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# SEQUENCE SPACES DEFINED BY MODULUS FUNCTIONS AND SUPERPOSITION OPERATORS

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# Contents

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# List of original publications

1. E. Kolk and A. Mölder, *Inclusion theorems for some sets of sequence spaces defined by $\varphi$-functions*, Math. Slovaca **54** (2004), no. 3, 267–279.

2. A. Mölder, *The topologization of sequence spaces defined by a matrix of moduli*, Proc. Estonian Acad. Sci. Phys. Math. **53** (2004), no. 4, 218–225.

3. E. Kolk and A. Mölder, *The continuity of superposition operators on some sequence spaces defined by moduli*, Czechoslovak Math. J. (submitted).

4. A. Mölder, *Boundedness of superposition operators on some sequence spaces defined by moduli*, Demonstratio Math. (accepted).

5. A. Mölder, *Boundedness of superposition operators on sequence space* $(w_0)_p(\Phi)$, Acta Comment. Univ. Tartuensis Math. (submitted).

# Introduction

The theory of sequence spaces deals with different spaces of sequences including sequence spaces defined by Orlicz functions and by moduli. For an Orlicz function $\varphi$ the Orlicz sequence space is determined by

$$\ell^{\exists}(\varphi) = \left\{ x = (x_k)\colon \sum_k \varphi\left(\frac{|x_k|}{\varrho}\right) < \infty \text{ for some } \varrho > 0 \right\} =$$
$$= \left\{ x = (x_k)\colon \left(\varphi\left(\frac{|x_k|}{\varrho}\right)\right) \in \ell \text{ for some } \varrho > 0 \right\}.$$

For a certain solid sequence space $\lambda$ and for a modulus $\varphi$ Ruckle [48] and Maddox [30] considered a new sequence space

$$\lambda(\varphi) = \{x = (x_k)\colon (\varphi(|x_k|)) \in \lambda\}.$$

The extension of this definition was given by Kolk [21]. For a sequence space $\lambda$ and a sequence of moduli $\Phi = (\varphi_k)$ he defined

$$\lambda(\Phi) = \{x = (x_k)\colon (\varphi_k(|x_k|)) \in \lambda\}.$$

In the special case from the definition of $\lambda(\Phi)$ we get the sequence spaces of Maddox type (see, for example, [16] and [28]), which generalize the corresponding classical sequence spaces.

To investigate all such spaces from a more general point of view, we introduce the notion of $\varphi$-function and generalize the results of [16, 21] to the case of $\varphi$-functions.

An essential problem in the theory of sequence spaces is the topologization of various vector spaces of sequences. For example, if $\Phi = (\varphi_k)$ is a sequence of moduli and $\lambda$ is a normed (or an F-seminormed) solid sequence space, then the linear space $\lambda(\Phi)$ can be topologized by an

F-seminorm (see [22, 23]) or by a paranorm (see [50]). We characterize the F-seminormability of the sequence space

$$\Lambda(\mathcal{F}) = \{x = (x_k)\colon (f_{ki}(|x_k|)) \in \Lambda\},$$

where $\mathcal{F} = (f_{ki})$ is a matrix of moduli and $\Lambda$ is a solid space of double sequences.

The topologization of the spaces $\lambda(\Phi)$ allows us to study different topological properties, as continuity, boundedness and so on, of operators on there sequence spaces. We are interested of the superposition operators, which form a subclass of all (linear and nonlinear) operators.

Superposition operators on sequence spaces are not studied so intensiv as on spaces of functions (see, for example, [1]). A *superposition operator* (sometimes called also *outer superposition operator*, *composition operator*, *substitution operator*, or *Nemytskij operator*) $P_f\colon \lambda \to \mu$ is defined by

$$P_f(x) = (f(k, x_k)) \in \mu \quad (x = (x_k) \in \lambda),$$

where $\lambda$ and $\mu$ are two sequence spaces and $f\colon \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ is a function with $f(k, 0) = 0$ $(k \in \mathbb{N})$. In general the superposition operator $P_f$ is nonlinear. Some properties of this operator can be found in [1].

Characterization of $P_f$ on Orlicz sequence spaces was given by Robert [47] and Šragin [51]. Superpositsion operators on sequence spaces $\ell_\infty$, $c_0$ and $\ell_p$ for $1 \le p < \infty$ have been completely studied by Dedagich and Zabreĭko [10] (see also [8, 44]). Płuciennik [45, 46] considered the superposition operators on $w_0$. Some authors [9, 44, 49, 52, 53] have been studied continuity and boundedness of superposition operators in various sequence spaces. Our purpose is give necessary and sufficent conditions for the continuity, local boundedness and boundedness of superposition operators on sequence spaces defined by a sequence of moduli. Main theorems generalize the results of Dedagich and Zabreĭko [10], Płuciennik [45, 46] and Kolk [21, 22].

The thesis is organized as follows.

In Chapter 1 we give necessary and sufficent conditions for some inclusions of type $\lambda \subset \mu(\Phi)$ and $\lambda(\Phi) \subset \mu$, where $\Phi = (\varphi_k)$ is a sequence of $\varphi$-functions and $\lambda, \mu \in \{\ell_\infty, c_0, \ell_p\}$ $(0 < p < \infty)$. The inclusions $\lambda \subset \mu^\forall(\Phi)$ and $\lambda^\exists(\Phi) \subset \mu$ are also considered. We apply our theorems to the Maddox sequence spaces.

In Chapter 2 we study the topologization of sequence space $\Lambda(\mathcal{F})$ under some restrictions to the matrix of moduli $\mathcal{F} = (f_{ki})$ or on the space $(\Lambda, g)$. Our results give known theorems about the topologization of $\lambda(\Phi)$. As the concrete examples we consider the spaces of strongly summable sequences.

In Chapter 3 we characterize the continuity, the local boundedness and the boundedness of superpostion operators on sequence spaces defined by a sequence of moduli. As an application we consider superposition operators on multiplier spaces of Maddox type.

Chapters 1 and 2 are based on [25] and [35], respectively. Chapter 3 develop results from [26, 36, 37].

# Chapter 1

# Sequence spaces defined by moduli and $\varphi$-functions

Main results of this chapter (see Section 1.3) are published in [25].

## 1.1 Sequence spaces, moduli and $\varphi$-functions

We use the symbol $\mathbb{N}$ to denote the set of all positive integers, and $\mathbb{K}$ to denote the set of all complex numbers $\mathbb{C}$ or the set of all real numbers $\mathbb{R}$. We write $\inf_k$, $\sup_k$, $\sum_k$ and $\lim_k$ instead of $\inf_{k\in\mathbb{N}}$, $\sup_{k\in\mathbb{N}}$, $\sum_{k\in\mathbb{N}}$ and $\lim_{k\to\infty}$, respectively.

Let $\omega$ be the vector space of all number sequences, i.e.,

$$\omega = \{x = (x_k) = (x_k)_{k\in\mathbb{N}} \colon x_k \in \mathbb{K} \quad (k \in \mathbb{N})\},$$

where vector space operations are defined coordinatewise, i.e.,

$$x + y = (x_k + y_k), \quad \alpha x = (\alpha x_k) \qquad (x = (x_k),\ y = (y_k) \in \omega,\ \alpha \in \mathbb{K}).$$

By the term *sequence space* we shall mean any linear subspace of $\omega$.

The sequence space $\lambda$ is called *solid* if $(y_k) \in \lambda$ whenever $(x_k) \in \lambda$ and $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$. Well-known solid sequence spaces are the space $\ell_\infty$ of all bounded sequences, the space $c_0$ of all convergent to zero sequences, the spaces

$$\ell_p = \left\{ x = (x_k) \in \omega \colon \sum_k |x_k|^p < \infty \right\}$$

and

$$(w_0)_p = \left\{ x = (x_k) \in \omega \colon \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\}$$

for $0 \leq p < \infty$. Moreover (see [31], p. 523),

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \Longleftrightarrow \lim_{i \to \infty} \frac{1}{2^i} \sum_{k=2^i}^{2^{i+1}-1} |x_k|^p = 0. \tag{1.1.1}$$

The sequences from $(w_0)_p$ are called strongly convergent (with index $p$) to zero. We write $\ell$ and $w_0$ instead of $\ell_1$ and $(w_0)_1$, respectively.

For example, the space $c$ of all convergent sequences is non-solid.

The idea of a modulus function was structured in 1953 by Nakano [38]. Following Ruckle [48] we formulate

**Definition 1.1.1.** A function $\varphi\colon [0, \infty) \to [0, \infty)$ is called a *modulus function* (or simply a *modulus*), if

(i) $\varphi(t) = 0 \quad \Leftrightarrow \quad t = 0$,

(ii) $\varphi(t + u) \leq \varphi(t) + \varphi(u) \qquad (t,\ u \geq 0)$,

(iii) $\varphi$ is nondecreasing,

(iv) $\varphi$ is continuous from the right at 0.

It follows from (i) – (iv) that $\varphi$ is continuous everywhere on $[0, \infty)$.

**Lemma 1.1.2** ([22], Lemma 1; [33], p. 221)**.** *Any modulus $\varphi$ satisfies the conditions*

$$|\varphi(t) - \varphi(u)| \leq \varphi(|t - u|) \qquad (t,\ u \geq 0), \tag{1.1.2}$$

$$\frac{1}{n}\varphi(t) \leq \varphi\left(\frac{t}{n}\right) \qquad (n \in \mathbb{N}). \tag{1.1.3}$$

*Proof.* If $t \geq u$, then $t - u \geq 0$ and by (ii) we have

$$\varphi(t) = \varphi((t - u) + u) \leq \varphi(|t - u|) + \varphi(u)$$

which gives

$$\varphi(t) - \varphi(u) \leq \varphi(|t - u|).$$

Further, by (iii), $\varphi(t) \geq \varphi(u)$ and so $\varphi(u) \leq \varphi(t) \leq \varphi(t) + \varphi(|t-u|)$, i.e.,

$$-\varphi(|t-u|) \leq \varphi(t) - \varphi(u).$$

Consequently, (1.1.2) holds for $t \geq u$.

If $t < u$, then $u - t > 0$ and by above-proved we get

$$|\varphi(u) - \varphi(t)| \leq \varphi(|u-t|)$$

which is equivalent to (1.1.2).

Further, we have $\varphi(nt) \leq n\varphi(t)$ for all $n \in \mathbb{N}$ by condition (ii). So

$$\varphi(t) = \varphi\left(nt\frac{1}{n}\right) \leq n\,\varphi\left(\frac{t}{n}\right)$$

which clearly gives (1.1.3). □

A modulus may be bounded or unbounded. For example, $\varphi(t) = t^p$ is an unbounded modulus for $0 < p \leq 1$ and $\varphi(t) = t/(1+t)$ is a bounded modulus.

It is interesting to remark that the moduli are the same as the moduli of continuity: a function $\varphi\colon [0,\infty) \to [0,\infty)$ is a modulus of continuity of a continuous function if and only if the conditions (i) – (iv) are satisfied (see [11], p. 866).

If in the definition of a modulus the condition (iii) is replaced by the condition of convexity

(v) $\varphi(\alpha t + (1-\alpha)u) \leq \alpha\varphi(t) + (1-\alpha)\varphi(u) \qquad (t, u \geq 0,\ 0 \leq \alpha \leq 1),$

then $\varphi$ is called an *Orlicz function.*

Provided a modulus $\varphi$, Ruckle [48] defined and studied the space

$$\ell(\varphi) = \left\{x = (x_k)\colon \sum_k \varphi(|x_k|) < \infty\right\} = \{x = (x_k) : (\varphi(|x_k|)) \in \ell\}.$$

For an Orlicz function $\varphi$, the *Orlicz sequence space* is determined by (see, [27], p. 137)

$$\ell^{\exists}(\varphi) = \left\{x = (x_k)\colon \exists\ \varrho > 0 \quad \sum_k \varphi\left(\frac{|x_k|}{\varrho}\right) < \infty\right\}.$$

If $\Phi = (\varphi_k)$ is a sequence of Orlicz functions, then the space

$$\ell^{\exists}(\Phi) = \left\{ x = (x_k) \colon \exists\ \varrho > 0 \quad \sum_k \varphi_k \left( \frac{|x_k|}{\varrho} \right) < \infty \right\}$$

is called a *modular space* or *Musielak–Orlicz sequence space* (see [34], p. 173). Together with $\ell^{\exists}(\varphi)$ and $\ell^{\exists}(\Phi)$ there are examined also the sets

$$\ell^{\forall}(\varphi) = \left\{ x = (x_k) \colon \sum_k \varphi \left( \frac{|x_k|}{\varrho} \right) < \infty \quad (\forall\ \varrho > 0) \right\},$$

$$\ell^{\forall}(\Phi) = \left\{ x = (x_k) \colon \sum_k \varphi_k \left( \frac{|x_k|}{\varrho} \right) < \infty \quad (\forall\ \varrho > 0) \right\}.$$

In the mathematical literature there exist various modifications of these definitions, where $\ell$ is replaced by another solid sequence space (see, for example, [5], [6], [12]–[15], [19]–[25], [30], [41]–[43], [50]). To investigate all such spaces from a more general point of view, we use the following notation.

**Definition 1.1.3.** A function $\varphi\colon [0, \infty) \to [0, \infty)$ is called a *$\varphi$-function* if the conditions (i) and (iii) are satisfied.

It should be noted that by our definition, a $\varphi$-function is not necessarily continuous and unbounded (cf. [34], p. 4).

## 1.2 Sets of sequences defined by $\varphi$-functions

Let $\Phi = (\varphi_k)$ be a sequence of $\varphi$-functions and let $\Phi(x) = (\varphi_k(|x_k|))$. For a sequence space $\lambda$ we define the sets

$$\lambda^{\varrho}(\Phi) = \{x = (x_k) \in \omega : \Phi(x/\varrho) \in \lambda\} \qquad (\varrho > 0),$$

$$\lambda^{\exists}(\Phi) = \{x = (x_k) \in \omega : (\exists \varrho > 0) \quad (\Phi(x/\varrho) \in \lambda)\} = \bigcup_{\varrho > 0} \lambda^{\varrho}(\Phi),$$

$$\lambda^{\forall}(\Phi) = \{x = (x_k) \in \omega : (\forall \varrho > 0) \quad (\Phi(x/\varrho) \in \lambda)\} = \bigcap_{\varrho > 0} \lambda^{\varrho}(\Phi).$$

We write $\lambda(\Phi)$ instead of $\lambda^1(\Phi)$. If $\varphi$ is a $\varphi$-function and $\varphi_k = \varphi \quad (k \in \mathbb{N})$, we write $\lambda^\varrho(\varphi)$, $\lambda^\exists(\varphi)$ and $\lambda^\forall(\varphi)$ instead of $\lambda^\varrho(\Phi)$, $\lambda^\exists(\Phi)$ and $\lambda^\forall(\Phi)$, respectively.

By definitions of $\lambda^\varrho(\Phi)$, $\lambda^\exists(\Phi)$ and $\lambda^\forall(\Phi)$ it is immediately clear that

$$\lambda^\forall(\Phi) \subset \lambda^\varrho(\Phi) \subset \lambda^\exists(\Phi). \tag{1.2.1}$$

The following examples show that these three sets are different in general.

**Example 1.2.1.** Let $\lambda = \ell_\infty$. We define the sequence of $\varphi$-functions $\Phi = (\varphi_k)$ by $\varphi_k(t) = t^k \quad (k \in \mathbb{N})$ and consider the sequence $e = (\epsilon_k) = (1, 1, 1, \ldots)$. Since

$$\sup_k \varphi_k(|\epsilon_k|) = \sup_k |\epsilon_k|^k = \sup_k 1 = 1 < \infty,$$

then $e \in \ell_\infty(\Phi)$. But for $\varrho \in (0, 1)$ we have

$$\sup_k \varphi_k \left( \frac{|\epsilon_k|}{\varrho} \right) = \sup_k \varphi_k \left( \frac{1}{\varrho} \right) = \sup_k \left( \frac{1}{\varrho} \right)^k = \infty,$$

i.e., $e \notin \ell_\infty^\forall(\Phi)$. Therefore, $\ell_\infty^\forall(\Phi) \subsetneqq \ell_\infty(\Phi)$.

**Example 1.2.2.** Let $\lambda = \ell_\infty$. For fixed $\varrho > 1$ we define the sequence of $\varphi$-functions $\Phi = (\varphi_k)$ by

$$\varphi_k(t) = \begin{cases} t & \text{if } t \in [0, 1), \\ k & \text{if } t \geq 1 \end{cases}$$

and consider the sequence $e = (\epsilon_k) = (1, 1, 1, \ldots)$. While

$$\sup_k \varphi_k(|\epsilon_k|) = \sup_k \varphi_k(1) = \sup_k k = \infty,$$

then $e \notin \ell_\infty(\Phi)$. On the other hand, since

$$\sup_k \varphi_k \left( \frac{|\epsilon_k|}{\varrho} \right) = \sup_k \varphi_k \left( \frac{1}{\varrho} \right) = \frac{1}{\varrho} < \infty,$$

then $e \in \ell_\infty^\exists(\Phi)$. So, $\ell_\infty(\Phi) \subsetneqq \ell_\infty^\exists(\Phi)$.

The sequence of $\varphi$-functions $\Phi = (\varphi_k)$ is said to have *uniform $\Delta_2$-condition* if there exists a constant $K > 0$ such that $\varphi_k(2t) \leq K\varphi_k(t) \quad (k \in \mathbb{N},\ t > 0)$ (*cf.* [27, *p.* 167]).

The following proposition shows that if $\Phi$ satisfies uniform $\Delta_2$-condition, then (1.2.1) takes the form

$$\lambda^{\forall}(\Phi) = \lambda^{\varrho}(\Phi) = \lambda^{\exists}(\Phi). \tag{1.2.2}$$

**Proposition 1.2.3.** *Let $\lambda$ be a solid sequence space. If the sequence of $\varphi$-functions $\Phi = (\varphi_k)$ satisfies uniform $\Delta_2$-condition, then* (1.2.2) *holds.*

*Proof.* By (1.2.1) it is sufficient to prove the inclusion

$$\lambda^{\exists}(\Phi) \subset \lambda^{\forall}(\Phi). \tag{1.2.3}$$

Let $x = (x_k) \in \lambda^{\exists}(\Phi)$. Then, there exists $\varrho > 0$ such that $\Phi(|x/\varrho|) = (\varphi_k(|x_k|/\varrho)) \in \lambda$.

Let $\mu > 0$. If $\mu \geq \varrho$, then

$$\frac{|x_k|}{\mu} \leq \frac{|x_k|}{\varrho} \qquad (k \in \mathbb{N}).$$

Since all $\varphi$-functions are nondecreasing,

$$\varphi_k\left(\frac{|x_k|}{\mu}\right) \leq \varphi_k\left(\frac{|x_k|}{\varrho}\right) \qquad (k \in \mathbb{N}).$$

Because of the solidity of $\lambda$ we have $\Phi(|x/\mu|) \in \lambda$.

If $\mu < \varrho$, then $1/\mu > 1/\varrho$. We choose a number $r > 0$ such that

$$\frac{1}{\mu} \leq 2^r\ \frac{1}{\varrho}.$$

Using the inequalities

$$\frac{|x_k|}{\mu} \leq 2^r\ \frac{|x_k|}{\varrho} \qquad (k \in \mathbb{N}),$$

by (iii) and uniform $\Delta_2$-condition, we get

$$\varphi_k\left(\frac{|x_k|}{\mu}\right) \leq \varphi_k\left(2^r\ \frac{|x_k|}{\varrho}\right) \leq K^r\varphi_k\left(\frac{|x_k|}{\varrho}\right) \qquad (k \in \mathbb{N}).$$

While $\lambda$ is a solid vector space, then $\Phi(|x/\mu|) \in \lambda$.

Consequently, $\Phi(|x/\mu|) \in \lambda$ for any $\mu > 0$, i.e., $x \in \lambda^{\forall}(\Phi)$. The inclusion (1.2.3) is proved. □

The following example shows that the sets $\lambda^{\varrho}(\Phi)$ ($\varrho > 0$) may not be linear, i.e., they may not be sequence spaces.

**Example 1.2.4.** Let $\lambda = \ell_\infty$ and $\varrho > 0$. We show that $\ell_\infty^{\varrho}(\Phi)$ is not a sequence space if the sequence of $\varphi$-functions $\Phi = (\varphi_k)$ is defined by

$$\varphi_k(t) = \begin{cases} \frac{t}{2} & \text{if } t \in \left[0, \frac{1}{\varrho}\right], \\ \frac{kt}{2} & \text{if } t > \frac{1}{\varrho}. \end{cases}$$

We consider the sequence $e = (\epsilon_k) = (1, 1, \ldots)$. Since

$$\sup_k \varphi_k\left(\frac{|\epsilon_k|}{\varrho}\right) = \sup_k \varphi_k\left(\frac{1}{\varrho}\right) = \frac{1}{2\varrho} < \infty,$$

then $e \in \ell_\infty^{\varrho}(\Phi)$. But $2e \notin \ell_\infty^{\varrho}(\Phi)$, because

$$\sup_k \varphi_k\left(\frac{|2\epsilon_k|}{\varrho}\right) = \sup_k \varphi_k\left(\frac{2}{\varrho}\right) = \sup_k \frac{2k}{2\varrho} = \sup_k \frac{k}{\varrho} = \infty.$$

Therefore, $\ell_\infty^{\varrho}(\Phi)$ is not a linear space.

At the end of this subsection we prove, that $\lambda^{\varrho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are sequence spaces under some restrictions on $\Phi$.

**Proposition 1.2.5.** *Let $\lambda$ be a solid sequence space and $\varrho > 0$. If the sequence of $\varphi$-functions $\Phi = (\varphi_k)$ satisfies either* (ii) *or* (v)*, then the sets $\lambda^{\varrho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are solid sequence spaces.*

*Proof.* Let $\Phi = (\varphi_k)$ be a sequence of $\varphi$-functions.

First we show, that the sets $\lambda^{\varrho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are solid whenever $\lambda$ is solid. Indeed, let $x = (x_k)$ and $\varrho > 0$ be such that $\Phi(|x/\varrho|) = (\varphi_k(|x_k|/\varrho)) \in \lambda$. If $|y_k| \le |x_k|$ ($k \in \mathbb{N}$), then also

$$\varphi_k\left(\frac{|y_k|}{\varrho}\right) \le \varphi_k\left(\frac{|x_k|}{\varrho}\right) \qquad (k \in \mathbb{N})$$

and by solidity of $\lambda$ we get $\Phi(|y/\varrho|) = (\varphi_k(|y_k|/\varrho)) \in \lambda$.

Next we prove, that $\lambda^{\varrho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are vector spaces.

1) Let $\Phi = (\varphi_k)$ satisfies the condition (ii) and let $x = (x_k)$ and $y = (y_k)$ be from $\lambda^{\exists}(\Phi)$. So, there exist $\varrho_1, \varrho_2 > 0$ such that $\Phi(|x/\varrho_1|) = (\varphi_k(|x_k|/\varrho_1)) \in \lambda$ and $\Phi(|y/\varrho_2|) = (\varphi_k(|y_k|/\varrho_2)) \in \lambda$. Let $\varrho_3 =:$

$\max\{\varrho_1, \varrho_2\}$ and let $\alpha, \beta \in \mathbb{K}$ be arbitrarily choosen. Using (ii), (iii) and the inequality $|\gamma| \le 1 + [|\gamma|]$ ($\gamma \in \mathbb{K}$), where $[|\gamma|]$ denotes the integer part of $|\gamma|$, for all $k \in \mathbb{N}$ we have

$$\begin{aligned}\varphi_k\left(\frac{|\alpha x_k + \beta y_k|}{\varrho_3}\right) &\le \varphi_k\left(\frac{|\alpha x_k|}{\varrho_3} + \frac{|\beta y_k|}{\varrho_3}\right) \le \varphi_k\left(\frac{|\alpha||x_k|\varrho_1}{\varrho_3\varrho_1}\right)\\ &+ \varphi_k\left(\frac{|\beta||y_k|\varrho_2}{\varrho_3\varrho_2}\right) \le \varphi_k\left(\frac{|\alpha||x_k|}{\varrho_1}\right) + \varphi_k\left(\frac{|\beta||y_k|}{\varrho_2}\right)\\ &\le (1 + [|\alpha|])\varphi_k\left(\frac{|x_k|}{\varrho_1}\right) + (1 + [|\beta|])\varphi_k\left(\frac{|y_k|}{\varrho_2}\right).\end{aligned}$$

While $\lambda$ is vector space, then

$$(1 + [|\alpha|])\Phi(|x/\varrho_1|) + (1 + [|\beta|])\Phi(|y/\varrho_2|) \in \lambda$$

and by solidity of $\lambda$ we get $\Phi(|(\alpha x + \beta y)/\varrho_3|) \in \lambda$. Hence, $\alpha x + \beta y \in \lambda^{\exists}(\Phi)$.

The same discussion with $\varrho_1 = \varrho_2 = \varrho$ proves also the linearity of $\lambda^{\varrho}(\Phi)$ and $\lambda^{\forall}(\Phi)$.

2) Let $\Phi = (\varphi_k)$ satisfies the condition (v) and let $x = (x_k)$ and $y = (y_k)$ belongs to $\lambda^{\exists}(\Phi)$. Then we can find $\varrho_1,\ \varrho_2 > 0$ such that $\Phi(|x/\varrho_1|) = (\varphi_k(|x_k|/\varrho_1)) \in \lambda$ and $\Phi(|y/\varrho_2|) = (\varphi_k(|y_k|/\varrho_2)) \in \lambda$. Let $\varrho_3 := \max\{2|\alpha|\varrho_1, 2|\beta|\varrho_2\}$ and $\alpha, \beta \in \mathbb{K}$. By (iii) and (v) we have

$$\begin{aligned}\varphi_k\left(\frac{|\alpha x_k + \beta y_k|}{\varrho_3}\right) &\le \varphi_k\left(\frac{|\alpha x_k|}{\varrho_3} + \frac{|\beta y_k|}{\varrho_3}\right) \le \varphi_k\left(\frac{|\alpha||x_k|}{\varrho_3} + \frac{|\beta||y_k|}{\varrho_3}\right)\\ &\le \varphi_k\left(\frac{|x_k|}{2\varrho_1} + \frac{|y_k|}{2\varrho_2}\right) \le \frac{1}{2}\varphi_k\left(\frac{|x_k|}{\varrho_1}\right) + \frac{1}{2}\varphi_k\left(\frac{|y_k|}{\varrho_2}\right)\end{aligned}$$

for all $k \in \mathbb{N}$. Since $1/2 \cdot \Phi(|x/\varrho_1|) + 1/2 \cdot \Phi(|y/\varrho_2|) \in \lambda$ and $\lambda$ is a solid sequence space, then $\Phi(|(\alpha x + \beta y)/\varrho_3|) \in \lambda$, i.e., $\alpha x + \beta y \in \lambda^{\exists}(\Phi)$. Consequently, $\lambda^{\exists}(\Phi)$ is a sequence space.

To prove the linearity of $\lambda^{\varrho}(\Phi)$ ($\varrho > 0$) and $\lambda^{\forall}(\Phi)$, it suffices to take $\varrho_1 = \varrho_2 = \varrho$ in our argument. □

**Remark 1.2.6.** Proposition 1.2.5 shows that, for a solid sequence space $\lambda$, the sets $\lambda^{\varrho}(\Phi)$, $\lambda^{\exists}(\Phi)$ and $\lambda^{\forall}(\Phi)$ are sequence spaces whenever $\varphi_k$ ($k \in \mathbb{N}$) are either moduli or Orlicz functions. Since uniform $\Delta_2$-condition holds (with $K = 2$) for every sequence of moduli $\Phi = (\varphi_k)$, we also conclude that (1.2.2) is true whenever all $\varphi_k$ are either moduli or Orlicz functions such that $\Phi$ satisfies uniform $\Delta_2$-condition.

## 1.3 Inclusion theorems

In this section we generalize the results of [21], where the inclusions $\lambda \subset \mu(\Phi)$ and $\lambda(\Phi) \subset \mu$ have been characterized for a sequence of moduli $\Phi = (\varphi_k)$ and $\lambda, \mu \in \{\ell_\infty, c_0\}$. Our investigations are also motivated by the work of Grinnell [16] which is devoted to the study of the inclusions $\lambda \subset \mu_\varphi$ for various sequence spaces $\lambda$ and $\mu$, by the assumptions that $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\mu_\varphi = \{x = (x_k)\colon (\varphi(x_k)) \in \mu\}$.

Throughout this work, by an *index sequence*, we mean any strictly increasing sequence of natural numbers and for a sequence space $\lambda$ we use the notation

$$\lambda^+ = \{(x_k) \in \lambda : x_k \geq 0 \quad (k \in \mathbb{N})\}.$$

Recall that the function $f\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ defines a *superposition operator* $P_f\colon \lambda \to \mu$ by

$$P_f(x) = (f(k, x_k)) \in \mu \qquad (x = (x_k) \in \lambda).$$

The characterizations of superposition operators on $\ell_\infty$, $c_0$ and $\ell_p$ $(0 < p < \infty)$ are contained in results of Dedagich and Zabreĭko [10], Petranuarat and Kemprasit [44] and Kolk [24].

**Proposition 1.3.1.** (1) Let $0 < p, q < \infty$. Then $P_f\colon \ell_p \to \ell_q$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \geq 0$, $\delta > 0$, $k_0 \in \mathbb{N}$ such that

$$|f(k,t)|^q \leq a_k + \gamma |t|^p \quad (|t| \leq \delta,\ k \geq k_0).$$

(2) Let $0 < p < \infty$ and $1 \leq q < \infty$. Then $P_f\colon \ell_p \to \ell_q$ if and only if there exist a sequence $(b_k) \in \ell_q^+$ and numbers $\gamma \geq 0$, $\delta > 0$, $k_0 \in \mathbb{N}$ such that

$$|f(k,t)| \leq b_k + \gamma |t|^{p/q} \quad (|t| \leq \delta,\ k \geq k_0).$$

**Proposition 1.3.2.** *Let* $0 < p < \infty$. *The following statements are equivalent:*

(a) $P_f\colon c_0 \to \ell_\infty$*;*

(b) $P_f\colon \ell_p \to \ell_\infty$*;*

(c) $\exists (a_k) \in \ell_\infty^+ \ \ \exists \delta > 0 \ \ \exists k_0 \in \mathbb{N} \quad |f(k,t)| \le a_k \quad (|t| \le \delta, k \ge k_0)$;

(d) $\exists \delta > 0 \ \ \exists k_0 \in \mathbb{N} \quad \sup_{|t|\le\delta, k\ge k_0} |f(k,t)| < \infty$.

**Proposition 1.3.3.** *Let $0 < p < \infty$. The following statements are equivalent:*

(a) $P_f \colon c_0 \to c_0$;

(b) $P_f \colon \ell_p \to c_0$;

(c) $\lim_{k\to\infty, t\to 0} |f(k,t)| = 0$;

(d) $\exists (a_k) \in c_0^+ \ \ \exists \delta > 0 \ \ \exists k_0 \in \mathbb{N} \quad |f(k,t)| \le a_k \quad (|t| \le \delta, k \ge k_0)$;

(e) $\exists k_0 \in \mathbb{N} \quad \lim_{t\to 0} \sup_{k\ge k_0} |f(k,t)| = 0$.

**Proposition 1.3.4.** *Let $0 < p < \infty$. Then $P_f \colon c_0 \to \ell_p$ if and only if*

$$\exists \delta > 0 \ \ \exists k_0 \in \mathbb{N} \quad \sum_{k\ge k_0} \sup_{|t|\le\delta} |f(k,t)|^p < \infty.$$

**Proposition 1.3.5.** *Let $0 < p < \infty$. Then $P_f \colon \ell_\infty \to \ell_p$ if and only if*

$$\sum_k \sup_{|t|\le\eta} |f(k,t)|^p < \infty \quad (\eta > 0).$$

**Proposition 1.3.6.** *$P_f \colon \ell_\infty \to c_0$ if and only if*

$$\lim_k \sup_{|t|\le\eta} |f(k,t)| = 0 \quad (\eta > 0).$$

### 1.3.1 Inclusions $\lambda \subset \mu(\Phi)$

Let $\Phi = (\varphi_k)$ be a sequence of $\varphi$-functions and $0 < p, q < \infty$. Necessary and sufficient conditions for the inclusions $\lambda \subset \mu(\Phi)$ in the case $\lambda, \mu \in \{\ell_\infty, c_0, \ell_p\}$ we derive from Propositions 1.3.1–1.3.6.

It is clear that $P_f \colon \lambda \to \mu$ if and only if $\lambda \subset \mu_f$, where $\mu_f = \{x = (x_k) : (f(k, x_k)) \in \mu\}$.

Now, if $\bar\varphi_k$ $(k \in \mathbb{N})$ are even extensions of our $\varphi$-functions $\varphi_k$, i.e.,

$$\bar\varphi_k(t) = \varphi_k(|t|) \quad (t \in \mathbb{R}),$$

and $\bar{\Phi} = (\bar{\varphi}_k)$, then we have

$$\lambda \subset \mu(\Phi) \iff P_{\bar{\Phi}} \colon \lambda \to \mu$$

because of $\mu_{\bar{\Phi}} = \mu(\Phi)$. So by Propositions 1.3.1–1.3.6 with $0 < p, q < \infty$ we may characterize the inclusions $\ell_q \subset \ell_p(\Phi)$, $\ell_p \subset c_0(\Phi)$, $c_0 \subset \ell_p(\Phi)$, $c_0 \subset c_0(\Phi)$, $\ell_\infty \subset \ell_p(\Phi)$, $\ell_\infty \subset c_0(\Phi)$ and $\ell_\infty \subset \ell_\infty(\Phi)$, using the following classes of $\varphi$-function sequences:

$$\begin{aligned}
C_0 &= \big\{\Phi = (\varphi_k) \colon \exists\ (a_k) \in \ell^+ \ \ \exists\ \gamma \geq 0 \ \ \exists\ k_0 \in \mathbb{N} \ \ \exists\ \delta > 0 \\
&\qquad\qquad (\varphi_k(t))^p \leq a_k + \gamma t^q \quad (k \geq k_0,\ t \in [0, \delta])\big\}, \\
C_1 &= \left\{\Phi = (\varphi_k) \colon \exists\ t_0 > 0 \quad \sum_k (\varphi_k(t_0))^p < \infty\right\}, \\
C_2 &= \left\{\Phi = (\varphi_k) \colon \sum_k (\varphi_k(t))^p < \infty \quad (t > 0)\right\}, \\
C_3 &= \left\{\Phi = (\varphi_k) \colon \exists\ k_0 \in \mathbb{N} \quad \lim_{t \to 0+} \sup_{k \geq k_0} \varphi_k(t) = 0\right\}, \\
C_4 &= \left\{\Phi = (\varphi_k) \colon \lim_k \varphi_k(t) = 0 \quad (t > 0)\right\}, \\
C_5 &= \left\{\Phi = (\varphi_k) \colon \sup_k \varphi_k(t) < \infty \quad (t > 0)\right\}, \\
C_6 &= \left\{\Phi = (\varphi_k) \colon \exists\ t_0 > 0 \quad \sup_k \varphi_k(t_0) < \infty\right\}.
\end{aligned}$$

**Theorem 1.3.7.** *Let $0 < p, q < \infty$. The following equivalences are true:*

(1) $\ell_q \subset \ell_p(\Phi) \iff \Phi \in C_0$;

(2) $c_0 \subset \ell_p(\Phi) \iff \Phi \in C_1$;

(3) $\ell_\infty \subset \ell_p(\Phi) \iff \Phi \in C_2$;

(4) $c_0 \subset c_0(\Phi) \iff \ell_p \subset c_0(\Phi) \iff \Phi \in C_3$;

(5) $\ell_\infty \subset c_0(\Phi) \iff \Phi \in C_4$;

(6) $\ell_\infty \subset \ell_\infty(\Phi) \iff \Phi \in C_5$;

(7) $c_0 \subset \ell_\infty(\Phi) \Longleftrightarrow \ell_p \subset \ell_\infty(\Phi) \Longleftrightarrow \Phi \in C_6$.

**Remark 1.3.8.** Proposition 1.3.1 (2) shows that if $1 \le p < \infty$ and $0 < q < \infty$, then $\ell_q \subset \ell_p(\Phi)$ if and only if $\Phi \in C_0'$, where

$$C_0' = \big\{\Phi = (\varphi_k)\colon \exists\ (a_k) \in \ell_p^+ \ \ \exists\ \gamma \ge 0 \ \ \exists\ k_0 \in \mathbb{N} \ \ \exists\ \delta > 0 \\ \varphi_k(t) \le a_k + \gamma t^{q/p} \quad (k \ge k_0,\ t \in [0,\delta])\big\}\,.$$

## 1.3.2 Inclusions $\lambda(\Phi) \subset \mu$

Let $\Phi = (\varphi_k)$ be a sequence of $\varphi$-functions and $1 \le p < \infty$. In this section we study the inclusions $\lambda(\Phi) \subset \mu$, where $\lambda \in \{\ell_\infty, c_0, \ell_p\}$ and $\mu \in \{\ell_\infty, c_0\}$. At it the following classes of $\varphi$-function sequences are important:

$$C_7 = \left\{\Phi = (\varphi_k)\colon \exists k_0 \in \mathbb{N} \quad \lim_{t\to\infty} \sup_{n \ge k_0} \inf_{k \ge n} \varphi_k(t) = \infty\right\},$$

$$C_8 = \left\{\Phi = (\varphi_k)\colon \exists t_0 > 0 \quad \inf_k \varphi_k(t_0) > 0\right\},$$

$$C_9 = \left\{\Phi = (\varphi_k)\colon \lim_k \varphi_k(t) = \infty \quad (t > 0)\right\},$$

$$C_{10} = \left\{\Phi = (\varphi_k)\colon \inf_k \varphi_k(t) > 0 \quad (t > 0)\right\}.$$

**Theorem 1.3.9.** *The inclusion $\ell_\infty(\Phi) \subset \ell_\infty$ holds if and only if $\Phi \in C_7$.*

*Proof. Necessity.* Let $\ell_\infty(\Phi) \subset \ell_\infty$. Suppose that $\Phi \notin C_7$. Since the functions

$$\psi(t) = \sup_{n \ge k_0} \inf_{k \ge n} \varphi_k(t)$$

are non-decreasing for every $k_0 \in \mathbb{N}$, there exists a number $H > 0$ such that $\inf_k \varphi_k(t) \le H$ for all $t > 0$. Thus, given $\varepsilon > 0$, we can choose an index sequence $(k_i)$ such that

$$\varphi_{k_i}(i) \le H + \varepsilon \quad (i \in \mathbb{N}).$$

So, taking

$$x_k = \begin{cases} i & \text{if } k = k_i \ \ (i \in \mathbb{N}), \\ 0 & \text{otherwise}, \end{cases}$$

we get $(x_k) \in \ell_\infty(\Phi)$. But $(x_k) \notin \ell_\infty$, contrary to $\ell_\infty(\Phi) \subset \ell_\infty$. Therefore $\Phi$ must be in $C_7$.

*Sufficiency.* Let $x \in \ell_\infty(\Phi)$, i.e., $\varphi_k(|x_k|) \leq M \quad (k \in \mathbb{N})$ for some $M > 0$. If $\Phi \in C_7$, then there exists a number $T > 0$ such that $t \geq T$ implies

$$\inf_{k \geq n} \varphi_k(t) \geq M \quad (n \geq k_0).$$

This yields

$$\varphi_n(t) \geq M \quad (n \geq k_0,\ t \geq T). \tag{1.3.1}$$

Assuming $x \notin \ell_\infty$, we can choose indices $k_i \geq k_0 \quad (i \in \mathbb{N})$ such that $|x_{k_i}| \geq T$, but

$$\varphi_{k_i}(|x_{k_i}|) \leq M \quad (i \in \mathbb{N}),$$

contrary to (1.3.1). Hence $x \in \ell_\infty$ and, consequently, $\ell_\infty(\Phi) \subset \ell_\infty$. □

**Theorem 1.3.10.** *The following statements are equivalent:*

(a) $c_0(\Phi) \subset \ell_\infty$;

(b) $\ell_p(\Phi) \subset \ell_\infty$;

(c) $\Phi \in C_8$.

*Proof.* (a) ⇒ (b) follows immediately.

(b) ⇒ (c). Let $\ell_p(\Phi) \subset \ell_\infty$. If $\Phi \notin C_8$, then $\inf_k \varphi_k(t) = 0$ for all $t > 0$. Thus we can choose an index sequence $(k_i)$ with

$$\varphi_{k_i}(i) \leq 2^{-i/p} \quad (i \in N).$$

So, if

$$x_k = \begin{cases} i & \text{for } k = k_i \quad (i \in \mathbb{N}), \\ 0 & \text{otherwise}, \end{cases}$$

we have $x \in \ell_p(\Phi)$. But $x \notin \ell_\infty$, contrary to $\ell_p(\Phi) \subset \ell_\infty$. Hence $\Phi \in C_8$.

(c) ⇒ (a). Suppose that $\Phi \in C_8$ and $x = (x_k)$ belongs to $c_0(\Phi)$. If we assume $x \notin \ell_\infty$, there exists an index sequence $(k_i)$ with $|x_{k_i}| \geq t_0 \quad (i \in \mathbb{N})$. This gives

$$\varphi_{k_i}(t_0) \leq \varphi_{k_i}(|x_{k_i}|) \quad (i \in \mathbb{N})$$

which by $x \in c_0(\Phi)$ shows that $\lim_i \varphi_{k_i}(t_0) = 0$, contrary to $\Phi \in C_8$. Consequently, $x \in \ell_\infty$ and the inclusion $c_0(\Phi) \subset \ell_\infty$ holds. □

**Theorem 1.3.11.** *The inclusion* $\ell_\infty(\Phi) \subset c_0$ *holds if and only if* $\Phi \in C_9$.

*Proof. Necessity.* Let $\ell_\infty(\Phi) \subset c_0$. Assuming that $\Phi \notin C_9$, we can find numbers $t_0 > 0$ , $M > 0$ and an index sequence $(k_i)$ such that $\varphi_{k_i}(t_0) \le M \ \ (i \in \mathbb{N})$. So the sequence $x = (x_k)$, where

$$x_k = \begin{cases} t_0 & \text{for } k = k_i \ \ (i \in \mathbb{N}), \\ 0 & \text{otherwise}, \end{cases}$$

belongs to $\ell_\infty(\Phi)$. But $x \notin c_0$. Consequently, $\Phi \in C_9$ is necessary for $\ell_\infty(\Phi) \subset c_0$.

*Sufficiency.* Let $\Phi \in C_9$ and let $x = (x_k)$ belongs to $\ell_\infty(\Phi)$. If $x \notin c_0$, there exist a number $\varepsilon_0 > 0$ and an index sequence $(k_i)$ such that $|x_{k_i}| \ge \varepsilon_0 \ \ (i \in \mathbb{N})$. Now, since the $\varphi$-functions are non-decreasing, by $x \in \ell_\infty(\Phi)$ we have, for some $M > 0$,

$$\varphi_{k_i}(\varepsilon_0) \le \varphi_{k_i}(|x_{k_i}|) \le M \quad (i \in \mathbb{N}),$$

contrary to $\Phi \in C_9$ . Hence $x \in c_0$, proving $\ell_\infty(\Phi) \subset c_0$. □

**Theorem 1.3.12.** *The following statements are equivalent:*

(a) $c_0(\Phi) \subset c_0$;

(b) $\ell_p(\Phi) \subset c_0$;

(c) $\Phi \in C_{10}$.

*Proof.* (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (c). Let $\ell_p(\Phi) \subset c_0$. If $\Phi \notin C_{10}$, there exists a number $t_0 > 0$ such that $\inf_k \varphi_k(t) = 0$ for all $t \le t_0$. Thus, letting $t_i = t_0 i/(i+1)$, by induction we can choose an index sequence $(k_i)$ such that

$$\varphi_{k_i}(t_i) \le 2^{-i/p} \quad (i \in \mathbb{N}).$$

Now, if $x = (x_k)$, where

$$x_k = \begin{cases} t_i & \text{for } k = k_i \ \ (i \in \mathbb{N}), \\ 0 & \text{otherwise}, \end{cases}$$

then $x \in \ell_p(\Phi)$. But by $\lim_i x_{k_i} = \lim_i t_i = t_0 > 0$ we have $x \notin c_0$ which contradicts $\ell_p(\Phi) \subset c_0$. So $\Phi$ must be in $C_{10}$.

(c) $\Rightarrow$ (a). Let $\Phi \in C_{10}$ and let $x = (x_k)$ belongs to $c_0(\Phi)$. If we suppose, that $x \notin c_0$, then there exist a number $\varepsilon_0 > 0$ and an index sequence $(k_i)$ such that $|x_{k_i}| \geq \varepsilon_0 \ \ (i \in \mathbb{N})$. This yields

$$0 < \varphi_{k_i}(\varepsilon_0) \leq \varphi_{k_i}(|x_{k_i}|) \quad (i \in \mathbb{N}),$$

and by $x \in c_0(\Phi)$ we have $\lim_i \varphi_{k_i}(\varepsilon_0) = 0$, contrary to $\Phi \in C_{10}$. Hence $x$ must belong to $c_0$. Consequently, $c_0(\Phi) \subset c_0$. □

### 1.3.3 The sets $\lambda^\varrho(\Phi)$, $\lambda^\exists(\Phi)$ and $\lambda^\forall(\Phi)$

Let $\Phi = (\varphi_k)$ be a sequence of $\varphi$-functions and $\lambda, \mu \in \{\ell_\infty, c_0, \ell_p\}$. For a fixed number $\varrho > 0$ we consider a new sequence of $\varphi$-functions $\Phi^\rho = (\varphi_k^\varrho)$, where

$$\varphi_k^\varrho(t) = \varphi_k(t/\varrho) \quad (k \in \mathbb{N}).$$

It is not difficult to see that $\lambda^\varrho(\Phi) = \lambda(\Phi^\varrho)$, $\mu^\varrho(\Phi) = \mu(\Phi^\varrho)$ and

$$\Phi^\varrho \in C_i \iff \Phi \in C_i \quad (i = 0, 1, 2, \ldots, 10).$$

Thus

$$\lambda \subset \mu(\Phi) \iff \lambda \subset \mu^\varrho(\Phi), \quad \lambda(\Phi) \subset \mu \iff \lambda^\varrho(\Phi) \subset \mu \tag{1.3.2}$$

and, therefore, all our Theorems 1.3.7 and 1.3.9–1.3.12 remain true if there $\lambda(\Phi)$ and $\mu(\Phi)$ are replaced by $\lambda^\varrho(\Phi)$ and $\mu^\varrho(\Phi)$, respectively.

Further, because of (1.2.1) it is clear that for a sequence of $\varphi$-functions $\Phi = (\varphi_k)$ we have

$$\lambda \subset \mu^\forall(\Phi) \implies \lambda \subset \mu(\Phi), \quad \lambda^\exists(\Phi) \subset \mu \implies \lambda(\Phi) \subset \mu.$$

It turns out that these implications are reversible.

**Theorem 1.3.13.** *For a sequence of $\varphi$-functions $\Phi = (\varphi_k)$ and a pair of sequence spaces $\lambda$, $\mu$ we have*

$$\lambda \subset \mu^\forall(\Phi) \iff \lambda \subset \mu(\Phi), \quad \lambda^\exists(\Phi) \subset \mu \iff \lambda(\Phi) \subset \mu.$$

*Proof.* It suffices to prove that

$$\lambda \subset \mu(\Phi) \implies \lambda \subset \mu^{\forall}(\Phi), \quad \lambda(\Phi) \subset \mu \implies \lambda^{\exists}(\Phi) \subset \mu.$$

But these implications follow immediately from the equalities $\mu^{\forall}(\Phi) = \cap_{\varrho>0}\mu^{\varrho}(\Phi)$, $\lambda^{\exists}(\Phi) = \cup_{\varrho>0}\lambda^{\varrho}(\Phi)$ because of the fact that $\lambda$ and $\mu$ as vector spaces contain together with an element $x$ also the element $x/\varrho$, and conversely. □

The equivalences (1.3.2) and Theorem 1.3.13 show that we can give extended versions of all Theorems 1.3.7, 1.3.9 – 1.3.12, replacing there $\lambda(\Phi)$ by $\lambda^{\varrho}(\Phi)$, $\mu(\Phi)$ by $\mu^{\varrho}(\Phi)$ and adding to each statement of the type $\lambda \subset \mu^{\varrho}(\Phi)$ or $\lambda^{\varrho}(\Phi) \subset \mu$ the equivalent statement $\lambda \subset \mu^{\forall}(\Phi)$ or $\lambda^{\exists}(\Phi) \subset \mu$, respectively. Here we formulate extended versions of Theorems 1.3.7 (7) and 1.3.12 only.

**Theorem 1.3.14.** *Let $0 < p < \infty$ and $\varrho > 0$. The following statements are equivalent:*

(a) $c_0 \subset \ell_{\infty}^{\varrho}(\Phi)$;

(b) $c_0 \subset \ell_{\infty}^{\forall}(\Phi)$;

(c) $\ell_p \subset \ell_{\infty}^{\varrho}(\Phi)$;

(d) $\ell_p \subset \ell_{\infty}^{\forall}(\Phi)$;

(e) $\Phi \in C_6$.

**Theorem 1.3.15.** *Let $1 \le p < \infty$ and $\varrho > 0$. The following statements are equivalent:*

(a) $c_0^{\exists}(\Phi) \subset c_0$;

(b) $c_0^{\varrho}(\Phi) \subset c_0$;

(c) $\ell_p^{\exists}(\Phi) \subset c_0$;

(d) $\ell_p^{\varrho}(\Phi) \subset c_0$;

(e) $\Phi \in C_{10}$.

### 1.3.4 Inclusion theorems for some sets of sequences defined by a matrix of $\varphi$-functions

Let $\mathcal{F} = (f_{ki})$ be a matrix of $\varphi$-functions such that

$$\tilde{f}_k(t) := \sup_i f_{ki}(t) < \infty \qquad (k \in \mathbb{N},\ t \geq 0). \tag{1.3.3}$$

By (1.3.3) it is clear that the functions $\tilde{f}_k$ $(k \in \mathbb{N})$ map $[0, \infty)$ into $[0, \infty)$. We claim, that $\tilde{f}_k$ $(k \in \mathbb{N})$ are $\varphi$-functions, i.e. they satisfy conditions (i) and (iii) of Definition 1.1.3.

Indeed, if $\tilde{f}_k(t) = 0$ $(k \in \mathbb{N})$, then $f_{ki}(t) = 0$ for all $i \in \mathbb{N}$ and since the functions $f_{ki}$ $(k, i \in \mathbb{N})$ are $\varphi$-functions, so $t = 0$. On the other hand, if $t = 0$, then

$$\tilde{f}_k(0) = \sup_i f_{ki}(0) = \sup_i 0 = 0 \quad (k \in \mathbb{N}).$$

Thus, the functions $\tilde{f}_k$ satisfy the condition (i).

Futher, let $0 \leq u \leq t$. While the functions $f_{ki}$ $(k, i \in \mathbb{N})$ are non-decreasing, we have

$$f_{ki}(u) \leq f_{ki}(t) \quad (k, i \in \mathbb{N}).$$

Consequently, for all $k \in \mathbb{N}$ we get

$$\tilde{f}_k(u) = \sup_i f_{ki}(u) \leq \sup_i f_{ki}(t) = \tilde{f}_k(t).$$

Therefore, the functions $\tilde{f}_k$ satisfy also the condition (iii).

Using a matrix of moduli $\mathcal{F} = (f_{ki})$, we define the sets

$$\ell_\infty(\mathcal{F}) = \left\{ x = (x_k) \in \omega\colon\ \sup_{k,i} f_{ki}\big(|x_k|\big) < \infty \right\},$$

$$c_0(\mathcal{F}) = \left\{ x = (x_k) \in \omega\colon\ \lim_k \sup_i f_{ki}\big(|x_k|\big) = 0 \right\},$$

$$\ell_p(\mathcal{F}) = \left\{ x = (x_k) \in \omega\colon\ \sum_k \left| \sup_i f_{ki}\big(|x_k|\big) \right|^p < \infty \right\} \quad (0 < p < \infty).$$

Since $\mathcal{F}$ satisfies (1.3.3), the sets $\ell_\infty(\mathcal{F})$, $c_0(\mathcal{F})$ and $\ell_p(\mathcal{F})$ we may consider as the sets $\ell_\infty(\widetilde{F})$, $c_0(\widetilde{F})$ and $\ell_p(\widetilde{F})$, where $\widetilde{F} = (\tilde{f}_k)$ is the sequence of $\varphi$-functions $\tilde{f}_k(t) = \sup_i f_{ki}(t)$.

Applying Theorems 1.3.7 and 1.3.9–1.3.12 for $\Phi = \widetilde{F}$, we get necessary and sufficent conditions for the inclusions $\lambda \subset \mu(\mathcal{F})$ and $\lambda(\mathcal{F}) \subset \mu$ in the case $\lambda, \mu \in \{\ell_\infty, c_0, \ell_p\}$ and $1 \le p, q < \infty$ (see Theorems 1.3.16 and 1.3.17). Thereby, every class $C_i$ $(i = 0, \dots, 10)$ alters to the corresponding class $\widetilde{C}_i$ as follows:

$$\widetilde{C}_0 = \Big\{\mathcal{F} = (f_{ki})\colon \exists\ (a_k) \in \ell^+ \ \ \exists\ \gamma \ge 0 \ \ \exists\ k_0 \in \mathbb{N} \ \ \exists\ \delta > 0 \\ \sup_i (f_{ki}(t))^p \le a_k + \gamma t^q \quad (k \ge k_0,\ t \in [0, \delta])\Big\},$$

$$\widetilde{C}_1 = \left\{\mathcal{F} = (f_{ki})\colon \exists\ t_0 > 0 \quad \sum_k \left|\sup_i f_{ki}(t_0)\right|^p < \infty\right\},$$

$$\widetilde{C}_2 = \left\{\mathcal{F} = (f_{ki})\colon \sum_k \left|\sup_i f_{ki}(t)\right|^p < \infty \quad (t > 0)\right\},$$

$$\widetilde{C}_3 = \left\{\mathcal{F} = (f_{ki})\colon \exists\ k_0 \in \mathbb{N} \quad \lim_{t \to 0+} \sup_{k \ge k_0} \sup_i f_{ki}(t) = 0\right\},$$

$$\widetilde{C}_4 = \left\{\mathcal{F} = (f_{ki})\colon \lim_k \sup_i f_{ki}(t) = 0 \quad (t > 0)\right\},$$

$$\widetilde{C}_5 = \left\{\mathcal{F} = (f_{ki})\colon \sup_{k,i} f_{ki}(t) < \infty \quad (t > 0)\right\},$$

$$\widetilde{C}_6 = \left\{\mathcal{F} = (f_{ki})\colon \exists\ t_0 > 0 \quad \sup_{k,i} f_{ki}(t_0) < \infty\right\},$$

$$\widetilde{C}_7 = \left\{\mathcal{F} = (f_{ki})\colon \exists k_0 \in \mathbb{N} \quad \lim_{t \to \infty} \sup_{n \ge k_0} \inf_{k \ge n} \sup_i f_{ki}(t) = \infty\right\},$$

$$\widetilde{C}_8 = \left\{\mathcal{F} = (f_{ki})\colon \exists t_0 > 0 \quad \inf_k \sup_i f_{ki}(t_0) > 0\right\},$$

$$\widetilde{C}_9 = \left\{\mathcal{F} = (f_{ki})\colon \lim_k \sup_i f_{ki}(t) = \infty \quad (t > 0)\right\},$$

$$\widetilde{C}_{10} = \left\{\mathcal{F} = (f_{ki})\colon \inf_k \sup_i f_{ki}(t) > 0 \quad (t > 0)\right\}.$$

**Theorem 1.3.16.** *Let $0 < p, q < \infty$. The following equivalences are true:*

(1) $\ell_q \subset \ell_p(\mathcal{F}) \Longleftrightarrow \mathcal{F} \in \widetilde{C}_0$*;*

(2) $c_0 \subset \ell_p(\mathcal{F}) \Longleftrightarrow \mathcal{F} \in \widetilde{C}_1$;

(3) $\ell_\infty \subset \ell_p(\mathcal{F}) \Longleftrightarrow \mathcal{F} \in \widetilde{C}_2$;

(4) $c_0 \subset c_0(\mathcal{F}) \Longleftrightarrow \ell_p \subset c_0(\mathcal{F}) \Longleftrightarrow \mathcal{F} \in \widetilde{C}_3$;

(5) $\ell_\infty \subset c_0(\mathcal{F}) \Longleftrightarrow \mathcal{F} \in \widetilde{C}_4$;

(6) $\ell_\infty \subset \ell_\infty(\mathcal{F}) \Longleftrightarrow \mathcal{F} \in \widetilde{C}_5$;

(7) $c_0 \subset \ell_\infty(\mathcal{F}) \Longleftrightarrow \ell_p \subset \ell_\infty(\mathcal{F}) \Longleftrightarrow \mathcal{F} \in \widetilde{C}_6$.

**Theorem 1.3.17.** *Let $0 < p, q < \infty$. The following equivalences are true:*

(1) $\ell_\infty(\mathcal{F}) \subset \ell_\infty \Longleftrightarrow \mathcal{F} \in \widetilde{C}_7$;

(2) $c_0(\mathcal{F}) \subset \ell_\infty \Longleftrightarrow \ell_p(\mathcal{F}) \subset \ell_\infty \Longleftrightarrow \mathcal{F} \in \widetilde{C}_8$;

(3) $\ell_\infty(\mathcal{F}) \subset c_0 \Longleftrightarrow \mathcal{F} \in \widetilde{C}_9$;

(4) $c_0(\mathcal{F}) \subset c_0 \Longleftrightarrow \ell_p(\mathcal{F}) \subset c_0 \Longleftrightarrow \mathcal{F} \in \widetilde{C}_{10}$.

### 1.3.5 Applications to Maddox sequence spaces

First let $\Phi = (\varphi_k)$ be a constant sequence of $\varphi$-functions, i.e., $\varphi_k = \varphi \quad (k \in \mathbb{N})$. In this case we write $\lambda(\varphi)$ instead of $\lambda(\Phi)$, and $\varphi \in C_i$ instead of $\Phi \in C_i$ for $i = 0, 1, 2, \ldots, 10$. It is clear that for an arbitrary $\varphi$-function $\varphi$ we have

$$\varphi \notin C_i \quad (i = 1, 2, 4, 9) \quad \text{and} \quad \varphi \in C_i \quad (i = 5, 6, 8, 10).$$

Moreover,

$$\begin{aligned}
\varphi \in C_0 &\iff \exists \alpha > 0 \ \ \exists \delta > 0 \quad (\varphi(t))^p \leq \alpha t^q \quad (t \in [0, \delta]), \\
\varphi \in C_3 &\iff \lim_{t \to 0+} \varphi(t) = 0, \\
\varphi \in C_7 &\iff \lim_{t \to \infty} \varphi(t) = \infty.
\end{aligned}$$

Thus our results permit to formulate:

**Corollary 1.3.18.** *Let $\varphi$ be a $\varphi$-function, $0 < p, q < \infty$ and $\varrho > 0$. The following statements are true:*

(1) $\ell_q \subset \ell_p^{\forall}(\varphi) \iff \ell_q \subset \ell_p^{\varrho}(\varphi)$
$\iff \exists \alpha > 0 \;\; \exists \delta > 0 \quad (\varphi(t))^p \le \alpha t^q \quad (t \in [0, \delta])$;

(2) $c_0^{\exists}(\varphi) \subset c_0$;

(3) $c_0 \subset c_0^{\forall}(\varphi) \iff c_0 = c_0^{\forall}(\varphi) = c_0^{\varrho}(\varphi) = c_0^{\exists}(\varphi)$
$\iff \lim\limits_{t\to 0+} \varphi(t) = 0$;

(4) $\ell_\infty \subset \ell_\infty^{\forall}(\varphi)$;

(5) $\ell_\infty^{\exists}(\varphi) \subset \ell_\infty \iff \ell_\infty^{\forall}(\varphi) = \ell_\infty^{\varrho}(\varphi) = \ell_\infty^{\exists}(\varphi) = \ell_\infty$
$\iff \lim\limits_{t\to\infty} \varphi(t) = \infty$.

It should be noted that the inclusion $\ell_\infty \subset \ell_\infty(\varphi)$ and the equivalences

$$\ell_q \subset \ell_p(\varphi) \iff \exists \alpha > 0 \;\; \exists \delta > 0 \quad (\varphi(t))^p \le \alpha t^q \quad (t \in [0, \delta]),$$
$$c_0 \subset c_0(\varphi) \iff \lim_{t\to 0+} \varphi(t) = 0$$

follow also from the corresponding results of Grinnell [16] because of $\mu(\varphi) = \mu_{\bar{\varphi}}$.

As an example of non-constant sequence of $\varphi$-functions we consider the sequence $\Phi^{(\boldsymbol{p})} = (\varphi_k^{(\boldsymbol{p})})$ of $\varphi$-functions $\varphi_k^{(\boldsymbol{p})}(t) = t^{p_k}$, where $\boldsymbol{p} = (p_k)$ is a bounded sequence of positive numbers, i.e.,

$$0 < p_k \le \sup_k p_k = P < \infty.$$

For $\Phi = \Phi^{(\boldsymbol{p})}$ the sequence spaces $\ell_\infty(\Phi)$, $c_0(\Phi)$ and $\ell(\Phi)$ are called as the sequence spaces of Maddox (see, for example, [17])

$$\ell_\infty(\boldsymbol{p}) = \{x = (x_k) \in \omega \colon \sup_k |x_k|^{p_k} < \infty\},$$
$$c_0(\boldsymbol{p}) = \{x = (x_k) \in \omega \colon \lim_k |x_k|^{p_k} = 0\},$$
$$\ell(\boldsymbol{p}) = \{x = (x_k) \in \omega \colon \sum_k |x_k|^{p_k} < \infty\},$$

respectively. Since the functions $\varphi_k^{(\boldsymbol{p}/r)}(t) = t^{p_k/r} \quad (k \in \mathbb{N})$ with $r = \max\{1,\ P\}$ are moduli, and for $\varrho > 0$ we have

$$\ell_\infty^{\varrho}(\Phi^{(\boldsymbol{p})}) = \ell_\infty^{\varrho}(\Phi^{(\boldsymbol{p}/r)}), \quad c_0^{\varrho}(\Phi^{(\boldsymbol{p})}) = c_0^{\varrho}(\Phi^{(\boldsymbol{p}/r)}), \quad \ell(\Phi^{(\boldsymbol{p})}) = \ell_r^{\varrho}(\Phi^{(\boldsymbol{p}/r)}),$$

the equalities (1.2.2) hold if $\Phi = \Phi^{(\boldsymbol{p})}$ and $\lambda \in \{\ell_\infty, c_0, \ell\}$.

To apply our theorems for sequence spaces of Maddox we must describe the classes of sequences $\boldsymbol{p} = (p_k)$ with $\Phi^{(\boldsymbol{p}/r)} \in C_0$ (for $\boldsymbol{p} = r$) and $\Phi^{(\boldsymbol{p})} \in C_i$ for $i = 1, 2, \ldots, 10$. By

$$\min\{1,\ t^P\} \le t^{p_k} \le \max\{1,\ t^P\}$$

it is easy to see that for any $\boldsymbol{p} = (p_k)$ we have

$$\Phi^{(\boldsymbol{p})} \in C_i \quad (i = 5, 6, 8, 10) \qquad \text{and} \qquad \Phi^{(\boldsymbol{p})} \notin C_i \quad (i = 1, 2, 4, 9).$$

Further, from the definitions of the sets $C_0$ and $C_3$ it follows that

$$\Phi^{(\boldsymbol{p}/r)} \in C_0 \iff \boldsymbol{p} \in \mathcal{P}_0^q \qquad \text{and} \qquad \Phi^{(\boldsymbol{p})} \in C_3 \iff \boldsymbol{p} \in \mathcal{P}_1,$$

where

$$\begin{aligned} \mathcal{P}_0^q = \big\{\boldsymbol{p} = (p_k) \colon \exists (a_k) \in \ell^+ \ \ \exists k_0 \in \mathbb{N} \ \ \exists \gamma \ge 0 \ \ \exists \delta > 0 \\ t^{p_k} \le a_k + \gamma t^q \quad (\forall k \ge k_0,\ t \in [0, \delta])\big\}, \\ \mathcal{P}_1 = \{\boldsymbol{p} = (p_k) \colon \inf_k p_k > 0\}. \end{aligned}$$

We claim that the $\varphi$-function sequence $\Phi^{(\boldsymbol{p})}$ from $C_7$ are also characterized by $\boldsymbol{p} \in \mathcal{P}_1$. Indeed, for $t \ge 1$ and $k_0 \in \mathbb{N}$ we have

$$\sup_{n \ge k_0} \inf_{k \ge n} t^{p_k} = t^{\sup\limits_{n \ge k_0} \inf\limits_{k \ge n} p_k},$$

which gives that $\Phi^{(\boldsymbol{p})} \in C_7$ if and only if

$$\exists k_0 \in \mathbb{N} \quad \sup_{n \ge k_0} \inf_{k \ge n} p_k > 0. \tag{1.3.4}$$

It is clear that $\inf_k p_k > 0$ yields (1.3.4). Indeed,

$$0 < \inf_k p_k \le \sup_{n \ge k_0} \inf_{k \ge n} p_k.$$

Conversely, let (1.3.4) be true. If $\boldsymbol{p} \notin \mathcal{P}_1$, then for some index sequence $(k_i)$ we have $\lim_i p_{k_i} = 0$, contrary to (1.3.4).

Consequently, from Theorems 1.3.7, 1.3.9 and 1.3.12 we get

**Corollary 1.3.19.** *Let $0 < q \leq \infty$ and let $\boldsymbol{p} = (p_k)$ be a bounded sequence of positive numbers. Then*

(1) $\ell_q \subset \ell(\boldsymbol{p}) \iff \boldsymbol{p} \in \mathcal{P}_0^q$;

(2) $\ell_q \subset c_0(\boldsymbol{p}) \iff \boldsymbol{p} \in \mathcal{P}_1$;

(3) $c_0(\boldsymbol{p}) \subset c_0$ *and* $\ell_\infty \subset \ell_\infty(\boldsymbol{p})$;

(4) $c_0(\boldsymbol{p}) = c_0 \iff \ell_\infty(r) = \ell_\infty \iff \boldsymbol{p} \in \mathcal{P}_1$.

Corollary 1.3.19 shows that $\ell \subset \ell(\boldsymbol{p})$ if and only if $\boldsymbol{p} \in \mathcal{P}_0^1$. A different necessary and sufficient condition for the inclusion $\ell \subset \ell(\boldsymbol{p})$ is contained in a (more general) result of Maddox (see [32], Theorem 1).

Let $\Phi = (\varphi_k)$ be a sequence of moduli. Kolk [21] considered the classes $C_4$, $C_5$, $C_9$ and $C_{10}$. It is clear by Lemmas 1 and 2 of [20], that the classes $C_5$ and $C_8$ coincide with the classes $C_6$ and $C_{10}$, respectively. The class $C_3$ can be formulated as follows

$$\left\{\Phi = (\varphi_k)\colon \lim_{t\to 0+} \sup_k \varphi_k(t) = 0\right\}.$$

So, from our Theorems 1.3.7 (4)–(7) and 1.3.10–1.3.12 it follows known Theorems 1, 2, 4, 5 and B of [21].

# Chapter 2

# Topologization of sequence spaces defined by moduli

Main results of this chapter (see Sections 2.3.1 and 2.4) are published in [35].

## 2.1 Topological sequence spaces

It is known that the classical sequence spaces $\ell_\infty$, $c_0$ and $\ell_p$ $(1 \leq p < \infty)$ are topologized by norms

$$\|x\|_{\ell_\infty} = \|x\|_{c_0} = \sup_k |x_k|$$

and

$$\|x\|_{\ell_p} = \left(\sum_k |x_k|^p\right)^{1/p},$$

respectively. By the topologization of sequence spaces defined by moduli there appear F-seminorms (or paranorms) instead of norms.

Recall that an F-*seminorm* $g$ on a vector space $V$ is a functional $g\colon V \to \mathbb{R}$ satisfying, for all $x, y \in V$, the axioms

(N1) $g(0) = 0$,

(N2) $g(x + y) \leq g(x) + g(y)$,

(N3) $g(\alpha x) \le g(x)$ for all scalars $\alpha$ with $|\alpha| \le 1$,

(N4) $\lim_n g(\alpha_n x) = 0$ for every scalar sequence $(\alpha_n)$ with $\lim_n \alpha_n = 0$.

A *paranorm* on $V$ is a functional $g\colon V \to \mathbb{R}$ satisfying (N1), (N2) and

(N5) $g(-x) = g(x)$,

(N6) $\lim_n g(\alpha_n x_n - \alpha x) = 0$ for every scalar sequence $(\alpha_n)$ with $\lim_n \alpha_n = \alpha$ and every sequence $(x_n)$ with $\lim_n g(x_n - x) = 0 \quad (x_n, x \in V)$.

An *Frechet norm* (or *F-norm*) is an F-seminorm with the condition

(N5) $g(x) = 0 \Rightarrow x = 0$.

A *Banach space* (or B-*space*) is a complete normed space. The topological sequence space in which all coordinate functionals $\pi_k$, $\pi_k(x) = x_k$, are continuous, is called a K-*space*. A BK-*space* is defined as a K-space which is also a B-space.

An F-seminorm $g$ on a sequence space $\lambda$ is said to be *absolutely monotone* if $g(y) \le g(x)$ for all $x = (x_k)$, $y = (y_k)$ from $\lambda$ with $|y_k| \le |x_k| \quad (k \in \mathbb{N})$.

An F-seminormed sequence space $(\lambda, g)$ is called an AK-*space* if $e^k \in \lambda$ $(k \in \mathbb{N})$ and for any $x = (x_k) \in \lambda$,

$$\lim_m \sum_{k=1}^{m} x_k e^k = x,$$

where $e^k = (\delta_{ki})_{i \in \mathbb{N}} \quad (k \in \mathbb{N})$ with $\delta_{ki} = 1$ if $k = i$ and $\delta_{ki} = 0$ otherwise.

## 2.2 Spaces of double sequences

Let $S$ be the vector space of all real or complex double sequences with the vector space operations defined coordinatewise. Vector subspaces of $S$ are called *double sequence spaces*. Some examples of such spaces can be found in [4].

A double sequence space $\Lambda$ is called *solid* if $(x_{ki}) \in \Lambda$ and $|y_{ki}| \le |x_{ki}| \quad (k, i \in \mathbb{N})$ yield $(y_{ki}) \in \Lambda$. For example, the sets

$$\mathcal{M}_b = \left\{ X = (x_{ki}) \in S\colon\ \sup_i |x_{ki}| < \infty\ (k \in \mathbb{N}) \right\},$$
$$\mathcal{M}_u = \left\{ X = (x_{ki}) \in S\colon\ \sup_{k,i} |x_{ki}| < \infty \right\},$$

$$W_\infty^p[\mathfrak{B}] = \left\{ X = (x_{ki}) \in S\colon\ \sup_{n,i} |\sigma_{ni}(X)| < \infty \right\},$$
$$W_0^p[\mathfrak{B}] = \left\{ X = (x_{ki}) \in W_\infty^p[\mathfrak{B}]\colon\ \lim_n \sigma_{ni}(X) = 0 \ \text{ uniformly in } i \right\}$$

are solid double sequence spaces, where $\mathfrak{B} = (B_i)$ is the sequence of infinite scalar matrices $B_i = (b_{nk}(i))$ with $b_{nk}(i) \geq 0 \quad (n, k, i \in \mathbb{N})$, $p > 0$ and

$$\sigma_{ni}(X) = \sum_k b_{nk}(i)|x_{ki}|^p.$$

Let $\mathcal{F} = (f_{ki})$ be a matrix of $\varphi$-functions and let $\mathcal{F}(|x|) = (f_{ki}(|x_k|)) = (f_{ki}(|x_k|))_{k,i\in\mathbb{N}}$. For a double sequence space $\Lambda$ we define the sets

$$\Lambda^\varrho(\mathcal{F}) = \{x = (x_k) \in \omega\colon \mathcal{F}(x/\varrho) \in \Lambda\} \qquad (\varrho > 0),$$
$$\Lambda^\exists(\mathcal{F}) = \{x = (x_k) \in \omega\colon \exists\ \varrho > 0 \quad \mathcal{F}(x/\varrho) \in \Lambda\},$$
$$\Lambda^\forall(\mathcal{F}) = \{x = (x_k) \in \omega\colon \mathcal{F}(x/\varrho) \in \Lambda \quad (\forall\ \varrho > 0)\}.$$

We write $\Lambda(\mathcal{F})$ instead of $\Lambda^1(\mathcal{F})$. It is clear that (cf. (1.2.1))

$$\Lambda^\forall(\mathcal{F}) \subset \Lambda^\varrho(\mathcal{F}) \subset \Lambda^\exists(\mathcal{F}).$$

**Definition 2.2.1.** A matrix of $\varphi$-functions $\mathcal{F} = (f_{ki})$ is said to satisfy *uniform $\Delta_2$-condition* if there exists a constant $K > 0$ such that

$$f_{ki}(2t) \leq K f_{ki}(t) \qquad (k, i \in \mathbb{N},\ t > 0).$$

Analogously to Proposition 1.2.3 we can prove

**Proposition 2.2.2.** *Let $\Lambda$ be a solid double sequence space and $\varrho > 0$. If the matrix of $\varphi$-functions $\mathcal{F} = (f_{ki})$ satisfies uniform $\Delta_2$-condition, then*

$$\Lambda^\forall(\mathcal{F}) = \Lambda^\varrho(\mathcal{F}) = \Lambda^\exists(\mathcal{F}).$$

Since the uniform $\Delta_2$-condition holds (with $K = 2$) for every matrix of moduli $\mathcal{F} = (f_{ki})$, by Proposition 2.2.2, in this case it is sufficient to consider only the set $\Lambda(\mathcal{F})$.

It is not difficult to see (cf. Proposition 1.2.5) that $\Lambda(\mathcal{F})$ is a solid sequence space whenever the double sequence space $\Lambda$ is solid.

## 2.3 The topologization of sequence spaces defined by a matrix of moduli

An essential problem in the theory of sequence spaces is the topologization of various vector spaces of sequences. For example, if $\Phi = (\varphi_k)$ is a sequence of moduli and $\lambda$ is an F-seminormed (paranormed) solid sequence space, then the linear space $\lambda(\Phi)$ may be topologized by an F-seminorm (paranorm) under some restrictions on the sequence $\Phi = (\varphi_k)$ or on the space $(\lambda, g)$ (see [22, 23, 50]).

Let $\Lambda$ be a double sequence space and $\mathcal{F} = (f_{ki})$ be a matrix of moduli. We consider the set

$$\Lambda(\mathcal{F}) = \{x = (x_k) \in \omega : \mathcal{F}(x) = (f_{ki}(|x_k|)) \in \Lambda\}.$$

Our purpose is to describe the topology of the sequence space $\Lambda(\mathcal{F})$.

### 2.3.1 Topologization of $\Lambda(\mathcal{F})$

Let $\Lambda$ be a double sequence space and let $g$ be an F-seminorm on $\Lambda$.

**Definition 2.3.1.** An F-seminorm $g$ on a double sequence space $\Lambda$ is said to be *absolutely monotone* if for all $X = (x_{ki})$ and $Y = (y_{ki})$ from $\Lambda$ with $|y_{ki}| \leq |x_{ki}|$ $(k, i \in \mathbb{N})$ we have $g(Y) \leq g(X)$.

Now we can describe the topology of the sequence space $\Lambda(\mathcal{F})$ defined by a matrix of moduli $\mathcal{F} = (f_{ki})$.

**Theorem 2.3.2.** *Let $(\Lambda, g)$ be an F-seminormed solid double sequence space. If $g$ is absolutely monotone and the matrix of moduli $\mathcal{F} = (f_{ki})$ satisfies the condition*

(M1) $\lim_{u\to 0+} \sup_{t>0} \sup_{k,i} \dfrac{f_{ki}(ut)}{f_{ki}(t)} = 0,$

*then the functional* $g_{\mathcal{F}}$ *defined by*

$$g_{\mathcal{F}}(x) = g(\mathcal{F}(x)) \quad (x \in \Lambda(\mathcal{F}))$$

*is an absolutely monotone F-seminorm on* $\Lambda(\mathcal{F})$.

*Proof.* Let $g$ be an absolutely monotone F-seminorm on $\Lambda$ and let $\mathcal{F} = (f_{ki})$ satisfy (M1).

First we prove that $g_{\mathcal{F}}$ is an F-seminorm, i.e., $g_{\mathcal{F}}$ satisfies the axioms (N1)–(N4). Since $g$ is an F-seminorm, (N1) holds by (i). The axiom (N2) follows immediately from the subadditivity of $g$ and $f_{ki}$ $(k, i \in \mathbb{N})$ because $g$ is an absolutely monotone F-seminorm and the functions $f_{ki}$ $(k, i \in \mathbb{N})$ satisfy the property (iii).

If $|\alpha| \leq 1$ $(\alpha \in \mathbb{K})$, then $|\alpha x_k| \leq |x_k|$ $(k \in \mathbb{N})$ and by (iii) we may write

$$f_{ki}(|\alpha x_k|) \leq f_{ki}(|x_k|) \quad (k, i \in \mathbb{N}).$$

So, since $g$ is absolutely monotone, we get

$$g_{\mathcal{F}}(\alpha x) = g\left(\left(f_{ki}(|\alpha x_k|)\right)\right) \leq g\left(\left(f_{ki}(|x_k|)\right)\right) = g_{\mathcal{F}}(x),$$

i.e., (N3) is valid.

To prove (N4), let $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$ and $x = (x_k) \in \Lambda(\mathcal{F})$. Since $f_{ki}(t) > 0$ $(k, i \in \mathbb{N})$ for $t > 0$ and $f_{ki}(|\alpha_n x_k|) = 0$ for $k \in K_0 = \{k \in \mathbb{N} : x_k = 0\}$, $i \in \mathbb{N}$, we have

$$f_{ki}(|\alpha_n x_k|) \leq h_n f_{ki}(|x_k|) \quad (k, i, n \in \mathbb{N}), \tag{2.3.1}$$

where

$$h_n = \sup_{k \notin K_0} \sup_i \frac{f_{ki}(|\alpha_n x_k|)}{f_{ki}(|x_k|)}.$$

While

$$h_n \leq \sup_{t>0} \sup_{k \notin K_0} \sup_i \frac{f_{ki}(|\alpha_n|\, t)}{f_{ki}(t)},$$

by condition (M1) we see that $h_n \longrightarrow 0$, as $n \to \infty$. Since $g$ is absolutely monotone, we get

$$g(\mathcal{F}(\alpha_n x)) = g\left(\left(f_{ki}(|\alpha_n x_k|)\right)\right) \leq g\left(h_n\left(f_{ki}(|x_k|)\right)\right) = g(h_n \mathcal{F}(x)) \tag{2.3.2}$$

by (2.3.1). Now, using that $g$ satisfies (N4), we have

$$\lim_n g(h_n \mathcal{F}(x)) = 0,$$

which, together with (2.3.2), gives

$$\lim_n g_{\mathcal{F}}(\alpha_n x) = \lim_n g(\mathcal{F}(\alpha_n x)) = 0.$$

Thus $g_{\mathcal{F}}$ is an F-seminorm on $\Lambda(\mathcal{F})$.

Finally, let $x = (x_k), \; y = (y_k)$ be in $\Lambda(\mathcal{F})$ and $|y_k| \leq |x_k| \quad (k \in \mathbb{N})$. Then

$$f_{ki}(|y_k|) \leq f_{ki}(|x_k|) \quad (k, i \in \mathbb{N})$$

and since $g$ is absolutely monotone,

$$g_{\mathcal{F}}(y) = g\left(\left(f_{ki}(|y_k|)\right)\right) \leq g\left(\left(f_{ki}(|x_k|)\right)\right) = g_{\mathcal{F}}(x).$$

Hence $g_{\mathcal{F}}$ is absolutely monotone F-seminorm and the proof is completed. □

In the following we apply Theorem 2.3.2 for the topologization of the sequence space

$$\lambda(\Phi) = \left\{x = (x_k) \in \omega \colon \Phi(x) = (\varphi_k(|x_k|)) \in \lambda\right\},$$

where $(\lambda, g)$ is an F-seminormed space and $\Phi = (\varphi_k)$ is a sequence of moduli. For this reason we consider the space $\Lambda_\lambda(\mathcal{F}_\Phi)$, where $\mathcal{F}_\Phi = (f_{ki}^\Phi)$ is the matrix with the elements

$$f_{ki}^\Phi(t) = \varphi_k(t) \quad (i \in \mathbb{N})$$

and $\Lambda_\lambda$ is the space of double sequences $X^x = (x_{ki}^x)$ with $x_{ki}^x = x_k$ $(i \in \mathbb{N}, \; x = (x_k) \in \lambda)$. If now $\lambda$ is solid and $g$ is absolutely monotone, then $\Lambda_\lambda$ is also solid and $g_\lambda$,

$$g_\lambda(X^x) = g(x) \quad (x \in \lambda),$$

clearly defines an absolutely monotone F-seminorm on $\Lambda_\lambda$. So from Theorem 2.3.2 we immediately get

**Proposition 2.3.3** ([50], Theorem 3; [22], Theorem 1)**.** *Let* $(\lambda, g)$ *be an F-seminormed space. If* $g$ *is absolutely monotone and the sequence of moduli* $\Phi = (\varphi_k)$ *satisfies one of equivalent conditions* (M) *and* (M′), *where*

(M) *there exists a function* $\nu$ *such that* $\varphi_k(ut) \le \nu(u)\varphi_k(t)$ $(0 \le u < 1, t \ge 0)$ *and* $\lim\limits_{u\to 0+} \nu(u) = 0$,

(M′) $\lim\limits_{u\to 0+} \sup\limits_{t>0} \sup\limits_{k} \dfrac{\varphi_k(ut)}{\varphi_k(t)} = 0$,

*then* $g_\Phi$ *is an absolutely monotone F-seminorm on* $\lambda(\Phi)$.

**Remark 2.3.4.** The equivalence of (M) and (M′) is proved in [22].

### 2.3.2 Topologization of $\Lambda(\mathcal{F})$ for AK-space $\Lambda$

Let $\Lambda$ be a double sequence space and $\mathcal{F} = (f_{ki})$ be a matrix of moduli. In Section 1.3.4 it was proved, that the functsions $\tilde{f}_k$, where

$$\tilde{f}_k(t) = \sup_i f_{ki}(t) < \infty \ (k \in \mathbb{N}, t > 0),$$

satisfy conditions (i) and (iii). Besides this we assume that

(vi) $\tilde{f}_k$ $(k \in \mathbb{N})$ is continuous from the right at zero.

It is not difficult to see that $\tilde{f}_k$ satisfies also the condition (ii). Indeed, since the functions $f_{ki}$ are moduli, for all $t, u \ge 0$ we get

$$\tilde{f}_k(t+u) \le \sup_i f_{ki}(t+u) \le \sup_i f_{ki}(t) + \sup_i f_{ki}(u) = \tilde{f}_k(t) + \tilde{f}_k(u).$$

Thus $\tilde{f}_k$ satisfy condition (ii). Hence $\tilde{f}_k$ $(k \in \mathbb{N})$ are moduli.

Let $\mathcal{E}^k = (e^k_{ji})$ $(k \in \mathbb{N})$ be a double sequence with the elements

$$e^k_{ji} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases} \quad (i \in \mathbb{N}).$$

For a double sequence $X = (x_{ki})$ we define its sections by

$$X^{[n]} = \sum_{k=1}^{n} (x_{ki})_i \ \mathcal{E}^k \qquad (n \in \mathbb{N}),$$

where $(x_{ki})_i \ \mathcal{E}^k = (x_{ki} \ e^k_{ij})_{i,j\in\mathbb{N}}$.

**Definition 2.3.5.** An F-seminormed double sequence space $\Lambda$ is called an AK-*space* if $\mathcal{E}^k \in \Lambda$ and for all $X = (x_{ki}) \in \Lambda$,

$$\lim_n X^{[n]} = X.$$

**Theorem 2.3.6.** *Let $(\Lambda, g)$ be a solid F-seminormed AK-space. If $g$ is absolutely monotone, then $g_{\mathcal{F}}$ is an absolutely monotone F-seminorm on $\Lambda(\mathcal{F})$ for every matrix of moduli $\mathcal{F} = (f_{ki})$ satisfying* (1.3.3) *and* (vi)*. If $(\Lambda, g)$ is an AK-space, then $(\Lambda(\mathcal{F}), g_{\mathcal{F}})$ is also an AK-space.*

*Proof.* The functional $g_{\mathcal{F}}$ satisfies the axioms (N1)–(N3) by Theorem 2.3.2. To prove (N4), let $\lim_n \alpha_n = 0 \ \ (\alpha_n \in \mathbb{K})$ and $x = (x_k) \in \Lambda(\mathcal{F})$. Since $\mathcal{F}(x) \in \Lambda$ and $(\Lambda, g)$ is an AK-space, then

$$\lim_n \mathcal{F}(x)^{[n]} = \mathcal{F}(x) \tag{2.3.3}$$

in $\Lambda$ by Definition 2.3.5, where

$$\mathcal{F}(x)^{[n]} = \sum_k (f_{ki}(|x_k|))_i \ \mathcal{E}^k \qquad (n \in \mathbb{N}).$$

Let $x^{[n]} = (x_k^{[n]}) \in \Lambda(\mathcal{F}) \ (n \in \mathbb{N})$. Using the equality

$$\mathcal{F}(x - x^{[n]}) = \mathcal{F}(x) - \mathcal{F}(x)^{[n]}, \tag{2.3.4}$$

for given $\varepsilon > 0$ we can find an index $m \in \mathbb{N}$ such that

$$g_{\mathcal{F}}(x - x^{[m]}) = g(\mathcal{F}(x - x^{[m]})) < \frac{\varepsilon}{2}. \tag{2.3.5}$$

For all $i \in \mathbb{N}$, by (1.3.3) we have

$$f_{ki}(|\alpha_n x_k|) \leq \tilde{f}_k(|\alpha_n x_k|).$$

While $\lim_n \alpha_n = 0$, the moduli $\tilde{f}_k$ $(k \in \mathbb{N})$ are continuous and (1.3.3) is true, we get

$$\lim_n \tilde{f}_k(|\alpha_n x_k|) = 0.$$

Therefore, since $g$ satisfies (N4),

$$\lim_n g(\tilde{f}_k(|\alpha_n x_k|) \ \mathcal{E}^k) = 0 \tag{2.3.6}$$

for all $k \in \mathbb{N}$. Moreover, since $g$ is absolutely monotone, then

$$g((f_{ki}(|\alpha_n x_k|))_i \mathcal{E}^k) \leq g(\tilde{f}_k(|\alpha_n x_k|)\ \mathcal{E}^k). \tag{2.3.7}$$

By (N2), (2.3.7) and $\mathcal{F}(x^{[m]}) = \mathcal{F}(x)^{[m]}$ we conclude

$$\begin{aligned} 0 \leq g_{\mathcal{F}}(\alpha_n x^{[m]}) &= g\left(\mathcal{F}(\alpha_n x^{[m]})\right) = g\left(\mathcal{F}(\alpha_n x)^{[m]}\right) \\ &= g\left(\sum_{k=1}^{m} (f_{ki}(|\alpha_n x_k|))_i\ \mathcal{E}^k\right) \leq \sum_{k=1}^{m} g\left((f_{ki}(|\alpha_n x_k|))_i\ \mathcal{E}^k\right) \\ &\leq \sum_{k=1}^{m} g(\tilde{f}_k(|\alpha_n x_k|)\mathcal{E}^k). \end{aligned}$$

From (2.3.6) it follows that

$$\lim_n g_{\mathcal{F}}(\alpha_n x^{[m]}) = 0.$$

So, there exists an index $n_0$ such that, for $n \geq n_0$, we have

$$g_{\mathcal{F}}(\alpha_n x^{[m]}) < \frac{\varepsilon}{2} \tag{2.3.8}$$

and $|\alpha_n| \leq 1 \quad (n \geq n_0)$ because $\lim_n \alpha_n = 0$. Now, since $g_{\mathcal{F}}$ satisfies (N3), we get

$$g_{\mathcal{F}}(\alpha_n x - \alpha_n x^{[m]}) = g_{\mathcal{F}}(\alpha_n (x - x^{[m]})) \leq g_{\mathcal{F}}(x - x^{[m]}) \tag{2.3.9}$$

for $n \geq n_0$. From (2.3.5), (2.3.8) and (2.3.9) by (F2) we deduce

$$\begin{aligned} g_{\mathcal{F}}(\alpha_n x) &= g_{\mathcal{F}}(\alpha_n x - \alpha_n x^{[m]} + \alpha_n x^{[m]}) \\ &\leq g_{\mathcal{F}}(\alpha_n x - \alpha_n x^{[m]}) + g_{\mathcal{F}}(\alpha_n x^{[m]}) \\ &\leq g_{\mathcal{F}}(x - x^{[m]}) + g_{\mathcal{F}}(\alpha_n x^{[m]}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for $n \geq n_0$. This implies

$$\lim_n g_{\mathcal{F}}(\alpha_n x) = 0.$$

So, (F4) is true for $g_{\mathcal{F}}$.

In Theorem 2.3.2 it was already proved that $g_{\mathcal{F}}$ is absolutely monotone.

Finally, using (2.3.3), by (2.3.4) we have

$$\lim_n g_{\mathcal{F}}(x - x^{[n]}) = \lim_n g(\mathcal{F}(x - x^{[n]})) = \lim_n g(\mathcal{F}(x) - \mathcal{F}(x^{[n]}))$$
$$= \lim_n g(\mathcal{F}(x) - \mathcal{F}(x)^{[n]}) = 0$$

for all $x \in \Lambda(\mathcal{F})$. Thus $\Lambda(\mathcal{F})$ is an AK-space and the proof is completed. □

If $\lambda$ is a solid AK-space and $g$ is an absolutely monotone F-seminorm on $\lambda$, then by definition of $\Lambda_\lambda$ it is clear that $g_\lambda$ is an absolutely monotone F-seminorm on $\Lambda_\lambda$ and $\Lambda_\lambda$ is a solid AK-space. Since $\mathcal{F}_\Phi = (f_{ki}^\Phi)$ is a matrix of moduli with

$$\sup_i f_{ki}^\Phi(t) = \sup_i \varphi_k(t) = \varphi_k(t) < \infty \quad (k \in \mathbb{N},\ t > 0),$$

condition (1.3.3) is satisfied. The condition (vi) also holds, because the moduli $\varphi_k$ $(k \in \mathbb{N})$ are continuous from the right at zero.

Now, from Theorem 2.3.6 we get

**Proposition 2.3.7** ([22], Theorem 2). *Let $(\lambda, g)$ be an F-seminormed AK-space. If $g$ is absolutely monotone, then $g_\Phi$ is an absolutely monotone F-seminorm on $\lambda(\Phi)$ for an arbitrary sequence of moduli $\Phi = (\varphi_k)$. Moreover, $(\lambda(\Phi), g_\Phi)$ is an AK-space.*

## 2.4 Spaces of strongly summable sequences

For a sequence $\mathfrak{B} = (B_i)$ of infinite scalar matrices $B_i = (b_{nk}(i))$ with $b_{nk}(i) \geq 0$ $(n, k, i \in \mathbb{N})$ we consider the spaces $W_\infty^p[\mathfrak{B}]$ and $W_0^p[\mathfrak{B}]$ of strongly $\mathfrak{B}$-bounded and strongly $\mathfrak{B}$-summable to zero double sequences, respectively, which were defined in Section 2.2.

It is easy to prove that for $p \geq 1$ the functional $g_{\mathfrak{B}}^p$, where

$$g_{\mathfrak{B}}^p(X) = \sup_{n,i} \left(\sigma_{ni}(X)\right)^{1/p},$$

is an absolutely monotone seminorm on $W_\infty^p[\mathfrak{B}]$ and $W_0^p[\mathfrak{B}]$.

Let $\mathcal{F} = (f_{ki})$ be a matrix of moduli and $p \geq 1$. We define the sequence spaces

$$w_\infty^p[\mathfrak{B}, \mathcal{F}] = \{x = (x_k) : \mathcal{F}(x) \in W_\infty^p[\mathfrak{B}]\},$$
$$w_0^p[\mathfrak{B}, \mathcal{F}] = \{x = (x_k) \in w_\infty^p[\mathfrak{B}, \mathcal{F}] : \mathcal{F}(x) \in W_0^p[\mathfrak{B}]\}.$$

A sequence $x = (x_k)$ from $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ ($w_0^p[\mathfrak{B}, \mathcal{F}]$) is called *strongly* $\mathfrak{B}$-*bounded* (*strongly* $\mathfrak{B}$-*summable to zero*) with respect to the matrix of moduli $\mathcal{F}$.

Our purpose is to characterize the F-seminormability of $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ and $w_0^p[\mathfrak{B}, \mathcal{F}]$.

For the topologization of $w_\infty^p[\mathfrak{B}, \mathcal{F}]$, $w_0^p[\mathfrak{B}, \mathcal{F}]$ we introduce the functional $g_{\mathfrak{B},\mathcal{F}}^p$ defined by

$$g_{\mathfrak{B},\mathcal{F}}^p(x) = \sup_{n,i} \left( \sum_k b_{nk}(i)(f_{ki}(|x_k|))^p \right)^{1/p}.$$

The sequence spaces $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ and $w_0^p[\mathfrak{B}, \mathcal{F}]$ are the spaces of type $\Lambda(\mathcal{F})$ with $\Lambda = W_\infty^p[\mathfrak{B}]$ and $\Lambda = W_0^p[\mathfrak{B}]$, respectively. In addition, $g_{\mathfrak{B},\mathcal{F}}^p = (g_{\mathfrak{B}}^p)_{\mathcal{F}}$. Since every seminorm is also an F-seminorm, from Theorem 2.3.2 we immediately get

**Corollary 2.4.1.** *Let $p \geq 1$. If the matrix of moduli $\mathcal{F} = (f_{ki})$ satisfies the condition* (M1), *then $g_{\mathfrak{B},\mathcal{F}}^p$ is an absolutely monotone F-seminorm on $w_\infty^p[\mathfrak{B}, \mathcal{F}]$ and $w_0^p[\mathfrak{B}, \mathcal{F}]$.*

**Remark 2.4.2.** It should be noted that $W_0^p[\mathfrak{B}_1]$ is not an AK-space in general. Indeed, let $\mathfrak{B}_1 = (B_i^1)$ be the sequence of infinite scalar matrices $B_i^1 = (b_{nk}^1(i))$ with the elements

$$b_{nk}^1(i) = \begin{cases} n^{-1} & \text{if } i \leq k < i + n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(W_0^p[\mathfrak{B}_1], g_{\mathfrak{B}_1}^p)$ is not an AK-space (cf. [23], p. 68).

For any constant sequence $\mathfrak{B} = (A)$, where $A = (a_{nk})$ is a non-negative matrix, and for a sequence of moduli $\Phi = (\varphi_k)$ we consider the space

$$w_0^p[A, \Phi] = \left\{ x = (x_k) \colon \lim_n \sum_k a_{nk}(\varphi_k(|x_k|))^p = 0 \right\} \quad (p \geq 1),$$

which is an one-dimensional analog of $W_0^p[\mathfrak{B}]$ and which can be topologized by the F-seminorm

$$g^p_{A,\Phi}(x) = \sup_n \left( \sum_k a_{nk} (\varphi_k(|x_k|))^p \right)^{1/p} .$$

Indeed, since

$$w_0^p[A] = \left\{ x = (x_k) \colon \lim_n \sum_k a_{nk} |x_k|^p = 0 \right\}$$

is an solid AK-space with respect to absolutely monotone seminorm

$$g_A^p(x) = \sup_n \sum_k a_{nk} |x_k|^p$$

and $g^p_{A,\Phi} = (g^p_A)_\Phi$, from Proposition 2.3.7 we get

**Corollary 2.4.3** ([23], Corollary 3)**.** *Let $p \geq 1$, $A = (a_{nk})$ be a non-negative matrix, and $\Phi = (\varphi_k)$ be a sequence of moduli. Then the space $w_0^p[A, \Phi]$ is an F-seminormed AK-space with respect to the absolutely monotone F-seminorm $g^p_{A,\Phi}$.*

Corollary 2.4.3 extends Theorem 1 of Bilgin [3].

Let $A = (a_{nk})$ be an infinite matrix of non-negative numbers, $\boldsymbol{p} = (p_k)$ a bounded sequence of positive numbers and $r = \max\{1, \ \sup_k p_k\}$. For a sequence of moduli $\Phi = (\varphi_k)$, following Esi [12], we consider the sequence spaces

$$w_\infty[A, \boldsymbol{p}, \Phi] = \left\{ x = (x_k) : \sup_{n,i} s_{ni}(x) < \infty \right\}$$

and

$$w_0[A, \boldsymbol{p}, \Phi] = \left\{ x \in w_\infty[A, \boldsymbol{p}, \Phi] : \lim_n s_{ni}(x) = 0 \text{ uniformly in } i \right\},$$

where

$$s_{ni}(x) = \sum_k a_{nk} \left(\varphi_k(|x_{k+i-1}|)\right)^{p_k} = \sum_{k=i}^{\infty} a_{n,k-i+1} \left(\varphi_{k-i+1}(|x_k|)\right)^{p_{k-i+1}} .$$

Nanda [39] examined similar to $w_\infty[A,\boldsymbol{p},\Phi]$ and $w_0[A,\boldsymbol{p},\Phi]$ sequence spaces. Theorem 3 of Esi [12] asserts that the functional $g_{A,\boldsymbol{p},\Phi}$, where

$$g_{A,\boldsymbol{p},\Phi}(x) = \sup_{n,i}(s_{ni}(x))^{1/r},$$

is a paranorm on $w_0[A,\boldsymbol{p},\Phi]$ for an arbitrary sequence of moduli $\Phi = (\varphi_k)$. But it seems that this is not true in general. In fact, if $A = C_1$, the matrix of arithmetical means, $\Phi = (\varphi_k)$ is a constant sequence of moduli, i.e., $\varphi_k = \varphi \quad (k \in \mathbb{N})$ and $p_k = 1 \quad (k \in \mathbb{N})$, then Corollary 2 of [23] shows that the functional $g_{A,\boldsymbol{p},\Phi}$ is not a paranorm on $w_0[A,\boldsymbol{p},\Phi]$ whenever $\varphi$ is bounded. Consequently, theorem of Esi can't be true without restrictions on the sequence of moduli $\Phi = (\varphi_k)$.

The sequence space $w_0[A,\boldsymbol{p},\Phi]$ can be considered as a space of type $\Lambda(\mathcal{F})$. Indeed, defining the matrix of moduli $\mathcal{F}^{\boldsymbol{p}} = (f^{\boldsymbol{p}}_{ki})$ by

$$f^{\boldsymbol{p}}_{ki}(t) = \begin{cases} (\varphi_{k-i+1}(t))^{(p_{k-i+1})/r} & \text{if } k \geq i, \\ t & \text{if } k < i, \end{cases} \tag{2.4.1}$$

we can write

$$w_0[A,\boldsymbol{p},\Phi] = \left(W_0^r[\mathfrak{B}]\right)(\mathcal{F}^{\boldsymbol{p}}),$$

where $B_i$ are matrices with the elements

$$b_{nk}(i) = \begin{cases} a_{n,k-i+1} & \text{if } k \geq i, \\ 0 & \text{if } k < i. \end{cases}$$

Since, moreover, $g_{A,\boldsymbol{p},\Phi} = (g_A^r)_{\mathcal{F}^{\boldsymbol{p}}}$, from Theorem 2.3.2 we get

**Corollary 2.4.4.** *If the sequence of moduli $\Phi = (\varphi_k)$ satisfies the condition*

$$\text{(M2)} \quad \lim_{u\to 0+} \sup_{t>0} \sup_{k} \left(\frac{\varphi_k(ut)}{\varphi_k(t)}\right)^{p_k} = 0,$$

*then $g_{A,\boldsymbol{p},\Phi}$ is an absolutely monotone F-seminorm on $w_0[A,\boldsymbol{p},\Phi]$.*

Our Corollary 2.4.4 shows that $w_0[A,\boldsymbol{p},\Phi]$ can be topologized by the F-seminorm $g_{A,\boldsymbol{p},\Phi}$ if the sequence of moduli $\Phi = (\varphi_k)$ satisfies the restriction (M2). Since, every F-seminorm is also a paranorm, Corollary 2.4.4 can be considered as a correction of Theorem 3 of Esi [12].

**Example 2.4.5.** Let $(\Lambda, g)$ be a solid F-seminormed double sequence space. Defining $p_k = \frac{1}{3}\left(1 + \frac{1}{k}\right)$ and $\varphi_k(t) = t$ $(k \in \mathbb{N})$, we get $r = \max\{1, \sup_k p_k\} = 1$. By (2.4.1) we have the matrix of moduli $\mathcal{F}^{\boldsymbol{p}} = (f_{ki}^{\boldsymbol{p}})$ with the elements

$$f_{ki}^{\boldsymbol{p}}(t) = \begin{cases} t^{1/3(1+1/(k-i+1))} & \text{if } k \geq i, \\ t & \text{if } k < i. \end{cases}$$

Since

$$\sup_{t>0} \sup_{k,i} \frac{f_{ki}^{\boldsymbol{p}}(ut)}{f_{ki}^{\boldsymbol{p}}(t)} = \max\{u^{2/3}, u\},$$

the condition (M1) is fulfilled. Therefore, the functional $g_{\mathcal{F}^{\boldsymbol{p}}}$ is an absolutely monotone F-seminorm on the sequence space $\Lambda(\mathcal{F}^{\boldsymbol{p}})$ by Theorem 2.3.2.

# Chapter 3

# Superposition operators on sequence spaces defined by moduli

## 3.1 Superposition operators

In an implicit form, the superposition operator can be found in any calculus textbook (in the old terminology, as "composite operator", "function of a function", etc.), where some of its properties are described. Typical examples are the continuity of the superposition of continuous functions, the differentiability of the superposition of differentiable functions, and so on. The superposition operator occurs everywhere: in mathematical analysis, functional analysis, differential and integral equations, probability theory and statistics, variational calculus, and other fields of mathematics.

Let $\lambda$ and $\mu$ be two sequence spaces and let $f\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a function with $f(k, 0) = 0$ $(k \in \mathbb{N})$. A *superposition operator* (sometimes called also *outer superposition operator*, *composition operator*, *substitution operator*, or *Nemytskij operator*) $P_f\colon \lambda \to \mu$ is defined by

$$P_f(x) = (f(k, x_k)) \in \mu \quad (x = (x_k) \in \lambda).$$

In general the superposition operator $P_f$ is nonlinear. Some properties of this operator can be found in [1].

Superposition operators on sequence spaces are not studied so intensiv as on spaces of functions (see, for example, [1]). Characterization of $P_f$ on Orlicz sequence spaces was given by Robert [47] and Šragin [51]. The complete investigation of superpositsion operators on sequence spaces $\ell_\infty$, $c_0$ and $\ell_p$ for $1 \le p < \infty$ was given by Dedagich and Zabreĭko [10] (see also [8, 44]). The acting conditions for $P_f : w_0 \to \ell$ are proved in [7] by the assumption that the functions $f(k, \cdot)$ are continuous. The results of Šragin [51] contain characterizations of superposition operators on $\ell^\varrho(\Phi)$ and $\ell^\exists(\Phi)$, where $\Phi = (\varphi_k)$ is a sequence of $\varphi$-functions. Some authors [9, 10, 44, 45, 46, 49, 52, 53] have been studied continuity and boundedness of superposition operators in various sequence spaces. Recently, the gilding hump property has been used in the study of the continuity and boundedness of superposition operators by Lee [28], and by Unoningsih, Płuciennik and Yee [53].

Basing on results of Dedagich and Zabreĭko [10], and Płuciennik [45, 46] we can give necessary and sufficient conditions for the continuity and boundedness of superposition operators on sequence spaces defined by a sequence of moduli. Main results (see Sections 3.3 and 3.4) are published in [26, 36, 37].

## 3.2 Auxiliary results

In this section we formulate some definitions and known propositions, and prove a few lemmas which are needed in the proofs of main results.

Let $\Phi = (\varphi_k)$ and $\Psi = (\psi_k)$ be two sequences of moduli. In addition, we assume that the moduli $\varphi_k$ $(k \in \mathbb{N})$ are unbounded.

In some results we need the following conditions:

(B) the functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are bounded on every bounded subset of real numbers;

(C) the functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are continuous.

We start with the following known acting conditions for superposition operators $P_f$.

**Proposition 3.2.1** ([24], Theorems 3 and 4(C)). *Let* $0 < p, q < \infty$ *and* $\lambda \in \{c_0, \ell_p\}$. *A superposition operator* $P_f$ *maps* $\lambda(\Phi)$ *into* $\ell_q(\Psi)$ *if* *and only if there exist* $(a_k) \in \ell^+$, *numbers* $\gamma \geq 0$, $\delta > 0$ *and* $k_0 \in \mathbb{N}$ *such that*

$$(\psi_k(|f(k,t)|))^q \leq a_k + \gamma(\varphi_k(|t|))^p \quad (\varphi_k(|t|) \leq \delta,\ k \geq k_0). \tag{3.2.1}$$

*Here* $\gamma = 0$ *if* $\lambda = c_0$.

**Proposition 3.2.2** ([24], Theorem 4(B))**.** *Let* $0 < p < \infty$ *and* $\lambda \in \{c_0, \ell_p\}$. *Then* $P_f\colon \lambda(\Phi) \to \ell_q(\Psi)$ *if and only if there exist a sequence* $(a_k) \in c_0^+$ *and numbers* $\delta > 0$, $k_0 \in \mathbb{N}$ *such that*

$$(\psi_k(|f(k,t)|))^q \leq a_k \quad (\varphi_k(|t|) \leq \delta,\ k \geq k_0). \tag{3.2.2}$$

**Proposition 3.2.3** ([24], Theorem 5)**.** *Let* $0 < p < \infty$.

(1) $P_f\colon \ell_\infty(\Phi) \to \ell_q(\Psi)$ *if and only if for any* $\varrho > 0$ *there exists a sequence* $(a_k) \in \ell^+$ *such that for all* $k \in \mathbb{N}$

$$(\psi_k(|f(k,t)|))^q \leq a_k \quad (\varphi_k(|t|) \leq \varrho). \tag{3.2.3}$$

(2) $P_f\colon \ell_\infty(\Phi) \to c_0(\Psi)$ *if and only if for any* $\varrho > 0$ *there exist a sequence* $(a_k) \in c_0^+$ *and number* $k_0 \in \mathbb{N}$ *such that*

$$\psi_k(|f(k,t)|) \leq a_k \quad (\varphi_k(|t|) \leq \varrho,\ k \geq k_0). \tag{3.2.4}$$

(3) $P_f\colon \ell_\infty(\Phi) \to \ell_\infty(\Psi)$ *if and only if for any* $\varrho > 0$ *there exists a sequence* $(a_k) \in \ell_\infty^+$ *such that for all* $k \in \mathbb{N}$

$$\psi_k(|f(k,t)|) \leq a_k \quad (\varphi_k(|t|) \leq \varrho). \tag{3.2.5}$$

Basing on Proposition 1 of [24], Propositions 3.2.1 and 3.2.2 we can reformulate in the following way.

**Proposition 3.2.4.** *Let* $1 \leq p, q < \infty$, $\lambda \in \{c_0, \ell_p\}$ *and* $\mu \in \{c_0, \ell_q, \ell_\infty\}$. *A superposition operator* $P_f$ *maps* $\lambda(\Phi)$ *into* $\mu(\Psi)$ *if and only if there exist a sequence* $(a_k) \in \mu^+$, *numbers* $\gamma \geq 0$, $\delta > 0$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\psi_k(|f(k,t)|) \leq a_k + \gamma(\varphi_k(|t|))^{p/q} \quad (\varphi_k(|t|) \leq \delta,\ k \geq k_0). \tag{3.2.6}$$

*Here* $\gamma = 0$ *for all pairs* $\lambda$, $\mu$ *with* $\lambda \in \{c_0, \ell_p\}$ *and* $\mu \in \{c_0, \ell_\infty\}$ *or* $\lambda = c_0$ *and* $\mu = \ell_q$.

Kolk [24] characterized the superposition operators $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ if $0 < p, q < \infty$ and (C) holds.

Using, in addition, the remarks of Płuciennik ([45], Remark 1; [46], Remark 1) we may formulate

**Proposition 3.2.5.** *Let $1 \le p, q < \infty$, $\Phi = (\varphi_k)$ be a sequence of strictly increasing moduli and $\Psi = (\psi_k)$ a sequence of moduli. If there exist a number $\delta > 0$ and sequences $(a_k) \in \ell^+$ and $(c_i)_{i=0}^{\infty} \in \ell^+$ such that*

$$(\psi_k(|f(k,t)|))^q \le a_k + c_i 2^{-i}(\varphi_k(|t|))^p, \tag{3.2.7}$$

*whenever $(\varphi_k(|t|))^p \le 2^i\delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0 = \{0, 1, \ldots\}$, then $P_f$ acts $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$. Condition* (3.2.7) *is necessary for $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ whenever* (B) *is satisfied.*

Proposition 3.2.5 may be modified as follows.

**Proposition 3.2.6.** *Let $1 \le p, q < \infty$, $\Phi = (\varphi_k)$ be a sequence of strictly increasing moduli and $\Psi = (\psi_k)$ a sequence of moduli. If there exist a number $\delta > 0$ and sequences $(b_k) \in \ell_q^+$ and $(d_i)_{i=0}^{\infty} \in \ell_q^+$ such that*

$$\psi_k(|f(k,t)|) \le b_k + d_i 2^{-i/q}(\varphi_k(|t|))^{p/q}, \tag{3.2.8}$$

*whenever $\varphi_k(|t|) \le 2^{i/p}\delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$, then $P_f$ acts $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$. Condition* (3.2.8) *is necessary for $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ whenever* (B) *is satisfied.*

*Proof.* Let $a_k = b_k^q$ and $c_i = d_i^q$. If $1 \le q < \infty$, then (3.2.7) gives

$$\psi_k(|f(k,t)|) \le a_k^{1/q} + c_i^{1/q} 2^{-i/q}(\varphi_k(|t|))^{p/q}, \tag{3.2.9}$$

whenever $\varphi_k(|t|) \le 2^{i/p}\delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$. So, we get (3.2.8).

Conversely, by (1.1.1) it is not difficult to see that (3.2.8) yields $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$. □

To investigate the continuity and boundedness of superposition operators we introduce certain F-seminorm topologies on the sequence spaces defined by moduli. If $(\lambda, g)$ is an F-seminormed space, then for the topologization of $\lambda(\Phi)$ it is natural to consider the functional $g_\Phi$ defined by

$$g_\Phi(x) = g(\Phi(x)).$$

Then the topology on $\lambda(\Phi)$ can be given by the F-seminorm $g_\Phi$ in view of Propositions 2.3.3 and 2.3.7 if $\lambda$ is solid and $g$ is absolutely monotone.

It is known that the spaces $c_0$, $\ell_p$ and $(w_0)_p$ $(1 \le p < \infty)$ are BK-AK-spaces with absolutely monotone norms $\|\cdot\|_{c_0}$, $\|\cdot\|_{\ell_p}$ (defined in Section 2.1), and

$$\|x\|_{(w_0)_p} = \sup_{i \ge 0} \left( \frac{1}{2^i} \sum_{k=2^i}^{2^{i+1}-1} |x_k|^p \right)^{1/p},$$

respectively. We remark that on the space $(w_0)_p$ is determined also the norm $\|x\| = \sup_n (1/n \sum_{k=1}^n |x_k|^p)^{1/p}$ which is equivalent to $\|\cdot\|_{(w_0)_p}$ (see, for example, [29], p. 39).

By Proposition 2.3.7, the topology on the sequence space $\lambda(\Phi)$ with $\lambda \in \{c_0, \ell_p, (w_0)_p\}$ is given by F-norm

$$g_\Phi(x) = \|\Phi(x)\|_\lambda.$$

Since $(\ell_\infty, \|.\|_{\ell_\infty})$ is not an AK-space, on $\ell_\infty(\Phi)$ the same F-norm topology can be given by Proposition 2.3.3 whenever $\Phi$ satisfies (M) or (M′).

Let $(\lambda, g)$ and $(\mu, h)$ be two F-seminormed spaces. Recall that the superposition operator $P_f \colon \lambda \to \mu$ is said to be *locally bounded* if for any $z \in \lambda$ there exist numbers $\alpha > 0$ and $\beta > 0$ such that for all $x \in \lambda$ with $g(x - z) \le \alpha$ we have $h(P_f(x) - P_f(z)) \le \beta$. The superposition operator $P_f$ is called *bounded* if $\sup\{h(P_f(x)) \colon g(x) \le \varrho\} < \infty$ for every $\varrho > 0$.

For the proof of main theorems we need the following lemmas.

**Lemma 3.2.7.** *Let $\varphi$ be an unbounded modulus. The function $\varphi^{-1}$, defined by*

$$\varphi^{-1}(t) = \sup\{u \colon \varphi(u) = t\},$$

*is continuous from the right at* 0. *Moreover,* $\varphi(\varphi^{-1}(t)) = t$ *and* $t \le \varphi^{-1}(\varphi(t))$.

*Proof.* The continuity of $\varphi^{-1}$ from the right at 0 follows from the fact that $\varphi(u) \to 0$ if and only if $u \to 0+$. The assertions $\varphi(\varphi^{-1}(t)) = t$ and $t \le \varphi^{-1}(\varphi(t))$ are clear by definition of $\varphi^{-1}$. □

**Lemma 3.2.8.** *Let $\lambda$, $\mu$ be two solid Banach sequence space and $\Phi = (\varphi_k)$, $\Psi = (\psi_k)$ be two sequences of unbounded moduli such that $\lambda(\Phi)$ and $\mu(\Psi)$ are topologized by Propositions* 2.3.3 *or* 2.3.7. *Assume that $e^k \in \lambda$ $(k \in \mathbb{N})$ and $P_f$ maps $\lambda(\Phi)$ into $\mu(\Psi)$.*

(1) *If the superposition operator $P_f$ is continuous, then the condition* (C) *holds, i.e., the functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are continuous.*

(2) *Let $\varphi_k$ and $\psi_k^{-1}$ be uniformly continuous in $k \in \mathbb{N}$ at the point* 0. *If $P_f$ is continuous, $e^k \in \mu$ $(k \in \mathbb{N})$, $\inf_k \|e^k\|_\lambda > 0$ and $\inf_k \|e^k\|_\mu > 0$, then the function $f(k, \cdot)$ is uniformly continuous in $k \in \mathbb{N}$.*

*Proof.* Let $i_k \colon \mathbb{R} \to \lambda(\Phi)$ be the embedding defined for every $u \in \mathbb{R}$ by the formula $i_k(u) = ue^k \in \lambda(\Phi)$. Then for every $k \in \mathbb{N}$ the function $f(k, \cdot)$ factors as follows

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f(k,\cdot)} & \mathbb{R} \\ {\scriptstyle i_k}\downarrow & & \uparrow{\scriptstyle \pi_k} \\ \lambda(\Phi) & \xrightarrow[P_f]{} & \mu(\Psi)\ . \end{array}$$

Let $\varepsilon > 0$ and $u_0 \in \mathbb{R}$.

(1) Suppose that the superposition operator $P_f \colon \lambda(\Phi) \to \mu(\Psi)$ is continuous. The coordinate functionals $\pi_k$ are continuous for every $k \in \mathbb{N}$, since by Proposition 3 from [22] the space $\mu(\Psi)$ is a K-space.

While the moduli $\varphi_k$ are continuous from the right at 0, there exists $\delta > 0$ such that $0 < t \leq \delta$ implies $|\varphi_k(t)| < \varepsilon(\|e^k\|_\lambda)^{-1}$. If now $|u - u_0| < \delta$ then by (iii) we have $\varphi_k(|u - u_0|) \leq \varphi_k(\delta) < \varepsilon(\|e^k\|_\lambda)^{-1}$. Thus

$$\begin{aligned} g_\Phi(i_k(u) - i_k(u_0)) &= \|\Phi(i_k(u) - i_k(u_0))\|_\lambda \\ &= \varphi_k(|u - u_0|)\|e^k\|_\lambda < \varepsilon. \end{aligned} \tag{3.2.10}$$

Hence $i_k$ is continuous.

Consequently, all functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are continuous as compositions of continuous functions $\pi_k$, $P_f$ and $i_k$.

(2) Suppose that the superposition operator $P_f \colon \lambda(\Phi) \to \mu(\Psi)$ is continuous and $\|e^k\|_\lambda \geq m > 0$ $(k \in \mathbb{N})$.

While the moduli $\varphi_k$ are uniformly continuous in $k$ from the right at 0, there exists $\delta > 0$ such that $0 < t \leq \delta$ implies $\varphi_k(t) < \varepsilon(\|e^k\|_\lambda)^{-1}$

for all $k \in \mathbb{N}$. If $|u - u_0| < \delta$ then by (iii) we have $\varphi_k(|u - u_0|) \leq \varphi_k(\delta) < \varepsilon(\|e^k\|_\lambda)^{-1} \leq \varepsilon m^{-1}$ $(k \in \mathbb{N})$. Hence $i_k$ is uniformly continuous in $k$.

By our assumption the functions $\psi_k^{-1}$ are uniformly continuous in $k$ at 0. So, there exists $\delta > 0$ such that if $0 < \psi_k(t) \leq \delta$ $(k \in \mathbb{N})$, then $t < \varepsilon$. Let $z^0 = (z_k^0) \in \mu(\Psi)$ be fixed, $z = (z_k) \in \mu(\Psi)$ and $\|e^k\|_\mu \geq r > 0$ $(k \in \mathbb{N})$. If now $\|\Psi(z - z^0)\|_\mu \leq r\delta$, then

$$\psi_k(|z_k - z_k^0|) \leq r^{-1}\|\psi_k(|z_k - z_k^0|)e^k\|_\mu \leq r^{-1}\|\Psi(z - z^0)\|_\mu \leq \delta \quad (k \in \mathbb{N}).$$

Consequently, $|z_k - z_k^0| < \varepsilon$ for all $k \in \mathbb{N}$. Thereby, $\pi_k$ is uniformly continuous in $k$.

Finally, $f(k, \cdot)$ $(k \in \mathbb{N})$ is uniformly continuous in $k \in \mathbb{N}$ as composition of $\pi_k$, $P_f$ and $i_k$. □

**Lemma 3.2.9.** *Let $\lambda$, $\mu$ be two solid BK-spaces and let $\Phi = (\varphi_k)$, $\Psi = (\psi_k)$ be two sequences of unbounded moduli such that $\lambda(\Phi)$ and $\mu(\Psi)$ are topologized by Propositions* 2.3.3 *or* 2.3.7. *Assume that $e^k \in \lambda$ $(k \in \mathbb{N})$ and $P_f$ maps $\lambda(\Phi)$ into $\mu(\Psi)$. If $P_f$ is locally bounded, then $f$ satisfies* (B).

*Proof.* Suppose that the superposition operator $P_f \colon \lambda(\Phi) \to \mu(\Psi)$ is locally bounded. Let $i_k \colon \mathbb{R} \to \lambda(\Phi)$ be the embedding defined for every $u \in \mathbb{R}$ by the formula $i_k(u) = ue^k \in \lambda(\Phi)$.

Since the map $i_k$ is continuous (see the proof of Lemma 3.2.8 (1)) and the operator $P_f$ is locally bounded, then for any $z \in \mathbb{R}$ there exists $\alpha, \beta > 0$ so that, for any $x \in \mathbb{R}$ with $|z - x| < \alpha$ we have

$$\|\psi_k(|f(k, z) - f(k, x)|)e^k\|_\mu < \beta,$$

hence

$$\psi_k(|f(k, z) - f(k, x)|) < \frac{\beta}{\|e^k\|_\mu}.$$

Now, since the moduli $\psi_k$ $(k \in \mathbb{N})$ are unbounded, for some $M > 0$ we get

$$|f(k, z) - f(k, x)| \leq M.$$

Thus the functions $f(k, \cdot)$ are locally bounded. Finally, it is enough to notice that local boundedness and boundedness for scalar functions of a scalar variable are equivalent. □

**Lemma 3.2.10.** *Let $\Phi = (\varphi_k)$ be a sequence of moduli and let $(\lambda, \|.\|_\lambda)$ be a solid Banach sequence space such that $\lambda \subseteq c_0$ and $|y_k| \le \|y\|_\lambda$ $(k \in \mathbb{N})$ for all $y = (y_k) \in \lambda$. For every fixed sequence $z = (z_k) \in \lambda(\Phi)$ and for a number $\delta > 0$ there exists $m \in \mathbb{N}$ such that*

$$\max\left\{\varphi_k\left(|z_k|\right),\ \varphi_k\left(|x_k|\right)\right\} < \delta \qquad (k > m). \tag{3.2.11}$$

*for all $x \in \lambda(\Phi)$ with $\|\Phi(x - z)\|_\lambda < \delta/2$.*

*Proof.* Let $z = (z_k) \in \lambda(\Phi)$ and let $\delta > 0$. Since $\Phi(z) = (\varphi_k(|z_k|)) \in \lambda \subseteq c_0$, so there exists $m \in \mathbb{N}$ with

$$\varphi_k\left(|z_k|\right) < \frac{\delta}{2} \qquad (k > m). \tag{3.2.12}$$

If $x = (x_k) \in \lambda(\Phi)$ satisfies $\|\Phi(x - z)\|_\lambda < \delta/2$, then

$$\begin{aligned}\varphi_k\left(|x_k|\right) &\le \varphi_k\left(|x_k + z_k|\right) + \varphi_k\left(|z_k|\right)\\ &< \|\Phi(x - z)\|_\lambda + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta\end{aligned} \tag{3.2.13}$$

for $k \ge m$. By (3.2.12) and (3.2.13) we get (3.2.11). □

**Lemma 3.2.11.** *Let $\Psi = (\psi_k)$ be the sequence of moduli. Let $z = (z_k)$ be a given sequence and $1 \le q < \infty$. If the functions $f(k, \cdot)$ $(k = 1, \ldots, m)$ are continuous, then for an arbitrary $\varepsilon > 0$ there exists a number $\delta' > 0$ such that*

$$\max_{k \le m} \psi_k(|f(k, t) - f(k, z_k)|) < \varepsilon m^{-1/q} \tag{3.2.14}$$

*whenever*

$$\varphi_k(|t - z_k|) < \delta' \quad (k = 1, \ldots, m).$$

*Proof.* Let $\varepsilon > 0$. Since the moduli $\psi_k$ $(k = 1, \ldots, m)$ are continuous from the right at 0, there exists $\alpha > 0$ such that

$$\psi_k(\alpha) < \varepsilon m^{-1/q} \tag{3.2.15}$$

for all $k = 1, \ldots, m$. By the continuity of functions $f(k, \cdot)$ $(k = 1, \ldots, m)$ there exists $\beta > 0$ such that

$$|t - z_k| < \beta$$

implies

$$|f(k,t) - f(k,z_k)| < \alpha. \tag{3.2.16}$$

Further, using Lemma 3.2.7, we can find $\delta' > 0$ such that

$$\varphi_k^{-1}(\delta') < \beta \quad (k = 1, \dots, m).$$

If now $\varphi_k(|t - z_k|) < \delta'$ $(k = 1, \dots, m)$, then for $k = 1, \dots, m$,

$$|t - z_k| \le \varphi_k^{-1}(\varphi_k(|t - z_k|)) \le \varphi_k^{-1}(\delta') < \beta.$$

Since the moduli $\psi_k$ $(k = 1, \dots, m)$ are nondecreasing, from (3.2.16) we deduce

$$\psi_k\left(|f(k,t) - f(k,z_k)|\right) \le \psi_k(\alpha),$$

which together with (3.2.15) gives (3.2.14). □

**Lemma 3.2.12.** *Let $\Psi = (\psi_k)$ be the sequence of moduli, $r \in \mathbb{N}$ and $\mu \in \{c_0, \ell_q, \ell_\infty\}$ with $1 \le q < \infty$. If the functions $f(k,\cdot)$ $(k = 1, \dots, r)$ are bounded on every bounded subset of real numbers $T \subset \mathbb{R}$, then there exists a number $M > 0$ such that*

$$\sup_{t_1,\dots,t_r \in T} \left\| \sum_{k=1}^{r} \psi_k(|f(k,t_k)|) e^k \right\|_\mu \le M. \tag{3.2.17}$$

*Proof.* While the functions $f(k,\cdot)$ $(k = 1, \dots, r)$ are bounded on every bounded subset of real numbers $T \subset \mathbb{R}$, there exist

$$m_k = \sup_{t \in T} |f(k,t)| \qquad (k = 1, \dots, r). \tag{3.2.18}$$

Since the moduli $\psi_k$ are nondecreasing, because of (3.2.18) we deduce

$$\begin{aligned}\sup_{t_1,\dots,t_r \in T} \left\| \sum_{k=1}^{r} \psi_k(|f(k,t_k)|) e^k \right\|_\mu &\le \sup_{t_1,\dots,t_r \in T} \sum_{k=1}^{r} \psi_k(|f(k,t_k)|) \|e^k\|_\mu \\ &\le \sum_{k=1}^{r} \psi_k(m_k) \|e^k\|_\mu.\end{aligned}$$

Putting $M = \sum_{k=1}^{r} \psi_k(m_k) \|e^k\|_\mu$, we have (3.2.17). □

By a *finite sequence* we mean a sequence $x = (x_k)$ for which there exists $k_0 \in \mathbb{N}$ such that $x_k = 0$ if $k \ge k_0$.

**Lemma 3.2.13.** *Let $1 \le p, q < \infty$. Assume that $f$ satisfies* (B) *and the moduli $\varphi_k$ ($k \in \mathbb{N}$) are unbounded. If for every $\beta > 0$ there is a number $\vartheta(\beta) > 0$ such that for every finite sequence $x = (x_k)$ we have*

$$\|\Psi(P_f(x))\|_{\ell_q} \le \vartheta(\beta), \tag{3.2.19}$$

*provided*

$$\sum_{k=1}^{\infty} (\varphi_k(|x_k|))^p \le \beta^p, \tag{3.2.20}$$

*then there exists a sequence $a(\beta) = (a_k(\beta)) \in \ell_q^+$ with $\|a(\beta)\|_{\ell_q} \le \vartheta(\beta)$ such that for each $k \in \mathbb{N}$,*

$$\psi_k(|f(k,t)|) \le a_k(\beta) + 2^{1/q}\beta^{-p/q}\vartheta(\beta)(\varphi_k(|t|))^{p/q} \tag{3.2.21}$$

*whenever $\varphi_k(|t|) \le \beta$.*

*Proof.* Let $\beta > 0$. By the assumption, there exists $\vartheta(\beta) > 0$ such that for any finite sequence $x = (x_k)$ the inequality (3.2.19) holds whenever (3.2.20) is satisfied. For every $k \in \mathbb{N}$ we define

$$\begin{aligned} h_\beta(k,t) &= \max\left\{0, \psi_k(|f(k,t)|) - 2^{1/q}\beta^{-p/q}\vartheta(\beta)(\varphi_k(|t|))^{p/q}\right\}, \\ a_k(\beta) &= \sup\left\{h_\beta(k,t) \colon \varphi_k(|t|) \le \beta\right\}. \end{aligned} \tag{3.2.22}$$

Since all sets $\{t \colon \varphi_k(|t|) \le \beta\}$ ($k \in \mathbb{N}$) are bounded subsets of $\mathbb{R}$, by (B) we clearly have $a_k(\beta) < \infty$ ($k \in \mathbb{N}$).

If $h_\beta(k,t) = 0$, then

$$\begin{aligned} \psi_k(|f(k,t)|) &\le 2^{1/q}\beta^{-p/q}\vartheta(\beta)(\varphi_k(|t|))^{p/q} \\ &\le a_k(\beta) + 2^{1/q}\beta^{-p/q}\vartheta(\beta)(\varphi_k(|t|))^{p/q}. \end{aligned}$$

Now, if $h_\beta(k,t) \ne 0$, then

$$\begin{aligned} \psi_k(|f(k,t)|) &= h_\beta(k,t) + 2^{1/q}\beta^{-p/q}\vartheta(\beta)(\varphi_k(|t|))^{p/q} \\ &\le a_k(\beta) + 2^{1/q}\beta^{-p/q}\vartheta(\beta)(\varphi_k(|t|))^{p/q}. \end{aligned}$$

Therefore, the inequality (3.2.21) holds for every $k \in \mathbb{N}$ if $\varphi_k(|t|) \le \beta$.

Next we show that $a(\beta) \in \ell_q^+$ and $\|a(\beta)\|_{\ell_q} \le \vartheta(\beta)$. By the definition of $a_k(\beta)$, for each $\varepsilon > 0$ there is a sequence $y(\beta,\varepsilon) = (y_k(\beta,\varepsilon))$ such that

$$\varphi_k(|y_k(\beta,\varepsilon)|) \le \beta \quad (k \in \mathbb{N})$$

and

$$a_k(\beta) \le h_\beta(k, y_k(\beta,\varepsilon))) + \frac{\varepsilon}{2^k} \quad (k \in \mathbb{N}). \tag{3.2.23}$$

Let $\tilde{y}(\beta,\varepsilon) = (\tilde{y}_k(\beta,\varepsilon))$ be the sequence with

$$\tilde{y}_k(\beta,\varepsilon) = \begin{cases} y_k(\beta,\varepsilon) & \text{if } h_\beta(k, y_k(\beta,\varepsilon)) \neq 0, \\ 0 & \text{if } h_\beta(k, y_k(\beta,\varepsilon)) = 0. \end{cases}$$

Then by (3.2.22), for every $k \in \mathbb{N}$, we conclude

$$h_\beta(k, \tilde{y}_k(\beta,\varepsilon)) = \psi_k(|f(k, \tilde{y}_k(\beta,\varepsilon))|) - 2^{1/q}\beta^{-p/q}\vartheta(\beta)(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^{p/q}. \tag{3.2.24}$$

Since $h_\beta(k, \tilde{y}_k(\beta,\varepsilon)) \ge 0$, using also (3.2.19), we get

$$(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^p \le \frac{1}{2}\beta^p \left(\frac{\psi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|)}{\vartheta(\beta)}\right)^q \le \frac{\beta^p}{2}.$$

Thus, for each $m \in \mathbb{N}$, we can choose the indices $m_1 = 1 < m_2 < \ldots < m_l = m$ such that

$$\sum_{k=1}^{m}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^p = \sum_{k=1}^{m_2-1}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^p + \sum_{k=m_2}^{m_3-1}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^p + \ldots + \sum_{k=m_{l-1}}^{m}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^p$$

and

$$\begin{aligned} \frac{\beta^p}{2} &\le \sum_{k=m_i}^{m_{i+1}-1}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^p \le \beta^p \quad (i = 1, 2, \ldots, l-2), \\ 0 &\le \sum_{k=m_{l-1}}^{m}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|))^p \le \beta^p. \end{aligned} \tag{3.2.25}$$

By (3.2.19) we have

$$\begin{aligned} &\left(\sum_{k=m_i}^{m_{i+1}-1}(\psi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|))^q\right)^{1/q} \le \vartheta(\beta) \quad (i = 1, 2, \ldots, l-2), \\ &\left(\sum_{k=m_{l-1}}^{m}(\psi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|))^q\right)^{1/q} \le \vartheta(\beta). \end{aligned} \tag{3.2.26}$$

Using Minkowski's inequality and (3.2.23) we get

$$\left(\sum_{k=1}^{m}(a_k(\beta))^q\right)^{1/q} \leq \left(\sum_{k=1}^{m}\left(h_\beta(k,\tilde{y}_k(\beta,\varepsilon))+\frac{\varepsilon}{2^k}\right)^q\right)^{1/q}$$
$$\leq \left(\sum_{k=1}^{m}(h_\beta(k,\tilde{y}_k(\beta,\varepsilon))^q\right)^{1/q} + \left(\sum_{k=1}^{m}\left(\frac{\varepsilon}{2^k}\right)^q\right)^{1/q}.$$

Now we use the inequality $(a-b)^q \leq a^q - b^q$ for $a \geq b \geq 0$ which we can deduce from the inequality $(c+d)^q \geq c^q + d^q$ $(c, d \geq 0)$. By (3.2.24) this gives, for $m \in \mathbb{N}$,

$$\sum_{k=1}^{m}(h_\beta(k,\tilde{y}_k(\beta,\varepsilon))^q \leq \sum_{k=1}^{m}(\psi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|))^q$$
$$- 2\beta^{-p}\vartheta(\beta)^q\sum_{k=1}^{m}(\varphi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|))^p$$
$$\leq \sum_{k=1}^{m_2-1}(\psi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|))^q + \sum_{k=m_2}^{m_3-1}(\psi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|))^q + \dots$$
$$+ \sum_{k=m_{l-1}}^{m}(\psi_k(|f(k,\tilde{y}_k(\beta,\varepsilon))|)^q - 2\beta^{-p}\vartheta(\beta)^q\left(\sum_{k=1}^{m_2-1}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|)^p\right.$$
$$\left.+\sum_{k=m_2}^{m_3-1}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|)^p + \dots + \sum_{k=m_{l-1}}^{m}(\varphi_k(|\tilde{y}_k(\beta,\varepsilon)|)^p\right).$$

Applying now (3.2.25) and (3.2.26) we have

$$\sum_{k=1}^{m}(h_\beta(k,\tilde{y}_k(\beta,\varepsilon))^q \leq (l-1)\vartheta(\beta)^q - 2\beta^{-p}\vartheta(\beta)^q(l-2)\beta^p 2^{-1} = \vartheta(\beta)^q$$

for all $m \in \mathbb{N}$. Therefore

$$\left(\sum_{k=1}^{m}(a_k(\beta))^q\right)^{1/q} \leq \vartheta(\beta)+\varepsilon \quad (m \in \mathbb{N}).$$

Thus

$$\|a(\beta)\|_{\ell_q} = \left(\sum_{k=1}^{\infty}(a_k(\beta))^q\right)^{1/q} = \lim_{m\to\infty}\left(\sum_{k=1}^{m}(a_k(\beta))^q\right)^{1/q} \leq \vartheta(\beta)+\varepsilon$$

which shows that $a(\beta) \in \ell_q^+$ with $\|a(\beta)\|_{\ell_q} \le \vartheta(\beta)$ because $\varepsilon > 0$ is arbitrary. □

**Lemma 3.2.14.** *Let $f$ satisfy* (B), $1 \le p < \infty$, $\lambda \in \{c_0, \ell_p, \ell_\infty\}$ *and* $\mu \in \{c_0, \ell_\infty\}$. *If for every $\beta > 0$ there is an $\vartheta(\beta) > 0$ such that for every finite sequence $x = (x_k)$ one has*

$$\|\Psi(P_f(x))\|_\mu \le \vartheta(\beta), \tag{3.2.27}$$

*provided*

$$\|\Phi(x)\|_\lambda \le \beta,$$

*then there exists a sequence $a(\beta) = (a_k(\beta)) \in \ell_\infty^+$ with $\|a(\beta)\|_{\ell_\infty} \le \vartheta(\beta)$ such that for each $k \in \mathbb{N}$,*

$$\psi_k(|f(k,t)|) \le a_k(\beta) \tag{3.2.28}$$

*whenever $\varphi_k(|t|) \le \beta$.*

*Proof.* Let $\beta > 0$. By the assumption, there exists $\vartheta(\beta) > 0$ such that the inequality (3.2.27) holds whenever $\|\Phi(x)\|_\lambda \le \beta$. For every $k \in \mathbb{N}$, we define

$$a_k(\beta) = \sup\{\psi_k(|f(k,t)|) \colon \varphi_k(|t|) \le \beta\}. \tag{3.2.29}$$

Since $f$ satisfies (B), we have $a_k(\beta) < \infty$ $(k \in \mathbb{N})$.

The inequality (3.2.28) is clear by (3.2.29).

Next we show that $a(\beta) \in \ell_\infty^+$ and $\|a(\beta)\|_{\ell_\infty} \le \vartheta(\beta)$. By (3.2.29), for each $\varepsilon > 0$ there is a sequence $y(\beta,\varepsilon) = (y_k(\beta,\varepsilon))$ such that

$$\varphi_k(|y_k(\beta,\varepsilon)|) \le \beta \quad (k \in \mathbb{N}) \tag{3.2.30}$$

and

$$a_k(\beta) \le \psi_k(|f(k, y_k(\beta,\varepsilon))|) + \varepsilon \quad (k \in \mathbb{N}). \tag{3.2.31}$$

Let $\tilde{y}(\beta,\varepsilon) = y_k(\beta,\varepsilon)\ e^k = (y_k(\beta,\varepsilon)\ \delta_{ki})_{i=1}^\infty$ for every fixed $k \in \mathbb{N}$. So, by (3.2.30),

$$\|\Phi(\tilde{y}(\beta,\varepsilon))\|_\lambda = \|\Phi(y_k(\beta,\varepsilon)\ e^k)\|_\lambda = \varphi_k(|y_k(\beta,\varepsilon)|) \le \beta$$

which yields

$$\|a(\beta)\|_{\ell_\infty} = \sup_k a_k(\beta) \le \vartheta(\beta) + \varepsilon < \infty$$

in view of (3.2.27) and (3.2.31). Thus $a(\beta) \in \ell_\infty^+$, and since $\varepsilon > 0$ is arbitrary, we also get $\|a(\beta)\|_{\ell_\infty} \le \vartheta(\beta)$. □

For $\lambda \in \{c_0, \ell_p, \ell_\infty\}$ and $\mu \in \{c_0, \ell_q, \ell_\infty\}$ $(1 \le p, q < \infty)$ we use the notation

$$\eta_{f,\mu}(\varrho) = \sup \{\|\Psi(P_f(x))\|_\mu \, : \, \|\Phi(x)\|_\lambda \le \varrho\}$$

for every $\varrho > 0$.

**Lemma 3.2.15.** *Let $1 \le p < \infty$, $\lambda \in \{c_0, \ell_p, \ell_\infty\}$, $\mu \in \{c_0, \ell_\infty\}$ and $P_f \colon \lambda(\Phi) \to \mu(\Psi)$, where $\Phi = (\varphi_k)$, $\Psi = (\psi_k)$ are two sequence of unbounded moduli. Assume that for $\lambda = \ell_\infty$ $(\mu = \ell_\infty)$ the sequence of moduli $\Phi$ $(\Psi)$ satisfies one of conditions* (M) *and* (M′)*. If for every $\varrho > 0$ there exists a sequence $a(\varrho) = (a_k(\varrho)) \in \ell_\infty^+$ such that*

$$\psi_k(|f(k,t)|) \le a_k(\varrho) \qquad (\varphi_k(|t|) \le \varrho, \ k \in \mathbb{N}), \tag{3.2.32}$$

*then $P_f$ is bounded. Moreover,*

$$\eta_{f,\mu}(\varrho) \le \nu_{f,\infty}(\varrho)$$

*for every $\varrho > 0$, where*

$$\nu_{f,\infty}(\varrho) = \inf \{\|a(\varrho)\|_{\ell_\infty} \colon \ (3.2.32) \text{ holds}\} .$$

*Proof.* Let $\varrho > 0$ and $x = (x_k) \in \lambda(\Phi)$ be such that $\|\Phi(x)\|_\lambda \le \varrho$. From (3.2.32), by

$$\varphi_k(|x_k|) \le \|\Phi(x)\|_\lambda \le \varrho \qquad (k \in \mathbb{N}),$$

it follows that

$$\psi_k(|f(k,x_k)|) \le a_k(\varrho)$$

for all $k \in \mathbb{N}$. Therefore,

$$\|\Psi(P_f(x))\|_\mu \le \|a(\varrho)\|_{\ell_\infty} < \infty \tag{3.2.33}$$

provided $\|\Phi(x)\|_\lambda \le \varrho$. Consequently, the superposition operator $P_f$ is bounded.

The inequality $\eta_{f,\mu}(\varrho) \le \nu_{f,\infty}(\varrho)$ is true because of (3.2.33) and $\|\Phi(x)\|_\lambda \le \varrho$. □

## 3.3 Continuity of superposition operators

In the following let $\Phi = (\varphi_k)$ be a sequence of unbounded moduli and $\Psi = (\psi_k)$ an arbitrary sequence of moduli.

First we characterize the continuity of superposition operators from $\ell_p(\Phi)$ and $c_0(\Phi)$ into $\ell_q(\Psi)$.

**Theorem 3.3.1.** *Let $1 \le p, q < \infty$. A superposition operator $P_f \colon \ell_p(\Phi) \to \ell_q(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

*Proof.* If $P_f$ is continuous, then all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous by Lemma 3.2.8 (1).

Conversely, suppose that $P_f$ maps $\ell_p(\Phi)$ into $\ell_q(\Psi)$ and all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous. Let $z = (z_k) \in \ell_p(\Phi)$ and $\varepsilon > 0$. If the numbers $\delta > 0$, $\gamma \ge 0$, $k_0 \in \mathbb{N}$ and the sequence $(a_k) \in \ell^+$ are determined by Proposition 3.2.1, then, basing also on Lemma 3.2.10, we may choose a number $m \in \mathbb{N}$ such that $m \ge k_0$,

$$\sum_{k=m+1}^{\infty} a_k < \varepsilon^q, \tag{3.3.1}$$

$$\sum_{k=m+1}^{\infty} (\varphi_k(|z_k|))^p < \frac{1}{2}\left(\frac{\varepsilon^q}{\gamma+1}\right)^{1/p} \tag{3.3.2}$$

and condition (3.2.11) is satisfied whenever

$$\|\Phi(x-z)\|_{\ell_p} < \varrho = \min\left\{\frac{\delta}{2}, \frac{1}{2}\left(\frac{\varepsilon^q}{\gamma+1}\right)^{1/p}\right\}.$$

Thus we get

$$\begin{aligned}\left(\sum_{k=m+1}^{\infty} (\varphi_k(|x_k|))^p\right)^{1/p} &\le \left(\sum_{k=m+1}^{\infty} (\varphi_k(|x_k - z_k|))^p\right)^{1/p} \\ &\quad + \left(\sum_{k=m+1}^{\infty} (\varphi_k(|z_k|))^p\right)^{1/p} \\ &\le \varrho + \frac{1}{2}\left(\frac{\varepsilon^q}{\gamma+1}\right)^{1/p} < \left(\frac{\varepsilon^q}{\gamma+1}\right)^{1/p}.\end{aligned} \tag{3.3.3}$$

Moreover, by inequality (3.2.1), because of (3.2.11), for all $k > m$ we have

$$\begin{aligned} (\psi_k(|f(k,x_k)|))^q \le a_k + \gamma(\varphi_k(|x_k|))^p, \\ (\psi_k(|f(k,z_k)|))^q \le a_k + \gamma(\varphi_k(|z_k|))^p. \end{aligned} \tag{3.3.4}$$

Further, since the functions $f(k,\cdot)$ are continuous, by Lemma 3.2.11 there exists $\delta' > 0$ with $\delta' \le \varrho$ such that $\|\Phi(x-z)\|_{\ell_p} < \delta'$ implies

$$\psi_k(|f(k,x_k) - f(k,z_k)|) < \varepsilon m^{-1/q} \quad (k = 1, 2, \dots, m). \tag{3.3.5}$$

Now, by (3.3.1)–(3.3.5) we get

$$\begin{aligned} \|\Psi(P_f(x) - P_f(z))\|_{\ell_q} &\le \left( \sum_{k=1}^{m} (\psi_k(|f(k,x_k) - f(k,z_k)|))^q \right)^{1/q} \\ &+ \left( \sum_{k=m+1}^{\infty} (\psi_k(|f(k,x_k)|))^q \right)^{1/q} + \left( \sum_{k=m+1}^{\infty} (\psi_k(|f(k,z_k)|))^q \right)^{1/q} \\ &\le \left( \sum_{k=1}^{m} \left(\varepsilon (m+1)^{-1/q}\right)^q \right)^{1/q} + 2 \left( \sum_{k=m+1}^{\infty} a_k \right)^{1/q} \\ &+ \left( \sum_{k=m+1}^{\infty} \gamma(\varphi_k(|x_k|))^p \right)^{1/q} + \left( \sum_{k=m+1}^{\infty} \gamma(\varphi_k(|z_k|))^p \right)^{1/q} \\ &< \varepsilon + 2\varepsilon + \varepsilon + \varepsilon = 5\varepsilon. \end{aligned}$$

Consequently, $\|\Psi(P_f(x) - P_f(z))\|_{\ell_q} < 5\varepsilon$ whenever $\|\Phi(x-z)\|_{\ell_p} < \delta'$. □

**Theorem 3.3.2.** *Let $1 \le q < \infty$. A superposition operator $P_f\colon c_0(\Phi) \to \ell_q(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

*Proof.* If $P_f$ is continuous, then the continuity of functions $f(k,\cdot)$ $(k \in \mathbb{N})$ is clear by Lemma 3.2.8 (1).

Conversely, if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous, $z = (z_k) \in c_0(\Phi)$ and $\varepsilon > 0$ is arbitrarily given, then, basing on Proposition 3.2.1 with $\gamma = 0$ and Lemmas 3.2.10 and 3.2.11, similarly to the proof of

Theorem 3.3.1 we may find a sequence $(a_k) \in \ell^+$ and numbers $m \in \mathbb{N}$, $\delta' > 0$ such that (3.3.1) holds and $\|\Phi(x-z)\|_{c_0} < \delta'$ yields (3.3.5) and

$$(\psi_k(|f(k,x_k)|))^q \le a_k, \quad (\psi_k(|f(k,z_k)|))^q \le a_k \quad (k > m). \tag{3.3.6}$$

Consequently, by (3.3.1), (3.3.5) and (3.3.6) we get

$$\begin{aligned}\|\Psi(P_f(x)-P_f(z))\|_{\ell_q} &\le \left(\sum_{k\le m} (\psi_k(|f(k,x_k)-f(k,z_k)|))^q\right)^{1/q} \\ &+ \left(\sum_{k>m} (\psi_k(|f(k,x_k)|))^q\right)^{1/q} + \left(\sum_{k>m} (\psi_k(|f(k,z_k)|))^q\right)^{1/q} \\ &\le \left(\sum_{k\le m} \left(\varepsilon m^{-1/q}\right)^q\right)^{1/q} + 2\left(\sum_{k>m} a_k\right)^{1/q} < \varepsilon + 2\varepsilon = 3\varepsilon\end{aligned}$$

whenever $\|\Phi(x-z)\|_{c_0} < \delta'$. □

The continuity of superposition operators from $\ell_p(\Phi)$ $(1 \le p < \infty)$ and $c_0(\Phi)$ into $c_0(\Psi)$ describes

**Theorem 3.3.3.** *Let $1 \le p < \infty$ and $\lambda \in \{c_0, \ell_p\}$. A superposition operator $P_f\colon \lambda(\Phi) \to c_0(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

*Proof.* Lemma 3.2.8 (1) shows that the continuity of functions $f(k,\cdot)$ $(k \in \mathbb{N})$ is necessary for the continuity of $P_f$.

Conversely, suppose that all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous and let $z = (z_k)$ be an element from $\ell_p(\Phi)$ or $c_0(\Phi)$. By Proposition 3.2.2 there exist numbers $\delta > 0$, $k_0 \in \mathbb{N}$ and a sequence $(a_k) \in c_0^+$ such that (3.2.2) holds. Now, in view of Lemma 3.2.10, for an arbitrary number $\varepsilon > 0$ we may choose an index $m \in \mathbb{N}$, $m \ge k_0$, such that

$$a_k < \frac{\varepsilon}{2} \quad (k > m)$$

and (3.2.11) is true whenever $\|\Phi(x-z)\|_\lambda < \delta/2$. So by (3.2.2) we have, for all $k > m$,

$$\begin{aligned}\psi_k(|f(k,x_k)-f(k,z_k)|) &\le \psi_k(|f(k,x_k)|) + \psi_k(|f(k,z_k)|) \\ &\le a_k + a_k < 2\cdot\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $\|\Phi(x-z)\|_\lambda < \delta/2$ yields

$$\sup_{k>m} \psi_k\left(|f(k,x_k) - f(k,z_k)|\right) < \varepsilon. \tag{3.3.7}$$

Further, using Lemma 3.2.11, we fix a number $\delta' \le \delta/2$ such that (3.3.5) holds for $\|\Phi(x-z)\|_\lambda < \delta'$. But (3.3.5) immediately gives

$$\sup_{k\le m} \psi_k\left(|f(k,x_k) - f(k,z_k)|\right) < \varepsilon. \tag{3.3.8}$$

Finally, by (3.3.7) and (3.3.8) we obtain

$$\|\Psi(P_f(x) - P_f(z))\|_{c_0} = \max\left\{\sup_{k\le m} \psi_k\left(|f(k,x_k) - f(k,z_k)|\right), \right.$$
$$\left. \sup_{k>m} \psi_k\left(|f(k,x_k) - f(k,z_k)|\right)\right\} < \varepsilon$$

whenever $\|\Phi(x-z)\|_\lambda < \delta'$. □

**Theorem 3.3.4.** *Let* $1 \le q < \infty$. *If the sequence of moduli* $\Phi = (\varphi_k)$ *satisfies one of conditions* (M) *and* (M′), *then* $P_f\colon \ell_\infty(\Phi) \to \ell_q(\Psi)$ *is continuous if and only if all functions* $f(k,\cdot)$ $(k \in \mathbb{N})$ *are continuous.*

*Proof.* If $P_f$ is continuous, then functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous by Lemma 3.2.8 (1).

Conversely, suppose that all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous. If $z = (z_k) \in \ell_\infty(\Phi)$, then for some $\eta > 0$ we have

$$\varphi_k(|z_k|) \le \frac{\eta}{2}. \tag{3.3.9}$$

By Proposition 3.2.3 (1), for this number $\eta > 0$ we can find a sequence $(a_k) \in \ell^+$ such that the condition (3.2.3) is valid for every $k \in \mathbb{N}$. Since $(a_k) \in \ell^+$, for a given $\varepsilon > 0$ we may choose $m \in \mathbb{N}$ such that (3.3.1) holds. On the other hand, (3.3.9) together with (3.2.13) (for $\delta = \eta$) gives

$$\varphi_k(|x_k|) \le \eta$$

if $\|\Phi(x-z)\|_{\ell_\infty} < \eta/2$. So (3.2.3) yields (3.3.6) whenever $\|\Phi(x-z)\|_{\ell_\infty} < \eta/2$.

Further, using the continuity of functions $f(k, \cdot) \quad (k = 1, \ldots, m)$, by Lemma 3.2.11 there exists $\delta' > 0$ with $\delta' \leq \eta/2$ such that (3.3.5) is true if

$$\varphi_k(|x_k - z_k|) < \delta'.$$

Now, as in Theorem 3.3.2, from (3.3.1), (3.3.5) and (3.3.6) we deduce the continuity of $P_f$ at $z$. □

**Theorem 3.3.5.** *If the sequence of moduli* $\Phi = (\varphi_k)$ *satisfies one of the conditions* (M) *and* (M′)*, then* $P_f \colon \ell_\infty(\Phi) \to c_0(\Psi)$ *is continuous if and only if all functions* $f(k, \cdot)$ $(k \in \mathbb{N})$ *are continuous.*

*Proof.* The continuity of functions $f(k, \cdot)$ $(k \in \mathbb{N})$ is necessary for the continuity of $P_f$ by Lemma 3.2.8 (1).

If all functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are continuous and $P_f \colon \ell_\infty(\Phi) \to c_0(\Psi)$, then by Proposition 3.2.3 (2) we can find, for $\eta = 1$, a sequence $(a_k) \in c_0^+$ and a number $k_0 \in \mathbb{N}$ such that (3.2.4) is satisfied. Now, putting $\delta = 1$, continuity of $P_f$ follows in the same way as in Theorem 3.3.3. □

Now we characterize the continuity of superposition operators into the space $\ell_\infty(\Psi)$.

**Theorem 3.3.6.** *Let the moduli* $\varphi_k$, $\psi_k$ *and the functions* $\varphi_k^{-1}$, $\psi_k^{-1}$ *be uniformly continuous in* $k \in \mathbb{N}$ *at the point* 0, $1 \leq p < \infty$ *and* $\lambda \in \{\ell_\infty, c_0, \ell_p\}$*. Assume that the sequence of moduli* $\Psi = (\psi_k)$ *and for* $\lambda = \ell_\infty$ *the sequence of moduli* $\Phi = (\varphi_k)$ *satisfies one of conditions* (M) *and* (M′)*. Then* $P_f \colon \lambda(\Phi) \to \ell_\infty(\Psi)$ *is continuous if and only if the function* $f(k, \cdot)$ *is uniformly continuous in* $k \in \mathbb{N}$*.*

*Proof.* The proof of necessity follows from Lemma 3.2.8 (2).

*Sufficiency.* Let the function $f(k, \cdot)$ be uniformly continuous in $k \in \mathbb{N}$, $\varepsilon > 0$ and $z = (z_k) \in \lambda(\Phi)$. Since the moduli $\psi_k$ are uniformly continuous in $k$ at the point 0, then there exist $\alpha > 0$ such that for all $k \in \mathbb{N}$

$$\psi_k(t) < \varepsilon \tag{3.3.10}$$

whenever $0 \leq t \leq \alpha$. Because the function $f(k, \cdot)$ is uniformly continuous in $k \in \mathbb{N}$, so there exists $\beta > 0$ such that

$$|x_k - z_k| < \beta \tag{3.3.11}$$

implies

$$|f(k, x_k) - f(k, z_k)| < \alpha \tag{3.3.12}$$

for all $k \in \mathbb{N}$. While $\varphi_k^{-1}$ is uniformly continuous in $k \in \mathbb{N}$ at the point 0, then there exist $\delta > 0$ such that (3.3.11) is satisfied whenever $0 < \varphi_k(|x_k - z_k|) \le \delta$ for all $k \in \mathbb{N}$. Let $\|\Phi(x - z)\|_\lambda \le \delta$, then

$$\varphi_k(|x_k - z_k|) \le \|\Phi(x - z)\|_\lambda \le \delta \quad (k \in \mathbb{N}).$$

Therefore, (3.3.11) holds. By (3.3.10) and (iii) from (3.3.12) we deduce

$$\psi_k(|f(k, x_k) - f(k, z_k)|) < \psi_k(\alpha) < \varepsilon \quad (k \in \mathbb{N}).$$

Consequently, we get

$$\|\Psi(P_f(x) - P_f(z))\|_{\ell_\infty} = \sup_k \psi_k(|f(k, x_k) - f(k, z_k)|) < \varepsilon.$$

□

Our last theorem describes the continuity of superposition operators on the space $(w_0)_p(\Phi)$.

**Theorem 3.3.7.** *Let $1 \le p, q < \infty$. If the moduli $\varphi_k$ $(k \in \mathbb{N})$ are strictly increasing, then a superposition operator $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ is continuous if and only if all functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are continuous.*

*Proof.* If $P_f$ is continuous, then the continuity of functions $f(k, \cdot)$ $(k \in \mathbb{N})$ follows by Lemma 3.2.8 (1).

Conversely, suppose that all functions $f(k, \cdot)$ $(k \in \mathbb{N})$ are continuous, $P_f$ maps $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$ and $z = (z_k) \in (w_0)_p(\Phi)$. By Proposition 3.2.5 there exist a number $\delta > 0$ and sequences $(c_k)_{k=0}^\infty \in \ell^+$ and $(d_k) \in \ell^+$ such that condition (3.2.7) holds whenever $\varphi_k(|t|)^p \le 2^i\delta$, $2^i \le k < 2^{i+1}$ $(i = 0, 1, \ldots)$. By (1.1.1) $z = (z_k) \in (w_0)_p(\Phi)$ if and only if

$$\lim_{i\to\infty} 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p = 0.$$

For a fixed $\varepsilon > 0$ we denote by $i_\varepsilon$ the least of all numbers $s$ such that

$$\sup_{i\ge s} 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p < \frac{\delta}{2^p}, \quad \sum_{k=2^s}^\infty d_k < \left(\frac{\varepsilon}{2}\right)^q \quad \text{and} \quad \sum_{i=s}^\infty c_i < \frac{\varepsilon^q}{\delta}.$$

Let $x \in (w_0)_p(\Phi)$ be such that

$$\|\Phi(x - z)\|_{(w_0)_p} < \frac{1}{2}(2^i\delta)^{1/p}. \tag{3.3.13}$$

Since in the case $i \geq i_\varepsilon$ we have

$$(\varphi_k(|z_k|))^p < 2^{-p}2^i\delta \quad (2^i \leq k < 2^{i+1}), \tag{3.3.14}$$

by (ii), Minkowski's inequality, (3.3.13) and (3.3.14), for $i \geq i_\varepsilon$, we get

$$\begin{aligned} \left(2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p\right)^{1/p} &\leq \left(2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k - z_k|))^p\right)^{1/p} \\ &+ \left(2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|z_k|))^p\right)^{1/p} \leq \|\Phi(x - z)\|_{(w_0)_p} \\ &+ \left(2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|z_k|))^p\right)^{1/p} \leq 2^{-1}\delta^{1/p} + 2^{-1}\delta^{1/p} = \delta^{1/p}. \end{aligned} \tag{3.3.15}$$

Thus, if $i \geq i_\varepsilon$, then

$$\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p \leq 2^i\delta,$$

so $(\varphi_k(|x_k|))^p \leq 2^i\delta \quad (2^i \leq k < 2^{i+1})$. Therefore, (3.2.7) implies

$$\begin{aligned} &(\psi_k(|f(k, z_k)|))^q \leq d_k + c_i 2^{-i}(\varphi_k(|z_k|))^p, \\ &(\psi_k(|f(k, x_k)|))^q \leq d_k + c_i 2^{-i}(\varphi_k(|x_k|))^p \quad (i \geq i_\varepsilon). \end{aligned} \tag{3.3.16}$$

Further, using the continuity of functions $f(k, \cdot)$, by Lemma 3.2.11 (for $m = 2^{i_\varepsilon}$) we may choose $\delta' > 0$ with $\delta' \leq 1/2\,(2^{i_\varepsilon}\delta)^{1/p}$ such that

$$\max_{k < 2^{i_\varepsilon}} \psi_k(|f(k, x_k) - f(k, z_k)|) < \varepsilon 2^{-i_\varepsilon/q} \tag{3.3.17}$$

if $\|\Phi(x-z)\|_{(w_0)_p} < \delta'$. Now, by (3.10) and (3.11) we conclude

$$\begin{aligned}
\|\Psi(P_f(x)-P_f(z))\|_{\ell_q} &\leq \left(\sum_{k=1}^{2^{i_\varepsilon}-1} \left(\psi_k\left(|f(k,x_k)-f(k,z_k)|\right)\right)^q\right)^{1/q} \\
&+ \left(\sum_{k=2^{i_\varepsilon}}^{\infty} \left(\psi_k\left(|f(k,x_k)|\right)\right)^q\right)^{1/q} + \left(\sum_{k=2^{i_\varepsilon}}^{\infty} \left(\psi_k\,|f(k,z_k)|\right)^q\right)^{1/q} \\
&\leq \left(\sum_{k=1}^{2^{i_\varepsilon}-1} \left(\varepsilon 2^{-i_\varepsilon/q}\right)^q\right)^{1/q} + \left(\sum_{i=i_\varepsilon}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k,x_k)|))^q\right)^{1/q} \\
&+ \left(\sum_{i=i_\varepsilon}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k,z_k)|))^q\right)^{1/q} \\
&< \varepsilon 2^{i_\varepsilon} 2^{-i_\varepsilon} + 2\left(\sum_{k=2^{i_\varepsilon}}^{\infty} d_k\right)^{1/q} + \left(\sum_{i=i_\varepsilon}^{\infty} c_i 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p\right)^{1/q} \\
&+ \left(\sum_{i=i_\varepsilon}^{\infty} c_i 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p\right)^{1/q} \\
&< \varepsilon + 2\frac{\varepsilon}{2} + 2\left(\frac{\varepsilon^q}{\delta}\delta\right)^{1/q} = 4\varepsilon.
\end{aligned}$$

Consequently, $\|\Psi(P_f(x)-P_f(z))\|_{\ell_q} < 4\varepsilon$ whenever $\|\Phi(x-z)\|_{(w_0)_p} < \delta'$. □

## 3.4 Boundedness of superposition operators

In this section we give necessary and sufficent conditions for local boundedness and boundedness of superposition operators on some sequence spaces defined by a sequence of modulus functions.

### 3.4.1 Local boundedness of $P_f$

In the following let $\Phi = (\varphi_k)$ and $\Psi = (\psi_k)$ be two sequences of unbounded moduli. By the definition of a modulus it is not difficult to see that, for a fixed sequence $z = (z_k)$, the set of real numbers

$$T_m(\varkappa) = \{t \in \mathbb{R}\colon \max_{1\le k\le m} \varphi_k(|t - z_k|) \le \varkappa\}$$

is bounded for every $m \in \mathbb{N}$ and $\varkappa > 0$.

Because of Theorems 3.3.1–3.3.7 a superposition operator $P_f\colon \lambda(\Phi) \to \mu(\Psi)$ is continuous for some sequence spaces $\lambda$ and $\mu$ if and only if all functions $f(k, \cdot)$ ($k \in \mathbb{N}$) are continuous, i.e., $f$ satisfies (C). By the investigation of local boundeness of $P_f$ the condition (B) is important.

Now we are able to describe the local boundedness of superposition operator $P_f$.

**Theorem 3.4.1.** *Let $1 \le p, q < \infty$, $\lambda \in \{c_0, \ell_p\}$ and $\mu \in \{c_0, \ell_q, \ell_\infty\}$. Assume that for $\mu = \ell_\infty$ the sequence $\Psi = (\psi_k)$ satisfies one of conditions* (M) *and* (M′)*. A superposition operator $P_f\colon \lambda(\Phi) \to \mu(\Psi)$ is locally bounded if and only if $f$ satisfies* (B)*.*

*Proof.* If $P_f$ is locally bounded, then $f$ satisfies (B) by Lemma 3.2.9.

Conversely, suppose that $P_f$ maps $\lambda(\Phi)$ into $\mu(\Psi)$ and $f$ satisfies (B). Let $z = (z_k) \in \lambda(\Phi)$. By Proposition 3.2.4 we determine the numbers $\delta > 0$, $\gamma \ge 0$, $k_0 \in \mathbb{N}$ and the sequence $(a_k) \in \mu^+$ such that (3.2.6) holds. Let $x = (x_k) \in \lambda(\Phi)$ with

$$\|\Phi(x - z)\|_\lambda < \frac{\delta}{2}. \tag{3.4.1}$$

We may choose a number $m \in \mathbb{N}$, $m > k_0$, such that

$$\|R_m\Phi(z)\|_\lambda \le \frac{\delta}{2}, \tag{3.4.2}$$

where $R_m\Phi(z) = (\varphi_k(|z_k|))_{k=m}^{\infty}$. Hence, for $k \ge m$ we get $\varphi_k(|z_k|) \le 2^{-1}\delta$.
Now, by (3.4.1) and (3.4.2), we have

$$\begin{aligned}\|R_m\Phi(x)\|_\lambda &\le \|R_m\Phi(x-z)\|_\lambda + \|R_m\Phi(z)\|_\lambda \\ &\le \|\Phi(x-z)\|_\lambda + \|R_m\Phi(z)\|_\lambda \le \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned} \tag{3.4.3}$$

Therefore, $\varphi_k(|x_k|) \le \delta$ for all $k \ge m$. From (3.2.6) we deduce that

$$\psi_k(|f(k,x_k)|) \le a_k + \gamma(\varphi_k(|x_k|))^{p/q} \qquad (k \ge m). \tag{3.4.4}$$

Further, since the functions $f(k,\cdot)$ $(k = 1,\dots,m)$ are bounded on every bounded subset of real numbers $T_m(\varkappa)$ with $\varkappa = 2^{-1}\delta$, by Lemma 3.2.12 there exists $M > 0$ such that

$$\left\|\sum_{k=1}^{m-1}\psi_k(|f(k,x_k)|)e^k\right\|_\mu \le M. \tag{3.4.5}$$

If $\mu = \ell_q$, then by Minkowski's inequality and (3.4.3)–(3.4.5) we get

$$\|\Psi(P_f(x) - P_f(z))\|_\mu \le L + \|\Psi(P_f(z))\|_\mu, \tag{3.4.6}$$

where $L = M + \gamma\delta^{p/q} + \|(a_k)\|_{\ell_q}$. Otherwise, using (3.4.4) and (3.4.5), because of $\gamma = 0$ we obtain (3.4.6) with $L = \max\{M, \|(a_k)\|_\mu\}$.

Putting $\beta = L + \|\Psi(P_f(z))\|_\mu$, we have $\|\Psi(P_f(x) - P_f(z))\|_\mu \le \beta$ whenever $\|\Phi(x-z)\|_\lambda \le 2^{-1}\delta$. □

Our last theorem in this subsection describes the local boundedness of superposition operators on the space $(w_0)_p(\Phi)$.

**Theorem 3.4.2.** *Let $1 \le p, q < \infty$. If the moduli $\varphi_k$ $(k \in \mathbb{N})$ are strictly increasing, then a superposition operator $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ is locally bounded if and only if $f$ satisfies* (B).

*Proof.* The necessity of condition (B) follows from Lemma 3.2.9.

Conversely, suppose that $f$ satisfies (B), $P_f$ maps $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$ and $z = (z_k) \in (w_0)_p(\Phi)$. By Proposition 3.2.5, there exist a number $\delta > 0$ and sequences $(a_k) \in \ell^+$ and $(c_i)_{i=0}^{\infty} \in \ell^+$ such that (3.2.7) is satisfied whenever $(\varphi_k(|t|))^p \le 2^i\delta$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$. Since by (1.1.1),

$$\lim_{i\to\infty} 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p = 0,$$

there exists $\tilde{r} \in \mathbb{N}$ with

$$2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p < 2^{-p}\delta \quad (i \ge \tilde{r}). \tag{3.4.7}$$

Let $x = (x_k) \in (w_0)_p(\Phi)$ be such that

$$\|\Phi(x-z)\|_{(w_0)_p} \le 2^{-1}\delta^{1/p}. \tag{3.4.8}$$

Then by (ii), Minkowski's inequality, (3.4.7) and (3.4.8), for $i \ge \tilde{r}$, we have

$$\begin{aligned} \left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p\right)^{1/p} &\le \left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k - z_k|))^p\right)^{1/p} \\ &+ \left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p\right)^{1/p} \le \|\Phi(x-z)\|_{(w_0)_p} \\ &+ \left(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|z_k|))^p\right)^{1/p} \le 2^{-1}\delta^{1/p} + 2^{-1}\delta^{1/p} = \delta^{1/p}. \end{aligned} \tag{3.4.9}$$

Consequently, if $i \ge \tilde{r}$, then

$$\sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p \le 2^i\delta$$

and so $(\varphi_k(|x_k|))^p \le 2^i\delta \quad (2^i \le k < 2^{i+1})$. Further, by (3.2.7) we get

$$(\psi_k(|f(k, x_k)|))^q \le a_k + c_i 2^{-i}(\varphi_k(|x_k|))^p \tag{3.4.10}$$

Since the functions $f(k, \cdot)$ $(i < \tilde{r},\ 2^i \le k < 2^{i+1})$ are bounded on every bounded subset of real numbers $T_m(\varkappa)$ with $\varkappa = 2^{-1}\delta^{1/p}$, by Lemma 3.2.12 there exists $M > 0$ such that

$$\sum_{k=1}^{2^{\tilde{r}}-1} (\psi_k(|f(k, x_k)|))^q = \sum_{i=0}^{\tilde{r}-1} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k, x_k)|))^q \le M. \tag{3.4.11}$$

Consequently, by (ii), Minkowski's inequality and (3.4.9)–(3.4.11) we conclude

$$\begin{aligned}
&\|\Psi(P_f(x) - P_f(z))\|_{\ell_q} \le \left( \sum_{k=1}^{2^{\tilde{r}}-1} (\psi_k(|f(k, x_k)|))^q \right)^{1/q} \\
&+ \left( \sum_{k=2^{\tilde{r}}}^{\infty} (\psi_k(|f(k, x_k)|))^q \right)^{1/q} + \left( \sum_{k=1}^{\infty} (\psi_k(|f(k, z_k)|))^q \right)^{1/q} \\
&\le M^{1/q} + \left( \sum_{i=\tilde{r}}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k, x_k)|))^q \right)^{1/q} + \|\Psi(P_f(z))\|_{\ell_q} \\
&\le M^{1/q} + \left( \sum_{i=\tilde{r}}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} a_k \right)^{1/q} + \left( \sum_{i=\tilde{r}}^{\infty} c_i 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p \right)^{1/q} \\
&+ \|\Psi(P_f(z))\|_{\ell_q} \le M^{1/q} + \|(a_k)\|_{\ell}^{1/q} + \left( \delta \sum_{i=\tilde{r}}^{\infty} c_i \right)^{1/q} + \|\Psi(P_f(z))\|_{\ell_q} \\
&\le M^{1/q} + \|(a_k)\|_{\ell}^{1/q} + (\delta\ \|(c_i)_{i=0}^{\infty}\|_{\ell})^{1/q} + \|\Psi(P_f(z))\|_{\ell_q}.
\end{aligned}$$

So, putting $\beta = M^{1/q} + \|(a_k)\|_{\ell}^{1/q} + (\delta\ \|(c_i)_{i=0}^{\infty}\|_{\ell})^{1/q} + \|\Psi(P_f(z))\|_{\ell_q}$, we have $\|\Phi(P_f(x) - P_f(z))\|_{\ell_p} \le \beta$ whenever $\|\Phi(x - z)\|_{(w_0)_p} \le 2^{-1}\delta^{1/p}$. □

Local boundedness of superposition operators on $\ell_\infty(\Psi)$ is treated in Corollary 3.4.5.

### 3.4.2 Boundedness of $P_f$

Let $\lambda$ be a solid sequence space with $e^k \in \lambda$ $(k \in \mathbb{N})$ and $\mu$ be a solid BK-space. It is easy to verify that if $P_f \colon \lambda(\Phi) \to \mu(\Psi)$ is bounded then it is also locally bounded. So, Lemma 3.2.9 shows that $f$ satisfies the condition (B) if $P_f$ is bounded.

The boundedness of superposition operators into $\ell_q(\Psi)$ can be described as follows.

**Theorem 3.4.3.** *Let $1 \le p, q < \infty$ and $\lambda \in \{c_0, \ell_p, \ell_\infty\}$. For $\lambda = \ell_\infty$ we assume that the sequence of moduli $\Phi = (\varphi_k)$ satisfies one of conditions* (M) *and* (M′)*. Then a superposition operator $P_f \colon \lambda(\Phi) \to \ell_q(\Psi)$ is bounded if and only if for every $\varrho > 0$ there exist a sequence $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and a number $\gamma(\varrho) \ge 0$ such that*

$$\psi_k(|f(k,t)|) \le a_k(\varrho) + \gamma(\varrho)(\varphi_k(|t|))^{p/q} \qquad (\varphi_k(|t|) \le \varrho,\ k \in \mathbb{N}). \tag{3.4.12}$$

*Here $\gamma(\varrho) = 0$, if $\lambda \in \{c_0, \ell_\infty\}$. Furthermore,*

$$\eta_{f,\mu}(\varrho) \le \nu_{f,q}(\varrho) \le (1 + 2^{1/q})\ \eta_{f,\mu}(\varrho) \tag{3.4.13}$$

*for every $\varrho > 0$, where*

$$\nu_{f,q}(\varrho) = \inf\left\{\|a(\varrho)\|_{\ell_q} + \gamma(\varrho)\varrho^{p/q} \ :\ (3.4.12) \text{ holds}\right\}.$$

*In the case $\gamma(\varrho) = 0$ we have*

$$\eta_{f,\mu}(\varrho) = \nu_{f,q}(\varrho).$$

*Proof. Sufficiency.* Suppose that for every $\varrho > 0$ there exist a sequence $a(\varrho) \in \ell_q^+$ and a number $\gamma(\varrho) \ge 0$ such that for each $k \in \mathbb{N}$ the inequality (3.4.12) is true whenever $\varphi_k(|t|) \le \varrho$. Let $\varrho > 0$ and $x = (x_k) \in \lambda(\Phi)$ be such that

$$\|\Phi(x)\|_\lambda \le \varrho. \tag{3.4.14}$$

Since

$$\varphi_k(|x_k|) \le \|\Phi(x)\|_\lambda \le \varrho \qquad (k \in \mathbb{N}),$$

by (3.4.12) we deduce

$$\psi_k(|f(k,x_k)|) \le a_k(\varrho) + \gamma(\varrho)(\varphi_k(|x_k|))^{p/q} \qquad (k \in \mathbb{N})$$

which gives, in the case $\lambda = \ell_p$,

$$\|\Psi(P_f(x))\|_{\ell_q} = \left(\sum_{k=1}^{\infty}(\psi_k(|f(k,x_k)|))^q\right)^{1/q} \leq \left(\sum_{k=1}^{\infty}(a_k(\varrho))^q\right)^{1/q}$$
$$+ \gamma(\varrho)\left(\sum_{k=1}^{\infty}(\varphi_k(|x_k|))^p\right)^{1/q} \leq \|a(\varrho)\|_{\ell_q} + \gamma(\varrho)\ \left(\|\Phi(x)\|_{\ell_p}^p\right)^{1/q}$$
$$\leq \|a(\varrho)\|_{\ell_q} + \gamma(\varrho)\ \varrho^{p/q} < \infty. \tag{3.4.15}$$

If $\lambda \in \{c_0, \ell_\infty\}$, then we have

$$\|\Psi(P_f(x))\|_{\ell_q} \leq \left(\sum_{k=1}^{\infty}(a_k(\varrho))^q\right)^{1/q} = \|a(\varrho)\|_{\ell_q} < \infty. \tag{3.4.16}$$

The inequality $\eta_{f,\mu}(\varrho) \leq \nu_{f,q}(\varrho)$ holds because of (3.4.14) and (3.4.15) or (3.4.14) and (3.4.16).

*Necessity.* Let $P_f$ be a bounded superposition operator acting from $\lambda(\Phi)$ into $\ell_q(\Psi)$ and $x = (x_k) \in \lambda(\Phi)$. By Lemma 3.2.9 $f$ satisfies (B). For a fixed $\varrho > 0$, we have

$$\|\Psi(P_f(x))\|_{\ell_q} \leq \eta_{f,\mu}(\varrho)$$

whenever $\|\Phi(x)\|_\lambda \leq \varrho$.

If $\lambda = \ell_p$, then by Lemma 3.2.13 there exist a sequence $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ with $\|a(\varrho)\|_{\ell_q} \leq \eta_{f,\mu}(\varrho)$ such that for every $k \in \mathbb{N}$,

$$\psi_k(|f(k,t)|) \leq a_k(\varrho) + 2^{1/q}\varrho^{-p/q}\eta_{f,\mu}(\varrho)(\varphi_k(|t|))^{p/q}$$

provided $\varphi_k(|t|) \leq \varrho$. Putting $\gamma(\varrho) = 2^{1/q}\varrho^{-p/q}\mu_{f,\mu}(\varrho)$, we have (3.4.12).

From Lemma 3.2.13 we also get $\|a(\varrho)\|_{\ell_q} \leq \eta_{f,\mu}(\varrho)$, so

$$\begin{aligned}\|a(\varrho)\|_{\ell_q} + \gamma(\varrho)\varrho^{p/q} &\leq \eta_{f,\mu}(\varrho) + \gamma(\varrho)\varrho^{p/q}\\ &\leq \eta_{f,\mu}(\varrho) + 2^{1/q}\varrho^{-p/q}\eta_{f,\mu}(\varrho)\varrho^{p/q}\\ &\leq (1+2^{1/q})\eta_{f,\mu}(\varrho).\end{aligned}$$

Hence we have $\nu_{f,q}(\varrho) \leq (1+2^{1/q})\eta_{f,\mu}(\varrho)$.

Otherwise, i.e., for $\lambda \in \{c_0, \ell_\infty\}$, we define

$$a_k(\varrho) = \sup\{\psi_k(|f(k,t)|) \colon \varphi_k(|t|) \le \varrho\} \qquad (k \in \mathbb{N}). \tag{3.4.17}$$

Since $f$ satisfies (B), then $a_k(\varrho) < \infty$ for every $k \in \mathbb{N}$. The inequality (3.4.12) (with $\gamma(\varrho) = 0$) is immediately clear.

To prove that $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$, let $\varepsilon > 0$. By (3.4.17) there exists a sequence $y(\varrho,\varepsilon) = (y_k(\varrho,\varepsilon))$ such that

$$\varphi_k(|y_k(\varrho,\varepsilon)|) \le \varrho \qquad (k \in \mathbb{N}) \tag{3.4.18}$$

and

$$a_k(\varrho) \le \psi_k(|f(k, y_k(\varrho,\varepsilon))|) + \frac{\varepsilon}{2^k} \tag{3.4.19}$$

for every $k \in \mathbb{N}$. From (3.4.18) we get

$$\|y(\varrho,\varepsilon)\|_{\lambda(\Phi)} = \sup_k \varphi_k(|y_k(\varrho,\varepsilon)|) \le \varrho.$$

Using (3.4.19), we have

$$\begin{aligned}\left(\sum_{k=1}^{\infty}(a_k(\varrho))^q\right)^{1/q} &\le \left(\sum_{k=1}^{\infty}(\psi_k(|f(k,y_k(\varrho,\varepsilon))|))^q\right)^{1/q} \\ &\quad + \left(\sum_{k=1}^{\infty}\left(\frac{\varepsilon}{2^k}\right)^q\right)^{1/q} = \|\Psi(P_f(y(\varrho,\varepsilon)))\|_{\ell_q} + \varepsilon \\ &\le \eta_{f,\mu}(\varrho) + \varepsilon.\end{aligned}$$

Hence, by the arbitrariness of $\varepsilon$, we conclude that $a(\varrho) \in \ell_q^+$ with $\|a(\varrho)\|_{\ell_q} \le \eta_{f,\mu}(\varrho)$. This also shows that $\nu_{f,q}(\varrho) \le \eta_{f,\mu}(\varrho)$. □

Next we characterize the boundedness of superposition operator acting from $c_0(\Phi)$, $\ell_p(\Phi)$ $(1 \le p < \infty)$ and $\ell_\infty(\Phi)$ into $c_0(\Psi)$ and $\ell_\infty(\Phi)$.

**Theorem 3.4.4.** *Let $1 \le p < \infty$, $\lambda \in \{c_0, \ell_p, \ell_\infty\}$ and $\mu \in \{c_0, \ell_\infty\}$. Assume that for $\lambda = \ell_\infty$ ($\mu = \ell_\infty$) the sequence of moduli $\Phi = (\varphi_k)$ ($\Psi = (\psi_k)$) satisfies one of conditions* (M) *and* (M′)*. Then a superposition operator $P_f \colon \lambda(\Phi) \to \mu(\Psi)$ is bounded if and only if for every $\varrho > 0$ there exists a sequence $a(\varrho) = (a_k(\varrho)) \in \ell_\infty^+$ such that* (3.2.32) *holds. Furthermore, for every $\varrho > 0$,*

$$\eta_{f,\mu}(\varrho) = \nu_{f,\infty}(\varrho).$$

*Proof.* The sufficiency follows from Lemma 3.2.15. Moreover, we get $\eta_{f,\mu}(\varrho) \leq \nu_{f,\infty}(\varrho)$.

*Necessity.* Let $P_f \colon \lambda(\Phi) \to \mu(\Psi)$ be bounded and $x = (x_k) \in \lambda(\Phi)$. By Lemma 3.2.9 $f$ satisfies (B). For any fixed $\varrho > 0$ we have

$$\|\Psi(P_f(x))\|_\mu \leq \eta_{f,\mu}(\varrho)$$

provided $\|\Phi(x)\|_\lambda \leq \varrho$. Applying Lemma 3.2.14 with $\vartheta(\beta) = \eta_{f,\mu}(\varrho)$ and $\beta = \varrho$, we can find a sequence $a(\varrho) \in \ell_\infty^+$ with $\|a(\varrho)\|_{\ell_\infty} \leq \eta_{f,\mu}(\varrho)$ such that for every $k \in \mathbb{N}$,

$$\psi_k(|f(k,t)|) \leq a_k(\varrho)$$

whenever $\varphi_k(|t|) \leq \varrho$. Therefore, (3.2.32) is true. From the inequality

$$\|a(\varrho)\|_{\ell_\infty} = \sup_k a_k(\varrho) \leq \eta_{f,\mu}(\varrho)$$

it follows that $\nu_{f,\infty}(\varrho) \leq \eta_{f,\mu}(\varrho)$. □

**Corollary 3.4.5.** *Let $1 \leq q < \infty$ and $\mu \in \{c_0, \ell_q, \ell_\infty\}$. Assume that the sequence of moduli $\Phi = (\varphi_k)$ and for $\mu = \ell_\infty$ the sequence of moduli $\Psi = (\psi_k)$ satisfies one of conditions* (M) *and* (M$'$)*.Superposition operators $P_f$ from $\ell_\infty(\Phi)$ into $\mu(\Psi)$ are always bounded and hence locally bounded.*

*Proof.* By Proposition 3.2.3 operator $P_f$ acts $\ell_\infty(\Phi)$ into $\mu(\Psi)$ if and only if for every $\varrho > 0$ there exists a sequence $a(\varrho) = (a_k(\varrho)) \in \mu^+$ such that

$$\psi_k(|f(k,t)|) \leq a_k(\varrho) \qquad (\varphi_k(|t|) \leq \varrho,\ k \in \mathbb{N}).$$

Since $\mu^+ \subseteq \ell_\infty^+$, it remains to apply Theorems 3.4.3 and 3.4.4. □

Finally, we consider the boundeness of superposition operator on the space $(w_0)_p(\Phi)$.

**Theorem 3.4.6.** *Let $1 \leq p, q < \infty$. A superposition operator $P_f \colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ is bounded if and only if for every $\varrho > 0$ there are sequences $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and $c(\varrho) = (c_i(\varrho))_{i=0}^\infty \in \ell_q^+$ such that*

$$\psi_k(|f(k,t)|) \leq a_k(\varrho) + c_i(\varrho) 2^{-i/q} (\varphi_k(|t|))^{p/q} \tag{3.4.20}$$

*whenever $\varphi_k(|t|) \leq 2^{i/p}\varrho$, $2^i \leq k < 2^{i+1}$, $i \in \mathbb{N}_0$.*

*Furthermore, for every $\varrho > 0$,*

$$\eta_{f,w_0}(\varrho) \le \nu_{f,w_0}(\varrho) \le (1 + 2^{1/q})\eta_{f,w_0}(\varrho),$$

*where*

$$\eta_{f,w_0}(\varrho) = \sup\left\{\|\Psi(P_f(x))\|_{\ell_q} \ : \ \|\Phi(x)\|_{(w_0)_p} \le \varrho\right\}$$

*and*

$$\nu_{f,w_0}(\varrho) = \inf\Big\{\|a(\varrho)\|_{\ell_q} + \varrho^{p/q}\|c(\varrho)\|_{\ell_q} \ : \ (3.4.20) \text{ holds } (\varphi_k(|t|) \le 2^{i/p}\varrho, 2^i \le k < 2^{i+1}, \ i \in \mathbb{N}_0)\Big\}. \quad (3.4.21)$$

*Proof. Sufficiency.* Suppose that for every $\varrho > 0$ there are sequences $a(\varrho)$ and $c(\varrho)$ from $\ell_q^+$ such that the inequality (3.4.20) holds if $\varphi_k(|t|) \le 2^{i/p}\varrho$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$. Let $\varrho > 0$ and $x = (x_k) \in (w_0)_p(\Phi)$ be such that

$$\|\Phi(x)\|_{(w_0)_p} \le \varrho.$$

Then $\varphi_k(|x_k|) \le 2^{i/p}\varrho \quad (2^i \le k < 2^{i+1}, \ i \in \mathbb{N}_0)$ and (3.4.20) yields

$$\psi_k(|f(k, x_k)|) \le a_k(\varrho) + c_i(\varrho)2^{-i/q}(\varphi_k(|x_k|))^{p/q}.$$

So we have

$$\|\Psi(P_f(x))\|_{\ell_q} = \left(\sum_{i=0}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}(\psi_k(|f(k, x_k)|))^q\right)^{1/q}$$

$$\le \left(\sum_{i=0}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}(a_k(\varrho))^q\right)^{1/q} + \left(\sum_{i=0}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}\left(c_i(\varrho)2^{-i/q}(\varphi_k(|x_k|))^{p/q}\right)^q\right)^{1/q}$$

$$\le \|a(\varrho)\|_{\ell_q} + \left(\sum_{i=0}^{\infty}(c_i(\varrho))^q 2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p\right)^{1/q}$$

$$\le \|a(\varrho)\|_{\ell_q} + \left(\sum_{i=0}^{\infty}(c_i(\varrho))^q\varrho^p\right)^{1/q} \le \|a(\varrho)\|_{\ell_q} + \varrho^{p/q}\|c(\varrho)\|_{\ell_q} < \infty$$

whenever $\|\Phi(x)\|_{(w_0)_p} \le \varrho$.

The inequality $\eta_{f,w_0}(\varrho) \le \nu_{f,w_0}(\varrho)$ is obvious because of

$$\|\Psi(P_f(x))\|_{\ell_q} \le \|a(\varrho)\|_{\ell_q} + \varrho^{p/q}\|c(\varrho)\|_{\ell_q}$$

and $\|\Phi(x)\|_{(w_0)_p} \le \varrho$.

*Necessity.* Let $P_f$ be a bounded superposition operator acting from $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$ and $x = (x_k) \in (w_0)_p(\Phi)$. For fixed $\varrho > 0$ we have

$$\|\Psi(P_f(x))\|_{\ell_q} = \left(\sum_{k=1}^{\infty}(\psi_k(|f(k,x_k)|))^q\right)^{1/q} \le \eta_{f,w_0}(\varrho)$$

whenever

$$\|\Phi(x)\|_{(w_0)_p} = \sup_{i\ge 0}\left(2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p\right)^{1/p} \le \varrho.$$

We define, for every $i \in \mathbb{N}_0$,

$$\tilde{c}_i(\varrho) = \sup\left\{\left(\sum_{k=2^i}^{2^{i+1}-1}(\psi_k(|f(k,x_k)|))^q\right)^{1/q} : 2^{-i}\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|x_k|))^p \le \varrho^p\right\}. \quad (3.4.22)$$

Since $f$ satisfies (B) by Lemma 3.2.9, we see that $\tilde{c}_i(\varrho) < \infty$ $(i \in \mathbb{N}_0)$. Therefore, by definition of $\tilde{c}_i(\varrho)$, for every $\varepsilon > 0$ there exists a sequence $y(\varrho,\varepsilon) = (y_k(\varrho,\varepsilon))$ such that

$$\sum_{k=2^i}^{2^{i+1}-1}(\varphi_k(|y_k(\varrho,\varepsilon)|))^p \le 2^i\varrho^p \quad (3.4.23)$$

and

$$\tilde{c}_i(\varrho) \le \left(\sum_{k=2^i}^{2^{i+1}-1}(\psi_k(|f(k,y_k(\varrho,\varepsilon))|))^q\right)^{1/q} + \frac{\varepsilon}{2^i} \quad (3.4.24)$$

for any $i \in \mathbb{N}_0$.

Let $\tilde{r} \in \{0,1,2,\dots\}$ and $\tilde{y}(\varrho,\varepsilon) = (\tilde{y}_k(\varrho,\varepsilon))$ be a sequence with

$$\tilde{y}_k(\varrho,\varepsilon) = \begin{cases} y_k(\varrho,\varepsilon) & \text{if } 1 \le k \le 2^{\tilde{r}}, \\ 0 & \text{if } k > 2^{\tilde{r}}. \end{cases}$$

Then, by (3.4.23), we have

$$\|\Phi(\tilde{y}(\varrho,\varepsilon))\|_{(w_0)_p} \le \varrho.$$

Next, we show that $\tilde{c}(\varrho) = (\tilde{c}_i(\varrho))_{i=0}^{\infty} \in \ell_q^+$ and $\|\tilde{c}(\varrho)\|_{\ell_q} \le \eta_{f,w_0}(\varrho)$. Indeed, using (3.4.24), we get

$$\begin{aligned}
&\left(\sum_{i=0}^{\tilde{r}} (\tilde{c}_i(\varrho))^q\right)^{1/q} \\
&\le \left(\sum_{i=0}^{\tilde{r}} \left(\left(\sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k,y_k(\varrho,\varepsilon))|))^q\right)^{1/q} + \frac{\varepsilon}{2^i}\right)^q\right)^{1/q} \\
&\le \left(\sum_{i=0}^{\tilde{r}} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k,y_k(\varrho,\varepsilon))|))^q\right)^{1/q} + \left(\sum_{i=0}^{\tilde{r}} \left(\frac{\varepsilon}{2^i}\right)^q\right)^{1/q} \\
&\le \|\Psi(P_f(\tilde{y}(\varrho,\varepsilon)))\|_{\ell_q} + \varepsilon \le \eta_{f,w_0}(\varrho) + \varepsilon < \infty.
\end{aligned}$$

Thus

$$\|\tilde{c}(\varrho)\|_{\ell_q} = \left(\sum_{i=0}^{\infty} (\tilde{c}_i(\varrho))^q\right)^{1/q} = \lim_{\tilde{r}\to\infty} \left(\sum_{i=0}^{\tilde{r}} (\tilde{c}_i(\varrho))^q\right)^{1/q} \le \eta_{f,w_0}(\varrho) + \varepsilon.$$

While $\varepsilon > 0$ is arbitrary, then $\tilde{c}(\varrho) \in \ell_q^+$ with $\|\tilde{c}(\varrho)\|_{\ell_q} \le \eta_{f,w_0}(\varrho)$.

On the other hand, for every $i \in \mathbb{N}_0$,

$$\left(\sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k,x_k)|))^q\right)^{1/q} \le \tilde{c}_i(\varrho)$$

whenever

$$2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p \le \varrho^p.$$

Applying Lemma 3.2.13 to the previous inequality with $\beta^p = 2^i \varrho^p$, $\vartheta(\beta) = \tilde{c}_i(\varrho)$ and $f(k,t) = 0$ for $k \ne 2^i, 2^i+1, \ldots, 2^{i+1}-1$, we can find numbers $a_k(\varrho)$ $(k = 2^i, 2^i+1, \ldots, 2^{i+1}-1)$ such that

$$\sum_{k=2^i}^{2^{i+1}-1} (a_k(\varrho))^q \le \|a(\varrho)\|_{\ell_q}^q \le (\tilde{c}_i(\varrho))^q,$$

$$\psi_k(|f(k,t)|) \leq a_k(\varrho) + 2^{1/q}\ 2^{-i/q}\varrho^{-p/q}\tilde{c}_i(\varrho)(\varphi_k(|t|))^{p/q} \tag{3.4.25}$$

provided $\varphi_k(|x_k|) \leq 2^{i/p}\varrho$, $2^i \leq k < 2^{i+1}$. Putting $c_i(\varrho) = 2^{1/q}\ \varrho^{-p/q}\tilde{c}_i(\varrho)$ we have (3.4.20).

So we get

$$\|a(\varrho)\|_{\ell_q}^q = \sum_{k=1}^{\infty}(a_k(\varrho))^q = \sum_{i=0}^{\infty}\sum_{k=2^i}^{2^{i+1}-1}(a_k(\varrho))^q \leq \sum_{i=0}^{\infty}(\tilde{c}_i(\varrho))^q = \|\tilde{c}(\varrho)\|_{\ell_q}^q$$

which yields

$$\|a(\varrho)\|_{\ell_q} \leq \|\tilde{c}(\varrho)\|_{\ell_q} \leq \eta_{f,w_0}(\varrho).$$

By (3.4.25) it follows

$$\begin{aligned}
a_k(\varrho) + 2^{-i/q}c_i(\varrho)(\varphi_k(|t|))^{p/q} &\leq a_k(\varrho) + 2^{-i/q}2^{1/q}\varrho^{-p/q}\tilde{c}_i(\varrho)(\varphi_k(|t|))^{p/q} \\
&\leq \|a(\varrho)\|_{\ell_q} + 2^{-i/q}2^{1/q}\varrho^{-p/q}\|c(\varrho)\|_{\ell_q}(2^{i/p}\varrho)^{p/q} \\
&\leq \eta_{f,w_0}(\varrho) + 2^{1/q}\|\tilde{c}(\varrho)\|_{\ell_q} \leq \eta_{f,w_0}(\varrho) + 2^{1/q}\eta_{f,w_0}(\varrho) \\
&= (1+2^{1/q})\eta_{f,w_0}(\varrho)
\end{aligned}$$

whenever $\varphi_k(|x_k|) \leq 2^{i/p}\varrho$ and $i \in \mathbb{N}_0$. Hence

$$\nu_{f,w_0}(\varrho) \leq (1+2^{1/q})\eta_{f,w_0}(\varrho).$$

□

## 3.5 Applications

The classical sequence spaces $c_0$, $\ell_p$, $\ell_\infty$ $(1 \leq p < \infty)$ can be considered as the spaces $c_0(\Phi)$, $\ell_p(\Phi)$, $\ell_\infty(\Phi)$, where $\Phi = (\varphi_k)$ with $\varphi_k(t) = t$ $(k \in \mathbb{N})$. For $\Psi = \Phi$ from Theorems 3.3.1–3.3.6 we conclude the continuity of superposition operators from $\ell_\infty$, $\ell_p$ and $c_0$ into $\ell_q$ and $c_0$ for $1 \leq p, q < \infty$ (see [10], Theorems 2, 7 and 8; [44], Theorems 2.4 and 2.5) and from Theorems 3.4.3, 3.4.4 and Corollary 3.4.5 we get known characterizations of the local boundedness and boundedness of superposition operators in sequence spaces $c_0$, $\ell_p$, $\ell_\infty$ ([10], Theorems 3, 7 and 8). We remark that Theorems 7 and 8 of [10] are formulated without proofs.

Theorems 3.3.7, 3.4.2 and 3.4.6 allows to formulate extensions of some results of Płuciennik ([45], Theorems 2, 3 and 5) about the continuity and the boundedness of superposition operator on $w_0$.

**Proposition 3.5.1.** *Let $1 \le p, q < \infty$. If the moduli $\varphi_k$ $(k \in \mathbb{N})$ are strictly increasing, then a superposition operator $P_f\colon (w_0)_p \to \ell_q$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous, i.e., $f$ satisfies* (C).

**Proposition 3.5.2.** *Let $1 \le p, q < \infty$. A superposition operator $P_f\colon (w_0)_p \to \ell_q$ is locally bounded if and only if $f$ satisfies* (B).

**Proposition 3.5.3.** *Let $1 \le p, q < \infty$. A superposition operator $P_f\colon (w_0)_p \to \ell_q$ is bounded if and only if for every $\varrho > 0$ there are sequences $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and $c(\varrho) = (c_i(\varrho))_{i=0}^{\infty} \in \ell_q^+$ such that*

$$|f(k,t)| \le a_k(\varrho) + c_i(\varrho) 2^{-i/q} |t|^{p/q} \tag{3.5.1}$$

*whenever $|t| \le 2^{i/p}\ \varrho$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$.*
*Furthermore,*

$$\overline{\eta}_{f,w_0}(\varrho) \le \overline{\nu}_{f,w_0}(\varrho) \le (1 + 2^{1/q}) \overline{\eta}_{f,w_0}(\varrho)$$

*for every $\varrho > 0$ with*

$$\overline{\eta}_{f,w_0}(\varrho) = \sup\left\{ \|P_f(x)\|_{\ell_q} \ :\ \|x\|_{(w_0)_p} \le \varrho^{1/p} \right\}$$

*and*

$$\overline{\nu}_{f,w_0}(\varrho) = \inf\left\{ \|a(\varrho)\|_{\ell_q} + \varrho^{p/q} \|c(\varrho)\|_{\ell_q} \colon (3.5.1) \text{ holds} \right.$$
$$\left. (|t| \le 2^{i/q}\varrho,\ 2^i \le k < 2^{i+1},\ i \in \mathbb{N}_0) \right\}. \tag{3.5.2}$$

As certain generalizations of the spaces $\ell_\infty$, $c_0$, $\ell_p$ and $w_0$ we consider the multiplier sequence spaces of Maddox type

$$\ell_\infty(p,u) = \left\{ x \in \omega : \sup_k |u_k x_k|^{p_k} < \infty \right\},$$
$$c_0(p,u) = \left\{ x \in \omega : \lim_k |u_k x_k|^{p_k} = 0 \right\},$$
$$\ell(p,u) = \left\{ x \in \omega : \sum_{k=1}^{\infty} |u_k x_k|^{p_k} < \infty \right\},$$
$$w_0(p,u) = \left\{ x \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} |u_k x_k|^{p_k} = 0 \right\},$$

where $u = (u_k)$ is a sequence with $u_k \neq 0$ $(k \in \mathbb{N})$ and $p = (p_k)$ is a bounded sequence of strictly positive numbers (cf. [18]). Some authors ([2, 52, 49]) consider the spaces $\ell_\infty(p,u)$, $c_0(p,u)$ and $\ell(p,v)$ for special multipliers

$$u_k = k^{-\alpha/p_k}, \quad v_k = k^{\alpha/p_k} \quad (\alpha > 0). \tag{3.5.3}$$

In the case $u_k = 1$ $(k \in \mathbb{N})$ the spaces $\ell_\infty(p,u)$, $c_0(p,u)$, $\ell(p,u)$ and $w_0(p,u)$ are known as the sequence spaces of Maddox type $\ell_\infty(p)$, $c_0(p)$, $\ell(p)$ and $w_0(p)$, respectively (see, for example, [17] and [29]). We note that the sequence spaces of type $\ell(p)$ were introduced much earlier by Orlicz [40].

To apply our theorems for the multiplier spaces of Maddox type, we put $r = \max\{1, \sup_k p_k\}$ and define the sequence of moduli $\Phi = (\varphi_k)$ by

$$\varphi_k(t) = (|u_k|t)^{p_k/r} \qquad (k \in \mathbb{N}).$$

Then the spaces $\ell_\infty(p,u)$, $c_0(p,u)$, $\ell(p,u)$ and $w_0(p,u)$ we may consider as the spaces $\ell_\infty(\Phi)$, $c_0(\Phi)$, $\ell_r(\Phi)$ and $(w_0)_r(\Phi)$, respectively. So, by Propositions 2.3.3 and 2.3.7, the F-norm

$$g_\Phi(x) = \sup_k |u_k x_k|^{p_k/r}$$

is defined on $c_0(p,u)$ for any $p$ and on $\ell_\infty(p,u)$ under the restriction $\inf_k p_k > 0$. We remark that if $\inf_k p_k > 0$, then $\ell_\infty(p) = \ell_\infty$ and $\ell_\infty(p,u)$ reduces to normed space

$$\ell_\infty(u) = \left\{ x \in \omega \colon \|x\| = \sup_k |u_k x_k| < \infty \right\}.$$

The corresponding F-norms on $\ell(p,u)$ and $w_0(p,u)$ are determined, respectively, by

$$g_\Phi(x) = \left( \sum_{k=1}^{\infty} |x_k|^{p_k} \right)^{1/r}$$

and

$$g_\Phi(x) = \sup_{i \geq 0} \left( \frac{1}{2^i} \sum_{k=2^i}^{2^{i+1}-1} |u_k x_k|^{p_k} \right)^{1/r}.$$

It is not difficult to formulate the acting conditions for superposition operators on multiplier sequence spaces of Maddox type based on Propositions 3.2.1–3.2.6. Thereby, for the multipliers (3.5.3) we get known characterizations of the operators $P_f : \ell_\infty(p,u) \to \ell$ and $P_f : \ell(p,v) \to \ell$ ([49], Theorems 1 and 8; [52], Theorems 2.1 and 2.2, the case $p_k = 1 \ \ (k \in \mathbb{N})$).

Let $q = (q_k)$ be another bounded sequence of strictly positive numbers and $v = (v_k)$ be a sequence such that $v_k \neq 0$ $(k \in \mathbb{N})$. Now, putting $s = \max\{1, \sup_k q_k\}$ and defining the sequence of moduli $\Psi = (\psi_k)$ by

$$\psi_k(t) = (|v_k|t)^{q_k/s} \quad (k \in \mathbb{N}),$$

from Theorems 3.3.1–3.3.7 we get the following statements about the continuity of superposition operators on multiplier sequence spaces of Maddox type.

**Proposition 3.5.4.** *Superposition operators $P_f\colon \ell(p,u) \to \ell(q,v)$, $P_f\colon \ell(p,u) \to c_0(q,v)$, $P_f\colon c_0(p,u) \to c_0(q,v)$, $P_f\colon c_0(p,u) \to \ell(q,v)$ and $P_f\colon w_0(p,u) \to \ell(q,v)$ are continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

**Proposition 3.5.5.** *If $\inf\limits_k p_k > 0$, then $P_f\colon \ell_\infty(p,u) \to \ell(q,v)$ and $P_f\colon \ell_\infty(p,u) \to c_0(q,v)$ are continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

Basing on Theorems 3.4.1–3.4.4, 3.4.6 and Corollary 3.4.5 we get the following statements about the boundedness of superposition operators on multiplier sequence spaces of Maddox type.

**Proposition 3.5.6.** *Let $\lambda \in \{c_0(p,u), \ell(p,u)\}$ and $\mu \in \{c_0(q,v), \ell(q,v), \ell_\infty(q,v)\}$. For $\mu = \ell_\infty(q,v)$ we assume that $\inf_k q_k > 0$. A superposition operator $P_f\colon \lambda \to \mu$ is locally bounded if and only if $f$ satisfies* (B).

**Proposition 3.5.7.** *A superposition operator $P_f$ acting $w_0(p,u)$ into $\ell(q,v)$ is locally bounded if and only if $f$ satisfies* (B).

We use the notation

$$\tilde{\eta}_{f,g_\Psi}(\varrho) = \sup\{g_\Psi(P_f(x)) \ : \ g_\Phi(x) \le \varrho\}$$

for every $\varrho > 0$.

**Proposition 3.5.8.** *Let $\lambda \in \{c_0(p,u), \ell(p,u), \ell_\infty(p,u)\}$. For $\lambda = \ell_\infty(p,u)$ we assume, in addition, that $\inf_k p_k > 0$. A superposition operator $P_f\colon \lambda \to \ell(q,v)$ is bounded if and only if for every $\varrho > 0$ there exist a sequence $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and a number $\gamma(\varrho) \geq 0$ such that*

$$|v_k f(k,t)|^{q_k/s} \leq a_k(\varrho) + \gamma(\varrho)|u_k t|^{p_k/s} \qquad (|u_k t|^{p_k/r} \leq \varrho,\ k \in \mathbb{N}). \tag{3.5.4}$$

*If $\lambda \in \{c_0(p,u), \ell_\infty(p,u)\}$, then $\gamma(\varrho) = 0$. Furthermore, for every $\varrho > 0$,*

$$\tilde{\eta}_{f,g_\Psi}(\varrho) \leq \tilde{\nu}_{f,q}(\varrho) \leq (1 + 2^{1/s})\tilde{\eta}_{f,g_\Psi}(\varrho),$$

*where*

$$\tilde{\nu}_{f,q}(\varrho) = \inf\left\{\|a(\varrho)\|_{\ell_q} + \gamma(\varrho)\ \varrho^{r/s}\ :\ (3.5.4) \text{ is satisfied}\right\}.$$

*In the case $\gamma(\varrho) = 0$, we have*

$$\tilde{\eta}_{f,g_\Psi}(\varrho) = \tilde{\nu}_{f,q}(\varrho).$$

**Proposition 3.5.9.** *Let $\lambda \in \{c_0(p,u), \ell(p,u), \ell_\infty(p,u)\}$ and $\mu \in \{c_0(q,v), \ell_\infty(q,v)\}$. For $\lambda = \ell_\infty(p,u)$ ($\mu = \ell_\infty(q,v)$) we assume, in addition, that $\inf_k p_k > 0$ ($\inf_k q_k > 0$). A superposition operator $P_f\colon \lambda \to \mu$ is bounded if and only if for every $\varrho > 0$ there exists a sequence $a(\varrho) = (a_k(\varrho)) \in \ell_\infty^+$ such that*

$$|v_k f(k,t)|^{q_k/s} \leq a_k(\varrho) \qquad (|u_k t|^{p_k/r} \leq \varrho,\ k \in \mathbb{N}). \tag{3.5.5}$$

*Furthermore, for every $\varrho > 0$,*

$$\tilde{\eta}_{f,g_\Psi} = \tilde{\nu}_{f,\infty}(\varrho),$$

*where*

$$\tilde{\nu}_{f,\infty}(\varrho) = \inf\left\{\|a(\varrho)\|_{\ell_\infty}\colon\ (3.5.5) \text{ is satisfied}\right\}.$$

**Corollary 3.5.10.** *Let $\mu \in \{c_0(q,v), \ell(q,v), \ell_\infty(q,v)\}$. For $\mu = \ell_\infty(q,v)$ we assume that $\inf_k q_k > 0$. Superposition operators $P_f$ from $\ell_\infty(p,u)$ into $\mu$ are always bounded and hence locally bounded.*

**Proposition 3.5.11.** *A superposition operator $P_f$ acting $w_0(p,u)$ into $\ell(q,v)$ is bounded if and only if for every $\varrho > 0$ there are sequences $a(\varrho) = (a_k(\varrho)) \in \ell_q^+$ and $c(\varrho) = (c_i(\varrho))_{i=0}^\infty \in \ell_q^+$ such that*

$$|v_k f(k,t)| \leq a_k(\varrho) + c_i(\varrho) 2^{-i/q_k} |u_k t|^{p_k/q_k} \tag{3.5.6}$$

*whenever* $|u_k t| \le 2^i\ \varrho$, $2^i \le k < 2^{i+1}$, $i \in \mathbb{N}_0$.
*Furthermore,*

$$\tilde{\eta}_{f,w_0}(\varrho) \le \tilde{\nu}_{f,w_0}(\varrho) \le (1 + 2^{1/s})\tilde{\eta}_{f,w_0}(\varrho),$$

*for every* $\varrho > 0$ *with*

$$\tilde{\eta}_{f,w_0}(\varrho) = \sup\left\{\|P_f(x)\|^s_{\ell(q,v)}\ :\ \|x\|_{w_0(p,u)} \le \varrho^{1/r}\right\}$$

*and*

$$\tilde{\nu}_{f,w_0}(\varrho) = \inf\Big\{\|a(\varrho)\|_\ell + \varrho\|c(\varrho)\|_\ell\ :\ \text{(3.5.6) holds } (|u_k t|^{p_k} \le 2^i\varrho,\ 2^i \le k < 2^{i+1},\ i \in \mathbb{N}_0)\Big\}. \quad (3.5.7)$$

Sama-ae ([52], Theorems 6 and 14) considered the continuity of superposition operators $P_f : \ell_\infty(p,u) \to \ell$ and $P_f : \ell(p,v) \to \ell$ for the multipliers (3.5.3). Suantai ([52], Theorems 3.1–3.3 and 3.5, the case $p_k = 1$ ($k \in \mathbb{N}$)) and Sama-ae ([49], Theorems 2, 9 and 13, Corollary 3) studied the local boundedness and boundedness of superposition operators $P_f : \ell_\infty(p,u) \to \ell$ and $P_f : \ell(p,v) \to \ell$ for the multipliers (3.5.3).

# Moodulfunktsioonide abil defineeritud jadaruumid ja superpositsioonoperaatorid

Jadaruumide teoorias on üheks uurimisobjektiks Orliczi jadaruumid. Orliczi funktsiooni $\varphi$ korral saab Orliczi jadaruumi defineerida võrdusega

$$\ell^{\exists}(\varphi) = \left\{ x = (x_k)\colon \sum_{k=1}^{\infty} \varphi\Big(|x_k|/\varrho\Big) < \infty \text{ mingi } \varrho > 0 \text{ korral} \right\} =$$
$$= \{x = (x_k)\colon (\varphi(|x_k|/\varrho)) \in \ell \text{ mingi } \varrho > 0 \text{ korral}\}.$$

Ruckle [48] ja Maddox [30] tõid antud soliidse jadaruumi $\lambda$ ja moodulfunktsiooni $\varphi$ korral sisse uue jadaruumi

$$\lambda(\varphi) = \{x = (x_k)\colon (\varphi(|x_k|)) \in \lambda\}.$$

Jadaruumi $\lambda(\varphi)$ mõistet üldistas Kolk [21], asendades moodulfunktsiooni $\varphi$ moodulite jadaga $\Phi = (\varphi_k)$ ja vaadeldes ruumi

$$\lambda(\Phi) = \{x = (x_k)\colon (\varphi_k(|x_k|)) \in \lambda\}.$$

Ruumi $\lambda(\Phi)$ definitsioon sisaldab erijuhuna Maddox'i tüüpi jadaruume [16, 28], mis omakorda üldistavad vastavaid klassikalisi jadaruume $\ell_\infty$, $c_0$, $\ell_p$ ja $(w_0)_p$ $(1 \leq p < \infty)$.

Et käsitleda selliseid ruume ühtsest ja üldisemast vaatepunktist, lähtume nn. $\varphi$-funktsiooni mõistest, mis üldistab moodul- ja Orliczi funktsiooni mõisteid. Artiklites [16, 21] vaadeldud sisalduvusi on võimalik $\varphi$-funktsioonide abil käsitleda üldisemal kujul. Saadud tulemused on esitatud peatükis 1.3.

Jadaruumide teoorias pakub olulist huvi ka ruumide $\lambda(\varphi)$ ja $\lambda(\Phi)$ topologiseerimine. Moodulfunktsioonide jada $\Phi = (\varphi_k)$ korral vektorruum $\lambda(\Phi)$ ei ole enamasti normeeritav, siin tuleb normi asemel kasutada üldisemaid funktsionaale, näiteks F-poolnormi (vt. [22, 23]) või paranormi (vt. [50]). Moodulfunktsioonide maatriksi $\mathcal{F} = (f_{ki})$ ja soliidse topeltjadade ruumi $\Lambda$ korral kirjeldame jadaruumi

$$\Lambda(\mathcal{F}) = \{x = (x_k) \colon (f_{ki}(|x_k|)) \in \Lambda\}$$

F-poolnormeeritavust. Saadud tulemused üldistavad artiklites [22, 23, 50] tõestatud teoreeme (vt. peatükk 2.3 ja 2.4).

Ruumide $\lambda(\Phi)$ topologiseerimisvõimalus lubab uurida sellistes ruumides tegutsevate operaatorite erinevaid omadusi, nt. pidevust, tõkestatust jne. Meid huvitavad nn. superpositsioonoperaatorid, mis moodustavad ühe alamklassi kõigi (lineaarsete ja mittelineaarsete) operaatorite hulgas.

Superpositsioonoperaatoreid ei ole jadaruumides uuritud nii põhjalikult kui funktsioonaalruumides (vt. [1]). *Superpositsioonoperaator* jadaruumist $\lambda$ jadaruumi $\mu$ defineeritakse seosega

$$P_f(x) = (f(k, x_k)) \in \mu \quad (x = (x_k) \in \lambda),$$

kus $f \colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ on mingi funktsioon omadusega $f(k, 0) = 0$ ($k \in \mathbb{N}$). Üldiselt superpositsioonoperaator $P_f$ on mittelineaarne. Mõned nimetatud operaatori omadused võib leida Appelli ja Zabreĭko raamatust [1].

Robert [47] ja Šragin [51] kirjeldasid operaatorit $P_f$ Orliczi jadaruumides. Superpositsioonoperaatoreid jadaruumides $\ell_\infty$, $c_0$ ja $\ell_p$, kui $1 \leq p < \infty$, on mitmekülgselt uurinud Dedagich ja Zabreĭko [10] (vt. ka [8, 44]). Płuciennik [45, 46] vaatles superpositsioonoperaatoreid jadaruumis $w_0$. Superpositsioonoperaatorite pidevust ja tõkestatust jadaruumides on uuritud artilites [9, 44, 49, 52, 53]. Käesolevas doktoritöös antakse tarvilikud ja piisavad tingimused superpositsioonoperaatorite pidevuseks, lokaalseks tõkestatuseks ja tõkestatuseks moodulfunktsioonide jada abil defineeritud jadaruumides. Saadud tulemuste rakendusena vaatleme superpositsioonoperaatoreid Maddox'i tüüpi jadaruumides (vt. peatükk 3.5). Põhitulemused on esitatud peatükkides 3.3 ja 3.4.

# References

[1] J. Appell and P. P. Zabreĭko, *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics, Vol. 95, Cambrige University Press, Cambrige, 1990.

[2] M. Başarir, *On some new sequence spaces and related matrix transformations*, Indian J. Pure Appl. Math. **26** (10) (1995), 1003–1010.

[3] T. Bilgin, *On strong A-summability defined by a modulus*, Chinese J. Math. **24** (1996), no. 2, 159–166.

[4] J. Boos, T. Leiger, and K. Zeller, *Consistency theory for SM-methods*, Acta Math. Hungar. **76** (1997), no. 1–2, 109–142.

[5] V. K. Bhardwaj and N. Singh, *Some sequence spaces defined by Orlicz functions*, Demonstratio Math. **33** (2000), 571–582.

[6] J. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull. **32** (1989), 194–198.

[7] T. S. Chew, *Superposition operators on $w_0$ and $W_0$*, Comment. Math. Prace Mat. **29** (1990), 149–153.

[8] T. S. Chew and P. Y. Lee, *Orthogonally additive functionals on sequence spaces*, Southeast Asian Bull. Math. **9** (1985), 81–85.

[9] B. Choudhary, *A note on boundedness of superposition operators on sequence spaces*, J. Anal. **8** (2000), 55–64.

[10] F. Dedagich and P. P. Zabreĭko, *On superposition operators in $\ell_p$ spaces*, Sibirsk. Mat. Zh. **28** (1987), no.1, 86–98 (Russian); English translation: Siberian Math. J. **28** (1987), no. 1, 63–73.

[11] *Encyclopedia of Mathematics*, Vol. 1, M. Hazewinkel Kluwer Academic Publishers, Dortrecht, 1995.

[12] A. Esi, *Some new sequence spaces defined by a sequence of moduli*, Turkish J. Math. **21** (1997), 61–68.

[13] A. Esi, *Some new sequence spaces defined by a modulus function*, Math. Slovaca **49** (1999), 53–61.

[14] A. Esi, *Some new sequence spaces defined by Orlicz functions*, Bull. Inst. Math. Acad. Sinica **27** (1999), 71–76.

[15] D. Ghosh and P. D. Srivastava, *On some vector valued sequence space using Orlicz function*, Glas. Math. Ser. III **34** (1999), 253–261.

[16] R. J. Grinnell, *Functions preserving sequence spaces*, Real Anal. Exchange **25** (1999/2000), 239–256.

[17] K.-G. Grosse-Erdmann, *The structure of the sequence spaces of Maddox*, Canad. J. Math. **4**4 (1992), 298–307.

[18] Mushir A. Khan, Qamaruddin, *Some generalized sequence spaces and related matrix transformations*, Far East J. Math. Sci. **5** (1997), 243–252.

[19] E. Kolk, *Sequence spaces defined by a sequence of moduli*, In: Abstracts of conference "Problems of Pure and Applied Mathematics", Tartu, 1990, p. 131–134.

[20] E. Kolk, *On strong boundedness and summability with respect to a sequence of moduli*, Tartu Ül. Toimetised **960** (1993), 41–50.

[21] E. Kolk, *Inclusion theorems for some sequence spaces defined by a sequence of moduli*, Tartu Ül. Toimetised **970** (1994), 65–72.

[22] E. Kolk, *F-seminormed sequence spaces defined by a sequence of modulus functions and strong summability*, Indian J. Pure Appl. Math. **28** (1997), 1547–1566.

[23] E. Kolk, *Counterexamples concerning topologization of spaces of strongly almost convergent sequences*, Acta Comment. Univ. Tartuensis Math. **3** (1999), 63–72.

[24] E. Kolk, *Superposition operators on sequence spaces defined by $\varphi$-functions*, Demostratio Math. **37** (2004), 159–175.

[25] E. Kolk and A. Mölder, *Inclusion theorems for some sets of sequence spaces defined by $\varphi$-functions*, Math. Slovaca **54** (2004), no. 3, 267–279.

[26] E. Kolk and A. Mölder, *The continuity of superposition operators on some sequence spaces defined by moduli*, Czechoslovak Math. J. (submitted).

[27] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I. Sequence Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92, Springer-Verlag, Berlin–New York, 1977.

[28] P. Y. Lee, *Sequence spaces and the gilding hump property*, Southeast Asian Bull. Math. **17** (1993), 65–72.

[29] Y. Luh, *Die Räume $\ell(p)$, $\ell_\infty(p)$, $c(p)$, $c_0(p)$, $w(p)$, $w_0(p)$ und $w_\infty(p)$*, Mitt. Math. Sem. Giessen **180** (1987), 35–37.

[30] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc. **100** (1986), 161–166.

[31] I. J. Maddox, *Inclusions between FK-spaces and Kuttner's theorem*, Math. Proc. Camb. Phil. Soc. **101** (1987), 523–527.

[32] I. J. Maddox, *A class of dual sequence spaces*, Tartu Ül. Toimetised **960** (1993), 67–74.

[33] E. Malkowsky and E. Savas, *Some $\lambda$-sequence spaces defined by a modulus*, Arch. Math. (Brno) **36** (2000), 219–228.

[34] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1983.

[35] A. Mölder, *The topologization of sequence spaces defined by a matrix of moduli*, Proc. Estonian Acad. Sci. Phys. Math. **53** (2004), no. 4, 218–225.

[36] A. Mölder, *Boundedness of superposition operators on some sequence spaces defined by moduli*, Demonstratio Math. (accepted).

[37] A. Mölder, *Boundedness of superposition operators on sequence space* $(w_0)_p(\Phi)$, Acta Comment. Univ. Tartuensis Math. (submitted).

[38] H. Nakano, *Concave modulus*, J. Math. Soc. Japan. **5** (1953), 29–49.

[39] S. Nanda, *Strongly almost summable and strongly almost convergent sequences*, Acta Math. Hungar. **49** (1987), 71–76.

[40] W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math. **3** (1931), 200–211.

[41] S. D. Parashar and B. Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math. **25** (1994), 419–428.

[42] S. Pehlivan, *On strong almost convergence and uniform statistical convergence*, Acta Comment. Univ. Tartuensis Math. **2** (1998), 19–22.

[43] S. Pehlivan and B. Fisher, *Some sequence spaces defined by a modulus*, Math. Slovaca, **45** (1995), 275–280.

[44] S. Petranuarat and Y. Kemprasit, *Superposition operators of* $\ell_p$ *and* $c_0$ *into* $\ell_q$ $(1 \leq p, q < \infty)$, Southeast Asian Bull. Math. **21** (1997), 139–147.

[45] R. Płuciennik, *Continuity of superposition operators on* $w_0$ *and* $W_0$, Comment. Math. Univ. Carolin. **31** (1990), 529–542.

[46] R. Płuciennik, *Boundedness of superposition operators on* $w_0$, Southeast Asian Bull. Math. **15** (1991), 145–151.

[47] J. Robert, *Continuité d'un opérateur non linéaire sur certains espaces de suites*, C. R. Acad. Sci. Paris, Ser. A. **259** (1964), 1287–1290.

[48] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25** (1973), 973–978.

[49] A. Sama-ae, *Boundedness and continuity of superposition operator on* $E_r(p)$ *and* $F_r(p)$, Songklanakarin J. Sci. Technol. **24** (2002), 451–466.

[50] V. Soomer, *On the sequence space defined by a sequence of moduli and on the rate-space*, Acta Comment. Univ. Tartuensis Math. **1** (1996), 71–74.

[51] I. V. Šragin, *Conditions for the embedding of classes of sequence spaces, and their consequences*, Mat. Zametki **20** (1976), 681–692 (Russian); English translation: Math. Notes **20** (1976), 942–948.

[52] S. Suantai, *Boundedness of superposition operators on $E_r$ and $F_r$*, Comment. Math. Prace Mat. **37** (1997), 249–259.

[53] S. D. Unoningsih, R. Płuciennik and L. P. Yee, *Boundedness of superposition operators on sequence spaces*, Comment. Math. Prace Mat. **35** (1995), 209–216.

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