DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

QUADRATIC AND CUBIC SPLINE COLLOCATION FOR VOLTERRA INTEGRAL EQUATIONS

DARJA SAVELJEVA

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Chapter 1

Introduction

The theory of integral equations has been an active research field for many years and is based on analysis, function theory, and functional analysis.

The theory of integral equations is interesting not only in itself, but its results are essential for the analysis of numerical methods. Besides existence and uniqueness statements, the theory concerns, in particular, questions of regularity and stability.

An integral equation is a functional equation in which the unknown function appears under one or several integral signs. In an integral equation of Volterra type the integrals containing the unknown function are characterized by a variable upper limit of integration. To be more precise, let $I = [0, T]$ denote a given closed and bounded interval, with $0 < T$, and set $S = \{(t, s) : 0 \leq s \leq t \leq T\}$.

The functional equation for the unknown function y of the form

$$y(t) = \int_0^t \mathcal{K}(t, s, y(s))ds + f(t), \qquad t \in I, \tag{1.1}$$

is called a nonlinear Volterra integral equation of the second kind; here, f and $\mathcal{K}$ (the kernel of the integral equation) are given real-valued functions.

In an integral equation of the first kind the unknown function occurs only under the integral sign; hence, a nonlinear Volterra integral equation of the first kind is given by

$$\int_0^t \mathcal{K}(t, s, y(s))ds = f(t), \qquad t \in I.$$

The name "integral equation" appears in 1888 in a paper on elliptic partial differential equations by German scientist Paul du Bois-Reymond. The name "Volterra integral equation" was first coined by Romanian mathematician Traian Lalesco in 1908, seemingly following a suggestion by his teacher French mathematician Emile Picard. The terminology "integral equation of the first (second, third) kind" was first used by German mathematician David Hilbert in connection with his study of Fredholm integral equations.

The origins of the quantitative theory of integral equations with variable (upper) limits of integration go back to the early 19th century. Norwegian mathematician Niels Hendrik Abel in his works in 1823 and in 1826 considered the problem of determining the equation of a curve in a vertical plane such that the time taken by a mass point to slide, under the influence of gravity, along this curve from a given positive height to the horizontal axis is equal to a prescribe (monotone) function of the height. He showed that this problem can be described by a first kind integral equation of the form

$$\int_0^t (t-s)^{-\alpha} y(s)ds = f(t), \quad t > 0, \tag{1.2}$$

with $\alpha = 1/2$, and he then proved that, for any $\alpha \in (0, 1)$, the solution of (1.2) is given by the "inversion formula",

$$y(t) = c_\alpha \frac{d}{dt}\Big(\int_0^t (t-s)^{\alpha-1} f(s)ds\Big), \quad t > 0, \tag{1.3}$$

with $c_\alpha = \sin(\alpha\pi)/\pi = 1/(\Gamma(\alpha)\Gamma(1-\alpha))$.

Three years after Abel's death, in 1832, the problem of inverting (1.2) was also studied by French mathematician Joseph Liouville (who seems to have been unaware of Abel's work), again in a purely formal manner. The discovery of the inversion formula (1.3) was the starting point for the systematic development of what is known now as *Fractional Calculus.*

In 1896 Italian mathematician Vito Volterra published his general theory on the inversion of first kind integral equations. He transformed

$$\int_0^t \mathcal{K}(t,s)y(s)ds = f(t), \quad t \in [0,T], \quad f(0) = 0 \tag{1.4}$$

into an integral equation of the second kind whose kernel and forcing function are, respectively,

$$\widetilde{\mathcal{K}}(t,s) = -\frac{\partial \mathcal{K}(t,s)}{\partial t} \cdot \frac{1}{\mathcal{K}(t,t)}, \quad \text{and} \quad \widetilde{f}(t) = \frac{f'(t)}{\mathcal{K}(t,t)}.$$

If $\mathcal{K}(t,t)$ does not vanish on I, and if the derivatives of $\mathcal{K}$ and f are continuous, then the unique solution of (1.4) is given by the "inversion formula"

$$y(t) = f(t) + \int_0^t \widetilde{R}(t,s)f(s)ds, \quad t \in I,$$

where $\widetilde{R}(t,s)$ denotes the so-called resolvent kernel of $\widetilde{\mathcal{K}}(t,s)$. It is defined in terms of the iterated kernels $\widetilde{\mathcal{K}}_n$ of $\widetilde{\mathcal{K}}$,

$$\widetilde{\mathcal{K}}_n(t,s) = \int_s^t \widetilde{\mathcal{K}}(t,u)\widetilde{\mathcal{K}}_{n-1}(u,s)du, \quad n \geq 2, \quad \widetilde{\mathcal{K}}_1(t,s) = \widetilde{\mathcal{K}}(t,s).$$

Volterra proved that this sequence converges absolutely and uniformly on S for any kernel $\mathcal{K}$ in (1.4).

Even though Volterra's result was new, his method was not entirely a novel one. In his thesis of 1894, French mathematician Joel Le Roux had already studied the problem of inverting the "definite integral" (1.4), using the same approach. But second kind integral equations with variable limit of integration occurred already in the work of Liouville in 1837.

The notion of the iterated kernels and the associated "Neumann series" were first used by French mathematician Joseph Caqué in 1864. Generalizing Liouville's idea, he studied the solution of the $(p+1)$-st order linear differential equation,

$$y^{(p+1)} = \sum_{j=0}^{p} A_j(t)y^{(j)} + A(t),$$

by rewriting the equation as a second kind integral equation of Volterra type with the kernel

$$\mathcal{K}(t,s) = \sum_{j=0}^{p} \frac{A_j(s)(t-s)^{p-j}}{(p-j)!}.$$

The existence of a solution was then established formally by introducing the iterated kernels and the corresponding Neumann series. At about the same time, in 1865, German mathematician August Beer used the same concepts, still in a purely formal way, in connection with the study of second kind integral equations with fixed limits of integration which arose in the analysis of Poisson's equation in Potential Theory. It was left to German mathematician Carl Gottfried Neumann to furnish the rigorous convergence analysis for the series of iterated kernels (associated with a second kind integral equation of Fredholm type) now named after him.

In another paper at the year 1896, Volterra extended his idea to linear integral equations of the first kind with weakly singular kernels. Using the approach employed by Abel to establish the inversion formula (1.3), he showed that

$$\int_0^t (t-s)^{-\alpha}\mathcal{K}(t,s)y(s)ds = f(t), \quad t \in [0,T], \quad 0 < \alpha < 1,$$

can be transformed into a first kind equation with regular kernel, to which the theory of his first work applies. The remaining two papers of Volterra from 1896 are concerned with the analysis of integral equations of the third kind.

The next forty years mainly a consolidation of Volterra's work took place. During this time the center stage belonged to the study of Fredholm integral equations and their implications for the development of functional analysis.

Since 1960s there has been renewed interest in qualitative and asymptotic properties of solutions of Volterra equations.

It is known that the Cauchy problem for ordinary differential equation is equivalent to Volterra integral equation. Thus, all approximate methods for solving Volterra equations could be applied to Cauchy problems as well.

The presented brief history of Volterra integral equations is mainly based on [14].

In scientific and engineering problems, Volterra integral equations are always encountered and have attracted much attention ([2, 3, 14, 18]). An application of Volterra integral equations arises on considering population dynamics ([5, 6]), spread of epidemics ([7, 20]), semi-conductor devices ([44]). Inverse problems related to wave propagation ([31]), superfluidity ([40]) and travelling wave analysis ([27]) provide examples from wave problems. Another application area is electrotechnics. One of its widespread problems is a transition process in (electric) circuit, which is introduced as an example of the application in [66]. Recently, authors of [22] found that the elastodynamic problems for piezoelectric and pyroelectric hollow cylinders under radial deformation can be successfully transformed into a second kind Volterra integral equation with respect to a function of time. In [21, 29] there are studied the elastodynamic problems for magneto-electric-elastic hollow cylinders and it is shown that in the state of axisymmetric plane strain case the problems can be transformed into two second kind Volterra integral equations with respect to two functions of time.

There are enormous list of publications on the numerical solutions of integral equations. An account of numerical methods (degenerate kernel methods, projection methods, the Nyström method, etc) for a large class of integral equations is provided in [2, 28, 66, 68]. The book [14] covers the numerical solution of Volterra integral equations. In [10] attention is concentrated to collocation methods for Volterra integral and integro-differential equations with a variety of kernels; this book is also a rich source of unsolved problems for future research.

Probably the most widely used in practice and theoretically studied class of methods for initial value problems of ordinary differential equations is Runge-Kutta methods. These methods were introduced at around the turn of the 20th century. The modern theory of this class of methods, due to J. C. Butcher, dates back to the mid-1960s. At about the same time, P. Pouzet [59] and B. A. Bel'tyukov [4] extended the Runge-Kutta idea, in different ways, to Volterra integral equations of the second kind with bounded kernel. In the late 1970s and the early 1980s, a systematic analysis of Runge-Kutta-type methods for second kind integral equations was given by H. Brunner, E. Hairer and S. Nørsett [12], see also [14].

Runge-Kutta methods are fully discretized and give a finite number of approximate values (in grid points) to the exact solution. Spline collocation for integral equations requires the evaluation of integrals but gives a function as approximate solution which is principal advantage in comparison with Runge-Kutta methods. In addition, spline method allows to speak about retainment of smoothness proper to exact solution. Let us mention that, with suitable evaluation of integrals by interpolatory quadratures, in some special cases like, e.g., the equations generated by ordinary differential equations, the spline collocation is equivalent to Runge-Kutta methods, for details, see [14].

In order to describe the relevant collocation framework for Volterra integral equations, let $\Delta_N : 0 = t_0 < t_1 < \ldots < t_N = T$ denote a mesh on the interval $I = [0, T]$ and set $h_i = t_i - t_{i-1}$. For given integers $m \geq 1$ and $d \geq -1$, define the space of splines (for more detailed notations see Section 2.3)

$$S^d_{m+d}(\Delta_N) = \{u : \ u \in C^d[0,T], \ u \in \mathcal{P}_{m+d}[t_{i-1}, t_i), \quad i = 1, \ldots, N\} .$$

Let the collocation parameters $0 < c_1 < \ldots < c_m \leq 1$ be fixed (independent of N). In order to determine the approximate solution $u \in S^d_{m+d}$ of the equation (1.1) we impose the collocation conditions

$$u(t) = \int_0^t \mathcal{K}(t, s, u(s)) ds + f(t), \quad t = t_{n-1} + c_i h_n, \qquad (1.5)$$

for $i = 1, \ldots, m$, $n = 1, \ldots, N$. Assuming that the equation (1.1) has the solution $y \in C^k$ for some $k \geq d$ it is quite natural to impose initial condition $u^{(j)}(0) = y^{(j)}(0)$ for $j = 0, \ldots, d$.

In the particular case of ordinary differential equations spline collocation methods, where instead of (1.5) the approximate solution $u \in S^d_{m+d}$ satisfies the equation $y'(t) = f(t, y(t))$ at the collocation points and appropriate initial conditions, have been studied for a long time. Loscalzo and

Talbot [41] showed that the collocation with smooth splines ($m = 1$, $d \geq 0$, $c_1 = 1$) is convergent for $m \leq 3$ and divergent for $m \geq 4$. In [67] it is shown that collocation with continuous splines without any smoothness ($d = 0$) is equivalent to implicit Runge-Kutta methods and is convergent for any values of collocation parameters c_i, $i = 1, \ldots, m$. A comprehensive convergence and divergence analysis of piesewice polynomial collocation for ordinary differential equations was provided by Mülthei [45, 46, 48, 47, 50, 49] between 1979 and 1982.

In 1970 Hung [30] showed that collocation in the classical cubic spline space S_3^2 is divergent. His analysis was extended by El Tom [25, 23, 24] to some more general smooth piecewise polynomial spaces. El Tom used equidistant collocation points (i.e. $c_i = i/m$, $i = 1, \ldots, m$). The reader may also wish to consult the related paper [51] and the survey paper [9].

A particular case of spline collocation methods for solving Volterra integral equations is the piecewise polynomial collocation method ($d = -1$), which is well investigated (see, e.g., [14, 8, 17]) and known to be always convergent. We refer to [13] where a wide class of nonlinear Volterra integral equations (including equations with weakly singular kernels) is considered. The stability analysis in this case (see [54]) give an supplementary explication why it is so. A number of additional convergence results in certain smoother spline spaces were recently established in [53, 54] and [33]. The numerical stability of the polynomial spline collocation in general form is investigated also in [15]. Unfortunately, the proof of the main result is not correct. A characterization of those smooth piecewise polynomial spaces and the set of collocation parameters, that lead to divergent collocation solutions for Volterra integral equations is presented in [11].

The works by Stechkin and Subbotin [63], Zav'jalov, Kvasov and Miroshnigenko [69] and Schumaker [61] present a comprehensive treatment of the theory and numerical analysis of polynomial spline functions (see also [16, 35]). The book [63] is intended as a supplement and complement to the book [1]. Thus, much room is given to detailed analysis of parabolic spline interpolation. Spline of degree higher than cubic appear only with uniformly spaced knots. The author of [61] states that his original intention was to cover both the theory and applications of spline function. This book covers the main algebraic, analytic and approximation-theoretic properties of various spaces of splines. The detailed study of approximation of functions, numerical differentiation and integration, and solution of boundary value problems for ordinary differential equations is given in [69].

In the following we give a brief overview of the dissertation by chapters. The present work consists of seven chapters and an appendix.

In the present Chapter 1 we already gave an overview of history of integral equations, examples of applications of Volterra integral equations, a brief survey of the papers on spline collocation methods and a short review of the main books on spline theory.

In Chapter 2 we present some general convergence theorems for linear and nonlinear operator equations. There is also described the standard step-by-step spline collocation method for Volterra integral equations.

Chapter 3 is devoted to study the collocation method with cubic splines for Volterra integral equations. We replace an initial condition in the standard step-by-step cubic spline collocation method by a not-a-knot boundary condition at the other end of the interval. Such a method is stable in the same region of collocation parameter as in the step-by-step implementation with linear splines. The results about stability and convergence are based on the uniform boundedness of corresponding cubic spline interpolation projections. The results of Chapter 3 are published in [56].

In Chapter 4 we investigate the collocation method with quadratic splines for Volterra integral equations. As in previous chapter we replace an initial condition in the traditional step-by-step collocation method with quadratic splines by a not-a-knot boundary condition at the other end of the interval. Such a nonlocal method gives the uniform boundedness of collocation projections for all parameters $c \in (0, 1)$ characterizing the position of collocation points between spline knots. For $c = 1$ the projection norms have linear growth and, therefore, for any choice of c some general convergence theorems may be applied to establish the convergence with two-sided error estimates. The results of Chapter 4 are published in [57].

Chapter 5 treats the superconvergence of the nonlocal collocation method with quadratic splines for linear Volterra integral equations. Using special collocation points, error estimates at the collocation points are derived showing a more rapid convergence of order $\mathcal{O}(h^4)$ than the global uniform convergence of order $\mathcal{O}(h^3)$ in the interval of integration. The results of this Chapter are published in [60].

In Chapter 6 the step-by-step and nonlocal subdomain methods with quadratic splines are considered. We prove that the first method is unstable. In the case of nonlocal method we replaced the first derivative condition by a not-a-knot boundary condition at the other end of the interval of integration. As a result, we get convergence of this method. The results of Chapter 6 are published in [19].

Three methods presented in this thesis (collocation methods with cubic and quadratic splines and subdomain method with quadratic splines) are described in detail in the case of a test equation. In particular, most part of numerical results which are given in Chapter 7 are done for this test equation. The numerical tests totaly support the theoretical analysis.

In Appendix we provide a formulation of the Schoenberg−Whitney theorem and a proof of convergence rate for quadratic spline histopolation.

Chapter 2

Convergence theorems for operator equations

In this chapter we present some general convergence theorems for operator equations, which we will use in our investigations, and describe the collocation method for Volterra integral equations.

2.1 Convergence theorems for linear equations

Let E and F be Banach spaces, $\mathcal{L}(E, F)$ and $\mathcal{K}(E, F)$ spaces of linear continuous and compact operators (later we will use these notations for operators in normed spaces, too). Suppose we have an equation

$$y = Ky + f \tag{2.1}$$

where $K \in \mathcal{K}(E, E)$ and $f \in E$. Let it be given a sequence of approximating operators $P_N \in \mathcal{L}(E, E)$, $N = 1, 2, \ldots$. Consider also equations

$$y_N = P_N K y_N + P_N f\,. \tag{2.2}$$

The following theorem for second kind equations may be called classical because it is one of the most important tools in the theory of approximate methods for integral equations (see [2, 28]).

Theorem 2.1 (General convergence theorem)**.** *Suppose $y = Ky$ only if $y = 0$ and $P_N y \to y$ for all $y \in E$ as $N \to \infty$. Then*

1) *equation* (2.1) *has the unique solution y^*;*

2) *there is N_0 such that, for $N \geq N_0$, the equation* (2.2) *has the unique solution y_N^*;*

3) *$y_N^* \to y^*$ as $N \to \infty$;*

4) *there are $C_1, C_2, C_3 > 0$ such that*

$$C_1 \|P_N y^* - y^*\| \le \|y_N^* - y^*\| \le C_2 \|P_N y^* - y^*\| \tag{2.3}$$

and

$$\|y_N^* - P_N y^*\| \le C_3 \|K(P_N y^* - y^*)\| \,. \tag{2.4}$$

This theorem can be deduced from more general ones [37, 65]. The reader can find the following notions and results, for instance, in [65].

Definition 2.2. The sequence of operators $A_N \in \mathcal{L}(E,F)$ is said to be *stably convergent* to the operator $A \in \mathcal{L}(E,F)$ if A_N converges to A pointwise (i.e. $A_N x \to Ax$ for all $x \in E$) and there is N_0 such that, for $N \ge N_0$, $A_N^{-1} \in \mathcal{L}(F,E)$ and $\|A_N^{-1}\| \le \text{const}$.

Definition 2.3. The sequence A_N is said to be *regularly convergent* to A if A_N converges to A pointwise and if x_N is bounded and $A_N x_N$ compact, then x_N is compact itself.

Definition 2.4. The sequence A_N is said to be *compactly convergent* to A if A_N converges to A pointwise and if x_N is bounded, then $A_N x_N$ is compact.

Theorem 2.5. *Having $P_N f \to f$ and compact convergence of $P_N K$ to K instead of $P_N y \to y$ for all $y \in E$, the assertions of Theorem* 2.1 *hold.*

Consider the equations

$$Ay = f \tag{2.5}$$

and

$$A_N y_N = f_N \tag{2.6}$$

with $A, A_N \in \mathcal{L}(E,F)$ and $f, f_N \in F$.

Theorem 2.6. *The following two conditions are equivalent:*

1) $\mathrm{Im}A = F$*, A_N converges to A stably;*

2) $\mathrm{Ker}A = \{0\}$*, A_N are Fredholm operators of index* 0 *for $N \ge N_0$ with some N_0, and A_N converges to A regularly.*

If one of them is satisfied, then

1) *equation* (2.5) *has the unique solution y^*;*

2) *there is* N_0 *such that, for* $N \geq N_0$*, the equations* (2.6) *are uniquely solvable*

and

3) *if* f_N *converges to* f *then* y_N *converges to* y *with the estimate*

$$C_1\|A_N y^* - f_N\| \leq \|y_N^* - y^*\| \leq C_2\|A_N y^* - f_N\|.$$

Note, that the assumption P_N converges strongly to I in Theorem 2.1 yields $\|P_N K - K\| \to 0$. Taking this assumption instead of $P_N y \to y$ for all $y \in E$, we get the assertions 1), 2) and the estimate (2.3) of Theorem 2.1. If, in addition, it holds $P_N y^* \to y^*$ then the convergence 3) takes place as well.

Another observation about the assumptions of Theorem 2.1 is that the convergence $\|P_N K - K\| \to 0$ implies the compact convergence $P_N K \to K$, and, with regard to this, remember Theorem 2.5.

The compact convergence, in turn, yields $I - P_N K \to I - K$ stably and $I - P_N K \to I - K$ regularly. In the case of Volterra integral operators the last two statements coincide. Again having $I - P_N K \to I - K$ stably or regularly, we get the two-sided estimate (2.3).

2.2 Convergence theorems for nonlinear equations

In this section we present a counterpart of Theorem 2.1 for the general equation (2.1) with a nonlinear operator K (see [38], Section 50.2).

Consider the equation

$$y = Ay \tag{2.7}$$

with a nonlinear operator $A : \overline{\Omega} \to E$, where Ω is a bounded subset of E. Let P_N be a sequence of linear continuous projections onto finite dimensional subspaces. Consider also the equations

$$y_N = P_N A y_N. \tag{2.8}$$

Theorem 2.7. *Suppose the equation* (2.7) *has the unique solution* $y^* \in \Omega$*. Let the completely continuous operator* A *be Fréchet differentiable at* y^* *and the number* 1 *not be the eigenvalue of operator* $A'(y^*)$*. Suppose also* $\|P_N y - y\| \to 0$ *for all* $y \in E$*. Then*

1) *there is* N_0 *such that, for* $N \geq N_0$*, the equation* (2.8) *has the unique solution* $y_N^* \in \Omega$*;*

2) $y_N^* \to y^*$ *as* $N \to \infty$*;*

3) *there are* $C_1, C_2 > 0$ *such that*

$$C_1 \|P_N y^* - y^*\| \le \|y_N^* - y^*\| \le C_2 \|P_N y^* - y^*\|.$$

This theorem requires the complete continuity, i.e. continuity and compactness, of the nonlinear operator A. As in this thesis we are going to consider integral operators, let us present sufficient conditions for complete continuity in this case (see [39] [36], Chapter 1, Section 3, or [68], Chapter 10, Section 1).

Consider the Uryson operator A such that

$$(Au)(t) = \int_\Omega \mathcal{K}(t, s, u(s))\, ds, \tag{2.9}$$

where Ω is a closed bounded subset of $\mathbb{R}$ and $\mathcal{K} : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$.

Theorem 2.8. *Suppose the kernel* $\mathcal{K}(t, s, u)$ *satisfies the following conditions:*

1) $\mathcal{K}(t, s, u)$ *is continuous with respect to* u *for* $t \in \Omega$ *and for almost all* $s \in \Omega$ *and measurable with respect to* s *for* $t \in \Omega$ *and* $-\infty < u < \infty$*;*

2) *for any* $\varepsilon > 0$*, it holds*

$$\int_\Omega \sup_{|u| \le \varepsilon} |\mathcal{K}(t, s, u)| < \infty\,;$$

3) *for any* $\varepsilon > 0$*, it holds*

$$\lim_{\|h\| \to 0,\ t+h \in \Omega} \int_\Omega \sup_{|u| \le \varepsilon} |\mathcal{K}(t + h, s, u) - \mathcal{K}(t, s, u)|\ ds = 0.$$

Then the Uryson operator (2.9) *operates in the space* $C(\Omega)$ *and is completely continuous.*

Note that the assumptions of Theorem 2.8 are satisfied, e.g., if the kernel $\mathcal{K}$ is continuous.

2.3 Description of collocation method for Volterra integral equations

Consider the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t, s, y(s))ds + f(t), \qquad t \in [0, T], \tag{2.10}$$

where $f : [0, T] \to \mathbb{R}$ and $\mathcal{K} : R \times \mathbb{R} \to \mathbb{R}$ are given functions and the set R is defined by $R = \{(t, s) : 0 \leq s \leq t \leq T\}$.

A mesh $\Delta_N : 0 = t_0 < t_1 < \ldots < t_N = T$ will be used representing the spline knots. As we consider the process $N \to \infty$, the knots t_i depend on N. Denote $h_i = t_i - t_{i-1}$.

For notation, let

$C[a, b]$ be the space of all continuous functions on the interval $[a, b]$;

$C^k[a, b]$ be the space of all functions that are k times continuously differentiable on $[a, b]$ with $k = 1, 2, \ldots$ and $C^0[a, b] = C[a, b]$;

$C^{-1}[a, b]$ be the space of all bounded functions on $[a, b]$, which can have finite number of jump discontinuities and functions are considered being continuous from the right;

$\mathcal{P}_k[a, b]$ be the space of all polynomials of degree not exceeding the number k ($k = 0, 1, \ldots$) on the interval $[a, b]$.

Definition 2.9. For given integers $m \geq 1$ and $d \geq -1$, *the space of polynomial spline functions* of degree $m+d$ and continuity class d is the set

$$S^d_{m+d}(\Delta_N) = \{u : \ u \in C^d[0, T], \ u \in \mathcal{P}_{m+d}[t_{i-1}, t_i), \quad i = 1, \ldots, N\}.$$

An element $u \in S^d_{m+d}$ as a spline of degree not greater than $m + d$ for all $t \in [t_{i-1}, t_i)$, $i = 1, \ldots, N$, can be represented in the form

$$u(t) = \sum_{k=0}^{m+d} b_{ik}(t - t_{i-1})^k, \quad t \in [t_{i-1}, t_i).$$

Thus, the spline $u \in S^d_{m+d}$ is well defined when we know the coefficients b_{ik} for all $i = 1, \ldots, N$ and $k = 0, \ldots, m + d$. In order to compute these coefficients we consider the set of collocation parameters

$$0 < c_1 < \ldots < c_m \leq 1$$

and define the collocation points

$$t_{ij} = t_{i-1} + c_j h_i, \quad i = 1, \ldots, N, \quad j = 1, \ldots, m.$$

For determining the approximate solution $u \in S^d_{m+d}$ of the equation (2.10), we impose the following collocation conditions

$$u(t_{ij}) = \int_0^{t_{ij}} \mathcal{K}(t_{ij}, s, u(s))ds + f(t_{ij}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, m.$$

To be able to start the calculations by this method, we assume that we can use the initial values

$$u^{(j)}(0) = y^{(j)}(0), \quad j = 0, \ldots, d,$$

which is justified by the requirement $u \in C^d[0, T]$. Thus, on every interval $[t_{i-1}, t_i)$ we have $d+1$ conditions of smoothness and m collocation conditions to determine $m + d + 1$ parameters b_{ik} of u. This allows to implement the method step-by-step going from an interval $[t_{i-1}, t_i)$ to the next one.

Described above method of collocation with step-by-step implementation is one of the most practical methods for solving Volterra integral equations of the second kind. It is known to be unstable for cubic and higher order smooth splines (see [53, 54]). In the case of quadratic splines of class C^1 the stability region consists only of one point [54]. The results of numerical tests about step-by-step collocation method in the case of cubic and quadratic splines are presented in Tables 6 and 12, they are taken from [54].

In this dissertation we will show how to get a collocation method with cubic splines having stability in the same interval of collocation parameter as in the case of linear splines [54] and with quadratic splines having stability in the whole interval of collocation parameter. In addition, the convergence of these methods will be proved.

Chapter 3

Cubic spline collocation

In this chapter we will study cubic spline collocation for Volterra integral equations.

Consider the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t, s, y(s))ds + f(t), \qquad t \in [0, T], \tag{3.1}$$

where $f : [0, T] \to \mathbb{R}$ and $\mathcal{K} : R \times \mathbb{R} \to \mathbb{R}$ are given functions and the set R is defined by $R = \{(t, s) : 0 \leq s \leq t \leq T\}$.

3.1 Description of the nonlocal method

Let us consider the collocation in the case of $d = 2$, $m = 1$. Then approximate solutions are in the space of cubic splines (denote it, for brevity, by $S_3(\Delta_N)$). For given $c \in (0, 1]$, as $m = 1$, define collocation points $\tau_i = t_{i-1} + ch_i$, $i = 1, \ldots, N$. In this case the collocation conditions are

$$u(\tau_i) = \int_0^{\tau_i} \mathcal{K}(\tau_i, s, u(s))ds + f(\tau_i), \quad i = 1, \ldots, N. \tag{3.2}$$

Since $\dim S_3(\Delta_N) = N + 3$, it is necessary to give three additional conditions. We replace an initial condition used in the standard step-by-step cubic collocation method by a not-a-knot boundary condition at the end of the interval, so we have

$$\begin{aligned} &u(0) = y(0), \\ &u'(0) = y'(0), \\ &u'''(t_{N-1} - 0) = u'''(t_{N-1} + 0). \end{aligned} \tag{3.3}$$

This method cannot be implemented step-by-step, unlike the one described in Section 2.3, and leads to a system of equations which is, as we will see, successfully solvable by Gaussian elimination.

Let us introduce the vector subspace of $C[0,T]$

$$C_0[0,T] = \left\{ f \in C[0,T] : \exists f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} \right\}$$

with the norm

$$\|f\|_{C_0[0,T]} = \|f\|_{C[0,T]} + |f'(0)|.$$

Proposition 3.1. *The space $C_0[0,T]$ is not complete.*

Proof. Consider the continuous function $u(t) = \sqrt{t}$, $t \in [0,T]$. Suppose $1/n < T$, $n \in \mathbb{N}$. Let $u_n(t)$ be such that

$$u_n(t) = \begin{cases} \sqrt{t}, & t \in [1/n, T], \\ n\sqrt{n}\, t^2, & t \in [0, 1/n]. \end{cases}$$

Obviously, this function is from the space $C_0[0,T]$. It is also clear that

$$\|u_n - u\|_{C[0,T]} = \max_{0\le t\le 1/n} |u_n(t) - u(t)| \le \frac{1}{\sqrt{n}} \to 0, \quad \text{when} \quad n\to\infty.$$

This gives that u_n is fundamental in $C[0,T]$. As $u_n'(0) = 0$ we have

$$\|u_n - u_m\|_{C_0[0,T]} = |u_n'(0) - u_m'(0)| + \|u_n - u_m\|_{C[0,T]} \to 0 \quad \text{as} \quad m,n\to\infty,$$

which means that u_n is Cauchy sequence in $C_0[0,T]$. Suppose there is a function $v \in C_0[0,T]$ such that $\|u_n - v\|_{C_0[0,T]} \to 0$. Then u_n converges to v also in the space $C[0,T]$, consequently, the function v coincides with u, which does not belong to $C_0[0,T]$. We have shown that the Cauchy sequence u_n does not converge in the space $C_0[0,T]$. □

For any $f \in C_0[0,T]$ let $P_N f \in S_3(\Delta_N)$ be such that

$$\begin{aligned} &(P_N f)(0) = f(0), \\ &(P_N f)'(0) = f'(0), \\ &(P_N f)(\tau_i) = f(\tau_i), \quad i = 1,\dots,N, \\ &(P_N f)'''(t_{N-1} - 0) = (P_N f)'''(t_{N-1} + 0). \end{aligned} \tag{3.4}$$

Let us introduce the vector of knots

$$\begin{aligned} \boldsymbol{s}\colon\ s_1 = \ldots = s_4 = t_0 < s_5 = t_1 < \ldots < s_{N+2} = t_{N-2} \\ < s_{N+3} = t_N = \ldots = s_{N+6} \end{aligned}$$

and corresponding B-splines $B_{1,3,\boldsymbol{s}}, \ldots, B_{N+2,3,\boldsymbol{s}}$. These B-splines being linearly independent form a basis in the spline space

$$S_3(\tilde{\Delta}_N) = S_3(\Delta_N) \cap \{f : f'''(t_{N-1} - 0) = f'''(t_{N-1} + 0)\},$$

where $\tilde{\Delta}_N$ means the grid $t_0 < t_1 < \ldots < t_{N-2} < t_N$. The vector of nodes $\boldsymbol{x}$ is

$$\boldsymbol{x}: \quad x_1 = x_2 = 0 < x_3 = \tau_1 < \ldots < x_{N+2} = \tau_N.$$

To ensure that the interpolation problem (3.4) to determine

$$P_N f = \sum_{i=1}^{N+2} c_i B_{i,3,\boldsymbol{s}}$$

has a unique solution, we check the Schoenberg – Whitney conditions (A.4) (for the new notations appearing here, see Appendix A.1). From the inclusion $x_{i+2} \in (s_{i+2}, s_{i+6})$, $i = 1, \ldots, N$, we get $\tau_i \in (t_{i-2}, t_{i+2})$, which takes place as $\tau_i = t_{i-1} + ch_i$. For $x_1 = 0$ and $x_2 = 0$ we have $\lambda_{\boldsymbol{x}}(1) = 0$ and $\lambda_{\boldsymbol{x}}(2) = 1$ accordingly. Using (A.1) and $s_1 = \ldots = s_4$, we calculate

$$B_{1,3,\boldsymbol{s}}(s_1) = B_{2,2,\boldsymbol{s}}(s_1) = B_{3,1,\boldsymbol{s}}(s_1) = B_{4,0,\boldsymbol{s}}(s_1) = 1.$$

To check whether $D_+ B_{2,3,\boldsymbol{s}}(s_2) \neq 0$, we use the formula (A.2) and

$$B_{3,2,\boldsymbol{s}}(s_2) = B_{4,1,\boldsymbol{s}}(s_2) = B_{5,0,\boldsymbol{s}}(s_2) = 0.$$

With this we have shown that the operator $P_N : C_0[0,T] \to C_0[0,T]$ is correctly defined. It is also clear that P_N is a linear projection onto the space $S_3(\tilde{\Delta}_N)$.

Lemma 3.2. *It holds*

$$P_N f = 0 \Leftrightarrow \begin{cases} f(0) = 0, \\ f'(0) = 0, \\ f(\tau_i) = 0, \quad i = 1, \ldots, N. \end{cases}$$

Proof. Taking $P_N f = 0$ and having (3.4), we get $f(0) = 0$, $f'(0) = 0$ and $f(\tau_i) = 0$, $i = 1, \ldots, N$. If, on the other hand, $f(0) = 0$, $f'(0) = 0$ and $f(\tau_i) = 0$, $i = 1, \ldots, N$, then, using the uniqueness of solution of (3.4), we have the zero spline $P_N f = 0$. □

Further, consider the integral operator defined by

$$(Ky)(t) = \int_0^t \mathcal{K}(t, s, y(s))ds, \quad t \in [0, T]. \tag{3.5}$$

Lemma 3.3. *The spline collocation problem* (3.2), (3.3) *is equivalent to the equation*

$$u = P_N K u + P_N f, \quad u \in S_3(\tilde{\Delta}_N), \tag{3.6}$$

provided the kernel $\mathcal{K}$ in (3.5) *has some continuity properties, for example, it is sufficient that $\mathcal{K}$ is continuous and differentiable with respect to t in some neighbourhood of* 0.

Proof. The proof is a standard calculation based on Lemma 3.2. Indeed, then (3.6) is equivalent to the equalities

$$\begin{aligned}
&(u - Ku - f)(0) = 0,\\
&(u - Ku - f)'(0) = 0,\\
&(u - Ku - f)(\tau_i) = 0, \quad i = 1, \dots, N.
\end{aligned}$$

The first one of them is equivalent to $u(0) = f(0)$ or $u(0) = y(0)$ because $y(0) = f(0)$. The equalities in τ_i are just (3.2). Taking into account (3.1), we get that $(u - Ku - f)'(0) = 0$ is equivalent to $u'(0) - (Ku)'(0) = y'(0) - (Ky)'(0)$ or $u'(0) - \mathcal{K}(0, 0, u(0)) = y'(0) - \mathcal{K}(0, 0, y(0))$ (in fact, we use the differentiability of $\mathcal{K}$ with respect to t in some neighbourhood of 0 to ensure the existence of $(Ku)'(0)$ and $(Ky)'(0)$). But the last equality, as $u(0) = y(0)$, is equivalent to $u'(0) = y'(0)$ which completes the proof. □

3.2 Method in the case of a test equation

Consider the test equation

$$y(t) = \lambda \int_0^t y(s)ds + f(t), \quad t \in [0, T], \quad \lambda \in \mathbb{C}.$$

It is typically regarded as the relevant test equation for investigation of numerical methods for general Volterra integral equations (3.1) (see [3, 14]). For approximate solution of this equation by collocation, described in Section 3.1, we rewrite (3.2) and (3.3) as follows

$$\begin{aligned}
&u'(0) = y'(0),\\
&u(0) = y(0),\\
&u(\tau_i) = \lambda \int_0^{\tau_i} u(s)ds + f(\tau_i), \quad i = 1, \dots, N,\\
&u'''(t_{N-1} - 0) = u'''(t_{N-1} + 0).
\end{aligned} \tag{3.7}$$

Calculate the first derivative $u'(0) = y'(0) = \lambda y(0) + f'(0) = \lambda f(0) + f'(0)$ and locate collocation conditions. More precisely, from

$$u(\tau_i) = \lambda \int_0^{\tau_i} u(s)ds + f(\tau_i),$$
$$u(\tau_{i-1}) = \lambda \int_0^{\tau_{i-1}} u(s)ds + f(\tau_{i-1}),$$

we get

$$u(\tau_i) - u(\tau_{i-1}) = \lambda \int_{\tau_{i-1}}^{\tau_i} u(s)ds + f(\tau_i) - f(\tau_{i-1}), \quad i = 2, \ldots, N. \tag{3.8}$$

Thus, we have received the system of equations

$$\begin{aligned} &u'(0) = \lambda f(0) + f'(0), \\ &u(0) = f(0), \\ &u(\tau_1) = \lambda \int_0^{\tau_1} u(s)ds + f(\tau_1), \\ &u(\tau_i) - u(\tau_{i-1}) = \lambda \int_{\tau_{i-1}}^{\tau_i} u(s)ds + f(\tau_i) - f(\tau_{i-1}), \\ &\qquad\qquad\qquad\qquad\qquad\qquad i = 2, \ldots, N, \\ &u'''(t_{N-1} - 0) = u'''(t_{N-1} + 0). \end{aligned} \tag{3.9}$$

Let us restrict ourselves to the case of uniform mesh, i.e. we suppose $t_i - t_{i-1} = h = T/N$, $i = 1, \ldots, N$. Assume that the mesh Δ_N is complemented with knots $t_i = ih$, $i = -3, -2, -1$, and $i = N+1, N+2, N+3$. For $i = -1, \ldots, N+1$, introduce the B-splines (see, e.g., [55])

$$B_i(t) = \frac{1}{h^3} \begin{cases} (t - t_{i-2})^3, & t \in [t_{i-2}, t_{i-1}], \\ h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3, & \\ & t \in [t_{i-1}, t_i], \\ 4h^3 - 6h(t - t_i)^2 + 3(t - t_i)^3, & t \in [t_i, t_{i+1}], \\ (t_{i+2} - t)^3, & t \in [t_{i+1}, t_{i+2}], \end{cases}$$

and $B_i(t) = 0$ if t does not belong to $[t_{i-2}, t_{i+2}]$. These functions form a basis in $S_3(\Delta_N)$ and therefore any cubic spline u can be represented as

$$u(t) = \sum_{i=-1}^{N+1} c_i B_i(t). \tag{3.10}$$

The spline u can be written on the interval $[t_{i-1}, t_i]$ as

$$\begin{aligned}
u(t) &= \sum_{j=i-2}^{i+1} c_j B_j(t) \\
&= c_{i-2}B_{i-2}(t) + c_{i-1}B_{i-1}(t) + c_i B_i(t) + c_{i+1}B_{i+1}(t) \\
&= \frac{1}{h^3}\Big(c_{i-2}(t_i - t)^3 \\
&\quad + c_{i-1}\left(4h^3 - 6h(t - t_{i-1})^2 + 3(t - t_{i-1})^3\right) \\
&\quad + c_i\left(h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3\right) \\
&\quad + c_{i+1}(t - t_{i-1})^3\Big).
\end{aligned}$$

Also the integral of u will be

$$\begin{aligned}
\int u(t)dt = \; & \frac{1}{h^3}\Bigg(- c_{i-2}\frac{(t_i - t)^4}{4} \\
& + c_{i-1}\big(4h^3(t - t_{i-1}) - 2h(t - t_{i-1})^3 + \frac{3}{4}(t - t_{i-1})^4\big) \\
& + c_i\big(h^3(t - t_{i-1}) + \frac{3}{2}h^2(t - t_{i-1})^2 + h(t - t_{i-1})^3 \\
& - \frac{3}{4}(t - t_{i-1})^4\big) + c_{i+1}\frac{(t - t_{i-1})^4}{4}\Bigg).
\end{aligned} \tag{3.11}$$

Taking into account $\tau_i = t_{i-1} + ch$, we get, for $i = 1, \ldots, N$,

$$u(\tau_i) = c_{i-2}(1-c)^3 + c_{i-1}(4-6c^2+3c^3) + c_i(1+3c+3c^2-3c^3) + c_{i+1}c^3 \tag{3.12}$$

and, for $i = 2, \ldots, N$,

$$\begin{aligned}
\int_{\tau_{i-1}}^{\tau_i} u(s)ds &= \int_{\tau_{i-1}}^{t_{i-1}} u(s)ds + \int_{t_{i-1}}^{\tau_i} u(s)ds \\
&= \frac{h}{4}\Big(c_{i-3}(1 - c)^4 \\
&\quad + c_{i-2}(11 - 12c - 6c^2 + 12c^3 - 4c^4)^3 \\
&\quad + c_{i-1}(11 + 12c - 6c^2 - 12c^3 + 6c^4) \\
&\quad + c_i(1 + 4c + 6c^2 + 4c^3 - 4c^4) + c_{i+1}c^4\Big).
\end{aligned}$$

Putting these formulae into located collocation conditions (3.8), we have

$$\alpha c_{i-3} + \beta c_{i-2} + \gamma c_{i-1} + \delta c_i + \varepsilon c_{i+1} = f_i \tag{3.13}$$

with $i = 2, \dots, N$ and

$$\begin{aligned}
\alpha &= -(1-c)^3 - \frac{\lambda h}{4}(1-c)^4, \\
\beta &= -(3+3c-9c^2+4c^3) - \frac{\lambda h}{4}(11-12c-6c^2+12c^3-4c^4), \\
\gamma &= 3-3c-9c^2+6c^3 - \frac{\lambda h}{4}(11+12c-6c^2-12c^3+6c^4), \\
\delta &= 1+3c+3c^2-4c^3 - \frac{\lambda h}{4}(1+4c+6c^2+4c^3-4c^4), \\
\varepsilon &= c^3 - \frac{\lambda h}{4}c^4, \\
f_i &= f(\tau_i) - f(\tau_{i-1}).
\end{aligned}$$

In the case of $i = 1$, using (3.12) and (3.11), we obtain

$$\beta_1 c_{-1} + \gamma_1 c_0 + \delta_1 c_1 + \varepsilon_1 c_2 = f_1, \tag{3.14}$$

where

$$\begin{aligned}
\beta_1 &= (1-c)^3 - \frac{\lambda h}{4}(4c-6c^2+4c^3-c^4), \\
\gamma_1 &= 4-6c^2+3c^3 - \frac{\lambda h}{4}(16c-8c^3+3c^4), \\
\delta_1 &= 1+3c+3c^2-3c^3 - \frac{\lambda h}{4}(4c+6c^2+4c^3-3c^4), \\
\varepsilon_1 &= c^3 - \frac{\lambda h}{4}c^4, \\
f_1 &= f(\tau_1).
\end{aligned}$$

It remains to express boundary conditions in terms of c and h. The equality $u(0) = f(0)$ gives

$$c_{-1} + 4c_0 + c_1 = f(0). \tag{3.15}$$

The derivative of u on $[t_{i-1}, t_i]$ is

$$\begin{aligned}
u'(t) = \frac{1}{h^3}\Big(&-3c_{i-2}(t_i-t)^2 + c_{i-1}\big(-12h(t-t_{i-1}) + 9(t-t_{i-1})^2\big) \\
&+ c_i\big(3h^2 + 6h(t-t_{i-1}) - 9(t-t_{i-1})^2\big) + 3c_{i+1}(t-t_{i-1})^2\Big).
\end{aligned}$$

Then from $u'(0) = \lambda f(0) + f'(0)$ it follows

$$-c_{-1} + c_1 = \frac{1}{3}h\big(\lambda f(0) + f'(0)\big).$$

Eliminating c_1 from the previous formula, using (3.15), we get the equation

$$c_{-1} + 2c_0 = \frac{1}{2}\left(f(0) - \frac{1}{3}\left(\lambda h f(0) + h f'(0)\right)\right). \tag{3.16}$$

Continuing with the calculation of derivatives for $t \in [t_{i-1}, t_i]$, we obtain

$$\begin{aligned} u''(t) &= \frac{1}{h^3}\Big(6c_{i-2}(t_i - t) + c_{i-1}\big(-12h + 18(t - t_{i-1})\big) \\ &\quad + c_i\big(6h - 18(t - t_{i-1})\big) + 6c_{i+1}(t - t_{i-1})\Big), \\ u'''(t) &= \frac{1}{h^3}\big(-6c_{i-2} + 18c_{i-1} - 18c_i + 6c_{i+1}\big). \end{aligned}$$

Hence, the not-a-knot condition $u'''(t_{N-1} - 0) = u'''(t_{N-1} + 0)$ gives

$$c_{N-3} - 4c_{N-2} + 6c_{N-1} - 4c_N + c_{N+1} = 0. \tag{3.17}$$

Using (3.16), (3.15), (3.14), (3.13) and (3.17), we can write (3.9) in the form of a linear system to determine the coefficients c_i as follows

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & \cdots & & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & \cdots & & 0 \\ \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & 0 & 0 & & & \\ \alpha & \beta & \gamma & \delta & \varepsilon & 0 & & & \\ 0 & \alpha & \beta & \gamma & \delta & \varepsilon & & & \\ & & \ddots & & & & \ddots & & \\ & & & & & & & & \\ 0 & & \cdots & 0 & \alpha & \beta & \gamma & \delta & \varepsilon \\ 0 & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ \\ c_N \\ c_{N+1} \end{pmatrix} = \begin{pmatrix} g_{-1} \\ g_0 \\ g_1 \\ g_2 \\ g_3 \\ \vdots \\ \\ g_N \\ 0 \end{pmatrix}$$

with

$$\begin{aligned} g_{-1} &= \frac{1}{2}\left(f(0) - \frac{1}{3}\big(\lambda h f(0) + h f'(0)\big)\right), \\ g_0 &= f(0), \quad g_i = f_i, \quad i = 1, \ldots, N. \end{aligned}$$

The matrix of the system has non-zero elements on six diagonals. To solve this system, we use the Gaussian elimination process. We start with elimination over the main diagonal from the bottom to the up and get the system $A\mathbf{c} = \widetilde{\mathbf{g}}$. The entries of A and $\widetilde{\mathbf{g}}$ are calculated as follows

$$a_{N,N-1} = \frac{\gamma - 6\varepsilon}{\delta + 4\varepsilon}, \quad a_{N,N-2} = \frac{\beta + 4\varepsilon}{\delta + 4\varepsilon}, \quad a_{N,N-3} = \frac{\alpha - \varepsilon}{\delta + 4\varepsilon},$$

$$a_{i-1,i-2} = \frac{\gamma - \varepsilon a_{i,i-2}}{\delta - \varepsilon a_{i,i-1}}, \quad a_{i-1,i-3} = \frac{\beta - \varepsilon a_{i,i-3}}{\delta - \varepsilon a_{i,i-1}}, \quad a_{i-1,i-4} = \frac{\alpha}{\delta - \varepsilon a_{i,i-1}},$$

$$i = N, \dots, 3,$$

$$a_{1,0} = \frac{\gamma_1 - \varepsilon_1 a_{2,0}}{\delta_1 - \varepsilon_1 a_{2,1}}, \quad a_{1,-1} = \frac{\beta_1 - \varepsilon_1 a_{2,-1}}{\delta_1 - \varepsilon_1 a_{2,1}}, \quad a_{0,-1} = \frac{1 - a_{1,-1}}{4 - a_{1,0}},$$

$$\widetilde{g}_N = \frac{g_N}{\delta + 4\varepsilon}, \quad \widetilde{g}_{i-1} = \frac{g_{i-1} - \varepsilon \widetilde{g}_i}{\delta - \varepsilon a_{i,i-1}}, \quad i = N, \dots, 3,$$

$$\widetilde{g}_1 = \frac{g_1 - \varepsilon \widetilde{g}_2}{\beta_1 - \varepsilon_1 a_{2,1}}, \quad \widetilde{g}_0 = \frac{g_0 - \widetilde{g}_1}{4 - a_{1,0}}.$$

Then the coefficients c_i are

$$c_{-1} = \frac{g_{-1} - 2\widetilde{g}_0}{1 - 2a_{0,-1}},$$

$$c_0 = \widetilde{g}_0 - a_{0,-1} c_{-1},$$

$$c_1 = \widetilde{g}_1 - a_{1,0} c_0 - a_{1,-1} c_{-1},$$

$$c_i = \widetilde{g}_i - a_{i,i-1} c_{i-1} - a_{i,i-2} c_{i-2} - a_{i,i-3} c_{i-3}, \quad i = 2, \dots, N,$$

$$c_{N+1} = 4c_N - 6c_{N-1} + 4c_{N-2} - c_{N-3}.$$

Thus, we get the coefficients in the representation of the approximate solution of the test equation as cubic spline (3.10) by B-splines.

3.3 Conditions for uniform boundedness of collocation projections

One of the assumptions in general convergence theorem (see Section 2.1) is the convergence of the sequence of approximating operators P_N to the identity or injection operator. This means that the uniform boundedness of the sequence P_N is the key problem in the study of the collocation method (3.2), (3.3).

Given a function $f \in C_0[0, T]$, let us consider the cubic spline $u = P_N f = \sum_{-1 \le i \le N+1} c_i B_i$, which is equivalent to the conditions

$$\begin{aligned} &u'(0) = f'(0), \\ &u(0) = f(0), \\ &u(\tau_i) = f(\tau_i), \quad i = 1, \dots, N, \\ &u'''(t_{N-1} - 0) = u'''(t_{N-1} + 0). \end{aligned} \tag{3.18}$$

Note that the problem (3.18) is in fact (3.2), (3.3) in the case $K = 0$ or the problem (3.7) in the case $\lambda = 0$.

We restrict ourselves, like in the previous section, to the case of uniform mesh. Taking into account the representation of cubic spline by B-splines (3.10), we get (3.12). Then, using (3.16) with $\lambda = 0$, (3.15), (3.12) and (3.17), write (3.18) in the form of a linear system to determine the coefficients c_i as follows

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & \dots & & 0 \\ 1 & 4 & 1 & 0 & 0 & \dots & & 0 \\ \alpha & \beta & \gamma & \delta & 0 & \dots & & 0 \\ 0 & \alpha & \beta & \gamma & \delta & \dots & & 0 \\ & & \ddots & & & \ddots & & \\ & & & & & & & \\ 0 & \dots & & 0 & \alpha & \beta & \gamma & \delta \\ 0 & \dots & 0 & 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ \\ c_N \\ c_{N+1} \end{pmatrix} = \begin{pmatrix} g_{-1} \\ g_0 \\ g_1 \\ g_2 \\ \vdots \\ \\ g_N \\ 0 \end{pmatrix}, \tag{3.19}$$

where

$$g_{-1} = \frac{1}{2} f(0) - \frac{h}{6} f'(0), \quad g_0 = f(0), \quad g_i = f(\tau_i), \quad i = 1, \dots, N, \tag{3.20}$$

and

$$\begin{aligned} \alpha &= (1-c)^3 = d^3, \\ \beta &= 4 - 6c^2 + 3c^3 = (1+d)^3 - 4d^3, \\ \gamma &= 1 + 3c + 3c^2 - 3c^3 = (1+c)^3 - 4c^3, \\ \delta &= c^3 \end{aligned}$$

with $d = 1 - c$.

Our main technique is the study of Gaussian elimination process for (3.19). We start with the elimination over the diagonal and get the system

$$A\mathbf{c} = B^{-1}\mathbf{g} = \tilde{\mathbf{g}}, \tag{3.21}$$

where $\mathbf{g}$ is the right hand side of (3.19). The entries of A and vector $\tilde{\mathbf{g}}$ are calculated as follows

$$a_{N,N-1} = \frac{\beta - 6\delta}{\gamma + 4\delta}, \quad a_{N,N-2} = \frac{\alpha + 4\delta}{\gamma + 4\delta}, \quad a_{N,N-3} = \frac{-\delta}{\gamma + 4\delta},$$

$$a_{N-1,N-2} = \frac{\beta - \delta a_{N,N-2}}{\gamma - \delta a_{N,N-1}}\,, \quad a_{N-1,N-3} = \frac{\alpha - \delta a_{N,N-3}}{\gamma - \delta a_{N,N-1}}\,,$$

$$a_{i-1,i-2} = \frac{\beta - \delta a_{i,i-2}}{\gamma - \delta a_{i,i-1}}\,, \quad a_{i-1,i-3} = \frac{\alpha}{\gamma - \delta a_{i,i-1}}\,, \quad i = N-1, \dots, 2,$$

$$a_{0,-1} = \frac{1 - a_{1,-1}}{4 - a_{1,0}},$$

$$\tilde{g}_N = \frac{g_N}{\gamma + 4\delta}\,, \quad \tilde{g}_{N-1} = \frac{g_{N-1} - \delta \tilde{g}_N}{\gamma - \delta a_{N,N-1}}\,,$$

$$\tilde{g}_{i-1} = \frac{g_{i-1} - \delta \tilde{g}_i}{\gamma - \delta a_{i,i-1}}, \quad i = N-1, \dots, 2\,,$$

$$\tilde{g}_0 = \frac{g_0 - \tilde{g}_1}{4 - a_{1,0}}\,.$$

Then the coefficients c_i are

$$\begin{aligned}
&c_{-1} = \frac{g_{-1} - 2\tilde{g}_0}{1 - 2a_{0,-1}}\big(= \tilde{g}_{-1}\big),\\
&c_0 = \tilde{g}_0 - a_{0,-1}c_{-1},\\
&c_i = \tilde{g}_i - a_{i,i-1}c_{i-1} - a_{i,i-2}c_{i-2}, \quad i = 1, \dots, N-1,\\
&c_N = \tilde{g}_N - a_{N,N-1}c_{N-1} - a_{N,N-2}c_{N-2} - a_{N,N-3}c_{N-3},\\
&c_{N+1} = 4c_N - 6c_{N-1} + 4c_{N-2} - c_{N-3}.
\end{aligned} \tag{3.22}$$

First, we are interested in the asymptotic behavior of the entries of A if $N \to \infty$.

Lemma 3.4. *For each $c \in (0,1]$, the elements $a_{i,i-1}$ and $a_{i,i-2}$ converge as $N - i \to \infty$ (considering $N \to \infty$).*

Proof. Denote $x_n = a_{i,i-1}$, $y_n = a_{i,i-2}$ and $x_{n+1} = a_{i-1,i-2}$, $y_{n+1} = a_{i-1,i-3}$. Then we have the iteration process

$$x_{n+1} = \frac{\beta - \delta y_n}{\gamma - \delta x_n}\,, \qquad y_{n+1} = \frac{\alpha}{\gamma - \delta x_n} \tag{3.23}$$

or $(x_{n+1}, y_{n+1}) = F(x_n, y_n)$ corresponding to the function

$$F(x,y) = \left(\frac{\beta - \delta y}{\gamma - \delta x}, \frac{\alpha}{\gamma - \delta x} \right)\,.$$

We begin with the values $a_{N,N-1}$ and $a_{N,N-2}$. Then the calculation of $a_{N-1,N-2}$ and $a_{N-1,N-3}$ uses formulae slightly different from (3.23), but after that we go on (3.23). Anyway, this iteration process, starting with the values $a_{N,N-1}$ and $a_{N,N-2}$, is completely determined by the value of c.

`Case` $1/2 \le c \le 1$. We show that $F : [0,2] \times [0,1] \to [0,2] \times [0,1]$ and find a metric which guarantees the contractivity of F.

We have $\alpha \in [0, 1/8]$, $\beta \in [1, 23/8]$, $\gamma \in [23/8, 4]$ and $\delta \in [1/8, 1]$. For $x \in [0,2]$, it holds $\gamma - \delta x \ge \gamma - 2\delta = (1+c)^3 - 6c^3$. Denote here $\varphi(c) = (1+c)^3 - 6c^3$. Then $\varphi(1/2) = 21/8$ and $\varphi(1) = 2$. We have also $\varphi'(c) = 3(1+2c-5c^2)$. This gives $\varphi'(1/2) > 0$ and $\varphi'(1) < 0$, which means that the function φ is increasing at the beginning of the interval $[1/2, 1]$ and then, at the other part of interval, is decreasing. Thus, $\varphi(c) \ge 2$ on $[1/2, 1]$ and we get $\gamma - \delta x \ge 2$. For $y \in [0,1]$, it holds $0 \le \beta - \delta \le \beta - \delta y \le \beta \le 23/8$. Hence,

$$0 \le \frac{\beta - \delta y}{\gamma - \delta x} \le \frac{23}{16} \le 2, \quad 0 \le \frac{\alpha}{\gamma - \delta x} \le \frac{1}{16} \le 1,$$

which ensures that F maps $[0,2] \times [0,1]$ into itself.

In addition, let us show $(x_0, y_0) = (a_{N-1,N-2}, a_{N-1,N-3}) \in [0,2] \times [0,1]$. Consider first the denominator and numerators of the fractions $a_{N,N-1}$, $a_{N,N-2}$ and $a_{N,N-3}$. Similarly to the investigation of $\varphi(c)$ above, we have $\gamma + 4\delta \in [27/8, 8]$, $\beta - 6\delta \in [-5, 17/8]$ and $\alpha + 4\delta \in [5/8, 4]$, which give $a_{N,N-1} < 1$, $a_{N,N-2} > 0$ and $a_{N,N-3} \in [-1/8, -1/27]$. From the inequalities $\gamma - \delta a_{N,N-1} \ge 22/8$, $\beta - \delta a_{N,N-2} \le \beta \le 23/8$ and $0 < \alpha - \delta a_{N,N-3} \le 1/4$, we get $a_{N-1,N-2} \le 2$ and $a_{N-1,N-3} \in [0,1]$. It holds $a_{N,N-2} = ((1-c)^3 + 4c^3)/(1+c)^3 \le 1/2$, therefore, $\beta - \delta a_{N,N-2} \ge \beta - \delta/2 \ge 0$ and $a_{N-1,N-2} \ge 0$.

To prove the contractivity, write, for $z^i = (x_i, y_i)$, $i = 1, 2$,

$$F(z^1) - F(z^2) = \left(\frac{\beta\delta(x_1 - x_2) + \gamma\delta(y_2 - y_1) + \delta^2(x_2 y_1 - x_1 y_2)}{(\gamma - \delta x_1)(\gamma - \delta x_2)}, \frac{\alpha\delta(x_1 - x_2)}{(\gamma - \delta x_1)(\gamma - \delta x_2)} \right).$$

We have already shown that $\gamma - \delta x \ge 2$, $\delta^2 \le 1$. It is easy to see that $\alpha\delta = c^3(1-c)^3 \le 1/64$ and $\gamma\delta \le 4$. Investigating the function $\varphi(c) = \beta\delta = c^3(4 - 6c^2 + 3c^3)$, we get $\beta\delta \le 1$. Rewrite $y_1 x_2 - y_2 x_1$ as $y_1(x_2 - x_1) + x_1(y_1 - y_2)$. Having $x_i \in [0,2]$, $y_i \in [0,1]$, we may estimate the components as follows:

$$\left| \left[F(z^1) - F(z^2) \right]_1 \right| \le \frac{1}{2} |x_1 - x_2| + \frac{3}{2} |y_1 - y_2|,$$

$$\left| \left[F(z^1) - F(z^2) \right]_2 \right| \le \frac{1}{256} |x_1 - x_2|.$$

To get the contractivity with a coefficient $q \in (0,1)$, choose a norm $\|z\| = \max\{|x|, K|y|\}$. Denote, for brevity, $a = |x_1 - x_2|$ and $b = |y_1 - y_2|$. We wish to have the inequality

$$\max\{\frac{1}{2}a + \frac{3}{2}b, \frac{Ka}{256}\} \leq q \max\{a, Kb\}. \tag{3.24}$$

Let $\max\{a, Kb\} = a$, then $b \leq a/K$, and the following conditions

$$\frac{K}{256} \leq q \quad \text{and} \quad \frac{1}{2} + \frac{3}{2K} \leq q\,. \tag{3.25}$$

imply (3.24). In the case $\max\{a, Kb\} = Kb$ the analysis is similar. Thus, in the chosen norm we may get the contractivity with a coefficient $q \in (0,1)$ if the inequalities (3.25) hold. Choosing, for example, $q = 3/4$, we may take $6 \leq K \leq 192$.

`Case` $0 < c < 1/2$. Now we have

$$\alpha \in (1/8, 1), \quad \beta \in (23/8, 4), \quad \gamma \in (1, 23/8) \quad \text{and} \quad \delta \in (0, 1/8).$$

Suppose $x \in [1,4]$ and $y \in [0,1]$. Then, using the same technique as in previous case, we get $\gamma - \delta x \in (1, 11/4)$ and $\beta - \delta y \in (11/4, 4)$. These inclusions allow to show that $F: [1,4] \times [0,1] \to [1,4] \times [0,1]$.

For proving $(x_0, y_0) = (a_{N-1,N-2}, a_{N-1,N-3}) \in [1,4] \times [0,1]$, consider again $a_{N,N-1}$, $a_{N,N-2}$ and $a_{N,N-3}$. We have $\gamma + 4\delta \in (1, 27/8)$, $\beta - 6\delta \in (17/8, 4)$ and $\alpha + 4\delta \in [4/9, 1)$, which give $a_{N,N-1} \leq 4$, $(2/3)^5 \leq a_{N,N-2} < 1$ and $-1/27 < a_{N,N-3} < 0$. This allows us to check that $a_{N-1,N-2} \leq 4$. On the other hand, detailed analysis of the function

$$a_{N-1,N-2} = \frac{(4 - 6c^2 + 3c^3)(1+c)^3 - c^3((1-c)^3 + 4c^3)}{((1+c)^3 - 4c^3)(1+c)^3 - c^3(4 - 6c^2 - 3c^3)}$$

leads to the estimate $a_{N-1,N-2} \geq 1$. Let us mention that separate estimation of $\beta - \delta a_{N,N-2}$ and $\gamma - \delta a_{N,N-1}$ is too rough to establish $a_{N-1,N-2} \geq 1$. The inclusion $a_{N-1,N-3} \in [0,1]$ follows from $\gamma - \delta a_{N,N-1} > 1$, $\alpha - \delta a_{N,N-3} > \alpha > 0$ and $\alpha - \delta a_{N,N-3} \leq \alpha + \delta \leq 1$.

Here the contractivity can be concluded from

$$\left|\left[F(z^1) - F(z^2)\right]_1\right| \leq \frac{3}{8}\,|x_1 - x_2| + \frac{27}{64}\,|y_1 - y_2|,$$
$$\left|\left[F(z^1) - F(z^2)\right]_2\right| \leq \frac{1}{64}\,|x_1 - x_2|$$

already in 1-norm $\|\mathbf{z}\|_1 = |x| + |y|$, for example, with the coefficient $q = 1/2$.

This completes the proof. □

Let us use in the sequel the notation $x = \lim a_{i,i-1}$ and $y = \lim a_{i,i-2}$ which means also that (x, y) is the fixed point of F. By Lemma 3.4 and (3.22) the coefficients c_i may be calculated from the difference equation

$$c_i = \tilde{g}_i - xc_{i-1} - yc_{i-2}$$

provided i is far enough from N.

Denote by $\lambda_{\max}$ the maximal by modulus root of the characteristic equation

$$\lambda^2 + x\lambda + y = 0\,. \tag{3.26}$$

Lemma 3.5. *It holds* $|\lambda_{\max}| > 1$ *for* $c < 1/2$*,* $\lambda_{\max} = -1$ *for* $c = 1/2$ *and* $|\lambda_{\max}| < 1$ *for* $c > 1/2$*.*

Proof. We know that

$$x = \frac{\beta - \delta y}{\gamma - \delta x}\,, \quad y = \frac{\alpha}{\gamma - \delta x}\,. \tag{3.27}$$

Using $p = \gamma - \delta x \geq 1$ (see the proof of Lemma 3.4), we have

$$\begin{aligned}
x^2 - 4y > 0 \quad &\Leftrightarrow \quad (\beta - \delta y)^2 > 4\alpha p \\
&\Leftrightarrow \quad \left(\beta - \frac{\alpha\delta}{p}\right)^2 > 4\alpha p \\
&\Leftrightarrow \quad (\beta p - \alpha\delta)^2 > 4\alpha p^3 \\
&\qquad\quad \text{(due to } \beta p \geq 1 \text{ and } \alpha\delta \leq 1/64) \\
&\Leftarrow \quad \beta p - \alpha\delta > 2p\sqrt{\alpha p} \\
&\Leftarrow \quad \beta > \frac{1}{64p} + 2\sqrt{\alpha p} \\
&\Leftarrow \quad \beta > \frac{1}{64} + 2\sqrt{\alpha p} \\
&\Leftrightarrow \quad \left(\beta - \frac{1}{64}\right)^2 > 4\alpha(\gamma - \delta x) \\
&\Leftarrow \quad \left(\beta - \frac{1}{64}\right)^2 > 4\alpha\gamma,
\end{aligned}$$

and the last holds for all $c \in (0, 1]$. Thus, the inequality $x^2 > 4y$ yields that $\lambda_{\max} = -(x + \sqrt{x^2 - 4y})/2$. If $x > 2$ (this can happen only for

$c \in (0,1/2)$), it always holds $\sqrt{x^2-4y} > 2-x$ and then $|\lambda_{\max}| > 1$. If $x \le 2$, then $\sqrt{x^2-4y} < 2-x$ is equivalent to $x^2-4y < 4-4x+x^2$ or $x-y<1$. Hence, we obtain

$$\begin{aligned} |\lambda_{\max}| < 1 \quad &\Leftrightarrow \quad x-y<1\,,\\ |\lambda_{\max}| = 1 \quad &\Leftrightarrow \quad x-y=1\,,\\ |\lambda_{\max}| > 1 \quad &\Leftrightarrow \quad x-y>1\,. \end{aligned}$$

From (3.27) it follows

$$x-y = \frac{\beta-\delta y-\alpha}{\gamma-\delta x}\,, \tag{3.28}$$

and $x-y=1$ if and only if $\beta-\alpha=\gamma-\delta$. But $\gamma-\delta=(1+c)^3-5c^3$ and $\beta-\alpha=(1+d)^3-5d^3$. Let $\varphi(c)=(1+c)^3-5c^3$. We have

$$\begin{aligned} &\varphi(c) < 11/4 \quad \text{for} \quad c<1/2\,,\\ &\varphi(1/2) = 11/4\,,\\ &\varphi(c) > 11/4 \quad \text{for} \quad c>1/2\,. \end{aligned}$$

Thus $x-y=1$ if and only if $c=1/2$. Then write (3.28) as

$$(x-y)(\gamma-\delta-\delta x+\delta) = \beta-\alpha-\delta x+\delta(x-y)$$

or

$$(x-y)(\gamma-\delta-\delta x) = \beta-\alpha-\delta x. \tag{3.29}$$

For $c>1/2$, we have $\beta-\alpha<\gamma-\delta$. In addition, $\gamma-\delta>11/4$, $\delta\le 1$, $x\le 2$ give $\gamma-\delta-\delta x>0$. Thus, taking into account $\beta-\alpha-\delta x<\gamma-\delta-\delta x$ and (3.29), we get also $x-y<1$. For $c<1/2$, we have $\beta-\alpha>\gamma-\delta$. In this case $\gamma-\delta>1$ and $\delta<1/8$, which also give $\gamma-\delta-\delta x>0$ and, by (3.29), $x-y>1$. □

We intend to estimate the vector **c** by **g**, using the system (3.21). Write $\mathbf{g}=B\tilde{\mathbf{g}}$ as follows:

$$\begin{aligned} &g_{N+1} = \tilde{g}_{N+1} = 0\,,\\ &g_N = (\gamma+4\delta)\tilde{g}_N\,,\\ &g_i = (\gamma-\delta a_{i,i-1})\tilde{g}_{i-1}+\delta\tilde{g}_i, \quad i=N,\ldots,2,\\ &g_0 = (4-a_{1,0})\tilde{g}_0+\tilde{g}_1\,,\\ &g_{-1} = (1-2a_{0,-1})\tilde{g}_{-1}+2\tilde{g}_0\,. \end{aligned}$$

First, consider all other equations without g_0 and g_{-1}. The matrix of this system has diagonal dominance by rows. Indeed, for $c \in [1/2, 1]$, we got $a_{N,N-1} \leq 1$ (see the proof of Lemma 3.4), implying $\gamma - \delta a_{N,N-1} \geq 11/4$. We know already that on next steps ($i < N$) $a_{i,i-1} \leq 2$ and $\gamma - \delta a_{i,i-1} \geq \gamma - 2\delta \geq 2$. This allows us to write $|\gamma - \delta a_{i,i-1}| - |\delta| = \gamma - \delta a_{i,i-1} - \delta \geq 2 - \delta \geq 1$, $i = N, \dots, 2$. For $c \in (0, 1/2)$, always $a_{i,i-1} \leq 4$ and, taking into account $\delta < 1/8$, we get directly $\gamma - \delta a_{i,i-1} \geq \gamma - 4\delta \geq 1/2$ and $|\gamma - \delta a_{i,i-1}| - |\delta| = \gamma - \delta a_{i,i-1} - \delta \geq 1/2 - 1/8 = 3/8$, $i = N. \dots, 2$. Hence

$$\max_{1\leq i\leq N} \left|\tilde{g}_i\right| \leq \text{const} \max_{1\leq i\leq N} \left|g_i\right| .$$

We know that $4 - a_{1,0} \neq 0$ and $1 - 2a_{0,-1} \neq 0$ as otherwise Gaussian elimination could not give the unique solution of (3.21). In addition, as $N \to \infty$, actually $a_{1,0} \to x$ and $x = 4$ is contradicting to (3.27), because in this case detailed study really give $\gamma - \delta x = \gamma - 4\delta > 1$ but $\beta - \delta y < 4$. Likewise, as $N \to \infty$, $a_{1,-1} \to y$ and

$$1 - 2a_{0,-1} \to \frac{2 - x + 2y}{4 - x} = \frac{2 - (x - y) + y}{4 - x} \geq \frac{1}{4} \text{ for } c \in \left[\frac{1}{2}, 1\right] .$$

Thus, at least in the case $c \in [1/2, 1]$, we may estimate $|\tilde{g}_0|$ and $|\tilde{g}_{-1}|$ uniformly by $\|\mathbf{g}\|_\infty$ which means uniform in N boundedness of $\|B^{-1}\|$. Note that we use here and in the sequel the matrix norm generated by uniform vector norm. Obviously $\|B\|$ is uniformly in N bounded for all $c \in (0, 1]$. We have proved

Lemma 3.6. *For all fixed* $c \in (0, 1]$*, we have* $\|B\| \leq \text{const}$*, and for all* $c \in [1/2, 1]$*, it holds* $\|B^{-1}\| \leq \text{const}$ *as* $N \to \infty$*.*

Lemma 3.7. *For all* $c \in (1/2, 1]$*, we have* $\|A^{-1}\| \leq \text{const}$*, and for all* $c \in (0, 1/2)$*, it holds* $\|A^{-1}\| \to \infty$ *as* $N \to \infty$*.*

Proof. Consider first the case $c \in (0, 1/2)$. The system $A\mathbf{c} = \tilde{\mathbf{g}}$ in developed form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ a_{0,-1} & 1 & 0 & 0 & \dots & 0 \\ y & x & 1 & 0 & \dots & 0 \\ 0 & y & x & 1 & \dots & 0 \\ & & \ddots & & \ddots & \\ 0 & \dots & 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N+1} \end{pmatrix} = \tilde{\mathbf{g}} \, .$$

Take $\tilde{g}_0 = \varepsilon > 0$ and $\tilde{g}_i = 0$, $i \neq 0$. Then $c_{-1} = 0$ and $c_0 = \varepsilon$. Let λ and μ be the roots of the equation (3.26). Look for $c_i = K_1\lambda^{i+1} + K_2\mu^{i+1}$, $-1 \leq i \leq N-k$, where k is such that the coefficients in the i-th equation are (approximately) y and x. For $i = -1$, we have $K_1+K_2 = c_{-1} = 0$. Then, for $i = 0$ it follows $K_1 = \varepsilon/(\lambda-\mu)$ and, consequently, $K_2 = \varepsilon/(\mu-\lambda)$. We may rewrite $c_i = K_1\lambda^{i+1} + K_2\mu^{i+1}$ as $c_i = \tilde{K}_1\lambda^i + \tilde{K}_2\mu^i$ with $\tilde{K}_1 = \varepsilon\lambda/(\lambda-\mu)$ and $\tilde{K}_2 = \varepsilon\mu/(\mu-\lambda)$. Then it holds $c_i = (\tilde{K}_1 + \tilde{K}_2(\mu/\lambda)^i)\lambda^i \sim \tilde{K}_1\lambda^i$, because of $|\lambda| > 1$ in the case $c \in (0, 1/2)$. This gives unboundedness of $\|\mathbf{c}\|_\infty$. Taking into account $\|A^{-1}\|\|\tilde{\mathbf{g}}\|_\infty \geq \|A^{-1}\tilde{\mathbf{g}}\|_\infty \geq \|\mathbf{c}\|_\infty$, we have unboundedness of $\|A^{-1}\|$ as well.

Let now $c \in (1/2, 1]$. Then $c_{-1} = \tilde{g}_{-1}$ and thus $|c_{-1}| \leq \|\tilde{\mathbf{g}}\|_\infty$. If N is large enough, then $a_{0,-1} = (1-y)/(4-x)$. From $x - y < 1$ (see the proof of Lemma 3.5) we get even $a_{0,-1} < 1$. Thus, $|c_0| = |\tilde{g}_0 - a_{0,-1}\tilde{g}_{-1}| \leq 2\|\tilde{\mathbf{g}}\|_\infty$. The next coefficients c_i are determined from

$$c_i + xc_{i-1} + yc_{i-2} = \tilde{g}_i, \quad i = 1, \ldots, N-k\,. \tag{3.30}$$

Last coefficients c_i, $i = N-k+1, \ldots, N$, are determined by equations (3.22), where $a_{i,i-1}$, $a_{i,i-2}$ and k do not depend on N. Consequently, we have to show only that c_i in (3.30) could be estimated by $\|\tilde{\mathbf{g}}\|_\infty$. For this, we will use the ideas from the general theory of equations in finite differences [26].

Let us look for the solution of (3.30) as

$$c_i = a_i\lambda^i + b_i\mu^i, \quad i = 1, 2, \ldots, \tag{3.31}$$

where λ and μ are the roots of (3.26). To the equation

$$a_1\lambda + b_1\mu = \tilde{g}_0 - xc_0 - yc_{-1} = q_1 \quad \text{(denote so)}$$

we add

$$a_1 + b_1 = 0\,.$$

Then

$$a_1 = \frac{-q_1}{\begin{vmatrix} 1 & 1 \\ \lambda & \mu \end{vmatrix}}, \qquad b_1 = \frac{q_1}{\begin{vmatrix} 1 & 1 \\ \lambda & \mu \end{vmatrix}}.$$

Similarly, to

$$a_2\lambda^2 + b_2\mu^2 = \tilde{g}_2 - xc_1 - yc_0 = q_2$$

we add

$$a_2\lambda + b_2\mu = a_1\lambda + b_1\mu = q_1 \tag{3.32}$$

and get

$$a_2 = \frac{\begin{vmatrix} q_1 & \mu \\ q_2 & \mu^2 \end{vmatrix}}{\begin{vmatrix} \lambda & \mu \\ \lambda^2 & \mu^2 \end{vmatrix}}, \qquad b_2 = \frac{\begin{vmatrix} \lambda & q_1 \\ \lambda^2 & q_2 \end{vmatrix}}{\begin{vmatrix} \lambda & \mu \\ \lambda^2 & \mu^2 \end{vmatrix}}.$$

Next, substituting (3.31) for c_i in (3.30), we obtain

$$\begin{aligned} a_i\lambda^i + b_i\mu^i &+ x(a_{i-1}\lambda^{i-1} + b_{i-1}\mu^{i-1}) \\ &+ y(a_{i-2}\lambda^{i-2} + b_{i-2}\mu^{i-2}) = \tilde{g}_i, \quad i = 3, 4, \ldots . \end{aligned} \tag{3.33}$$

Add

$$a_i\lambda^i + b_i\mu^i = a_{i-1}\lambda^i + b_{i-1}\mu^i \tag{3.34}$$

or

$$\Delta a_i\lambda^{i-1} + \Delta b_i\mu^{i-1} = 0, \quad i = 3, 4, \ldots, \tag{3.35}$$

where $\Delta a_i = a_i - a_{i-1}$. Using (3.32) or (3.34), rewrite (3.33) as

$$a_i\lambda^i + b_i\mu^i + x(a_{i-1}\lambda^{i-1} + b_{i-1}\mu^{i-1}) + y(a_{i-1}\lambda^{i-2} + b_{i-1}\mu^{i-2}) = \tilde{g}_i$$

or

$$a_i\lambda^i + b_i\mu^i + a_{i-1}(x\lambda^{i-1} + y\lambda^{i-2}) + b_{i-1}(x\mu^{i-1} + y\mu^{i-2}) = \tilde{g}_i.$$

Taking into account $\lambda^i + x\lambda^{i-1} + y\lambda^{i-2} = 0$, we have

$$a_i\lambda^i + b_i\mu^i - a_{i-1}\lambda^i - b_{i-1}\mu^i = \tilde{g}_i$$

or

$$\Delta a_i\lambda^i + \Delta b_i\mu^i = \tilde{g}_i.$$

The last equation with (3.35) yields

$$\Delta a_i = \frac{\begin{vmatrix} 0 & \mu^{i-1} \\ \tilde{g}_i & \mu^i \end{vmatrix}}{\begin{vmatrix} \lambda^{i-1} & \mu^{i-1} \\ \lambda^i & \mu^i \end{vmatrix}}, \qquad \Delta b_i = \frac{\begin{vmatrix} \lambda^{i-1} & 0 \\ \lambda^i & \tilde{g}_i \end{vmatrix}}{\begin{vmatrix} \lambda^{i-1} & \mu^{i-1} \\ \lambda^i & \mu^i \end{vmatrix}}.$$

As $a_i = a_{i-1} + \Delta a_i = a_{i-2} + \Delta a_{i-1} + \Delta a_i = \ldots = a_2 + \sum_{j=3}^{i} \Delta a_j$, we get

$$a_i = a_2 + \sum_{j=3}^{i} \frac{\begin{vmatrix} 0 & \mu^{j-1} \\ \tilde{g}_j & \mu^j \end{vmatrix}}{\begin{vmatrix} \lambda^{j-1} & \mu^{j-1} \\ \lambda^j & \mu^j \end{vmatrix}}, \quad b_i = b_2 + \sum_{j=3}^{i} \frac{\begin{vmatrix} \lambda^{j-1} & 0 \\ \lambda^j & \tilde{g}_j \end{vmatrix}}{\begin{vmatrix} \lambda^{j-1} & \mu^{j-1} \\ \lambda^j & \mu^j \end{vmatrix}}$$

and

$$c_i = a_2\lambda^i + b_2\mu^i + \Sigma \tag{3.36}$$

with

$$\Sigma = \sum_{j=3}^{i} \tilde{g}_j \frac{\begin{vmatrix} \lambda^{j-1} & \mu^{j-1} \\ \lambda^i & \mu^i \end{vmatrix}}{\begin{vmatrix} \lambda^{j-1} & \mu^{j-1} \\ \lambda^j & \mu^j \end{vmatrix}}.$$

We have

$$\Sigma = \sum_{j=3}^{i} \frac{\lambda^{j-1}\mu^i - \lambda^i\mu^{j-1}}{\lambda^{j-1}\mu^j - \lambda^j\mu^{j-1}}\tilde{g}_j = \sum_{j=3}^{i} \frac{\mu^{i-j+1} - \lambda^{i-j+1}}{\mu - \lambda}\tilde{g}_j$$

$$= \sum_{j=3}^{i} (\mu^{i-j} + \mu^{i-j-1}\lambda + \ldots + \lambda^{i-j})\tilde{g}_j.$$

Taking into account $|\lambda| < 1$, $|\mu| < 1$, we obtain

$$|\Sigma| \leq \|\tilde{\mathbf{g}}\|_\infty (1 + |\mu| + |\lambda| + |\mu|^2 + |\mu||\lambda| + |\lambda|^2 + \ldots)$$

$$= \|\tilde{\mathbf{g}}\|_\infty (1 + |\mu| + |\mu|^2 + \ldots)(1 + |\lambda| + |\lambda|^2 + \ldots).$$

Hence,

$$|c_i| \leq |a_2| + |b_2| + \frac{1}{(1 - |\lambda|)(1 - |\mu|)}\|\tilde{\mathbf{g}}\|_\infty.$$

We have obtained earlier that $a_2 = (q_1\mu - q_2)/\lambda(\mu - \lambda)$ and $b_2 = (q_2 - q_1\lambda)/\mu(\mu - \lambda)$. To estimate a_2 and b_2 by $\|\tilde{\mathbf{g}}\|_\infty$, we have to keep in mind that $x^2 - 4y > 0$ (proof of Lemma 3.5) or $|\mu - \lambda| = \sqrt{x^2 - 4y} > 0$.

The proof is complete. □

Corollary 3.8. *For each $c \in (0, 1/2)$, there is $\nu > 1$, $C > 0$ and N_0 such that $\|A^{-1}\| \geq C\nu^N$ for $N \geq N_0$.*

Proof. In fact, we may take $\nu = |\lambda_{\max}| > 1$. In addition, it is immediate to check that $-1 < \mu < 0$. It is now clear from the proof of Lemma 3.7 that, for large N,

$$\left|c_{N-k}\right| \geq \left|\frac{\varepsilon\lambda}{\lambda-\mu}\lambda^{N-k} + \frac{\varepsilon\mu}{\mu-\lambda}\mu^{N-k}\right|$$

$$= \left(\frac{\varepsilon\lambda}{\lambda-\mu} + \frac{\varepsilon\mu}{\mu-\lambda}\left(\frac{\mu}{\lambda}\right)^{N-k}\right)\lambda^{N-k}$$

$$\sim \frac{\varepsilon\nu^N}{|\lambda-\mu|\nu^k}.$$

From $\|A^{-1}\|\|\tilde{\mathbf{g}}\|_\infty \geq \|A^{-1}\tilde{\mathbf{g}}\|_\infty = \|\mathbf{c}\|_\infty$ it follows immediately the statement of the corollary. □

Theorem 3.9. *For $N \to \infty$, the following is true:*

1) *if $c \in (1/2, 1]$, then $\|P_N\|_{C_0[0,T]\to C[0,T]} \leq \text{const}$,*

2) *if $c \in (0, 1/2)$, then $\|P_N\|_{C_0[0,T]\to C[0,T]}$ increases exponentially,*

3) *if $c = 1/2$, then $\|P_N\|_{C^1[0,T]\to C[0,T]} \leq \text{const}$ and*

   $\|P_N\|_{C_0[0,T]\to C_0[0,T]} \leq \text{const}\cdot N$.

Proof. By C_i we denote here (different) generic constants, not depending on N.

1) Let $c \in (1/2, 1]$. Take arbitrary $f \in C_0[0,T]$. We apply the stability of B-splines [61], Lemma 3.7, Lemma 3.6 and get

$$\begin{aligned}
\|P_N f\|_{C[0,T]} &= \|\sum_i c_i B_i\|_{C[0,T]} \\
&\leq C_1\|\mathbf{c}\|_\infty \\
&\leq C_1\|A^{-1}\|\|\tilde{\mathbf{g}}\|_\infty \\
&\leq C_1\|A^{-1}\|\|B^{-1}\|\|\mathbf{g}\|_\infty \\
&\leq C_2\|\mathbf{g}\|_\infty \leq C_3\|f\|_{C_0[0,T]}.
\end{aligned}$$

2) Consider the case $c \in (0, 1/2)$. For particular number N, the operator P_N can be estimated as in the previous part, hence

$$\|P_N f\|_{C_0[0,T]} \leq C_1(N)\|f\|_{C_0[0,T]} \quad \forall\, f \in C_0[0,T]$$

where we denote by $C_1(N)$ the norm of P_N. Let us use Corollary 3.8 and for given $N \geq N_0$ take $\tilde{\mathbf{g}} \neq 0$ with $\tilde{g}_{N+1} = 0$. We have $\mathbf{c} = A^{-1}\tilde{\mathbf{g}}$ such that $\|\mathbf{c}\|_\infty \geq C_2\nu^N\|\tilde{\mathbf{g}}\|_\infty$. Then $\mathbf{g} = B\tilde{\mathbf{g}}$ is such that $g_{N+1} = 0$. Applying uniform boundedness of B, we get $\|\mathbf{g}\|_\infty = \|B\tilde{\mathbf{g}}\|_\infty \leq \|B\|\|\tilde{\mathbf{g}}\|_\infty \leq C_3\|\tilde{\mathbf{g}}\|_\infty$. Hence,

$$\|\mathbf{c}\|_\infty \geq C_4\nu^N\|\mathbf{g}\|_\infty \tag{3.37}$$

with $C_4 = C_2/C_3$.

On the other hand, using again the stability of B-splines, for each $f \in C_0[0,T]$, such that $P_N f = \sum_i c_i B_i$, we get

$$\begin{aligned}\|\mathbf{c}\|_\infty \leq\ & C_5\|\sum_i c_i B_i\|_{C[0,T]} = C_5\|P_N f\|_{C[0,T]} \\ \leq\ & C_1(N)C_5\|f\|_{C_0[0,T]}\,.\end{aligned} \tag{3.38}$$

We shall show how to construct appropriate f by $\mathbf{g}$. Let f be the linear spline interpolant on $[\tau_1, T]$ with $f(\tau_i) = g_i$, $i = 1,\ldots,N$, and $f(T) = g_{N+1} = 0$, where we may also assume (according to the proof of Lemma 3.7) that $g_{N-1} = g_N = 0$. Let f be quadratic polynomial on $[0,\tau_1]$ determined by $f(0) = g_0$, $f'(0) = 3(g_0 - 2g_{-1})/h$ (from (3.20)), and, of course, $f(\tau_1) = g_1$. Thus, using divided differences $f(t_0,t_0) = f'(t_0)$ and $f(t_0,t_0,t_1) = (f'(t_0)(t_0-t_1) - f(t_0) + f(t_1))/(t_0-t_1)^2$, we obtain

$$\begin{aligned} f(t) &= f(x_0) + f(t_0,t_0)(t-t_0) + f(t_0,t_0,t_1)(t-t_0)(t-t_1) \\ &= g_0\left(1 + \frac{(t-t_0)(t_1-t)}{(t_0-t_1)^2}\right) + g_1\frac{(t-t_0)(t-t_1)}{(t_0-t_1)^2} \\ &\quad + \frac{3}{h}(g_0 - 2g_{-1})\left((t-t_0) + \frac{(t-t_0)(t_1-t)}{(t_0-t_1)}\right). \end{aligned}$$

Estimating the function f on the intervals $[\tau_1, T]$ and $[0,\tau_1]$, we have, respectively,

$$\begin{aligned} &\max_{\tau_1\leq t\leq T}|f(t)| \leq \|\mathbf{g}\|_\infty, \\ &\max_{0\leq t\leq\tau_1}|f(t)| \leq (1+\frac{1}{4})|g_0| + \frac{1}{4}|g_1| + 3(1+\frac{1}{4})|g_0 - 2g_{-1}| \leq C_6\|\mathbf{g}\|_\infty. \end{aligned}$$

Taking also into account $|f'(0)| = 3|g_0 - 2g_{-1}|/h \leq C_7\|\mathbf{g}\|_\infty/h$, it holds $\|f\|_{C_0[0,T]} \leq C_8\|\mathbf{g}\|_\infty/h$. This with (3.37) and (3.38) yields

$$C_4\nu^N h \leq C_1(N)C_5C_8$$

or $C_1(N) \geq C_9\nu^N/N$, which means the exponential growth of $C_1(N)$.

3) Assume that $c = 1/2$. Then $\lambda = \lambda_{\max} = -1$ and $-1 < \mu < 0$. In this case we can also deduce the representation (3.36), so $c_i = a_2(-1)^i + b_2\mu^i + \Sigma$, where

$$\Sigma = \sum_{j\le i}(-1)^{i-j}(1 + |\mu| + \ldots + |\mu|^{i-j})g_j$$

$$= \frac{1}{1-|\mu|}\sum_{j\le i}(-1)^{i-j}(1 - |\mu|^{i-j+1})g_j$$

$$= \frac{1}{1-|\mu|}\big((\tilde{g}_i - \tilde{g}_{i-1}) + (\tilde{g}_{i-2} - \tilde{g}_{i-3}) + \ldots\big)$$

$$- \frac{|\mu|}{1-|\mu|}\sum_{j\le i}(-1)^{i-j}|\mu|^{i-j}\tilde{g}_j\,.$$

The last sum could be estimated by $|\mu|\|\tilde{\mathbf{g}}\|_\infty/(1-|\mu|)^2$. So here we have to study only $\Sigma_1 = (\tilde{g}_i - \tilde{g}_{i-1}) + (\tilde{g}_{i-2} - \tilde{g}_{i-3}) + \ldots$. Denote $\omega = \gamma - \delta x$. It holds $\omega \approx \gamma - \delta a_{i,i-1} \approx 11/4$ because of $\delta = 1/8$, $\gamma = 23/8$ and $x \approx 1$. Then

$$\tilde{g}_i = \frac{g_i - \delta\tilde{g}_{i+1}}{\gamma - \delta a_{i+1,i}} \approx \frac{g_i - \delta\tilde{g}_{i+1}}{\omega}\,.$$

We have for any ν

$$\Sigma_1 = \nu\left((\tilde{g}_i - \tilde{g}_{i-1}) + (\tilde{g}_{i-2} - \tilde{g}_{i-3}) + \ldots\right) + (1-\nu)\tilde{g}_i$$
$$- (1-\nu)\left((\tilde{g}_{i-1} - \tilde{g}_{i-2}) + (\tilde{g}_{i-3} - \tilde{g}_{i-4}) + \ldots\right).$$

Then, using the representation

$$\tilde{g}_{i-1} - \tilde{g}_{i-2} = \frac{g_{i-1} - g_{i-2} - \delta(\tilde{g}_i - \tilde{g}_{i-1})}{\omega}\,,$$

we continue by

$$\Sigma_1 = \left(\nu + (1-\nu)\frac{\delta}{\omega}\right)\Sigma_1$$

$$- \frac{1-\nu}{\omega}\big((g_{i-1} - g_{i-2}) + (g_{i-3} - g_{i-4}) + \ldots\big) + O(\|\tilde{\mathbf{g}}\|_\infty)\,.$$

Let us choose $\nu < 0$ such that $\nu + (1-\nu)\delta/\omega = 0$ (in fact, $\delta/\omega \approx 1/22$). Because of the estimates

$$\big|g_{i-1} - g_{i-2}\big| = |f(\tau_{i-1}) - f(\tau_{i-2})| \le h\,\|f'\|_{C[0,T]}$$

and the boundedness of $\|B^{-1}\|$ (Lemma 3.6), for some $C > 0$ we get:

$$|\Sigma_1| \leq \frac{1-\nu}{\omega} T\|f'\|_{C[0,T]} + C\|f\|_{C[0,T]} \,.$$

A finite number (not depending on N) of equations (3.22) where $a_{i,i-1}$ and $a_{i,i-2}$ can considerably differ from x and y, does not spoil the estimations of c_i by $\|\tilde{\mathbf{g}}\|_\infty$ and $\|f\|_{C^1[0,T]}$. Thus, we have

$$\|P_N f\|_{C[0,T]} = \Big\| \sum_i c_i B_i \Big\|_{C[0,T]} \leq \text{const}\, \|f\|_{C^1[0,T]} \,.$$

The last estimate of Theorem is easy to establish if we keep in view $|\Sigma| \leq \text{const} \cdot i\|\tilde{\mathbf{g}}\|_\infty$ which, in turn, gives $|c_i| \leq \text{const} \cdot i\|\tilde{\mathbf{g}}\|_\infty$ and $\|\mathbf{c}\|_\infty \leq \text{const} \cdot N\|\tilde{\mathbf{g}}\|_\infty$.

The proof is complete. □

Remark 3.10. It is clear that, in the assertions of Theorem 3.9, the space $C[0,T]$ may be replaced by $C_0[0,T]$.

Remark 3.11. In our analysis the uniformness of the mesh is essential. Of course, there is no theoretical problem in using B-splines on a non-uniform mesh, since the stability results of B-splines remain applicable (see [61]). But in (3.19) the entries of the matrix are no more stationary and the technique of difference equations with constant coefficients based on Lemma 3.4 and, especially, on the quite sharp assertion of Lemma 3.5, cannot be exploited. It seems that a possible extension of the results of this section to non-uniform meshes should use different ideas.

Remark 3.12. Let us mention that the behaviour of $\|P_N\|$ cannot be obtained from the general considerations presented in [62]. The results therein, applied in our case, give only the uniform boundedness of $\|P_N\|$ from below.

Cubic spline interpolation projections with not-a-knot boundary conditions in both ends of the interval are studied in [64]. Graded meshes and the interpolation at mesh points (the case $c = 1$) are considered and convergence results are established. Note that cubic spline interpolation projections on arbitrary sequence of non-linear meshes with the interpolation at mesh points may be not convergent in the space of continuous functions (see [70]).

3.4 Application of the classical convergence theorem

We shall analyse here what gives the general convergence theorem (Theorem 2.1) applied to Volterra integral equations in the light of results about boundedness of interpolation projections.

Lemma 3.13. *Suppose that the projections P_N defined by* (3.4) *are uniformly bounded as operators in $C_0[0,T]$. Then $P_N f \to f$ in $C_0[0,T]$ as $N \to \infty$ for all $f \in C_0[0,T]$.*

Proof. Choose $f \in C_0[0,T]$. Let $S \in S_3(\tilde{\Delta}_N)$ be such that $S'(0) = f'(0)$, $S(t_i) = f(t_i)$, $i = 0, \ldots, N$. Then $S \to f$ in $C_0[0,T]$ (see [69, p.102]).

Clearly, $P_N S = S$. We have

$$\|S - P_N f\|_{C[0,T]} = \|P_N(S - f)\|_{C[0,T]} \leq \text{const}\, \|S - f\|_{C_0[0,T]} \to 0$$

and

$$\|P_N f - f\|_{C[0,T]} \leq \|P_N f - S\|_{C[0,T]} + \|S - f\|_{C[0,T]}$$

ensures required convergence. □

Let $\bar{E}$ and $\bar{F}$ be the completions of normed spaces E and F, respectively, and let $\bar{K}$ be the prolongation by continuity of an operator $K \in \mathcal{L}(E,F)$ on the space $\bar{E}$. We will need the next auxiliary result.

Proposition 3.14. *Suppose $K \in \mathcal{K}(E,F)$. Then it holds $\bar{K} \in \mathcal{K}(\bar{E},F)$.*

Proof. First, show that $\bar{K} : \bar{E} \to F$. Assume, there is $x \in \bar{E}$ such that $\bar{K}x \in \bar{F} \setminus F$. We can find a sequence $x_n \in E$ which converges to x in the space $\bar{E}$ and is therefore bounded. As K is compact operator, the sequence Kx_n is compact in F, i.e. there are $N' \subset \mathbb{N}$ and $z \in F$ so that $\|Kx_n - z\|_F \to 0$ for $n \in N'$. It is clear that Kx_n converges to z in the space $\bar{F}$ as well. Because of convergence of x_n to x in $\bar{E}$ and $\bar{K} \in \mathcal{L}(\bar{E},\bar{F})$, it holds $\|\bar{K}x_n - \bar{K}x\|_{\bar{F}} \to 0$. On the other hand, we have $\bar{K}x_n = Kx_n \to z$ in the space $\bar{F}$, which means that $\bar{K}x = z \in F$. Hence, our assumption was wrong and the operator $\bar{K}$ maps from $\bar{E}$ to F.

Next, check the compactness of the operator $\bar{K}$. Consider a bounded sequence $x_n \in \bar{E}$. Let $y_n \in E$ be a sequence $\|y_n - x_n\|_{\bar{E}} \to 0$. Then, y_n is also bounded due to $\|y_n\|_E = \|y_n\|_{\bar{E}} \leq \|y_n - x_n\|_{\bar{E}} + \|x_n\|_{\bar{E}}$. As K is compact, for any $N' \subset \mathbb{N}$ there are a subsequence $N'' \subset N'$ and $z \in F$ such that the sequence Ky_n, $n \in N''$, converges to z in the space F. Having

$\|y_n - x_n\|_{\bar{E}} \to 0$ and $\bar{K} \in \mathcal{L}(\bar{E}, F)$, we get the convergence of $\bar{K}(y_n - x_n)$ to zero in F. Thus, it holds

$$\|\bar{K}x_n - z\|_F \leq \|\bar{K}x_n - \bar{K}y_n\|_F + \|\bar{K}y_n - z\|_F \to 0 \quad \text{as} \quad n \in N''.$$

This gives the compactness of operator $\bar{K} : \bar{E} \to F$ and therefore completes the proof. □

Theorem 3.15. *Suppose that the kernel $\mathcal{K}$ in* (3.1) *is such that*

1) $K \in \mathcal{K}(C_0[0,T], C_0[0,T])$;

2) $y = Ky$, $y \in C[0,T]$, *implies* $y = 0$

and

3) $c \in (1/2, 1]$.

Then the method (3.2), (3.3) *is convergent in* $C_0[0,T]$ *and the estimates* (2.3) *and* (2.4) *hold.*

Proof. We know that $C_0[0,T]$ is not complete (see Proposition 3.1). In order to apply the general convergence theorem choose $E = \overline{C_0[0,T]}$, the completion of $C_0[0,T]$. The operators K and P_N can be prolonged by continuity on the space E. Due to finite dimension of $\operatorname{Im} P_N$ and Proposition 3.14, the prolongations of P_N and K have the values in $C_0[0,T]$. From the assumption that $\operatorname{Ker}(I - K) = \{0\}$ in $C[0,T]$, we conclude that $\operatorname{Ker}(I - K) = \{0\}$ in E, too. Because of Lemma 3.13 and the Banach-Steinhaus theorem, the prolongations of P_N strongly converge to I in E. Thus, the assertions of Theorem 3.15 follow from the general convergence theorem. □

The assumptions of Theorem 3.15 about the kernel $\mathcal{K}$ are satisfied, e.g., if $\mathcal{K}$ is continuous.

Let us note that the convergence rate of $\|P_N y^* - y^*\|$ is known to be $O(h^4)$ if $y^* \in C^4[0,T]$.

Conjecture. In the case $c = 1/2$, there is no compact convergence of operators $P_N K$ to K and therefore there is no convergence $\|P_N K - K\| \to 0$. There is stable and regular convergence $I - P_N K \to I - K$ in the case of arbitrary operator $(Ku)(t) = \int_0^t \mathcal{K}(t,s)u(s)ds$, $u \in C[0,T]$, with continuous kernel $\mathcal{K}$.

Chapter 4

Quadratic spline collocation

In this chapter we will study quadratic spline collocation for Volterra integral equations.

Consider again the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t, s, y(s))ds + f(t), \qquad t \in [0, T], \tag{4.1}$$

where $f : [0, T] \to \mathbb{R}$ and $\mathcal{K} : R \times \mathbb{R} \to \mathbb{R}$ are given functions and the set R is defined by $R = \{(t, s) : 0 \le s \le t \le T\}$.

4.1 Description of the method

Let us consider the collocation in the case of $d = 1, \quad m = 1$. Then approximate solutions are in the space of quadratic splines (denote it by $S_2(\Delta_N)$). For given $c \in (0, 1]$, as $m = 1$, we define collocation points $\tau_i = t_{i-1} + ch_i$, $i = 1, \dots, N$. In this case the collocation conditions are

$$u(\tau_i) = \int_0^{\tau_i} \mathcal{K}(\tau_i, s, u(s))ds + f(\tau_i), \quad i = 1, \dots, N\,. \tag{4.2}$$

Since $\dim S_2(\Delta_N) = N + 2$ it is necessary to give two additional conditions which we choose

$$\begin{aligned} &u(0) = y(0), \\ &u''(t_{N-1} - 0) = u''(t_{N-1} + 0). \end{aligned} \tag{4.3}$$

Let $P_N : C[0, T] \to C[0, T]$ be such that for any $f \in C[0, T]$ we have $P_N f \in S_2(\Delta_N)$ and

$$\begin{aligned} &(P_N f)(0) = f(0), \\ &(P_N f)(\tau_i) = f(\tau_i), \quad i = 1, \dots, N, \\ &(P_N f)''(t_{N-1} - 0) = (P_N f)''(t_{N-1} + 0). \end{aligned} \tag{4.4}$$

Like in Section 3.1, let us introduce, the vector of knots

$$\boldsymbol{s} : s_1 = s_2 = s_3 = t_0 < s_4 = t_1 < \ldots < s_{N+1} = t_{N-2}$$
$$< s_{N+2} = t_N = s_{N+3} = s_{N+4}$$

and corresponding B-splines $B_{1,2,\boldsymbol{s}}, \ldots, B_{N+1,2,\boldsymbol{s}}$, which are linearly independent functions. These B-splines form a basis in the spline space

$$S_2\ (\tilde{\Delta}_N)\ = S_2(\Delta_N) \bigcap \{f : f''(t_{N-1} - 0) = f''(t_{N-1} + 0)\}.$$

The Schoenberg – Whitney conditions (A.4) applied in the quadratic case ensure that the interpolation problem (4.4) has a unique solution. Thus, the operator P_N is correctly defined. It is clear that P_N is a linear projection onto the space $S_2(\tilde{\Delta}_N)$.

We consider also the integral operator defined by

$$(Ku)(t) = \int_0^t \mathcal{K}(t, s, u(s))ds, \quad t \in [0, T].$$

Similarly to Lemma 3.2, we can prove

Lemma 4.1. *It holds*

$$P_N f = 0 \Leftrightarrow \begin{cases} f(0) = 0, \\ f(\tau_i) = 0, \quad i = 1, \ldots, N. \end{cases}$$

Basing on Lemma 4.1, a standard calculation like in the proof of Lemma 3.3 enables to establish

Lemma 4.2. *The spline collocation problem* (4.2), (4.3) *is equivalent to the equation*

$$u = P_N K u + P_N f, \quad u \in S_2(\tilde{\Delta}_N).$$

By Lemma 4.2 the method (4.2), (4.3) is a projection method for the equation (4.1).

4.2 Method in the case of a test equation

Consider the test equation

$$y(t) = \lambda \int_0^t y(s)ds + f(t), \quad t \in [0, T], \quad \lambda \in \mathbb{C}.$$

We will look for approximate solution by collocation, stated in Section 4.1. Like in Section 3.2, using the boundary conditions (4.3) and locating the collocation conditions from (3.7), we obtain

$$
\begin{aligned}
&u(0) = f(0),\\
&u(\tau_1) = \lambda \int_0^{\tau_1} u(s)ds + f(\tau_1),\\
&u(\tau_i) - u(\tau_{i-1}) = \lambda \int_{\tau_{i-1}}^{\tau_i} u(s)ds + f(\tau_i) - f(\tau_{i-1}), \quad i = 2, \ldots, N,\\
&u''(t_{N-1} - 0) = u''(t_{N-1} + 0).
\end{aligned}
$$

Assume that the mesh Δ_N is uniform and complemented with knots $t_i = ih$, $i = -2, -1$, and $i = N+1, N+2$. For $i = -1, \ldots, N$. Let us consider the B-splines

$$
B_i(t) = \frac{1}{h^2} \begin{cases} (t - t_{i-1})^2, & t \in [t_{i-1}, t_i],\\ 2h^2 - (t_{i+1} - t)^2 - (t - t_i)^2, & t \in [t_i, t_{i+1}],\\ (t_{i+2} - t)^2, & t \in [t_{i+1}, t_{i+2}]. \end{cases} \tag{4.5}
$$

They are normalized with the condition

$$
\sum_{i=-1}^{N} B_i(t) = 2, \quad t \in [0, T].
$$

These functions form a basis in $S_2(\Delta_N)$ and therefore any quadratic spline u is representable by them

$$
u(t) = \sum_{i=-1}^{N} c_i B_i(t). \tag{4.6}
$$

The spline u can be written on the interval $[t_{i-1}, t_i]$ as

$$
\begin{aligned}
u(t) = &\; c_{i-2}B_{i-2}(t) + c_{i-1}B_{i-1}(t) + c_i B_i(t)\\
= &\; \frac{1}{h^2}\Big(c_{i-2}(t_i - t)^2\\
&+ c_{i-1}(2h^2 - (t_i - t)^2 - (t - t_{i-1})^2)\\
&+ c_i(t - t_{i-1})^2\Big).
\end{aligned}
$$

The integral of u will be

$$
\begin{aligned}
\int u(t)dt = \ & \frac{1}{h^2}\Bigg(-c_{i-2}\frac{(t_i-t)^3}{3} \\
& + c_{i-1}\left(2h^2t + \frac{(t_i-t)^3}{3} - \frac{(t-t_{i-1})^3}{3}\right) \\
& + c_i\frac{(t-t_{i-1})^3}{3}\Bigg).
\end{aligned}
\tag{4.7}
$$

Taking $\tau_i = t_{i-1} + ch$, we get, for $i = 1, \ldots, N$,

$$
u(\tau_i) = c_{i-2}(1-c)^2 + c_{i-1}(1+2c-2c^2) + c_ic^2, \tag{4.8}
$$

and, for $i = 2, \ldots, N$,

$$
\begin{aligned}
\int_{\tau_{i-1}}^{\tau_i} u(s)ds = & \int_{\tau_{i-1}}^{t_{i-1}} u(s)ds + \int_{t_{i-1}}^{\tau_i} u(s)ds \\
= & \frac{h}{3}\Big(c_{i-3}(1-c)^3 + c_{i-2}(4-6c^2+3c^3) \\
& + c_{i-1}(1+3c+3c^2-3c^3) + c_ic^3\Big).
\end{aligned}
$$

Using these formulae in the located collocation conditions (3.8), we have

$$
\alpha c_{i-3} + \beta c_{i-2} + \gamma c_{i-1} + \delta c_i = f_i \tag{4.9}
$$

with $i = 2, \ldots, N$ and

$$
\begin{aligned}
\alpha &= -(1-c)^2 - \frac{\lambda h}{3}(1-c)^3, \\
\beta &= -4c + 3c^2 - \frac{\lambda h}{3}(4-6c^2+3c^3), \\
\gamma &= 1 + 2c - 3c^2 - \frac{\lambda h}{3}(1+3c+3c^2-3c^3), \\
\delta &= c^2 - \frac{\lambda h}{3}c^3, \\
f_i &= f(\tau_i) - f(\tau_{i-1}).
\end{aligned}
$$

In the case of $i = 1$, using (4.8) and (4.7), we obtain

$$
\beta_1 c_{-1} + \gamma_1 c_0 + \delta_1 c_1 = f_1, \tag{4.10}
$$

where

$$\begin{aligned}
\beta_1 &= (1-c)^2 - \frac{\lambda h}{3}(3c - 3c^2 + c^3),\\
\gamma_1 &= 1 + 2c - 2c^2 - \frac{\lambda h}{3}(3c + 3c^2 - 2c^3),\\
\delta_1 &= c^2 - \frac{\lambda h}{3}c^3,\\
f_1 &= f(\tau_1).
\end{aligned}$$

The condition $u(0) = f(0)$ yields

$$c_{-1} + c_0 = f(0). \tag{4.11}$$

The derivatives of u on $[t_{i-1}, t_i]$ are

$$\begin{aligned}
u'(t) &= \frac{1}{h^2}\big(-2c_{i-2}(t_i - t) + 2c_{i-1}((t_i - t) - (t - t_{i-1})) + 2c_i(t - t_{i-1})\big),\\
u''(t) &= \frac{1}{h^2}\big(2c_{i-2} - 4c_{i-1} + 2c_i\big).
\end{aligned}$$

Thus, the not-a-knot condition $u''(t_{N-1} - 0) = u''(t_{N-1} + 0)$ gives

$$-c_{N-3} + 3c_{N-2} - 3c_{N-1} + c_N = 0. \tag{4.12}$$

Combining (4.11), (4.10), (4.9) and (4.12), we can write

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots & & & \\
\beta_1 & \gamma_1 & \delta_1 & 0 & 0 & & & & \\
\alpha & \beta & \gamma & \delta & 0 & & & & \\
0 & \alpha & \beta & \gamma & \delta & & & & \\
 & & \ddots & & \ddots & \ddots & & & \\
 & & & & & & & & \\
 & & \cdots & 0 & \alpha & \beta & \gamma & \delta \\
 & & \cdots & 0 & -1 & 3 & -3 & 1
\end{pmatrix}
\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \\ \\ c_{N-1} \\ c_N \end{pmatrix}
=
\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ \\ f_N \\ 0 \end{pmatrix}.$$

The matrix of the system has non-zero elements on five diagonals. To solve this system, we use the Gaussian elimination process. As in Section 3.2, we start with elimination over the main diagonal from the bottom to the top and get the system $A\mathbf{c} = \mathbf{g}$. The entries of A and $\mathbf{g}$ are calculated as follows

$$a_{N-1,N-2} = \frac{\beta - 3\delta}{\gamma + 3\delta}, \quad a_{N-1,N-3} = \frac{\alpha + \delta}{\gamma + 3\delta},$$

$$a_{i-1,i-2} = \frac{\beta - \delta a_{i,i-2}}{\gamma - \delta a_{i,i-1}}\,, \quad a_{i-1,i-3} = \frac{\alpha}{\gamma - \delta a_{i,i-1}}\,, \quad i = N-1, \ldots, 2,$$

$$a_{0,-1} = \frac{\beta_1 - \delta_1 a_{1,-1}}{\gamma_1 - \delta_1 a_{1,0}}\,,$$

$$g_{N-1} = \frac{f_N}{\gamma + 3\delta}\,, \quad g_{i-1} = \frac{f_{i-1} - \delta g_i}{\gamma - \delta a_{i,i-1}}\,, \quad i = N-1, \ldots, 2,$$

$$g_0 = \frac{f_1 - \delta_1 g_1}{\gamma_1 - \delta_1 a_{1,0}}\,, \quad g_{-1} = \frac{f_0 - g_0}{1 - a_{0,-1}}.$$

Then the coefficients c_i are

$$\begin{aligned}
&c_{-1} = g_{-1}, \\
&c_0 = g_0 - a_{0,-1} c_{-1}, \\
&c_i = g_i - a_{i,i-1} c_{i-1} - a_{i,i-2} c_{i-2}, \quad i = 1, \ldots, N, \\
&c_N = 3c_{N-1} - 3c_{N-2} + c_{N-3}.
\end{aligned}$$

Thus, we get the coefficients in the representation of the approximate solution of the test equation as quadratic spline (4.6) by B-splines.

4.3 Uniform boundedness of projections

In this section we will study the uniform boundedness of the sequence P_N.

Fix a number $c \in (0,1)$. Given any function $f \in C[0,T]$, let us consider $S = P_N f \in S_2(\Delta_N)$ determined by the conditions

$$\begin{aligned}
&S(0) = f(0), \\
&S(\tau_i) = f(\tau_i), \quad i = 1, \ldots, N, \\
&S''(t_{N-1} - 0) = S''(t_{N-1} + 0).
\end{aligned} \tag{4.13}$$

Denote $S_{i,c} = S(\tau_i)$ and $m_i = S'(t_i)$. Taking $t = t_{i-1} + \tau h_i$, we have the following representation of S:

$$S(t) = S_{i,c} + \frac{h_i}{2}(\tau - c)\big(\,(2 - (c+\tau))\,m_{i-1} + (c+\tau) m_i\big). \tag{4.14}$$

The continuity of S in the knots, i.e. the conditions $S(t_i - 0) = S(t_i + 0)$, $i = 1, \ldots, N-1$, gives

$$\begin{aligned}
(1-c)^2 h_i m_{i-1} \;+\; &\big((1-c^2)h_i + c(2-c)h_{i+1}\big)\, m_i + c^2 h_{i+1} m_{i+1} \\
=\; &2\,(f(\tau_{i+1}) - f(\tau_i))\,, \quad i = 1, \ldots, N-1.
\end{aligned} \tag{4.15}$$

The initial condition $S(0) = f(0)$ adds the equation

$$c(2-c)h_1 m_0 + c^2 h_1 m_1 = 2\left(f(\tau_1) - f(0)\right) \tag{4.16}$$

and the not-a-knot requirement at t_{N-1} could be written in the form

$$h_N m_{N-2} - (h_{N-1} + h_N) m_{N-1} + h_{N-1} m_N = 0. \tag{4.17}$$

The equation (4.17) yields

$$m_N = \left(1 + \frac{h_N}{h_{N-1}}\right) m_{N-1} - \frac{h_N}{h_{N-1}} m_{N-2}. \tag{4.18}$$

Then, using (4.18), eliminate m_N in (4.15). Write (4.16) together with (4.15) as follows

$$\begin{aligned} &\beta_0 m_0 + \gamma_0 m_1 = g_0, \\ &\alpha_i m_{i-1} + \beta_i m_i + \gamma_i m_{i+1} = g_i, \quad i = 1, \ldots, N-2, \\ &\alpha_{N-1} m_{N-2} + \beta_{N-1} m_{N-1} = g_{N-1}, \end{aligned} \tag{4.19}$$

where we denote

$$\beta_0 = \frac{2-c}{2(1-c)}\,, \quad \gamma_0 = \frac{c}{2(1-c)}\,, \quad g_0 = \frac{1}{c(1-c)}\,\frac{f(\tau_1) - f(0)}{h_1}\,,$$

and

$$\lambda_i = \frac{h_i}{h_i + h_{i+1}}\,, \quad \mu_i = 1 - \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}\,, \quad i = 1, \ldots, N-1,$$

$$\alpha_i = \frac{1-c}{2c}\,\lambda_i\,, \quad \beta_i = \frac{1+c}{2c}\,\lambda_i + \frac{2-c}{2(1-c)}\,\mu_i\,, \quad \gamma_i = \frac{c}{2(1-c)}\,\mu_i\,,$$

$$i = 1, \ldots, N-2,$$

$$\alpha_{N-1} = \frac{1-c}{2c}\,\lambda_{N-1} - \frac{c}{2(1-c)}\,\frac{h_N}{h_{N-1}}\,\mu_{N-1}\,,$$

$$\beta_{N-1} = \frac{1+c}{2c}\,\lambda_{N-1} + \frac{2-c}{2(1-c)}\,\mu_{N-1} + \frac{c}{2(1-c)}\,\frac{h_N}{h_{N-1}}\,,$$

and finally

$$g_i = \frac{1}{c(1-c)}\,\frac{f(\tau_{i+1}) - f(\tau_i)}{h_i + h_{i+1}}\,, \quad i = 1, \ldots, N-1.$$

It is easy to check that the matrix of the system (4.19) has a diagonal domination. Indeed, the difference of domination in rows is $|\beta_i| - (|\alpha_i| + |\gamma_i|) = 1$, $i = 0, \ldots, N-2$. Rewrite $\alpha = \alpha_{N-1}$ and $\beta = \beta_{N-1}$ as

$$\alpha = \frac{(1-c)^2 h_{N-1}^2 - c^2 h_N^2}{2c(1-c)h_{N-1}(h_{N-1}+h_N)},$$

$$\beta = \frac{(1-c^2)h_{N-1}^2 + 2c h_{N-1} h_N + c^2 h_N^2}{2c(1-c)h_{N-1}(h_{N-1}+h_N)} > 0.$$

If $\alpha \geq 0$, then

$$\beta - \alpha = \frac{h_{N-1}}{h_{N-1}+h_N} + \frac{h_N}{(1-c)(h_{N-1}+h_N)} + \frac{c\, h_N^2}{(1-c)h_{N-1}(h_{N-1}+h_N)} > 1.$$

If $\alpha < 0$, then

$$\beta - |\alpha| = \frac{h_{N-1}}{c(h_{N-1}+h_N)} + \frac{h_N}{(1-c)(h_{N-1}+h_N)} > 1.$$

Hence,

$$\max_{0 \leq i \leq N-1} |m_i| \leq \max_{0 \leq i \leq N-1} |g_i|. \tag{4.20}$$

In addition, from (4.18) and (4.20), we get

$$|m_N| \leq \left(1 + 2\frac{h_N}{h_{N-1}}\right) \max_{0 \leq i \leq N-1} |g_i|. \tag{4.21}$$

Our aim now is to estimate the norms of projections P_N in the space $C[0,T]$. In this section, in the sequel, we assume the sequence of meshes Δ_N is quasi-uniform, i.e. there is a constant r such that $h_{\max}/h_{\min} \leq r$ where $h_{\max} = \max_{1 \leq i \leq N} h_i$ and $h_{\min} = \min_{1 \leq i \leq N} h_i$. Then, for any function $f \in C[0,T]$, we have

$$|g_i| \leq \frac{1}{c(1-c)h_{\min}} \, \|f\|_{C[0,T]}, \quad i = 1, \ldots, N-1,$$

and

$$|g_0| \leq \frac{2}{c(1-c)h_{\min}} \, \|f\|_{C[0,T]}.$$

The representation (4.14), quasi-uniformity of the meshes and obtained estimates (4.20), (4.21) allow to get

$$\begin{aligned} \|P_N f\|_{C[0,T]} &= \max_{1\le i\le N} \max_{t\in[t_{i-1},t_i]} |S(t)| \\ &\le \|f\|_{C[0,T]} + h_{\max} \max_{0\le i\le N} |m_i| \\ &\le \text{const } \|f\|_{C[0,T]} , \end{aligned}$$

where the constant is independent of N and h, but depends on c and r. We have proved the following

Proposition 4.3. *For $c \in (0,1)$, in the case of quasi-uniform meshes, the projections P_N defined by* (4.4) *are uniformly bounded in the space $C[0,T]$.*

Note that similar quadratic spline projections are studied in [34] on quasi-uniform meshes and in [58] on graded meshes. Let us mention that quadratic spline projections on arbitrary sequence of meshes could be not uniformly bounded in the space $C[0,T]$ (see [70]).

Next, we will study the behavior of $\|P_N\|$ in the space $C[0,T]$ for $c = 1$. We restrict ourselves to the case of uniform mesh. We will use the B-splines (4.5) from Section 4.2. Given any function $f \in C[0,T]$, let us consider $u = P_N f = \sum_{-1\le j\le N} c_j B_j$, which is equivalent to the conditions

$$\begin{aligned} &u(t_i) = f(t_i), \quad i = 0,\dots,N, \\ &u''(t_{N-1} - 0) = u''(t_{N-1} + 0). \end{aligned} \tag{4.22}$$

Write (4.22) in the form of a linear system to determine the coefficients c_j as follows

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ 0 & & \cdots & 0 & 0 & 1 & 1 \\ 0 & & \cdots & -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ \\ c_{N-1} \\ c_N \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ \\ f_N \\ 0 \end{pmatrix} \tag{4.23}$$

with $f_i = f(t_i), \quad i = 0,\dots,N$. The system (4.23) has the unique solution because the determinant of its matrix is different from zero. Actually, the solution of (4.23) is

$$
\begin{aligned}
c_N &= \frac{1}{8}(f_{N-2} - 4f_{N-1} + 7f_N),\\
c_{N-1} &= \frac{1}{8}(-f_{N-2} + 4f_{N-1} + f_N),\\
c_{N-2} &= \frac{1}{8}(f_{N-2} + 4f_{N-1} - f_N),\\
c_{N-3} &= \frac{1}{8}(7f_{N-2} - 4f_{N-1} + f_N),\\
c_{N-4} &= f_{N-3} - c_{N-3},\\
c_i &= f_{i+1} - f_{i+2} + \ldots + (-1)^{N-i} f_{N-3} + (-1)^{N-i+1} c_{N-3},\\
&\qquad\qquad\qquad\qquad i = N-5, \ldots, -1.
\end{aligned}
\tag{4.24}
$$

This allows to get

$$\|P_N f\|_{C[0,T]} = \Big\| \sum_{i=-1}^{N} c_i B_i \Big\|_{C[0,T]} \leq 2 \max_{-1 \leq i \leq N} |c_i| \leq 2N \|f\|_{C[0,T]}.$$

Consider the function $f \in C[0,T]$ (see fig. 4.1) such that $f(t_i) = (-1)^i$,

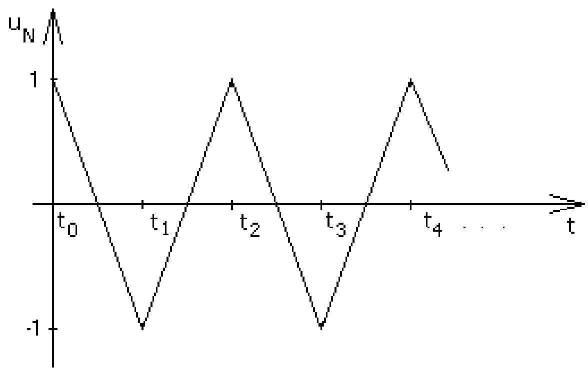

Figure 4.1

$i = 0, \ldots, N$, being linear between the knots t_i. Then, for example, for

$i = 2$ with $N \to \infty$, we have

$$\begin{aligned}\|P_N f\|_{C[0,T]} &\geq |(P_N f)(t_i)| \\ &= |c_{i-1}B_{i-1}(t_i) + c_i B_i(t_i) + c_{i+1}B_{i+1}(t_i)| \approx N\|f\|_{C[0,T]}.\end{aligned}$$

Thus, $\|P_N\| \geq \text{const} \cdot N$. It is established that the sequence $\|P_N\|$ has order N as $N \to \infty$.

In [52] the norm of the quadratic spline interpolation operators is explicitly calculated for the case of interpolation conditions which are actually the same as in the classical step-by-step collocation. This formula implies the order N of these projections. Let us mention that the results in [62] do not yield this asymptotics of the projection norms.

4.4 Application of the classical convergence theorem

In this section we show that, for $c \in (0, 1)$, Theorem 2.1 is applicable. Suppose that the sequence of meshes Δ_N is quasi-uniform.

Lemma 4.4. *For $c \in (0, 1)$, the projection operators P_N defined by* (4.4) *converge pointwise to the identity, i.e. $P_N f \to f$ in $C[0, T]$ for all $f \in C[0, T]$ as $N \to \infty$.*

Proof. Choose any $c \in (0, 1)$. For given $f \in C^1[0, T]$, let S and z be quadratic splines satisfying (4.13) for the chosen c and for $c = 1/2$ respectively. Taking into account $S = P_N f$, $\|P_N\| \leq \text{const}$ and $\|z - f\|_C \to 0$ (see [32]), we get

$$\begin{aligned}\|S - f\|_C &\leq \|S - z\|_C + \|z - f\|_C \\ &= \|P_N(f - z)\|_C + \|z - f\|_C \\ &\leq \text{const}\, \|f - z\|_C + \|z - f\|_C \to 0.\end{aligned}$$

This means that $\|P_N f - f\|_C \to 0$ for all $f \in C^1[0, T]$. Using the Banach-Steinhaus theorem, we get the convergence of the sequence of operators P_N to the identity operator everywhere in the space $C[0, T]$, since $C^1[0, T]$ is dense in $C[0, T]$. □

Let $E = C[0, T]$ and

$$(Ku)(t) = \int_0^t \mathcal{K}(t, s)u(s)ds, \quad u \in C[0, T]. \tag{4.25}$$

With the help of Lemma 4.4, Theorem 2.1 yields

Theorem 4.5. *Suppose the kernel $\mathcal{K}$ is such that K is compact, the mesh is quasi-uniform and $c \in (0,1)$. Then the method* (4.2), (4.3) *is convergent in $C[0,T]$ and the estimates* (2.3), (2.4) *hold.*

4.5 Compact convergence

We have already shown that our method is convergent for $c \in (0,1)$. In the case $c = 1$ the sequence of operators P_N is unbounded, so we cannot apply the classical convergence theorem. Taking into consideration Theorem 2.5 it is justified to ask whether there is the compact convergence of $P_N K$ to K.

First we state

Proposition 4.6. *Suppose the operator K is given by* (4.25), *where $\mathcal{K}(t,s)$ is continuous in $\{(t,s)|\ 0 \le s \le t \le T\}$ and continuously differentiable with respect to t. Then the sequence $P_N K$ converges strongly to K in $C[0,T]$.*

Proof. For given $u \in C[0,T]$, denote $f = Ku$. Then $f \in C^1[0,T]$. Let S be the quadratic spline interpolant determined by (4.13) in the case $c = 1/2$. It is known that $\|S - f\|_{C[0,T]} \le \text{const} \cdot h\ \omega(f')$ (see [32]), where $\omega(f)$ is the modulus of continuity of the function f. We obtain

$$\|P_N f - f\|_C \le \|P_N\| \|(f - S)\|_C + \|S - f\|_C.$$

The last norm converges to zero and

$$\|P_N\| \|f - S\|_C \le (\text{const} \cdot N)(\text{const} \cdot h\ \omega(f')) \le \text{const}\ \omega(f') \to 0.$$

Hence, $\|P_N K u - Ku\|_C \to 0$ for all $u \in C[0,T]$, which completes the proof. □

Let us focus our attention on the operator $(Ku)(t) = \int_0^t u(s)ds$. Consider in the rest of this section (and in the next section, too) the uniform mesh. Choose the sequence of functions $u_N \in C[0,T]$ (fig. 4.2) such that, for $i = 1, \dots, N$ and sufficiently small $\delta = \delta(N) > 0$,

$$u_N(t) = \begin{cases} 1, & \text{for } t \in [t_{i-1} + \delta, t_i - \delta] \text{ and } i \text{ is even,} \\ -1, & \text{for } t \in [t_{i-1} + \delta, t_i - \delta] \text{ and } i \text{ is odd,} \end{cases}$$

u_N being linear for $t \in [t_i - \delta, t_i + \delta]$, $i = 1, \dots, N-1$, and constant in $[t_0, t_0 + \delta]$ and $[t_N - \delta, t_N]$. Obviously, $\|u_N\|_C = 1$. Define $f_i = f(t_i) = (Ku_N)(t_i)$. We have $f_0 = 0$, $f_i = -h + \delta/2$ for $i = 1, 3, \dots$, $f_i = -\delta/2$ for

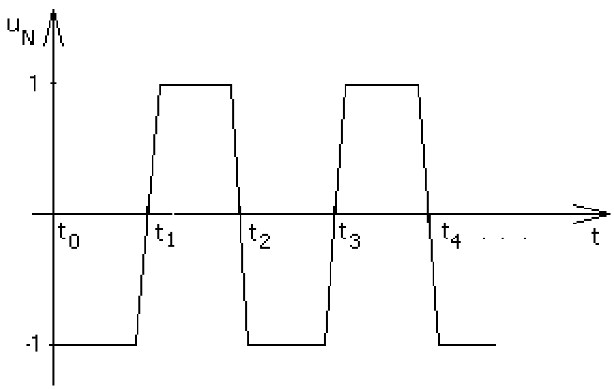

Figure 4.2

$i = 2, 4, \ldots$, and $f_N = 0$ for N even or $f_N = -h$ for N odd. Taking, for example, $\delta = \mathcal{O}(h^2)$, and using the equalities (4.24), we calculate for large N and relatively small i the coefficients

$$c_i = \frac{T}{2} + \mathcal{O}(h), \quad i = -1, 1, \ldots,$$

$$c_i = -\frac{T}{2} + \mathcal{O}(h), \quad i = 0, 2, \ldots \; .$$

The values of B-splines

$$B_i\left(t_i + \frac{h}{2}\right) = \frac{3}{2} \text{ and } B_{i-1}\left(t_i + \frac{h}{2}\right) = B_{i+1}\left(t_i + \frac{h}{2}\right) = \frac{1}{4}$$

allow us to obtain for odd values of i, that

$$(P_N K u_N)\left(t_i + \frac{h}{2}\right) = \sum_{j=i-1}^{i+1} c_j B_j\left(t_i + \frac{h}{2}\right) = \frac{T}{2} + \mathcal{O}(h),$$

$$(P_N K u_N)\left(t_{i-1} + \frac{h}{2}\right) = \sum_{j=i-2}^{i} c_j B_j\left(t_{i-1} + \frac{h}{2}\right) = -\frac{T}{2} + \mathcal{O}(h).$$

Hence,

$$|\,(P_N K u_N)\left(t_i + \frac{h}{2}\right) - (P_N K u_N)\left(t_{i-1} + \frac{h}{2}\right)| = T + \mathcal{O}(h),$$

which means that the functions $P_N K u_N$, as $N \to \infty$ or $h \to 0$, are not equicontinuous and, therefore, the sequence $P_N K u_N$ is not compact. We have proved

Proposition 4.7. *For $(Ku)(t) = \int_0^t u(s)ds$, the sequence $P_N K$ does not converge compactly to K in the case $c = 1$ as $N \to \infty$.*

4.6 Regular convergence

Our purpose in this section is to prove the regular convergence of operators $I - P_N K$ to $I - K$ in the case $c = 1$, using a new representation of quadratic splines. Recall that the mesh is assumed to be uniform.

Given a function $f \in C[0, T]$, let $S = P_N f \in S_2(\Delta_N)$ be such that

$$S(t_i) = f(t_i), \quad i = 0, \dots, N,$$
$$S''(t_{N-1} - 0) = S''(t_{N-1} + 0).$$

Denote $S_i = S(t_i)$ and $S_{i-1/2} = S(t_{i-1} + h/2)$. Using $t = t_{i-1} + \tau h$, we get the representation of S:

$$S(t) = (1 - \tau)(1 - 2\tau)S_{i-1} + 4\tau(1 - \tau)S_{i-1/2} + \tau(2\tau - 1)S_i. \tag{4.26}$$

The continuity of S' in the knots t_i, i.e. $S'(t_i - 0) = S'(t_i + 0)$, leads to the equations

$$S_{i-1} + 6S_i + S_{i+1} = 4(S_{i-1/2} + S_{i+1/2}), \quad i = 1, \dots, N - 1.$$

The not-a-knot boundary condition gives

$$S_N - S_{N-2} = 2(S_{N-1/2} - S_{N-3/2}).$$

Considering the values $S_i = f_i = f(t_i), \quad i = 0, \dots, N$, as known data we have the system

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & & \cdots & 0 & 1 & 1 \\ 0 & & \cdots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} S_{1/2} \\ S_{3/2} \\ \vdots \\ S_{N-3/2} \\ S_{N-1/2} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{N-1} \\ d_N \end{pmatrix}, \tag{4.27}$$

where $d_i = (f_{i-1} + 6f_i + f_{i+1})/4$, $i = 1, \dots, N-1$, and $d_N = (f_N - f_{N-2})/2$. However, the matrix of the system (4.27) is regular because its determinant is equal to 2. By direct calculation we obtain

$$\begin{aligned} S_{N-1/2} &= \frac{1}{8}\left(-f_{N-2} + 6f_{N-1} + 3f_N\right), \\ S_{N-3/2} &= \frac{1}{8}\left(3f_{N-2} + 6f_{N-1} - f_N\right), \\ S_{N-5/2} &= \frac{1}{8}\left(2f_{N-3} + 9f_{N-2} - 4f_{N-1} + f_N\right), \\ S_{k-1/2} &= \frac{1}{4}\left(f_{k-1} + 5f_k\right) - f_{k+1} + f_{k+2} - \dots \\ &\quad + \frac{(-1)^{N-k}}{8}\left(7f_{N-2} - 4f_{N-1} + f_N\right), \quad k = N-3, \dots, 1. \end{aligned} \tag{4.28}$$

Now, having S_i, $i = 0, \dots, N$, and $S_{i-1/2}$, $i = 1, \dots, N$, the spline S is determined by (4.26).

The next theorem is the main result of this section.

Theorem 4.8. *Suppose that the functions $\mathcal{K}$ and $\partial\mathcal{K}/\partial t$ are continuous in $\{(t,s)|\ 0 \le s \le t \le T\}$. Then, for $c = 1$ and for K defined by (4.25), the sequence $I - P_N K$ converges to $I - K$ regularly.*

Proof. In the assumptions of theorem we have the strong convergence of $I - P_N K$ to $I - K$ due to Proposition 4.6.

Choose a sequence $g_N \in C[0,T]$ such that $\|g_N\|_C \le 1$. Assume that the sequence $(I - P_N K)g_N$ is compact. We have to show the compactness of g_N.

Denote here $S = P_N K g_N$ and use the values S_i and $S_{i-1/2}$ of S as before. However, we have to keep in mind that they depend on N. The continuity of $\mathcal{K}$ and $\partial\mathcal{K}/\partial t$ ensures also the uniform continuity and boundedness of them. Thus, there are numbers M and M_1 such that $|\mathcal{K}(t,s)| \le M$ and $|(\partial\mathcal{K}/\partial t)(t,s)| \le M_1$ on $\{(t,s)|\ 0 \le s \le t \le T\}$.

For $t \in [t_{k-1}, t_k]$, $k = 1, \dots, N$, we have

$$\begin{aligned} g_N(t) - (P_N K g_N)(t) &= g_N(t) - S(t) \\ &= g_N(t) - \big((1-2\tau)(1-\tau)S_{k-1} + 4\tau(1-\tau)S_{k-1/2} + \tau(2\tau-1)S_k\big). \end{aligned}$$

The difference

$$\begin{aligned} S_{k-1} - (Kg_N)(t) &= \int_0^{t_{k-1}} \mathcal{K}(t_{k-1}, s) g_N(s) ds - \int_0^t \mathcal{K}(t,s) g_N(s) ds \\ &= \int_0^{t_{k-1}} \left(\mathcal{K}(t_{k-1}, s) - \mathcal{K}(t,s)\right) g_N(s) ds - \int_{t_{k-1}}^t \mathcal{K}(t,s) g_N(s) ds \end{aligned}$$

goes to zero uniformly on $[0,T]$ as $h \to 0$ because of the uniform continuity and boundedness of $\mathcal{K}$. Similarly, $S_k - (Kg_N)(t) \to 0$ uniformly on $[0,T]$ as $h \to 0$. Using $(1-2\tau)(1-\tau) + 4\tau(1-\tau) + \tau(2\tau-1) = 1$, we obtain

$$\begin{aligned} g_N(t) - S(t) = {} & g_N(t) - (Kg_N)(t) \\ & - 4\tau(1-\tau)\big(S_{k-1/2} - (Kg_N)(t)\big) + (G_N^1 g_N)(t), \end{aligned}$$

where $G_N^1 g_N \to 0$ in $C[0,T]$. Here the sequence Kg_N is compact and to establish the compactness of g_N, we will study the term $S_{k-1/2} - (Kg_N)(t)$.

Taking $f_i = (Kg_N)(t_i)$, let us write $S_{k-1/2}$ from (4.28) in the form

$$\begin{aligned} S_{k-1/2} = {} & f_k + \frac{1}{2}\Big((f_k - f_{k-1}) - (f_{k+1} - f_k) + \ldots + (-1)^{N-k}(f_N - f_{N-1})\Big) \\ & + \frac{1}{4}(f_{k-1} - f_k) + \frac{(-1)^{N-k}}{8}\big((f_{N-1} - f_{N-2}) + 3(f_{N-1} - f_N)\big) \\ = {} & f_k + \frac{1}{2}\sum_{i=k}^{N}(-1)^{i-k}(f_i - f_{i-1}) + \mathcal{O}(h). \end{aligned}$$

Using again $S_k - (Kg_N)(t) \to 0$ as $h \to 0$ uniformly on $[0,T]$, we get

$$S_{k-1/2} - (Kg_N)(t) = \frac{1}{2}\sum_{i=k}^{N}(-1)^{i-k}(f_i - f_{i-1}) + (G_N^2 g_N)(t)$$

with $G_N^2 g_N \to 0$ in $C[0,T]$. Denote

$$I_k(g_N) = \int_{t_{k-1}}^{t_k} \mathcal{K}(t_k, s) g_N(s)\,ds.$$

Then

$$\begin{aligned} f_k - f_{k-1} = {} & \int_{t_{k-1}}^{t_k} \mathcal{K}(t_k, s) g_N(s)\,ds + \int_0^{t_{k-1}} \big(\mathcal{K}(t_k, s) - \mathcal{K}(t_{k-1}, s)\big) g_N(s)\,ds \\ = {} & I_k(g_N) + h\int_0^{t_{k-1}} \frac{\partial \mathcal{K}}{\partial t}(\xi_k, s) g_N(s)\,ds \end{aligned}$$

where $\xi_k \in [t_{k-1}, t_k]$. Again, we have

$$\begin{aligned} (f_k - f_{k-1}) - (f_{k+1} - f_k) = {} & I_k(g_N) - I_{k+1}(g_N) \\ & - h\int_0^{t_{k-1}} \left(\frac{\partial \mathcal{K}}{\partial t}(\xi_{k+1}, s) - \frac{\partial \mathcal{K}}{\partial t}(\xi_k, s)\right) g_N(s)\,ds \\ & - h\int_{t_{k-1}}^{t_k} \frac{\partial \mathcal{K}}{\partial t}(\xi_{k+1}, s) g_N(s)\,ds. \end{aligned}$$

In this expression, the uniform continuity of $\partial\mathcal{K}/\partial t$ allows to estimate the first integral by $\varepsilon_N h$ with $\varepsilon_N \to 0$ as $h \to 0$ and the second one by $M_1 h^2$. Summing up all the differences $(f_k - f_{k-1}) - (f_{k+1} - f_k)$ (if there are an odd number of terms $f_k - f_{k-1}$ it suffices to observe that, in fact, $f_k - f_{k-1} = I_k(g_N) + \mathcal{O}(h)$), we get

$$\sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) + r_N$$

with $r_N \to 0$ as $h \to 0$. We arrived at

$$\begin{aligned} g_N(t) \; &- (P_N K g_N)(t) = g_N(t) - (K g_N)(t) \\ &- 2\tau(1-\tau) \sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) + (G_N^3 g_N)(t) \end{aligned} \tag{4.29}$$

where $G_N^3 g_N \to 0$ in $C[0,T]$.

Denoting $\alpha_N(t) = 2\tau(1-\tau)$, define the operators $Q_N : C[0,T] \to C[0,T]$ and the functions $\varphi_N \in C[0,T]$ by

$$(Q_N g_N)(t) = \alpha_N(t) \sum_{i=k}^{N} (-1)^{i-k} I_i(g_N), \tag{4.30}$$

$$\varphi_N = g_N - Q_N g_N. \tag{4.31}$$

Clearly, (4.29) yields the compactness of φ_N. For $t \in [t_{N-1}, t_N]$, from $\varphi_N(t) = g_N(t) - \alpha_N(t) I_N(g_N)$ we get $I_N(\varphi_N) = I_N(g_N) - I_N(\alpha_N) I_N(g_N)$. Denoting $\lambda_k = I_k(\alpha_N)$, $k = 1, \ldots, N$, we see (here we keep in mind the equality $s = t_{k-1} + \sigma h$) that

$$|\lambda_k| \le \int_{t_{k-1}}^{t_k} |\mathcal{K}(t_k, s)| |2\sigma(1-\sigma)| ds \le \frac{Mh}{2}.$$

Thus, for sufficiently small h, taking $\mu_k = 1/(1-\lambda_k)$, we have $I_N(g_N) = \mu_N I_N(\varphi_N)$. Now, the induction leads to

$$\sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) = \sum_{i=k}^{N} (-1)^{i-k} \left(\prod_{j=k}^{i} \mu_j \right) I_i(\varphi_N).$$

Indeed, assume that

$$\begin{aligned} &I_{k+1}(g_N) - I_{k+2}(g_N) + \ldots + (-1)^{N-k-1} I_N(g_N) \\ &= \mu_{k+1} I_{k+1}(\varphi_N) - \mu_{k+1}\mu_{k+2} I_{k+1}(\varphi_N) + \ldots \\ &\quad +(-1)^{N-k-1} \mu_{k+1} \ldots \mu_N I_N(\varphi_N) \end{aligned} \tag{4.32}$$

takes place. From (4.31) and (4.30) we obtain

$$I_k(\varphi_N) = I_k(g_N) - \lambda_k\big(I_k(g_N) - I_{k+1}(g_N) + \ldots + (-1)^{N-k} I_N(g_N)\big).$$

Subtracting the equality (4.32) from the previous one, we get

$$\begin{aligned} &I_k(\varphi_N) - \mu_{k+1} I_{k+1}(\varphi_N) + \ldots + (-1)^{N-k}\mu_{k+1}\ldots\mu_N I_N(\varphi_N) \\ &= (1-\lambda_k)\Big(I_k(g_N) - I_{k+1}(g_N) + \ldots + (-1)^{N-k} I_N(g_N)\Big) \end{aligned}$$

or

$$\begin{aligned} &I_k(g_N) - I_{k+1}(g_N) + \ldots + (-1)^{N-k} I_N(g_N) \\ &= \mu_k\Big(I_k(\varphi_N) - \mu_{k+1} I_{k+1}(\varphi_N) + \ldots + (-1)^{N-k}\mu_{k+1}\ldots\mu_N I_N(\varphi_N)\Big). \end{aligned}$$

Let $R_N : C[0,T] \to C[0,T]$ be defined by

$$(R_N\varphi_N)(t) = \alpha_N(t)\sum_{i=k}^{N}(-1)^{i-k}\left(\prod_{j=k}^{i}\mu_j\right) I_i(\varphi_N). \tag{4.33}$$

Taking into account $(Q_N g_N)(t) = (R_N\varphi_N)(t)$, write (4.31) in the form

$$g_N = \varphi_N + R_N\varphi_N. \tag{4.34}$$

Hence,

$$Q_N g_N = Q_N\varphi_N + Q_N R_N\varphi_N.$$

Replacing $Q_N g_N$ in (4.31) by the last formula, we obtain

$$g_N = \varphi_N + Q_N\varphi_N + Q_N R_N\varphi_N. \tag{4.35}$$

Now we establish three lemmas to complete the proof of Theorem 4.8.

Lemma 4.9. *The convergence $\varphi_N \to \psi$ in the space $C[0,T]$ implies $Q_N\varphi_N \to 0$ in $C[0,T]$.*

Proof. Based on (4.29) we have

$$Q_N\varphi_N = P_N K\varphi_N - K\varphi_N + G_N^3\varphi_N.$$

By Proposition 4.6, $P_N K\varphi_N - K\varphi_N \to 0$. Since φ_N is bounded, we get also $G_N^3\varphi_N \to 0$ which completes the proof. □

Lemma 4.10. *The operators R_N are uniformly bounded.*

Proof. For a function $\varphi \in C[0,T]$, consider

$$(R_N\varphi)(t) = \alpha_N(t) \sum_{i=k}^{N} (-1)^{i-k} \left(\prod_{j=k}^{i} \mu_j \right) I_i(\varphi).$$

Taking into account the estimates $|\alpha_N(t)| \leq 1/2$, $|I_k(\varphi)| \leq Mh\|\varphi\|_C$ and (for small h) $\mu_k = 1/(1-\lambda_k) \leq 1/(1-Mh/2) = \mu$ (denote so), we have

$$\|R_N\varphi\|_C \leq \frac{1}{2} Mh \|\varphi\|_C \left(\sum_{i=1}^{N-k+1} \mu^i \right) = \frac{h}{2} M \frac{\mu(\mu^{N-k+1}-1)}{\mu-1} \|\varphi\|_C$$
$$\sim \mu(\mu^{N-k+1}-1)\|\varphi\|_C \leq \mathrm{const}\|\varphi\|_C,$$

which completes the proof. □

Lemma 4.11. *It holds* $Q_N R_N = R_N Q_N$.

Proof. From (4.31) and (4.34) we get $(I+R_N)(I-Q_N) = I$. To prove the lemma, it is sufficient to check that $(I-Q_N)(I+R_N) = I$.

Choose an arbitrary $\varphi_N \in C[0,T]$ and determine $g_N = (I+R_N)\varphi_N$. We will prove that $(I-Q_N)g_N = \varphi_N$. Using (4.34) and (4.33), we calculate

$$I_i(g_N) = I_i(\varphi_N) + \lambda_i \sum_{j=i}^{N} (-1)^{j-i} \left(\prod_{l=i}^{j} \mu_l \right) I_j(\varphi_N),$$
$$i = k, \ldots, N.$$

Since $\varphi_N = g_N - R_N\varphi_N$ and, on the other hand, $\varphi_N = g_N - Q_N g_N$, we show that $R_N\varphi_N = Q_N g_N$ or, taking into account the definitions of R_N and Q_N,

$$\sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) = \sum_{i=k}^{N} (-1)^{i-k} \left(\prod_{j=k}^{i} \mu_j \right) I_i(\varphi_N). \tag{4.36}$$

However,

$$\sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) = \sum_{i=k}^{N} (-1)^{i-k} I_i(\varphi_N + R_N\varphi_N)$$
$$= \sum_{i=k}^{N} (-1)^{i-k} I_i(\varphi_N) + \sum_{i=k}^{N} (-1)^{i-k} \lambda_i \sum_{j=i}^{N} (-1)^{j-i} \left(\prod_{l=i}^{j} \mu_l \right) I_j(\varphi_N)$$

and, using $\lambda_k \mu_k = \mu_k - 1$, $k = 1, \ldots, N$, it is straightforward to check that here the coefficients of $I_i(\varphi_N)$ coincide with those in the right hand side of (4.36). Namely, the coefficient before $I_i(\varphi_N)$ is

$$\begin{aligned}
&(-1)^{i-k}(1 + \lambda_k \mu_k \ldots \mu_i + \lambda_{k+1}\mu_{k+1} \ldots \mu_i + \ldots + \lambda_i \mu_i) \\
&= (-1)^{i-k}(1 + (\mu_k - 1)\mu_{k+1} \ldots \mu_i + (\mu_{k+1} - 1)\mu_{k+2} \ldots \mu_i + \ldots + \mu_i - 1) \\
&= (-1)^{i-k}\mu_k \ldots \mu_i.
\end{aligned}$$

This proves the lemma. □

Finally, taking into account Lemma 4.11, we may write (4.35) in the form

$$g_N = \varphi_N + Q_N \varphi_N + R_N Q_N \varphi_N.$$

Remembering that φ_N was compact and, using Lemmas 4.9 and 4.10, we establish the compactness of g_N.

The proof of Theorem 4.8 is complete. □

Now, we are ready to close this section with the following

Theorem 4.12. *Let $\mathcal{K}$ and $\partial\mathcal{K}/\partial t$ be continuous in the closed triangle $\{(t,s)|\ 0 \le s \le t \le T\}$. Then, for $c = 1$ and for the uniform mesh, there is a number N_0 such that, for $N \ge N_0$, the problem* (4.2), (4.3) *has the unique solution, and the estimate* (2.3) *holds. If $P_N f$ converges to f then the solutions of* (4.2), (4.3) *converge to the solution of* (4.1) *in $C[0,T]$.*

Proof. Note that, in the assumptions about $\mathcal{K}$, the operator K defined by (4.25) is such that $u = Ku$ only for $u = 0$. Making use of Theorem 2.6, take $E = F = C[0,T]$, $A = I - K$, $A_N = I - P_N K$ and refer to Theorem 4.8. □

Remark 4.13. The rate of convergence of the method (4.2), (4.3) for linear equations, as stated in Theorems 4.5 and 4.12, is determined by the two-sided estimate (2.3). It is well known that quadratic spline interpolation projections P_N have the property $\|P_N y^* - y^*\| = \mathcal{O}(h^3)$ for smooth functions y^* (see [43]). The rate $\mathcal{O}(h^3)$ is confirmed by the numerical tests presented in Section 7.2.

4.7 The method in the space of continuously differentiable functions

We will focus our attention to the study of the method (4.2), (4.3) in the space $C^1[0,T]$. Similarly to the beginning of Section 4.3, fix $c \in (0,1)$ and define the projections P_N by (4.13). Without any additional assumption we may establish the estimates (4.20) and (4.21). Suppose now $f \in C^1[0,T]$. Then, for $i = 1, \dots, N-1$, and for some $\xi_i \in (\tau_i, \tau_{i+1})$, we have

$$\begin{aligned}
|g_i| &= \frac{|f'(\xi_i)|(\tau_{i+1}-\tau_i)}{c(1-c)(h_i+h_{i+1})} \\
&= |f'(\xi_i)| \left(\frac{1}{c}\,\frac{h_i}{h_i+h_{i+1}} + \frac{1}{1-c}\,\frac{h_{i+1}}{h_i+h_{i+1}} \right) \\
&\le \max\left\{ \frac{1}{c}\,, \frac{1}{1-c} \right\} \|f'\|_{C[0,T]}
\end{aligned}$$

and

$$|g_0| \le \frac{1}{1-c}\, \|f'\|_{C[0,T]}\,.$$

Taking into account (4.20), (4.21) and basing on the representation (4.14), we obtain $\|P_N\|_{C[0,T]} \le \text{const}\, \|f\|_{C^1[0,T]}$ only in the assumption $h_N^2/h_{N-1} = \mathcal{O}(1)$. The derivative of (4.14) ($S'(t) = (1-\tau)m_{i-1} + \tau m_i$) with the help of (4.20) and (4.21) allows to get the estimate

$$\|(P_N f)'\|_{C[0,T]} \le \max_{0\le i\le N} |m_i| \le \text{const}\, \|f\|_{C^1[0,T]}$$

provided $h_N/h_{N-1} = \mathcal{O}(1)$. We have proved the following

Proposition 4.14. *For $c \in (0,1)$, in the assumption $h_N/h_{N-1} = \mathcal{O}(1)$, the projections P_N are uniformly bounded in the space $C^1[0,T]$.*

It holds also

Lemma 4.15. *For $c \in (0,1)$ and $h_N/h_{N-1} = \mathcal{O}(1)$, the projections P_N converge pointwise to the identity in the space $C^1[0,T]$, i.e. $P_N f \to f$ in $C^1[0,T]$ for all $f \in C^1[0,T]$ as $h_{\max} \to 0$.*

Proof. Similarly to the proof of Lemma 4.4, for given $f \in C^2[0,T]$, construct the splines $P_N f$ and z. Then $\|z - f\|_{C^1} \to 0$ (see [32]) and

$$\|P_N f - f\|_{C^1} \le \|P_N\|_{C^1 \to C^1} \|f - z\|_{C^1} + \|z - f\|_{C^1} \to 0.$$

As $C^2[0,T]$ is dense in $C^1[0,T]$ it remains to use the Banach-Steinhaus theorem. □

Lemma 4.15 and Theorem 2.1 yield

Theorem 4.16. *Suppose the kernel $\mathcal{K}(t,s)$ is such that the operator K defined by* (4.25) *is compact in $C^1[0,T]$. Then the method* (4.2), (4.3) *with $c \in (0,1)$ and $h_N/h_{N-1} = \mathcal{O}(1)$ is convergent in $C^1[0,T]$ and the estimates* (2.3), (2.4) *hold.*

Let us add that the compactness of K in $C^1[0,T]$ takes place, for example, in the assumptions of Proposition 4.6, but they could be weakened so that weakly singular equations could be also included.

Next, we will study the method in $C^1[0,T]$ for $c = 1$. In the sequel, consider only the uniform mesh. As well as in Section 4.3 represent $P_N f$ by B-splines. The coefficients of the representation could be estimated from formulae (4.24) as

$$|c_i| \le \text{const}\left(\|f'\|_C + \|f\|_C\right) = \text{const}\,\|f\|_{C^1}\,.$$

Then it holds $\|P_N f\|_C \le \text{const}\,\|f\|_{C^1}$. Having also $|B_i'(t)| = \mathcal{O}(1/h)$, we get

$$\|(P_N f)'\|_C \le \|\sum_{i=-1}^{N} c_i B_i'\| \le \text{const}\cdot N\,\|f\|_{C^1}$$

which cannot be improved as we will see later. Thus, $\|P_N\|_{C^1\to C^1} = \mathcal{O}(N)$.

As in the case of the space of continuous functions and $c = 1$, we cannot apply the classical convergence theorem (Theorem 2.1). Wishing to apply Theorems 2.5 and 2.6, we have to prove that $P_N K$ converges strongly to K in $C^1[0,T]$. Assume that the kernel $\mathcal{K}$ in (4.25) is continuous and twice continuously differentiable with respect to t. Take $u \in C^1[0,T]$, then $f = Ku \in C^2[0,T]$. Likewise in the proof of Proposition 4.6, let S be the quadratic spline interpolant determined by (4.13) with $c = 1/2$. It is known (see [32]) that $\|S - f\|_{C^1} \le \text{const}\cdot h\,\omega(f'')$. Hence,

$$\|P_N f - f\|_{C^1} \le \|P_N\|_{C^1\to C^1}\|f - S\|_{C^1} + \|S - f\|_{C^1} \to 0$$

and we have the pointwise convergence $P_N K \to K$ in the space $C^1[0,T]$.

Let us show that there is no compact convergence $P_N K \to K$ in the space $C^1[0,T]$ even for the operator $(Ku)(t) = \int_0^t u(s)ds$. Take the functions $u_N \in C^1[0,T]$ (fig. 4.3) such that, for sufficiently small $\delta = \delta(N) > 0$,

$$u_N(t) = \begin{cases} t, & \text{for } t \in [0, \frac{h}{2} - \delta], \\ (-1)^i (t - t_i), & \text{for } t \in [t_i - \frac{h}{2} + \delta, t_i + \frac{h}{2} - \delta], \\ & \qquad i = 1, \ldots, N-1, \\ (-1)^N (t - t_N), & \text{for } t \in [t_N - \frac{h}{2} + \delta, t_N], \end{cases}$$

u_N being (uniquely determined by the continuity of u'_N) quadratic polynomial for $t \in [t_i - h/2 - \delta, t_i - h/2 + \delta]$, $i = 1, \ldots, N$. Clearly $\|u_N\|_C \le h/2$

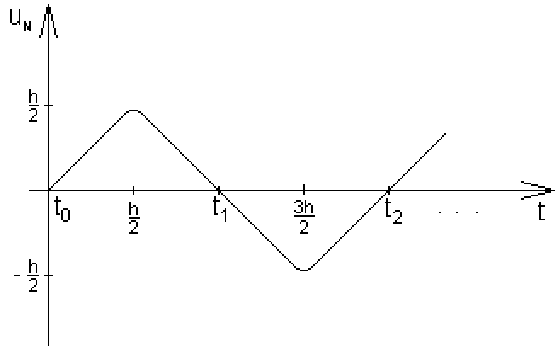

Figure 4.3

and $\|u'_N\|_C = 1$ which means that u_N is bounded in $C^1[0,T]$. Taking $\delta = \mathcal{O}(h^2)$ and defining $f_i = (Ku_N)(t_i)$, we calculate $f_i = h^2/4 + \mathcal{O}(h^3)$ for i odd and $f_i = 0$ for i even. Considering relatively small i and large N, we get

$$c_i = -\frac{Th}{8} + \mathcal{O}(h^2), \quad i = -1, 1, \ldots,$$

$$c_i = \frac{Th}{8} + \mathcal{O}(h^2), \quad i = 0, 2, \ldots \ .$$

Since $B'_i(t_i) = 2/h$, $B'_{i-1}(t_i) = -2/h$ and $B'_j(t_i) = 0$ hold for $j > i$ and $j < i - 1$, for odd values of i we have:

$$(P_N K u_N)'(t_i) = c_{i-1} B'_{i-1}(t_i) + c_i B'_i(t_i) = -\frac{T}{2} + \mathcal{O}(h),$$

$$(P_N K u_N)'(t_{i+1}) = c_i B'_i(t_{i+1}) + c_{i+1} B'_{i+1}(t_{i+1}) = \frac{T}{2} + \mathcal{O}(h).$$

Hence,

$$|\, (P_N K u_N)'\,(t_{i+1}) - (P_N K u_N)'\,(t_i)\,| = T + \mathcal{O}(h),$$

which means that $(P_N K u_N)'$ are not equicontinuous and, consequently, the sequence $P_N K u_N$ is not compact in $C^1[0,T]$. We have proved

Proposition 4.17. *For $(Ku)(t) = \int_0^t u(s)ds$, the sequence $P_N K$ does not converge compactly to K in the space $C^1[0,T]$ in the case $c = 1$ as $N \to \infty$.*

Note that, actually, in the last proof we established the inequality $\|P_N\|_{C^1 \to C^1} \geq \text{const} \cdot N$. Indeed, we have

$$\|P_N\|_{C^1 \to C^1} \geq \frac{\|P_N K u_N\|_{C^1}}{\|K u_N\|_{C^1}},$$

but

$$\begin{aligned} \|K u_N\|_{C^1} &= \|K u_N\|_C + \|(K u_N)'\|_C \\ &\leq T\|u_N\|_C + \|u_N\|_C \leq \frac{T+1}{2}\, h \end{aligned}$$

and

$$\begin{aligned} \|P_N K u_N\|_{C^1} &\geq \|(P_N K u_N)'\|_C \\ &\geq \|(P_N K u_N)'(t_i)| \geq \text{const}\frac{T}{2}. \end{aligned}$$

We state as an open problem, for $c = 1$, the regular convergence of $I - P_N K$ to $I - K$ in the space $C^1[0,T]$. Numerical results presented in Section 7.2 suggest the positive solution of this problem.

Chapter 5

Superconvergence of quadratic spline collocation

In this chapter we will study the convergence rate of the nonlocal collocation method with quadratic splines (see Chapter 4) at the collocation points for Volterra integral equations.

Consider the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t,s)y(s)ds + f(t), \qquad t \in [0,T], \tag{5.1}$$

where $f : [0,T] \to \mathbb{R}$ and $\mathcal{K} : R \to \mathbb{R}$ are given functions and the set R is defined by $R = \{(t,s) : 0 \le s \le t \le T\}$. The approximate solution $u_N \in S_2(\Delta_N)$ of this equation is determined by the collocation method (4.2), (4.3).

5.1 Superconvergence in the case $c = 1/2$

In this section we show the superconvergence of the spline collocation method in collocation points for $c = 1/2$ and uniform mesh Δ_N. As we have already seen, in this case projections P_N, defined by (4.4), are uniformly bounded. Thus, Theorem 2.1 is applicable and the estimates (2.3) and (2.4) hold. We assume in this section that the kernel $\mathcal{K}$ of the equation (5.1) and $\partial\mathcal{K}/\partial s$ are continuous. For the solution y of (5.1) suppose that $y''' \in \texttt{Lip}\,1$.

Using (4.4) and (2.4), for $\tau_i = t_{i-1} + h/2$ we have

$$\begin{aligned} |u_N(\tau_i) - y(\tau_i)| = |u_N(\tau_i) - P_N y(\tau_i)| &\le \|u_N - P_N y\|_C \\ &\le \text{const}\|K(P_N y - y)\|_C. \end{aligned}$$

Therefore the rate of $\|K(P_N y-y)\|_C$ is the key problem in our investigation.

First of all we find a suitable representation of quadratic splines. Given any function $y \in C[0,T]$, let us consider $S = P_N y \in S_2(\Delta_N)$ determined by the conditions

$$\begin{aligned} &S(0) = y(0), \\ &S(t_{i-1}+h/2) = y(t_{i-1}+h/2),\ i=1,\dots,N, \\ &S''(t_{N-1}-0) = S''(t_{N-1}+0). \end{aligned}$$

Denote $S_{i-1/2} = S(t_{i-1}+h/2)$ and $m_i = S'(t_i)$. Consider the representation (4.14), i.e., for $t \in [t_{i-1}, t_i]$, using $t = t_{i-1}+\tau h$, we have

$$S(t) = S_{i-1/2} + \frac{h}{8}(2\tau-1)\left((3-2\tau)m_{i-1} + (2\tau+1)m_i\right).$$

The initial condition $S(0) = y(0)$, the continuity of S in the knots and the not-a-knot requirement at t_{N-1} give

$$\begin{aligned} &3m_0 + m_1 = \frac{8}{h}(S_{1/2} - S_0), \\ &m_{i-1} + 6m_i + m_{i+1} = \frac{8}{h}(S_{i+1/2} - S_{i-1/2}),\ i=1,\dots,N-1, \\ &m_{N-2} - 2m_{N-1} + m_N = 0. \end{aligned} \tag{5.2}$$

The system of equations (5.2) has a unique solution. It will be calculated as $m_i = y_i' + \alpha_i h^2 y_i''' + \beta_i$, $i = 0,\dots,N$, where $y_i' = y'(t_i)$ and $y_i''' = y'''(t_i)$. Using a Taylor expansion in t_i, $i = 0,\dots,N$, we get

$$\begin{aligned} &\left(3\alpha_0 + \alpha_1 + \frac{1}{3}\right) h^2 y_0''' + \alpha_1\gamma_0 + 3\beta_0 + \beta_1 = \mathcal{O}(h^3), \\ &\left(\alpha_{i-1} + 6\alpha_i + \alpha_{i+1} + \frac{2}{3}\right) h^2 y_i''' + (\alpha_{i-1}+\alpha_{i+1})\gamma_i \\ &\qquad +\beta_{i-1} + 6\beta_i + \beta_{i+1} = \mathcal{O}(h^3),\ i=1,\dots,N-1, \\ &(\alpha_{N-2} - 2\alpha_{N-1} + \alpha_N + 1)h^2 y_N''' + (\alpha_{N-2} - 2\alpha_{N-1})\gamma_N \\ &\qquad +\beta_{N-2} - 2\beta_{N-1} + \beta_N = \mathcal{O}(h^3) \end{aligned}$$

with $\gamma_i = \mathcal{O}(h^3)$, $i = 0,\dots,N$. Let us impose the conditions

$$\begin{aligned} &3\alpha_0 + \alpha_1 = -\frac{1}{3}, \\ &\alpha_{i-1} + 6\alpha_i + \alpha_{i+1} = -\frac{2}{3},\ i=1,\dots,N-1, \\ &\alpha_{N-2} - 2\alpha_{N-1} + \alpha_N = -1. \end{aligned} \tag{5.3}$$

The unique solution of the system (5.3) is $\alpha_i = -1/12$, $i = 0, \dots, N-4$, and $\alpha_{N-3} = -67/840$, $\alpha_{N-2} = -11/105$, $\alpha_{N-1} = 1/24$, $\alpha_N = -341/420$. Then the unknowns β_i are determined from

$$
\begin{aligned}
&3\beta_0 + \beta_1 = \mathcal{O}(h^3),\\
&\beta_{i-1} + 6\beta_i + \beta_{i+1} = \mathcal{O}(h^3), \quad i = 1, \dots, N-1,\\
&\beta_{N-2} - 2\beta_{N-1} + \beta_N = \mathcal{O}(h^3).
\end{aligned}
$$

Thus, the numbers β_i are uniquely defined and $\beta_i = \mathcal{O}(h^3)$, $i = 0, \dots, N$. Therefore, for $t \in [t_{i-1}, t_i]$, using a Taylor expansion in t, we obtain the following representation of the spline S

$$
\begin{aligned}
S(t) =& y(t) + y'''(t)\frac{h^3}{24}(-4\tau^3 + 6\tau^2 - 1) + \mathcal{O}(h^4),\ i = 1, \dots, N-4,\\
S(t) =& y(t) + y'''(t)\frac{h^3}{48}(1-2\tau)\left((1-2\tau)^2 - 6(3-2\tau)\alpha_{i-1} - 6(2\tau+1)\alpha_i\right)\\
&+ \mathcal{O}(h^4),\ i = N-3, \dots, N.
\end{aligned}
$$

Then, for $t \in [t_{i-1}, t_i]$, $i = 1, \dots, N-4$, we get

$$
\begin{aligned}
&(K(P_N y - y))(t) = \int_0^t \mathcal{K}(t,s)(P_N y - y)(s)ds \qquad (5.4)\\
&= \frac{h^3}{24}\left(\sum_{k=1}^{i-1}\int_{t_{k-1}}^{t_k} \mathcal{K}(t,s)y'''(s)\varphi(\sigma)ds + \int_{t_{i-1}}^{t} \mathcal{K}(t,s)y'''(s)\varphi(\sigma)ds\right) + \mathcal{O}(h^4),
\end{aligned}
$$

where $\varphi(\sigma) = -4\sigma^3 + 6\sigma^2 - 1$ and we keep in mind the equality $s = t_{k-1} + \sigma h$. The sum of integrals is of order $\mathcal{O}(h)$. Indeed, we have

$$
\begin{aligned}
&\mathcal{K}(t,s)y'''(s) = \mathcal{K}(t, t_{k-1} + \sigma h)y'''(t_{k-1} + \sigma h)\\
&= \mathcal{K}(t, t_{k-1})y'''(t_{k-1}) + \sigma h\ \rho(t,s)|_{s=\xi_k},\ \xi_k \in [t_{k-1}, t_k].
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\int_{t_{k-1}}^{t_k} \mathcal{K}(t,s)y'''(s)\varphi(\sigma)ds = h\,\mathcal{K}(t, t_{k-1})\,y'''(t_{k-1})\int_0^1 \varphi(\sigma)d\sigma\\
&+ h^2\int_0^1 \rho(t,s)|_{s=\xi_k}\,\sigma\varphi(\sigma)d\sigma = \mathcal{O}(h^2),
\end{aligned}
$$

as $\int_0^1 \varphi(\sigma)d\sigma = 0$ and $\rho(t,s)$ is bounded, last due to $\mathcal{K}(t,\cdot)y''' \in \mathtt{Lip}\,1$ with the Lipschitz constant common for all t. The last integral in (5.4) can be estimated by const h. In the case $t \in [t_{k-1}, t_k]$, $k = N-3, \dots, N$, there are a bounded number of integrals, each of order $\mathcal{O}(h)$. Hence, we have proved

Theorem 5.1. *Suppose that $\mathcal{K}$ and $\partial\mathcal{K}/\partial s$ are continuous in $\{(t,s)|\,0 \le s \le t \le T\}$ and $y''' \in \mathtt{Lip}\,1$. Then, for $c = 1/2$, it holds*

$$\max_{1\le i\le N} |u_N(t_{i-1} + h/2) - y(t_{i-1} + h/2)| = \mathcal{O}(h^4).$$

5.2 Superconvergence in the case $c = 1$

According to Section 4.3 the sequence of projections P_N is not bounded when $c = 1$. Nevertheless, operators $I - P_N K$ converge regularly to $I - K$. In our case the regular and stable convergence coincide, so to prove the superconvergence, we can use a modified estimate (2.4).

By the definition of stable convergence the sequence of operators $(I - P_N K)^{-1}$ is bounded. Then, using (4.4) and the equality

$$(I - P_N K)(u_N - P_N y) = P_N K(P_N y - y),$$

we have

$$\begin{aligned} |u_N(t_i) - y(t_i)| &\le \|u_N - P_N y\|_C \le \|(I - P_N K)^{-1}\| \|P_N K(P_N y - y)\|_C \\ &\le \text{const}\, \|P_N K(P_N y - y)\|_C. \end{aligned}$$

In this section we shall show that $\|P_N K(P_N y - y)\|_C = \mathcal{O}(h^4)$ provided the mesh Δ_N is uniform and $c = 1$.

First, as above, we are going to find an appropriate representation of the spline. Given any function $y \in C[0,T]$, let us consider $S = P_N y \in S_2(\Delta_N)$ determined by the conditions

$$\begin{aligned} &S(t_i) = y(t_i), \quad i = 0, \dots, N, \\ &S''(t_{N-1} - 0) = S''(t_{N-1} + 0). \end{aligned}$$

Denote $S_i = S(t_i)$ and $S_{i-1/2} = S(t_{i-1} + h/2)$. Consider the representation of S from Section 4.6, i.e. for $t \in [t_{i-1}, t_i]$, $t = t_{i-1} + \tau h$, we have

$$S(t) = (1-\tau)(1-2\tau)S_{i-1} + 4\tau(1-\tau)S_{i-1/2} + \tau(2\tau - 1)S_i. \tag{5.5}$$

We will find the unknowns $S_{i-1/2}$, $i = 1, \dots, N$, from the system

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ & & & & & \\ 0 & & \cdots & 0 & 1 & 1 \\ 0 & & \cdots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} S_{1/2} \\ S_{3/2} \\ \vdots \\ \\ S_{N-3/2} \\ S_{N-1/2} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ \\ d_{N-1} \\ d_N \end{pmatrix}, \tag{5.6}$$

where $d_i = (y_{i-1}+6y_i+y_{i+1})/4$, $i = 1,\dots,N-1$, and $d_N = (y_N - y_{N-2})/2$. The matrix of (5.6) is regular because its determinant is equal to 2. Then, by direct calculation we obtain

$$\begin{aligned}
S_{N-1/2} &= \frac{1}{8}\big(-y_{N-2} + 6y_{N-1} + 3y_N\big),\\
S_{N-3/2} &= \frac{1}{8}\big(3y_{N-2} + 6y_{N-1} - y_N\big),\\
S_{N-5/2} &= \frac{1}{8}\big(2y_{N-3} + 9y_{N-2} - 4y_{N-1} + y_N\big),\\
S_{k-1/2} &= \frac{1}{4}\big(y_{k-1} + 5y_k\big) - y_{k+1} + y_{k+2} - \dots\\
&\quad + \frac{(-1)^{N-k}}{8}\big(7y_{N-2} - 4y_{N-1} + y_N\big), \quad k = N-3,\dots,1.
\end{aligned}$$

Suppose now that $y''' \in \mathtt{Lip}\,1$. Consider the case when $N-k$ is even, except $k = N, N-2$. Using Simpson's rule, i.e.

$$\begin{aligned}
&\frac{h}{3}\left(y_{k-1} + 4y_k + 2y_{k+1} + 4y_{k+2} + \dots + 4y_{N-2} + y_{N-1}\right)\\
&\qquad = \int_{t_{k-1}}^{t_{N-1}} y(t)dt + \frac{h^4}{180}\left(y'''_{N-1} - y'''_{k-1}\right) + \mathcal{O}(h^5)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{h}{3}\left(y_k + 4y_{k+1} + 2y_{k+2} + 4y_{k+3} + \dots + 4y_{N-1} + y_N\right)\\
&\qquad = \int_{t_k}^{t_N} y(t)dt + \frac{h^4}{180}\left(y'''_N - y'''_k\right) + \mathcal{O}(h^5),
\end{aligned}$$

we get

$$\begin{aligned}
S_{k-1/2} &= \frac{3}{2h}\left(\frac{h}{3}\left(y_{k-1} + 4y_k + 2y_{k+1} + 4y_{k+2} + \dots + 4y_{N-2} + y_{N-1}\right)\right.\\
&\quad \left. - \frac{h}{3}\left(y_k + 4y_{k+1} + 2y_{k+2} + 4y_{k+3} + \dots + 4y_{N-1} + y_N\right)\right)\\
&\quad - \frac{1}{4}y_{k-1} - \frac{1}{4}y_k - \frac{1}{8}y_{N-2} + y_{N-1} + \frac{5}{8}y_N\\
&= \frac{3}{2h}\left(\int_{t_{k-1}}^{t_k} y(t)dt + \frac{h^4}{180}\left(y'''_k - y'''_{k-1}\right)\right) - \frac{1}{4}(y_{k-1} + y_k)\\
&\quad - \frac{3}{2h}\left(\int_{t_{N-1}}^{t_N} y(t)dt - \frac{h^4}{180}\left(y'''_{N-1} - y'''_N\right)\right) - \frac{1}{8}y_{N-2} + y_{N-1} + \frac{5}{8}y_N.
\end{aligned}$$

Performing a Taylor expansion in $t_{k-1/2}$ and t_N for the last two rows of the preceding formula, respectively, we obtain

$$S_{k-1/2} = y_{k-1/2} + \frac{h^3}{16} y_N''' + \mathcal{O}(h^4). \tag{5.7}$$

Likewise, for $N-k$ odd, except $k = N-1$, we get

$$S_{k-1/2} = y_{k-1/2} - \frac{h^3}{16} y_N''' + \mathcal{O}(h^4). \tag{5.8}$$

In the cases $k = N, N-1, N-2$, using a Taylor expansion in $t_{k-1/2}$ and taking into account $f_{k-1/2}''' - f_N''' = \mathcal{O}(h)$, we get (5.7) and (5.8) as well. Now, substitute (5.7) or (5.8) for $S_{i-1/2}$ in (5.5) and use a Taylor expansion in $t = t_{i-1} + \tau h$, this gives

$$S(t) = y(t) - \frac{h^3}{12} y'''(t)\varphi(\tau) + (-1)^{N-i}\frac{h^3}{4} y'''(t_N)\psi(\tau) + \mathcal{O}(h^4), \tag{5.9}$$

where $\varphi(\tau) = \tau(1-\tau)(1-2\tau)$ and $\psi(\tau) = \tau(1-\tau)$.

Similarly to the case $c = 1/2$, we can show $(K(P_N y - y))(t) = \mathcal{O}(h^4)$. Namely, for $t \in [t_{i-1}, t_i]$, $i = 1, \dots, N$, we have

$$(K(P_N y - y))(t) = \int_0^t \mathcal{K}(t,s)(P_N y - y)(s)ds \tag{5.10}$$

$$= -\frac{h^3}{12}\int_0^t \mathcal{K}(t,s) y'''(s)\varphi(\sigma)ds + \frac{h^3}{4} y_N''' \int_0^t (-1)^{N-i}\mathcal{K}(t,s)\psi(\sigma)ds + \mathcal{O}(h^4).$$

Using the same technique as in Section 5.1, assuming that $\mathcal{K}(t,s)$ is continuous and continuously differentiable with respect to s and taking into account that $\int_0^1 \varphi(\sigma)d\sigma = 0$, we get that the first integral is of order $\mathcal{O}(h)$. Rewrite the second integral as follows

$$\int_0^t \mathcal{K}(t,s)\psi(\sigma)ds = \sum_{k=1}^{i-1}\int_{t_{k-1}}^{t_k} \mathcal{K}(t,s)\psi(\sigma)ds + \int_{t_{i-1}}^t \mathcal{K}(t,s)\psi(\sigma)ds.$$

Taking into account

$$(-1)^{N-k}\int_{t_{k-1}}^{t_k} \psi(\sigma)ds + (-1)^{N-k-1}\int_{t_k}^{t_{k+1}} \psi(\sigma)ds = 0,$$

and considering the pairs of integrals

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} \mathcal{K}(t,s)\psi(\sigma)ds - \int_{t_k}^{t_{k+1}} \mathcal{K}(t,s)\psi(\sigma)ds \\
&= \mathcal{K}(t,t_k)\left(\int_{t_{k-1}}^{t_k} \psi(\sigma)ds - \int_{t_k}^{t_{k+1}} \psi(\sigma)ds\right) \\
&\quad + \gamma(t,s)\left(\int_{t_{k-1}}^{t_k} (\sigma-1)\psi(\sigma)ds - \int_{t_k}^{t_{k+1}} \sigma\psi(\sigma)ds\right) = \mathcal{O}(h^2)
\end{aligned}$$

with $\gamma(t,s) = \mathcal{O}(h)$, it is straightforward to check that the last integral in (5.10) has also the order $\mathcal{O}(h)$. Thus, $(K(P_N y - y))(t) = \mathcal{O}(h^4)$.

Finally, apply the operator P_N to $K(P_N y - y)$. Denote $v = P_N y - y$. Then, using (5.9) for $t \in [t_{i-1}, t_i]$, we have

$$\begin{aligned}
(P_N K v)(t) = (Kv)(t) - \frac{h^3}{12}(Kv)'''(t)\varphi(\tau) \\
+ (-1)^{N-i}\frac{h^3}{4}(Kv)'''(t_N)\psi(\tau) + \mathcal{O}(h^4).
\end{aligned}$$

Our aim now is to show $(Kv)'''(t) = \mathcal{O}(h)$. Using a Taylor expansion in $t = t_{i-1} + \tau h$, we calculate

$$\begin{aligned}
v'(t) &= (P_N y)'(t) - y'(t) = \frac{h^2}{12}(-1 + 6\tau - 6\tau^2)y'''(t), \\
v''(t) &= \frac{h}{2}(1 - 2\tau)y'''(t).
\end{aligned}$$

Assume that $\mathcal{K}$ is continuous and three times continuously differentiable with respect to the first variable on $\{(t,s) : 0 \le s \le t \le T\}$ and the function $t \mapsto \mathcal{K}(t,t)$ is two times continuously differentiable on $[0,T]$. Then,

$$\begin{aligned}
(Kv)'''(t) = &\frac{d^3}{dt^3}\left(\int_0^t \mathcal{K}(t,s)v(s)ds\right) \\
= &\frac{d^2}{dt^2}\left(\mathcal{K}(t,t)\right)v(t) + 2\frac{d}{dt}\left(\mathcal{K}(t,t)\right)v'(t) + \mathcal{K}(t,t)v''(t) \\
&+ \frac{d}{dt}\left(\frac{\partial}{\partial t}\mathcal{K}(t,s)\Big|_{s=t}\right)v(t) + \frac{\partial}{\partial t}\mathcal{K}(t,s)\Big|_{s=t} v'(t) \\
&+ \frac{\partial^2}{\partial t^2}\mathcal{K}(t,s)\Big|_{s=t} v(t) + \int_0^t \frac{\partial^3}{\partial t^3}\mathcal{K}(t,s)v(s)ds \\
= &\mathcal{O}(h).
\end{aligned}$$

Thus, we have proved $\|P_N K(P_N y - y)\|_C = \mathcal{O}(h^4)$ and established

Theorem 5.2. *Suppose that $\mathcal{K}$, $\partial\mathcal{K}/\partial s$, $\partial\mathcal{K}/\partial t$, $\partial^2\mathcal{K}/\partial t^2$ and $\partial^3\mathcal{K}/\partial t^3$ are continuous on $\{(t,s) : 0 \le s \le t \le T\}$. Suppose also the function $t \mapsto \mathcal{K}(t,t)$ is twice continuously differentiable on $[0,T]$ and $y''' \in \mathtt{Lip}\,1$. Then,*

$$\max_{0\le i\le N} |u_N(t_i) - y(t_i)| = \mathcal{O}(h^4)$$

in the case of uniform mesh.

For $c = 1$ and $c = 1/2$, the numerical experiments in Section 7.3 confirm the convergence rate $\mathcal{O}(h^4)$ predicted by theory. The results in Tables 13−17 also show the superconvergence in collocation points with $c = \mathcal{O}(h^2)$ for the test equations. It leads us to the conjecture that the superconvergence holds also for this case. We state as an open problem the superconvergence of the spline collocation method (4.2), (4.3) in collocation points with $c = \mathcal{O}(h^2)$.

Chapter 6

Quadratic spline subdomain method for Volterra integral equations

In this chapter we will study the step-by-step and nonlocal subdomain methods with quadratic splines. It is natural to use a subdomain method when equation's free term is given by average values in subintervals. In addition, the collocation method could be considered as a subdomain method's approximation. Thus, the subdomain method is a basic method in collocation methods theory.

6.1 Description of the method

Consider the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t,s,y(s))ds + f(t), \qquad t \in [0,T], \tag{6.1}$$

with given functions $f : [0,T] \to \mathbb{R}$, $\mathcal{K} : R \times \mathbb{R} \to \mathbb{R}$ and the set $R = \{(t,s) : 0 \le s \le t \le T\}$.

There will be used a mesh $\Delta_N : 0 = t_0 < t_1 < \ldots < t_N = T$ which represents spline knots. As we consider the process $N \to \infty$, the knots t_i depend on N. In order to determine the approximate solution $u \in S_2(\Delta_N)$ of the equation (6.1), we impose the following subdomain conditions

$$\int_{t_{i-1}}^{t_i} u(t)dt = \int_{t_{i-1}}^{t_i} \int_0^t \mathcal{K}(t,s,u(s))ds\,dt + \int_{t_{i-1}}^{t_i} f(t)dt, \tag{6.2}$$

for all $i = 1, \ldots, N$.

Since $\dim S_2(\Delta_N) = N + 2$ it is necessary to give two additional conditions. We consider here two particular cases: the traditional step-by-step method, and the nonlocal method where one of initial conditions is replaced by the not-a-knot condition at the other end of the interval of integration.

6.1.1 Step-by-step method

To be able to start the calculations of the subdomain method with step-by-step implementation, assume that we can use the initial values

$$\begin{aligned} u(0) &= y(0), \\ u'(0) &= y'(0), \end{aligned} \tag{6.3}$$

which is justified by the requirement $u \in C^1[0, T]$. Thus, on every interval $[t_{i-1}, t_i]$ we have two conditions of smoothness and one subdomain condition to determine three parameters of u as a polynomial of degree two on $[t_{i-1}, t_i]$. This allows us to implement the method step-by-step, progressing from the interval $[t_{i-1}, t_i]$ to the next one.

We need the vector space

$$C_0[0, T] = \left\{ f \in C[0, T] : \exists\, f'(0) = \lim_{h \to 0+} \frac{(f(h) - f(0)}{h} \right\}.$$

For any $f \in C_0[0, T]$, let $P_N f \in S_2(\Delta_N)$ be such that

$$\begin{aligned} &(P_N f)(0) = f(0), \\ &(P_N f)'(0) = f'(0), \\ &\int_{t_{i-1}}^{t_i} (P_N f)(t) dt = \int_{t_{i-1}}^{t_i} f(t) dt, \; i = 1, \dots, N. \end{aligned} \tag{6.4}$$

Let the mesh Δ_N be complemented with knots $t_{-2} < t_{-1} < t_0$ and $t_{N+2} > t_{N+1} > t_N$. Denote $h_i = t_i - t_{i-1}$, $i = -1, \dots, N + 2$. This enables to introduce the normalized B-splines, for $i = -1, \dots, N$,

$$B_i(t) = \begin{cases} \dfrac{(t - t_{i-1})^2}{h_i(h_i + h_{i+1})}, & t \in [t_{i-1}, t_i), \\ \dfrac{(t_{i+1} - t)(t - t_{i-1})}{h_{i+1}(h_i + h_{i+1})} + \dfrac{(t_{i+2} - t)(t - t_i)}{h_{i+1}(h_{i+1} + h_{i+2})}, & t \in [t_i, t_{i+1}), \\ \dfrac{(t_{i+2} - t)^2}{h_{i+2}(h_{i+1} + h_{i+2})}, & t \in [t_{i+1}, t_{i+2}). \end{cases}$$

If we use the representation

$$(P_N f)(t) = \sum_{i=-1}^{N+1} c_i B_i(t), \quad t \in [0,T],$$

the conditions (6.4) could be written equivalently as

$$c_{-1} = f(0) - \frac{h_0}{2} f'(0),$$

$$h_1 c_{-1} + h_0 c_0 = (h_0 + h_1) f(0),$$

$$c_{i-2} \int_{t_{i-1}}^{t_i} B_{i-2}(t)dt + c_{i-1} \int_{t_{i-1}}^{t_i} B_{i-1}(t)dt + c_i \int_{t_{i-1}}^{t_i} B_i(t)dt = \int_{t_{i-1}}^{t_i} f(t)dt,$$

$$i = 1, \ldots, N.$$

This linear system to determine the coefficients c_i has a lower tridiagonal matrix with nonzero entries on main diagonal. Thus, the projector P_N is correctly defined.

Consider the integral operator defined by

$$(Ku)(t) = \int_0^t \mathcal{K}(t, s, u(s))ds, \quad t \in [0,T]. \tag{6.5}$$

Lemma 6.1. *The spline subdomain problem* (6.2), (6.3) *is equivalent to the equation*

$$u = P_N K u + P_N f, \quad u \in S_2(\Delta_N), \tag{6.6}$$

provided the kernel $\mathcal{K}$ *in* (6.5) *is continuous in* R *and differentiable with respect to* t *in some neighbourhood of* 0.

Proof. Note first that in the assumptions about the kernel $\mathcal{K}$ it holds $K : C[0,T] \to C_0[0,T]$. The proof of Lemma 6.1 is analogous to that of Lemma 3.3. Here we used a standard calculation based on the property of P_N that $P_N f = 0$ if and only if

$$f(0) = 0, \quad f'(0) = 0,$$

$$\int_{t_{i-1}}^{t_i} f(t)dt = 0, \ i = 1, \ldots, N.$$

Indeed, then (6.6) is equivalent to the equalities

$$(u - Ku - f)(0) = 0, \quad (u - Ku - f)'(0) = 0,$$

$$\int_{t_{i-1}}^{t_i} (u - Ku - f)(t)dt = 0, \quad i = 1, \ldots, N.$$

The first one of them is equivalent to $u(0) = f(0)$ or $u(0) = y(0)$ because $y(0) = f(0)$. The integral equalities are just (6.2). Taking into account

$$(Kv)'(t) = \int_0^t \frac{\partial \mathcal{K}}{\partial t}(t, s, v(s))ds + \mathcal{K}(t, t, v(t)), \quad v \in C[0, T],$$

and using the differentiability of $\mathcal{K}$ with respect to t in some neighbourhood of 0, we get $(Ku)'(0) = \mathcal{K}(0, 0, u(0))$ and $(Ky)'(0) = \mathcal{K}(0, 0, y(0))$. Thus, with the help of (6.1), the equality $(u - Ku - f)'(0) = 0$ is equivalent to $u'(0) - (Ku)'(0) = y'(0) - (Ky)'(0)$ or $u'(0) - \mathcal{K}(0, 0, u(0)) = y'(0) - \mathcal{K}(0, 0, y(0))$. But the last equality, as $u(0) = y(0)$, is equivalent to $u'(0) = y'(0)$ which completes the proof. □

6.1.2 Nonlocal method

We will see in Section 6.3 that the subdomain method with step-by-step implementation is unstable. A hopeful reparation which works in collocation could be the replacement of the first derivative condition by a not-a-knot boundary condition at the other end of the interval of integration, so in this case additional conditions are

$$\begin{aligned} &u(0) = y(0), \\ &u''(t_{N-1} - 0) = u''(t_{N-1} + 0). \end{aligned} \tag{6.7}$$

Let the operator $P_N : C[0, T] \to C[0, T]$ be such that, for any $f \in C[0, T]$, we have $P_N f \in S_2(\Delta_N)$ and

$$\begin{aligned} &(P_N f)(0) = f(0), \\ &\int_{t_{i-1}}^{t_i} (P_N f)(t)dt = \int_{t_{i-1}}^{t_i} f(t)dt, \ i = 1, \ldots, N, \\ &(P_N f)''(t_{N-1} - 0) = (P_N f)''(t_{N-1} + 0). \end{aligned} \tag{6.8}$$

First we have to verify that such a spline $P_N f$ exists and is uniquely determined.

Denote

$$z_i = \frac{1}{h_i} \int_{t_{i-1}}^{t_i} f(t)dt, \quad i = 1, \ldots, N.$$

Look for a cubic interpolating spline $S \in S_3(\Delta_N)$ satisfying

$$\begin{aligned} &S(0) = 0, \quad S'(0) = f(0), \\ &S(t_i) = \sum_{j=1}^{i} h_j z_j, \ i = 1, \ldots, N, \\ &S'''(t_{N-1} - 0) = S'''(t_{N-1} + 0). \end{aligned} \tag{6.9}$$

This problem has the unique solution $S \in S_3(\Delta_N)$, see Section 3.1. Taking $P_N f = S'$, we get a solution of (6.8), because

$$(P_N f)(0) = S'(0) = f(0),$$
$$\int_{t_{i-1}}^{t_i} (P_N f)(t)dt = S(t_i) - S(t_{i-1}) = h_i z_i = \int_{t_{i-1}}^{t_i} f(t)dt, \; i = 1, \dots, N.$$

Vice versa, let $u = P_N f \in S_2(\Delta_N)$ be a solution of (6.8). Then we have $S(t) = \int_0^t u(s)ds$ as a solution of (6.9). This means that projections P_N are correctly defined by (6.8).

Lemma 6.2. *Suppose the kernel* $\mathcal{K}$ *in* (6.5) *is such that the operator* K *maps the space* $C[0,T]$ *into itself. Then the subdomain problem* (6.2), (6.7) *is equivalent to the equation*

$$u = P_N K u + P_N f, \quad u \in S_2(\Delta_N). \tag{6.10}$$

The proof is analogous to that of Lemma 6.1.

6.2 Method in the case of a test equation

Consider already appeared test equation

$$y(t) = \lambda \int_0^t y(s)ds + f(t), \quad t \in [0,T], \quad \lambda \in \mathbb{C}. \tag{6.11}$$

We are looking for an approximate solution $u \in S_2(\Delta_N)$ which will be determined by subdomain method with different implementations, stated in previous section.

Assume that the mesh Δ_N is uniform. Similarly to Section 4.2, the quadratic spline u will be represented by B-splines

$$u(t) = \sum_{i=-1}^{N} c_i B_i(t), \tag{6.12}$$

where B_i are determined by (4.5).

Here we are going to express the subdomain conditions (6.2), which in this case are

$$\int_{t_{i-1}}^{t_i} u(t)dt = \lambda \int_{t_{i-1}}^{t_i} \int_0^t u(s)ds \; dt + \int_{t_{i-1}}^{t_i} f(t)dt, \quad i = 1, \dots, N, \tag{6.13}$$

in the form of linear equations.

First, find the left-hand side of (6.13). Using (4.7), we have

$$\int_{t_{i-1}}^{t_i} u(t)dt = \frac{h}{3}(c_{i-2} + 4c_{i-1} + c_i), \quad i = 1, \dots, N. \tag{6.14}$$

Secondly, calculate the double integral

$$\int_{t_{i-1}}^{t_i} \int_0^t u(s)ds\, dt, \quad i = 1, \dots, N.$$

Taking into account (6.12), write

$$\int_0^t u(s)ds = \sum_{j=-1}^{N} c_j \int_0^t B_j(s)ds = \sum_{j=-1}^{i} c_j \int_{[t_{j-1},t_{j+2}] \bigcap [0,t]} B_j(s)ds.$$

The support of the B-spline B_j can be divided into three intervals: $[t_{j-1}, t_j]$, $[t_j, t_{j+1}]$ and $[t_{j+1}, t_{j+2}]$. So we have to find the following integrals

$$S_k = \int_{t_{j+k-2}}^{t_{j+k-1}} B_j(s)ds \quad \text{and} \quad T_k = \int_{t_{j+k-2}}^{t} B_j(s)ds, \quad k = 1, 2, 3.$$

Using (4.5) and $t = t_{i-1} + \tau h$, $\tau \in [0, 1]$, we obtain

$$S_1 = \frac{h}{3}, \quad S_2 = \frac{4h}{3}, \quad S_3 = \frac{h}{3},$$

$$T_1 = \frac{h}{3}\tau^3, \quad T_2 = \frac{h}{3}(3\tau + 3\tau^2 - 2\tau^3), \quad T_3 = \frac{h}{3}(3\tau - 3\tau^2 + \tau^3).$$

Then for $t \in [t_0, t_1]$, $t \in [t_1, t_2]$ and $t \in [t_2, t_3]$, the corresponding integrals are

$$\begin{aligned}\int_0^t u(s)ds &= T_3 c_{-1} + T_2 c_0 + T_1 c_1 \\ &= \frac{h}{3}\left(c_{-1}(3\tau - 3\tau^2 + \tau^3) + c_0(3\tau + 3\tau^2 - 2\tau^3) + c_1\tau^3\right),\end{aligned}$$

$$\begin{aligned}\int_0^t u(s)ds &= S_3 c_{-1} + (S_2 + T_3)c_0 + (S_1 + T_2)c_1 + T_1 c_2 \\ &= \frac{h}{3}\left(c_{-1} + (a - 1)c_0 + bc_1 + cc_2\right),\end{aligned}$$

$$\begin{aligned}\int_0^t u(s)ds &= S_3 c_{-1} + (S_2 + S_3)c_0 + (S_1 + S_2 + T_3)c_1 + (S_1 + T_2)c_2 + T_1 c_3 \\ &= \frac{h}{3}\Big(c_{-1} + 5c_0 + ac_1 + bc_2 + cc_3\Big),\end{aligned}$$

and for $t \in [t_{i-1}, t_i],\ i = 4, \ldots, N,$

$$\begin{aligned}\int_0^t u(s)ds =& S_3 c_{-1} + (S_2 + S_3)c_0 + (S_1 + S_2 + S_3)(c_1 + \ldots + c_{i-3}) \\ &+ (S_1 + S_2 + T_3)c_{i-2} + (S_1 + T_2)c_{i-1} + T_1 c_i \\ =& \frac{h}{3}\left(c_{-1} + 5c_0 + 6(c_1 + \ldots + c_{i-3}) + ac_{i-2} + bc_{i-1} + cc_i\right),\end{aligned}$$

where $a = 5 + 3\tau - 3\tau^2 + \tau^3$, $b = 1 + 3\tau + 3\tau^2 - 2\tau^3$ and $c = \tau^3$. Now we are able to find the double integrals

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^t u(s)ds\,dt =& \frac{h^2}{12}(3c_{-1} + 8c_0 + c_1), \\ \int_{t_2}^{t_3}\int_0^t u(s)ds\,dt =& \frac{h^2}{12}(4c_{-1} + 19c_0 + 12c_1 + c_2), \\ \int_{t_2}^{t_3}\int_0^t u(s)ds\,dt =& \frac{h^2}{12}(4c_{-1} + 20c_0 + 23c_1 + 12c_2 + c_3), \\ \int_{t_{i-1}}^{t_i}\int_0^t u(s)ds\,dt =& \frac{h^2}{12}\big(4c_{-1} + 20c_0 + 24(c_1 + \ldots + c_{i-3}) \\ &+ 23c_{i-2} + 12c_{i-1} + c_i\big), \quad i = 4, \ldots, N.\end{aligned} \tag{6.15}$$

Replacing (6.14) and (6.15) in subdomain conditions (6.13), we get

$$\begin{aligned}&\alpha_1 c_{-1} + \alpha_2 c_0 + \beta_5 c_1 = \frac{12F_1}{h}, \\ &\beta_1 c_{-1} + \alpha_3 c_0 + \beta_4 c_1 + \beta_5 c_2 = \frac{12F_2}{h}, \\ &\beta_1 c_{-1} + \beta_2 c_0 + \beta_3 c_1 + \beta_4 c_2 + \beta_5 c_3 = \frac{12F_3}{h}, \\ &\beta_1 c_{-1} + \beta_2 c_0 + \gamma(c_1 + \ldots + c_{i-3}) + \beta_3 c_{i-2} + \beta_4 c_{i-1} \\ &\qquad + \beta_5 c_i = \frac{12F_i}{h}, \quad i = 4, \ldots, N,\end{aligned} \tag{6.16}$$

where

$$F_i = \int_{t_{i-1}}^{t_i} f(t)dt,\ i = 1, \ldots, N,$$

$$\alpha_1 = 4 - 3\lambda h, \quad \alpha_2 = 16 - 8\lambda h, \quad \alpha_3 = 4 - 19\lambda h, \quad \gamma = -24\lambda h,$$

$$\beta_1 = -4\lambda h, \quad \beta_2 = -20\lambda h, \quad \beta_3 = 4 - 23\lambda h, \quad \beta_4 = 16 - 12\lambda h, \quad \beta_5 = 4 - \lambda h.$$

Thus, the subdomain conditions (6.13) are explicitly written as linear equations.

Next we will explain how to find the coefficients c_i in the representation (6.12) of the approximate solution u.

6.2.1 Step-by-step method

Consider the initial conditions (6.3). From Section 4.2 (see formula (4.11)) and $u'(0) = y'(0) = \lambda y(0) + f'(0) = \lambda f(0) + f'(0)$, we obtain

$$\begin{aligned} c_{-1} + c_0 &= f(0), \\ c_{-1} - c_0 &= \frac{h}{2}\left(\lambda f(0) + f'(0)\right). \end{aligned} \tag{6.17}$$

Denote $g_0 = f(0)$, $g_1 = -h(\lambda f(0) + f'(0))/2$, then

$$\begin{aligned} c_{-1} &= \frac{g_0 + g_1}{2}, \\ c_0 &= \frac{g_0 - g_1}{2}. \end{aligned}$$

Using (6.16), we get the unknowns c_i

$$\begin{aligned} c_1 &= \frac{g_2 - \alpha_1 c_{-1} - \alpha_2 c_0}{\beta_5}, \\ c_2 &= \frac{g_3 - \beta_1 c_{-1} - \alpha_3 c_0 - \beta_4 c_1}{\beta_5}, \\ c_3 &= \frac{g_4 - \beta_1 c_{-1} - \beta_2 c_0 - \beta_3 c_1 - \beta_4 c_2}{\beta_5}, \\ c_i &= \frac{g_{i+1} - \beta_1 c_{-1} - \beta_2 c_0 - \gamma \sum_{j=1}^{i-3} c_j - \beta_3 c_{i-2} - \beta_4 c_{i-1}}{\beta_5}, \quad i = 4, \ldots, N, \end{aligned}$$

where $g_i = 12F_{i-1}/h$, $i = 2, \ldots, N+1$.

6.2.2 Nonlocal method

Collecting into a system boundary conditions (4.11), (4.12) and subdomain conditions (6.16), we have

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & & \dots & & 0 \\ \alpha_1 & \alpha_2 & \beta_5 & 0 & 0 & 0 & & \dots & & 0 \\ \beta_1 & \alpha_3 & \beta_4 & \beta_5 & 0 & 0 & & \dots & & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & 0 & & \dots & & 0 \\ \beta_1 & \beta_2 & \gamma & \beta_3 & \beta_4 & \beta_5 & & \dots & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & & \\ \beta_1 & \beta_2 & \gamma & \gamma & \dots & \gamma & \gamma & \beta_3 & \beta_4 & \beta_5 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{N-1} \\ c_N \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ \vdots \\ g_N \\ 0 \end{pmatrix},$$

with $g_0 = f(0)$ and $g_i = 12F_i/h$, $i = 1, \dots, N$. Solving this system, we get the coefficients in the representation of the quadratic spline (6.12) by B-splines.

6.3 Instability of the step-by-step method

In this section we show that the traditional step-by-step implementation of subdomain method gives an unstable method.

Suppose throughout this section that the mesh is uniform.

Definition 6.3. We say that the spline subdomain method with quadratic splines is *stable* if, for any $\lambda \in \mathbb{C}$ and any $f \in C^1[0,T]$, the approximate solution $u \in S_2(\Delta_N)$ of the test equation (6.11) remains bounded in $C[0,T]$ as $h \to 0$.

Proposition 6.4. *The quadratic spline subdomain method is not stable.*

Proof. Suppose that the method is stable. For $\lambda = 0$, the approximate solution u of the test equation (6.11) is just $P_N f$. Then the principle of uniform boundedness yields that the sequence $\|P_N\|_{C^1 \to C}$ is bounded, i.e., for all $f \in C^1[0,T]$,

$$\|u\|_C = \|P_N f\|_C \le \text{const} \, \|f\|_{C^1} \,, \tag{6.18}$$

where the constant may depend on T and λ, but not on N.

Given any function $f \in C_0[0,T]$, the coefficients c_i of $u = P_N f = \sum_{-1\le i\le N} c_i B_i$ are determined by the system (6.17), (6.16) in the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & & & & \\ 0 & \cdots & 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ \\ c_N \end{pmatrix} = \begin{pmatrix} g_{-1} \\ g_0 \\ g_1 \\ \vdots \\ \\ g_N \end{pmatrix}, \tag{6.19}$$

where

$$g_{-1} = f(0) - \frac{h}{2} f'(0), \quad g_0 = 2f(0), \quad g_i = \frac{6}{h} \int_{t_{i-1}}^{t_i} f(t)dt, \quad i = 1, \dots, N.$$

Let us consider the equations of (6.19) (except first two of them) as the difference equation (see [26])

$$c_{i-2} + 4c_{i-1} + c_i = g_i, \quad i = 1, \dots, N.$$

Its characteristic equation $\lambda^2 + 4\lambda + 1 = 0$ has the roots $\lambda = -2 - \sqrt{3}$ and $\mu = -2 + \sqrt{3}$.

Take the vector $g = (\varepsilon/2, \varepsilon, 0, \dots, 0)^T$ with $\varepsilon > 0$ and look for

$$c_i = K_1 \lambda^{i+1} + K_2 \mu^{i+1}, \quad i = -1, \dots, N,$$

as the corresponding solution of the system (6.19). First two equations of (6.19) allow to determine the coefficients

$$K_1 = \frac{\varepsilon(2-\mu)}{2(\lambda-\mu)}, \quad K_2 = \frac{\varepsilon(2-\lambda)}{2(\mu-\lambda)}.$$

Also, for $i \ge 1$, we have

$$|c_i| = \left| \lambda^i \left(K_1\lambda + K_2\mu \left(\frac{\mu}{\lambda}\right)^i \right) \right| \ge \text{const } \varepsilon|\lambda|^i, \tag{6.20}$$

because $(\mu/\lambda)^i \to 0$ as $i \to \infty$.

Consider the function $f(t) = h\cos(\pi t/h)$, $t \in [0,T]$. Then

$$f(0) = h, \quad f'(0) = 0 \quad \text{and} \quad \int_{t_{i-1}}^{t_i} f(t)dt = 0, \quad i = 1, \dots, N.$$

It is also easy to verify that $\|f\|_{C^1} \le \text{const}$. Letting $\varepsilon = h$, the use of (6.20) and the stability of B-splines (see [61]) gives

$$\|u\|_C \ge \text{const}\,|c_N| \ge \text{const}\,\varepsilon|\lambda|^N = \text{const}\,\frac{|\lambda|^N}{N} \to \infty\,,$$

as $N \to \infty$, which contradicts (6.18). Proposition is proved. □

Remark 6.5. The proof of Proposition 6.4 shows that the influence of round-off errors (whose role may be played, e.g., by ε) increases unboundedly when N goes to infinity.

6.4 Convergence of the nonlocal method

We would like to apply general convergence theorem (Theorem 2.1) for operator equations (6.10). One of the assumptions in this theorem is the convergence of the sequence of approximating operators P_N to the identity or injection operator. This means that the uniform boundedness of the sequence P_N is the key problem in the study of the collocation method (6.2), (6.7).

Throughout this section, consider the general (not necessarily uniform) mesh Δ_N with $h = \max_{1\le i\le N} h_i \to 0$ as $N \to \infty$. Given any function $f \in C[0,T]$, let us consider $u = P_N f \in S_2(\Delta_N)$ determined by the conditions

$$u(0) = f(0),$$
$$\int_{t_{i-1}}^{t_i} u(t)dt = \int_{t_{i-1}}^{t_i} f(t)dt,\ i = 1,\dots,N,$$
$$u''(t_{N-1} - 0) = u''(t_{N-1} + 0).$$

Using the notation $u_i = u(t_i)$, $z_i = h_i^{-1}\int_{t_{i-1}}^{t_i} f(t)dt$ and $t = t_{i-1} + \tau h_i$, $\tau \in [0,1]$, we have the following representation of u

$$u(t) = 6z_i\tau(1-\tau) + u_{i-1}(1 - 4\tau + 3\tau^2) + u_i\tau(3\tau - 2) \quad (6.21)$$

where $t \in [t_{i-1}, t_i]$. The continuity of the derivative u' in the knots gives

$$\mu_i u_{i-1} + 2u_i + \lambda_i u_{i+1} = w_i,\quad i = 1,\dots,N-1, \quad (6.22)$$

where

$$\mu_i = 1 - \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}},\quad w_i = 3(\mu_i z_i + \lambda_i z_{i+1}).$$

The initial condition $u(0) = f(0)$ adds the equation $u_0 = f(0)$, and the not-a-knot requirement at t_{N-1} could be written in the form

$$h_N^2 u_{N-2} + (h_N^2 - h_{N-1}^2)u_{N-1} - h_{N-1}^2 u_N = 2(h_N^2 z_{N-1} - h_{N-1}^2 z_N). \quad (6.23)$$

Then, eliminating u_N in (6.22) with the help of (6.23), we write the equalities to determine u_i, $i = 0, \dots, N-1$, as follows:

$$
\begin{aligned}
&u_0 = f(0), \\
&\mu_i u_{i-1} + 2u_i + \lambda_i u_{i+1} = w_i, \quad i = 1, \dots, N-2, \qquad (6.24) \\
&\frac{h_N}{h_{N-1}} u_{N-2} + \left(1 + \frac{h_N}{h_{N-1}}\right) u_{N-1} = \left(3 + 2\frac{h_N}{h_{N-1}}\right) \mu_{N-1} z_{N-1} + \lambda_{N-1} z_N.
\end{aligned}
$$

It is clear now that in the system (6.24) the difference of domination in rows is 1. Hence, if $h_N/h_{N-1} \le \text{const}$, it holds

$$\max_{0 \le i \le N-1} |u_i| \le \text{const} \max_{0 \le i \le N} |z_i| \le \text{const} \, \|f\|_C$$

and, in addition, with the help of (6.23),

$$|u_N| \le \text{const} \max_{0 \le i \le N} |z_i| \le \text{const} \, \|f\|_C \, .$$

The representation (6.21), assumption $h_N/h_{N-1} \le \text{const}$ and obtained estimates allow to get

$$\|P_N f\|_C = \max_{1 \le i \le N} \max_{t \in [t_{i-1}, t_i]} |u(t)| \le \text{const} \, \|f\|_C,$$

for any $f \in C[0,T]$ with a constant independent of N. We have proved the following

Proposition 6.6. *Assuming that* $h_N/h_{N-1} \le \text{const}$, *the projections* P_N *determined by* (6.8) *are uniformly bounded in the space* $C[0,T]$.

Remark 6.7. Note that the operators P_N, determined by (6.8), are correctly defined. Namely, the unknowns u_i, $i = 1, \dots, N$, in the representation (6.21) of $P_N f$ are uniquely determined from the system (6.24) and the condition (6.23).

We will apply Theorem 2.1 to the equation (6.1) which is possible due to the following

Lemma 6.8. *The projections* P_N *defined by* (6.8) *converge pointwise to the identity, i.e.* $P_N f \to f$ *in* $C[0,T]$ *for all* $f \in C[0,T]$ *as* $N \to \infty$.

Proof. The proof is similar to that of Lemma 4.4. For given $f \in C^1[0,T]$, let S be the quadratic spline satisfying

$$
\begin{aligned}
&S(0) = f(0), \\
&S\left(t_{i-1} + \frac{h_i}{2}\right) = f\left(t_{i-1} + \frac{h_i}{2}\right), \; i = 1, \dots, N, \\
&S''(t_{N-1} - 0) = S''(t_{N-1} + 0).
\end{aligned}
$$

Taking into account $\|P_N\| \le \text{const}$ and $\|S - f\|_C \to 0$ (see [32]), we get

$$\begin{aligned}\|P_N f - f\|_C &\le \|P_N f - S\|_C + \|S - f\|_C \\ &= \|P_N(f - S)\|_C + \|S - f\|_C \\ &\le \text{const}\, \|f - S\|_C + \|S - f\|_C \to 0.\end{aligned}$$

This means that $\|P_N f - f\|_C \to 0$ for all $f \in C^1[0,T]$. Basing on the Banach-Steinhaus theorem, we get the convergence of the sequence P_N to the identity operator everywhere in the space $C[0,T]$, since $C^1[0,T]$ is dense in $C[0,T]$. The proof is completed. □

Let $E = C[0,T]$ and the operator K be defined by (4.25). Using Lemma 6.8, Theorem 2.1 directly yields

Theorem 6.9. *Suppose the kernel $\mathcal{K}$ is such that K is compact and the mesh Δ_N has the property $h_N/h_{N-1} = \mathcal{O}(1)$. Then the method* (6.2), (6.7) *is convergent in $C[0,T]$ and the estimates* (2.3) *and* (2.4) *hold.*

Remark 6.10. The rate of convergence of the method (6.2), (6.7) for linear equations, as stated in Theorem 6.9, is determined by the two-sided estimate (2.3). In Appendix A.2 it is shown that quadratic spline histopolation projections P_N have the property $\|P_N u - u\|_C = \mathcal{O}(h^3)$ for smooth functions u (actually, it suffices $u'' \in \texttt{Lip}\ 1$). The rate $\mathcal{O}(h^3)$ is confirmed by the numerical tests presented in Section 7.4.

Chapter 7

Numerical tests

In numerical tests we chose the test equation

$$y(t) = \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0,1]\,, \tag{7.1}$$

with the exact solution $y(t) = (\sin t + \cos t + e^t)/2$. We also implemented collocation and subdomain methods for the equation in the linear case with

$$\mathcal{K}(t,s) = t - s \;\text{ and }\; f(t) = \sin t$$

whose exact solution is $y(t) = (2\sin t + e^t - e^{-t})/4$ on the interval $[0,1]$. This equation is used in [3, 14]. Actually, we calculated the error of the methods $\|u - y\|_C$ approximately as

$$\max_{1\le n\le N}\max_{0\le k\le 10}\left|(u-y)\left(t_{n-1} + \frac{kh}{10}\right)\right|.$$

The results are presented in the following sections and tables.

7.1 Cubic spline collocation

The approximate solution u is calculated by the cubic collocation method (3.2), (3.3). The convergence rate of $\|u-y\|_C = O(h^4)$ for smooth solutions in the case appropriate collocation parameter $c \in [1/2, 1]$ supports the theory.

Numerical results for $y(t) = \lambda \int\limits_0^t y(s)ds + f(t)$

Table 1. $\lambda = -1, \quad f(t) = \sin t + e^t$

N	*4*	*16*	*64*	*256*
$c = 1$	$1.94 \cdot 10^{-4}$	$8.63 \cdot 10^{-7}$	$3.48 \cdot 10^{-9}$	$1.37 \cdot 10^{-11}$
$c = 0.75$	$4.00 \cdot 10^{-4}$	$1.65 \cdot 10^{-6}$	$6.52 \cdot 10^{-9}$	$2.56 \cdot 10^{-11}$
$c = 0.5$	$8.68 \cdot 10^{-4}$	$3.73 \cdot 10^{-6}$	$1.49 \cdot 10^{-8}$	$5.85 \cdot 10^{-11}$

Table 2. $\lambda = 2, \quad f(t) = (3\cos t - \sin t - e^t)/2$

N	*4*	*16*	*64*	*256*
$c = 1$	$1.65 \cdot 10^{-4}$	$6.81 \cdot 10^{-7}$	$2.70 \cdot 10^{-9}$	$1.06 \cdot 10^{-11}$
$c = 0.75$	$3.12 \cdot 10^{-4}$	$1.55 \cdot 10^{-6}$	$6.39 \cdot 10^{-9}$	$2.53 \cdot 10^{-11}$
$c = 0.5$	$7.22 \cdot 10^{-4}$	$3.32 \cdot 10^{-6}$	$1.35 \cdot 10^{-8}$	$5.31 \cdot 10^{-11}$

Table 3. $\lambda = 1, \quad f(t) = \cos t$

N	*8*	*32*	*128*	*512*	*2048*
$c = 0.49$	$5.53 \cdot 10^{-5}$	$2.42 \cdot 10^{-7}$	$1.93 \cdot 10^{-9}$	$4.25 \cdot 10^{-8}$	$4.16 \cdot 10^{5}$
$c = 0.4$	$8.44 \cdot 10^{-5}$	$3.12 \cdot 10^{-5}$	$1.34 \cdot 10^{3}$	$7.96 \cdot 10^{40}$	$5.64 \cdot 10^{198}$
$c = 0.1$	$1.05 \cdot 10^{-2}$	$1.77 \cdot 10^{6}$	$2.86 \cdot 10^{46}$	$3.35 \cdot 10^{214}$	–

Table 4. $\lambda = -2, \quad f(t) = (3\sin t - \cos t + 3e^t)/2$

N	*8*	*32*	*128*	*512*	*2048*
$c = 0.49$	$6.23 \cdot 10^{-5}$	$2.79 \cdot 10^{-7}$	$3.14 \cdot 10^{-9}$	$8.98 \cdot 10^{-8}$	$3.53 \cdot 10^{6}$
$c = 0.4$	$1.11 \cdot 10^{-4}$	$6.48 \cdot 10^{-5}$	$2.83 \cdot 10^{3}$	$1.70 \cdot 10^{41}$	$1.99 \cdot 10^{199}$
$c = 0.1$	$1.94 \cdot 10^{-2}$	$3.62 \cdot 10^{6}$	$6.00 \cdot 10^{46}$	$6.73 \cdot 10^{214}$	–

It was impossible to compute the entries corresponding to the cases $c = 0.1$ with $N = 2048$ because of numerical overflow in the used computational packages (the highest representable value is 10^{307}).

Table 5. `Numerical results for` $y(t) = \int_0^t (t-s)y(s)ds + f(t)$

N	*8*	*32*	*128*	*512*
$c = 1.0$	$5.94 \cdot 10^{-6}$	$2.59 \cdot 10^{-8}$	$1.04 \cdot 10^{-10}$	$4.07 \cdot 10^{-13}$
$c = 0.5$	$2.37 \cdot 10^{-5}$	$1.04 \cdot 10^{-7}$	$4.15 \cdot 10^{-10}$	$1.64 \cdot 10^{-12}$
$c = 0.4$	$2.76 \cdot 10^{-5}$	$3.54 \cdot 10^{-8}$	1.71	$2.74 \cdot 10^{37}$
$c = 0.1$	$3.20 \cdot 10^{-4}$	$1.26 \cdot 10^{4}$	$5.16 \cdot 10^{43}$	$1.51 \cdot 10^{211}$

In addition, we present the results of numerical tests for the test equation (7.1) by step-by-step collocation method with cubic splines in the next table, taken from [54]. Given numbers are the approximate values of $\|u\|_C$, calculated as $\max_{1 \le n \le N} \max_{0 \le k \le 10} |u(t_{n-1} + kh/10)|$.

Table 6. $\lambda = 1, \quad f(t) = \cos t$

N	*4*	*8*	*16*	*32*	*64*
$c = 1$	2.0498	2.0491	33.56	$3.10 \cdot 10^{9}$	$3.85 \cdot 10^{26}$
$c = 0.5$	3.089	$1.42 \cdot 10^{4}$	$4.65 \cdot 10^{13}$	$8.33 \cdot 10^{33}$	$4.39 \cdot 10^{75}$
$c = 0.1$	$3.33 \cdot 10^{5}$	$5.99 \cdot 10^{16}$	$3.42 \cdot 10^{40}$	$1.88 \cdot 10^{89}$	$9.26 \cdot 10^{187}$

7.2 Quadratic spline collocation

In this section we found the quadratic spline u by the collocation method (4.2), (4.3). In the following tables we present $\|u - y\|_C$ in the upper row and $\|u' - y'\|_C$ in the lower row for particular values of N and c. The approximate value of $\|u' - y'\|_C$ is calculated similarly to that of $\|u - y\|_C$. The results confirm the rate $\|u - y\|_C = O(h^3)$ and $\|u' - y'\|_C = O(h^2)$ for smooth solutions predicted by the theory.

`Numerical results for` $y(t) = \lambda \int_0^t y(s)ds + f(t)$

Table 7. $\lambda = -2, \quad f(t) = (3\sin t - \cos t + 3e^t)/2$

N	*4*	*16*	*64*	*256*
$c = 1$	$1.12 \cdot 10^{-3}$	$2.21 \cdot 10^{-5}$	$3.63 \cdot 10^{-7}$	$5.74 \cdot 10^{-9}$
	$2.68 \cdot 10^{-2}$	$1.90 \cdot 10^{-3}$	$1.22 \cdot 10^{-4}$	$7.66 \cdot 10^{-6}$
$c = 0.7$	$2.66 \cdot 10^{-3}$	$4.79 \cdot 10^{-5}$	$7.66 \cdot 10^{-7}$	$1.20 \cdot 10^{-8}$
	$4.60 \cdot 10^{-2}$	$3.51 \cdot 10^{-3}$	$2.29 \cdot 10^{-4}$	$1.45 \cdot 10^{-5}$
$c = 0.5$	$4.59 \cdot 10^{-3}$	$8.94 \cdot 10^{-5}$	$1.46 \cdot 10^{-6}$	$2.31 \cdot 10^{-8}$
	$5.69 \cdot 10^{-2}$	$4.48 \cdot 10^{-3}$	$2.95 \cdot 10^{-4}$	$1.86 \cdot 10^{-5}$

Table 8. $\lambda = -1, \quad f(t) = \sin t + e^t$

N	*4*	*16*	*64*	*256*
$c = 0.1$	$7.71 \cdot 10^{-3}$ $7.11 \cdot 10^{-2}$	$1.62 \cdot 10^{-4}$ $5.81 \cdot 10^{-3}$	$2.70 \cdot 10^{-6}$ $3.85 \cdot 10^{-4}$	$4.29 \cdot 10^{-8}$ $2.44 \cdot 10^{-5}$
$c = 10^{-3}$	$8.0866 \cdot 10^{-3}$ $7.2391 \cdot 10^{-2}$	$1.6899 \cdot 10^{-4}$ $5.8971 \cdot 10^{-3}$	$2.8187 \cdot 10^{-6}$ $3.9151 \cdot 10^{-4}$	$4.4751 \cdot 10^{-8}$ $2.4834 \cdot 10^{-5}$
$c = 10^{-6}$	$8.0891 \cdot 10^{-3}$ $7.2399 \cdot 10^{-2}$	$1.6902 \cdot 10^{-4}$ $5.8972 \cdot 10^{-3}$	$2.8184 \cdot 10^{-6}$ $3.9148 \cdot 10^{-4}$	$4.4618 \cdot 10^{-8}$ $2.4900 \cdot 10^{-5}$

Table 9. $\lambda = 1, \quad f(t) = \cos t$

N	*4*	*16*	*64*	*256*
$c = 1$	$1.32 \cdot 10^{-3}$ $2.29 \cdot 10^{-2}$	$2.90 \cdot 10^{-5}$ $1.85 \cdot 10^{-3}$	$4.90 \cdot 10^{-7}$ $1.25 \cdot 10^{-4}$	$7.80 \cdot 10^{-9}$ $7.98 \cdot 10^{-6}$
$c = 0.7$	$2.19 \cdot 10^{-3}$ $4.20 \cdot 10^{-2}$	$4.73 \cdot 10^{-5}$ $3.42 \cdot 10^{-3}$	$7.98 \cdot 10^{-7}$ $2.27 \cdot 10^{-4}$	$1.27 \cdot 10^{-8}$ $1.44 \cdot 10^{-5}$
$c = 0.5$	$4.03 \cdot 10^{-3}$ $5.38 \cdot 10^{-2}$	$8.63 \cdot 10^{-5}$ $4.41 \cdot 10^{-3}$	$1.45 \cdot 10^{-6}$ $2.93 \cdot 10^{-4}$	$2.30 \cdot 10^{-8}$ $1.86 \cdot 10^{-5}$

Table 10. $\lambda = 2, \quad f(t) = (-\sin t + 3\cos t - e^t)/2$

N	*4*	*16*	*64*	*256*
$c = 0.1$	$7.24 \cdot 10^{-3}$ $6.99 \cdot 10^{-2}$	$1.56 \cdot 10^{-4}$ $5.79 \cdot 10^{-3}$	$2.62 \cdot 10^{-6}$ $3.85 \cdot 10^{-4}$	$4.17 \cdot 10^{-8}$ $2.44 \cdot 10^{-5}$
$c = 10^{-3}$	$7.8735 \cdot 10^{-3}$ $7.1893 \cdot 10^{-2}$	$1.6810 \cdot 10^{-4}$ $5.8948 \cdot 10^{-3}$	$2.8145 \cdot 10^{-6}$ $3.9150 \cdot 10^{-4}$	$4.4722 \cdot 10^{-8}$ $2.4834 \cdot 10^{-5}$
$c = 10^{-6}$	$7.8789 \cdot 10^{-3}$ $7.1906 \cdot 10^{-2}$	$1.6819 \cdot 10^{-4}$ $5.8950 \cdot 10^{-3}$	$2.8158 \cdot 10^{-6}$ $3.9150 \cdot 10^{-4}$	$4.4719 \cdot 10^{-8}$ $2.4847 \cdot 10^{-5}$

Table 11. `Numerical results for` $y(t) = \int_0^t (t-s)y(s)ds + f(t)$

N	*8*	*32*	*128*	*512*
$c = 1$	$4.81 \cdot 10^{-5}$	$9.20 \cdot 10^{-7}$	$1.51 \cdot 10^{-8}$	$2.38 \cdot 10^{-10}$
	$2.15 \cdot 10^{-3}$	$1.56 \cdot 10^{-4}$	$1.01 \cdot 10^{-5}$	$6.36 \cdot 10^{-7}$
$c = 0.5$	$1.95 \cdot 10^{-4}$	$3.71 \cdot 10^{-6}$	$6.07 \cdot 10^{-8}$	$9.59 \cdot 10^{-10}$
	$5.02 \cdot 10^{-3}$	$3.75 \cdot 10^{-4}$	$2.45 \cdot 10^{-5}$	$1.55 \cdot 10^{-6}$
$c = 0.1$	$3.56 \cdot 10^{-4}$	$6.84 \cdot 10^{-6}$	$1.12 \cdot 10^{-7}$	$1.77 \cdot 10^{-9}$
	$6.53 \cdot 10^{-3}$	$4.92 \cdot 10^{-4}$	$3.22 \cdot 10^{-5}$	$2.03 \cdot 10^{-6}$

In addition, as in previous section, we present the results of numerical tests for the test equation (7.1) by step-by-step collocation method with quadratic splines taken from [54]. Given numbers are the approximate values of $\|u\|_C$, calculated as $\max_{1\le n\le N} \max_{0\le k\le 10} |u(t_{n-1} + kh/10)|$.

Table 12. $\lambda = 1, \quad f(t) = \cos t$

N	*4*	*8*	*16*	*32*	*64*
$c = 1$	2.050080	2.050031	2.050028	2.050028	2.050028
$c = 0.7$	2.050162	2.052011	2.597786	$6.02 \cdot 10^{5}$	$1.13 \cdot 10^{19}$
$c = 0.1$	$2.43 \cdot 10^{2}$	$2.64 \cdot 10^{9}$	$5.70 \cdot 10^{24}$	$4.50 \cdot 10^{56}$	$4.61 \cdot 10^{121}$

7.3 Superconvergence of quadratic spline collocation

Here, as in Section 7.2, we used the collocation method (4.2), (4.3). We calculated the error at the collocation points as follows

$$\max_{1\le i\le N} |u_N(t_{i-1} + ch) - y(t_{i-1} + ch)|.$$

The numerical experiments confirm the convergence rate $\mathcal{O}(h^4)$ predicted by theory.

`Numerical results for` $y(t) = \lambda \int\limits_0^t y(s)ds + f(t)$

Table 13. $\lambda = -2,\ f(t) = (3\sin t - \cos t + 3e^t)/2$

N	*4*	*16*	*64*	*256*
$c = 1$	$2.11 \cdot 10^{-4}$	$9.75 \cdot 10^{-7}$	$3.93 \cdot 10^{-9}$	$1.55 \cdot 10^{-11}$
$c = 0.5$	$1.59 \cdot 10^{-4}$	$9.33 \cdot 10^{-7}$	$3.99 \cdot 10^{-9}$	$1.59 \cdot 10^{-11}$
$c = N^{-2}$	$5.72 \cdot 10^{-5}$	$1.71 \cdot 10^{-7}$	$5.32 \cdot 10^{-10}$	$1.95 \cdot 10^{-12}$

Table 14. $\lambda = -1,\ f(t) = \sin t + e^t$

N	*4*	*16*	*64*	*256*
$c = 1$	$1.16 \cdot 10^{-4}$	$6.25 \cdot 10^{-7}$	$2.62 \cdot 10^{-9}$	$1.04 \cdot 10^{-11}$
$c = 0.5$	$8.04 \cdot 10^{-5}$	$4.53 \cdot 10^{-7}$	$1.92 \cdot 10^{-9}$	$7.63 \cdot 10^{-12}$
$c = N^{-2}$	$3.56 \cdot 10^{-5}$	$1.15 \cdot 10^{-7}$	$3.66 \cdot 10^{-10}$	$1.34 \cdot 10^{-12}$

Table 15. $\lambda = 1,\ f(t) = \cos t$

N	*4*	*16*	*64*	*256*
$c = 1$	$2.78 \cdot 10^{-4}$	$1.79 \cdot 10^{-6}$	$7.87 \cdot 10^{-9}$	$3.16 \cdot 10^{-11}$
$c = 0.5$	$7.29 \cdot 10^{-5}$	$3.68 \cdot 10^{-7}$	$1.52 \cdot 10^{-9}$	$5.98 \cdot 10^{-12}$
$c = N^{-2}$	$6.09 \cdot 10^{-5}$	$2.47 \cdot 10^{-7}$	$8.57 \cdot 10^{-10}$	$3.19 \cdot 10^{-12}$

Table 16. $\lambda = 2,\ f(t) = (-\sin t + 3\cos t - e^t)/2$

N	*4*	*16*	*64*	*256*
$c = 1$	$1.76 \cdot 10^{-3}$	$1.09 \cdot 10^{-5}$	$5.16 \cdot 10^{-8}$	$2.11 \cdot 10^{-10}$
$c = 0.5$	$1.19 \cdot 10^{-4}$	$6.87 \cdot 10^{-7}$	$3.11 \cdot 10^{-9}$	$1.26 \cdot 10^{-11}$
$c = N^{-2}$	$1.69 \cdot 10^{-4}$	$8.03 \cdot 10^{-7}$	$2.95 \cdot 10^{-9}$	$1.12 \cdot 10^{-11}$

Table 17. `Numerical results for` $y(t) = \int\limits_0^t (t-s)y(s)ds + f(t)$

N	*4*	*16*	*64*	*256*
$c = 1$	$2.39 \cdot 10^{-5}$	$1.53 \cdot 10^{-7}$	$1.03 \cdot 10^{-10}$	$2.67 \cdot 10^{-12}$
$c = 0.5$	$8.39 \cdot 10^{-7}$	$7.67 \cdot 10^{-9}$	$3.59 \cdot 10^{-11}$	$1.47 \cdot 10^{-13}$
$c = N^{-2}$	$1.99 \cdot 10^{-6}$	$1.17 \cdot 10^{-8}$	$4.46 \cdot 10^{-11}$	$1.71 \cdot 10^{-13}$

7.4 Subdomain method

In Table 18 there are presented the numerical results for the test equation (7.1) determined by the step-by-step subdomain method (6.2), (6.3). In Tables 19 and 20 we present results obtained by the nonlocal subdomain method (6.2), (6.7). The rate $\mathcal{O}(h^3)$ in the last two tables supports the theory.

Numerical results for $y(t) = \lambda \int_0^t y(s)ds + f(t)$

Table 18. $f(t) = ((1-\lambda)\sin t + (1+\lambda)\cos t + (1-\lambda)e^t)/2$

N	*4*	*16*	*64*	*256*
$\lambda = -2$	$3.28 \cdot 10^{-3}$	98.02	$1.10 \cdot 10^{27}$	$2.79 \cdot 10^{134}$
$\lambda = -1$	$2.59 \cdot 10^{-3}$	76.56	$8.54 \cdot 10^{26}$	$2.18 \cdot 10^{134}$
$\lambda = 1$	$1.46 \cdot 10^{-3}$	45.99	$5.17 \cdot 10^{26}$	$1.32 \cdot 10^{134}$
$\lambda = 2$	$1.02 \cdot 10^{-3}$	35.36	$4.01 \cdot 10^{26}$	$1.03 \cdot 10^{134}$

Table 19. $f(t) = ((1-\lambda)\sin t + (1+\lambda)\cos t + (1-\lambda)e^t)/2$

N	*4*	*16*	*64*	*256*
$\lambda = -2$	$3.27 \cdot 10^{-3}$	$6.27 \cdot 10^{-5}$	$1.02 \cdot 10^{-6}$	$1.61 \cdot 10^{-8}$
$\lambda = -1$	$3.11 \cdot 10^{-3}$	$6.18 \cdot 10^{-5}$	$1.02 \cdot 10^{-6}$	$1.61 \cdot 10^{-8}$
$\lambda = 1$	$2.77 \cdot 10^{-3}$	$5.99 \cdot 10^{-5}$	$1.01 \cdot 10^{-6}$	$1.61 \cdot 10^{-8}$
$\lambda = 2$	$2.60 \cdot 10^{-3}$	$5.91 \cdot 10^{-5}$	$1.01 \cdot 10^{-6}$	$1.60 \cdot 10^{-8}$

Table 20. Numerical results for $y(t) = \int_0^t (t-s)y(s)ds + f(t)$

N	*4*	*16*	*64*	*256*
	$5.70 \cdot 10^{-3}$	$9.02 \cdot 10^{-5}$	$1.41 \cdot 10^{-6}$	$2.20 \cdot 10^{-8}$

Appendix A

A.1 B-splines and Schoenberg – Whitney theorem

To prove the existence and uniqueness of solution of the spline interpolation problem, we use a basic theorem in spline theory: the Schoenberg – Whitney theorem.

Let d be a nonnegative integer and let

$$\boldsymbol{s}: \; s_1 \leq \ldots \leq s_{n+d+1}$$

be the vector of knots such that $s_i < s_{i+d+1}$ for $i = 1, \ldots, n$.

Definition A.1. The i-th *B-spline* of degree d with knots $\boldsymbol{s}$ is defined by

$$B_{i,d,\boldsymbol{s}}(x) = \frac{x - s_i}{s_{i+d} - s_i} B_{i,d-1,\boldsymbol{s}}(x) + \frac{s_{i+d+1} - x}{s_{i+d+1} - s_{i+1}} B_{i+1,d-1,\boldsymbol{s}}(x), \qquad \text{(A.1)}$$

for all real numbers x, with

$$B_{i,0,\boldsymbol{s}}(x) = \begin{cases} 1, & \text{if } \; s_i \leq x < s_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, the convention that "0/0 = 0" is assumed. There is another equivalent definition of B-spline, using truncated power functions. Denote

$$(s - x)_+^d = \begin{cases} (s - x)^d, & \text{if } \; s - x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Considering the function $\varphi_d(s, x) = (-1)^{d+1}(d+1)(s-x)_+^d$, we can establish that

$$B_{i,d,\boldsymbol{s}}(x) = \varphi_d[x; s_i, \ldots, s_{i+d+1}], \quad i = 1, \ldots, n,$$

where $\varphi_d[x; s_i, \ldots, s_{i+d+1}]$ is $(d+1)$-st order divided difference for the knots $s_i, \ldots, s_{i+d+1}$ to the function $\varphi_d(s, x)$ with respect to the variable x.

The functions $B_{i,d,\boldsymbol{s}}(x), \quad i = 1, \ldots, n$, are linearly independent (see [42, 61]). Let us denote their span by $S_{d,\boldsymbol{s}}$. Thus, any spline S from the space $S_{d,\boldsymbol{s}}$ can be written as

$$S(x) = \sum_{i=1}^{n} c_i B_{i,d,\boldsymbol{s}}(x).$$

Let us denote, for $r \geq 1$, the right derivative of order r by

$$(D_+^r f)(x) = \lim_{h \to 0+} \frac{(D^{r-1} f)(x+h) - (D^{r-1} f)(x)}{h},$$

accepting $D_+^0 f = f$.

Theorem A.2 ([42, 61])**.** *The r-th right derivative of the i-th B-spline of degree d on $\boldsymbol{s}$ is given by*

$$(D_+^r B_{i,d,\boldsymbol{s}})(x) = d \left(\frac{(D_+^{r-1} B_{i,d-1,\boldsymbol{s}})(x)}{s_{i+d} - s_i} - \frac{(D_+^{r-1} B_{i+1,d-1,\boldsymbol{s}})(x)}{s_{i+d+1} - s_{i+1}} \right), \quad \text{(A.2)}$$

for $d \geq 1$ and for any real number x.

Suppose a sequence of interpolation points $\boldsymbol{x} : x_1 \leq \ldots \leq x_n$ with $x_i < x_{i+d+1}$ for $i = 1, \ldots, n-d-1$ is given. Consider the following interpolation problem: find $S \in S_{d,\boldsymbol{s}}$ such that, for given f_i,

$$\left(D_+^{\lambda_{\boldsymbol{x}}(i)} S \right) (x_i) = f_i, \quad i = 1, \ldots, n, \quad \text{(A.3)}$$

where $\lambda_{\boldsymbol{x}}(i) = \max\{j |\ x_{i-j} = x_i\}$ is the left multiplicity of the node x_i.

Next theorem is one of the most fundamental results in the study of spline functions.

Theorem A.3 (Schoenberg – Whitney theorem)**.** *The interpolation problem* (A.3) *is uniquely solvable for any f_i if and only if*

$$x_i \in (s_i, s_{i+d+1}) \bigcup \{s_i, \quad if \quad D_+^{\lambda_{\boldsymbol{x}}(i)} B_{i,d,\boldsymbol{s}}(s_i) \neq 0\}, \quad i = 1, \ldots, n. \quad \text{(A.4)}$$

See [42, 61] for the proof.

A.2 Quadratic spline histopolation

This part of Appendix is written to present a probably known result about convergence rate for quadratic spline histopolation. Unfortunately, we cannot give the precise reference where it was proved.

Let it be given a continuous function f on $[0,T]$. Consider a mesh

$$\Delta_N : \ 0 = t_0 < t_1 < \ldots < t_N = T.$$

Denote $h_i = t_i - t_{i-1}$. Here we will study the following histopolation problem: find a quadratic spline $u \in S_2(\Delta_N)$ determined by

$$\begin{aligned} &u(0) = f(0), \\ &\int_{t_{i-1}}^{t_i} u(t)dt = \int_{t_{i-1}}^{t_i} f(t)dt, \quad i = 1, \ldots, N, \\ &u''(t_{N-1} - 0) = u''(t_{N-1} + 0). \end{aligned} \tag{A.5}$$

As we have already shown, this problem has the unique solution (see Sections 6.1.2).

Using the notation $u_i = u(t_i)$, $t = t_{i-1} + \tau h_i$, $\tau \in [0,1]$, and

$$z_i = \frac{1}{h_i} \int_{t_{i-1}}^{t_i} f(t)dt, \quad i = 1, \ldots, N,$$

we have the representation of u for $t \in [t_{i-1}, t_i]$

$$u(t) = 6z_i\tau(1-\tau) + u_{i-1}(1 - 4\tau + 3\tau^2) + u_i\tau(3\tau - 2).$$

The continuity of the derivative u' in the knots gives

$$\mu_i u_{i-1} + 2u_i + \lambda_i u_{i+1} = w_i, \quad i = 1, \ldots, N-1, \tag{A.6}$$

where

$$\mu_i = 1 - \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad w_i = 3(\mu_i z_i + \lambda_i z_{i+1}).$$

The initial condition $u(0) = f(0)$ adds the equation $u_0 = f(0)$, and the not-a-knot requirement at t_{N-1} could be written in the form

$$h_N^2 u_{N-2} + (h_N^2 - h_{N-1}^2)u_{N-1} - h_{N-1}^2 u_N = 2(h_N^2 z_{N-1} - h_{N-1}^2 z_N). \tag{A.7}$$

Then, eliminating u_N in (A.6) with the help of (A.7), we write the equalities to determine u_i, $i = 0, \dots, N-1$, as follows:

$$
\begin{aligned}
&u_0 = f(0),\\
&\mu_i u_{i-1} + 2u_i + \lambda_i u_{i+1} = w_i, \quad i = 1, \dots, N-2,\\
&\frac{h_N}{h_{N-1}} u_{N-2} + \left(1 + \frac{h_N}{h_{N-1}}\right) u_{N-1}\\
&\qquad = \left(3 + 2\frac{h_N}{h_{N-1}}\right) \mu_{N-1} z_{N-1} + \lambda_{N-1} z_N.
\end{aligned}
\tag{A.8}
$$

First, investigate $q_i = u_i - f_i$, $i = 1, \dots, N$, with $f_i = f(t_i)$. Using (A.8), we have

$$
\begin{aligned}
&q_0 = 0,\\
&\mu_i q_{i-1} + 2q_i + \lambda_i q_{i-1} = d_i, \quad i = 1, \dots, N-2,\\
&\frac{h_N}{h_{N-1}} q_{N-2} + \left(1 + \frac{h_N}{h_{N-1}}\right) q_{N-1} = d_{N-1},
\end{aligned}
\tag{A.9}
$$

where

$$
\begin{aligned}
d_i &= 3(\mu_i z_i + \lambda_i z_{i+1}) - (\mu_i f_{i-1} + 2f_i + \lambda_i f_{i-1}), \quad i = 1, \dots, N-2,\\
d_{N-1} &= \left(3 + 2\frac{h_N}{h_{N-1}}\right) \mu_{N-1} z_{N-1} + \lambda_{N-1} z_N\\
&\quad - \left(\frac{h_N}{h_{N-1}} f_{N-2} + \left(1 + \frac{h_N}{h_{N-1}}\right) f_{N-1}\right).
\end{aligned}
$$

It is clear that, in the system (A.9), the difference of domination in rows is 1. Hence,

$$
\max_{0 \le i \le N-1} |q_i| \le \max_{0 \le i \le N-1} |d_i|.
$$

Let $f'' \in \mathtt{Lip}\ 1$ and $h_N / h_{N-1} \le \text{const}$. Using a Taylor expansion of z_j and f_j in t_i for d_i, $i = 0, \dots, N-1$, we get

$$
u_i - f_i = \mathcal{O}(h^3), \quad i = 1, \dots, N-1.
$$

In addition, using (A.7), we obtain

$$
\begin{aligned}
u_N - f_N &= \frac{h_N^2}{h_{N-1}^2}(u_{N-2} - f_{N-2}) + \left(\frac{h_N^2}{h_{N-1}^2} - 1\right)(u_{N-1} - f_{N-1})\\
&\quad - 2\left(\frac{h_N^2}{h_{N-1}^2} z_{N-1} - z_N\right) + \frac{h_N^2}{h_{N-1}^2} f_{N-2} + \left(\frac{h_N^2}{h_{N-1}^2} - 1\right) f_{N-1} - f_N.
\end{aligned}
$$

Expanding z_j and f_j in the last row by a Taylor formula, we also establish

$$u_N - f_N = \mathcal{O}(h^3).$$

Secondly, consider for $t \in [t_{i-1}, t_i]$

$$u(t) - f(t) = 6z_i\tau(1-\tau) - f(t) + f_{i-1}(1 - 4\tau + 3\tau^2) + f_i\tau(3\tau - 2)$$
$$+ (u_{i-1} - f_{i-1})(1 - 4\tau + 3\tau^2) + (u_i - f_i)\tau(3\tau - 2).$$

Again, using in the first row of the last expansion a Taylor expansion of z_i, f_{i-1} and f_i in t, we get

$$u(t) - f(t) = \mathcal{O}(h^3).$$

We have proved the following

Theorem A.4. *Suppose* $h_N/h_{N-1} \leq \text{const}$ *and the funcion* f *is such that* $f'' \in \mathtt{Lip}\ 1$. *Then*

$$\|u - f\|_C = \mathcal{O}(h^3),$$

where u *is the histopolating quadratic spline determined by* (A.5).

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KOKKUVÕTE

Volterra integraalvõrrandite lahendamine ruut- ja kuupsplainidega kollokatsioonimeetodil

Integraalvõrrandite teooria uurimine on tunduvalt intensiivistunud viimasel paarikümnel aastal. Nende võrrandite rakendusi võib leida erinevates eluvaldkondades: meditsiinis, bioloogias, majanduses. Praktikas esinevad integraalvõrrandid lahendatakse reeglina ligikaudselt ehk kasutades erinevaid diskretisatsioonimeetodeid.

Teatavasti on kõige levinumad praktikas Volterra II liiki integraalvõrrandite lahendamiseks Runge-Kutta meetodid ja splainidega kollokatsioonimeetod, mida saab realiseerida sammhaaval. On teada, et kollokatsioonimeetod splainidega ruumist C^{-1} on alati stabiilne ja annab head koonduvusteoreemid. Lineaarsete splainide korral on sammumeetod stabiilne, kui kollokatsiooniparameeter $c \in (0, 1]$ (mis iseloomustab kollokatsioonipunktide asendit splaini sõlmede vahel) on lõigust $[1/2, 1]$. Ruutsplainide korral on meetod stabiilne ainult siis, kui $c = 1$. Siledate kuup- ja kõrgemat järku splainide korral on kollokatsioonimeetod alati mittestabiilne, olenemata sellest, kuidas valitakse kollokatsiooniparameeter. Sellest kerkib loomulik küsimus, kuidas lahendada kollokatsioonimeetodil Volterra integraalvõrrandit, kui soovitakse kasutada ruut- ja kuupsplaine.

Käesolevas doktoritöös on käsitletud mittelokaalseid meetodeid kuup- ja ruutsplainidega, mille korral on välja selgitatud kollokatsiooniparameetri stabiilsuspiirkond. Samuti on tõestatud rida koonduvusteoreeme.

Töö esimeses peatükis antakse lühike ülevaade integraalvõrrandite teooria ajaloost, tuuakse näiteid Volterra integraalvõrrandite rakendamisest ja tutvustatakse teiste autorite töid, mis käsitlevad splainidega kollokatsioonimeetodit.

Töös esitatud kollokatsioonimeetodeid saab kõiki vaadelda erinevate projektsioonimeetoditena. Seega saab meetodite koonduvuse tõestamiseks kasutada üldisi projektsioonimeetodite koonduvusteoreeme, mida vaadeldakse teises peatükis. Samuti kirjeldatakse seal sammhaaval realiseeritavat splain-kollokatsioonimeetodit.

Kolmandas peatükis on vaadeldud kuupsplain-kollokatsioonimeetodit, kus üks algtingimustest asendatakse "puuduva sõlme" rajatingimusega integreerimislõigu teises otsas. Saadud lähisülesanne nõuab võrrandisüsteemi lahendamist, seejuures ei suurene tehete arvu järk võrreldes sammumeetodiga. Sel juhul saadakse meetodi stabiilsus, kui kollokatsiooniparameeter kuulub lõiku [1/2,1]. Põhitulemused on seotud vastavate projektorite uurimisega. Näidatakse lähendusoperaatorite tugevat koondumist ühtlase võrgu korral. Selleks kasutatakse kuupsplainide esitust B-splainide kaudu ja mitmeid erinevaid tehnilisi vahendeid, nagu näiteks Banachi püsipunkti printsiip, diferentsvõrrandite teooria ja funktsioonide käitumise uurimine. Juhul $c = 1/2$ on tõestatud, et kuupsplain-projektorid ei ole ühtlaselt tõkestatud, kuid nende normid on järguga $\mathcal{O}(N)$. Siis on lisaeeldustel võimalik saada koonduv meetod sileda lahendi korral.

Neljandas peatükis on kirjeldatud ruutsplain-kollokatsioonimeetodit, kus üks algtingimustest asendatakse "puuduva sõlme" rajatingimusega integreerimislõigu teises otsas, nagu eelmises peatükis. Tõestatakse meetodi koonduvus ruumides C ja C^1 suvalise kollokatsiooniparameetri $c \in (0, 1)$ ja mitteühtlase võrgu korral. Koondumine saadakse üldisest koonduvusteoreemist operaatorvõrrandite jaoks. Kui meetodis esineb kollokatsiooniparameeter $c = 1$, siis on näidatud, et ruutsplain-projektorid ei ole ühtlaselt tõkestatud ühtlase võrgu korral, kuid nende normid on $\mathcal{O}(N)$ järku. Sellisel juhul tõestatakse lisaeeldustel vastavate projektorite regulaarne koondumine ja saadakse koonduv meetod, tuginedes projektsioonimeetodite koonduvusteoreemile operaatorvõrrandite korral, mis kasutab operaatorite regulaarset koondumist.

Viiendas peatükis on vaadeldud ruutsplain-kollokatsioonimeetodi tähelepanuväärset omadust lineaarse Volterra integraalvõrrandi korral. Nimelt on tõestatud superkoonduvus kollokatsioonisõlmedes üksikute kollokatsiooniparameetri väärtuste korral. Võrreldes ruutsplain-kollokatsioonimeetodit koonduvusjärguga $\mathcal{O}(h^3)$ tervel lõigul, saadakse järk $\mathcal{O}(h^4)$ küllalt sileda lahendi ja ühtlase võrgu korral.

Kuues peatükk käsitleb ruutsplainidega osapiirkondade meetodit. Osapiirkondade meetodit on loomulik rakendada näiteks siis, kui võrrandi vabaliikmest on teada keskmised väärtused osalõikudel. Kollokatsioonimeetodit võib seejuures käsitleda kui osapiirkondade meetodi ligikaudset realiseeringut näiteks ühesõlmelise kvadratuurvalemi abil. Seetõttu on osapiirkondade meetod vaadeldav vahepealse baasmeetodina ka kollokatsioonimeetodite teoorias ning selle uurimine on väga loomulik ja vajalik. Töös tõestatakse, et osapiirkondade sammumeetod on ebastabiilne, mis on kooskõlas sammhaaval realiseeritava kollokatsioonimeetodi mittestabiilsusega. Vaadeldakse ka modifitseeritud osapiirkondade meetodit (rajatingimuse üleviimisega) ning on tõestatud selle meetodi stabiilsus ja näidatud lähismeetodite klassikalise koonduvusteoreemi rakendatavust avaratel eeldustel.

Töö seitsmendas peatükis on esitatud rida arvulisi tulemusi üldtunnustatud testvõrrandite korral, valides erinevaid testvõrrandi parameetri ning võrgu sõlmede ja kollokatsiooniparameetri väärtusi. Saadud tulemused on täielikus kooskõlas töö teoreetiliste tulemustega.

Lisas on ära toodud Schoenberg – Whitney teoreem, mis annab splainidega interpolatsiooniülesannete lahendi olemasolu ja ühesuse tarvilikud ja piisavad tingimused. Samuti on leitud ruutsplainidega histopolatsiooniprotsessi koonduvuskiirus.

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Peamine uurimisvalkond on Volterra integraalvõrandite lahendamine splain-kollokatsiooni- ja osapiirkondade meetodil. Tulemused on publitseeritud neljas artiklis. Esinenud rakendusmatemaatika instituudi talvekoolides Käärikul (2001, 2002) ja matemaatilise analüüsi seminaris Riias Läti Ülikoolis (2001) ning konverentsidel "The Seventh International Conference Mathematical Modelling and Analysis" Käärikul (2002), "The Second International Conference Approximation Methods and Orthogonal Expansions" Käärikul (2003), "Algebra ja Analüüsi Meetodid IV" Tartus (2003), "The Ninth International Conference Mathematical Modelling and Analysis" Jurmalas (2004), "The Tenth International Conference Mathematical Modelling and Analysis" Trakais (2005), "The Fifth International Conference Algorithms for Approximation" Chesteris (Inglismaa) (2005) ja "The Second Finnish-Estonian Mathematical Colloquium" Tamperes (2006).

LIST OF PUBLICATIONS

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