$$

the Banach space of all functions $f : \mathcal{A} \to \cup_{\alpha \in \mathcal{A}} X_\alpha$ such that $f(\alpha) \in X_\alpha$ whenever $\alpha \in \mathcal{A}$, for which the norm

$$\|f\|_\infty = \sup_\alpha \|f(\alpha)\| < \infty,$$

and for every $\varepsilon > 0$ the set

$$\{\alpha \in \mathcal{A} : \|f(\alpha)\| > \varepsilon\}$$

is finite. The space $(\sum_\alpha X_\alpha)_p$ will be called the direct sum of Banach spaces (X_α) in the sense of ℓ_p (where ℓ_∞ denotes the space c_0).

Recall that $(\sum_\alpha X_\alpha)_p$ is reflexive whenever $1 < p < \infty$ and the spaces X_α are all reflexive (see, e.g., [Day, page 36]).

The basic notions and theorems of the theory of Banach spaces and locally convex spaces, that we shall use, can be found, for instance, in [D2], [HHZ1], [HHZ2], [LT], and [SW].

Chapter 2

Isometric version of the Davis-Figiel-Johnson-Pełczyński factorization

All the main results of this thesis are based on, and some of them are formulated in terms of, the famous Davis-Figiel-Johnson-Pełczyński factorization construction and lemma (see [DFJP, pages 313–314]) from 1974. More precisely, we shall not rely on the classical Davis-Figiel-Johnson-Pełczyński factorization, but we shall rely on its isometric version obtained by Lima, Nygaard, and Oja [LNO] in 2000.

Because of the seminal importance of the Lima-Nygaard-Oja isometric version of the Davis-Figiel-Johnson-Pełczyński factorization for the results in our thesis, in this chapter, we shall give a rather detailed treatment of it, basing on [LNO, pages 328-329] and [DFJP, pages 313-314].

Let us notice that monographical treatments of the Davis-Figiel-Johnson-Pełczyński factorization construction are contained, for instance, in [D1, pages 160-162], [D2, page 228], [DU, pages 250-251], [HHZ2, pages 227-228], and [W, pages 51-52]. In particular, the proofs of the reflexivity of the factorization space given, for instance, in [D2] and [W] are different from the original one in [DFJP] (see Remark 2.2.2 below).

2.1 Quantitative version of the Davis-Figiel-Johnson-Pełczyński factorization construction

Let $a > 1$. Let X be a Banach space and let K be a closed absolutely convex subset of B_X, the closed unit ball of X. For each $n \in \mathbb{N}$, put

$$B_n = a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_X.$$

Remark 2.1.1. In the classical Davis-Figiel-Johnson-Pełczyński construction, $a = 4$, that is,

$$B_n = 2^n K + 2^{-n} B_X,$$

and K is a bounded absolutely convex subset of X.

Proposition 2.1.2. *The set B_n is absolutely convex and absorbing.*

The proof of Proposition 2.1.2 is a straightforward easy verification.

Recall that if A is an absorbing subset of a vector space X, then the non-negative real function p_A on X, defined by

$$p_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}, \ x \in X,$$

is called the Minkowski functional of A.

Let us denote the Minkowski functional of the set B_n by $\|\cdot\|_n$ and the norm of the Banach space X by $\|\cdot\|$.

Proposition 2.1.3. *The Minkowski functional $\|\cdot\|_n$ is a norm and it is equivalent to the norm $\|\cdot\|$.*

The proof of Proposition 2.1.3 uses that

$$B_n \subset a^{\frac{n}{2}} B_X + a^{-\frac{n}{2}} B_X \subset a^{\frac{n}{2}} B_X + a^{\frac{n}{2}} B_X = 2a^{\frac{n}{2}} B_X,$$

$$B_X = \{0\} + a^{\frac{n}{2}} a^{-\frac{n}{2}} B_X \subset a^{\frac{n}{2}} a^{\frac{n}{2}} K + a^{\frac{n}{2}} a^{-\frac{n}{2}} B_X = a^{\frac{n}{2}} B_n,$$

and the following well-known fact.

Lemma 2.1.4. *Every bounded absolutely convex absorbing subset A of a normed space X defines a norm on X; this norm is the Minkowski functional p_A of the set A.*

Put

$$\|x\|_K = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{1/2}$$

and define $X_K = \{x \in X : \|x\|_K < \infty\}$.

Proposition 2.1.5. *X_K is a linear subspace of the Banach space X and $(X_K, \|\cdot\|_K)$ is a normed space.*

Proposition 2.1.5 can be easily proved relying on Proposition 2.1.3 and using the triangle inequality in the sequence space ℓ_2.

The function f described in the next result was introduced in [LNO, page 328] as important ingredient of the isometric factorization construction.

Proposition 2.1.6. *Function $f : (1, \infty) \to (0, \infty)$,*

$$f(a) = \Big(\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2}\Big)^{\frac{1}{2}}, \ a \in (1, \infty),$$

is continuous, strictly decreasing, $\lim_{a \to 1+} f(a) = \infty$, and $\lim_{a \to \infty} f(a) = 0$.

Proof. Let $a \in (1, \infty)$. Then

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2} = \sum_{n=1}^{\infty} \frac{a^n}{a^{2n} + 2a^n + 1} \leq \sum_{n=1}^{\infty} \frac{a^n}{a^{2n}} = \sum_{n=1}^{\infty} \frac{1}{a^n} < \infty,$$

hence the series defining the function f converges. Therefore $f : (1, \infty) \to (0, \infty)$.

Let $a_1, a_2 \in (1, \infty)$ be such that $a_1 < a_2$. For every $n \in \mathbb{N}$,

$$\frac{a_1^n}{(a_1^n + 1)^2} > \frac{a_2^n}{(a_2^n + 1)^2}. \tag{1}$$

Indeed, the inequality (1) holds if and only if

$$\frac{(a_2^n + 1)^2}{a_2^n} > \frac{(a_1^n + 1)^2}{a_1^n},$$

i.e.,

$$a_2^n + \frac{1}{a_2^n} > a_1^n + \frac{1}{a_1^n},$$

i.e.,

$$a_2^n - a_1^n > \frac{1}{a_1^n} - \frac{1}{a_2^n},$$

i.e.,

$$a_2^n - a_1^n > \frac{a_2^n - a_1^n}{a_1^n a_2^n},$$

which holds because $a_1^n a_2^n > 1$ and $a_2^n - a_1^n > 0$. Hence the function f is strictly decreasing.

Let us fix $a \in (1, \infty)$ and let $a_0 \in (1, a)$. Then $a \in (a_0, \infty)$. Let us show that f is continuous in (a_0, ∞). Since for every $n \in \mathbb{N}$,

$$\frac{a^n}{(a^n + 1)^2} \leq \frac{1}{a^n} \leq \frac{1}{a_0^n}, \quad a \in (a_0, \infty),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{a_0^n} < \infty,$$

the series, defining f, converges uniformly in (a_0, ∞). Consequently the function f is continuous in (a_0, ∞) and therefore also in a.

Using then that

$$0 \leq f(a) = \Big(\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2} \Big)^{\frac{1}{2}} \leq \Big(\sum_{n=1}^{\infty} \frac{1}{a^n} \Big)^{\frac{1}{2}}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{a^n} = \frac{1}{a - 1} \to 0$$

whenever $a \to \infty$, we have $f(a) \to_{a \to \infty} 0$.

Let $a \in (1, \infty)$ and notice that

$$\int_1^{\infty} \frac{a^t}{(a^t + 1)^2} dt \leq (f(a))^2.$$

Denoting $a^t = x$, we have

$$\begin{aligned} \int_1^{\infty} \frac{a^t}{(a^t + 1)^2} dt &= \int_a^{\infty} \frac{1}{(x + 1)^2 \ln a} dx = \frac{1}{\ln a} \int_a^{\infty} \frac{1}{(x + 1)^2} dx \\ &= \frac{1}{(1 + a) \ln a} \to \infty \end{aligned}$$

whenever $a \to 1+$. Therefore $f(a) \to_{a \to 1+} \infty$. □

Corollary 2.1.7. *There exists exactly one number $a \in (1, \infty)$ such that $f(a) = 1$ or, equivalently,*

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2} = 1.$$

Proof. By Proposition 2.1.6 there exist $a_0 > 1$ and $b_0 > a_0$ such that $f(a_0) \geq 1$ and $f(b_0) \leq 1$, respectively. By the Bolzano-Cauchy theorem, the function f attains each value in $[f(b_0), f(a_0)]$. Hence there exists $a \in [a_0, b_0] \subset (1, \infty)$ such that $f(a) = 1$. □

Remark 2.1.8. By [LNO] a "good" estimate of a in Corollary 2.1.7 is $\exp(4/9)$. Hence
$$\frac{1}{4} + \frac{1}{\ln a} \approx \frac{5}{2}.$$

2.2 Isometric version of the Davis-Figiel-Johnson-Pełczyński factorization lemma

Let a be the unique solution of the equation
$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1, \ a > 1$$
(see Corollary 2.1.7).

Lemma 2.2.1 (Davis-Figiel-Johnson-Pełczyński-Lima-Nygaard-Oja). *Let X be a Banach space and let K be a closed absolutely convex subset of B_X, the closed unit ball of X. Let X_K be the normed space described in Section 2.1, let B_{X_K} be the closed unit ball of X_K, and let $J_K : X_K \to X$ denote the identity embedding. Then the following holds.*

(i) $X_K = (X_K, \|\cdot\|_K)$ *is a Banach space and* $\|J_K\| \le 1$.

(ii) $K \subset B_{X_K} \subset B_X$.

(iii) *If* $x \in K$, *then* $\|x\|_K^2 \le (1/4 + 1/\ln a)\|x\|$.

(iv) *The X-norm and X_K-norm topologies coincide on K.*

(v) J_K^{**} *is injective.*

(vi) J_K *is a compact operator if and only if K is a compact subset of X; in this case X_K is separable.*

(vii) X_K *is reflexive if and only if K is a weakly compact subset of X.*

Proof. (i) Let $X_n = (X, \|\cdot\|_n)$. By Proposition 2.1.3 the norm $\|\cdot\|_n$ is equivalent to the norm $\|\cdot\|$ of the space X and therefore X_n is a Banach space. Recall that (see Section 1.3 or [W, page 43])
$$\Big(\sum_n X_n\Big)_2 = \{(x_1, x_2, ...) : \ x_n \in X_n, \ \sum_{n=1}^{\infty} \|x_n\|_n^2 < \infty\}$$
is a Banach space with
$$\|(x_n)\| = \Big(\sum_{n=1}^{\infty} \|x_n\|_n^2\Big)^{\frac{1}{2}}.$$

Let us define a mapping $\varphi : \ X_K \to (\sum_n X_n)_2$ by
$$\varphi x = (x, \ x, \ ...), \ x \in X_K.$$

Since for every $x \in X_K$

$$\|x\|_K^2 = \sum_{n=1}^{\infty} \|x\|_n^2 < \infty,$$

we have $\varphi x \in (\sum_n X_n)_2$ and

$$\|\varphi x\| = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{\frac{1}{2}} = \|x\|_K.$$

Since φ is also linear, we get that φ is a linear isometry from the normed space X_K into the Banach space $(\sum_n X_n)_2$.

We know that $\varphi(X_K)$ is a subspace of the space $(\sum_n X_n)_2$. Consider a sequence $(x_n,\ x_n,\ \ldots),\ \ x_n \in X_K$. Let us assume that

$$(x_n,\ x_n,\ \ldots) \to_n (a_1,\ a_2,\ \ldots) \in \Big(\sum_n X_n\Big)_2.$$

Then,

$$x_n \to_n a_1,$$

$$x_n \to_n a_2,$$

$$\ldots\ .$$

Hence $a_1 = a_2 = \ldots =: x$ and $x \in X_K$, i.e., $\sum_{n=1}^{\infty} \|x\|_n^2 < \infty$, because $(x,\ x,\ \ldots) \in (\sum_n X_n)_2$. Consequently $\varphi(X_K)$ is a closed subspace of $(\sum_n X_n)_2$. Now, $\varphi(X_K)$ is a Banach space. Since X_K is isometrically isomorphic to the Banach space $\varphi(X_K)$, we have that X_K is a Banach space.

Let $J_K :\ X_K \to X$ be the identity embedding. Let us prove that

$$\|J_K x\| \leq \|x\|_K,\ x \in X_K,$$

i.e.,

$$\|x\|_K \geq \|x\|,\ x \in X_K.$$

Using that $K \subset B_X$, B_X is convex, and

$$\frac{a^{\frac{n}{2}}}{a^{\frac{n}{2}} + a^{-\frac{n}{2}}} + \frac{a^{-\frac{n}{2}}}{a^{\frac{n}{2}} + a^{-\frac{n}{2}}} = 1,$$

we have for every $n \in \mathbb{N}$

$$\frac{a^{\frac{n}{2}}}{a^n + 1} B_n = \frac{1}{a^{\frac{n}{2}} + a^{-\frac{n}{2}}} (a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_X) \subset B_X.$$

Hence

$$\|x\|_n \geq \frac{a^{\frac{n}{2}}}{a^n+1} p_{B_X}(x) = \frac{a^{\frac{n}{2}}}{a^n+1}\|x\|$$

and

$$\|x\|_K = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{\frac{1}{2}} \geq \Big(\sum_{n=1}^{\infty} \Big(\frac{a^{\frac{n}{2}}}{a^n+1}\Big)^2 \|x\|^2\Big)^{\frac{1}{2}}$$

$$= \Big(\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2}\Big)^{\frac{1}{2}} \|x\| = \|x\|.$$

(ii) Let $x \in K$. Since $x \in B_X$, we have

$$x \in \frac{1}{a^{\frac{n}{2}} + a^{-\frac{n}{2}}} B_n$$

and

$$\|x\|_n \leq \frac{1}{a^{\frac{n}{2}} + a^{-\frac{n}{2}}} = \frac{a^{\frac{n}{2}}}{a^n+1}$$

for every $n \in \mathbb{N}$. Hence

$$\|x\|_K = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{\frac{1}{2}} \leq \Big(\sum_{n=1}^{\infty} \Big(\frac{a^{\frac{n}{2}}}{a^n+1}\Big)^2\Big)^{\frac{1}{2}} = 1.$$

Consequently $K \subset B_{X_K}$. The inclusion $B_{X_K} \subset B_X$ is clear from (i).

(iii) We include the proof from [LNO, page 335]. Let $x \in K$, $x \neq 0$. Then we have

$$a^{n/2}x + a^{-n/2}\frac{x}{\|x\|} \in B_n,$$

so that

$$\|x\|_K^2 \leq \sum_{n=1}^{\infty} \frac{1}{(a^{n/2} + a^{-n/2}\|x\|^{-1})^2} = \|x\| \sum_{n=1}^{\infty} \frac{a^n\|x\|}{(a^n\|x\|+1)^2}.$$

Let $h(t) = a^t\|x\|/(a^t\|x\|+1)^2$, $1 \leq t < \infty$. The graph of h has a bell-shaped form and $\max h(t) = 1/4$. Let $k \in \mathbb{N}$ be such that

$$h(1) \leq h(2) \leq \cdots \leq h(k-1) \leq h(k) \geq h(k+1) \geq \cdots.$$

Then

$$\frac{\|x\|_K^2}{\|x\|} \leq \sum_{n=1}^{\infty} h(n) \leq h(k) + \int_1^{\infty} h(t)\,dt$$

$$\leq \frac{1}{4} + \frac{1}{\ln a}\int_{1+a\|x\|}^{\infty} \frac{du}{u^2}$$

$$= \frac{1}{4} + \frac{1}{\ln a}\Big(\frac{1}{1+a\|x\|}\Big) \leq \frac{1}{4} + \frac{1}{\ln a}.$$

(iv) Let $x, y \in K$. Then $(x-y)/2 \in K$. By (i) and (iii),

$$\|x-y\|^2 = \|J_K(x-y)\|^2 \le \|x-y\|_K^2 \le \Big(\frac{1}{2} + \frac{2}{\ln a}\Big)\|x-y\|.$$

This proves (iv).

(v) For proving that $J_K^{**} : X_K^{**} \to X^{**}$ is injective, we shall use the fact that the operator φ^{**} is injective, where φ is the linear isometric operator defined in (i) of the proof. (Actually φ^{**} is also isometric, since φ is isometric, but we only need the injectivity of φ^{**}.)

Recall that

$$\varphi^* : \Big(\Big(\sum_n X_n\Big)_2\Big)^* \to X_K^*$$

and

$$\varphi^{**} : X_K^{**} \to \Big(\Big(\sum_n X_n\Big)_2\Big)^{**}.$$

Similarly to the proof of $\ell_2^* = \ell_2$ and $\ell_2^{**} = \ell_2$, one can prove (and this is a well-known fact; see, e.g., [W, page 44]) that

$$\Big(\Big(\sum_n X_n\Big)_2\Big)^* = \Big(\sum_n X_n^*\Big)_2$$

and

$$\Big(\Big(\sum_n X_n\Big)_2\Big)^{**} = \Big(\sum_n X_n^{**}\Big)_2.$$

Let $j_n : X_K \to X_n$ be the natural embedding for every $n \in \mathbb{N}$. With this notation, $\varphi : X_K \to (\sum_n X_n)_2$ is defined by

$$\varphi x = (j_1 x,\ j_2 x,\ \ldots),\ x \in X_K.$$

Using the definition of a dual operator, it is immediate that for every $z^* = (x_n^*) \in ((\sum_n X_n)_2)^* = (\sum_n X_n^*)_2$ and for every $x \in X_K$

$$(\varphi^* z^*)(x) = \sum_{n=1}^{\infty} (j_n^* x_n^*)(x).$$

Now, the series $\sum_{n=1}^{\infty} j_n^* x_n^*$ converges in X_K^*. Indeed, for $q \in \mathbb{N}$, we have

$$
\begin{aligned}
\|\sum_{n=p}^{p+q} j_n^* x_n^*\| &= \sup_{\|x\|_K \le 1} \Big| \Big(\sum_{n=p}^{p+q} j_n^* x_n^* \Big)(x) \Big| \le \sup_{\|x\|_K \le 1} \sum_{n=p}^{p+q} |x_n^*(j_n x)| \\
&\le \sup_{\|x\|_K \le 1} \sum_{n=p}^{p+q} \|x_n^*\| \|j_n x\| \le \sup_{\|x\|_K \le 1} \Big(\sum_{n=p}^{p+q} \|j_n x\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=p}^{p+q} \|x_n^*\|^2 \Big)^{\frac{1}{2}} \\
&\le \sup_{\|x\|_K \le 1} \Big(\sum_{n=1}^{\infty} \|x\|_n^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=p}^{p+q} \|x_n^*\|^2 \Big)^{\frac{1}{2}} \\
&= \sup_{\|x\|_K \le 1} \|x\|_K \Big(\sum_{n=p}^{p+q} \|x_n^*\|^2 \Big)^{\frac{1}{2}} = \Big(\sum_{n=p}^{p+q} \|x_n^*\|^2 \Big)^{\frac{1}{2}} \to_p 0,
\end{aligned}
$$

because $(x_n^*) \in (\sum_n X_n^*)_2$.

Thus for every $z^* = (x_n^*) \in ((\sum_n X_n)_2)^*$,

$$\varphi^* z^* = \sum_{n=1}^{\infty} j_n^* x_n^*$$

in X_K^*. This implies that

$$\varphi^{**} x^{**} = (j_1^{**} x^{**},\ j_2^{**} x^{**},\ ...),\ x^{**} \in X_K^{**}.$$

It is clear that $J_K = I_n j_n,\ n \in \mathbb{N}$, where $I_n :\ X_n \to X$ is the identity embedding. By Proposition 2.1.3, $I_n :\ X_n \to X$ is an isomorphism. Thus I_n^{**} is injective.

Now, the injectivity of $J_K^{**} :\ X_K^{**} \to X^{**}$ follows from the equalities

$$J_K^{**} = I_n^{**} j_n^{**},\ n \in \mathbb{N};$$

$$\varphi^{**} x^{**} = (j_1^{**} x^{**},\ j_2^{**} x^{**},\ ...),\ x^{**} \in X_K^{**};$$

and from the injectivity of I_n^{**}, j_n^{**}, and φ^{**}.

(vi) The compactness of J_K means that B_{X_K} is relatively compact in X. If B_{X_K} is relatively compact, then K is compact, because $K \subset B_{X_K}$ (see (ii)). On the other hand, since $\|\cdot\|_n$ is the Minkowski functional of the set $a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_X$, we have

$$\{x \in X : \|x\|_n < 1\} \subset a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_X,\ n \in \mathbb{N}.$$

Since $\|x\|_n \neq 0$ for all $n \in \mathbb{N}$ whenever $\|x\| \neq 0$, this clearly implies that

$$B_{X_K} = \{x \in X : \sum_{n=1}^{\infty} \|x\|_n^2 \le 1\} \subset a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_X,\ n \in \mathbb{N}.$$

Consequently, for each $\varepsilon > 0$, choosing n big enough, $a^{\frac{n}{2}}K$ is an ε-net for B_{X_K}. Hence, if K is compact, then B_{X_K} is relatively compact.

By (v), $J_K^*(X^*)$ is norm dense in X_K^*. If J_K is a compact operator, then also J_K^* is. But compact operators have separable ranges. Hence, X_K^* is separable, implying the separability of X_K.

(vii) Let X_K be reflexive. Then B_{X_K} is weakly compact in X_K. By (i) $J_K: X_K \to X$ is continuous and consequently $J_K : (X_K, \sigma(X_K, X_K^*)) \to (X, \sigma(X, X^*))$ is continuous. Thereby B_{X_K} is weakly compact in X. Then, by (ii), the weak closure of K is compact in the weak topology. Since the norm and weak closures of a convex set coincide, K is weakly compact in X.

Now we shall prove that X_K is reflexive whenever K is weakly compact in X.

We begin with the general observation that always

$$(J_K^{**})^{-1}(X) = X_K.$$

Indeed, since $J_K^{**}|_{X_K} = J_K$, we have

$$\begin{aligned}(J_K^{**})^{-1}(X) &= \{x^{**} \in X_K^{**} : \ J_K^{**}(x^{**}) \in X\} \\ &\supset \{x \in X_K : \ J_K x \in X\} = X_K.\end{aligned}$$

Let $x^{**} \in X_K^{**}$ be such that $J_K^{**}x^{**} \in X$. Let us denote $x = J_K^{**}x^{**}$. Then using the notation of the proof of (v), we have $x = I_n^{**}(j_n^{**}x^{**})$ for every $n \in \mathbb{N}$. Therefore for every $n \in \mathbb{N}$

$$j_n^{**}x^{**} = (I_n^{**})^{-1}x = (I_n^{-1})^{**}x = I_n^{-1}x = x.$$

Recall that

$$(j_1^{**}x^{**},\ j_2^{**}x^{**},\ \ldots) = \varphi^{**}x^{**} \in \Big(\sum_n X_n^{**}\Big)_2.$$

Hence

$$\sum_{n=1}^{\infty} \|j_n^{**}x^{**}\|_n^2 < \infty,$$

i.e.,

$$\sum_{n=1}^{\infty} \|x\|_n^2 < \infty,$$

i.e., $x \in X_K$. Consequently

$$x = J_K x = J_K^{**}x.$$

On the other hand, $x = J_K^{**}x^{**}$. Since J_K^{**} is injective by (v), we have $x^{**} = x$. Hence $x^{**} \in X_K$. This proves that

$$(J_K^{**})^{-1}(X) \subset X_K.$$

We also need the following general observation that

$$J_K^{**}(B_{X_K^{**}}) \subset \overline{B_{X_K}}^{\sigma(X^{**},\, X^*)}$$

in X^{**}. Indeed, by Goldstine's theorem

$$B_{X_K^{**}} = \overline{B_{X_K}}^{\sigma(X_K^{**},\, X_K^*)}.$$

Using the fact that $J_K^{**} : (X_K^{**}, \sigma(X_K^{**}, X_K^*)) \to (X^{**}, \sigma(X^{**}, X^*))$ is continuous and $J_K^{**}(B_{X_K}) = B_{X_K}$, we get

$$\begin{aligned} J_K^{**}(B_{X_K^{**}}) &= J_K^{**}(\overline{B_{X_K}}^{\sigma(X_K^{**},\, X_K^*)}) \subset \overline{J_K^{**}(B_{X_K})}^{\sigma(X^{**},\, X^*)} \\ &= \overline{B_{X_K}}^{\sigma(X^{**},\, X^*)}. \end{aligned}$$

Let K be weakly compact in X. Since $K \subset X$, the $\sigma(X, X^*)$- and $\sigma(X^{**}, X^*)$-topologies coincide on K. The set K is $\sigma(X, X^*)$-compact and also $\sigma(X^{**}, X^*)$-compact.

Recall that (see the proof of (vi))

$$B_{X_K} \subset a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_X,$$

for every $n \in \mathbb{N}$. Since $B_X \subset B_{X^{**}}$, we have

$$B_{X_K} \subset a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_{X^{**}}.$$

We know that K is $\sigma(X^{**}, X^*)$-compact and therefore $a^{\frac{n}{2}}K$ also is. By Alaoglu's theorem $a^{-\frac{n}{2}} B_{X^{**}}$ is $\sigma(X^{**}, X^*)$-compact. Hence $a^{\frac{n}{2}}K + a^{-\frac{n}{2}} B_{X^{**}}$ is $\sigma(X^{**}, X^*)$-compact and therefore $\sigma(X^{**}, X^*)$-closed. Consequently

$$\overline{B_{X_K}}^{\sigma(X^{**},\, X^*)} \subset a^{\frac{n}{2}} K + a^{-\frac{n}{2}} B_{X^{**}}$$

for every $n \in \mathbb{N}$. Using that $a^{\frac{n}{2}}K \subset X$, we have

$$J_K^{**}(B_{X_K^{**}}) \subset \overline{B_{X_K}}^{\sigma(X^{**},\, X^*)} \subset X + a^{-\frac{n}{2}} B_{X^{**}}$$

for every $n \in \mathbb{N}$. Hence

$$J_K^{**}(B_{X_K^{**}}) \subset \bigcap_{n=1}^{\infty} (X + a^{-\frac{n}{2}} B_{X^{**}}) = \overline{X} = X$$

in X^{**}. Consequently

$$J_K^{**}(X_K^{**}) \subset X$$

and

$$X_K^{**} \subset (J_K^{**})^{-1}(X).$$

Using that $(J_K^{**})^{-1}(X) = X_K$, we have

$$X_K^{**} \subset (J_K^{**})^{-1}(X) = X_K \subset X_K^{**}.$$

Hence X_K is reflexive. $\square$

Remark 2.2.2. Probably the shortest proof of the reflexivity of X_K is given in [D2, page 228]. It relies on Grothendieck's lemma (see, e.g., [D2, page 227]) to show that B_{X_K} is weakly compact in X, and uses the fact that on B_{X_K} the weak topologies from X and X_K coincide. Probably the most elementary proof is contained in [W, page 52]. It shows that the subspace $\varphi(X_K)$ of $(\sum_n X_n)_2$ is reflexive relying on the fact that

$$B_{X_n^{**}} = a^{\frac{n}{2}}K + a^{-\frac{n}{2}}B_{X^{**}}.$$

2.3 Isometric version of the Davis-Figiel-Johnson-Pełczyński factorization theorem

In 1973, Figiel [F] wrote: "It is not known whether every compact operator can be factorized through a reflexive space." Recall that the Figiel-Johnson theorem (see Theorem 1.1.1) gives the factorization of a compact operator through a reflexive space. Factorization of weakly compact operators through reflexive spaces was proved by Davis, Figiel, Johnson, and Pełczyński [DFJP] in 1974. Recall that if $S = u \circ v$ then $\|S\| \leq \|u\|\|v\|$. The Davis-Figiel-Johnson-Pełczyński factorization theorem enables to choose operators v and u such that $\|u\|\|v\| \leq 4\|S\|$ (see, e.g., [W, page 51]). In 1980, Pietsch [P, 2.4.3] essentially established the following isometric version of the Davis-Figiel-Johnson-Pełczyński factorization theorem. We shall present its proof, due to Lima, Nygaard, and Oja [LNO], which is based on the isometric version of the Davis-Figiel-Johnson-Pełczyński factorization (see Lemma 2.2.1).

Theorem 2.3.1. *Let $X \neq \{0\}$ and Y be Banach spaces. For every weakly compact operator $S : Y \to X$ there exist a reflexive Banach space Z and weakly compact operators $v : Y \to Z$ and $u : Z \to X$ such that $S = u \circ v$, $\|v\| = \|S\|$, and $\|u\| = 1$.*

Proof. Let $S \in \mathcal{W}(Y, X)$. We clearly may assume that $S \neq 0$. Let

$$K = \overline{\frac{1}{\|S\|} S(B_Y)}.$$

Then $K \subset B_X$ is closed, absolutely convex, and weakly compact in X.

Let us denote the space X_K in Lemma 2.2.1 by Z. Recall that, by (i) and (vii) of Lemma 2.2.1, Z is a reflexive Banach space.

Define operator $v : Y \to Z$ by $vy = Sy$, $y \in Y$. Then v is a linear operator. Let u denote the identity embedding J_K in Lemma 2.2.1. Then $u \in \mathcal{L}(Z, X)$ and $\|u\| \leq 1$. Moreover, clearly, $S = u \circ v$.

Since $K \subset B_Z$, we have

$$S(B_Y) \subset \|S\| B_Z$$

and therefore

$$\begin{aligned} \|v\| &= \sup_{y \in B_Y} \|vy\| = \sup_{y \in B_Y} \|Sy\| = \sup_{z \in S(B_Y)} \|z\| \leq \sup_{z \in \|S\| B_Z} \|z\| \\ &= \sup_{z \in B_Z} \|S\| \|z\| = \|S\| \sup_{z \in B_Z} \|z\| = \|S\|. \end{aligned}$$

Hence $v \in \mathcal{L}(Y, Z)$ and

$$\|S\| = \|u \circ v\| \leq \|u\| \|v\| \leq 1 \|S\| = \|S\|.$$

Therefore

$$\|S\| = \|u\| \|v\|.$$

Recall that $\|S\| \neq 0$, hence $\|u\| \neq 0$, and

$$\frac{1}{\|u\|} \|S\| = \|v\| \leq \|S\|.$$

Consequently $\|u\| = 1$ and $\|v\| = \|S\|$.

Note that $v(B_Y)$ and B_Z are relatively weakly compact in Z (because Z is reflexive). Hence $u(B_Z)$ is relatively weakly compact in X. Thus v and u are weakly compact operators. $\square$

Chapter 3

Uniform factorization for compact sets of weakly compact operators

Main results of this chapter are published in [MO1].

3.1 Main factorization lemmas for compact subsets of weakly compact operators

To get the main results of this thesis, in particular, the quantitative strengthenings of the Graves-Ruess theorem (Theorem 1.1.4) and the Aron-Lindström-Ruess-Ryan theorem (Theorem 1.1.5), we shall rely on Lemmas 3.1.1 and 3.1.2 below. In these Lemmas we construct a mapping $S \to A_S$ from a compact subset $\mathcal{C}$ of weakly compact operators that preserves compact operators, as well as finite rank operators. This mapping is Hölder continuous, being also bijective and having a 1-Lipschitz continuous inverse, and $\operatorname{diam}\{A_S : S \in \mathcal{C}\} = \operatorname{diam} \mathcal{C}$ whenever $0 \in \mathcal{C}$. Our construction will be based on Lemma 2.2.1, which is the isometric version of the Davis-Figiel-Johnson-Pełczyński factorization lemma [DFJP] due to Lima, Nygaard, and Oja [LNO].

From Chapter 2, let us recall that $a > 1$ is a unique solution of the equation

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2} = 1.$$

If Y is a Banach space and K is a weakly compact absolutely convex subset of B_Y, then Y_K is the factorization space described in Section 2.1. Recall that Y_K is a linear subspace of Y and $J_K : Y_K \to Y$ denotes the identity embedding.

Lemma 3.1.1. *Let X and Y be Banach spaces. Let $\mathcal{C}$ be a compact subset of $\mathcal{L}_{w^*}(X^*, Y)$. Then there exist a weakly compact absolutely convex subset K of B_Y, which is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, and a linear mapping $\Phi : \text{span } \mathcal{C} \to \mathcal{L}_{w^*}(X^*, Y_K)$ such that $S = J_K \circ \Phi(S)$, for all $S \in \text{span } \mathcal{C}$, and $\|J_K\| = 1$. Moreover, if $S \in \text{span } \mathcal{C}$, then*

(i) *S has finite rank if and only if $\Phi(S)$ has finite rank,*

(ii) *S is compact if and only if $\Phi(S)$ is compact.*

The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying

$$\|S - T\| \leq \|\Phi(S) - \Phi(T)\|$$

$$\leq \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S - T\|^{1/2}\Big\}, \quad S, T \in \mathcal{C} \cup \{0\},$$

where

$$\mathrm{d} = \ \text{diam } \mathcal{C} \cup \{0\}.$$

In particular, if $-S \in \mathcal{C}$ for some $S \in \mathcal{C}$, then

$$\|\Phi(S)\| \leq \min\Big\{ \frac{\mathrm{d}}{2}, \Big(\frac{\mathrm{d}}{2}\Big)^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S\|^{1/2}\Big\}.$$

Proof. Let

$$K = \overline{\text{conv}}\Big\{\frac{1}{\mathrm{d}}(S - T)x^* : \ S, T \in \mathcal{C} \cup \{0\}, \ x^* \in B_{X^*}\Big\}.$$

Then K is contained in B_Y, K is closed and absolutely convex, hence weakly closed.

For proving that K is weakly compact, let us fix an arbitrary $\varepsilon > 0$. We shall find a weakly compact subset K_ε of Y such that $K \subset K_\varepsilon + \varepsilon B_Y$. Then the weak compactness of K will be immediate from Grothendieck's lemma (see, e.g., [D2, page 227]). Let $\{U_1, ..., U_n\}$ be an ε-net in the compact subset

$$\Big\{\frac{1}{\mathrm{d}}(S - T) : \ S, T \in \mathcal{C} \cup \{0\}\Big\}$$

of $\mathcal{L}_{w^*}(X^*, Y)$. Denoting by K_ε the closed convex hull of the weakly compact set $\overline{U_1(B_{X^*})} \cup ... \cup \overline{U_n(B_{X^*})}$, which is weakly compact by a classical result of Krein and Šmulian, it is straightforward to verify that $K \subset K_\varepsilon + \varepsilon B_Y$ as desired.

If $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then K_ε is compact (by a theorem of Mazur), implying that also K is compact.

Let the Banach space Y_K and the identity embedding $J_K : Y_K \to Y$ with $\|J_K\| \leq 1$ be given as in Lemma 2.2.1. Since $K \subset B_{Y_K}$,

$$\|J_K\| = \sup_{z \in B_{Y_K}} \|z\| \geq \sup_{z \in K} \|z\| \geq \frac{1}{\mathrm{d}} \sup_{S,T \in \mathcal{C} \cup \{0\}} \|S - T\| = 1.$$

Hence $\|J_K\| = 1$.

Let $S \in \operatorname{span} \mathcal{C}$. Then

$$\begin{aligned} \operatorname{ran} S &\subset \operatorname{span} \{Sx^* : \; S \in \mathcal{C}, \; x^* \in X^*\} \\ &\subset \operatorname{span} \{(S - T)x^* : \; S, T \in \mathcal{C} \cup \{0\}, \; x^* \in B_{X^*}\} \\ &\subset \operatorname{span} K \subset \operatorname{span} B_{Y_K} = Y_K. \end{aligned}$$

This permits us to define $\Phi(S) : \; X^* \to Y_K$ by

$$\Phi(S)x^* = Sx^*, \; x^* \in X^*.$$

Since $\Phi(S)$ is algebraically the same operator as S, we see that $\Phi(S)$ is linear, and $S = J_K \circ \Phi(S)$.

Let $S, T \in \mathcal{C} \cup \{0\}$. Then $\mathrm{d}^{-1}(S - T)x^* \in K \subset B_{Y_K}$ for all $x^* \in B_{X^*}$. Hence

$$\|\Phi(S - T)\| = \sup_{x^* \in B_{X^*}} \|(S - T)x^*\|_K \leq \mathrm{d}, \; S, T \in \mathcal{C} \cup \{0\}. \tag{2}$$

This implies, in particular, that $\|\Phi(S)\| < \infty$ for all $S \in \operatorname{span} \mathcal{C}$. Every $\Phi(S)$, $S \in \operatorname{span} \mathcal{C}$, is also weak*-weak continuous because, $J_K^*(Y^*)$ being norm dense in Y_K^* (since J_K^{**} is injective by Lemma 2.2.1), we have

$$\begin{aligned} (\Phi(S))^*(Y_K^*) &= (\Phi(S))^*(\overline{J_K^*(Y^*)}) \subset \overline{((\Phi(S))^* \circ J_K^*)(Y^*)} \\ &= \overline{S^*(Y^*)} \subset \overline{X} = X. \end{aligned}$$

Consequently, Φ is a linear mapping from $\operatorname{span} \mathcal{C}$ to $\mathcal{L}_{w^*}(X^*, Y_K)$.

Since $S \in \operatorname{span} \mathcal{C}$ and $\Phi(S)$ are algebraically the same operators, (i) clearly holds. Condition (ii) holds by Lemma 2.2.1, (iv), (and by the linearity of Φ) because $\mathrm{d}^{-1}S(B_{X^*}) \subset K$ for all $S \in \mathcal{C}$.

Finally, let $S, T \in \mathcal{C} \cup \{0\}$. Then, by (2),

$$\|S - T\| \leq \|J_K\| \|\Phi(S - T)\| = \|\Phi(S) - \Phi(T)\| \leq \mathrm{d}.$$

Since $\mathrm{d}^{-1}(S-T)x^* \in K$ for all $x^* \in B_{X^*}$, using 2.2.1, (iii), we also have

$$\begin{aligned}\|\Phi(S)-\Phi(T)\| &= \sup_{x^*\in B_{X^*}} \|(S-T)x^*\|_K \\ &\le \mathrm{d}^{1/2}(1/4+1/\ln a)^{1/2} \sup_{x^*\in B_{X^*}} \|(S-T)x^*\|^{1/2} \\ &= \mathrm{d}^{1/2}(1/4+1/\ln a)^{1/2}\|S-T\|^{1/2}.\end{aligned}$$

If, in particular, $S, -S \in \mathcal{C}$, then the desired estimate for the norm of $\Phi(S) = (\Phi(S) - \Phi(-S))/2$ immediately follows from the above. $\square$

Lemma 3.1.2. *Let X and Y be Banach spaces. Let $\mathcal{C}$ be a compact subset of $\mathcal{L}_{w^*}(X^*, Y)$. Then there exist a reflexive Banach space Z, a norm one operator $J \in \mathcal{L}_{w^*}(X^*, Z)$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}(Z, Y)$ satisfying conditions* (i) *and* (ii) *of Lemma 3.1.1 such that $S = \Phi(S) \circ J$, for all $S \in \operatorname{span} \mathcal{C}$. Moreover, $Z = X_K^*$ and $J = J_K^*$ for some weakly compact absolutely convex subset K of B_X, and if $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then Z is separable and $J \in \mathcal{K}_{w^*}(X^*, Z)$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Proof. Applying Lemma 3.1.1 to the compact subset $\mathcal{C}^* = \{S^* : S \in \mathcal{C}\}$ of $\mathcal{L}_{w^*}(Y^*, X)$, we can find a weakly compact absolutely convex subset K of B_X, which is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$ (since S^* is compact if and only if S is). We can also find a linear mapping

$$\Psi : \operatorname{span} \mathcal{C}^* \to \mathcal{L}_{w^*}(Y^*, X_K)$$

satisfying the conclusions of Lemma 3.1.1 such that $S^* = J_K \circ \Psi(S^*)$, for all $S \in \operatorname{span} \mathcal{C}$, and we know that $\|J_K\| = 1$.

Let $Z = X_K^*$ and $J = J_K^*$. Then Z is reflexive by Lemma 2.2.1, (vii), $\|J\| = 1$, and $J \in \mathcal{L}_{w^*}(X^*, Z)$ since $\operatorname{ran} J_K^{**} \subset X$ because Z is reflexive. The reflexive space Z is separable and the operator J is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$ (see Lemma 2.2.1, (vi)).

Let us define $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}(Z, Y)$ by

$$\Phi(S) = (\Psi(S^*))^*, \ S \in \operatorname{span} \mathcal{C}.$$

The properties of Ψ clearly imply that Φ is a linear mapping satisfying conditions (i) and (ii) of Lemma 3.1.1. If $S \in \operatorname{span} \mathcal{C}$, then $S^{**} = S$ (because $S^* \in \mathcal{L}_{w^*}(Y^*, X)$) and therefore $S = (J_K \circ \Psi(S^*))^* = \Phi(S) \circ J$. Since $\|S - T\| = \|S^* - T^*\|$ and $\|\Phi(S) - \Phi(T)\| = \|\Psi(S^*) - \Psi(T^*)\|$, for $S, T \in \operatorname{span} \mathcal{C}$, the mapping Φ restricted to $\mathcal{C} \cup \{0\}$ obviously satisfies the conclusions of Lemma 3.1.1.

$\square$

Remark 3.1.3. Observe that $\operatorname{diam}\Phi(\mathcal{C}\cup\{0\}) = \operatorname{diam}\mathcal{C}\cup\{0\}$ in Lemmas 3.1.1 and 3.1.2.

3.2 Quantitative versions of the uniform factorization for compact sets of weakly compact operators

For Banach spaces X and Y, let us consider the following infinite direct sum in the sense of ℓ_2 (see Section 1.3):

$$Z_{(X,Y)} = \Big(\sum_K X_K^*\Big)_2 \oplus_2 \Big(\sum_L Y_L\Big)_2$$

where K and L run through the weakly compact absolutely convex subsets of B_X and B_Y, respectively. The space $Z_{(X,Y)}$ is reflexive (see Lemma 2.2.1, (vii)). In Theorems 3.2.1–3.2.3 below, $Z_{(X,Y)}$ will serve as a universal factorization space for all compact sets of the space $\mathcal{L}_{w^*}(X^*, Y)$.

Theorem 3.2.1. *Let X and Y be Banach spaces. For every compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*, Y)$, there exist a linear mapping $\Phi : \operatorname{span}\mathcal{C} \to \mathcal{L}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ which preserves finite rank and compact operators and a norm one operator $v \in \mathcal{L}\big(Z_{(X,Y)}, Y\big)$ such that $S = v \circ \Phi(S)$, for all $S \in \operatorname{span}\mathcal{C}$. The mapping Φ restricted to $\mathcal{C}\cup\{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then $v \in \mathcal{K}\big(Z_{(X,Y)}, Y\big)$.*

Proof. Let L and φ be, respectively, the weakly compact absolutely convex subset of B_Y and the linear mapping from $\operatorname{span}\mathcal{C}$ to $\mathcal{L}_{w^*}(X^*, Y_L)$ given by Lemma 3.1.1. Let $I_L : Y_L \to Z_{(X,Y)}$ denote the natural norm one embedding and $P_L : Z_{(X,Y)} \to Y_L$ the natural norm one projection.

Let us define mapping Φ by $\Phi(S) = I_L \circ \varphi(S)$, $S \in \operatorname{span}\mathcal{C}$, and mapping v by $v = J_L \circ P_L$. Every $\Phi(S)$, $S \in \operatorname{span}\mathcal{C}$, is weak*-weak continuous because, $J_L^*(Y^*)$ being norm dense in Y_L^* (since J_L^{**} is injective by Lemma 2.2.1), we have

$$\begin{aligned}
(\Phi(S))^*(Z^*_{(X,Y)}) &= (I_L \circ \varphi(S))^*(Z^*_{(X,Y)}) \subset (\varphi(S))^*(Y_L^*) \\
&= (\varphi(S))^*(\overline{J_L^*(Y^*)}) \subset \overline{(\varphi(S)^* \circ J_L^*)(Y^*)} \\
&= \overline{S^*(Y^*)} \subset \overline{X} = X.
\end{aligned}$$

Consequently, Φ is a linear mapping from $\operatorname{span}\mathcal{C}$ to $\mathcal{L}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ and Φ preserves the finite rank and compact operators because φ does. Note that v is a linear mapping and $\|v\| \leq \|J_L\|\|P_L\| \leq 1$. On the other hand, $1 = \|J_L\| = \|J_L \circ P_L \circ I_L\| = \|v \circ I_L\| \leq \|v\|$, and we have $\|v\| = 1$.

Now, $S = J_L \circ \varphi(S) = J_L \circ P_L \circ I_L \circ \varphi(S) = v \circ \Phi(S)$, for all $S \in \operatorname{span} \mathcal{C}$.

Finally, let $S, T \in \mathcal{C} \cup \{0\}$. Then, φ is a homeomorphism satisfying

$$\|\varphi(S) - \varphi(T)\| \leq \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S - T\|^{1/2}\Big\},$$

where $\mathrm{d} = \operatorname{diam} \mathcal{C} \cup \{0\}$. Since

$$\|\Phi(S) - \Phi(T)\| = \|I_L(\varphi(S) - \varphi(T))\| \leq \|\varphi(S) - \varphi(T)\|$$

and

$$\|S - T\| = \|v \circ \Phi(S - T)\| \leq \|\Phi(S) - \Phi(T)\|,$$

we get

$$\|S - T\| \leq \|\Phi(S) - \Phi(T)\|$$

$$\leq \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S - T\|^{1/2}\Big\}.$$

For the "moreover" part, recall that L is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$ (see Lemma 3.1.1) and use Lemma 2.2.1, (vi), to see that J_L is compact.

□

Theorem 3.2.2. *Let X and Y be Banach spaces. For every compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*, Y)$, there exist a norm one operator $u \in \mathcal{L}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}\big(Z_{(X,Y)}, Y\big)$ which preserves finite rank and compact operators such that $S = \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$, then $u \in \mathcal{K}_{w^*}\big(X^*, Z_{(X,Y)}\big)$.*

Proof. Let K and φ be, respectively, the weakly compact absolutely convex subset of B_X and the linear mapping from $\operatorname{span} \mathcal{C}$ to $\mathcal{L}(X_K^*, Y)$ given by Lemma 3.1.2. Let $I_K : X_K^* \to Z_{(X,Y)}$ denote the natural norm one embedding and $P_K : Z_{(X,Y)} \to X_K^*$ the natural norm one projection. Let us define mapping Φ by $\Phi(S) = \varphi(S) \circ P_K$, $S \in \operatorname{span} \mathcal{C}$, and mapping u by $u = I_K \circ J_K^*$. Then $\Phi(S)$ is a linear mapping from $Z_{(X,Y)}$ to Y and $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}\big(Z_{(X,Y)}, Y\big)$ preserves the finite rank and compact operators because φ does. The mapping $u : X^* \to Z_{(X,Y)}$ is linear and $\|u\| = \|I_K \circ J_K^*\| \leq 1$. On the other hand, $1 = \|J_K^*\| = \|P_K \circ I_K \circ J_K^*\| =$

$\|P_K \circ u\| \leq \|u\|$, and we have $\|u\| = 1$. The mapping u is weak*-weak continuous because, using that X_K is reflexive (see Lemma 2.2.1, (vii)),

$$\begin{aligned} u^*(Z^*_{(X,Y)}) &= (I_K \circ J^*_K)^*(Z^*_{(X,Y)}) = (J^{**}_K \circ I^*_K)(Z^*_{(X,Y)}) \\ &\subset J^{**}_K(X^{**}_K) = J_K(X_K) \subset X. \end{aligned}$$

Now, $S = \varphi(S) \circ J^*_K = \varphi(S) \circ P_K \circ I_K \circ J^*_K = \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. Finally, let $S, T \in \mathcal{C} \cup \{0\}$. Then φ is a homeomorphism satisfying

$$\|\varphi(S) - \varphi(T)\| \leq \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2}\|S - T\|^{1/2}\Big\},$$

where $\mathrm{d} = \operatorname{diam} \mathcal{C} \cup \{0\}$. Since

$$\|\Phi(S) - \Phi(T)\| = \|(\varphi(S) - \varphi(T))P_K\| \leq \|\varphi(S) - \varphi(T)\|$$

and

$$\|S - T\| = \|\Phi(S - T) \circ u\| \leq \|\Phi(S) - \Phi(T)\|,$$

we get

$$\begin{aligned} \|S - T\| &\leq \|\Phi(S) - \Phi(T)\| \\ &\leq \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2}\|S - T\|^{1/2}\Big\}. \end{aligned}$$

For the "moreover" part, recall that J^*_K is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}_{w^*}(X^*, Y)$ (see Lemma 3.1.2).

$\square$

Theorem 3.2.3. *Let X and Y be Banach spaces. For every compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*, Y)$, there exist norm one operators $u \in \mathcal{L}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ and $v \in \mathcal{L}\big(Z_{(X,Y)}, Y\big)$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{L}\big(Z_{(X,Y)}, Z_{(X,Y)}\big)$ which preserves finite rank and compact operators such that $S = v \circ \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying*

$$\begin{aligned} \|S - T\| &\leq \|\Phi(S) - \Phi(T)\| \\ &\leq \min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4}\|S - T\|^{1/4}\Big\}, \quad S, T \in \mathcal{C} \cup \{0\}, \end{aligned}$$

where

$$\mathrm{d} = \operatorname{diam} \mathcal{C} \cup \{0\}.$$

In particular, if $-S \in \mathcal{C}$ *for some* $S \in \mathcal{C}$*, then*

$$\|\Phi(S)\| \leq \min\Big\{\frac{\mathrm{d}}{2}, \Big(\frac{\mathrm{d}}{2}\Big)^{3/4}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{3/4}\|S\|^{1/4}\Big\}.$$

Moreover, if $\mathcal{C}$ *is contained in* $\mathcal{K}_{w^*}(X^*, Y)$*, then* $u \in \mathcal{K}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ *and* $v \in \mathcal{K}\big(Z_{(X,Y)}, Y\big)$*.*

Proof. Let $K \subset B_X$, $J = J_K^* \in \mathcal{L}_{w^*}(X^*, X_K^*)$, and $\varphi : \operatorname{span}\mathcal{C} \to \mathcal{L}(X_K^*, Y)$ be, respectively, the weakly compact absolutely convex subset, the norm one operator, and the linear mapping given by Lemma 3.1.2.

Since $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{L}(X_K^*, Y) = \mathcal{L}_{w^*}(X_K^*, Y)$ (recall that X_K^* is reflexive), we can apply Lemma 3.1.1. Let $L \subset B_Y$ and $\psi : \operatorname{span}\varphi(\mathcal{C}) \to \mathcal{L}_{w^*}(X_K^*, Y_L)$ be, respectively, the weakly compact subset and the linear mapping given by Lemma 3.1.1.

Let $I_K : X_K^* \to Z_{(X,Y)}$ and $I_L : Y_L \to Z_{(X,Y)}$ denote the natural norm one embeddings, and let $P_K : Z_{(X,Y)} \to X_K^*$ and $P_L : Z_{(X,Y)} \to Y_L$ denote the natural norm one projections.

Let us define the mappings $u = I_K \circ J$, $\Phi(S) = I_L \circ \psi(\varphi(S)) \circ P_K$, $S \in \operatorname{span}\mathcal{C}$, and $v = J_L \circ P_L$. Then $u \in \mathcal{L}_{w^*}(X^*, Z_{(X,Y)})$ and $\|u\| = 1$ by the proof of Theorem 3.2.2, and $v \in \mathcal{L}(Z_{(X,Y)}, Y)$, $\|v\| = 1$ by the proof of Theorem 3.2.1. The mapping $\Phi(S)$ is a linear mapping from $Z_{(X,Y)}$ to $Z_{(X,Y)}$. The linear mapping $\Phi : \operatorname{span}\mathcal{C} \to \mathcal{L}\big(Z_{(X,Y)}, Z_{(X,Y)}\big)$ preserves the finite rank and compact operators because φ and ψ do. Moreover, for all $S \in \operatorname{span}\mathcal{C}$,

$$\begin{aligned} S &= \varphi(S) \circ J = J_L \circ \psi(\varphi(S)) \circ J \\ &= J_L \circ P_L \circ I_L \circ \psi(\varphi(S)) \circ P_K \circ I_K \circ J \\ &= v \circ \Phi(S) \circ u. \end{aligned}$$

Finally, let $S, T \in \mathcal{C} \cup \{0\}$. Then, φ is a homeomorphism satisfying

$$\|\varphi(S) - \varphi(T)\| \leq \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{1/2}\|S-T\|^{1/2}\Big\},$$

where $\mathrm{d} = \operatorname{diam}\mathcal{C} \cup \{0\}$, and ψ is a homeomorphism satisfying

$$\|\psi(\varphi(S)) - \psi(\varphi(T))\| \leq \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4}+\frac{1}{\ln a}\Big)^{1/2}\|\varphi(S)-\varphi(T)\|^{1/2}\Big\},$$

because $\operatorname{diam}\varphi(\mathcal{C} \cup \{0\}) = \mathrm{d}$. Since

$$\|S - T\| = \|v \circ \Phi(S-T) \circ u\| \leq \|\Phi(S) - \Phi(T)\|$$

and

$$\|\Phi(S) - \Phi(T)\| = \|I_L \circ \psi(\varphi(S) - \varphi(T)) \circ P_K\| \leq \|\psi(\varphi(S)) - \psi(\varphi(T))\|,$$

we get

$$\|S - T\| \leq \|\Phi(S) - \Phi(T)\|$$

$$\leq \min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} \|S - T\|^{1/4}\Big\}.$$

We can easily prove the "moreover" part by using that $J \in \mathcal{K}_{w^*}(X^*, X_K^*)$ whenever $\mathcal{C} \subset \mathcal{K}_{w^*}(X^*, Y)$ (see Lemma 3.1.2) and that, in this case, $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{K}_{w^*}(X_K^*, Y)$, implying (by Lemma 3.1.1 and Lemma 2.2.1, (vi)) the compactness of the operator J_L. $\square$

Remark 3.2.4. Observe that diam $\Phi(\mathcal{C} \cup \{0\}) =$ diam $\mathcal{C} \cup \{0\}$ in Theorems 3.2.1–3.2.3.

Remark 3.2.5. Theorem 3.2.3 represents a quantitative strengthening of the following result by Aron, Lindström, Ruess, and Ryan (see [ALRR, Proposition 2]): *for Banach spaces X and Y, there exists a reflexive Banach space $Z = Z(X, Y)$ such that, for every relatively compact subset $\mathcal{C}$ of $\mathcal{L}_{w^*}(X^*, Y)$, there exist operators $u \in \mathcal{L}_{w^*}(X^*, Z)$ and $v \in \mathcal{L}(Z, Y)$, and a relatively compact subset $\{A_S : S \in \mathcal{C}\}$ of $\mathcal{L}(Z, Z)$ such that $S = v \circ A_S \circ u$ for all $S \in \mathcal{C}$.* Note that our definition of $Z_{(X,Y)}$ is much simpler than that of $Z(X, Y)$, but similar.

3.3 Applications to weakly compact and compact operators

Since $\mathcal{W}(X, Y)$ and $\mathcal{L}_{w^*}(X^{**}, Y) = \mathcal{W}_{w^*}(X^{**}, Y)$, and also $\mathcal{K}(X, Y)$ and $\mathcal{K}_{w^*}(X^{**}, Y)$, are canonically isometrically isomorphic under the mapping $S \to S^{**}$, Theorems 3.2.1–3.2.3 yield immediate applications to factoring compact subsets of $\mathcal{W}(X, Y)$ and $\mathcal{K}(X, Y)$ (recall that $\mathcal{L}(X, Z) = \mathcal{W}(X, Z)$ and $\mathcal{L}(Z, Y) = \mathcal{W}(Z, Y)$ whenever Z is reflexive). We state the corresponding applications.

Corollary 3.3.1. *Let X and Y be Banach spaces, and let $Z = Z_{(X^*,Y)}$. For every compact subset $\mathcal{C}$ of $\mathcal{W}(X, Y)$, there exist a linear mapping Φ :* span $\mathcal{C} \to \mathcal{W}(X, Z)$ *which preserves finite rank and compact operators and a norm one operator $v \in \mathcal{W}(Z, Y)$ such that $S = v \circ \Phi(S)$, for all $S \in$* span $\mathcal{C}$*. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}(X, Y)$, then $v \in \mathcal{K}(Z, Y)$.*

Corollary 3.3.2. *Let X and Y be Banach spaces, and let $Z = Z_{(X^*,Y)}$. For every compact subset $\mathcal{C}$ of $\mathcal{W}(X,Y)$, there exist a norm one operator $u \in \mathcal{W}(X,Z)$ and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{W}(Z,Y)$ which preserves finite rank and compact operators such that $S = \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}(X,Y)$, then $u \in \mathcal{K}(X,Z)$.*

Corollary 3.3.3. *Let X and Y be Banach spaces, and let $Z = Z_{(X^*,Y)}$. For every compact subset $\mathcal{C}$ of $\mathcal{W}(X,Y)$, there exist norm one operators $u \in \mathcal{W}(X,Z)$ and $v \in \mathcal{W}(Z,Y)$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{W}(Z,Z)$ which preserves finite rank and compact operators such that $S = v \circ \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Theorem 3.2.3. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}(X,Y)$, then $u \in \mathcal{K}(X,Z)$ and $v \in \mathcal{K}(Z,Y)$.*

Chapter 4

Quantitative versions of the uniform factorization for compact sets of compact operators

Main results of this chapter (see Sections 4.2–4.6) are published in [MO1].

4.1 Universal factorization spaces C_p and Z_{FJ}

We recall the definition of the universal factorization spaces C_p, $1 \leq p \leq \infty$, which were introduced by Johnson [J].

It is well known (for a detailed proof, see [Si, pages 422–426]) that there exists a sequence $(G_n) = (G_n)_{n=1}^{\infty}$ of finite-dimensional Banach spaces such that for every finite-dimensional Banach space X and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\mathrm{d}_{\mathrm{BM}}(X, G_n) < 1 + \varepsilon$$

(one says that (G_n) is dense in the collection of all finite-dimensional Banach spaces). Moreover, the sequence (G_n) can be chosen such that every G_m, $m \in \mathbb{N}$, occurs in (G_n) an infinity of times, meaning that for every $m \in \mathbb{N}$, the set

$$\{n : \mathrm{d}_{\mathrm{BM}}(G_m, G_n) = 1\}$$

is infinite.

Let $1 \leq p \leq \infty$. The space C_p is defined as the infinite direct sum in the sense of ℓ_p of the spaces G_n, that is

$$C_p = \Big(\sum_{n=1}^{\infty} G_n\Big)_p .$$

(Recall that ℓ_∞ denotes the space c_0.)

Johnson [J] proved that, for any fixed p, every approximable operator factors through C_p. More precisely, by the proof of [J, Theorem 1], the following holds.

Theorem 4.1.1 (Johnson). *Let $1 \le p \le \infty$. Let X and Y be Banach spaces. If $S \in \overline{\mathcal{F}}(X, Y)$ is a norm one operator, then there exist operators $u \in \overline{\mathcal{F}}(X, C_p)$ and $v \in \overline{\mathcal{F}}(C_p, Y)$ such that $S = v \circ u$. Moreover, for every $\varepsilon > 0$, the operators u and v can be chosen such that $\|u\| = 1$ and $1 \le \|v\| \le 1 + \varepsilon$.*

Basing on Johnson's theorem, Figiel [F, Proposition 3.1] proved that every compact operator factors through some closed subspace of C_p. More precisely, by the proofs of [J, Theorem 1] and [F, Proposition 3.1], the following holds.

Theorem 4.1.2 (Figiel-Johnson). *Let $1 \le p \le \infty$. Let X and Y be Banach spaces. If $S \in \mathcal{K}(X, Y)$ is a norm one operator, then there exist a closed subspace W of C_p and operators $u \in \mathcal{K}(X, W)$ and $v \in \mathcal{K}(W, Y)$ such that $S = v \circ u$. Moreover, for every $\varepsilon > 0$, the operators u and v can be chosen such that $\|u\| = 1$ and $1 \le \|v\| \le 1 + \varepsilon$.*

Aron, Lindström, Ruess, and Ryan [ALRR] observed that in the Figiel-Johnson theorem, W may be replaced by a factorization space which is a universal for all X, Y, and $S \in \mathcal{K}(X, Y)$. Let us call this space the Figiel-Johnson universal factorization space and denote it by Z_{FJ}. The space Z_{FJ} can be defined, for instance, as

$$Z_{FJ} = \Big(\sum_{W \subset C_p} W\Big)_p$$

where W runs through the closed subspaces of C_p for any fixed p, $1 \le p \le \infty$.

4.2 Factorization through Z_{FJ}

We shall combine Theorem 3.2.3 with the well-known factorization methods by Johnson [J] and Figiel [F] to get the following quantitative strengthening of the Aron-Lindström-Ruess-Ryan theorem (see Theorem 1.1.5).

Theorem 4.2.1. *Let X and Y be Banach spaces and let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*, Y)$. Then, for every $\varepsilon > 0$, there exist operators $u \in \mathcal{K}_{w^*}(X^*, Z_{FJ})$ and $v \in \mathcal{K}(Z_{FJ}, Y)$ with $1 \le \|u\|, \|v\| \le 1 + \varepsilon$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z_{FJ}, Z_{FJ})$ such that $S = v \circ \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Theorem 3.2.3.*

The proof of Theorem 4.2.1 uses the following results.

Lemma 4.2.2 (Johnson). *Let $1 \le p \le \infty$. Let X and Y be Banach spaces. The subspace of $\overline{\mathcal{F}}_{w^*}(X^*, Y)$ consisting of operators T which admit a factorization $T = \beta \circ \alpha$ for some operators $\alpha \in \overline{\mathcal{F}}_{w^*}(X^*, C_p)$ and $\beta \in \overline{\mathcal{F}}(C_p, Y)$, is a Banach space under the norm*

$$\|T\|_{C_p} = \inf\{\|\beta\|\|\alpha\| : T = \beta \circ \alpha,\ \alpha \in \overline{\mathcal{F}}_{w^*}(X^*, C_p),\ \beta \in \overline{\mathcal{F}}(C_p, Y)\}.$$

Proof. It is almost identical to the proof of [J, Proposition 1]. □

Lemma 4.2.3 (Figiel-Johnson). *Let $1 \le p \le \infty$. Let X and Y be Banach spaces. If $S \in \mathcal{K}_{w^*}(X^*, Y)$ is a norm one operator, then there exist a closed subspace W of C_p and operators $u \in \mathcal{K}_{w^*}(X^*, W)$ and $v \in \mathcal{K}(W, Y)$ such that $S = v \circ u$. Moreover, for every $\varepsilon > 0$, the operators u and v can be chosen such that $\|v\| = 1$ and $1 \le \|u\| \le 1 + \varepsilon$.*

Proof. It relies on Lemma 4.2.2 and is almost identical to the proof of Theorem 4.1.2 (see the proofs of [J, Theorem 1] and [F, Proposition 3.1]). □

Proof of Theorem 4.2.1. Let $A \in \mathcal{K}_{w^*}\big(X^*, Z_{(X,Y)}\big)$, $B \in \mathcal{K}\big(Z_{(X,Y)}, Y\big)$, and $\varphi : \operatorname{span} \mathcal{C} \to \mathcal{K}\big(Z_{(X,Y)}, Z_{(X,Y)}\big)$ be the norm one operators and the linear mapping given by Theorem 3.2.3.

By Theorem 4.1.2 there exist a closed subspace W of C_p and operators $C \in \mathcal{K}(Z_{(X,Y)}, W)$ and $D \in \mathcal{K}(W, Y)$ such that $\|C\| = 1$, $1 \le \|D\| \le 1 + \varepsilon$, and $B = D \circ C$. Let $I_W : W \to Z_{FJ}$ denote the natural norm one embedding, and let $P_W : Z_{FJ} \to W$ denote the natural norm one projection. Let us define the mappings $V = I_W \circ C$ and $v = D \circ P_W$. Then $V \in \mathcal{K}\big(Z_{(X,Y)}, Z_{FJ}\big)$ and $v \in \mathcal{K}\big(Z_{FJ}, Y\big)$, $\|V\| = \|C\| = 1$, $1 \le \|D\| = \|D \circ P_W \circ I_W\| \le \|v\| \le 1 + \varepsilon$, and $B = v \circ V$ since $D \circ C = D \circ P_W \circ I_W \circ C$.

Arguing similarly, by Lemma 4.2.3 we can obtain for $A \in \mathcal{K}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ with $\|A\| = 1$, two operators $u \in \mathcal{K}_{w^*}(X^*, Z_{FJ})$ and $U \in \mathcal{K}\big(Z_{FJ}, Z_{(X,Y)}\big)$ such that $1 \le \|u\| \le 1 + \varepsilon$, $\|U\| = 1$, and $A = U \circ u$.

Since, for all $S \in \operatorname{span} \mathcal{C}$,

$$S = B \circ \varphi(S) \circ A = v \circ V \circ \varphi(S) \circ U \circ u,$$

the mapping Φ defined by $\Phi(S) = V \circ \varphi(S) \circ U$, $S \in \operatorname{span} \mathcal{C}$, has the desired properties. □

In the same vein like the space $Z_{(X,Y)}$ was "replaced" by Z_{FJ} in Theorem 3.2.3 to obtain Theorem 4.2.1, one can "replace" $Z_{(X,Y)}$ by Z_{FJ} also in Theorems 3.2.1 and 3.2.2 (or, equivalently, one may base on Lemmas 3.1.1 and 3.1.2 instead of Theorems 3.2.1 and 3.2.2). The corresponding results are following.

Theorem 4.2.4. *Let X and Y be Banach spaces and let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*, Y)$. Then, for every $\varepsilon > 0$, there exist operator $v \in \mathcal{K}(Z_{FJ}, Y)$ with $1 \leq \|v\| \leq 1 + \varepsilon$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(X^*, Z_{FJ})$ such that $S = v \circ \Phi(S)$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Theorem 4.2.5. *Let X and Y be Banach spaces and let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*, Y)$. Then, for every $\varepsilon > 0$, there exist operator $u \in \mathcal{K}_{w^*}(X^*, Z_{FJ})$ with $1 \leq \|u\| \leq 1 + \varepsilon$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z_{FJ}, Y)$ such that $S = \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Since $\mathcal{K}(X, Y)$ and $\mathcal{K}_{w^*}(X^{**}, Y)$ are canonically isometrically isomorphic under the mapping $S \to S^{**}$, Theorems 4.2.1, 4.2.4, and 4.2.5 yield immediate applications to factoring compact subsets of $\mathcal{K}(X, Y)$. We state the corresponding applications.

Corollary 4.2.6. *Let X and Y be Banach spaces and let $\mathcal{C}$ be a compact subset of $\mathcal{K}(X, Y)$. Then, for every $\varepsilon > 0$, there exist operators $u \in \mathcal{K}(X, Z_{FJ})$ and $v \in \mathcal{K}(Z_{FJ}, Y)$ with $1 \leq \|u\|, \|v\| \leq 1 + \varepsilon$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z_{FJ}, Z_{FJ})$ such that $S = v \circ \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Theorem 3.2.3.*

Corollary 4.2.7. *Let X and Y be Banach spaces and let $\mathcal{C}$ be a compact subset of $\mathcal{K}(X, Y)$. Then, for every $\varepsilon > 0$, there exist operator $v \in \mathcal{K}(Z_{FJ}, Y)$ with $1 \leq \|v\| \leq 1 + \varepsilon$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(X, Z_{FJ})$ such that $S = v \circ \Phi(S)$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Corollary 4.2.8. *Let X and Y be Banach spaces and let $\mathcal{C}$ be a compact subset of $\mathcal{K}(X, Y)$. Then, for every $\varepsilon > 0$, there exist operator $u \in \mathcal{K}(X, Z_{FJ})$ with $1 \leq \|u\| \leq 1 + \varepsilon$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z_{FJ}, Y)$ such that $S = \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

4.3 Factorization through C_p

Let us now point out an important case when $Z_{(X,Y)}$ may be "replaced" by any C_p, $1 \le p \le \infty$. We present the results that rely on Theorems 3.2.1, 3.2.2, and 3.2.3.

Recall that a Banach space X has the approximation property if, for every compact set $K \subset X$ and every $\varepsilon > 0$, there exists a finite rank operator $S \in \mathcal{F}(X)$ such that $\|Sx - x\| < \varepsilon$, for all $x \in K$.

Recall also the next well-known result due to Grothendieck [G, Chapter I, page 165]; for a recent proof, see [OPe, Section 3].

Theorem 4.3.1 (Grothendieck). *Let X and Y be Banach spaces. Then the following assertions are equivalent.*
(i) *X has the approximation property.*
(ii) *For every Banach space Y, one has $\mathcal{K}(Y,X) = \overline{\mathcal{F}}(Y,X)$.*
(iii) *For every Banach space Y, one has $\mathcal{K}_{w^*}(X^*,Y) = \overline{\mathcal{F}}_{w^*}(X^*,Y)$.*
(iv) *For every Banach space Y, one has $\mathcal{K}_{w^*}(Y^*,X) = \overline{\mathcal{F}}_{w^*}(Y^*,X)$.*

Theorem 4.3.2. *Let X and Y be Banach spaces such that Y has the approximation property and let $1 \le p \le \infty$. Let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*,Y)$. Then, for every $\varepsilon > 0$, there exist a linear mapping Φ : $\operatorname{span} \mathcal{C} \to \mathcal{K}_{w^*}(X^*,C_p)$ and an operator $v \in \mathcal{K}(C_p,Y)$ with $1 \le \|v\| \le 1+\varepsilon$ such that $S = v \circ \Phi(S)$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Proof. Let φ : $\operatorname{span} \mathcal{C} \to \mathcal{K}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ and $B \in \mathcal{K}\big(Z_{(X,Y)}, Y\big)$ be the linear mapping and the norm one operator given by Theorem 3.2.1. Since Y has the approximation property, we have by Theorem 4.3.1 that $\mathcal{K}\big(Z_{(X,Y)}, Y\big) = \overline{\mathcal{F}}\big(Z_{(X,Y)}, Y\big)$. Therefore, by Theorem 4.1.1, operator B admits a factorization $B = v \circ V$ with $V \in \mathcal{K}\big(Z_{(X,Y)}, C_p\big)$, $\|V\| = 1$, and $v \in \mathcal{K}(C_p,Y)$, $1 \le \|v\| \le 1+\varepsilon$. The mapping Φ defined by $\Phi(S) = V \circ \varphi(S)$, $S \in \operatorname{span} \mathcal{C}$, has the needed properties. □

For the next two theorems, we shall also need the following result.

Lemma 4.3.3 (Johnson). *Let $1 \le p \le \infty$. Let X and Y be Banach spaces. If $S \in \overline{\mathcal{F}}_{w^*}(X^*,Y)$ is a norm one operator, then there exist operators $u \in \overline{\mathcal{F}}_{w^*}(X^*,C_p)$ and $v \in \overline{\mathcal{F}}(C_p,Y)$ such that $S = v \circ u$. Moreover, for every $\varepsilon > 0$, the operators u and v can be chosen such that $\|v\| = 1$ and $1 \le \|u\| \le 1+\varepsilon$.*

Proof. It relies on Lemma 4.2.2 and is almost identical to the proof of [J, Theorem 1]. □

Theorem 4.3.4. *Let X and Y be Banach spaces such that X has the approximation property and let $1 \le p \le \infty$. Let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*, Y)$. Then, for every $\varepsilon > 0$, there exist an operator $u \in \mathcal{K}_{w^*}(X^*, C_p)$ with $1 \le \|u\| \le 1 + \varepsilon$ and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(C_p, Y)$ such that $S = \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Proof. Let $A \in \mathcal{K}_{w^*}\big(X^*, Z_{(X,Y)}\big)$ and $\psi : \operatorname{span} \mathcal{C} \to \mathcal{K}\big(Z_{(X,Y)}, Y\big)$ be the norm one operator and the linear mapping given by Theorem 3.2.2. Since X has the approximation property, we have by Theorem 4.3.1 that $\mathcal{K}_{w^*}(X^*, Z_{(X,Y)}) = \overline{\mathcal{F}}_{w^*}(X^*, Z_{(X,Y)})$. Therefore by Lemma 4.3.3 A admits a factorization $A = U \circ u$ with $u \in \mathcal{K}_{w^*}(X^*, C_p)$, $1 \le \|u\| \le 1 + \varepsilon$, and $U \in \mathcal{K}(C_p, Z_{(X,Y)})$ with $\|U\| = 1$. The mapping Φ defined by $\Phi(S) = \psi(S) \circ U$, $S \in \operatorname{span} \mathcal{C}$, has the needed properties. $\square$

Theorem 4.3.5. *Let X and Y be Banach spaces having the approximation property and let $1 \le p \le \infty$. Let $\mathcal{C}$ be a compact subset of $\mathcal{K}_{w^*}(X^*, Y)$. Then, for every $\varepsilon > 0$, there exist operators $u \in \mathcal{K}_{w^*}(X^*, C_p)$ and $v \in \mathcal{K}(C_p, Y)$ with $1 \le \|u\|, \|v\| \le 1 + \varepsilon$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(C_p, C_p)$ such that $S = v \circ \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Theorem 3.2.3.*

Proof. Let $A \in \mathcal{K}_{w^*}\big(X^*, Z_{(X,Y)}\big)$, $B \in \mathcal{K}\big(Z_{(X,Y)}, Y\big)$, and $\varphi : \operatorname{span} \mathcal{C} \to \mathcal{K}\big(Z_{(X,Y)}, Z_{(X,Y)}\big)$ be the norm one operators and the linear mapping given by Theorem 3.2.3. Let $A = U \circ u$ and $B = v \circ V$ be the factorizations obtained in the proofs of Theorems 4.3.4 and 4.3.2. Then the mappings u and v above and the mapping Φ defined by $\Phi(S) = V \circ \varphi(S) \circ U$, $S \in \operatorname{span} \mathcal{C}$, have the desired properties. $\square$

Since $\mathcal{K}(X, Y)$ and $\mathcal{K}_{w^*}(X^{**}, Y)$ are canonically isometrically isomorphic under the mapping $S \to S^{**}$, Theorems 4.3.2, 4.3.4, and 4.3.5 yield immediate applications to factoring compact subsets of $\mathcal{K}(X, Y)$. We state the corresponding applications.

Corollary 4.3.6. *Let X and Y be Banach spaces such that Y has the approximation property and let $1 \le p \le \infty$. Let $\mathcal{C}$ be a compact subset of $\mathcal{K}(X, Y)$. Then, for every $\varepsilon > 0$, there exist a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(X, C_p)$ and an operator $v \in \mathcal{K}(C_p, Y)$ with $1 \le \|v\| \le 1 + \varepsilon$ such that $S = v \circ \Phi(S)$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Corollary 4.3.7. *Let X and Y be Banach spaces such that X^* has the approximation property and let $1 \le p \le \infty$. Let $\mathcal{C}$ be a compact subset of*

$\mathcal{K}(X,Y)$. *Then, for every* $\varepsilon > 0$, *there exist an operator* $u \in \mathcal{K}(X, C_p)$ *with* $1 \le \|u\| \le 1+\varepsilon$ *and a linear mapping* $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(C_p, Y)$ *such that* $S = \Phi(S) \circ u$, *for all* $S \in \operatorname{span} \mathcal{C}$. *The mapping* Φ *restricted to* $\mathcal{C} \cup \{0\}$ *is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Corollary 4.3.8. *Let* X *and* Y *be Banach spaces such that* X^* *and* Y *have the approximation property and let* $1 \le p \le \infty$. *Let* $\mathcal{C}$ *be a compact subset of* $\mathcal{K}(X,Y)$. *Then, for every* $\varepsilon > 0$, *there exist operators* $u \in \mathcal{K}(X, C_p)$ *and* $v \in \mathcal{K}(C_p, Y)$ *with* $1 \le \|u\|, \|v\| \le 1+\varepsilon$, *and a linear mapping* $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(C_p, C_p)$ *such that* $S = v \circ \Phi(S) \circ u$, *for all* $S \in \operatorname{span} \mathcal{C}$. *The mapping* Φ *restricted to* $\mathcal{C} \cup \{0\}$ *is a homeomorphism satisfying the conclusions of Theorem 3.2.3.*

4.4 Factorization through ℓ_p

If we use, in the proof of Theorem 4.3.2 (recall that the $\mathcal{L}_p$-spaces have the approximation property), the factorization argument [J, Theorem 2] by Johnson instead of Theorem 4.1.1, then we immediately get the following quantitative strengthening of the symmetric version of Theorem 1.1.4 of Graves and Ruess for all $\mathcal{L}_p$-spaces, $1 \le p \le \infty$. Recall that ℓ_∞ denotes the space c_0.

Theorem 4.4.1. *Let* X *be a Banach space and let* Y *be an* $\mathcal{L}_{p,\lambda}$*-space* ($1 \le p \le \infty$, $1 \le \lambda < \infty$). *If* $\mathcal{C}$ *is a compact subset of* $\mathcal{K}_{w^*}(X^*, Y)$, *then, for every* $\varepsilon > 0$, *there exist a linear mapping* $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}_{w^*}(X^*, \ell_p)$ *and an operator* $v \in \mathcal{K}(\ell_p, Y)$ *with* $1 \le \|v\| \le \lambda + \varepsilon$ *such that* $S = v \circ \Phi(S)$, *for all* $S \in \operatorname{span} \mathcal{C}$, *and the mapping* Φ *restricted to* $\mathcal{C} \cup \{0\}$ *is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Since $\mathcal{K}(X,Y)$ and $\mathcal{K}_{w^*}(X^{**}, Y)$ are canonically isometrically isomorphic under the mapping $S \to S^{**}$, Theorem 4.4.1 yields the following immediate application to factoring compact subsets of $\mathcal{K}(X,Y)$.

Corollary 4.4.2. *Let* X *be a Banach space and let* Y *be an* $\mathcal{L}_{p,\lambda}$*-space* ($1 \le p \le \infty$, $1 \le \lambda < \infty$). *If* $\mathcal{C}$ *is a compact subset of* $\mathcal{K}(X,Y)$, *then, for every* $\varepsilon > 0$, *there exist a linear mapping* $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(X, \ell_p)$ *and an operator* $v \in \mathcal{K}(\ell_p, Y)$ *with* $1 \le \|v\| \le \lambda + \varepsilon$ *such that* $S = v \circ \Phi(S)$, *for all* $S \in \operatorname{span} \mathcal{C}$, *and the mapping* Φ *restricted to* $\mathcal{C} \cup \{0\}$ *is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

We conclude with a quantitative strengthening of the Graves-Ruess theorem (see Theorem 1.1.4) for all $\mathcal{L}_p$-spaces, $1 \le p \le \infty$, which is a symmetric version of Theorem 4.4.1. Let us recall that X is an $\mathcal{L}_p$-space if and only if

X^* is an $\mathcal{L}_q$-space where $1/q + 1/p = 1$ with $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$.

Theorem 4.4.3. *Let $1 \le p \le \infty$ and $1 \le \lambda < \infty$. Let X be an $\mathcal{L}_p$-space such that X^* is an $\mathcal{L}_{q,\lambda}$-space (where $1/q+1/p = 1$ with $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$) and let Y be a Banach space. If $\mathcal{C}$ is a compact subset of $\mathcal{K}(X,Y)$, then, for every $\varepsilon > 0$, there exist a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(\ell_p, Y)$ and an operator $u \in \mathcal{K}(X, \ell_p)$ with $1 \le \|u\| \le \lambda + \varepsilon$ such that $S = \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$, and the mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Proof. Suppose first that $1 \le p < \infty$. Observe that $\mathcal{C}^* = \{S^* : S \in \mathcal{C}\}$ is a compact subset of $\mathcal{K}_{w^*}(Y^*, X^*)$ (recall that $\operatorname{ran} S^{**} \subset Y$ whenever $S \in \mathcal{K}(X,Y)$) and apply Theorem 4.4.1. Let $\varepsilon > 0$ and let $\varphi : \operatorname{span} \mathcal{C}^* \to \mathcal{K}_{w^*}(Y^*, \ell_q)$ and $v \in \mathcal{K}(\ell_q, X^*)$ be given by Theorem 4.4.1. Then $\ell_q^* = \ell_p$ and $(\varphi(S^*))^* \in \mathcal{K}(\ell_p, Y)$ if $S \in \operatorname{span} \mathcal{C}$. Define $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(\ell_p, Y)$ by $\Phi(S) = (\varphi(S^*))^*$, $S \in \operatorname{span} \mathcal{C}$, and $u \in \mathcal{K}(X, \ell_p)$ by $u = v^*|_X$. These mappings have the desired properties. In particular, if $S \in \operatorname{span} \mathcal{C}$, then

$$\Phi(S) \circ u = ((\varphi(S^*))^* \circ v^*)|_X = (v \circ \varphi(S^*))^*|_X = S^{**}|_X = S.$$

Suppose now that $p = \infty$. Let Z be the reflexive space, $U \in \mathcal{K}(X, Z)$ the norm one operator, and $\varphi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z, Y)$ the linear mapping given by Corollary 3.3.2. Observe that $U \in \overline{\mathcal{F}}(X, Z)$ because X^*, being an $\mathcal{L}_1$-space, has the approximation property. In fact, $\mathcal{K}(X, Z) = \mathcal{K}_{w^*}(X^{**}, Z) = \overline{\mathcal{F}}_{w^*}(X^{**}, Z) = \overline{\mathcal{F}}(X, Z)$ by the canonical identifications and Theorem 4.3.1.

Let $\varepsilon > 0$. Consider any $T \in \mathcal{F}(X, Z)$. Then $T^* \in \mathcal{F}(Z^*, X^*)$ and, similarly to the proof of [J, Theorem 2], we can choose a finite-dimensional subspace E of X^* with $\operatorname{ran} T^* \subset E$, a positive integer n, and an isomorphism L from E onto ℓ_1^n such that, e.g., $\|L\| = 1$ and $1 \le \|L^{-1}\| < \lambda + \varepsilon/2$. Denoting by $V : Z^* \to E$ the astriction of T^* and by $j : E \to X^*$ the identity embedding, we have $V^* \in \mathcal{F}(E^*, Z)$ and $T = V^* \circ j^*|_X$. Hence, T admits a factorization $T = \beta \circ \alpha$ for some operators $\alpha \in \mathcal{F}(X, \ell_\infty^n)$ and $\beta \in \mathcal{F}(\ell_\infty^n, Z)$ with $\|\alpha\| \le \lambda + \varepsilon/2$ and $\|\beta\| \le \|T\|$. Since c_0 is isometrically isomorphic to the infinite direct sum $(\sum_n \ell_\infty^n)_\infty$ in the sense of c_0, we have, for the norm $\|\cdot\|_{c_0}$ introduced in [J, Proposition 1],

$$\begin{aligned}\|T\|_{c_0} &= \inf\{\|\beta\|\|\alpha\| : T = \beta \circ \alpha,\ \alpha \in \overline{\mathcal{F}}(X, c_0),\ \beta \in \overline{\mathcal{F}}(c_0, Z)\} \\ &\le (\lambda + \varepsilon/2)\|T\|.\end{aligned}$$

Consequently, $\|\cdot\|_{c_0}$ is equivalent to the operator norm on $\mathcal{F}(X, Z)$ and, since $\mathcal{F}(X, Z)$ is dense in $\overline{\mathcal{F}}(X, Z)$, it follows from [J, Proposition 1] that, in particular, U admits a factorization $U = v \circ u$ with $u \in \overline{\mathcal{F}}(X, c_0)$, $v \in$

$\overline{\mathcal{F}}(c_0, Z)$, and $\|U\| = 1 \leq \|v\|\|u\| \leq (\lambda + \varepsilon)\|U\| = \lambda + \varepsilon$. We may clearly assume that $1 \leq \|u\| \leq \lambda + \varepsilon$ and $\|v\| = 1$. Since, for all $S \in \operatorname{span} \mathcal{C}$,

$$S = \varphi(S) \circ U = \varphi(S) \circ v \circ u,$$

the mapping Φ defined by $\Phi(S) = \varphi(S) \circ v$, $S \in \operatorname{span} \mathcal{C}$, has the desired properties. □

4.5 Compact subsets of injective tensor products of Banach spaces

In this section, we point out applications of Theorems 4.3.2 and 4.4.1 to representing compact subsets of the injective tensor product $X \check{\otimes} Y$ of Banach spaces X and Y (see Corollaries 4.5.1 and 4.5.2).

Let $X \otimes Y$ denote the algebraic tensor product of Banach spaces X and Y. Recall that any element $u = \sum_{n=1}^m x_n \otimes y_n$ of the algebraic tensor product $X \otimes Y$ can be algebraically identified with the finite-rank operator

$$\sum_{n=1}^{m} x_n \otimes y_n : y^* \to \sum_{n=1}^{m} y^*(y_n) x_n$$

from Y^* to X. Thus $X \otimes Y$ may always be viewed as a linear subspace of $\mathcal{F}(Y^*, X)$. In fact, $X \otimes Y = \mathcal{F}_{w^*}(Y^*, X)$.

The injective tensor product $X \check{\otimes} Y$ of Banach spaces X and Y is the completion of the algebraic tensor product $X \otimes Y$ in the injective tensor norm $\| \cdot \|_\varepsilon$ (or ε-norm) defined as

$$\|\sum_{n=1}^{m} x_n \otimes y_n\|_\varepsilon = \sup\Big\{ |\sum_{n=1}^{m} x^*(x_n) y^*(y_n)| : \ x^* \in B_{X^*}, y \in B_{Y^*} \Big\}.$$

Clearly, the operator norm on $X \otimes Y = \mathcal{F}_{w^*}(Y^*, X)$ coincides with the injective tensor norm. Therefore the injective tensor product $X \check{\otimes} Y$ may be identified with the Banach space $\overline{\mathcal{F}}_{w^*}(Y^*, X)$ of weak*-weak continuous approximable operators.

The name "injective" stems from the fact that injective tensor products respect the subspace structure: if $Z \subset X$ and $W \subset Y$ are closed subspaces, then $Z \check{\otimes} W$ can be canonically identified with a closed subspace of $X \check{\otimes} Y$.

The basic references for the tensor products of Banach spaces are [DU] and [Ryan].

Corollary 4.5.1. *Let $1 \leq p \leq \infty$ and let X and Y be Banach spaces such that X has the approximation property. Let $\mathcal{C}$ be a compact subset of*

$X\check{\otimes}Y$. *Then, for every* $\varepsilon > 0$, *there exist a linear mapping* Φ *from* span $\mathcal{C}$ *to* $C_p\check{\otimes}Y$ *and an operator* $A \in \mathcal{K}(C_p, X)$ *with* $1 \le \|A\| \le 1+\varepsilon$ *such that* $u = (A \otimes Id)(\Phi u)$, *for all* $u \in$ span $\mathcal{C}$. *The mapping* Φ *restricted to* $\mathcal{C} \cup \{0\}$ *is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Proof. Recall that $\overline{\mathcal{F}}_{w^*}(Y^*, X) = \mathcal{K}_{w^*}(Y^*, X)$ whenever X or Y has the approximation property (see Theorem 4.3.1). Consequently, in this case $X\check{\otimes}Y$ can be canonically identified with $\mathcal{K}_{w^*}(Y^*, X)$. Recall now that the spaces C_p have the approximation property, and apply Theorem 4.3.2.

To verify the equality $u = (A \otimes Id)(\Phi u)$, we rely on the following easy fact. If $v \in C_p\check{\otimes}Y$ is canonically identified with $\widehat{v} \in \mathcal{K}_{w^*}(Y^*, C_p)$, then $(A \otimes Id)v \in X\check{\otimes}Y$ is canonically identified with the operator $A \circ \widehat{v} \in \mathcal{K}_{w^*}(Y^*, X)$. Indeed, it clearly suffices to show the claim for $v = z \otimes y$ with $z \in C_p$ and $y \in Y$ (because the linear span of these tensors is dense in $C_p\check{\otimes}Y$). Consider any $y^* \in Y^*$. Then

$$((A \otimes Id)v)(y^*) = (Az \otimes y)(y^*) = y^*(y)Az$$

and

$$(A \circ \widehat{v})(y^*) = A(y^*(y)z) = y^*(y)Az$$

as needed.

Finally, let $u \in$ span $\mathcal{C}$. Then $\Phi u \in C_p\check{\otimes}Y$, and $(A \otimes Id)(\Phi u)$ is canonically identified with the operator $A \circ (\widehat{\Phi u}) = \widehat{u} \in \mathcal{K}_{w^*}(Y^*, X)$ as desired. $\square$

Corollary 4.5.2. *Let* $1 \le p \le \infty$, *let* X *be an* $\mathcal{L}_{p,\lambda}$*-space, and let* Y *be a Banach space. Let* $\mathcal{C}$ *be a compact subset of* $X\check{\otimes}Y$. *Then, for every* $\varepsilon > 0$, *there exist a linear mapping* Φ *from* span $\mathcal{C}$ *to* $\ell_p\check{\otimes}Y$ *and an operator* $A \in \mathcal{K}(\ell_p, X)$ *with* $1 \le \|A\| \le \lambda+\varepsilon$ *such that* $u = (A \otimes Id)(\Phi u)$, *for all* $u \in$ span $\mathcal{C}$. *The mapping* Φ *restricted to* $\mathcal{C} \cup \{0\}$ *is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Proof. The proof is similar to the proof of Corollary 4.5.1. It applies Theorem 4.4.1 instead of Theorem 4.3.2 and uses the fact that the $\mathcal{L}_p$-spaces and the spaces ℓ_p have the approximation property. It also uses the easy fact that if $v \in l_p\check{\otimes}Y$ is canonically identified with $\widehat{v} \in \mathcal{K}_{w^*}(Y^*, \ell_p)$, then $(A \otimes Id)v \in X\check{\otimes}Y$ is canonically identified with the operator $A \circ \widehat{v} \in \mathcal{K}_{w^*}(Y^*, X)$. $\square$

Let us consider the particular case of Corollaries 4.5.1 and 4.5.2 when $Y = C(K)$, the Banach space of continuous functions on a compact Hausdorff space K. It is well known that $X\check{\otimes}C(K)$ can be identified with $C(K; X)$, the Banach space of continuous X-valued functions on K. This canonical identification is given by the linear isometry from $X \otimes C(K)$

(equipped with the injective tensor norm) into $C(K;X)$ which is defined by

$$\sum_{n=1}^{m} x_n \otimes f_n \to \sum_{n=1}^{m} f_n(t)x_n, \; t \in K.$$

Therefore Corollaries 4.5.1 and 4.5.2 yield immediately the following representation of $\mathcal{C} \subset C(K;X)$ through the subset $\Phi(\mathcal{C})$ of $C(K;C_p)$ (respectively, of $C(K;\ell_p)$).

Corollary 4.5.3. *Let $1 \le p \le \infty$ and let X be a Banach space having the approximation property. Let K be a compact Hausdorff space. If $\mathcal{C}$ is a compact subset of $C(K;X)$, then for every $\varepsilon > 0$, there exist a linear mapping Φ from* span $\mathcal{C}$ *to $C(K;C_p)$ and an operator $A \in \mathcal{K}(C_p, X)$ with $1 \le \|A\| \le 1+\varepsilon$ such that $f = A \circ (\Phi f)$, for all $f \in$* span $\mathcal{C}$. *The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Corollary 4.5.4. *Let $1 \le p \le \infty$ and let X be an $\mathcal{L}_{p,\lambda}$-space. Let K be a compact Hausdorff space. If $\mathcal{C}$ is a compact subset of $C(K;X)$, then for every $\varepsilon > 0$, there exist a linear mapping Φ from* span $\mathcal{C}$ *to $C(K;\ell_p)$ and an operator $A \in \mathcal{K}(\ell_p, X)$ with $1 \le \|A\| \le 1+\varepsilon$ such that $f = A \circ (\Phi f)$, for all $f \in$* span $\mathcal{C}$. *The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 3.1.1.*

Corollaries 4.5.1 and 4.5.2 may also be applied to identifications of $X \check{\otimes} Y$ as spaces of X-valued measures (e.g., when $Y = L_1(\mu)$ or $Y = ba(\mathcal{B})$, $\mathcal{B}$ being a Boolean algebra; see [DU, pages 223–224] and [GR1]) to deduce results similar to Corollaries 4.5.3 and 4.5.4.

Chapter 5

Uniform factorization for compact sets of operators acting from a Banach space to its dual space

Results of this chapter are published in [MO2].

5.1 Introduction and the main result

Main results of this chapter relies on Lemma 2.2.1 which is the isometric version of the Davis-Figiel-Johnson-Pełczyński factorization construction [DFJP] due to Lima, Nygaard, and Oja [LNO].

Recall that a is the unique solution of the equation

$$\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1, \ a > 1,$$

and K is a closed absolutely convex subset of B_X, the closed unit ball of a Banach space X. For each $n \in \mathbb{N}$, put $B_n = a^{n/2}K + a^{-n/2}B_X$. The Minkowski functional of B_n gives an equivalent norm $\|\cdot\|_n$ on X (see Proposition 2.1.3). Set

$$\|x\|_K = \Big(\sum_{n=1}^{\infty} \|x\|_n^2\Big)^{1/2},$$

define $X_K = \{x \in X : \|x\|_K < \infty\}$, and let $J_K : X_K \to X$ denote the identity embedding. Then $X_K = (X_K, \|\cdot\|_K)$ is a Banach space and $\|J_K\| \leq 1$ by Lemma 2.2.1 (i). Moreover X_K is reflexive if and only if K is weakly compact (see Lemma 2.2.1 (vii)), and J_K is compact if and only if K is compact; in this case X_K is separable (see Lemma 2.2.1 (vi)).

For a Banach space X, let us consider the following infinite direct sums in the sense of ℓ_2:

$$W_X = \Big(\sum_K (X^*)^*_K\Big)_2 \text{ and } Z_X = \Big(\sum_L (X^*)^*_L\Big)_2,$$

where K and L run, respectively, through the weakly compact and compact absolutely convex subsets of B_{X^*}. The spaces W_X and Z_X are reflexive. In Theorem 5.1.1 below, which is the main result of this chapter, they will, respectively, serve as universal factorization spaces for all compact sets of the spaces $\mathcal{W}(X, X^*)$ and $\mathcal{K}(X, X^*)$.

Theorem 5.1.1. *Let X be a Banach space. Let $W = W_X$ and $Z = Z_X$. Then, for every compact subset $\mathcal{C}$ of $\mathcal{W}(X, X^*)$, there exist norm one operators $u, v \in \mathcal{W}(X, W)$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{W}(W, W^*)$ which preserves finite rank and compact operators such that $S = v^* \circ \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying*

$$\|S - T\| \le \|\Phi(S) - \Phi(T)\|$$

$$\le \min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} \|S - T\|^{1/4}\Big\}, \quad S, T \in \mathcal{C} \cup \{0\},$$

where

$$\mathrm{d} = \operatorname{diam} \mathcal{C} \cup \{0\}.$$

In particular, if $-S \in \mathcal{C}$ for some $S \in \mathcal{C}$, then

$$\|\Phi(S)\| \le \min\Big\{\frac{\mathrm{d}}{2}, \Big(\frac{\mathrm{d}}{2}\Big)^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} \|S\|^{1/4}\Big\}.$$

Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}(X, X^*)$, then W is everywhere replaced by Z, and u and v are compact operators.

Remark 5.1.2. Observe that $\operatorname{diam} \Phi(\mathcal{C} \cup \{0\}) = \operatorname{diam} \mathcal{C} \cup \{0\}$ in Theorem 5.1.1.

5.2 Proof of the main result

The proof of Theorem 5.1.1 uses Lemmas 5.2.1 and 5.2.2 below. These lemmas are, respectively, immediate consequences of Lemmas 3.1.1 and 3.1.2 because $\mathcal{W}(X, Y)$ and $\mathcal{K}(X, Y)$ are canonically isometrically isomorphic (under the mapping $S \to S^{**}$) with the spaces of the weak*-weak continuous operators $\mathcal{W}_{w^*}(X^{**}, Y)$ and $\mathcal{K}_{w^*}(X^{**}, Y)$, respectively.

Lemma 5.2.1. *Let X and Y be Banach spaces. Let $\mathcal{C}$ be a compact subset of $\mathcal{W}(Y, X^*)$. Then there exist a weakly compact absolutely convex subset K of B_{X^*}, which is compact whenever $\mathcal{C}$ is contained in $\mathcal{K}(Y, X^*)$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{W}(Y, (X^*)_K)$ such that $S = J_K \circ \Phi(S)$, for all $S \in \operatorname{span} \mathcal{C}$, and $\|J_K\| = 1$. Moreover, if $S \in \operatorname{span} \mathcal{C}$, then*

(i) *S has finite rank if and only if $\Phi(S)$ has finite rank,*

(ii) *S is compact if and only if $\Phi(S)$ is compact.*

The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying

$$\|S - T\| \le \|\Phi(S) - \Phi(T)\|$$

$$\le \min\Big\{\mathrm{d}, \mathrm{d}^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S - T\|^{1/2}\Big\}, \ \ S, T \in \mathcal{C} \cup \{0\},$$

where

$$\mathrm{d} = \ \operatorname{diam} \mathcal{C} \cup \{0\};$$

in particular, if $-S \in \mathcal{C}$ for some $S \in \mathcal{C}$, then

$$\|\Phi(S)\| \le \min\Big\{\ \frac{\mathrm{d}}{2},\ \Big(\frac{\mathrm{d}}{2}\Big)^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S\|^{1/2}\Big\}.$$

Lemma 5.2.2. *Let X be a Banach space. Let $\mathcal{C}$ be a compact subset of $\mathcal{W}(X, X^*)$. Then there exist a weakly compact absolutely convex subset K of B_{X^*}, a norm one operator $J \in \mathcal{W}(X, (X^*)_K^*)$, and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{W}((X^*)_K^*, X^*)$ satisfying conditions* (i) *and* (ii) *of Lemma 5.2.1 such that $S = \Phi(S) \circ J$, for all $S \in \operatorname{span} \mathcal{C}$. Moreover, if $\mathcal{C}$ is contained in $\mathcal{K}(X, X^*)$, then K is compact and $J \in \mathcal{K}(X, (X^*)_K^*)$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying the conclusions of Lemma 5.2.1.*

Proof of Theorem 5.1.1. Let $K \subset B_{X^*}$, $J \in \mathcal{W}(X, (X^*)_K^*)$, and

$$\varphi : \operatorname{span} \mathcal{C} \to \mathcal{W}((X^*)_K^*, X^*),$$

respectively, be the weakly compact absolutely convex subset, the norm one operator, and the linear mapping given by Lemma 5.2.2.

Since $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{W}((X^*)_K^*, X^*)$, we can apply Lemma 5.2.1. Let $L \subset B_{X^*}$ and

$$\psi : \operatorname{span} \varphi(\mathcal{C}) \to \mathcal{W}((X^*)_K^*, (X^*)_L),$$

respectively, be the weakly compact subset and the linear mapping given by Lemma 5.2.1.

Let $I_K : (X^*)^*_K \to W$ and $I_L : (X^*)^*_L \to W$ denote the natural norm one embeddings, and let $P_K : W \to (X^*)^*_K$ and $P_L : W \to (X^*)^*_L$ denote the natural norm one projections. It is straightforward to verify (observing that $\operatorname{diam} \varphi(\mathcal{C} \cup \{0\}) = \mathrm{d}$) that the mappings $u = I_K \circ J$, Φ, $v = I_L \circ J^*_L|_X$, and Φ defined by $\Phi(S) = P^*_L \circ \psi(\varphi(S)) \circ P_K,\ S \in \operatorname{span} \mathcal{C}$, have desired properties. In particular, for all $S \in \operatorname{span} \mathcal{C}$,

$$\begin{aligned} S &= \varphi(S) \circ J = J_L \circ \psi(\varphi(S)) \circ J \\ &= J_L \circ (P_L \circ I_L)^* \circ \psi(\varphi(S)) \circ P_K \circ I_K \circ J \\ &= J_L \circ I^*_L \circ P^*_L \circ \psi(\varphi(S)) \circ P_K \circ u \\ &= v^* \circ \Phi(S) \circ u, \end{aligned}$$

and therefore

$$\|S - T\| \leq \|\Phi(S) - \Phi(T)\|, \ \ S, T \in \mathcal{C} \cup \{0\}.$$

The "moreover" part uses that φ and ψ preserve compact operators. It also uses that K is a compact set and $J \in \mathcal{K}(X, (X^*)^*_K)$ whenever $\mathcal{C} \subset \mathcal{K}(X, X^*)$ (see Lemma 5.2.2) and that, in this case, $\varphi(\mathcal{C})$ is a compact subset of $\mathcal{K}((X^*)^*_K, X^*)$, implying (see Lemma 5.2.1) the compactness of the set L and of the operator J_L. □

Let us point out the following immediate consequence of Theorem 5.1.1.

Corollary 5.2.3. *Let X be a Banach space and let $Z = Z_X$. Then, for every compact subset $\mathcal{C}$ of $\mathcal{K}(X, X^*)$, there exist norm one operators $u, v \in \mathcal{K}(X, Z)$ and a linear mapping $\Phi : \operatorname{span} \mathcal{C} \to \mathcal{K}(Z, Z^*)$ such that $S = v^* \circ \Phi(S) \circ u$, for all $S \in \operatorname{span} \mathcal{C}$. The mapping Φ restricted to $\mathcal{C} \cup \{0\}$ is a homeomorphism satisfying*

$$\|S - T\| \leq \|\Phi(S) - \Phi(T)\|$$

$$\leq \min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} \|S - T\|^{1/4}\Big\}, \ \ S, T \in \mathcal{C} \cup \{0\},$$

where

$$\mathrm{d} = \ \operatorname{diam} \mathcal{C} \cup \{0\}.$$

In particular, if $-S \in \mathcal{C}$ for some $S \in \mathcal{C}$, then

$$\|\Phi(S)\| \leq \min\Big\{\frac{\mathrm{d}}{2}, \Big(\frac{\mathrm{d}}{2}\Big)^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} \|S\|^{1/4}\Big\}.$$

Chapter 6

On polynomials that are weakly uniformly continuous on the unit ball of a Banach space

Main results of this chapter (see Sections 6.2–6.4) are published in [M1] and [MO2].

6.1 Preliminaries and notation

This introductory section is based on the monograph [Di] by Dineen and the article [AP] by Aron and Prolla. Let $n \in \mathbb{N}$. Let $\mathcal{L}(^nX)$ denote the Banach space of all continuous n-linear forms on X, with the norm given by

$$\|A\| = \sup\{|A(x_1, \ldots, x_n)| : x_1, \ldots, x_n \in B_X\}.$$

An n-linear form $A \in \mathcal{L}(^nX)$ is said to be symmetric if

$$A(x_1, \ldots, x_n) = A(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for any $x_1, \ldots, x_n \in X$ and any permutation π of the first n natural numbers. Let $\mathcal{L}^s(^nX)$ denote the subspace of $\mathcal{L}(^nX)$ consisting of the symmetric n-linear forms.

Denote by $s : \mathcal{L}(^nX) \to \mathcal{L}^s(^nX)$ the symmetrization operator, defined by

$$s(A)(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\pi} A(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

where π runs over all permutations of the first n natural numbers. It can be easily verified that s is a linear norm one projection from $\mathcal{L}(^nX)$ onto $\mathcal{L}^s(^nX)$.

A continuous mapping $P : X \to \mathbb{K}$ is called an n-homogeneous polynomial if there exists $A \in \mathcal{L}(^nX)$ such that $P(x) = A(x, \dots, x)$ for every $x \in X$. Let $\mathcal{P}(^nX)$ denote the Banach space of continuous n-homogeneous polynomials on X, with the norm given by

$$\|P\| = \sup\{|P(x)| : x \in B_X\}.$$

Proposition 6.1.1. *For each $P \in \mathcal{P}(^nX)$ there is a unique $A_P \in \mathcal{L}^s(^nX)$ satisfying*

$$P(x) = A_P(x, \dots, x)$$

for each $x \in X$.

Recall that $P \in \mathcal{P}(^nX)$ is *weakly uniformly continuous* on the closed unit ball B_X of X if for each $\varepsilon > 0$ there are $x_1^*, \dots, x_n^* \in X^*$ and $\delta > 0$ such that if $x, y \in B_X$, $|x_i^*(x-y)| < \delta$ for $i = 1, \dots, n$, then $|P(x) - P(y)| < \varepsilon$.

Let $\mathcal{P}_{wu}(^nX)$ denote the subspace of $\mathcal{P}(^nX)$ consisting of the polynomials that are weakly uniformly continuous on B_X. The corresponding subspace of $\mathcal{L}^s(^nX)$ is denoted by $\mathcal{L}^s_{wu}(^nX)$. In [AP, Proposition 2.4] the following result is proved.

Proposition 6.1.2. *The subspace $\mathcal{P}_{wu}(^nX)$ of $\mathcal{P}(^nX)$, with the norm induced from $\mathcal{P}(^nX)$, is a Banach space.*

For each $P \in \mathcal{P}(^nX)$ there is a linear operator $T_P : X \to \mathcal{L}^s(^{n-1}X)$ defined by

$$(T_P x_1)(x_2, \dots, x_n) = A_P(x_1, x_2, \dots, x_n),\ x_1, \dots, x_n \in X.$$

Proposition 6.1.3. *The correspondence $A_P \to T_P$ is linear and $\|T_P\| = \|A_P\|$.*

According to [AP] the following holds.

Proposition 6.1.4. *Let $P \in \mathcal{P}(^nX)$. Then $P \in \mathcal{P}_{wu}(^nX)$ if and only if $T_P \in \mathcal{K}(X, \mathcal{L}^s(^{n-1}X))$. Moreover, if $P \in \mathcal{P}_{wu}(^nX)$, then $T_P \in \mathcal{K}(X, \mathcal{L}^s_{wu}(^{n-1}X))$.*

The relation between homogeneous polynomials and symmetric n-linear forms is described by the following polarization formula (see [Di, Theorem 1.5]) and its application (see [Di, Corollary 1.6 and Theorem 1.7]).

Proposition 6.1.5 (polarization formula). *Let* $P \in \mathcal{P}_{wu}(^nX)$ *and* $x_1, \dots, x_n \in X$. *Then*

$$A_P(x_1, \dots, x_n) = \frac{1}{2^n n!} \sum_{\substack{\varepsilon_i = \pm 1 \\ 1 \le i \le n}} \varepsilon_1 \dots \varepsilon_n P(\sum_{j=1}^n \varepsilon_j x_j).$$

Proposition 6.1.6. *The correspondence* $P \to A_P$ *is an isomorphism between* $\mathcal{P}(^nX)$ *and* $\mathcal{L}^s(^nX)$ *satisfying*

$$\|P\| \le \|A_P\| \le \frac{n^n}{n!}\|P\|, \;\; P \in \mathcal{P}(^nX).$$

6.2 Uniform factorization for compact sets of 2-homogeneous polynomials

Main result of this section is published in [MO2].

In this section we shall be interested in the case $n = 2$ of $\mathcal{P}(^nX)$, i.e., we shall be interested in 2-homogeneous polynomials. In this case, clearly, $\mathcal{L}^s_{wu}(^1X) = \mathcal{L}^s(^1X) = X^*$ and therefore $\mathcal{K}(X, \mathcal{L}^s_{wu}(^1X)) = \mathcal{K}(X, \mathcal{L}^s(^1X)) = \mathcal{K}(X, X^*)$. Basing on Section 6.1, this enables us to apply Corollary 5.2.3 to get the following uniform factorization result for compact sets of 2-homogeneous polynomials. Recall that

$$s(A)(x_1, x_2) = \frac{1}{2}\Big(A(x_1, x_2) + A(x_2, x_1)\Big), \;\; x_1, x_2 \in X, \;\; A \in \mathcal{L}(^2X).$$

Recall also that the space Z_X was defined in Section 5.1.

Theorem 6.2.1. *Let* X *be a Banach space and let* $Z = Z_X$.*Then, for every compact subset* $\mathcal{C}$ *of* $\mathcal{P}_{wu}(^2X)$, *there exist norm one operators* $u, v \in \mathcal{K}(X, Z)$, *and linear mappings* $\Psi : \operatorname{span} \mathcal{C} \to \mathcal{P}_{wu}(^2Z)$ *and* $\psi : \operatorname{span} \mathcal{C} \to \mathcal{L}(^2Z)$ *such that, for all* $P \in \operatorname{span} \mathcal{C}$,

$$P(x) = \psi(P)(ux, vx), \;\; x \in X,$$

and

$$s(\psi(P)) = A_{\Psi(P)}.$$

The mappings Ψ *and* ψ *restricted to* $\mathcal{C} \cup \{0\}$ *satisfy*

$$\max\Big\{\|P - Q\|, \|\Psi(P) - \Psi(Q)\|\Big\} \le \|\psi(P) - \psi(Q)\|$$

$$\le 2 \min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4}\|P - Q\|^{1/4}\Big\}, \;\; P, Q \in \mathcal{C} \cup \{0\},$$

where

$$\mathrm{d} = \ \operatorname{diam} \mathcal{C} \cup \{0\}.$$

In particular, if $-P \in \mathcal{C}$ for some $P \in \mathcal{C}$, then

$$\|\Psi(P)\| \le \|\psi(P)\| \le \min\Big\{\mathrm{d}, 2^{1/4}\mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4}\|P\|^{1/4}\Big\}.$$

Proof. Let $\mathcal{C}$ be a compact subset of $\mathcal{P}_{wu}(^2X)$. Then

$$\mathcal{K} := \{T_P : \ P \in \mathcal{C}\} \subset \mathcal{K}(X, X^*).$$

The set $\mathcal{K}$ is compact because the correspondence $P \to A_P \to T_P$ is continuous. Notice that

$$\operatorname{diam} \mathcal{K} \cup \{0\} \le 2\mathrm{d}$$

because $\|T_P - T_Q\| = \|A_P - A_Q\| \le 2\|P - Q\|$ for all $P, Q \in \mathcal{P}(^2X)$.

Applying Corollary 5.2.3 to the compact subset $\mathcal{K} \subset \mathcal{K}(X, X^*)$, there are norm one operators $u, v \in \mathcal{K}(X, Z)$ and a linear mapping $\Phi : \operatorname{span} \mathcal{K} \to \mathcal{K}(Z, Z^*)$ such that $T_P = v^* \circ \Phi(T_P) \circ u$, for all $T_P \in \operatorname{span} \mathcal{K}$. Now, $\Phi(T_P) \in \mathcal{K}(Z, Z^*)$, but $\Phi(T_P)$ need not be of the form T_Q for some $Q \in \mathcal{P}(^2Z)$. Let us therefore consider the mapping $\sigma \in \mathcal{L}(\mathcal{K}(Z, Z^*), \mathcal{L}^s(^2Z))$ defined by

$$\sigma(S)(z_1, z_2) = \frac{1}{2}\Big((Sz_1)(z_2) + (Sz_2)(z_1)\Big), \ S \in \mathcal{K}(Z, Z^*), \ z_1, z_2 \in Z.$$

Observe that, in fact, $\sigma(S) \in \mathcal{L}^s_{wu}(^2Z)$ for all $S \in \mathcal{K}(Z, Z^*)$. Indeed, let $S \in \mathcal{K}(Z, Z^*)$. Then $\sigma(S) = A_Q$ for some $Q \in \mathcal{P}(^2Z)$. Since

$$(\sigma(S))(z_1, z_2) = \frac{1}{2}\Big((Sz_1)(z_2) + (S^*z_1)(z_2)\Big), \ z_1, z_2 \in Z,$$

we have $T_Q = (S + S^*)/2$. Hence $T_Q \in \mathcal{K}(Z, Z^*)$ and therefore $Q \in \mathcal{P}_{wu}(^2Z)$ meaning that $\sigma(S) \in \mathcal{L}^s_{wu}(^2Z)$.

This permits us to define a linear mapping $\Psi : \operatorname{span} \mathcal{C} \to \mathcal{P}_{wu}(^2Z)$ by

$$\Psi(P)(z) = \sigma(\Phi(T_P))(z, z), \ z \in Z, \ P \in \operatorname{span} \mathcal{C},$$

meaning that

$$A_{\Psi(P)} = \sigma(\Phi(T_P)), \ P \in \operatorname{span} \mathcal{C}.$$

We also define a linear mapping $\psi : \operatorname{span} \mathcal{C} \to \mathcal{L}(^2Z)$ by

$$\psi(P)(z_1, z_2) = (\Phi(T_P)z_1)(z_2), \quad z_1, z_2 \in Z, \ P \in \operatorname{span} \mathcal{C}.$$

Let now $P \in \operatorname{span} \mathcal{C}$. We have for all $x \in X$

$$P(x) = (T_P x)(x) = (v^* \Phi(T_P) u x)(x) = \psi(P)(ux, vx)$$

and we have for all $z_1, z_2 \in Z$

$$\begin{aligned} s(\psi(P))(z_1, z_2) &= \frac{1}{2}\Big(\psi(P)(z_1, z_2) + \psi(P)(z_2, z_1)\Big) \\ &= \sigma(\Phi(T_P))(z_1, z_2) = A_{\Psi(P)}(z_1, z_2). \end{aligned}$$

Let us finally consider the mappings Ψ and ψ restricted to $\mathcal{C} \cup \{0\}$. For all $P, Q \in \operatorname{span} \mathcal{C}$, we have, since $\|u\| = \|v\| = 1$,

$$\begin{aligned} \|P - Q\| &= \sup_{\|x\| \le 1} \|(P - Q)(x)\| = \sup_{\|x\| \le 1} |(\psi(P) - \psi(Q))(ux, vx)| \\ &\le \|\psi(P) - \psi(Q)\|. \end{aligned}$$

We also have

$$\begin{aligned} \|\Psi(P) - \Psi(Q)\| &\le \|A_{\Psi(P)} - A_{\Psi(Q)}\| = \|s(\psi(P) - \psi(Q))\| \\ &\le \|\psi(P) - \psi(Q)\|. \end{aligned}$$

For all $P, Q \in \mathcal{C} \cup \{0\}$, using the definition of ψ and Corollary 5.2.3, we have

$$\begin{aligned} \|\psi(P) - \psi(Q)\| &= \|\Phi(T_P) - \Phi(T_Q)\| \\ &\le \min\Big\{2\mathrm{d}, 2^{3/4}\mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4}\|T_P - T_Q\|^{1/4}\Big\}. \end{aligned}$$

Since

$$\|T_P - T_Q\| = \|A_P - A_Q\| \le 2\|P - Q\|,$$

we have

$$\begin{aligned} \|\psi(P) - \psi(Q)\| &\le \min\Big\{2\mathrm{d}, 2^{3/4}\mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4} 2^{1/4}\|P - Q\|^{1/4}\Big\} \\ &= 2\min\Big\{\mathrm{d}, \mathrm{d}^{3/4}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{3/4}\|P - Q\|^{1/4}\Big\} \end{aligned}$$

as needed.

If, in particular, $P, -P \in \mathcal{C}$, then the desired estimate for the norm of $\psi(P) = (\psi(P) - \psi(-P))/2$ immediately follows from the above.

□

6.3 Quantitative strengthening of a result of Aron, Lindström, Ruess, and Ryan concerning polynomials

Results of this section are published in [M1].

In 1999, Aron, Lindström, Ruess, and Ryan (see [ALRR, Proposition 5]) proved the following result.

Theorem 6.3.1 (Aron-Lindström-Ruess-Ryan). *Let X be a Banach space and let $n = 2, 3, \dots$. Let C_n be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}({}^{n-1}X))$. Then there is a compact subset C of X^* such that for all $S \in C_n$ and all $x \in X$*

$$|(Sx)(x, ..., x)| \leq \sup_{x^* \in\ C} |x^*(x)|^n.$$

Remark 6.3.2. Notice, that by Propositions 6.1.4, 6.1.2, and 6.1.6, the set C_n can be canonically identified with a relatively compact subset of $\mathcal{P}_{wu}({}^nX)$.

Theorem 6.3.1 together with its proof in [ALRR] gives no information about the size of the set C corresponding to the size of C_n.

In this section we shall prove the following quantitative strengthening of Theorem 6.3.1. We denote $|C| = \sup\{\|x\| : x \in C\}$, where C is a bounded set in a Banach space.

Theorem 6.3.3. *Let X be a Banach space and let $n = 2, 3, \dots$. Let C_n be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}({}^{n-1}X))$. Then there is a compact circled subset C of X^* with $|C| = \max\{|C_n|, 1\}$ such that for all $S \in C_n$ and all $x \in X$*

$$|(Sx)(x, \dots, x)| \leq \sup_{x^* \in\ C} |x^*(x)|^n.$$

The proof of Theorem 6.3.3 will be based on a factorization result that easily follows from Corollary 3.3.1.

Lemma 6.3.4. *Let X and Y be Banach spaces. For every relatively compact subset C of $\mathcal{K}(X, Y)$, there exist a reflexive Banach space Z, a linear mapping $\Phi : \operatorname{span} C \to \mathcal{K}(X, Z)$, and a norm one operator $v \in \mathcal{K}(Z, Y)$ such that $S = v \circ \Phi(S)$ for all $S \in \operatorname{span} C$. The mapping Φ restricted to C is a homeomorphism and satisfies*

$$\|S\| \leq \|\Phi(S)\| \leq \min\{|C|,\ |C|^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S\|^{1/2}\},$$

$S \in C$.

Proof. Since $\overline{\text{circ } C}$ is a compact subset of $\mathcal{K}\big(X, Y\big)$, by Corollary 3.3.1, there exist a reflexive Banach space Z, a linear mapping Φ : span $C \to \mathcal{K}\big(X, Z\big)$, and a norm one operator $v \in \mathcal{K}\big(Z, Y\big)$ such that $S = v \circ \Phi(S)$, for all $S \in$ span C. Moreover, the mapping Φ restricted to circ C is a homeomorphism satisfying

$$\|S\| \leq \|\Phi(S)\| \leq \min\Big\{ \frac{\mathrm{d}}{2}, \Big(\frac{\mathrm{d}}{2}\Big)^{1/2}\Big(\frac{1}{4} + \frac{1}{\ln a}\Big)^{1/2} \|S\|^{1/2}\Big\},$$

$S \in$ circ C, where d $=$ diam circ C.

Since for all $S \in C$

$$\|S\| = \frac{1}{2}\|2S\| = \frac{1}{2}\|S - (-S)\| \leq \frac{\mathrm{d}}{2},$$

we get $|C| \leq \mathrm{d}/2$. On the other hand, for all $S, T \in$ circ C, we have $S = \lambda S_0$ and $T = \mu T_0$ for some $S_0, T_0 \in C$ and for some $\lambda, \mu \in \mathbb{K}$ with $|\lambda|, |\mu| \leq 1$. Hence

$$\begin{aligned}\|S - T\| &\leq \|S\| + \|T\| = \|\lambda S_0\| + \|\mu T_0\| \\ &= |\lambda|\|S_0\| + |\mu|\|T_0\| \leq \|S_0\| + \|T_0\| \leq |C| + |C|,\end{aligned}$$

$S, T \in C$. Therefore $\mathrm{d}/2 \leq |C|$. Consequently, $\mathrm{d}/2 = |C|$, and we are done. □

The proof of Theorem 6.3.3 follows the idea of the proof in [ALRR, Proposition 5].

Proof of Theorem 6.3.3. We proceed by induction on $n = 2, 3, \ldots$. Let C_2 be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}(^1X)) = \mathcal{K}(X, X^*)$. By Lemma 6.3.4 there are a Banach space Z, a linear mapping Φ : span $C_2 \to \mathcal{K}(X, Z)$, and a norm one operator $v \in \mathcal{K}(Z, X^*)$ such that $S = v \circ \Phi(S)$ for all $S \in$ span C_2. Then for all $S \in C_2$ and for all $x \in X$,

$$|(Sx)(x)| = |v(\Phi(S)x)(x)| = |(v^*x)(\Phi(S)x)|,$$

hence

$$|(Sx)(x)| \leq \|v^*x\|\|\Phi(S)x\|.$$

Put

$$C_\Phi = \overline{\{(\Phi(S))^*(z^*) : S \in C_2, z^* \in B_{Z^*}\}} \subset X^*.$$

Then C_Φ is circled. For proving that it is also compact, let us fix an arbitrary $\varepsilon > 0$. Let $\{\Phi(S_1), \ldots, \Phi(S_n)\}$, $S_k \in C_2$, be an ε-net in the relatively compact set $\{\Phi(S) : S \in C_2\}$. Since $\Phi(S_k)$ is a compact operator, $(\Phi(S_k))^*$ is also a compact operator and therefore $(\Phi(S_k))^*(B_{Z^*})$ is a relatively compact set. Since $\bigcup_{k=1}^n (\Phi(S_k))^*(B_{Z^*})$ is clearly a relatively compact ε-net in the set $\{(\Phi(S))^*(z^*) : S \in C_2, z^* \in B_{Z^*}\}$, this set is relatively compact. Hence, C_Φ is a compact set.

Moreover, we get

$$\begin{aligned}\|\Phi(S)x\| &= \sup_{z^*\in B_{Z^*}} |z^*(\Phi(S)x)| = \sup_{z^*\in B_{Z^*}} |((\Phi(S))^*(z^*))(x)| \\ &\le \sup_{x^*\in C_\Phi} |x^*(x)|\end{aligned}$$

for all $S \in C_2$ and for all $x \in X$.

Denoting

$$C_v = \overline{v(B_Z)} \subset X^*,$$

we have that C_v is circled and compact, and

$$\|v^*x\| = \sup_{z\in B_Z} |(v^*x)(z)| = \sup_{z\in B_Z} |(vz)(x)| \le \sup_{x^*\in C_v} |x^*(x)|$$

for all $x \in X$.

Finally, let $C = C_\Phi \cup C_v$. Then C is circled and compact, and

$$\begin{aligned}|(Sx)(x)| &\le \|v^*x\|\|\Phi(S)x\| \le \sup_{x^*\in C_v} |x^*(x)| \sup_{x^*\in C_\Phi} |x^*(x)| \\ &\le \sup_{x^*\in C} |x^*(x)|^2\end{aligned}$$

for all $S \in C_2$ and all $x \in X$.

By the definition of $|C|$,

$$\begin{aligned}|C| &= \sup_{x^*\in C} \|x^*\| = \sup_{x^*\in C_\Phi\cup C_v} \|x^*\| = \max\{\sup_{x^*\in C_\Phi} \|x^*\|, \sup_{x^*\in C_v} \|x^*\|\} \\ &= \max\{|C_\Phi|, |C_v|\}.\end{aligned}$$

Let us first estimate

$$\begin{aligned}|C_\Phi| &= \sup_{x^*\in C_\Phi} \|x^*\| = \sup_{\substack{S\in C_2 \\ z^*\in B_{Z^*}}} \|(\Phi(S))^*(z^*)\| \\ &= \sup_{S\in C_2} \|(\Phi(S))^*\| = \sup_{S\in C_2} \|\Phi(S)\|.\end{aligned}$$

Using the conclusion of Lemma 6.3.4, we have for all $S \in C_2$,

$$\|S\| \le \|\Phi(S)\| \le \sup_{S\in C_2} \|\Phi(S)\| = |C_\Phi|$$

and

$$\|\Phi(S)\| \le |C_2|.$$

Hence

$$|C_2| \leq |C_\Phi| \leq |C_2|,$$

meaning that $|C_\Phi| = |C_2|$. Let us now compute

$$|C_v| = \sup_{x^* \in C_v} \|x^*\| = \sup_{z \in B_Z} \|vz\| = \|v\| = 1.$$

Consequently,

$$|C| = \max \{|C_\Phi|,\ |C_v|\} = \max \{|C_2|,\ 1\}.$$

Assume that the result is true for $n-1$, where $n \in \{3, 4, \ldots\}$. Let C_n be a relatively compact subset of the space $\mathcal{K}(X, \mathcal{L}^s_{wu}({}^{n-1}X))$. By Lemma 6.3.4 there are a reflexive Banach space Z, a linear mapping $\Phi : \operatorname{span} C_n \to \mathcal{K}(X, Z)$, and a norm one operator $v \in \mathcal{K}(Z, \mathcal{L}^s_{wu}({}^{n-1}X))$ such that $S = v \circ \Phi(S)$ for all $S \in \operatorname{span} C_n$. Then for all $S \in C_n$ and for all $x \in X$, considering $(x, \ldots, x) \in (\mathcal{L}^s_{wu}({}^{n-1}X))^*$ (note that if $A \in \mathcal{L}^s_{wu}({}^{n-1}X)$, then $\langle (x, \ldots, x), A \rangle = A(x, \ldots, x)$),

$$|(Sx)(x, \ldots, x)| = |v(\Phi(S)x)(x, \ldots, x)| = |(v^*(x, \ldots, x))(\Phi(S)x)|,$$

hence

$$|(Sx)(x, \ldots, x)| \leq \|v^*(x, \ldots, x)\| \|\Phi(S)x\|.$$

Put, as above,

$$C_\Phi = \overline{\{(\Phi(S))^*(z^*) : S \in C_n, z^* \in B_{Z^*}\}} \subset X^*.$$

Then C_Φ is circled and compact, and we get

$$\|\Phi(S)x\| = \sup_{z^* \in B_{Z^*}} |z^*(\Phi(S)x)| = \sup_{z^* \in B_{Z^*}} |((\Phi(S))^*(z^*))(x)| \leq \sup_{x^* \in C_\Phi} |x^*(x)|$$

for all $S \in C_n$ and for all $x \in X$. Recall that $v(B_Z)$ is a relatively compact subset of $\mathcal{L}^s_{wu}({}^{n-1}X)$. Hence

$$C_{n-1} := \{T_P :\ P \in \mathcal{P}_{wu}({}^{n-1}X),\ A_P \in v(B_Z)\} \subset \mathcal{L}(X, \mathcal{L}^s({}^{n-2}X))$$

is also relatively compact. According to Proposition 6.1.4,

$$C_{n-1} \subset \mathcal{K}(X, \mathcal{L}^s({}^{n-2}X)).$$

Therefore, by the induction hypothesis, there is a circled and compact subset $C_v \subset X^*$ with $|C_v| = \max\{|C_{n-1}|, 1\}$ such that

$$|(T_P x)(x, \ldots, x)| \leq \sup_{x^* \in C_v} |x^*(x)|^{n-1}$$

for all $P \in \mathcal{P}_{wu}({}^{n-1}X)$ with $A_P \in v(B_Z)$. Since $v(B_Z) \subset \mathcal{L}^s_{wu}({}^{n-1}X)$, for all $z \in B_Z$ there is $P \in \mathcal{P}_{wu}({}^{n-1}X)$ such that $vz = A_P$. By definition, $A_P(x, x, \dots, x) = (T_P x)(x, \dots, x)$, $x \in X$. Hence, for all $z \in B_Z$ and all $x \in X$,

$$|(vz)(x, \dots, x)| = |A_P(x, x, \dots, x)| = |(T_P x)(x, \dots, x)| \leq \sup_{x^* \in\ C_v} |x^*(x)|^{n-1}.$$

Therefore

$$\begin{aligned}\|v^*(x, \dots, x)\| &= \sup_{z \in B_Z} |(v^*(x, \dots, x))(z)| \\ &= \sup_{z \in B_Z} |(vz)(x, \dots, x)| \leq \sup_{x^* \in C_v} |x^*(x)|^{n-1}.\end{aligned}$$

Finally, let $C = C_\Phi \cup C_v$. Then C is circled and compact, and

$$\begin{aligned}|(Sx)(x, \dots, x)| &\leq \|v^*(x, \dots, x)\| \|\Phi(S)x\| \\ &\leq \sup_{x^* \in C_v} |x^*(x)|^{n-1} \sup_{x^* \in C_\Phi} |x^*(x)| \leq \sup_{x^* \in C} |x^*(x)|^n\end{aligned}$$

for all $S \in C_n$ and all $x \in X$.

To complete the proof, let us show that $|C| = \max\{|C_n|,\ 1\}$. Similarly to the case $n = 2$, we have

$$\begin{aligned}|C| &= \sup_{x^* \in C} \|x^*\| = \sup_{x^* \in C_\Phi \cup C_v} \|x^*\| = \max\{\sup_{x^* \in C_\Phi} \|x^*\|,\ \sup_{x^* \in C_v} \|x^*\|\} \\ &= \max\{|C_\Phi|, |C_v|\}\end{aligned}$$

and

$$\begin{aligned}|C_\Phi| &= \sup_{x^* \in C_\Phi} \|x^*\| = \sup_{\substack{S \in C_n \\ z^* \in B_{Z^*}}} \|(\Phi(S))^*(z^*)\| \\ &= \sup_{S \in C_n} \|(\Phi(S))^*\| = \sup_{S \in C_n} \|\Phi(S)\|.\end{aligned}$$

Using the conclusion of Lemma 6.3.4, we have for all $S \in C_n$,

$$\|S\| \leq \|\Phi(S)\| \leq |C_\Phi|$$

and

$$\|\Phi(S)\| \leq |C_n|.$$

Hence

$$|C_n| \leq |C_\Phi| \leq |C_n|,$$

meaning that $|C_\Phi| = |C_n|$. Let us show that $|C_v| = 1$. Recall that $|C_v| = \max\{|C_{n-1}|, 1\}$. Since

$$|C_{n-1}| = \sup_{T_P \in C_{n-1}} \|T_P\| = \sup_{A_P \in v(B_Z)} \|A_P\| \leq \sup_{z \in B_Z} \|vz\| = \|v\| = 1,$$

we clearly have $|C_v| = 1$. $\square$

6.4 Quantitative version of the Toma theorem

The result of this section is published in [M1].

The next characterization theorem is proved by Toma [T] (an alternative proof is given in [ALRR]).

Theorem 6.4.1 (Toma). *Let X be a Banach space, let $n = 2, 3, \ldots$, and let $P \in \mathcal{P}(^nX)$. The polynomial $P \in \mathcal{P}_{wu}(^nX)$ if and only if there is a compact subset C of X^* such that for all $x \in X$*

$$|P(x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

The following is a quantitative version of Theorem 6.4.1.

Corollary 6.4.2. *Let X be a Banach space, let $n = 2, 3, \ldots$, and let $P \in \mathcal{P}(^nX)$. The following are equivalent:*
(a) *$P \in \mathcal{P}_{wu}(^nX)$,*
(b) *there is a compact subset C of X^* such that for all $x \in X$*

$$|P(x)| \leq \sup_{x^* \in C} |x^*(x)|^n,$$

(c) *there is a compact circled subset C of X^* with*

$$\max\{\|P\|, 1\} \leq |C| \leq \max\{\frac{n^n}{n!}\|P\|, 1\}$$

such that for all $x \in X$

$$|P(x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

Proof. (a) $\Rightarrow$ (c). Let $P \in \mathcal{P}_{wu}(^nX)$, then $\{T_P\} \subset \mathcal{K}(X, \mathcal{L}^s_{wu}(^{n-1}X))$. Applying Theorem 6.3.3 to $C_n = \{T_P\}$, we get that there is a compact circled subset C of X^* with $|C| = \max\{\|T_p\|, 1\}$ such that for all $x \in X$

$$|P(x)| = |A_P(x, x, \ldots, x)| = |(T_P x)(x, \ldots, x)| \leq \sup_{x^* \in C} |x^*(x)|^n.$$

Applying Propositions 6.1.6 and 6.1.3, we have

$$\|P\| \leq \|T_P\| \leq \frac{n^n}{n!}\|P\|.$$

Hence

$$\max\{\|P\|, 1\} \leq |C| \leq \max\{\frac{n^n}{n!}\|P\|, 1\}.$$

(c) $\Rightarrow$ (b). This is obvious.

(b) $\Rightarrow$ (a). This is immediate from Theorem 6.4.1. $\square$

Banachi operaatorruumide kompaktsete alamhulkade ühtlane faktorisatsioon

Olgu X ja Y Banachi ruumid. Kõikide ruumist X ruumi Y tegutsevate pidevate lineaarsete operaatorite Banachi ruumi tähistamiseks kasutame sümbolit $\mathcal{L}(X,Y)$ ning sümbolitega $\mathcal{F}(X,Y)$, $\overline{\mathcal{F}}(X,Y)$, $\mathcal{K}(X,Y)$ ja $\mathcal{W}(X,Y)$ tähistame alamruume, mis on vastavalt lõplikumõõtmeliste, aproksimeeritavate, kompaktsete ja nõrgalt kompaktsete operaatorite ruumid. Kui $\mathcal{A}$ on $\mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{K}$, $\mathcal{W}$ või $\mathcal{L}$, siis $\mathcal{A}_{w^*}(X^*,Y)$ tähistab ruumi $\mathcal{A}(X^*,Y)$ *nõrk-nõrkade operaatorite alamruumi.

Olgu antud operaator $S \in \mathcal{L}(X,Y)$. Kui Banachi ruumi Z ning operaatorite $u \in \mathcal{L}(X,Z)$ ja $v \in \mathcal{L}(Z,Y)$ korral $S = v \circ u$, siis öeldakse, et operaator S faktoriseerub läbi ruumi Z.

Alates Grothendiecki ja Pietschi töödest, vastavalt 1950-ndatel aastatel ja 1960-ndate aastate lõpus, on operaatorite faktoriseerimist läbi klassikaliste ruumide Z või ruumidest X ja Y lihtsamate ruumide Z uurinud paljud matemaatikud (vt. näiteks monograafiat [DJT] aastast 1995).

Aastal 1971 tõestas Johnson [J], et iga aproksimeeritav operaator faktoriseerub läbi ruumi C_p, $1 \le p \le \infty$.

Tuginedes Johnsoni teoreemile, tõestas Figiel [F] aastal 1973, et iga kompaktne operaator faktoriseerub läbi ruumi C_p kinnise alamruumi.

Teoreem (Figiel-Johnson). *Olgu $1 \le p \le \infty$. Olgu X ja Y Banachi ruumid. Kui $S \in \mathcal{K}(X,Y)$, siis leidub ruumi C_p kinnine alamruum W ning operaatorid $u \in \mathcal{K}(X,W)$ ja $v \in \mathcal{K}(W,Y)$ nii, et $S = v \circ u$.*

Kompaktsete operaatorite faktoriseeruvuse läbi klassikaliste jadaruumide ℓ_1 ja c_0 tõestasid Randtke [R, järeldus 7], Terzioğlu [Te, lk 252] ja Dazord [Da, lause 5.12].

Teoreem (Randtke). *Olgu X $\mathcal{L}_1$-ruum ja Y Banachi ruum. Kui $S \in$*

$\mathcal{K}(X,Y)$, siis leiduvad operaatorid $u \in \mathcal{K}(X,\ell_1)$ ja $A \in \mathcal{K}(\ell_1,Y)$ nii, et $S = A \circ u$.

Teoreem (Terzioğlu-Dazord). *Olgu X $\mathcal{L}_\infty$-ruum ja Y Banachi ruum. Kui $S \in \mathcal{K}(X,Y)$, siis leiduvad operaatorid $u \in \mathcal{K}(X,c_0)$ ja $A \in \mathcal{K}(c_0,Y))$ nii, et $S = A \circ u$.*

Randtke ja Terzioğlu-Dazord'i teoreemid annavad ühe kompaktse operaatori faktorisatsiooni läbi ruumide ℓ_1 ja c_0. Enam kui kümme aastat hiljem, aastal 1987 tõestasid Graves ja Ruess [GR2, teoreem 2.1] järgneva kompaktsete operaatorite kompaktsete alamhulkade faktorisatsiooniteoreemi läbi ruumide ℓ_1 ja c_0.

Teoreem (Graves-Ruess). *Olgu X $\mathcal{L}_1$-ruum (vastavalt $\mathcal{L}_\infty$-ruum) ja Y Banachi ruum. Olgu $\mathcal{C}$ suhteliselt kompaktne alamhulk ruumis $\mathcal{K}(X,Y)$. Siis leidub operaator $u \in \mathcal{K}(X,\ell_1)$ (vastavalt $u \in \mathcal{K}(X,c_0)$) ja suhteliselt kompaktne alamhulk $\{A_S : S \in \mathcal{C}\}$ ruumis $\mathcal{K}(\ell_1,Y)$ (vastavalt ruumis $\mathcal{K}(c_0,Y)$) nii, et $S = A_S \circ u$ iga $S \in \mathcal{C}$ korral.*

Kompaktsete operaatorite ühtlase faktoriseerumise üldisemal juhul annab järgmine Aron-Lindström-Ruess-Ryani teoreem (vt. [ALRR, teoreem 1]) aastast 1999, kus Z_{FJ} tähistab Figiel-Johnsoni universaalset faktorisatsiooniruumi (näiteks $Z_{FJ} = (\sum_{W \subset C_p} W)_p$, kus lõpmatu otsesumma on võetud üle ruumi C_p kõikide kinniste alamruumide W).

Teoreem (Aron-Lindström-Ruess-Ryan). *Olgu X ja Y Banachi ruumid ja olgu $\mathcal{C}$ suhteliselt kompaktne alamhulk ruumis $\mathcal{K}_{w^*}(X^*,Y)$. Siis leiduvad operaatorid $u \in \mathcal{K}_{w^*}(X^*,Z_{FJ})$ ja $v \in \mathcal{K}(Z_{FJ},Y)$ ning suhteliselt kompaktne alamhulk $\{A_S : S \in \mathcal{C}\}$ ruumis $\mathcal{K}(Z_{FJ},Z_{FJ})$ nii, et $S = v \circ A_S \circ u$ iga $S \in \mathcal{C}$ korral.*

Artiklis [ALRR, järeldus 4]) on näidatud, et Aron-Lindström-Ruess-Ryani teoreemist järeldub Graves-Ruessi teoreem.

Aron-Lindström-Ruess-Ryani ning Graves-Ruessi teoreemid koos tõestustega ei anna mingit informatsiooni hulkade vastavust kirjeldavate kujutuste omaduste kohta. Nimetagem mõned antud kontekstis kerkivad loomulikud küsimused. Kas need kujutused on homöomorfismid? Millised on nende pidevusomadused? Kuidas on omavahel seotud vastavate hulkade diameetrid?

Käesoleva doktoritöö põhitulemusena on tõestatud Graves-Ruessi ning Aron-Lindström-Ruess-Ryani teoreemide kvantitatiivsed versioonid, mis kirjeldavad operaatorite suhteliselt kompaktsete hulkade faktorisatsiooni Hölderi mõttes pidevate homöomorfismide kaudu, mille pöördkujutused on Lipschitzi mõttes pidevad, ning on leitud tõhusaid hinnanguid vastavuses olevate hulkade diameetrite kohta. Väitekirja neljandas peatükis on esitatud eelpool nimetatud teoreemide kvantitatiivsed versioonid (teoreemid 4.2.1 ja 4.4.3). Tõestamisel on tuginetud kolmandas peatükis saadud kompaktsete ja nõrgalt kompaktsete operaatorite kompaktsete alamhulkade ühtlasele faktorisatsioonile. Tõestuse idee (lemmad 3.1.1 ja 3.1.2 ning teoreemid 3.2.1, 3.2.2 ja 3.2.3) seisneb niisuguse kujutuse $S \to A_S$, kus $S \in \mathcal{C}$ ja $\mathcal{C}$ on nõrgalt kompaktsete operaatorite kompaktne alamhulk, konstrueerimises, mis säilitab kompaktsed ja lõplikumõõtmelised operaatorid, on Hölderi mõttes pidev ning mille pöördkujutus on Lipschitzi mõttes pidev. Seejuures $\operatorname{diam}\{A_S : S \in \mathcal{C}\} = \operatorname{diam} \mathcal{C}$ niipea, kui $0 \in \mathcal{C}$.

Kolmandas peatükis esitatud konstruktsioon tugineb kuulsa Davis-Figiel-Johnson-Pełczyński faktorisatsioonilemma [DFJP] Lima, Nygaardi ja Oja [LNO] isomeetrilisele versioonile, mis on esitatud teises peatükis (vt lemma 2.2.1). Võrdluseks märgime, et Graves-Ruessi teoreemi tõestus artiklis [GR2] on vägagi tehniline, tugineb Ruessi artiklis [Ru] tuletatud hulkade suhtelise kompaktsuse kriteeriumitele ruumis $\mathcal{K}_{w^*}(X^*, Y)$ ning kasutab Saphari tensorkorrutiste aparatuuri [S]. Artikkel [ALRR] annab Aron-Lindström-Ruess-Ryani teoreemile kaks erinevat tõestust, milledest üks toetub suurel määral Grothendiecki memuaaris [G] antud suhteliselt kompaktsete hulkade iseloomustusele Banachi ruumide projektiivses tensorkorrutises ning teine Banach-Dieudonné teoreemile.

Viiendas peatükis on tõestatud ühtlase faktorisatsiooni teoreem, mis kirjeldab ruumist X kaasruumi X^* tegutsevate kompaktsete ja nõrgalt kompaktsete operaatorite kompaktsete hulkade faktorisatsiooni Hölderi mõttes pidevate homöomorfismide, mille pöördkujutus on Lipschitzi mõttes pidev, kaudu. Kuuendas peatükis on peatükkides 4 ja 5 saadud tulemusi rakendatud polünoomidele. On tõestatud faktorisatsiooniteoreem 2-homogeensete polünoomide kompaktsete hulkade jaoks ja Aron-Lindström-Ruess-Ryani ning Toma teoreemide kvantitatiivsed versioonid Banachi ruumi ühikkeral nõrgalt ühtlaselt pidevate polünoomide jaoks.

Kolmandas ja neljandas peatükis esitatud tulemused on artiklist [MO1], viies peatükk tugineb artiklile [MO2] ja kuuenda peatüki peamised tulemused on artiklitest [M1] ja [MO2].

Bibliography

[ALRR] R. Aron, M. Lindström, W. M. Ruess, R. Ryan, *Uniform factorization for compact sets of operators*, Proc. Amer. Math. Soc. **127** (1999), 1119–1125.

[AP] R. M. Aron, J. B. Prolla, *Polynomial approximation of differentiable functions on Banach spaces*, J. Reine Angew. Math. **313** (1980), 195–216.

[DFJP] W. J. Davis, T. Figiel, W. B. Johnson, A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.

[Day] M. M. Day, *Normed Linear Spaces*, Springer, Berlin-Heidelberg-New York (1973).

[Da] J. Dazord, *Factoring operators through* c_0, Math. Ann. **220** (1976), 105–122.

[D1] J. Diestel, *Geometry of Banach Spaces – Selected Topics*, Lect. Notes in Math. **485**, Springer, Berlin-Heidelberg (1975).

[D2] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. **92**, Springer, New York (1984).

[DJT] J. Diestel, H. Jarchow, A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Math. **43**, Cambridge Univ. Press, Cambridge (1995).

[DU] J. Diestel, J. J. Uhl, Jr., *Vector Measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, RI (1977).

[Di] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Studies, **57**, Notas de Matematica **83**, North-Holland, Amsterdam-New York (1981).

[F] T. Figiel, *Factorization of compact operators and applications to the approximation problem*, Studia Math. **45** (1973), 191–210.

[GR1] W. H. Graves, W. M. Ruess, *Compactness and weak compactness in spaces of compact-range vector measures*, Canad. J. Math. **36** (1984), 1000–1020.

[GR2] W. H. Graves, W. M. Ruess, *Representing compact sets of compact operators and of compact range vector measures*, Arch. Math. (Basel) **49** (1987), 316–325.

[G] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[HHZ1] P. Habala, P. Hájek, V. Zizler, *Introduction to Banach Spaces* I, Charles University, Prague (1996).

[HHZ2] P. Habala, P. Hájek, V. Zizler, *Introduction to Banach Spaces* II, Charles University, Prague (1996).

[J] W. B. Johnson, *Factoring compact operators*, Israel J. Math. **9** (1971), 337–345.

[JL] W. B. Johnson, J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. 1, W. B. Johnson and J. Lindenstrauss (eds.), Elsevier, Amsterdam (2001), 1–84.

[LNO] Å. Lima, O. Nygaard, E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. **119** (2000), 325–348.

[LP] J. Lindenstrauss, A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. **29** (1968), 275–326.

[LR] J. Lindenstrauss, H.P. Rosenthal, *The $\mathcal{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349.

[LT] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces*, Lect. Notes in Math. **338**, Springer, Berlin-Heidelberg-New York (1973).

[M1] K. Mikkor, *On polynomials that are weakly uniformly continuous on the unit ball of a Banach space*, Proc. Estonian Acad. Sci. Phys. Math. **55** (2006), 16–23.

[MO1] K. Mikkor, E. Oja, *Uniform factorization for compact sets of weakly compact operators*, Studia Math. (to appear).

[MO2] K. Mikkor, E. Oja, *Uniform factorization for compact sets of operators acting from a Banach space to its dual space*, Acta Comment. Univ. Tartuensis Math. **9** (2005), 87–93.

[OPe] E. Oja, A. Pelander, *The approximation property in terms of the approximability of weak*-weak continuous operators*, J. Math. Anal. Appl. **286** (2003), 713–723.

[P] A. Pietsch, *Operator Ideals*, North-Holland Math. Library **20**, North-Holland, Amsterdam-New York-Oxford (1980).

[R] D. J. Randtke, *A compact operator characterization of* ℓ_1, Math. Ann. **208** (1974), 1–8.

[Ru] W. M. Ruess, *Compactness and collective compactness in spaces of compact operators*, J. Math. Anal. Appl. **84** (1981), 400–417.

[Ryan] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer, London-Berlin-Heidelberg (2002).

[S] P. D. Saphar, *Produits tensoriels d'espaces de Banach et classes d'applications linéaires*, Studia Math. **38** (1970), 71–100.

[Si] I. Singer, *Bases in Banach spaces* II, Editura Acad. R. S. România, Bucureşti, and Springer, Berlin (1981).

[SW] H. H. Schaefer, M. P. Wolff, *Topological Vector Spaces*, Grad. Texts in Math. **3**, Springer, New York (1999).

[Te] T. Terzioğlu, *Remarks on compact and infinite-nuclear mappings*, Math. Balkanica **2** (1972), 251–255.

[T] E. Toma, *Aplicações holomorfas e polinômios* τ*-contínuous*, Thesis, Univ. Federal do Rio de Janeiro (1993).

[W] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Studies in Advanced Math. **25**, Cambridge Univ. Press, Cambridge (1991).

Curriculum vitae

Name: Kristel Mikkor
Citizenship: Estonian Republic
Born: April 04, 1978, Kadrina, Estonia
Martial status: cohabit
Address: J. Liivi 2, 50409 Tartu, Estonia, Institute of Pure Mathematics, Faculty of Mathematics and Computer Science, University of Tartu
Telephone: (+372) 56691395
E-mail: Kristel.Mikkor@ut.ee

Education
1996-2000: Faculty of Mathematics, University of Tartu, *baccalaureus scientiarum* in mathematics
2000-2002: Faculty of Mathematics and Computer Science, University of Tartu, *magister scientiarum* in mathematics
2002-2006: Faculty of Mathematics and Computer Science, University of Tartu, PhD studies in mathematics,

Professional employment
2000-2002: Draftsman-CAD specialist, Tartu Maamõõdubüroo Llc.
2005- : Surveyor general-archivist, Omandi Llc.

Scientific work
Field of interest: functional analysis – geometry of Banach spaces and spaces of operators.

Curriculum vitae

Nimi: Kristel Mikkor
Kodakondsus: Eesti Vabariik
Sünniaeg ja -koht: 04. aprill 1978, Kadrina, Eesti
Perekonnaseis: vabaabielus
Aadress: J. Liivi 2, 50409 Tartu, Eesti, Puhta matemaatika instituut, Matemaatika-informaatikateaduskond, Tartu Ülikool
Telefon: (+372) 56691395
E-mail: Kristel.Mikkor@ut.ee

Hariduskäik
1996-2000: Tartu Ülikooli matemaatikateaduskond, *baccalaureus scientiarum* matemaatika erialal
2000-2002: Tartu Ülikooli matemaatika-informaatikateaduskond, *magister scientiarum* matemaatika erialal
2002-2006: Tartu Ülikooli matemaatika-informaatikateaduskond, doktoriõpingud matemaatika erialal

Teenistuskäik
2000-2002: arvutijoonestaja-spetsialist, OÜ Tartu Maamõõdubüroo
2005- **:** peageodeet-arhivaar, OÜ Omandi

Teadustegevus
Peamine uurimisvaldkond: funktsionaalanalüüs – Banachi ruumide ja operaatorite ruumide geomeetria.