DE
SUMMATIONE SERIERUM,
SECUNDUM
DATAM LEGEM DIFFERENTIATARUM.

CONSENSU

AMPLISSIMI PHILOSOPHORUM ORDINIS

IN UNIVERSITATE CAESAREA LITERARUM DORPATENSI,

MODERANTE

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§. 1.

Doctrina serierum campum late patentem praebet, magnam copiam meditationum evolvere; atque cum tanti momenti sit in Analysi, contemplatio serierum tantum utilitatis quantum voluptatis adfert. Quare meas meditationes huc spectantes, quamvis exiguae sint, in hac Dissertatiuncula exponere constitui: ac ne hujus libelli fines excedam, ad eas series me restringam, quae secundum quandam legem differentiatae evadunt — Sit scilicet proposita series.

$$a + b\,x + c\,x^2 + d\,x^3 + e\,x^4 + f\,x^5 + \text{etc.}$$

cujus summa $= \varphi x$ habeatur

quae multiplicata per $x^{n'}$ et differentiata, perque dx divisa, dabit

$$a\,n'\,x^{n'-1} + b(n'+1)\,x^{n} + c(n'+2)\,x^{n'+1} + d(n'+3)\,x^{n+2} + e(n'+4)\,x^{n'+3}\;\ldots\ldots$$

quae multiplicata per $x^{n''}$ ac denuo differentiata, dabit

$$a.\,n'.\,(n'-1+n'')\,x^{n'-1+n''-1} + b(n'+1)\,(n'+n'')\,x^{n'+n''-1} + c(n'+2)\,(n'+n''+1)\,x^{n'+n''} + \text{etc.}$$

quae rursus multiplicata per $x^{n'''}$ ac differentiata, et sic continuo multiplicata per $x^{n''''}$, $x^{n^{V}}$, $x^{n^{VI}}\ldots$ antequam differentiatio suscipiatur, perducet ad seriem hujus formae:

$$a.\,n'.\,(n'-1+n'')\,(n'-1+n''-1+n''')\,(n'-1+n''-1+n'''-1+n^{IV})\;\ldots\ldots\;(\Sigma(n-1)+1)\,x^{\Sigma(n-1)+}$$
$$+\; b(n'+1)\,(n'+n'')\,(n'-1+n''+n''')\,(n'-1+n''-1+n'''+n^{IV})\;\ldots\ldots\;(\Sigma(n-1)+2)\,x^{\Sigma(n-1)+1}$$
$$+\;\text{etc.}\ldots\ldots + p(1+n'+m)\,(n'+n''+m)\,(n'-1+n''+n'''+m)\ldots\ldots(\Sigma(n-1)+2+m)\,x^{\Sigma(n-1)+m+1} + \ldots\textit{etc.}$$

Quum factores, quibus Coefficientes componuntur, ita sint comparati, ut, pluribus n', n'', $n'''\ldots$ positis $= 1$, aequales evadant, haec forma seriei prodibit:

$$a.\,m^{\alpha}.\,m.^{\beta}\cdot m,^{\gamma}.\,m_3^{\delta}\ldots m_n^{\mu}\cdot\; x^{\Sigma(n-1)} + b(m+1)^{\alpha}\,(m_1+1)^{\beta}\,(m_2+1)^{\gamma}\ldots\ldots(m_n+1)^{\mu}\; x^{\Sigma(n-1)+1} + \text{etc.}$$
$$+\; p(m+k)^{\alpha}\;(m_1+k)^{P}\ldots(m_n+k)^{\mu}\; x^{\Sigma(n-1)+k} + \ldots\;\text{etc.}\ldots \qquad (R)$$

existentes α, β, $\gamma\ldots$ m; m_1; m_2; ... numeri postitivi et integri.

Tradidit illustris Euler in ejus Institutione Calculi Differentialis methodos, quarum ope series et hac forma contentae summari possint, dummodo α, β, $\gamma\ldots$ sint numeri determinati, plerumque tamen illae, ab evolutione differentiarum finitarum vel differentialium pendentes, gravioribus difficultatibus obvolutae sunt, dum series generalioris indolis sint, earumque summae ita assignandae, ut ad nullas amplius evolutiones neque differentiarum, neque differentialium perducant. Quare cum ad methodum, nondum in summatione serierum, quantum equidem sciam, adhibitam, pervenerim, cujus ope facilius summae istarum serierum generalioris indolis eruuntur, operis pretium fore videtur, eam in hoc libello exponere. Primo quidem, ut sponte appareat, quaenam artificia in usum vocanda sint, paulo accuratius perpendam seriem:

$$1 + 2^n x + 3^n x^2 + 4^n x^3 + 5^n x^4 + 6^n x^5 \dots + t^n x^{t-1} \qquad (R^1)$$

cujus summa per x multiplicata constat esse =

$$x^t \left(\frac{x}{x-1} t^n - \frac{x}{(x-1)^2} n t^{n-1} + \frac{x^2+x}{(x-1)^3} \cdot \frac{n(n-1)}{2} t^{n-2} - \frac{x^3+4x^2+x}{(x-1)^4} \cdot \frac{n(n-1)(n-2)}{2.\ 3} t^{n-3} + \text{etc.} \right) \pm C$$

quae reddat summam = o, si ponatur t = o.

Haec vero expressio reductionem admittit, quam quidem formam reductam e seriei contemplatione delineabo. Et primo quidem summam assignabo, si ponatur $x < 1$ et series in infinitum excurrat, ac deinde si pro libitu abrumpatur, et x quemcunque induat valorem. Quare duae hujus investigationis partes erunt.

§. 2.

Investigatio nova seriei $1 + 2^n x + 3^n x^2 \dots\dots \quad (R^1)$

Pars prima.

Si ponatur $x < 1$ et series in infinitum excurrat.

1. Quo casu constat esse

$$1 + x + x^2 + x^3 + x^4 \dots + \text{etc.} \dots = \frac{1}{1-x}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots \text{etc.} \dots$$

$$\frac{1}{(1-x)^3} = 1 + (1+2)x + (1+2+3)x^2 + (1+2+3+4)x^3 + (1+2+3+4+5)x^4 \dots\dots$$

quae series multiplicata per $(1+x)$ reddit

$$\frac{1+x}{(1-x)^3} = \begin{array}{l} 1 + (1+2)x + (1+2+3)x^2 + (1+2+3+4)x^3 + (1+2+3+4+5)x^4 \dots\dots \\ \quad\ + 1 \qquad + (1+2) \qquad + (1+2+3) \qquad + (1+2+3+4) \dots\dots \end{array}$$

$$= 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + 5^2 x^4 \dots \text{etc.} \dots$$

quia coefficienti praecedenti continuo additur 1 + 2; 2 + 3; 3 + 4; etc. qui componunt numeros impares, quare quadrati numerorum naturalium evadunt.

2. Jam inde supponi poterit, seriei

$$1 + 2^3 x + 3^3 x^2 + 4^3 x^3 + 5^3 x^4 \dots + \text{etc.} \dots$$

fore summam $= \frac{\varphi x}{(1-x)^4}$; ac simili modo

$$1 + 2^4 x + 3^4 x^2 + 4^4 x^3 + 5^4 x^4 \dots \text{etc.} = \frac{Fx}{(1-x)^5};$$

sicque porro, ita ut sit seriei (R¹) summa sub forma $\frac{fx}{(1-x)^{n+1}}$ contenta; denotantes φx, Fx, fx functiones ipsius x; si series pro lubito abrumpatur, istis functionibus aliquid addendum erit, quod sit = X. Posito x = 1 series potestatum numerorum abtinsbitur

$$1 \text{ CE } 2'' + 3'' + 4^n + 5^n \dots\dots + p^n.$$

Quodsi vero ponatur $x = 1$ erit $(1 - x) = 0$, inde, sumto finito terminorum numero, fieri oportet $\frac{fx + X}{(1 - x)^{n+1}} = \frac{0}{0}$, cujus ergo valor differentiatione usque ad ordinem $(n+1)$ instituta, et tum ponendo $x = 1$ eruetur: differentiatione vero producta, ex $(n+1)$ factoribus composita, evadent, quem numerum factorum et functio, qua $\sum x^n$ exprimitur, continet. Quare convenit concludere, denominatorem habiturum esse Exponentem $(n+1)$.

3. Si fx definito terminorum numero constare accipiatur, e serie (R') multiplicata per $(1 - x)^{n+1}$ expressionem finitam prodire oportet, quametsi series in infinitum excurrat, vel, quod idem est, coefficientes, datum terminum excedentes, in producto evanescere debent.

Instituatur Multiplicatio:

$$\begin{array}{llllll}
1 & + 2^n x & + 3^n x^2 & + 4^n x^3 & + 5^n x^4 & + 6^n x^5 \\
 & - (n+1) & - (n+1)2^n & - (n+1)3^n & - (n+1)4^n & - (n+1)5^n \\
 & & + \frac{(n+1)n}{2} & + \frac{(n+1)n}{2}2^n & + \frac{(n+1)n}{2}3 & + \frac{(n+1)n}{2}4'' \\
 & & & - \frac{(n+1)n(n-1)}{2\cdot 3} & - \frac{(n+1)n(n-1)}{2\cdot 3}2^n & - \frac{(n+1)n(n-1)}{2\cdot 3}3^n \\
 & & & & + \frac{(n+1)n(n-1)(n-2)}{2\cdot 3\cdot 4} & + \frac{(n+1)n(n-1)(n-2)}{2\cdot 3\cdot 4}2^n \\
 & & & & & - \frac{(n+1)n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4\cdot 5}
\end{array}$$

$$\begin{array}{lll}
+ 7^n x^6 & + \text{etc.} \;\ldots & + p^n x^{p-1} \\
- (n+1)6^n & & -(n+1)(p-1)^n \\
+ \frac{(n+1)n}{2}5^n & & + \frac{(n+1)n}{2}(p-2)^n \\
- \frac{(n+1)n(n-1)}{2\cdot 3}4^n & & - \frac{(n+1)n(n-1)}{2\cdot 3}(p-3)^n \\
+ \frac{(n+1)n(n-1)(n-2)}{2\cdot 3\cdot 4}3^n & & + \frac{(n+1)n(n-1)(n-2)}{2\cdot 3\cdot 4}(p-4)^n \\
- \frac{(n+1)n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4\cdot 5}2^n & & \vdots \\
+ \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6} & & + (p-(n+1))^n
\end{array}$$

Quod productum littera Q designabo.

Ejusmodi Coefficientes, quibus quantitates x^{n+1}, x^{n+2} et sequentes affectae, utique evanescere, jam e doctrina serierum recurrentium apparet, quod attamen nova demonstratione illustrabo.

Sit p^n Coefficiens quicunque seriei (R') inde a termino $(n+2)$, erit, existente $k > n$, Coefficiens ipsius x^k in producto Q hujus formae:

$$p^n - (n+1)(p-1)^n + \frac{(n+1)n}{2}(p-2)^n - \frac{(n+1)n(n-1)}{2\cdot 3}(p-3)^n \ldots \pm (p-(n+1))^n$$

qui, evolutis $(p-1)^n$; $(P-2)^n$; $(p-3)^n$; etc. formam sequentem induet:

$$p^n$$

$$-(n+1)\left[p^n - np^{n-1} + \frac{(n-1)}{2}p^{n-2} - \frac{n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \ldots\ldots \pm p^{n-n}\right]$$

$$+\frac{(n+1)n}{2}\left[p^n - 2np^{n-1} + \frac{2^2 n(n-1)}{2}p^{n-2} - \frac{2^3 n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{2^4 n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \cdots \pm 2^n p^{n-n}\right]$$

$$-\frac{(n+1)n(n-1)}{2\cdot 3}\left(p^n - 3np^{n-1} + \frac{3^2 n(n-1)}{2}p^{n-2} - \frac{3^3 n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{3^4 n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \ldots \pm 3^n p^{n-n}\right]$$

$$+\frac{(n+1)\ldots(n-2)}{2\cdot 3\cdot 4}\left(p^n - 4np^{n-1} + \frac{4^2 n(n-1)}{2}p^{n-2} - \frac{4^3 n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{4^4 n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \ldots\ldots \pm 4^n p^{n-n}\right]$$

$$-\frac{(n+1)\ldots(n-3)}{2\cdot 3\cdot 4\cdot 5}\left(p^n - 5np^{n-1} + \frac{5^2 n(n-1)}{2}p^{n-2} - \frac{5^3 n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{5^4 n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \ldots\ldots \pm 5^n p^{n-n}\right]$$

$$+\frac{(n+1)\ldots(n-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}\left[p^n - 6np^{n-1} + \frac{6^2 n(n-1)}{2}p^{n-2} - \frac{6^3 n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{6^4 n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \ldots\ldots \pm 6^n p^{n-n}\right]$$

$$-\frac{(n+1)\ldots(n-5)}{2\cdot 3\ldots 7}\left(\quad np^{n-1} + \frac{7^2 n(n-1)}{2}p^{n-2} - \frac{7^3 n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{7^4 n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \ldots\ldots \pm 7^n p^{n-n}\right]$$

$$\frac{(n+1)\ldots(n-6)}{2\cdot 3\ldots\ldots 8}\left(p^n - 8np^{n-1} + \frac{8^2 n(n-1)}{2}p^{n-2} - \frac{8^3 n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \frac{8^4 n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^{n-4} \ldots\ldots \pm 8^n p^{n-n}\right]$$

etc. etc. etc. etc. etc.

$$+\frac{(n+1)\cdots 1}{2\cdot 3\ldots(n+1)}\left[p^n - (n+1)np^{n-1} + (n+1)^2\frac{n(n-1)}{2}p^{n-2} - (n+1)^3\frac{n(n-1)(n-2)}{2\cdot 3}p^{n-3} + \ldots\ldots \pm (n+1)^n p^{n-n}\right]$$

Considerentur nunc jam Coefficientes, quibus quantitates p^n, p^{n-1}, etc. affectae.

Coefficiens ipsius p^n est $= 1 - (n+1) + \frac{(n+1)n}{2} \ldots \pm 1 = (1-1)^{n+1} = 0$.

Coefficiens ipsius $p^{n-1} = (n+1)n\left(1 - n + \frac{n(n-1)}{2} \ldots \pm 1\right) = (n+1)n(1-1)^n = 0$
sit modo $n > 0$.

Huc usque ergo Coefficientes evanescere patet; erit demonstrandum, idem in sequentibus locum habere. Facile autem perspicitur ex expressione generali, omnes prodituros esse $= 0$, si in genere

$$1 - 2^{n'}\frac{n}{2} + 3^{n'}\frac{n(n-1)}{2\cdot 3} - 4^{n'}\frac{n(n-1)(n-2)}{2\cdot 3\cdot 4} \ldots\ldots = 0$$

denotante n' numerum quemcunque positivum, integrum nec majorem ipso n.

Quod manifestum, existente $n' = 1$, quare videndum erit quomodo aequationes sequentes a praecedentibus pendeant. Accipiatur ergo aequatio

$$1 - 2^{n'}\frac{n}{2} + 3^{n'}\frac{n(n-1)}{2.\,3} - 4^{n'}\frac{n(n-1)(n-2)}{2.\,3.\,4} \dots\dots \pm \frac{(n+1)^{n'}}{n+1} = 0$$

quae locum habeat, dummodo ne sit $n' > n$

quae, multiplicata per $(n+1)$, simili modo erit $= 0$. Quare multiplicetur per $(n+1)$, vel, quod idem est, terminus primus per $(1+n)$, secundus per $(2+(n-1))$, tertius per $(3+(n-2))$, quartus per $(4+(n-3))$ et sic porro; obtinebitur

$$\left.\begin{array}{l} 1 - 2^{n'+1}.\frac{n}{2} + 3^{n'+1}.\frac{n(n-1)}{2.\,3} - 4^{n'+1}.\frac{n(n-1)(n-2)}{2.\,3.\,4} \dots\dots \pm \frac{(n+1)^{n'+1}}{n+1} \\ + \left[n - 2^{n'}\frac{n(n-1)}{2} + 3^{n'}\frac{n(n-1)(n-2)}{2.\,3} - 4^{n'}.\frac{n(n-1)(n-2)(n-3)}{2.\,3.\,4} \pm \frac{n^{n'}}{n}.\,n\right] \end{array}\right\} = 0$$

Cum pars secunda, sub parenthesi contenta, sit $=$

$$n\left[1 - 2^{n'}.\frac{n-1}{2} + 3^{n'}\frac{(n-1)(n-2)}{2.\,3} - 4^{n'}\frac{(n-1)(n-2)(n-3)}{2.\,3.\,4}\dots\right] = 0 \quad (2)$$

propterea quod evanescit, si demum $(1-1)^{n-n'} = 0$, inde sequitur, et primam partem

$$1 - 2^{n'+1}\frac{n}{2} + 3^{n'+1}\frac{n(n-1)}{2.\,3} - 4^{n'+1}\frac{n(n-1)(n-2)}{2.\,3.\,4}\ \text{etc. esse} = 0 \qquad (1)$$

Quo modo vero pendeat (2) ab aequatione $(1-1)^{n-n'} = 0$ sequenti modo clarius apparebit:

Designent $(1-1)_2^m$; $(1-1)_3^m$; $(1-1),''$; $(1-1)_n^m$ quantitates

$$1 - 2^2\frac{m}{2} + 3^2\frac{m(m-1)}{2.\,3} - 4^2\frac{m(m-1)(m-2)}{2.\,3.\,4}.\text{ etc.};\quad 1 - 2^3\frac{m}{2} + 3^3\frac{m(m-1)}{2.\,3} + \text{etc.};$$

$$1 - 2^4\frac{m}{2} + 3^4\frac{m(m-1)}{2.\,3} + \text{etc.} - ;\quad 1 - 2^n\frac{m}{2} + 3^n\frac{m(m-1)}{2.\,3} - \text{etc.}\dots$$

Quibus signis adhibitis, ratio, quam inter se quantitates constituant, sequenti tabula repraesentari poterit

$(1-1)_n^n$.

$(1-1)_{n-1}^n$; $(1-1)_{n-1}^{n-1}$.

$(1-1)_{n-2}^n$; $(1-1)_{n-2}^{n-1}$; $(1-1)_{n-2}^{n-2}$.

$(1-1)_{n-3}^n$; $(1-1)_{n-3}^{n-1}$; $(1-1)_{n-3}^{n-2}$; $(1-1)_{n-3}^{n-3}$.

$(1-1)_{n-4}^n$; $(1-1)_{n-4}^{n-1}$; $(1-1)_{n-4}^{n-2}$; $(1-1)_{n-4}^{n-3}$; $(1-1)_{n-4}^{n-4}$.

$(1-1)_{n-5}^n$; $(1-1)_{n-5}^{n-1}$; $(1-1)_{n-5}^{n-2}$; $(1-1)_{n-5}^{n-3}$; $(1-1)_{n-5}^{n-4}$; $(1-1)_{n-5}^{n-5}$.

etc. etc. etc. etc.

$(1-1)_2^n$; $(1-1)_2^{n-1}$; $(1-1)_2^{n-3}$; $(1-1)_2^{n-4}$; $(1-1)_2^{n-5}$; $(1-1)_2^{n-6}\dots(1-1)_2^2$.

$(1-1)^n$; $(1-1)^{n-1}$; $(1-1)^{n-2}$; $(1-1)^{n-3}$; $(1-1)^{n-4}$; $(1-1)^{n-5}$; $(1-1)^{n-6}\dots\dots(1-1)$

Quilibet terminus superior $(1-1)_q^p$ a duobus terminis inferioribus $(1-1)_{q-1}^p$ et $(1-1)_{q-1}^{p-1}$ pendet, ac cum quivis seriei infimae terminus evanescat, omnes usque ad supremum termin oportet fiant $= 0$. Ulterius autem progredi non licet, propterea quod $(1 - 1)_n^{n+1}$ ab aequatione $(1 - 1)^0 = 1$ demum pendeat.

Ex modo allatis perspicitur, omnes Coefficientes ipsarum p^n, p^{n-1}, p^{n-2} etc. evanescere, quare et quemcunque Coefficientem in Producto Q inde a termino $(n + 2)$ evanescere conseqiiitur.

4. Quodsi series (R') pro lubitu abrumpatur, hancce demonstrationem postremos Coefficientes producti Q non admittere, sed diversos valores induere, perspicuum.

Si fingatur seriem in infinitum excurrere, Coefficientes postremi producti Q in infinitum excrescent, ita ut fiant numeri infiniti ordinis n. Qui tamen termini evanescunt, cum affecti sint dignitate infinita ipsius $x < 1$. Sit enim $x = \frac{1}{1 + q}$ denotante q quemcunque valorem positivum, etsi minimum, ne tamen in infinitum decrescat, erit terminus, quem evanescere statuimus $= \frac{A \infty^n}{(1 + q)^\infty} = \frac{A. \infty^n}{1 + \infty q + \frac{\infty(\infty - 1)}{2} q^2 \quad \frac{\infty(\infty - 1)(\infty - 2)}{2.\ 3} q^3 + \text{etc.}}$

denotante A numerum finitum;

qui terminus, denominatore existente infinite magno respectu numeratoris, utique evanescit.

5. Ex allatis consequitur, fx definito terminorum, secundum potestates ipsius x progredientium, numero constare; erit ergo

$$fx = 1 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 \ldots\ldots + A_{n-1} x^{n-1}.$$

qui valores A_1, A_2, A_3, etc.... facillime ex producto Q inveniuntur.

$$A_1 = 2^n - (n + 1).$$

$$A_2 = 3^n - (n + 1) 2^n + \frac{(n + 1)n}{2}.$$

$$A_3 = 4^n - (n + 1) 3^n + \frac{(n + 1)n}{2} 2^n - \frac{(n + 1)n(n - 1)}{2.\ 3}.$$

$$A_4 = 5^n - (n + 1) 4^n + \frac{(n + 1)n}{2} 3^n - \frac{(n + 1)n(n - 1)}{2.\ 3} 2^n + \frac{(n + 1)n(n - 1)(n - 2)}{2.\ 3.\ 4}.$$

$$A_5 = 6^n - (n + 1) 5^n + \frac{(n + 1)n}{2} 4^n - \frac{(n + 1)n(n - 1)}{2.\ 3} 3^n + \frac{(n + 1)n(n - 1)(n - 2)}{2.\ 3.\ 4} 2^n - \frac{(n + 1)n(n - 1)(n - 2)(n - 3)}{2.\ 3.\ 4.\ 5}$$

$$A_6 = 7^n - (n + 1) 6^n + \frac{(n + 1)n}{2} 5^n - \frac{(n + 1)n(n - 1)}{2.\ 3} 4^n + \text{etc.}$$

Perspicuum, esse $A_{n-1} = 1$, $A_{n-2} = A_1$, $A_{n-3} = A_2$, $A_{n-4} = A_3$, etc.; Coefficientem enim termini $(n + 2)$ in Producto Q evanescere ostendi, quare, $\frac{(p - p)^n}{x}$, $\frac{(p - (p + 1))^n}{x^2}$, $\frac{(p - (p + 2))^n}{x^3}$; $\frac{(p - (p + 3))^n}{x^4}$ etc. antepositis seriei (R') ita ut hanc formam induat

$$\ldots + \ldots \frac{(p-(p+3))^n}{x^4} + \frac{(p-(p+2))^n}{x^3} + \frac{(p-(p+1))^n}{x^2} + \frac{(p-p)^n}{x} + 1 + 2^n x + 3^n x^2 \ldots$$

ac postea continuata multiplicatione, evadet in hoc novo producto et termini $(n+1)$ Coefficiens $= 0$, atque simili modo terminorum n, $n-1$, $n-1$, etc. Coefficientes evanescent Hinc sequitur Coefficientem termini $(n+1)$ in ipso Q esse $= 0 + 0$, termini $n = 0 + 1$, termini $(n-1) = 0 + 2^n - (n+1)$, termini $(n-2) = 0 + 3^n - (n+1)2^n + \frac{(n+1)n}{2}$ etiii genere A_{n-2}, A_{n-3}, A_{n-4} etc. eosdem valores induere quam A_1, A_2, A_3 etc.

6. Ex allatis nunc colligitur, seriei propositae (R′) esse summam

$$= \frac{1 + A_1 x + A_2 x^2 + A_3 x^3 \ldots + A_{n-1} x^{n-1}}{(1-x)^{n+1}}$$ existente $x < 1$ et serie in infinitum excurrente.

Pars secunda.

Si series pro lubitu abramparur, et x quemcunque induat valorem.

Decignet X functionem, functioni f x addendam, quo seriei summa abruptae obtineatur; sit $t^n x^{t-1}$ terminus seriei ultimus, et consideretur pars ultima Producti Q, quae erit:

$$\begin{array}{llll}
 & t^n.\; x^{t-1} & & \\
 & -(n+1)(t-1)^n & -(n+1)t^n x^t & \\
 & +\frac{(n+1)n}{2}(t-2)^n & +\frac{(n+1)n}{2}(t-1)^n & +\frac{(n+1)n}{2}t^n x^{t+1} \\
-\frac{(n+1)n(n-1)}{2.\ 3}(t-3)^n & -\frac{(n+1)n(n-1)}{2.\ 3}(t-2)^n & -\frac{(n+1)n(n-1)}{2.\ 3}(t-1)^n & -\frac{(n+1)n(n-1)}{2.\ 3}t^n x^{t+2} + \text{etc.} \\
\text{etc.} & \text{etc.} & \text{etc.} & \text{etc.}
\end{array}$$

Inde sequitur fore $X =$

$$- (t+1)^n x^t + [(n+1)(t+1)^n - (t+2)^n] x^{t+1} - \left[\frac{(n+1)n}{2}(t+1)^n - (n+1)(t+2)^n + (t+3^n)\right] x^{t+2}$$
$$+ \left[\frac{(n+1)n(n-1)}{2.\ 3}(t+1)^n - \frac{(n+1)n}{2}(t+2)^n + (n+1)(t+3)^n - (t+4)^n\right] x^{t+3}$$
$$- \left[\frac{(n+1)n(n-1)(n-2)}{2.\ 3.\ 4}(t+1)^n - \frac{(n+1)n(n-1)}{2.\ 3}(t+2)^n + \frac{(n+1)n}{2}(t+3)^n\right.$$
$$\left. - (n+1)(t+4)^n + (t+5)^n\right] x^{t+4} + \ldots \text{etc.}$$

Quam functionem X e producto quidem valores t^n, $(t-1)^n$, $(t-2)^n$ etc. continere invenitur, sed facile perspicitur, inde ejusmodi, ad primos terminos ipsius X magis accommodatam, formam, sicut eam expressi, pendere.

§. 3.

Designet in genere fx fiinctionem, definito terminorum numero constantem; et Φx functionem quamcunque ipsius x. Manifestum est, functionem fx facillime erui, si summa seriei propositae T ad formam $\frac{fx}{\Phi x} + U$, denotante $\mathfrak{A}_1$ functionem cognitam ipsiiis x, reduci queat; prodibit enim aequatio $fx = (T - \mathfrak{A}_1)\,\Phi x$. Jam nunc, quomodo haec forma inveniatur, investigabo.

Res eo redit, ut inveniatur ejus modi Φx, quae coefficientes Producti $(T - \mathfrak{A}_1)\,\Phi x$ evoluti inde a quodam termino assignabili reddat $= 0$. Sit φx ita coinparata, ut ejus differentiale ordinis p determinati sit $=$
$(d + d_1 x + d_2 x' + d_3 x^3 + d_4 x \;\ldots\ldots + d_m x^m)^v = (f'x)^v$, cujus φx differentiatione evadant series formae (R) (§. 1); denotante v quemcunque numerum positivum, negativum, vel fractum. Instituatur differentiatio secundum legem (§. 1) indicatam, et obtinebitur:

$$\frac{d.\;x^{n'}.\;\varphi x}{dx} = n' x^{n'-1}\,\varphi x + x^{n'}\,\frac{d.\varphi x}{dx}$$

$$\frac{d.\,x^{n''}.\,d.\,x^{n'}.\,\varphi x}{dx^2} = e''\,x^{n'-1+n''-1}\,\varphi x + e_1''\,x^{n'-1+n''}\,\frac{d.\,\varphi x}{dx} + e_2''\,x^{n'+n''}\,\frac{d^2.\,\varphi x}{dx^2}$$

$$\frac{d.\;x^{n'''}.\,d.x^{n''}.\,d.x^{n'}.\;\varphi x}{dx^3} = e'''\,x^{n'-1+n''-1+n'''-1}\,\varphi x + e_1'''x^{n'-1+n''-1+n'''}\frac{d.\,\varphi x}{dx} + e_2'''\,x^{n'-1+n''+n'''}\frac{d^2.\varphi x}{dx^2}$$

$$+ e_3'''x^{n'+n''+n'''}.\quad \frac{d^3.\,\varphi x}{dx^3}$$

etc. etc.

Designet $d^m\left(\begin{matrix}\varphi x\\ x^{n}{}^{m}\end{matrix}\right)$ ejusmodi differentiale ordinis m, eritque

$$\frac{d^p\left(\begin{matrix}\varphi x\\ x^{n}{}^{p}\end{matrix}\right)}{dx^p} = e^p\;x^\sigma\;\varphi x + e_1^p\;x^{\sigma+1}.\;\;\frac{d.\,\varphi x}{dx} + e_2^p\;x^{\sigma+2}.\;\frac{d^2.\varphi x}{d\;x^2}$$

$$+ \text{etc.}\ldots\ldots + e_p^p\,x^{\sigma+p}\;(d + d_1\,x + d_2\,x^2\ldots\ldots + d_m\,x^m)^v$$

$$\frac{d^{p+1}\left(\begin{matrix}\varphi x\\ x^{n}{}^{p+1}\end{matrix}\right)}{dx^{p+1}} = e^{p+1}.\;x^{\sigma_1}.\;\varphi x + e_1^{(p+1)}\;x^{\sigma_1+1}.\;\frac{d.\,\varphi x}{dx}$$

$$+ \text{etc.}\ldots\ldots + e^{(p+1)}\;\;x^{\sigma_1+p}.\;(f'x)^v$$

$$+ (f'x)^{v-1}\,(d_1^1\,x^{\sigma_1+p+1} + d_2^1\,x^{\sigma_1+p+2} + d_3^1\,x^{\sigma_1+p+3}\;\ldots\ldots\ldots + d_m^1\,x^{\sigma_1+p+m})$$

$$\frac{d^{p+2}\left(\dfrac{\varphi x}{x^{n^{p+2}}}\right)}{dx^{p+2}} = e^{(p+2)}\; x^{\sigma 2}.\; \varphi x + e_1^{(p+2)} \frac{d.\varphi x}{dx} + \text{etc.}$$

$$+ e_p^{(p+1)}\; x^{\sigma 2+p}\; (f'x)^{V} + (d_1^{('')} x^{\sigma 2+p+1} + \dots + d_m^{('')} x^{\sigma 2+p+m})\; (f'x)^{V-1}$$

$$+ (d_1^{('')}\, 1\; x^{\sigma 2+p+2} + d_2^{('')}\, 1\; x^{\sigma 2+p+3} \;.\; ,\; . + d_{2m-1}^{('')}\, 1\; x^{\sigma 2+p+2m})\; (f'x)^{V-2}$$

$$\frac{d^{p+3}\left(\dfrac{\varphi x}{x^{n^{p+3}}}\right)}{dx^{p+3}} = e^{(p+3)} \dots \text{etc.} \;.\;.\;.\;.\;.\;,\;.\;.\;.\;.\;,\;.\;,\;.\;.$$

$$\vdots$$

etc. etc.

$$\frac{d^{n}\left(\dfrac{\varphi x}{x^{n^{n}}}\right)}{dx^{n}} = e^{n}.\; x^{\sigma n-p}\; \varphi x + e_1^{n}\; x^{\sigma n-p+1}.\, \frac{d.\varphi x}{dx^2}\; \text{etc.} \;.\;.\; + e_p^{n}\, x^{\sigma n-p.+p}\; (f'x)^{V}$$

$$+ (d_1^{(n-p)}\; x^{\sigma n-p+p+1} + \text{etc.} \;,\;.\;.\; + d_m^{(n-p)}\; x^{\sigma n-p+p+m})\; (f'x)^{V-1}$$

$$+ (d_1^{(n-p)}\, 1\; x^{\sigma n-p+p+2} + \;.\;.\;.\;,\;.\;)\; (f'x)^{V-2} + \text{etc.} \;.\;.\;.$$

$$+ (d_1^{\mu}\, x^{\sigma n-p+p+(n-p)} + \text{etc.} \;.\;.\;.\; + d_{\lambda}^{\mu}\, x^{\sigma n-p+p+(n-p)m})\; (f'x)^{V-(n-p)}$$

quod sit $= \mathfrak{A} + F\,(f'x)^{V} + F'\;(f'x)^{V-1} + F''\;(f'x)^{V-2} \;.\;.\;.\;.\; F^{n-p}\,(f'x)^{V-(n-p)}$

quae forma ad hanc reducitur

$$\mathfrak{A} + (F.\;(f'x)^{n-p} + F'\;(f'x)^{n-p+1} + F''\;(f'x)^{n-p+2}. + \text{etc.} \dots + F^{n-p-1}.\; f'x + F^{n-p})\,\frac{1}{fx^{n-p-V}}$$

quae evoluta hanc debit formam generalem:

$$U + \frac{Cx^{\sigma n-p+p} + C_{,}x^{\sigma n-p+1} + C_{,,}x^{\sigma n-p+2} + \text{etc.} \;.\;.\;.\; + C_{(n-p)\,m}x^{\sigma n-p+p+(n-p)m}}{(f'x)^{n-p-V}}$$

quae sit $= \dfrac{fx}{(f'x)^{p-n-V}}$

Litterae e, d, σ, C scilicet cum eorum signis denotant quantitates, per differentiationem et multiplicationem introductas.

Si nunc sit proposita series ejusmodi formae, ut (R) ejus summa sequenti modo assignari poterit:

E differentione ipsius φx *deducatur forma summae, quae sit* $\dfrac{fx}{\Phi x} + \mathfrak{A}$; *ac deinde ponatur*

$$fx = ((R) - U)\;\Phi x \;;\; (1)$$

Comparentur singuli termini, cumque ita pro quocunque Coefficiente nisi unicus detur valor, et x *in ipsa* fx *potestatem* $\sigma n - p + p + (n - p) m$ *non excedat, omnes Coefficientes Producti* $((R) - \mathfrak{A})\ \Phi x$ *ultra terminum, potestate* $\sigma n - p + p + (n - p) m$ *ipsius* x *affectum, progredientes neussario evanescere debent.*

Quod attinet ad functionem $\mathfrak{A}$, ea sine difficultate inveniri poterit, nam, cum p sit numerus determinatus, differentialia $\frac{d\varphi}{dx}$; $\frac{d^2\varphi}{dx^2}$, $\frac{d^3\varphi}{dx^3}$ . . . $\frac{d^p\varphi}{dx^p}$ assignari parsunt; ad quantitates e vero, quibus sunt affecta, abtinendas, consideretur, eosdem Coefficientes evadere, si loco φx differentietur e^{-x} simili modo. Designet E functionem quantitates e involventem, $d^n\left(\frac{e^{-x}}{\varepsilon x^n}\right)$ seriem, qua e^{-x} exprimitur, ita differentiatarn, prodibit $E = d^n\left(\frac{e^{-x}}{\varepsilon x^n}\right)e^x$, quae aequatio dabit valores e.

Jam nunc qnantitates C; C,; C,; C,;. . etc. ex aequatione (1) sine difficiiltate definiri possunt, si modo Φx ita comparata, ut ejus evolutio generalis vires Analyseos non superet.

Exemplum.

Proposita sit series cognita.

$$\text{Arc Sin } y = y + \tfrac{1}{2}\frac{y^3}{3} + \frac{1.3}{2.4}\frac{y^5}{5} + \frac{1.3.5}{2.4.6}\frac{y^7}{7} + \text{etc.}$$

e qua assignetur summa s seriei

$$\pm \frac{y}{Y} + \frac{1}{2.3} + \frac{1.3^n}{2.4}\frac{y^2}{5} + \frac{1.3.5^n}{2.4.6}\frac{y^4}{7} + \frac{1.3.5.7^n}{1.4.6.8}\frac{y^6}{9} + \text{etc.}$$

Differentietur $\frac{1}{y^2}$ Arc y, quo facto obtinebitur

$-\frac{2}{y^3}\text{Arc } y + \frac{1}{y^2(1-y^2)^{\frac{1}{2}}}$. Multiplicetur per x et denuo differentietur, sicque porro.

Erit $\mathfrak{A} = \pm 2^n \frac{1}{Y}\text{Arc } y$, + si n par, − si impar. $p = 1$; $\sigma n - p = -3$, quare $\sigma n - p + p = -2$; $m = 2$; $\sigma n - p + p + (n - p) m = 2(n - 2)$; $v = -\frac{1}{2}$; erit igitur forma summae $s = \pm 2^n \frac{1}{y^3}\text{Arc } y + \frac{C\frac{1}{y^2} + C_1 + C_2 y^2 \ldots + C_{n-1} y^{2(n-2)}}{(1-y^2)^{\frac{2(n-1)+1}{2}}}$ Ponatur $1 \pm 2'' = a$; $\frac{1 + 2^n}{2.3} = b$ $\frac{1.3(3^{n-1} \pm 2^n)}{2.4.5} = c$ etc. $\frac{2(n-1)+1}{2} = \mu$, prodibit ad determinandos Cotfficientes C, C, etc. aequatio.

$$\begin{array}{llllllll}
C\frac{1}{y^2} & & + & C_1 & + & C_2 y' & + & C_3 y' \quad \ldots\ldots + C_{n-1} y^{2(n-2)} = \\
\frac{a}{y''} & + b & + & cy^2 & + & dy' & + & ey' \quad \ldots\ldots \\
 & + \mu a & + & \mu b & + & \mu c & + & \mu d \quad \ldots\ldots \\
 & & + & \frac{\mu(\mu-1)}{2}a & + & \frac{\mu(\mu-1)}{2}b & + & \frac{1}{2} \quad \ldots\ldots \\
 & & & & + & \frac{\mu(\mu-1)(\mu-2)}{2.3}a & + & \frac{\mu(\mu-1)(\mu-2)}{2.3}b \ldots\ldots \\
 & & & & & & + & \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{2.3.4}a \ldots\ldots
\end{array}$$

§. 4.

Saepenumero series occurrunt, de quibus non statim perspicitur, quomodo earum summae ad formam (§. 3.) exhibitam reduci queant. In usum tum erunt artificia vocanda, quorum ope aequationes obtineantur, quae ita comparatae sint, ut quantitates determinandas in altera parte signi aequalitatis separatim contineant. Quod quidem in genere interdum comparatione plurium serierum, interdum introducendo novas series, quarum summae quantitatis quaesitas continent, efficitur. Quem in finem consideretur series:

$$1 - \frac{3^n}{2.3}x^3 + \frac{5^n}{2.3.4.5}x^5 - \frac{7^n}{2.3.4.5.6.7}x^7 + \frac{9^n}{2.3.4.5.6.7.8.9}x^9 + \ldots \text{ in infinitum}$$

functio $x \operatorname{Cos} x$ differentiata dabit

$$\operatorname{Cos} x - x \operatorname{Sin} x = 1 - \frac{3^2}{2.3}x^2 + \frac{5^2}{2.3.4.5}x^4 - \text{etc.} \ldots\ldots$$

quae multiplicata per x, et differentiata, perque dx divisa dabit

$$\operatorname{Cos} x - 3x \operatorname{Sin} x - x^2 \operatorname{Cos} x = 1 - \frac{3^3}{2.3}x^2 + \frac{5^3}{2.3.4.5}x^4 - \text{etc.} \ldots\ldots$$

eritque in genere

$$(1 + Cx^2 + C_1x^4 + C_2x^6 \ldots\ldots C_m x^{n-1}) \operatorname{Cos} x + (C'x + C'_1 x^3 \ldots\ldots + C'_m x^{n-2}) \operatorname{Sin} x$$

$$= 1 - \frac{3^n}{2.3}x^2 + \frac{5^n}{2.3.4.5}x^4 - \frac{7^n}{2.3.4.5.6.7}x^6 + \text{etc.} \ldots\ldots \quad (1)$$

Erit que simili modo

$$\mathrm{Sin}x + x\,\mathrm{Cos}x = 2x - \frac{4x^3}{2.\,3} + \frac{6}{2.\,3.\,4.\,5}x^5 - \text{etc.} \ldots\ldots$$

$$\mathrm{Sin}x + 3x\,\mathrm{Cos}x. - x^2\,\mathrm{Sin}x = 2^2x - \frac{4^2}{2.\,3}x^3 + \frac{6^2}{2.\,3.\,4.\,5}x^5 - \text{etc.} \ldots\ldots$$

sicque porro

$$(1 + Cx^2 + C_1x^4 \ldots\ldots + C_mx^{n-1})\,\mathrm{Sin}x - (C'x + C'_1x^3 \ldots\ldots + C_mx^{n-2})\,\mathrm{Cos}x = 2^{n-1}x - \frac{4^{n-1}}{2.\,3}x^3 + \frac{6^{n-1}}{2.\,3.\,4}x^5 - \text{etc.} \ldots\ldots \quad (2)$$

denotante scilicet n numerum positivum integrum imparem. Facile perspicitur quomodo se habeat, si n par.

Aequationes (1) per Cosx, (2) per Sin x multiplicatae et additae, dabunt
$1 + Cx^a + C,x^4 \ldots\ldots + C,\ x^{n-1}$, (quod sit T)

$$= \left\{ \begin{array}{lllllllll} 1 & - \frac{3^{n-1}}{1.\,2}x^2 & + & \frac{5^{n-1}}{1.\,2.\,3.\,4}x^4 & - & \frac{7^{n-1}}{1.\,2.\,3.\,4.\,5.\,6}x^6 & + & \text{etc.} \ldots\ldots \\ & - \frac{1}{2} & + & \frac{3^{n-1}}{1.\,2.\,1.\,2} & - & \frac{5^{n-1}}{1.\,2.\,3.\,4.\,1.\,2} & + & \text{etc.} \ldots\ldots \\ & & + & \frac{1}{1.\,2.\,3.\,4} & - & \frac{3^{n-1}}{1.\,2.\,1.\,2.\,3.\,4} & + & \text{etc.} \ldots\ldots \\ & & & & - & \frac{1}{1.\,2.\,3.\,4.\,5.\,6} & + & \text{etc.} \ldots\ldots \\ & + 2^{n-1}x^2 & - & \frac{4^{n-1}}{1.\,2.\,3}x^4 & + & \frac{6^{n-1}}{1.\,2.\,3.\,4.\,5}x^6 & + & \text{etc.} \ldots\ldots \\ & & - & \frac{2^{n-1}}{1.\,2.\,3} & \text{Cf.} & \frac{4^{n-1}}{1.\,2.\,3.\,2.\,3} & + & \text{etc.} \ldots\ldots \\ & & & & + & \frac{2^{n-1}}{1.\,2.\,3.\,4.\,5} & + & \text{etc.} \ldots\ldots \end{array} \right.$$

Inde quo lex, secundum quam Coefficientes C, C, etc. progrediuntur, facile perspicitur. Ponatur $n - 1 = n'$, eritque .

$$C = - \frac{3^{n'} + 1}{2} + 2^{n'}$$

$$C_1 = \frac{5^{n'} + 1}{1.\,2.\,3.\,4} + \frac{3^{n'}}{1.\,2.\,1.\,2} - \frac{4^{n'}\ \text{Cf.}\ 2^{n'}}{1.\,2.\,3}$$

$$C_2 = - \frac{7^{n'} + 1}{1.\,2.\,3.\,4.\,5.\,6} - \frac{5^{n'} + 3^{n'}}{1.\,2.\,3.\,4.\,1.\,2} + \frac{6^{n'} + 2^{n'}}{1.\,2.\,3.\,4.\,5} + \frac{4^{n'}}{1.\,2.\,3.\,2.\,3}$$

etc. . . . etc. . . . .

Erit simili modo $C^{1}x + C^{1}_{1}x^{3} + \ldots + C^{1}_{m}x^{n-2} = Z^{1}$

$$= \left\{ \begin{array}{lllllllll}
x & - & \frac{3^{n'}}{1.2}x^{3} & + & \frac{5^{'''}}{1.2.3.4}x^{5} & - & \frac{7^{n'}}{1.2.3.4.5.6}x^{7} & + & \text{etc.}\\
 & - & \frac{1}{2.3.} & + & \frac{3^{n'}}{1.2.2.3} & - & \frac{5^{n'}}{1.2.3.4.2.3} & & \ldots\ldots\\
 & & & + & \frac{1}{1.2.3.4.5} & - & \frac{3^{n'}}{1.2.2.3.4.5} & & \ldots\ldots\\
 & & & & & - & \frac{1}{1.2.3.4.5.6.7} & & \\
-2^{n'}x & + & \frac{4^{n'}}{2.3}x^{3} & - & \frac{6^{n'}}{2.3.4.5}x^{5} & + & \frac{8^{'''}}{2.3.4.5.6.7}x^{7} & - & \text{etc.}\\
 & + & \frac{2^{n'}}{1.2} & - & \frac{4^{n'}}{2.3.1.2} & \text{CE} & \frac{6^{'''}}{2.3.4.5.1.2} & - & \text{etc.}\\
 & & & - & \frac{2^{n'}}{1.2.3.4} & + & \frac{4^{n'}}{2.3.1.2.3.4} & & \ldots\ldots\\
 & & & & & + & \frac{2^{n'}}{1.2.3.4.5.6} & &
\end{array}\right.$$

quare

$$C^{1} = 1 - 2^{n'}$$

$$C^{1}_{1} = -\frac{3^{n'}}{1.2} - \frac{1}{2.3} + \frac{4^{n'}}{1.2.3} + \frac{2^{n'}}{1.2}$$

$$C^{1}_{2} = +\frac{5^{n'}}{1.2.3.4} + \frac{3^{n'}}{1.2.2.3} + \frac{1}{2.3.4.5} - \frac{6^{n'}}{1.2.3.4.5} - \frac{4^{n'}}{1.2.3.1.2} - \frac{2^{n'}}{1.2.3.4}$$

$$C^{1}_{3} = -\frac{7^{n'}}{1.2.3.4.5.6} - \text{etc.} \;\ldots\ldots\ldots$$

$$\begin{array}{ll} \vdots & \vdots \\ \vdots & \vdots \end{array}$$

$$C^{1}_{m} = \pm\left[\frac{(n-2)^{n'}}{1.2.3\ldots\ldots(n-3)} + \frac{(n-4)^{n'}}{1.2.3\ldots\ldots(n-5)2.3} + \frac{(n-6)^{n'}}{1.2.3\ldots\ldots(n-7).2.3.4.5} + \ldots + \frac{1}{1.2\ldots\ldots(n-2)}\right]$$
$$\mp\left[\frac{(n-1)^{n'}}{1.2.3\ldots(n-2)} + \frac{(n-3)^{n'}}{1.2.3\ldots(n-4).1.2} + \frac{(n-5)^{n'}}{1.2.3\ldots(n-6).1.2.3.4} \ldots\ldots + \frac{2^{n'}}{1.2.3\ldots(n-3)}\right]$$

Praecedentibus, ut corollarium, adjici potest, esse seriei

$$1 + 2^{n}x - 3^{n}x^{2} - 4^{n}x^{3} + 5^{n}x^{4} + 6^{n}x^{5} \ldots\ldots$$

summam $= Z\,(\text{Sin}x + \text{Cos}x) + Z^{1}\,(\text{Sin}x - \text{Cos}x)$.

Si Coefficienter C; C'; $C,$; C'_1 etc. expressionibus recurremibus definiri debeane, consideretur functio e^x, quae per x multiplicata, antequam differentiatio suscipiatur, dabit

$$xe^x = x + x^2 + \frac{x^3}{1.\,2} + \frac{x^4}{1.\,2.\,3} + \text{etc.} \;.\;.\;.$$

$$e^x + xe^x = 1 + 2x + \frac{3x^2}{1.\,2} + \frac{4x^3}{1.\,2.\,3} + \text{etc.} \;.\;.$$

$$e^x + 3xe^x + x^2e^x = 1 + 2^2x + \frac{3^2x^2}{1.\,2} + \frac{4^2}{1.\,2.\,3}x^3 + \text{etc.} \;.\;.\;.\;.$$

$$e^x + 7xe^x + 6x^2e^x + x^3e^x = 1 + 2^3x + \frac{3^3}{1.\,2}x^2 + \frac{4^3}{1.\,2.\,3}x^3 + \text{etc.} \;.\;.\;.$$

sicque porro; unde, si in Z, Z^1 Coefficientes C; $C,$; C'; C'_1; etc. quicunque positivi assumantur

$$(Z + Z^1)\,e^x = 1 + 2^nx + \frac{3^n}{1.\,2}x^2 + \frac{4^n}{1.\,2.\,3}x^3 + \text{etc.} \;.\;.\;.$$

quare, si α, β, γ, δ . . . . . designent Coefficientes positivos assumtos

$$\left.\begin{array}{lllllll}
1 & + x & + \frac{x^2}{1.\,2} & + \frac{1}{1.\,2.\,3}x^3 & + \frac{1}{1.\,2.\,3.\,4}x^4 & + \frac{1}{1.\,2.\,3.\,4.\,5}x^5 & + \text{etc.} \\
 & + \alpha & + \alpha & + \frac{\alpha}{1.\,2} & + \frac{\alpha}{1.\,2.\,3} & + \frac{\alpha}{1.\,2.\,3.\,4} & \\
 & & + \beta & + \beta & + \frac{\beta}{1.\,2} & + \frac{\beta}{1.\,2.\,3} & \\
 & & & + \gamma & + \gamma & + \frac{\gamma}{1.\,2} & \\
 & & & & + \delta & + \delta & \\
 & & & & & + \varepsilon &
\end{array}\right\} =$$

$$1 + 2^nx + \frac{3^n}{1.\,2}x^2 + \frac{4^n}{1.\,2.\,3}x^3 + \frac{5^n}{1.\,2.\,3.\,4}x^4 + \frac{6^n}{1.\,2.\,3.\,4.\,5}x^5 + \text{etc.} \;.\;.\;.$$

Inde sequitur

$$\alpha = 2^n - 1$$

$$\beta = \frac{3^n}{1.\,2} - \alpha - \frac{1}{1.\,2}$$

$$\gamma = \frac{4^n}{1.\,2.\,3} - \beta - \frac{\alpha}{1.\,2} - \frac{1}{1.\,2.\,3}$$

$$\delta = \frac{5^n}{1.\,2.\,3.\,4} - \gamma - \frac{\beta}{1.\,2} - \frac{\alpha}{1.\,2.\,3} - \frac{1}{1.\,2.\,3.\,4}$$

$$\varepsilon = \frac{6^n}{1.\,2.\,3.\,4.\,5} - \delta - \frac{\gamma}{1.\,2} - \frac{\beta}{1.\,2.\,3} - \frac{\alpha}{1.\,2.\,3.\,4} - \frac{1}{1.\,2.\,3.\,4}$$

etc. . . . . . . etc. . . . .

Apparet, si ponatur $d^u \left(\begin{matrix} e^{-x} \\ \varepsilon x \end{matrix}\right) = 1 + 2^n x + \frac{3^n}{1.2} x^2 - \frac{4^n}{1.2.3} x^3 + \frac{5''}{1.2.3.4} x^4 + \text{etc.}$ quantitates C; C^I; $C_{,}$; C^I_I etc. et obtineri aequatione $Z + Z^I = d^u \left(\begin{matrix} e^{-x} \\ \varepsilon x \end{matrix}\right) e^{-x}$.

Ad valores α, β, y, δ . . . calculandos inde ab $n = 1$, differentiatione continua commode uti licet. Quos valores calculatos usque ad $n = 8$ repraesentabit istud triangulum.

		α	β	γ	δ	ε	ζ	η
	1							
1	1	1^2						
2	1	3	1^3					
3	1	7	6	1^4				
4	1	15	25	10	1^5			
5	1	31	90	65	15	1^6		
6	1	63	301	350	140	21	1^7	
7	1	127	966	1701	1050	266	28	1^8

Litteris α, ß, γ . . . numeri in columnis verticalibus subjacentes, diversos valores ipsarum, α, β, γ..., post primam, secundain, tertiam etc. differentiationem prodeuntes significant. Numeri, 1; 2; 3; etc. laeva manu designant ordinem differentiationis, a quo pendet valor ipsius n; numeri, hypotheniisae inscripti, sunt factores, per quos si quilibet valorum α, ß, γ . . . multiplicentur, evadunt differentiae sei-ierum, quas numeri parallelli cum hypothenusa progredientes constituunt.

§. 5.

Sic, idoneis artificiis adhibitis, ulterius progredi licebit. Niinc quidem, antequam finem investigationi imponam, dabo seriem

$$1 - 2^n + \frac{3''}{1.\ 2} - \frac{4^n}{1.\ 2.\ 3} + \frac{5''}{1.\ 2.\ 3.\ 4} \ldots\ldots\ldots$$

calculatam inde ab $n = 0$ usque $n = 7$

$$\frac{1}{e} = 1 - 1 + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \frac{1}{1.2.3.4.5} + \text{etc.}$$

$$0 = 1 - 2 + \frac{3}{1.2} - \frac{4}{1.2.3} + \frac{5}{1.2.3.4} - \frac{6}{1.2.3.4.5} + \text{etc.}$$

$$-\frac{1}{e} = 1 - 2^2 + \frac{3'}{1.2} - \frac{4^2}{1.2.3} + \frac{5^2}{1.2.3.4} - \frac{6^2}{1.2.3.4.5} + \text{etc.}$$

$$-\frac{1}{e} = 1 - 2^3 + \frac{3^3}{1.2} - \frac{4^3}{1.2.3} + \frac{5^3}{1.2.3.4} - \frac{6^3}{1.2.3.4.5} + \text{etc.}$$

$$\frac{2}{e} = 1 - 2^4 + \frac{3}{1.2} - \frac{4^4}{1.2.3} + \frac{5^4}{1.2.3.4} - \frac{6^4}{I.2.3.4.5} + \text{etc.}$$

$$\frac{9}{e} = 1 - 2^5 + \frac{3^5}{1.2} - \frac{4^5}{1.2.3} + \frac{5^5}{1.2.3.4} - \frac{6^5}{1.2.3.4.5} + \text{etc.}$$

$$\frac{9}{e} = 1 - 2^6 + \frac{3''}{1.2} - \frac{4^6}{1.2.3} + \frac{5^6}{1.2.3.4} - \frac{6^6}{1.2.3.4.5} + \text{ac.}$$

$$-\frac{50}{e} = 1 - 2^7 + \frac{3^7}{1.2} - \frac{4^7}{1.2.3} + \frac{5^7}{1.2.3.4} - \frac{6^7}{1.2.3.4.5} + \text{etc.}$$

Finis.

Theses

ad Disputandum propositae.

I.

Nulla datur lineae rectae definitio realis.

II.

Differentiale est differentiae status, medio arithmetico omnium differentiae valorum assignabilium correspondens.

III.

Luminis reflexio est purum elasticitatis phaenomenon.

IV.

Amor prodigii est magni momenti in amplificandis scientiis.

V.

Est spatium vacuum in mundo.

VI.

Telluris et planetarum orbitas perpetuo coargui probabile est.

VII.

Quae sit vera elasticitatis causa adhuc nescitur.

VIII.

Planetae moventur in medio, quod eorum motibus non resistit.

IX.

Telluris figura accurate cognosci nequit.

Emendanda.

			Errata.	Corrige.
Pag.	1	linea	10. x^n	$x^{n\,1}$
—	1	—	10. x^{n+2}	$x^{n\,1+2}$
—	2	—	penult. abtinebitur	obtinebitur
—	5	—	4 ante ult. $(1-1)_2^{n-3}$	$(1-1)_2^{n-2}$; $(1-1)_2^{n-3}$
—	5	—	ult. termin	terminum
—	7	—	10 abrampatur	abrumpatur
—	8 et 9		ubique σ	σ—
—	8 et 9	—	$(x)^v$	$(x)^\nu$
—	9	—	13. debit	dabit
—	9	—	antepenult. differentione	differentiatione
—	1	—	4. neussario	necessario
—	10	—	6. parsunt	possunt
—	10	—	19. x	y
—	12	—	9. T	Z
—	13	—	penult. $1 + 2^n x - 3^n + \dots$	$1 + \frac{}{1.2}x - \frac{3^n}{1.2.3}x^2 - \frac{4^n}{1.2.3.4}x^3 + \frac{5^n}{1.2.3.4.5}x^4 + \frac{6^n}{1.2.3.4.5.6}x^5 \dots$
Theses VI.			coargui	coarctari