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*To my family*

# ABSTRACT

It is possible to solve most mathematical problems, including equations of school mathematics, with the help of Computer Algebra Systems (CAS). However, while many answers offered by CAS (*CAS answers*) do not differ from the answers expected in a school context (*school answers*), there are some exceptions.

Such unexpected CAS answers are often correct and could offer opportunities for learning by utilising the computational power provided by CAS, but students and teachers need to be able to understand them. A systematic review of the differences between CAS answers and school answers is missing from research and such review and organisation of resources could greatly add to CAS assisted teaching practices in schools.

A review of the differences between CAS answers and school answers in the case of equations is provided in this dissertation. The spectrum of differences is explained by using two kinds of classifications. A key criterion of the first classification is comparing whether or not CAS answers include a larger or a smaller number of solutions than the expected answers. The other classification is more content-oriented, highlighting the issues of the form, completeness, dependence on the number domain, and branching of answers and automatic simplification of equations.

The differences caused by the number domain and branching are discussed separately in greater depth in separate chapters. The possibilities of determining real or complex domains in different CAS are presented. Branching is described by evaluating the diversities between CAS answers, school answers and mathematically branch-complete answers.

Although the differences between school answers and CAS answers are often thought to be confusing they can also serve as opportunities for teaching and learning mathematics. Moreover, it is possible to make a productive use the differences between CAS answers and students' answers, which may also differ from school answers. This dissertation proposes a pedagogical approach to utilise CAS-assisted teaching opportunities offered by the differences between various answers.

The topic of trigonometric equations, which has several properties to exemplify answer diversity, was chosen for testing the pedagogical approach in the mathematics classes. The proposed pedagogical approach is based on comparative discussions on students' answers and CAS answers in pairs, and provides also opportunities for collecting data on students' understandings and misunderstandings. The focus of the research was on analyzing whether or not students can adequately identify the equivalence/non-equivalence and correctness of their own answer compared to CAS answer. I found that even if the students' solutions look to be correct, students may have misunderstandings and knowledge gaps.

The systematic review of differences between CAS answers and school answers provided in this dissertation could support CAS developers to improve their

software, teachers to utilise opportunities offered by CAS in their classrooms, and curriculum designers in organizing the teaching process based on CAS assisted teaching. The proposed pedagogical approach based on comparative discussions on students' answers and CAS answers in pairs of students complements repertoire of teachers and researchers.

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# 1. INTRODUCTION

This dissertation investigates the differences between answers offered by Computer Algebra Systems (CAS) and answers expected from students in mathematics classes at schools. The locations of these answers in the curriculum, their characteristics, underlying reasons for appearance, and possibilities for using these differences in teaching and learning are examined in the following chapters. Answers offered by CAS are referred to here as *CAS answers* and answers expected from students are referred to as *school answers*. The dissertation focuses on CAS and school answers in case of solving equations. The standards for school answers can vary depending on the country, the curriculum, the textbook, the teacher. As far as possible, different variants are taken into account in this study. Although the dissertation focuses on school mathematics, some examples and discussions go beyond the school level in the interests of a slightly broader overview.

In the last part of the dissertation, students' expectations based on their own solutions play an important role. These answers are referred to as *students' answers*. Both school answers, as answers expected from students, and students' answers, as the basis of their expectations for CAS answers, are referred to as 'expected answers' in the title of the dissertation.

The purpose of this chapter is to describe the background, the motivation and structure of this dissertation. Firstly, the use of CAS in teaching and learning in school mathematics will be briefly introduced, followed by an overview of expectations for the answers in schools and the differences between answers offered by CAS and expected school answers. Secondly, the motivations for research and the identification of research issues and research questions will be described. Thirdly, at the end of the chapter, the structure of the dissertation will be outlined.

## 1.1. Context: Computer Algebra Systems and their relations to school mathematics

Section 1.1 points out the possible advantages of CAS for teaching and learning mathematics as well as the ways of using CAS in school mathematics (Section 1.1.1). Section 1.1.2 introduces the differences between CAS answers and school answers and their possible use in teaching and learning.

### 1.1.1. Computer Algebra Systems in school

The use of CAS in teaching and learning school mathematics is briefly discussed in this section, but a more thorough overview of the specific CAS systems and their uses is provided in Chapter 2. Cohen (2003) defined CAS as follows:

> A computer algebra system (CAS) or symbol manipulation system is a computer program that performs symbolic mathematical operations.

Although CAS generate graphs and perform numeric calculation in addition to symbolic manipulation (Heid, 2005), this dissertation concentrates on symbolic manipulation, particularly solving of equations and simplification of expressions.

There are many computer programs that can be classified as CAS. This dissertation examines (to a greater or lesser extent) Axiom, Derive, GeoGebra, Maple, Mathematica, Maxima, MuPAD, Sage, TI-92+, TI-nspire, WIRIS, WolframAlpha. CAS are often primarily designed for professional users just for getting the answer but there are also systems (e.g., Derive, GeoGebra, TI-92+, TI-nspire, WIRIS) that are designed for teaching purposes.

The operations that can be performed by CAS include most of the algebraic operations of school and university mathematics courses (Geddes & Labahn, 1992). Therefore, CAS provide students with an opportunity to outsource routine work to the CAS (Heid & Edwards, 2001). Artigue (2002) stressed that, as CAS technology could take charge of most of the taught techniques, it is commonly considered that the use of CAS could allow students work directly at conceptual level. Relationship and balance between procedural skills and conceptual understanding in teaching and learning algebra is crucial to debates on development of algebra education (Drijvers, Goddijn, & Kindt, 2011).

Heid, Thomas, and Zbiek (2013) argued that using CAS could allow users to explore mathematical invariants, active linking of dynamic representations, engagement with real data, and simulations of real and mathematical relationships. More specifically, Pierce, Bardini, et al. (2015) listed 12 implementations for mathematics classes: *Checking answers*, *Obtaining results faster than with pen & paper*, *Doing calculations that students might find hard*, *Doing algebra that students might find hard*, *Doing application problems*, *Showing the impact of varying coefficients, powers etc.*, *Creating tables*, *Graphing functions*, *Solving*, *Expanding or factorising*, *Differentiating or integrating*, *Doing matrix operations*.

This dissertation focuses on answers offered by CAS to equations because solving of equations has a very important position in school mathematics curriculum and is also one of the main activities performable by CAS. Although solving equations can also be related to some other items on the above list, *Solving* and *Checking answers* are in the focus of this study.

CAS answers often coincide with school answers, but not always. Moreover, different CAS can offer different answers. Wester (1999b) has provided probably the most extensive review of how CAS solve different problems (542 problems were included), but only some of them were from school mathematics. I have been interested in CAS since before 2000 and have also found a few bugs in different CAS. Wester's review (1999b) gave me inspiration to offer something similar with regard to school mathematics.

Drijvers (2002) stressed that differences between CAS answers and school answers can elicit a feeling of irritation and frustration in students. Similarly, Ball (2014) described a case where a teacher felt that students experienced frustration when CAS offered an unexpected output. On the other hand, Drijvers (2002) and

Buteau, Marshall, Jarvis, and Lavicza (2010) noted that differences between CAS answers and school answers can serve as pedagogical opportunities. The unexpected response of CAS can be a catalyst for rich mathematical discussion (Pierce & Stacey, 2010).

In the early years of my research I concentrated on the differences of CAS and school answers in order to alert teachers, curriculum designers and CAS authors. Later, my focus shifted on the possible constructive ways of using these differences. My motivation for concrete research questions of the dissertation is described in Section 1.2. Before that, the differences between CAS and school answers are discussed in Section 1.1.2.

### 1.1.2. Differences between CAS and school answers

As the differences between CAS and school answers constitute the main object of this dissertation, they are introduced in this section before identification of research questions in Section 1.2. The issues of number domain and branching are highlighted because of their important role in causing the differences.

This dissertation is focused on the solving of equations, a leitmotif of school mathematics. There are different types of equations in school curricula (referred to here as *school mathematics equations*): linear, quadratic and fractional equations, equations that contain an absolute value of an expression, irrational (radical), exponential, logarithmic and trigonometric equations and literal equations (equation with parameters). Some equations can also be combined from different types. Equation solving is closely related to functions (Kolyagin, Lukankin, & Mokrushin, 1977; Usiskin, 1988). The introduction of a new operation or function is usually also associated with a new type of equation in school curricula. The operations and functions in equations determine the form of the solution set, the requirements for solution and solving technique. In addition, possible difficulties and mistakes in the solving process are also related to the properties of operations and functions. As the type of equations has a significant bearing on possible difficulties, mistakes and standards, all mentioned types of equations from school mathematics are observed separately in the dissertation.

The CAS and school approaches can be different and here the school standards are presented first. The standards for school answers are shaped by requirements and examples from study material (e.g., textbooks) and teachers. The standards can somewhat vary depending on the country, the curriculum, the textbook, the teacher, etc. School mathematics can differ from university mathematics and mathematicians' mathematics, for example, in the number domain. As CAS are often designed for tertiary (university) and disciplinary (mathematicians') mathematics, different standards can cause differences between CAS and school answers.

Firstly, the issue of the number domain is introduced because difference of domains is one of the significant reasons for the differences between CAS and school answers. The complex domain is the default domain in some and the real domain in other CAS. An answer offered by a CAS may be unexpected for a student because it belongs to domain different from the domain used at school (Alonso, Garcia, Garcia, Hoya, et al., 2001). For example, in the case of entering `solve(x^2+1=0,x)`, a CAS (WolframAlpha) offers $x = \pm i$. An imaginary answer can be unexpected for students as school mathematics does not (at least usually) deal with imaginary numbers. Wester (1999b) mentioned that the elementary mathematics option for a CAS (all variables are declared to be real, $\sqrt{-1}$ is undefined, etc.) should be useful. CAS can have features for confining the work in the real domain. Sometimes, CAS features for determination of domain can be inconsistent and present a real solution that is considered as extraneous in school: $-1$ in the case of $\sqrt{2x} = \sqrt{x-1}$. $-1$ is the correct answer in the complex domain, but $\sqrt{-2}$ is undefined in the real domain.

Secondly, branching of solutions is under consideration. Namely, a solution can consist of different branches in some manner and CAS and school answers can differ in their presentation of the branches. However, CAS and school answers coincide in the following case: Kadijevich (2014) was concerned that CAS usually simplifies $\frac{x^2}{x}$ to $x$, not adding the constraint $x \neq 0$. Similar examples are provided in Berger (2009) and Olive et al. (2010). The distinguishing of such 'forbidden' branches is also often discarded in the school context. The 'forbidden' operations are legalized by hidden assumptions to avoid branching. For example, a textbook Barnett, Ziegler, and Byleen (1999) says:

> Even though not always explicitly stated, we always assume that variables are restricted so that division by 0 is excluded.

Such transformation can lead to extraneous solutions in equation solving. The common strategy for excluding extraneous solutions is checking the provisional solution in the initial equation.

The differences in presentation of branches occur with different types of equations. Examples of quadratic, literal and trigonometric equations are presented here. Some textbooks say that there are two equal roots, some say one real root (a repeated root), and some say just one real solution if the discriminant is zero in case of solving quadratic equation (e.g., $x^2 + 2x + 1 = 0$). Sangwin (2015) pointed out the issue of repeated root in CAS answers and illustrated it by solving the equation $x^2 - 6x + 9 = 0$ with six CAS. Two of the CAS indicated multiplicity and four did not.

Literal equations (equations with parameters) offer different levels for treatment of branches. For example, only the main branch is often assumed in applied problems in the school context, e.g., in physics: $A = P + Prt$; *please express r*. These are cases where $P = 0$ or $t = 0$ are not particularly handled. On the other hand, for example, all cases are expected to present in the answer of equation $(a^2 - 1)x - (2a^2 + a - 3) = 0$.

Stoutemyer (1991) pointed out that most CAS return only $x = 0$ when solving

$cx = 0$. Similarly, Bernardin (1999) mentioned that almost all CAS provide only the main branch (in case of equation $ax = 1$, only the branch where $a \neq 0$).

There are different sources for branching in case of trigonometric equations — periodicity and the families of solutions. Textbooks may or may not provide general solutions. In case of general solutions, different forms are possible. For example, in case of $\sin x + \cos 2x = 0$ textbooks can give the answer

$$x_1 = \frac{\pi}{2} + 2n\pi \text{ and } x_2 = \frac{\pi}{6} \pm \frac{\pi}{3} + \frac{4}{3}n\pi$$

or

$$x_1 = (-1)^n \ \frac{\pi}{2} + n\pi \text{ and } x_2 = (-1)^{n+1} \ \frac{\pi}{6} + n\pi.$$

CAS can present the general solution or a set of particular solutions in case of trigonometric equations. Stacey and Ball (2001) showed CAS answers in case of

$$5\cos^2 x + 2\sin^2 x = 2$$

- 1.5707 (by HP-49G),
- $x = ((2n-1)\pi)/2$ (by TI-89),
- $x = (2\pi)k - \pi/2$ and $x = (2\pi)k + \pi/2$ (by FX-2.0),
- $x = -\pi/2$ and $x = \pi/2$ (Mathematica).

Trigonometric equations are very interesting because of a variety of possible presentations of solutions, units of measurement, general and particular solutions, and they are used frequently in this dissertation.

The differences can be noticed already in relatively simple examples. Sometimes CAS can provide answers that are not fully simplified. Drijvers (2002) pointed out that students can experience difficulties in recognizing equivalence of a CAS answer and the answer that the student had in mind. The examples were $-(x-12)$ vs $12-x$ and $\sqrt{\frac{s}{4}}$ vs $\frac{1}{2}\sqrt{s}$. Similar examples were noted by Stacey (2003), $(b+a)^2$ vs $a^2+b^2+2ab$, and Meagher (2005), $-(x-3x^2)$ vs $3x^2-x$. All these authors mentioned that recognition of equivalence/non-equivalence of answers can be difficult for some students but not for others.

Although the differences between CAS answers and school answers are often thought to be confusing and undesirable as an obstacle, they can also serve as opportunities for teaching and learning. Alonso et al. (2001) listed some unexpected answers and proposed not to skip such exercises but use them in classroom, as it is necessary for developing students' critical ability, and CAS offer the possibility of contrasting all the results, using a variety of representations. They are convinced that the unexpected results can be used to emphasize concepts and increase the critical perspective. Pierce and Stacey (2010) stated that it is possible to exploit the contrast between ideal and machine mathematics. Teachers could make deliberate use of 'unexpected' error messages, expression formats and graphical displays as a catalyst for rich mathematical discussion.

There are quite a few concrete examples about the use of the differences between CAS answers and school answers in literature. In some cases, students are asked to explain CAS answers. Kieran et al. (2006) described the justification of CAS simplification $(2-x)(1-2x)$ to $(x-2)(2x-1)$ by students. Artigue (2005) presented a case where students were explicitly asked to record the result, their remarks, comments and interpretations of the result in case of an unexpected or incomprehensible CAS result. For example, in case of $(x\sqrt{2}-\sqrt{3})(x\sqrt{3})-\sqrt{2}$, CAS did not transform $\sqrt{2}\sqrt{3}$ into $\sqrt{6}$ as would be done with pen and paper.

Ball (2014) and Pantzare (2012) presented some questions from the written examination in Mathematical Methods in Victoria (Australia). For example, students were asked to solve a trigonometric equation by CAS and choose the correct answer from four alternatives. The correct answer was equivalent to CAS answer but looked different.

Lagrange (1999) described a case where students had to find their answer themselves and then by CAS and explain the equivalence of results. The students were asked to differentiate trigonometric function $\cos(3x-\frac{\pi}{6})$. Guzmán, Kieran, and Martínez (2010, 2011); Martínez, Kieran, and Guzmán (2012); Stayton (2016) described the case where students worked in pairs and compared CAS answers to their own pen-and-paper answers. The examples were chosen to guide students to notice important differences between the following expressions:

$$\frac{x(3+x)}{x}, \frac{4x+4y}{x+y}, \frac{3x+4y}{x+y}.$$

Section 1.1 introduced CAS use briefly in teaching and learning mathematics in schools. Special attention was paid to the differences between CAS answers and school answers — their possible causes and opportunities for teaching. Although possible uses of the differences between CAS answers and school answers are mentioned in literature, only a few concrete cases or practices could be found. However, these examples encouraged me to become engaged with the differences between CAS and school answers.

## 1.2. Motivation, research issues and questions

In this section, the central issues of this dissertation are identified, with some references to my personal motivation. Four research issues are introduced, each in a separate section from 1.2.1 to 1.2.4. Six research questions are formulated as well.

### 1.2.1. The first issue — differences between CAS and school answers

I have been interested in using CAS for nearly 20 years. The main focus of my research has been on identifying the differences between CAS and school answers.

Although I found a few bugs, I understood that usually these differences are related to the choices made by CAS authors. The choices have been influenced by the fact that CAS were initially developed for professional users and even now professional users are the main user group of CAS. Unfinished answers, exceptional solutions and particular solutions vs. general solutions are probably not so confusing for professional users but they can be confusing for students. It is worth mentioning that some systems (e.g., GeoGebra and WIRIS) are developed primarily for educational purposes and more school-friendly choices have been made in these systems.

It is notable that CAS answers to a particular problem can vary between different CAS, commands and settings. Wester (1999b) described a large experiment aimed at discovering how different CAS solve problems. Wester (1999b) as well as Bernardin (1999) did not provide many examples from school mathematics. There are also some examples of school mathematics discussed in different papers (e.g., Alonso et al., 2001; Drijvers, 2002) but no large systematic review has been compiled.

The first research issue can be formulated as follows:

*Computer output provided by CAS can, on some occasions, be different than the answers expected from students in schools. These differences can be confusing but can also serve as opportunities for teaching and learning. Therefore, a systematic review of these differences in relation to the curriculum and the applied CAS software would be important and could also be useful for teachers, curriculum designers, and CAS authors. In particular, the review should describe the place in curricula, the characteristics, the reason of appearance of such differences, as well as the characteristics of CAS software and its output style.*

Two research questions (RQ1 and RQ2) could be formulated based on the outlined issue. As the dissertation focuses on school mathematics equations, the first research question is:

**RQ1. Where differences between CAS and school answers could be detected in equations within the school curriculum?**

RQ1 focuses on the location of the differences between CAS and school answers. However, a systematic review should not be limited to location alone — the contents and causes of the differences between CAS and school answers should be examined as well. Different authors mention various phenomena in connection with the differences. Wester (1999b) noted that some CAS answers are incomplete or incompletely simplified. Böhm (2009) and Kadijevich (2009) found that automatic simplification can be confusing. Aslaksen (1999) highlighted the difference of transformation rules in different number domains. Bradford, Corless, Davenport, Jeffrey, and Watt (2002); Corless and Jeffrey (1996); Rich and Jeffrey (1996) discussed multivalued functions. These sources give ideas and inspiration for description and classification of the differences between CAS and school answers.

The second research question is:

**RQ2. How can the detected differences between CAS and school answers to equations in the school curriculum be described and classified?**

### 1.2.2. The second issue — number domain

Both literature and the answers to RQ1 and RQ2 show that the issue of the number domain is one of the important reasons for the differences between CAS and school answers. As the issue of the number domain is, in case of equations, closely related to simplification of expressions, it is justified to discuss them together. The solution sets of equations and the validity of transformation rules in the manipulation of expressions may depend on the domain in case of some types — they are domain-sensitive. For example, $\sqrt{-1}$ is $i$ in case of complex numbers but not defined in case of real numbers. Furthermore, the equation $x^2+1=0$ has solutions in the complex domain but not in the real (or rational) domain.

A CAS answer may be unexpected to students because the domain may differ from the domain used at school (Alonso et al., 2001). Wester (1999b) mentioned that an elementary mathematics option for a CAS should be useful. CAS have various possibilities to determine the domain of a result, variable value or equation solution. Such features can sometimes be inconsistent, for example, $-1$ is offered as answer to the equation $x+\sqrt{x}=\sqrt{x}-1$ even in the real domain.

The issue here is formulated based on the results of RQ1 and RQ2:

*Different CAS software provide various real and/or complex answers; it would be important to have an overview of the features used for determination of the domain of a calculation result, variable value or equation solution and of the quality of these features. Such overview would be useful for teachers who teach (or plan to teach) with CAS, but also for curriculum designers and CAS authors.*

Some CAS work in the complex domain by default, others work in the real domain. CAS have different possibilities for the determination of the domain of a calculation result, variable value or equation solution. The next research question is posed for assessing the performance of CAS and it is formulated as follows:

**RQ3. When do CAS outputs offer correct and incorrect answers for domain-sensitive examples, specifically for expression simplification and equation solving?**

### 1.2.3. The third issue — branching solutions

As presented in literature and in the answers to RQ1 and RQ2, a solution can, in many cases, consist of different branches. Solutions of quadratic, literal and trigonometric equations and cancellation of the fraction with variables in the denominator ($\frac{x^2}{x}$ to $x$) introduced in Section 1.1.2 serve as examples of branching. Some branches can be absent from school and CAS answers compared to mathematically branch-complete answers. For example, the mathematically branch-complete answer for $\frac{x^2}{x}$ is

$$\begin{cases} x & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

An answer is mathematically branch-complete if the multiplicity of roots is shown; for example, if the discriminant is zero in case of a quadratic equation. A mathematically branch-complete answer to a literal equation presents all cases and a mathematically branch-complete answer to a trigonometric equation provides a general solution.

The next research issue is formulated as follows:

*As school answers and CAS answers could differ from mathematically branch-complete answers, it is reasonable to consider besides the CAS answer and the school answer also the mathematically complete answer in the case of branching. An overview of branching would be useful for teachers, curriculum designers and CAS authors.*

Based on the outlined issue, the following research question RQ4 can be formulated:

**RQ4. How can branching be described for answers provided by different CAS software; by different school solutions and textbooks; by the possibilities of mathematical approaches for expressions simplifications and equations solving?**

### 1.2.4. The fourth issue — using of differences

In this section the last issue and research questions will be formulated. Although CAS answers different from school answers are often thought to be confusing and undesirable as an obstacle, they could also serve as opportunities for teaching and learning, especially to stimulate the students' analysis and discussion. Finding such opportunities has been particularly important to me as mere identification of the differences between CAS and school answers seemed to be biased.

In case of the examples by Guzmán et al. (2010, 2011); Lagrange (1999); Martínez et al. (2012); Stayton (2016), presented in Section 1.1.2, students were directed to compare their own answers to CAS answers. This approach seems reasonable to me, as it involves simultaneous solving with pen-and-paper and by CAS. Balance between procedural skills and conceptual understanding is taken into account. The use of students' own answers can add a personal touch. Moreover, students' discussion on the equivalence and correctness of a CAS answer and their own answer can produce better insights into students' understandings and misunderstandings.

Unfortunately, there are only few examples of comparison CAS and students' answers found in literature. Therefore, the last research issue is formulated as follows:

*There is a lack of examples where the differences between CAS answers and students' answers are used in education to support students' understanding. Additionally, the research potential in studying students' discussions has been largely untapped.*

Two activities can be identified here. First, there is a need for a pedagogical approach where the differences between CAS answers and students' answers are used in education to support students' understanding. Then it is possible to examine the approach as well as students' identification of equivalence/non-equivalence and correctness of their answers and CAS answers to particular types of equations.

At first, the research question RQ5 is formulated as follows:

**RQ5. What pedagogical approaches could be proposed to utilize the teaching opportunities offered by the differences between CAS and students' answers?**

For the experimental part with students, solving of trigonometric equations was chosen as the main topic because of the natural variation of possible presentations of solutions, units of measurement, general and particular solutions, etc. The variety of answers of trigonometric equations is highlighted by Abramovich (2005, 2014); Kieran and Saldanha (2005); Lagrange (1999); Pantzare (2012). The variations give a good basis for students' discussion on equivalence and correctness of their own answers and CAS answers.

A pedagogical approach where pairs of students were charged with the task of comparing the answers offered by a CAS with their own answers was suggested and implemented. This approach is referred to here as a lesson scenario based on comparative discussion on students' answers and computer algebra system answers in pairs of students.

The students' discussion when comparing their answers and CAS answers also provides data about students' understandings and misunderstandings related to correctness and equivalence of their answers and CAS answers. Equivalence is one of the important issues in mathematics but is somewhat hidden in school mathematics. Kieran, Boileau, Tanguay, and Drijvers (2013) mentioned the crucial role of equivalence of algebraic expressions in expression simplification and equation solving and in a broader context.

Recognition of equivalence and correctness is crucial when students compare CAS and their own answers. The experimentation of the lesson scenario lead to the research question RQ6 which is formulated as follows:

**RQ6. How can students identify 1) the equivalence and non-equivalence between CAS and their own answers; 2) correctness of CAS and their own answers of trigonometric equations in lessons based on comparative discussions on students' answers and CAS answers in pairs of students?**

The research design will be described by individual research questions in Section 3.1, after the chapter on related works. Research based on the issues and questions will be presented in Chapters 4–8.

## 1.3. Outline

The structure of the dissertation is outlined in this section.

The dissertation is divided into nine chapters. An overview of related works is presented in Chapter 2, after the Introduction (Chapter 1). Chapter 3 gives an overview of research design and instruments.

The dissertation is a monograph based on six papers (Tonisson, 2007, 2008, 2011, 2013, 2015; Tonisson & Lepp, 2015). Five of them are single-authored; in case of Tonisson and Lepp (2015) the co-author helped with the preparation of the experiment and data analyses. Although the papers generally deal with quite connected topics, there was no direct plan for a monograph beforehand. The monograph was composed after the papers were published.

Chapter 4 is based on the paper *Differences between Expected Answers and the Answers Given by Computer Algebra Systems to School Equations* (Tonisson, 2015) and demonstrates where the differences between CAS and school answers can be detected in equations within the school curriculum (RQ1). Over 120 equations from school mathematics are solved using 8 different CAS. The detected differences between CAS and school answers are described and classified (RQ2). The classification consists of 6 types.

Chapter 5 provides a different classification of the differences between CAS answers and school answers than Chapter 4 (RQ2). The classification is more content-oriented and the types are based on the form, completion, dependence on the number domain, and branching of answers and automatic simplification of equations. Some phenomena seem to have more didactic value for treatment of certain topics. The scenario where students compare CAS answers with their own answers is outlined here but the experiments are described in Chapter 8. The chapter is based on the paper *Unexpected answers offered by Computer Algebra Systems to school equations* (Tonisson, 2011) but is considerably improved.

The issues of domain and branches are so distinguished that special chapters are justified. Chapter 6 (based on *Issues of Domain in School Mathematics and in Computer Algebra Systems* (Tonisson, 2008)) focuses on the issues of domain (mainly the real and complex domain). The question when do CAS outputs offer correct and incorrect answers for domain-sensitive examples (RQ3) is answered.

The area of Chapter 7 (based on *the paper Branch Completeness in School Mathematics and in Computer Algebra Systems* (Tonisson, 2007)) is limited to the problems where a solution is separable in some manner. Branching diversities (RQ4) between CAS answers (CAS), school answers (SCH) and mathematically branch-complete answers (MATH) are described using special notation, e.g., $CAS < SCH = MATH$.

Unlike the preceding chapters, Chapter 8 (based on the papers *Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System* (Tonisson, 2013) and *Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System in Terms of Equivalence* *and Correctness* (Tonisson & Lepp, 2015)) includes an experimental part with students. Firstly, a lesson scenario based on comparative discussion on students' answers and computer algebra system answers in pairs of students is proposed in order to utilize the teaching opportunities offered by the differences between CAS and students' answers (RQ5). Secondly, the chapter analyses whether students can adequately identify the equivalence/non-equivalence and correctness of their answer and CAS answer (RQ6).

The concluding part is numbered as Chapter 9. The conclusion includes an overview of the results by research questions, an outline of the contribution to the work of different user groups, and ideas on possible future work. The Appendix includes tables of equations of the test suite used in Chapter 4 and worksheets and questionnaires used in the lessons described in Chapter 8.

# 2. RELATED WORKS

The purpose of Chapter 2 is to introduce related works from different areas. Firstly, the works where CAS answers have been analysed are presented in Section 2.1. Section 2.2 introduces some aspects of teaching and learning equation solving. Thirdly, the use of the differences between expected answers and CAS answers is in the focus in Section 2.3. Finally, the chapter concludes with some ideas, which are useful in the context of this dissertation.

## 2.1. Analysis of CAS answers

This section focuses on works about CAS answers, their possible shortcomings, and their differences from one another and (school) mathematics answers. Section 2.1.1 is based on Wester (1999b) which contains probably the thoroughest overview of CAS answers. Bugs and limitations are discussed in Section 2.1.2. Issues related to the number domain, branching and the simplest form of CAS answers are discussed in Sections 2.1.3–2.1.5 respectively. Section 2.1.6 is devoted to answers to trigonometric equations.

### 2.1.1. Wester's review of CAS answers

CAS answers have been analysed in different works. Section 2.1.1 is based on *A critique of the mathematical abilities of CA systems* (Wester, 1999b) which is a part of the collection of papers *Computer algebra systems: A Practical Guide* (Wester, 1999a). Wester's (1999b) paper is probably the most extensive review of how CAS solve different problems (542 are included). Wester compared how seven different computer algebra systems (Axiom, Derive, Macsyma, Maple, Mathematica, MuPAD, Reduce) deal with a chosen set of problems. All results were collected into one table showing the overall tendencies, shortages and patterns. Figure 1 and Figure 2 present small parts of this table. The answers were evaluated by special 'grades':

- • success;
- ⋆ success, but a little fudging or subtlety required,
  or the answer could be just a little nicer or more complete;
- ◦ success but indirectly, incomplete or unsimplified;
- ♮ tricky, very inelegant or minimal success;
- # incompletely simplified, but some useful transformations were
  performed;
- could not do the problem;
- – lack the capability to do or state the problem;

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| # | PROBLEM | Ax | De | Mc | Mp | Mm | Mu | Re |
|---|---|---|---|---|---|---|---|---|
| | M. Equations | | | | | | | |
| *M1* | $\frac{x=2}{2} + (1=1) \Rightarrow \frac{x}{2} + 1 = 2$ | • | | • | • | ○ | • | • |
| M2 | solve$(3x^3 - 18x^2 + 33x - 19 = 0, \mathbf{R})$ | ○ | • | ⋆ | ⋆ | ○ | ⋆ | • |
| M3 | solve$(x^4 + x^3 + x^2 + x + 1 = 0)$ | ♮ | • | • | • | ○ | • | • |
| M4 | verify a solution of the above | • | • | • | • | • | ⋆ | • |
| M5 | solve$(x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23)$ | | | | | | | |
| M6 | solve$(x^7 - 1 = 0) \Rightarrow x = \{1, \{e^{\pm 2k\pi i/7}\}_{k=1}^{3}\}$ | ♮ | ⋆ | • | • | ⋆ | ♮ | ♮ |
| M7 | solve$(x^8 - 8x^7 + \cdots - 140x + 46 = 0)$ | | | • | • | • | • | • |
| M8 | solve$(e^{2x} + 2e^x + 1 = z, x)$ | $\circ^{\mathbf{r}}$ | $\circ^{\mathbf{r}}$ | • | $\circ^{\mathbf{r}}$ | $\circ^{\mathbf{r}}$ | $\star^{\mathbf{18}}$ | • |
| M9 | solve$(e^{2-x^2} = e^{-x}) \Rightarrow x = \{-1, 2\}[+\mathbf{C}]$ | | $\circ^{\mathbf{r}}$ | • | $\circ^{\mathbf{r}}$ | $\circ^{\mathbf{r}}$ | $\star^{\mathbf{18}}$ | |
| M10 | solve$(e^x = x) \Rightarrow x = -W_n(-1) \quad (n \in \mathbf{Z})$ | | | $\circ^{\mathbf{r}}$ | $\star^{\mathbf{r}}$ | $\circ^{\mathbf{r}}$ | | $\circ^{\mathbf{r}}$ |
| M11 | solve$(x^x = x) \Rightarrow x = \{-1, 1\}$ | | | | ○ | ○ | | |

**Figure 1.** Review of CAS (Wester, 1999b)

- $\varepsilon$ an error message was generated;
- $\varepsilon$ a surprising error occurred;
- Ⓔ a fatal program error occurred;
- $\tau$ very slow;
- $\boxed{\star}$ almost correct;
- $\otimes$ partial success, but also partially incorrect;
- ⊎ mostly, but not completely wrong;
- $\times$ produced the wrong answer;
- $^m$ does not indicate some or all solutions may have a multiplicity > 1;
- $^r$ only provides solutions within a restricted interval;
- $^s$ one or more spurious solutions were produced;
- $^{18}$ unsuccessful using E^x, but successful using exp(x).

Grading variants also indicate that the picture is not contrastingly black and white. There were cases where somewhat partial or dubious success was expressed — the evaluation of CAS answers is not always straightforward. There were not very many school problems reviewed in Wester, 1999b. It is understandable as CAS were initially designed mainly to help professional users of mathematics. However, there were some equations in school level (like $\sin x = \cos x$, $\tan x = 1$, $\sin x = \tan x$, $\sqrt{x^2+1} = x - 2$, $x + \sqrt{x} = 2$, $|2x+5| = |x-2|$).

Besides of the grade 'success', also 'success but indirectly, incomplete or unsimplified', 'success, but a little fudging or subtlety required, or the answer could be just a little nicer or more complete' and 'partial success, but also partially incorrect' were noted. The comments 'only provides solutions within a restricted interval' (marked by r) and 'does not indicate some or all solutions may have multiplicity > 1' (marked by m) were added in case of some answers. For example, the answers to trigonometric equations (see Figure 2) where a CAS gave particular solutions were classified as 'success but indirectly, incomplete or unsimplified'.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| *M13* | solve$(\sin x = \cos x) \Rightarrow x = \frac{\pi}{4}\,[+n\pi]$ | $\circ^{\mathbf{r}}$ | $\circ^{\mathbf{r}}$ | $\star$ | $\bullet$ | $\circ^{\mathbf{r}}$ | $\star$ | $\star$ |
| *M14* | solve$(\tan x = 1) \Rightarrow x = \frac{\pi}{4}\,[+n\pi]$ | $\circ^{\mathbf{r}}$ | $\circ^{\mathbf{r}}$ | $\bullet$ | $\bullet$ | $\circ^{\mathbf{r}}$ | $\bullet$ | $\bullet$ |
| *M15* | solve$(\sin x = \frac{1}{2}) \Rightarrow \frac{\pi}{6}, \frac{5\pi}{6}\,[+n2\pi,\ +n2\pi]$ | $\natural^{\mathbf{r}}$ | $\circ^{\mathbf{r}}$ | $\star$ | $\bullet$ | $\circ^{\mathbf{r}}$ | $\bullet$ | $\bullet$ |
| M16 | solve$(\sin x = \tan x) \Rightarrow 0, 0\,[+n\pi,\ +n2\pi]$ | $\circ^{\mathbf{m,r}}$ | $\circ^{\mathbf{r}}$ | $\bullet$ | $\circ^{\mathbf{m,r}}$ | $\circ^{\mathbf{m}}$ | $\bullet$ | $\otimes^{\mathbf{s}}$ |
| M17 | solve$(\sin^{-1} x = \tan^{-1} x) \Rightarrow x = \{0, 0, 0\}$ | $\circ^{\mathbf{m}}$ | $\circ^{\mathbf{m}}$ | $\boxed{\star}^{\mathbf{s}}$ | $\circ^{\mathbf{m}}$ | $\circ^{\mathbf{m}}$ | | $\bullet$ |

**Figure 2.** Trigonometric equations (Wester, 1999b)

Wester did not classify the grades on the basis of equivalence. It is hard to say what was exact meaning of 'almost correct', for example. Is the answer equivalent to the expected answer or not? The equivalence issues are important for students and teachers and also in the context of this dissertation.

There are some grades that could seem somewhat subjective, for example, 'the answer could be just a little nicer or more complete' or 'success, but indirectly, incomplete or unsimplified'. It should be noted that some differences that are not hindrances for professional users could confuse students and teachers.

Wester's review provided the inspiration for using similar tables with different CAS in this dissertation. His gradings were quite emotional and diverse, and different classifications were used. Grading variants also indicate that the picture is not plainly black and white. Wester's grading style and test suite are not directly suitable for school mathematics. However, some highlighted issues are also important and interesting in the school context.

The next section discusses bugs and limitations.

### 2.1.2. Bugs and limitations

Section 2.1.2 is focused on classification of some unexpected answers.

Stoutemyer (1991) differentiated between bugs and limitations. He considered a bug as something bad that a program does contrary to the programmers' intent and a limitation as something a program cannot do that a user wishes it could. Stoutemyer divided the limitations in turn to theoretical, resource and algorithmic limitations. Theoretical (incl. undecidability results from logic and the theory of computation) and resource (e.g., computer memory, computational time) limitations are not relevant for this dissertation. Algorithmic limitations that arise from the fact that it is impractical to implement all possible details are, on the contrary, pertinent.

Talking about bugs, Stoutemyer suggested that a CAS result should be checked on more than one CAS, because it is unlikely that two systems have the same bug. Trying more than one system would also provide an overview of the best features of each system.

Some of the Stoutemyer's (1991) sentences from have been somewhat inspiring for this dissertation:

- *The goal here is to inspire caution. These systems can be extraordinarily useful if users are aware of underlying assumptions and of their responsibility to verify results.*
- *It is important for users to be aware of some of the limitations of such systems to use them wisely.*

The topic of Alonso et al., 2001 (*Some Unexpected Results Using Computer Algebra Systems*) is very pertinent for this dissertation. Alonso et al. (2001) presented the following classification of unexpected results:

- Bugs, incorrect results, contrary to the programmers' intent.
- Limitations imposed by the programmers with a view to avoiding problems or facilitating other situations.
- Correct results that are surprising to the user, who implicitly assumes certain hypotheses or believes that the variables lie in a given specific domain.
- Others. The range of situations is very broad. Thus, it is something surprising that the system is unable to perform an apparently simple calculation, in which case it is necessary to "help the computer".

The last three classes are focused on in this dissertation. Alonso et al. (2001) presented an example of an incorrect result — the approximation of the fraction $110781/10000$ to $11.0780$ when using Derive 4.11. Derive 5 offers the correct answer. It shows that a bug can be corrected in future versions of a CAS.

Bugs are not very important in the context of this dissertation, as they probably would be fixed in subsequent versions. On the other hand, Lavicza (2008) pointed out that compatibility of different CAS versions can be problematic for mathematicians as worksheets should be redeveloped after the upgrade. For this dissertation, intentional behavior is more interesting as regards to the possibility of use in teaching and learning. The next sections are devoted to different aspects of intentional behavior.

### 2.1.3. Number domain

The choice of number domain is one of the intentional choices. Section 2.1.3 presents the works that highlight the differences related to different number domains. Alonso et al. (2001) noted that a CAS answer can be unexpected for students because the domain may differ from the domain used at school. Wester (1999b) mentioned a possible special need for elementary mathematics:

> It certainly would be wonderful if CASs were adjustable to the mind-sets of their users (mathematicians, physicists, engineers, etc.) One could invoke `mindset(elementary_math_student)` to initially declare all variables to be real, make $\sqrt{-1}$ undefined, etc., for example.

Stoutemyer (1991) discussed the issue of imaginary numbers and assumptions, for example, being real or nonnegative. In case of simplification of $\ln(1)$ the users could expect 0 as the answer but actually the correct answer (in the complex domain) is $2n\pi \mid n \in \mathbb{Z}$. The multiple-valued complex functions in the CAS context are the topic of many papers. For example, Aslaksen (1999) listed some tests for the checking of correctness. One of them specified that

- $\sqrt{zw} - \sqrt{z}\sqrt{w}$ *should not simplify when $z$ and $w$ are complex.*
- $\sqrt{zw} - \sqrt{z}\sqrt{w}$ *should simplify to* 0 *when $z$ and $w$ are both positive.*

Kaltofen said:

> One of the fundamental difficulties in dealing with "the square root bug" is that because of multivaluedness, some cherished algebraic identities, such as
>
> $$\ln z_1 z_2 = \ln z_1 + \ln z_2$$
>
> no longer hold — they are not true for all specializations of the variables. People have been willing to try to keep these identities, at almost any cost. (Kaltofen, 2000)

Multivalued functions and branch cuts were the main topic of several works of Jeffrey and his co-authors (Bradford et al., 2002; Corless & Jeffrey, 1996; Corless, Jeffrey, Watt, & Davenport, 2000; Rich & Jeffrey, 1996). Jeffrey (2014) described the new approach to multivalued elementary inverse functions ($\log(z)$, $\arcsin(z)$, $\arccos(z)$, $\arctan(z)$, $\operatorname{arcsinh}(z)$, $\operatorname{arccosh}(z)$, $\operatorname{arctanh}(z)$, fractional powers $z^{1/n}$). It is based on an idea of the branch of an inverse function and defining of an index for each branch. He also observed that

> ... in the 1980s, errors resulting from the incorrect application of these transformations were common. Since then, systems have improved and now they usually avoid simplification errors, although the price paid is often that no simplification is made when it could be.

Although, for instance, inverse trigonometric functions are important in the context of this dissertation, the issues of multivalued function are not discussed in greater depth here.

The correct but unexpected results are related to the fact that a CAS itself is not limited to the real domain. For example, Alonso et al. (2001) mentioned that the user expects $-2$ instead of $2(-1)^{\frac{1}{3}}$ as the cube root of $-8$. Unwritten assumptions leads to expected simplifications that are not generally correct. $(x^n)^{\frac{1}{n}}$ is not $x$ in general, $\log(xy)$ is equivalent to $\log(x) + \log(y)$ only when $x$ and $y$ are positive. Furthermore, $\ln(\exp(z)) = z$ holds only if the imaginary part of $z$ is in $(-\pi, \pi)$.

Domain issues are very important for this dissertation as the complex domain is often used in CAS but not in school mathematics. The above examples give ideas for a test suite. The branching of solutions, which is also related to the number domain, will be discussed in the next section.

### 2.1.4. Branching of solution

Section 2.1.4 includes examples where a solution consists of different branches in some manner. The issues of literal equation, extraneous solutions, repeated roots and simplification without comments are discussed.

*Literal equations.* Most of the Stoutemyer's (1991) examples were beyond school mathematics but some topics were related to school equations. For example, CAS tend to ignore special cases in some context. Stoutemyer said:

> David Jeffrey pointed out to me another example of ignoring a set of measure 0 in solving an equation such $cx = 0$ for $x$. Most systems return only $x = 0$, but if declarations don't exclude $c = 0$, then another solution is $c = 0$. Since we requested the values of $x$ that satisfy the equation, we could express the solution set somewhat awkwardly as
>
> $$x = \text{ if } c = 0 \text{ then @ else } 0$$
>
> where @ is a unique new variable designating "anything". In contrast, there is less need to worry about the case $c = 0$ in solving $cx = 1$ for x, because the solution $x = 1/c$ contains a manifest indication of a limit solution at $x = 1/0 =$ complex $\infty$. It is the invisible failures of a formula that are most dangerous. (Stoutemyer, 1991)

Bernardin (1999) (an earlier version is Bernardin, 1996) focused on algebraic and transcendental equation solving, which is closely related to this dissertation. Although many of the reviewed equations were beyond school mathematics, there were several that are appropriate for the school level. For example, the correctness of answer

$$x = \frac{b}{a}$$

was discussed in case of the equation

$$ax = b.$$

*Extraneous solutions.* Stoutemyer (1991) touched on the issue of extraneous solutions. Human beings and CAS sometimes take steps that introduce extraneous solutions. The equation

$$\sqrt{x} = 1 - x$$

was explained as an example where squaring both sides generates a quadratic equation with two solutions, only one of which is a solution of the original equation. The problem is much deeper in case of cubic or quartic equations. The candidates should be verified but sometimes CAS could skip it. Stoutemyer suggested that at least a warning should be used in such cases.

In addition, Bernardin (1999) presented some school level irrational equations (like $x + \sqrt{x} = 1$ and $\sqrt{x} - \sqrt{x-1} = 3$).

*Repeated root.* The issue of repeated root could arise in case of equations. For example, Sangwin (2015) pointed out the issue of repeated root and illustrated it by solving the equation $x^2 - 6x + 9 = 0$ with several CAS. Two of the CAS indicated multiplicity while four did not. Furthermore, depending on the particular standard applied in a school, one could speak about one root, two equal roots or a repeated root.

*Simplification without comments.* Alonso et al. (2001) presented the simplification of $\frac{x}{x}$ to 1 without showing the special case of $x = 0$ as an example of a limitation . As similar examples, $\frac{x^2}{x}$ is in Kadijevich, 2014 and $\frac{x^2-1}{x-1}$ in Berger, 2009 and $\frac{(x-1)^2}{x^2-1}$ in Olive et al., 2010.

Actually, this behaviour is common in the school context as the identification of such 'forbidden' branches is also often discarded in the school context. For example, Barnett et al. (1999) said: "*Even though not always explicitly stated, we always assume that variables are restricted so that division by 0 is excluded*".

Similar 'hidden assumptions' can also be used in other cases. For example, $(\sqrt{x})^2$ is not $x$ in case of the real domain, as for negative numbers square root is not defined. Kadijevich (2009) approved a CAS (TI-Nspire) that adds the warning "Domain of the result might be larger than domain of the input". Furthermore, the warning "Non-real calculation" is used in case of $\sqrt{-1}$. Kadijevich and also Böhm (2009) found that it could be fine if a user could turn automatic simplification on or off. It seems that the question is not so much about automatic simplification in itself but about the lack of comments in case of automatic simplification.

The examples of different ways of presenting of branches inspire to compare them in terms of completeness of the solution set. The choice of branches is important for the final answer. The next section discusses the issues of the final form in even more detail.

### 2.1.5. Final form of CAS answer

Choices for the final form of CAS answers are reported in this section.

It is not always clear what is the simplest correct form. Carette (2004) argued that there are cases where is clear what is simpler. For example,

- 0 is simpler than $(x+3)^3 - x^3 - 9x^2 - 27x - 27$,
- $1 + \sqrt{2}$ is simpler than $\sqrt[3]{7 + 5\sqrt{2}}$,
- 4 is simpler than $2 + 1 + 1$.

In other cases, it is not so clear. For example, what is simpler $2^{2^{1024}} - 1$ or a very long equivalent integer? It should be noted that some valuable information could be lost in the simplest form.

The issue of the simplest form is very important in case of computer aided assessment (CAA). Bradford, Davenport, and Sangwin (2009) and Bradford, Davenport, and Sangwin (2010) distinguished between mathematical, pedagogical and aesthetical correctness. For example, in the task *Expand out* $(x+1)^2$, the answer $2x+x^2+1$ is:

- mathematically correct, in that it is equal to the question posed;
- pedagogically correct, in that the student has done the task required;
- aesthetically incorrect, in that there are more conventional ways of writing the answer.

If the answers were equivalent the notation $f_P =_{CAS} f_T$ was used. The main idea of the experiment described in Chapter 8 of this dissertation is that students themselves should analyze the equivalence of their and CAS answers.

Drijvers (2002) emphasised that students could experience difficulties in recognizing equivalence of a CAS answer and the answer that the student had in mind. The examples were $-(x-12)$ and $12-x$ and $\sqrt{\frac{s}{4}}$ and $\frac{1}{2}\sqrt{s}$.

The final form of the answer is not always uniquely determined. Checking equivalence of different answers can be instructive.

### 2.1.6. Answers to trigonometric equation

Section 2.1.6 is devoted to trigometric equations because they are particularly interesting as answers to these equations are more sophisticated and different possible forms of an answer are common in solving the equations with or without CAS.

Fateman (2005) focused on the equation from British A-level examination $\cos(x)+\cos(3x)+\cos(5x)=0$ for $x$. The solutions of this equation could be presented, for example, as

$$\frac{(2n+1)\pi}{6},\ n \in \mathbb{Z}$$
$$\frac{(3n\pm 1)\pi}{3},\ n \in \mathbb{Z}.$$

The CAS (Fateman tested the CAS in 1991, and again in 2005) used different styles for the solutions. A CAS can present the general solution or a set of particular solutions. In case of particular solutions the question is, how many and which solutions should be presented. For example, one CAS presented 5 (Figure 3) and another 10 solutions (Figure 4) in case of $\cos(x)+\cos(3x)+\cos(5x)=0$.

```
1/2 Pi, 2/3 Pi, 1/3 Pi, 1/6 Pi, 5/6 Pi
```

**Figure 3.** Maple 1991, 2005 (Fateman, 2005)

Bernardin (1999) classified the CAS answers as *ok* answers, partial answers and wrong answers. The issue of general solution and particular solution was dis-

$$\{\{x \to \frac{-5\pi}{6}\}, \{x \to \frac{-2\pi}{3}\}, \{x \to \frac{-\pi}{2}\}, \{x \to \frac{-\pi}{3}\}, \{x \to \frac{-\pi}{6}\}, \{x \to \frac{\pi}{6}\}, \{x \to \frac{\pi}{3}\}, \{x \to \frac{\pi}{2}\}, \{x \to \frac{2\pi}{3}\}, \{x \to \frac{5\pi}{6}\}\}$$

**Figure 4.** Mathematica 2005 (Fateman, 2005)

cussed in case of trigonometric equations. If the CAS only gave a set of particular solutions it was marked as a partial answer.

The equation $\frac{1}{\tan x} = 0$ is an example of the topic of indeterminacy and infinity. Bernardin (1999) classified as *ok* answers both, the family of solutions $x = (\frac{1}{2} + u)\pi$, $u \in \mathbb{Z}$ and 'no solutions'.

The answers of equivalent equations could be different in appearance. For example, Pantzare (2012) observed that answers of $\sin 2x = 0.5$ and $\sin 2x = \frac{1}{2}$ are different in a CAS calculator. The answer to $\sin 2x = 0.5$ is

$$x = \frac{180 \cdot (n1 \cdot \pi + 1.309)}{\pi} \text{ or } x = \frac{180 \cdot (n2 \cdot \pi + 0.2617)}{\pi}$$

and to $\sin 2x = \frac{1}{2}$ is

$$x = 15 \cdot (12 \cdot n2 + 5) \text{ or } x = 15 \cdot (12 \cdot n2 + 1).$$

Abramovich (2014) described the case of solving the trigonometric equation $a \cos\varphi + b \sin\varphi = c$ and the corresponding inequalities. Besides other examples he took notice of the equation $4\cos\varphi + 3\sin\varphi = 2$. The answer offered by WolframAlpha

$$x = 2(\pi n + \tan^{-1}(\frac{1}{6}(3 - \sqrt{21}))) \approx 2.0000(3.1416n - 0.25789), n \in Z$$

$$x = 2(\pi n + \tan^{-1}(\frac{1}{6}(3 + \sqrt{21}))) \approx 2.0000(3.1416n + 0.90139), n \in Z$$

differs from the answer of human solution

$$\varphi = \arcsin\frac{6 - 4\sqrt{21}}{25} + 2\pi k, k \in Z$$

$$\varphi = \arccos\frac{8 - 3\sqrt{21}}{25} + 2\pi k, k \in Z.$$

If one uses a CAS for solving school mathematics problems, say equations, he or she will probably get an answer quicker than with pen and paper. However, this is not always the case. Stacey and Ball (2001) presented the trigonometric equation $5\cos^2 x + 2\sin^2 x = 2$ that can be easier to solve by hand than with a CAS, as the substitution of $\cos^2 x + \sin^2 x = 1$ is recognizable. The use of trigonometric features is different in different CAS. Furthermore, the answers can also be different and would need interpretation. The CAS answers in case of

$$5\cos^2 x + 2\sin^2 x = 2$$

are

- 1.5707 (by HP-49G),
- $x = ((2n-1)\pi)/2$ (by TI-89),
- $x = (2\pi)k - \pi/2$ and $x = (2\pi)k + \pi/2$ (by FX-2.0),
- $x = -\pi/2$ and $x = \pi/2$ (Mathematica).

The notation could be unfamiliar for students (Drijvers, 2000). Lokar (2009) said that students can often adapt to different notation faster than we think they can. Sometimes, the same notation could be used for several things, for example, $\tan^{-1}$ in $\tan^2(x) + \tan^{-1}(x)$ for 'cotangent' or 'arctangent' (Smirnova & Watt, 2006).

The answers to trigonometric equations are very interesting in their variability — general or particular solutions, format of answer, etc.

Section 2.1 provides several useful examples and topics that are interesting also in the school context. Some concluding remarks are listed in the final section (2.4) of Chapter 2.

## 2.2. Teaching and learning equation solving

This dissertation investigates the differences between CAS answers and school answers. When Section 2.1 described the works on CAS answers, this section focuses on the school side. On the one hand, the purpose is to explore the background for test suites of CAS answers as they are studied according to the school context. On the other hand, the current situation in teaching and learning equation solving is discussed in order to find a suitable way of utilizing the differences in education.

The place of equation solving in curricula and related problematic issues are described in Section 2.2.1. The procedural character of equation solving is pointed out in Section 2.2.2 and an overview of school answers is provided in Section 2.2.3). Comparing alternative solutions (Section 2.2.4) and discussion amongst students (Section 2.2.5) are also covered in this chapter. CAS issues are not directly presented in Section 2.2, contrary to Section 2.3.

### 2.2.1. Equations at school

The dissertation focuses on solving equations and Section 2.2.1 describes the place of equations in school mathematics and the issues that are related to teaching and learning equation solving.

Cai, Nie, and Moyer (2010) noted that the first thing that people remember from school algebra is usually equations and equation solving. Jakobsson-Åhl (2006) pointed out in her dissertation that school algebra is concerned with formulating and solving equations.

The place of equations in curricula can vary in different countries (or educational jurisdictions). In some countries, for example, in the Russian Federation, the Czech Republic and Hungary, transforming and manipulating expressions, variables and equations are introduced earlier than in many other countries (Kendal & Stacey, 2004 in ICMI 12th study (Stacey, Chick, & Kendal, 2006)). On the other hand, curricula can have different accents — some curricula are restricted to formal solutions to given equations, others also involve formulation equations as models of various types of situations (Jakobsson-Åhl, 2006). The concept of equation itself is defined in different ways at different school levels and, therefore, may be hard to understand as mentioned by Attorps in her dissertation (2006).

However, in all curricula, students get the first idea about equations through linear equations. Learning and teaching the linear equation is a part of transforming from arithmetic to algebra and it has been studied in many works. Kieran (2007) noted that understanding of expressions, variables and equivalence is necessary when working with equations. Booth and Koedinger (2008) pointed out that an incomplete or incorrect understanding of the role of the equals sign and negative signs is injurious to performance and learning of equation-solving procedures.

Different types of equations provide different issues. For example, Zakaria, Maat, et al. (2010) discussed the errors of quadratic factorization, completing the square and quadratic formula in case of quadratic equation. In case of the trigonometric equation, Chigonga (2016) mentioned among other issues checking the validity of the solutions and lack of knowledge about the periodicity of trigonometric functions, which prevents linking revolutions to integers. Equation solving is closely related to functions (Usiskin, 1988; Kolyagin et al., 1977). For example, the quadratic equation is related to the square root, trigonometric equations to trigonometric functions and inverse trigonometric functions, etc.

Both, issues related to a particular type of equations and general issues of equation solving have been highlighted in different works. For example, Kieran (2004) stressed equivalence and the notion of solution to an equation. Knuth, Alibali, McNeil, Weinberg, and Stephens (2005) claimed that equivalence is one of the two most fundamental key ideas (alongside the variable). Kieran et al. (2013) mentioned the crucial role of equivalence of algebraic expressions in expression simplification and equation solving and in a broader context. They were concerned that equivalence, one of the big ideas of algebra, is somewhat cursory in school algebra. Furthermore, they noted the small number of research reports that address explicitly the mathematical concept of equivalence.

From the general issues, the topics of equivalence and solution of equation are especially interesting for this dissertation. Furthermore, linking the periodicity of trigonometric functions and integers in solutions to trigonometric equations is one of the issues observed in Chapter 8.

### 2.2.2. Procedural character of solving equations

Many works emphasised the procedural (rule-based, technical) character of equation solving in school. Some of them are presented in Section 2.2.2. Kieran et al. (2006) claimed that school algebra has been an area where technique and theory collide and technique usually wins over theory. Kaput (1999) wrote that *school algebra has traditionally been taught and learned as a set of procedures disconnected both from other mathematical knowledge and from students' real worlds*.

Kieran (2004) divided the activities of school algebra into three types: generation, transformational, and global/meta-level. Solving equations, simplifying expressions, working with equivalent expressions and equations are main examples of the transformational (rule-based) type. Attorps (2006) pointed out that the concept of equations has a strong process-oriented or operational character because it is used as an effective problem-solving tool at all school levels. Attorps (2006) also noted that teachers perceive equation teaching as a study of procedures rather than as a study of central ideas and concepts of algebra. Moreover, Doerr (2004) pointed that teachers tend to emphasise procedural knowledge for solving equations as the core of algebra.

There are several descriptions of cases where students have the procedural skills and can solve mathematical problems without having an actual understanding of what they do. For example, the cases of computation with integers and solving two-step equations (Reyes, 2012), solving quadratic equations (Didiş, Baş, & Erbaş, 2011), solving word problems (Reusser, 1988), and physics (Ohlsson, 1992) have been presented.

Moreover, Chevallard (2006) and Vinner (2013) discussed the blind compliance with the textbook algorithm in case of the quadratic equation. Attorps (2006) described a study (originally in Wagner, 1981) where only 38% of 29 interviewed students from middle and high school gave a correct answer to the question: *For the equations,* $7 \times W + 22 = 109$ *and* $7 \times N + 22 = 109$*, which would be larger, W or N?* Part of the remaining students had a procedural concept of equations and wanted to solve the equations first. Also, Attorps (2006) found that some teachers have an operational interpretation of algebraic expressions like $x^2 - 5x + 10$, $\sin^2 x + 3\sin x - 4$ and they see these expressions as equations.

Attorps (2006) pointed out that *mathematical understanding of the concept of equation is a complicated interaction between both the operational and the structural aspects of the concept where both aspects are equally important for the acquisition of mathematical knowledge*.

The idea that procedural skills and conceptual understanding should be balanced is expressed, for example, in the preambles of curricula or standards and many different works. *Mathematics Standards* (2017) prescribes that mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness. Hiebert and Carpenter (1992) argued that both, conceptual and procedural knowledge, are necessary in mathematics but conceptual knowledge is essential for understanding. Drijvers et al. (2011) stressed that the aim should be integration instead of polarization — a balanced equilibrium between procedural skills and conceptual understanding is to be preferred. Schoenfeld (2002) argued that it is possible, with well-designed curricula, to teach for understanding without sacrificing procedural skills. The relationship between procedural skills and conceptual understanding in teaching and learning algebra is crucial to debates on development of algebra education (Drijvers et al., 2011).

For this dissertation, it is important to take into account that procedural skills have been and still are very important in teaching and learning equation-solving. A balance between procedural skills and conceptual understanding should be achieved in possible new approaches for the use of CAS answers.

After slightly more general topics, the next section discusses school answers.

### 2.2.3. School answers

This dissertation investigates the relations between CAS answers and school answers. The term school answer was defined as *an answer expected from students* in the introduction of the dissertation. This section gives a brief overview of school answers — their place in textbooks and some different formats of answers.

There are many exercises (for example, equations) in mathematics textbooks. Many textbooks also provide the answers to the exercises. The answers can be provided for all exercises or, for some exercises, the answers can be listed in another book. The expected answers are also presented in worked examples present expected answers.

The differences between answers in different countries (educational jurisdictions) are not often highlighted in literature. However, some issues are listed here.

*Decimal and thousands separators.* There are different ways to notate numbers, for example, different decimal and thousands separators are used in different countries. The United Kingdom and the United States use a period to indicate the decimal place, while many other countries use a comma. The United Kingdom and the United States use a comma to separate groups of thousands, while many other countries use a period and some countries a thin space (*Decimal and Thousands Separators*, 2017).

- 4,294,967,295.00 (e.g. GB-English, US-English)
- 4.294.967.295,00 (e.g. Italian, Spanish)
- 4 294 967 295,00 (e.g. German, French, Finnish, and also Estonian)

CAS use numbers without separation of thousands and a period as decimal separator.

*Mixed number.* The issue of mixed numbers (such as $3\frac{1}{2}$) is one of the differences between countries listed in Paditz (2011). Mixed numbers are used in many countries but are unusual, for example, in France.

*Functions.* Paditz (2011) pointed out some function notations that can have different meanings. For example, $\tan^{-1}$ can stand for the arctan-function or the cot-function. In case of $y = \log(x)$ a base could be 10 or $e$.

*Order of terms.* There can be somewhat different expectations for form of the final answer to simplification of expressions. For example, it is often suggested that the terms of an expression should be sorted from the highest degree to the lowest and variables should be in an alphabetic order. However, several textbooks present, for example, $4+4x$ (Safier, 1990), or $\frac{10}{3}t - \frac{13}{4}s$ (Abramsom, 2017) as the final answers.

*Repeated roots.* Textbooks can include different presentations of the repeated root, for example, in case of quadratic equation where the discriminant is zero. Some textbooks say that there are two equal roots, some say one real root (a repeated root), and some say just one real solution.

*Solutions to trigonometric equations.* The expectations regarding the answers to trigonometric equations can be variable. Sometimes a general solution is expected, sometimes one solution, or solutions in a specified interval. The question of unit, radian or degree is also possible. Moreover, the general solution to $\sin x = m$ can be expressed in the form of two series:

$$x = \arcsin m + 2n\pi, n \in Z$$

$$x = \pi - \arcsin m + 2n\pi, n \in Z$$

or (as in some Estonian textbooks, for example, Lepmann, Lepmann, and Velsker (1996))

$$x = (-1)^n \arcsin m + n\pi, n \in Z.$$

The answers are certainly connected to particular tasks, say equations, but they also provide information about the expectations for the answer format for certain types of task. Textbook answers support students and comparing their answers with textbook answers can be an important part of learning. The next section presents this comparison in more detail.

#### 2.2.4. Comparing alternative solutions

Comparing their own answers to textbook answers is a customary practice for students. Comparison in general is a topic of several works and some of them are presented in this section. Comparison is interesting for the dissertation as the differences between CAS and school answer naturally lead to comparison.

Booth, Lange, Koedinger, and Newton (2013) examined whether correct and incorrect examples with prompts for self-explanation can be effective for learning. Their results showed that incorrect examples (alone or in combination with correct examples) may promote conceptual understanding.

Rittle-Johnson, Star, Durkin and colleagues have written many papers on the use of comparison in case of learning algebra. Star et al. (2015) claimed that comparison is an important part of best practices in mathematics education and sharing solution procedures and discussion of similarities and differences are at the core of reform pedagogy. They distinguished between five types of comparisons. In case of one of them, the same problem solved with two different correct methods and in case of another, the same problem is solved with a correct and an incorrect method (Rittle-Johnson & Star, 2011). Rittle-Johnson and Star (2007) suggested that it pays to compare in learning to solve equations.

Some types of equations are more suitable for comparison tasks. One of them is the topic of trigonometric equations. The trigonometric equation has a somewhat special position, as in case of general solutions it is often possible to produce different-looking correct solutions. Different solution strategies may lead to different-looking but still equivalent answers. Abramovich (2005) presented a classroom discourse in case of the equation

$$2+\cos^2 2x = (2-\sin^2 x)^2.$$

Four different answers were under consideration.

Abramovich (2014) stressed that different solutions to trigonometric equations can be used in activities that can be called 'check the result' and are in style of Polya (2014).

Although Section 2.2 does not provide direct links to CAS it should be noted here that the use of instant answers as feedback when learning pen-and-paper skills is one of the examples of pedagogical opportunities of using CAS as identified (and mapped) by Pierce and Stacey (2010).

Guzmán et al. (2010); Martínez et al. (2012); Guzmán et al. (2011); Stayton (2016) found that differences between students' incorrect answers and correct CAS answers provoked students to reflect on the mathematics involved. The researchers also found that teachers' intervention is inevitable, as students miss important mathematical connections when they find ways to correct their errors.

A part of the inspiration for this dissertation stems from the importance of comparison and from idea of comparing answers to trigonometric equations.

### 2.2.5. Discussion among students

Section 2.2.4 focused on comparing alternative solutions. Every student can compare the solutions by himself/herself but it can also be done in discussion with other students. Discussion among students can be valuable for learning. Discussion among students on comparing CAS answers and students' own answers forms the basis of the pedagogical approach that is the focus of Chapter 8.

Mercer (2000); Mercer and Littleton (2007); Mercer (2008) stressed the important contribution of student-to-student interaction for developing conceptual understanding. In this section, some aspects of discussion that can be related to the use of the differences between CAS and expected answers are highlighted.

Vygotsky (1980) stressed that when students are asked to work collaboratively they are capable of performing at higher intellectual level than in case of individual work. Gokhale (1995) said that proponents of collaborative learning claim that the active exchange of ideas within small groups promotes critical thinking. In her experiment the students had to compare their own answers with a given solution sheet. Sfard (2008) emphasised the interconnection of doing mathematics, thinking mathematically and communicating mathematically.

Working in pairs or in groups also provides new possibilities for data collection. Analyzing working sheets (paper or CAS files) is also possible in case of individual solving. Interviews can be individual as well. (Activity sheets and audio and video-recorded interviews were used, for example, in the study described in Martínez et al., 2012.) Still, analysing of discussion gives opportunity better understanding of misconceptions.

It should be noted that discussion and collaboration do not result in success by default. Mercer and Littleton (2007) suggested that the specific design of activities can significantly impact students' collaboration. Artigue (2005) warned that students could fulfill only minimum obligations.

The guidance provided with a task should lead the student straight to the point. Lagrange (2005) warned for too abstract questions — the instruction 'observe what happens' may not lead to an interesting observation.

The idea that discussion among students can be useful for conceptual learning and data collection is interesting in the context of this dissertation. Furthermore, it should be noted that the specific design of activities (for example, worksheets) can be useful as well.

It is possible to highlight the following keywords from Section 2.2 for this dissertation.

- The concepts of equation, solution of equation and equivalence as general issues of equation solving.
- Answers to trigonometric equations, including general and particular solutions, linking of the periodicity of trigonometric functions and integers in solutions to trigonometric equations as a potentially fruitful topic for experiments.
- The procedural character of equation solving and a desired balance between procedural skills and conceptual understanding as the background for pedagogical approaches.
- The value of discussion and comparison for teaching and learning, but also for research.

Several of the issues highlighted in this section are considered in the CAS context in Section 2.3.

## 2.3. Use of the differences between expected answers and CAS answers

Section 2.3 is devoted to works that are related to the use of the differences between expected answers and CAS answers. Section 2.3.1 highlights some more general aspects of using CAS in teaching and learning of school mathematics. Section 2.3.2 describes the reactions to unexpected answers. Possible ways of developing expectations are presented in Section 2.3.3. Finally, Section 2.3.4 presents some examples where recognition of equivalence/non-equivalence of different answers was utilized.

### 2.3.1. CAS in teaching and learning school mathematics

Before the direct discussion on the use of the differences between CAS answers and expected answers, some more general aspects of the use of CAS in school mathematics are highlighted in Section 2.3.1.

The use of CAS provides an opportunity to reorganize teaching and learning school mathematics and particularly expression simplification and equation solving.

The operations that can be performed by CAS include most of the algebraic operations of school and university mathematics courses (Geddes & Labahn, 1992). Watson (2008) has even said that all school algebra could be solved by CAS. CAS provide students with opportunity to outsource routine work to the CAS (Heid & Edwards, 2001).

Teachers can be assisted by CAS *to enhance students' opportunity to acquire insightful problem solving skills, develop deep conceptual understanding, develop higher levels of thinking, and gain an understanding of how to validate and interpret solutions* (Kendal, Stacey, & Pierce, 2005). The conceptual issues were also important in the 17th ICMI study (Hoyles & Lagrange, 2010). Hoyles and Lagrange observed in the introduction to the proceedings of ICMI Study 17 that almost all papers deal with the issue of how students could focus on conceptual rather than procedural or technical issues.

Artigue (2002) described the theoretical frameworks for thinking about learning issues in CAS environments — anthropological approach (roots from Chevallard) and the theory of instrumentation. They are developed in order to overcome some research traps, as 'technical-conceptual cut'. The 'technical-conceptual cut' refers to an epistemological position, which opposed conceptual and technical work in mathematical activity. The balance between task, technique and theory in acquiring skills is crucial in anthropological terms (Bokhove, 2008). The interaction between the use of ICT tools and conceptual understanding has been demonstrated in research within the framework of anthropological and instrumental approaches (Bokhove, 2011).

There are two 'cover' examples of the theory of instrumentation that are related to this dissertation. Solving of parametric equations is the topic of one of the three schemes of instrumented action in Drijvers et al., 2010 (also described in Drijvers & Gravemeijer, 2005). One element of the scheme is: *Being able to interpret the result, particularly when it is an expression.*

Another example of the scheme of instrumented actions in Drijvers et al., 2010 (from Kieran & Drijvers, 2006; Kieran et al., 2006) is about the notion of equivalence of algebraic expressions. It includes a number of techniques (substituting numerical values, common form by factoring, common form by expanding, common form by automatic simplification). Each of them affects the understanding of the notion of equivalence.

Artigue (with reference to Balacheff, 1994) emphasized that techniques and the mathematical needs of techniques change when computer technology is used. New needs are linked with the computer implementation of mathematics. There is no easy way to identify these needs if the mathematical needs of the technical work are not taken into account.

Stacey (2003) argued that CAS could be used as a pragmatic, epistemic or pedagogical tool. Kadijevich (2014) described these pragmatic, epistemic and pedagogical uses accordingly as 'calculate it', 'understand it' and 'practice it'.

When one discusses possible replacement of pen-and-paper skills with CAS skills it is important to think about the value of pen-and-paper skills. Ball (2014) wrote that pen-and-paper skills are needed for developing mathematical understanding in a broader sense and not just for calculation. Pen-and-paper skills have epistemic value. Berger (2009) presented the equation $x^5 - 10x^3 + 9x = -4x^4 + 40x^2 - 36$ which is easily solvable by CAS and this way has high pragmatic value. However, the epistemic value of manual solving (insight into the nature of roots of a polynomial, the Factor Theorem) is lost.

One idea is replacing pen-and-paper work with programming. Kadijevich suggested programming own functions or programs by CAS for deeper insight (Kadijevich, 2014). The use of programming in the context of learning mathematics has been discussed by many authors; only Papert (1980) is mentioned here. Stoutemyer (1979) compared CAS and programming languages. He argued that CAS is far more relevant to math education than the commonly taught programming languages. He emphasized exact solution of algebraic equations.

For this dissertation, it is interesting that, although CAS provide an opportunity to delegate procedural work to CAS, pen-and-paper activities are still important.

The next section focuses, in particular, on the reactions to unexpected CAS answers.

### 2.3.2. Reactions to unexpected CAS answers

This section presents some reactions to unexpected CAS answers, which have been described in literature.

Drijvers (2002) listed different obstacles that students often encounter in case of working with CAS. The Drijvers' examples were mainly connected with the concept of parameter but the findings are more general. Some obstacles were closely related to CAS answers. The most relevant for this dissertation are the following:

- The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'.
- The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.
- The tendency to accept only numerical solutions and no algebraic solutions. (Numerical solutions and algebraic solutions are here in sense of answer, not as solution process.)

The examples were correspondingly

- $-(x-12)$ vs $12-x$ and $\sqrt{\frac{s}{4}}$ vs $\frac{1}{2}\sqrt{s}$
- $\sqrt{2}$ vs 1.41
- $x=\frac{1}{2}s-\frac{1}{2}v$ as an unexpected answer for those who expect numerical solutions.

Drijvers suggested two reasons why obstacles should be taken seriously in the classroom.

- Encountering obstacles can elicit feelings of irritation and frustration by the students. Although dealing with frustration is a part of doing mathematics, in some cases it can be counterproductive. Ignoring the obstacles in teaching can amplify this effect.
- Obstacles often integrate a technical and a conceptual aspect. Therefore, working on overcoming an obstacle often also means working on the conceptual development of the mathematics involved. (Drijvers, 2002)

Different feelings that students can get in case of unexpected answers were also discussed in other works. Ball described in her dissertation (Ball, 2014) a case where a teacher felt that students experienced frustration when CAS offered an unexpected display for solution of trigonometric equations. However, the teacher mentioned that it was good for developing an appreciation of domain and types of answers. Unexpected results could highlight important aspects of mathematics to be understood.

Artigue (2005) described the effect of surprise produced by unexpected results. It could be useful *to destabilize erroneous conceptions, to promote questioning, to motivate mathematical work*. Artigue described that, in case of computations with radicals where different interpretations were possible, students only noted the results but no more. She stresses that *surprise effects and the resulting motivation for understanding can only exist if there is some expectation*. The familiarity of those students with the topic was too limited to induce spontaneous predictions. Artigue wrote that such a prediction should be partly done mentally without carrying out very detailed calculations. As it was too complicated for the majority of those students, they fulfilled the minimum obligations without deeper discussion.

Stayton (2016) focused on reactions of pre-service secondary mathematics teachers to the differences of outputs of different technologies. Her results suggest that there are opportunities for rich discussion in teacher education for deepening content knowledge and broadening perspectives on technology use.

Alonso et al. (2001) listed some unexpected answers and proposed not to skip such exercises but use them in classroom, as it *is necessary for developing students' critical ability and CAS offer the possibility of contrasting all the results, using a variety of representations*. They were *convinced that the unexpected results presented in this paper can be used to emphasize concepts and increase the critical perspective that every scientist should have*. Pierce and Stacey (2010) stated that it is possible to exploit the contrast of ideal and machine mathematics. Teachers could deliberately use 'unexpected' error messages, expression formats and graphical displays as a catalyst for rich mathematical discussion.

Interpretation and explanation, which is important in case of unexpected answers, could be unaccustomed for students. Meagher (2005) wrote in his dissertation about a student who was bothered by an unexpected answer of a CAS (Mathematica) (e.g., giving an answer as $-(x-3x\hat{}2)$) rather than $3x\hat{}2-x$). The student had the idea that authentic mathematical activity involves working with numbers and expressions and not interpretation and explanation.

Unexpected answers can be a source for whole class discussion.Beaudin, Picard and their colleagues at École de technologie supérieure (ÉTS) have experiences with extensive use of CAS in several mathematics courses over many years. CAS calculators have been mandatory in their courses since 1999 (Beaudin, 2008). They have found surprising or incorrect answers of CAS in different areas, for example, antiderivatives (Beaudin, 2011) or systems of equations (Beaudin, Picard, & Savard, 2013). It is important that these answers are also used in whole class discussions with students.

The reactions described in this section demonstrate that unexpected answers can be useful for teaching and learning, which is a source of inspiration for this dissertation.

### 2.3.3. Expectations for CAS answers

Artigue (2005) stressed that certain expectations are necessary to elicit a sense of surprise about unexpected answers. Some examples of how expectations can be developed are presented in this section.

The first situation refers to students having expectations for the general look of the answer.

Drijvers' (2002) example $x=\frac{1}{2}s-\frac{1}{2}v$ which is an unexpected answer for those who expect numerical solutions is one of them. A similar example, $y=31-x$ as unexpected answer to equation solving, was described by Drijvers and Gravemeijer (2005). It is conceptually difficult for many students, as 'solve' (also by CAS) means finding a solution but the possible answer $31 - x$ to the equation is seen as an expression, not a solution. The idea that the same *Solve*-command is used for isolating an unknown (the result is an algebraic expression) as well as for solving an 'ordinary' equation (the result is a number (or numbers)) points that these operations (that could seems different for the student) are identical in the world of CAS and are also mathematically equivalent (Drijvers & Gravemeijer, 2005).

The inconsistency with the general look of the answer (for example, algebraic expression instead of expected numerical answer to a parametric equation) is quite noticeable even without exact formulation of the expected answer. Often, it is not obvious to students that the answer is unexpected (as they do not have any expectations). The question is how the student could be provided with (or guided to) an expected answer.

Lagrange (2005) said that the results, which were expected by the teacher to be surprising to students, did not alone create surprise. The learning situation had to point to puzzling peculiarities and to challenge students' anticipations. He suggested preparation of examples *where students' predictions will very probably be wrong and to ask them to compare these predictions with the calculator's answer*.

It is possible to provide some answers to students and they have to choose the correct answer. Multiple-choice answers are used in written examination of Mathematical Methods in Victoria (Australia) (*Mathematical Methods — Exams and Examination Reports*, 2016). The students are permitted to use an approved CAS calculator in Part 2 of the examination. Two examples of questions are presented here. The first (Question 1 (2013) (Figure 5)) is quite typical to the Victorian examination. Multiple-choice answers are rare in equation solving (as in Question 4 (2009) (Figure 6)). Both of them are close to the examples used in the experiments described in Chapter 8 of this dissertation.

The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\frac{2}{\pi}$

B. $2$

C. $\frac{1}{2}$ (correct)

D. $\frac{1}{4}$

E. $2\pi$

**Figure 5.** Question 1 (2013) (*Mathematical Methods — Exams and Examination Reports*, 2016)

Similarly, Pantzare (2012) described a case where *the students were asked to solve a trigonometric equation* $\sin 2x = \frac{1}{2}$*, and choose the correct answer from four alternatives*.

There is an example where explanation of one prescribed answer is needed in

The general solution to the equation $\sin(2x) = -1$ is

A. $x = n\pi - \frac{\pi}{4}, n \in Z$ (correct)

B. $x = 2n\pi + \frac{\pi}{4}$ or $x = 2n\pi - \frac{\pi}{4}, n \in Z$

C. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{2}, n \in Z$

D. $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{4}, n \in Z$

E. $x = n\pi + \frac{\pi}{4}$ or $x = 2n\pi + \frac{\pi}{4}, n \in Z$

**Figure 6.** Question 4 (2009) (*Mathematical Methods — Exams and Examination Reports*, 2016)

the question (Figure 7) (Kieran & Saldanha, 2005). There are also some guiding questions asked. The importance of guiding questions will be discussed in Section 2.3.

Q.5 The following equation has $x = 2$ and $x = 2/3$ as solutions:

$$x(2x-4) + (-x+2)^2 = -3x^2 + 8x - 4$$

(i) Precisely what does it mean to say that, "the values 2 and 2/3 are solutions of this equation"?

(ii) Use the CAS to show that:

(a) the two values above are indeed solutions, and

(b) there are no other solutions.

What I entered into the CAS:

What the CAS displays and my interpretation of it:

(iii) Are the expressions on the left- and right-hand sides of this equation equivalent? Please explain.

**Figure 7.** Question 5 (Kieran & Saldanha, 2005)

Lagrange (1999) described a case where the students had to find their answer themselves. The students were asked to differentiate the trigonometric function $x \to \cos(3x - \frac{\pi}{6})$ by hand and with TI-92. After that they had to explain why the resulting expressions were equivalent. The students' expected answer was $-3\sin(3x - \frac{\pi}{6})$ and TI-92 gave $3\cos(3x + \frac{\pi}{3})$. The researchers expected that students could give an explanation like $\cos(\alpha + \frac{\pi}{3}) = -\sin(\alpha + \frac{\pi}{6})$ because the property $\cos(\alpha + \frac{\pi}{2}) = -\sin(\alpha)$ was known to them. Actually, only 8 students of 26 were able to do that. Others only mentioned something like "the calculator doesn't work like we do".

The experiment by Guzman, Kieran and Martinez was described in Guzmán et al., 2010, 2011; Martínez et al., 2012; Stayton, 2016. The topic was simplification of rational expressions. The students worked in pairs and compared CAS answers to their own pen-and-paper answers. The students had to simplify the following expressions by themselves in the beginning and with a CAS afterwards:

$$\frac{x(3+x)}{x}, \frac{4x+4y}{x+y}, \frac{3x+4y}{x+y}.$$

The examples were chosen to guide students to notice important differences between the two last expressions. The latter was manually quite often simplified to 7, after 'canceling' $x+y$.

Artigue (2005) described a case where students were explicitly asked to record the following in case of an unexpected or incomprehensible CAS result:

- the expression concerned;
- the nature of the algebraic manipulation involved: expansion or factorization;
- the result given by TI-92;
- their remarks and comments, as well as their interpretation of the result, if any.

Artigue (2005) also described a case where the students had to compare the results of apparently similar computations, for example, $(\sqrt{3}-\sqrt{6})^2$ and $(\sqrt{3}-\sqrt{5})^2$.

Lagrange (1999) presented a case where the students had to discuss different automatic answers of TI-92 to obviously equivalent expressions.

$$1-(1-x)(3+2x)-x \longrightarrow x^2-2$$

$$1-x-(1-x)(3+2x) \longrightarrow 2(x-1)(x+1)$$

The students also had to use (and learn to use) the items of the algebra menu (*Factor*, *Expand*, *ComDenom*) in order to get the needed output.

Recognition of equivalence/non-equivalence is discussed in the next section.

### 2.3.4. Recognition of equivalence/non-equivalence of different answers

Section 2.3.4 highlights examples from literature where recognition of equivalence/non-equivalence of different answers is presented.

Artigue (2005) said that simplification, which is performed automatically with CAS, could often produce a result different from school answer. In case of those results, students have to develop competences for dealing with the problems of equivalence efficiently, both mentally and with the help of the calculator.

Kieran et al. (2006) described a task where students had to justify CAS simplification $(2-x)(1-2x)$ to $(x-2)(2x-1)$. Students used different techniques. For example, one of them (after some thinking) used substitution, adding a number to both expressions.

With regard to Victorian examination, Ball (2014) affirmed that students need to recognize the equivalence of their own or CAS answers with the correct multiple-choice response. The phase of ascertaining equivalence (or non-equivalence) can be completed with the help of a CAS or with paper-and-pen. Ball (2014) described the comments of teachers about working with pen-and-paper. Working initially with pen-and-paper helps students to understand how a CAS works. Pen-and-paper techniques also seem to be important in transforming an unexpected answer into a multiple-choice response in an examination. Use of CAS in that examination is also discussed in other works (Ball, Pierce, & Stacey, 2003; Pierce, 2001; Pierce & Stacey, 2004).

It should be noted that some of the examples from literature can seem very simple but are actually frustrating for several students. For example, Stacey (2003) talked about recognizing the equivalence of $(b+a)^2$ and $a^2+b^2+2ab$. Only 54% of Year 11 students, who participated in the study, were able to quickly recognize the equivalence. Simplicity is quite subjective and depends on different aspects. For example, the equivalence of $(b+a)^2$ and $a^2+b^2+2ab$ is probably obvious to first-year university students. However, students can have problems with equivalence of trigonometric expressions, as presented in Chapter 8 of this dissertation.

Differently-looking (but equivalent) answers are more common in case of algebraic expressions. Nevertheless, there are numerical answers where equivalence is not obvious. Trigonometry and roots are the most interesting topics to that effect. It is possible that answers of different systems differ. For example, Stacey and Ball (2001) introduced a case where the calculator FX-2.0 offered

$$\frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

and Mathematica gave

$$\frac{-1+\sqrt{3}}{2\sqrt{2}}$$

while $\cos(75^\circ)$ is entered. They argued that *students will frequently need basic algebra skills, if they are to match output to given forms (e.g. for checking answers in a textbook, working with others using a different system, getting a specified answer on a test etc.)*. Trigonometry is also mentioned in other works. A CAS could give immediate results for problems that are very laborious or impossible to solve manually. These cases could help students to become an efficient user of the CAS and also develop the intelligence of algebraic computation (Artigue, 2005).

Recognition of equivalence/non-equivalence of different answers is used in experiments described in Chapter 8 of the dissertation. The related works described in Section 2.3 encouraged the use of the differences between CAS an school (or student) answers in teaching and learning.

## 2.4. Summary of related works

Section 2.4 summarises the chapter on related works in order to highlight the ideas that are particularly important for this dissertation.

An overview of the works that analysed CAS answers was presented in Section 2.1. Wester's review (Wester, 1999b) was inspiring for this dissertation as similar tables can be used for analysing CAS answers in the school context. However, the grading style used by Wester for the answers is not directly applicable in the school context. CAS answers that could differ from school answers were found in different works (for example, in Abramovich, 2014; Alonso et al., 2001; Kadijevich, 2014; Sangwin, 2015; Stoutemyer, 1991) but the exact locations and categories of the differences have not been mapped. Providing a systematic review of the differences between CAS and school answers is one of the main goals of this dissertation.

The works referred to in Section 2.1 presented different examples that can be used in the test suites of the dissertation. Moreover, some interesting issues were highlighted, namely, the issues of the number domain, branching of solutions, and the final form. The domain issues and branching of solutions are discussed in dedicated chapters of this dissertation.

The domain issues have been discussed in literature, for example, in Alonso et al., 2001; Aslaksen, 1999; Jeffrey, 2014; Kaltofen, 2000; Stoutemyer, 1991. Wester (1999b) mentioned a possible special need case for elementary mathematics where all variables should be declared to be real. It provided the inspiration for investigating such possibilities in different CAS in this dissertation.

Examples where a solution consists of different branches in some manner have been presented in several works, for example, the issues of literal equation and extraneous solutions in Bernardin (1999); Stoutemyer (1991); repeated roots in Sangwin (2015); and simplification without comments in Alonso et al. (2001); Berger (2009); Kadijevich (2014); Olive et al. (2010).

Furthermore, this dissertation proposes a pedagogical approach where the differences between CAS answers and students' answers are used in education to support students' understanding. Sections 2.2 and 2.3 mainly described the background for this goal.

Section 2.2 introduced some aspects of teaching and learning equation solving. For this dissertation, it is important to take into account the procedural character of equation solving and try to maintain a balance between procedural skills and conceptual understanding in a pedagogical approach. The concepts of equation, solution of equation and equivalence were highlighted as general issues of equation solving.

Rittle-Johnson and Star (2007) suggested that, in learning to solve equations, it pays to compare. The differences between CAS and school answers provide an opportunity for comparing them. Guzmán et al. (2010, 2011); Martínez et al. (2012); Stayton (2016) found that the differences between students' incorrect answers and correct CAS answers provoked students to reflect on the mathematics involved.

Comparison can be connected to discussion among students. Mercer (2000); Mercer and Littleton (2007); Mercer (2008) stressed the important contribution of student-to-student interaction for developing conceptual understanding. The specific design of activities can significantly impact students' collaboration (Mercer & Littleton, 2007). Moreover, analysing the discussion provides an opportunity for better understanding of any misconceptions.

The use of the differences between expected answers and CAS answers was the focus of Section 2.3. Different works (for example, Alonso et al., 2001; Artigue, 2005; Ball, 2014; Drijvers, 2002; Pierce & Stacey, 2010 ) have discussed students' reactions to unexpected CAS answers and their possible value in teaching and learning.

Students can have expectations for the general look of the answer (Drijvers, 2002). It is possible to provide some answers to students and they have to choose the correct answer (*Mathematical Methods — Exams and Examination Reports*, 2016). The cases where students had to find their answer themselves and compare it with the CAS answer (Lagrange, 1999; Guzmán et al., 2010, 2011; Martínez et al., 2012; Pantzare, 2012; Stayton, 2016) are closest to this dissertation.

Although different topics were highlighted in the works, the answers to trigonometric equations seem to be particularly interesting due to their variability — general or particular solutions, format of answer, etc. Linking of the periodicity of trigonometric functions and integers in solutions to trigonometric equations has been identified as a problematic issue for students. The answers to trigonometric equations are potentially a fruitful topic for comparison and discussion.

Publications from different decades were referenced in this chapter. It is possible that some examples of CAS answers are no longer relevant in case of new CAS or new versions. Unfortunately, there has been no large-scale review of CAS answers after Wester, 1999b. The highlighted issues (for example, domain, branching and automatic simplification) are still relevant for the most part. The mathematical results are, certainly, ageless.

The discussion of related works concluded in this section. Some references to literature are also provided in Chapter 3.

# 3. RESEARCH DESIGN AND INSTRUMENTS

The purpose of Chapter 3 is to describe the research design and instruments. The research questions were proposed in Section 1.2; the research design is introduced in Sections 3.1–3.5. A description of ethical considerations (Section 3.6) and the reasoning behind the choice of CAS (Section 3.7) is provided as well.

## 3.1. Research questions 1 and 2. Differences between CAS and school answers

Research questions 1 and 2 are discussed together because both of them explain the differences between CAS and school answers. To state the research questions again:

> RQ1. Where differences between CAS and school answers could be detected in equations within the school curriculum?
>
> RQ2. How can the detected differences between CAS and school answers to equations in the school curriculum be described and classified?

Wester's work (1999b) was used as a model to investigate these research questions. Firstly, a test suite of different types of equations was composed similar to Wester's (1999b) test suite consisting numerous problems from different types of equations. The equations were chosen for the test suite systematically in order to cover all intriguing equation types taught in schools. Most examples were in accord with textbook contents as much as it was possible to keep diversity. Some examples were taken from textbooks or from research papers and some were developed myself to cover all important types of equations. The examples from the test suite were solved by different CAS and every answer was analyzed afterwards.

The school answers are observed on the basis of some textbooks from Estonia, the USA, and Norway. Some rather old books on teaching mathematics in Russian language were also used. It is possible to say that small differences exist, but such a passing observation is not sufficient for a general description of the differences between countries.

The tables in Chapter 4 outlined the answers to RQ1 indicating where differences between CAS and school answers were possible to detect. Each row of a table includes an equation, the expected school answer and codes that express the type of difference between CAS and school answers in the case of particular CAS used in the solution. The form of the tables was also inspired by Wester (1999b). The descriptions and classification (as answer to RQ2) presented in Chapter 4 mirror the possible set-theoretic relations between answers depending whether the CAS answer includes more or less solutions than the school answer.

Secondly, the content-oriented classification is presented in Chapter 5. The test suite (partially different from the test suite used in Chapter 4) consisted of equations of different types. Answers were analyzed in order to describe and classify differences between CAS and school answers. The classification is based on the form, completion, dependence on the number domain, and branching of answers and automatic simplification of equations. Every type was divided into subtypes (phenomena) and at the end 29 subtypes were identified.

The (second) answer to RQ2 was offered by equation types in Section 5.2. Each row of a table includes an equation, a CAS answer, the name of the CAS, and the identified phenomenon. The types are discussed in Section 5.3 where, in addition to a description of the types, a few brief ideas on the potential didactical use of the types are presented. The connections between the type and Drijvers' (2002) list of obstacles are presented as well.

The limitations of the study on RQ1 and RQ2 are related to the choice of examples for the test suites and the choice of CAS. Although the equations for the test suite were systematically chosen, it is possible to add specific examples.

The sets of compared CAS differ from paper to paper, both in general and in my papers. For example, different sets of CAS were used in the tests described in the papers in the collection Wester (1999a). Also, the publications underlying this dissertation were written in different times and different CAS and versions were used. It causes a limitation for repeating the tests because the old CAS or versions, especially if they are web-based can be inaccessible. The free CAS (or demo versions) are preferred for testing in this study. Usually, the CAS are used with the main settings (unless otherwise specified).

### 3.2. Research question 3. Number domain

> RQ3. When do CAS outputs offer correct and incorrect answers for domain-sensitive examples, specifically for expression simplification and equation solving?

Examples related to square root, natural logarithm and arcsine were chosen for the test suite. The choice was inspired by Rich and Jeffrey (1996) who discussed the calculations $\sqrt{-1}$, $\ln(-1)$ and $\arcsin(5/4)$. In addition to these calculations, the test suite contained 'cancellation expressions', like $\sqrt{x^2}$, identities by simplifying the difference of two expressions, like $\ln xy - (\ln x + \ln y)$, equations without solutions in school context, like $\sin x = 5/4$, and equations with a disputable real solution, like $\sqrt{x} = \sqrt{2x+1}$ where, in case of $-1$, a negative number appears under square root.

The examples from the test suite were solved by different CAS using features suitable for school-like situations. Different CAS have different possibilities (commands, packages, buttons, etc.) for determining the domain of a calculation result, variable values or equation solutions. The correctness of CAS answers was evaluated and answers to RQ3 is presented as a table in Chapter 6. In some cases, CAS does not provide desirable restrictions of domains and in some other cases CAS does not simplify the expression and causing problems for understanding in schools.

## 3.3. Research question 4. Branching solutions

> RQ4. How can branching be described for answers provided by different CAS software; by different school solutions and textbooks; by the possibilities of mathematical approaches for expressions simplifications and equations solving?

The topics where branching solutions appear are collected from answers to RQ1 and RQ2 and from the literature (Bernardin, 1999; Bradford et al., 2002; Stoutemyer, 1991). As in RQ3, simplification of expressions is closely connected to solving equations, and both of them are studied.

The treatment of branching in textbooks and CAS, and mathematical branch-completeness are explained for each topic. Comparison of school, CAS and mathematically branch-complete answers is presented by giving evaluations of branching diversities (EBD). For example, the evaluation CAS < SCH = MATH indicates that CAS answer is less complete (<) than the school and mathematically branch-complete answers, which present branches similarly (=). It is the case, for example, when a CAS provides only $x = 1/a$ as the answer to the literal equation $ax = 1$ but also the case where $a = 0$ should be included in a school answer and in a mathematically branch-complete answer.

In case of previous research questions, tables showed information on CAS answers based on examples from the test suite. Here, in case of answering RQ4, the answer is formed of evaluations presented by topics with comments.

## 3.4. Research question 5. Lesson scenario

> RQ5. What pedagogical approaches could be proposed to utilize the teaching opportunities offered by the differences between CAS and students' answers?

Looking at the balance between procedural skills and conceptual understanding, equation solving is biased toward procedural skills. A desirable pedagogical approach should develop students' conceptual understanding and also produce better insights into students' understandings and misunderstandings.

It is reasonable to use the comparison of CAS answers and students' answers and their discussions. For example, Guzmán et al. (2010, 2011); Lagrange (1999); Martínez et al. (2012); Stayton (2016) introduced examples where students compared their own answers to CAS answers. Discussion among students can provide an important contribution for developing a conceptual understanding (Mercer, 2000; Mercer & Littleton, 2007; Mercer, 2008). An analysis of students' discussion, in which they explain solutions (on paper or verbally), can produce better insights into students' understandings and misunderstandings as conceptual gaps can become evident during discussion even in case of procedurally correct solutions.

Research question 5 focuses on the utilization of differences between CAS and students' answers. Although some other pedagogical approaches may be possible, a lesson scenario was developed and tested. The scenario was based on comparative discussion on students' answers and CAS answers in student pairs. Students were paired up and supplied with given worksheets with tasks and questions. Initially, students solved a task (correctly or not) without a CAS and then with a particular CAS. The worksheets guided students to analyze the differences, equivalence and correctness of their own and CAS answers.

The approach was tested in two rounds of lessons in 2012 and 2013. The lessons were taught by myself (I was not the regular teacher of the course). The scenario was also discussed with the actual teachers of the groups. The task of comparing their own answers and CAS answers was new for students. The students filled out a pre-questionnaire at the beginning and a post-questionnaire at the end of the lesson. The questionnaires were analyzed afterwards and it can be said that the students rather appreciated such lessons.

The description of a lesson scenario with an analysis of students' feedback is provided as an answer to RQ5. The answer to RQ5 is mainly limited to one approach while other approaches are certainly possible.

## 3.5. Research question 6. Equivalence and correctness of CAS and students' answers

> RQ6. How can students identify 1) the equivalence and non-equivalence between CAS and their own answers; 2) correctness of CAS and their own answers of trigonometric equations in lessons based on comparative discussions on students' answers and CAS answers in pairs of students?

A developed lesson scenario was also implemented to collect data for answering RQ6. The lessons were a part of a course in elementary mathematics for first-year university students. As the lessons took place in the beginning of their university studies the situation was similar to secondary school examples. The worksheets of student pairs were used as the main source of data. The audio-tapes were used only for understanding questionable places of the worksheets. Equivalence/non-equivalence and correctness of the students' answers and CAS answers were evaluated. Students' opinions about the equivalence/non-equivalence and correctness were derived from the analysis of worksheets. The worksheet analysis (in case of lessons 2013) was performed in discussion with Marina Lepp, co-author of Tonisson & Lepp, 2015. The collaboration helped to make decisions in problematic cases and improved trustworthiness.

Answers to RQ6 are presented in the form of tables. In case of the equivalence issue, the rows in the tables refer to equivalence/non-equivalence of students' and CAS answers, while the columns refer to students' opinions about equivalence/non-equivalence of their and CAS answers. In case of the correctness issue, the rows in the tables refer to the correctness of students' answers and the columns refer to students' opinion about the correctness of their answer.

The course "Elementary mathematics I" was chosen because the course included topic of trigonometric equations and was conducted at the appropriate time in the university. As the course was one of the first courses for the students at the university, it was probably quite similar to a secondary school situation. Moreover, advanced students were dismissed from the course. However, computer science students (constituting the main group of participants) are probably better in mathematics than secondary school students in general.

## 3.6. Ethical Considerations

Students who participated in the lessons answered RQ5 and RQ6 were informed about the study and they signed a consent form for using their work in research. Participants were free to withdraw their participation at any time. Anonymity was guaranteed for every participant.

## 3.7. Choice of CAS

There are dozens of CAS available nowadays; for example, Grabmeier, Kaltofen, and Weispfenning (2003) gave brief descriptions of more than 60 CAS packages. The sets of compared CAS differ from paper to paper of different authors. Also, the choice of CAS varied in different chapters of this dissertation, as studies were carried out at different times (publications 2007-2015).

CAS packages used in the dissertation were, by chapters:

Chapter 4 (2015): GeoGebra, Maple, Mathematica, Maxima, Sage, WIRIS, WolframAlpha, Xcas;

Chapter 5 (2011): Maxima, OpenAxiom, Sage, WIRIS, WolframAlpha;

Chapter 6 (2008): Derive, Maple, Mathematica, Maxima, MuPAD, TI-92+, TI-nspire, WIRIS;

Chapter 7 (2007): Derive, Maple, Mathematica, Maxima, MuPAD, TI-92+, TI-nspire, WIRIS;

Chapter 8 (2013, 2015): Maxima, WIRIS, WolframAlpha.

The choice of the particular set (which was still subjective) was influenced by various circumstances, some of which were general but some were related to the Estonian situation. The general issue was that support of the authors for some CAS (Derive, MuPAD) ended during the research period and these CAS were not used afterwards. WIRIS (designed for schools) is used as it has been available in Estonian schools from 2006 to 2017. As CAS calculators (like TI-92+, TI-nspire) are not popular in Estonia, they were not used in later publications. Although the heavyweight CAS (Maple and Mathematica) packages have also been used, the main focus is still on CAS available free of charge (Maxima, OpenAxiom, Sage, WolframAlpha, Xcas). GeoGebra was included in later research as it is popular software in schools (also in Estonia) and CAS possibilities were included in recent versions.

Chapters 1, 2 and 3 were introductory chapters. The following chapters that form main part of dissertation are based on different papers and may somewhat vary in style.

# 4. SPECTRUM OF DIFFERENCES BETWEEN EXPECTED SCHOOL ANSWERS AND CAS ANSWERS

## 4.1. Introduction

Chapter 4 is devoted to answering the research questions RQ1 and RQ2:

> RQ1. Where differences between CAS and school answers could be detected in equations within the school curriculum?
>
> RQ2. How can the detected differences between CAS and school answers of equations in the school curriculum be described and classified?

The chapter is based on Tonisson, 2015. The answer to RQ1 is provided in Tables 1–10. Each row of a table includes an equation, the expected school answer, and the type of difference between CAS and school answer in case of particular CAS. Such form of tables was inspired by Wester (1999b). The types with descriptions offer one possible answer to RQ2.

The test suite consisted of 127 equations. Linear, quadratic and fractional equations, equations that contain an absolute value of an expression, irrational, exponential, logarithmic and trigonometric equations and literal equations were included. Each equation was solved by 8 different CAS. The systems GeoGebra (GGB in tables, webpage in the list of references), Maple (MPL), Mathematica (MTH), Maxima (MXM), Sage (SAGE), Wiris (WRS), WolframAlpha (WA) and Xcas (XCS) were used for the chapter. The versions accessible in 2014 were used. Most of them were accessible for free, enabling even parallel use of the systems. The trial versions of Maple and Mathematica were used as well.

The choice of example equations and CAS commands is introduced in Section 4.2. The types of differences and the tables with typed CAS answers are presented in Section 4.3. Some remarks on making use of the differences in the classroom can be found in Section 4.4. Finally, a conclusion is provided in Section 4.5.

## 4.2. Choice of equations and commands

Section 4.2 describes, firstly, the selection of equations for the test suite and, secondly, the selection of commands.

Equations play an important role in school mathematics. There are hundreds of equations in the textbooks and choosing a reasonable set of examples can be quite complicated. In rough terms, the test suite consists of equations selected for the following reasons:

- Common equations that are or could be in textbooks, for example, $2x^2 - 4x - 5 = 0$, in order to cover all basic types.

- Equations developed from trivial equations by attaching an 'intriguing behaviour', for example, $x+\frac{1}{x-1}=1+\frac{1}{x-1}$ instead of $x=1$. In some cases, triviality was disguised, for example, $x+1=x$ or $x-(x-1)=0$ instead of $1=0$.
- Equations from literature, for example, $\tan(x+\frac{\pi}{4})=2\cot x-1$.

The test suite itself, with reasons given by equations, is presented in Appendix A. The reasons are described now in more detail.

The aim was to include all basic types of equations from school mathematics. The most general level of classification of equations (linear, quadratic and fractional equations, equations that contain an absolute value of an expression, irrational, exponential, logarithmic, trigonometric and literal equations) is presented quite well in different sources. The types at more refined levels are usually less clearly distinguished in textbooks or elsewhere. However, the aim was to find examples from a more exact level than just the general level. For example, in case of quadratic equations, the test suite included equations from types $ax^2+c=0$, $ax^2+bx=0$, $b^2-4ac>0$, $b^2-4ac=0$, $b^2-4ac<0$ but also some equations chosen for other reasons, like an equation with same factor in both sides $(x-1)x=(2x+1)(x-1)$.

The idea was to provide an example for every significant solving technique. The examples were chosen from the textbooks or composed so that the observed (somewhat problematic) issue is as exposed as possible. Many equations have solutions with 'simple' numbers, like $-1$, 0, 1, 2, for this reason. On the other hand, there are also 'ugly' cases, like $\sin x=\frac{1}{10}$. As domains and ranges of functions are essential topics, there are special examples focusing on that, for example $2^x=-1$. The issue of extraneous solutions is very important as some solving algorithms lead to possible extraneous solutions. Besides 'normal' equations with possible extraneous solutions, like $x+\sqrt{x}=2$, there are examples of type $\frac{f^2(x)}{f(x)}$, like $\frac{\tan^2 x}{\tan x}=0$, and also examples where an 'intriguing' term is inserted by addition, like $x+\sqrt{x}=\sqrt{x}-1$, or by division, like $\frac{x}{\log x}=\frac{1}{\log x}$. On the other hand, examples of type $f^3(x)=f(x)$, like $x\sqrt{x}=\sqrt{x}$, provoke missing of the solution due to oversight.

Several examples in the test suite were taken from or inspired by related works. Wester's review (1999b) did not focus on school mathematics and included only a few examples from school. Nevertheless, 12 examples from his review are suitable for our test suite.

Two equations ($ax=0$ and $ax=1$) are included in our test suite from Stoutemyer (1991). Kadijevich (2014) mentioned simplification of $\frac{x^2}{x}$ to $x$ not adding the constraint $x\neq 0$. Several examples of this type are tested in this chapter. Although the main topic of Sangwin (2015) was inequalities, some equations are also examined. For example, he pointed out the issue of repeated root and illustrated it by solving equation $x^2 - 6x + 9 = 0$ with several CAS. This issue is also touched upon in the tests.

The test suite includes examples of trigonometric equations that illustrate issues covered in the chapter on related works. The variety of correct answers to trigonometric equation is discussed, for example, by Abramovich (2005) and Pantzare (2012). The possibility that a CAS does not indicate some solutions or only provides solutions within a restricted interval is noted by Wester (1999b). It depends mainly on the CAS (and command) but not on particular equation.

The solving of equations is certainly a topic in very many textbooks. Some rather old textbooks of didactics of mathematics in Russian (for example, Bloh, Gusev, Dorofeev, et al., 1987; Boltyanskii, Sidorov, & Shabunin, 1972; Kolyagin et al., 1977) are good resources for interesting examples and elaborated classification. For example, $\tan(x + \frac{\pi}{4}) = 2\cot x - 1$ is taken from (Kolyagin et al., 1977).

Some examples (like $x^x = x$, $(x-6)^x = 2^x$) are actually beyond school mathematics.

As our interest is focused on the answers offered by CAS, the choice of commands of CAS is also important. The solution process (and the answer) can be sensitive to a change of command, additional arguments, the form of an argument, etc. In this study, we mainly use the most natural method of solution — the *solve* command. However, sometimes it is reasonable to use other commands or additional options. For example, the additional *to_poly_solve* command is used in Maxima and Sage and the *R*educe command in Mathematica. There are also possibilities to determine domain in some CAS, for example, *RealDomain* in Maple. (However, even more suitable variants may exist.) The parallel use of different commands or options of the same CAS could be instructive as the differences between them could reveal important issues. For example, if one command presents an imaginary solution and the other not, this could serve as a lead-in to a discussion on number domains.

The syntax of input differs in different CAS; for instance, a logarithm can be very multifarious (discussed also in Section 5.2.8). Although output notation can be confusing sometimes, we do not focus on that and presume that the user can understand the answer. For example, response to command

`to_poly_solve(sin(x)=1/2,x);` (shown in Figure 8)
could be confusing at first.

$$\%union\left( [x = 2\,\pi\,\%z12 + \frac{\pi}{6}],\ [x = 2\,\pi\,\%z14 + \frac{5\,\pi}{6}] \right)$$

**Figure 8.** Output of command *to_poly_solve* (Maxima)

In some cases, CAS deliberately offer answers that are not equivalent to the general mathematically correct answer. For example, particular solutions could be presented instead of general solutions in case of trigonometric equations. (It is even possible that the methods used can only produce particular solutions.) Some CAS also generate warning messages if this is the case.

There reasoning behind the selection was described in this section. The next section introduces the types of differences and proposes a respective classification of the differences.

## 4.3. Types of differences between CAS answers and school answers

Section 4.3 introduces the types of differences between CAS answers and school answers, and then $127 \cdot 8$ answers (127 equations, 8 CAS) are classified according to these types.

RQ1 asks where the differences between CAS and school answers can be detected in equations. For this question, it is enough just to mark whether a particular CAS answer differs from the school answer or not. It is reasonable to observe the answers somewhat more thoroughly at same time. The answers were analysed and the differences between CAS answers and expected school answers were classified in six types in this chapter.

(1) No difference.

(2) Equivalent but different.

(3) More solutions than expected.

(4) Fewer solutions than expected.

(5) Did not solve or only some transformations were completed. Error message.

(6) Very complicated answer.

If a CAS answer is not exactly the same as the school answer it is reasonable to ask whether the CAS answer is equivalent to the school answer. In addition to checking equality or equivalence of single solutions, there are several cases where a set of solutions should be compared. As a solution set of CAS answers can contain more or fewer solutions than the expected school answer, the types (2), (3), (4) can be established. The types (5) and (6) are needed for extraordinary cases.

This classification is quite simple and only one possible way of doing this, while more elaborate classifications are possible as well. Another classification is presented in Chapter 5.

Now the types are described in more detail.

(1) No difference. The answer is in the expected form. If there are any questions, they could be about the expected form. For example, in case of trigonometric equations the 'cultural' issue is important, as the expectations for answers

can be different in different cases. For instance, the solution to $\sin x = m$ could be expressed in the form of two series:

$$x = \arcsin m + 2n\pi, n \in Z$$

$$x = \pi - \arcsin m + 2n\pi, n \in Z$$

or (as in Estonian textbooks)

$$x = (-1)^n \arcsin m + n\pi, n \in Z.$$

The examples where this is the only difference are also marked as (1). Trigonometric equations are discussed in Section 5.2.9 and Chapter 8.

(2) Equivalent but different.

The expected answer (or, more specifically, a response) could also be an error message or warning message. For example, in case of $x^2 + 1 = 0$ Wiris generates the message *Warning, difficulty: It is not possible to find a result or solution.*

The type *Equivalent but different* is very extensive — from answers that are almost the same as expected to answers that look very different and require hard (and possibly instructive) work to check the equivalence. This category also includes answers that could be called unfinished but the unknown should be already isolated. For example, we treat $\frac{\ln 8}{\ln 2}$ as unfinished answer but not $\frac{\ln 7}{\ln 2}$.

The next types are roughly identified by comparing whether the CAS answer includes more or less solutions than the expected answer.

(3) More solutions than expected.

A CAS answer can have more solutions if extraneous solutions are included. This is closely related to the domain questions of equation and function, as extraneous solutions are outside the domain of the functions that are included in the equation. The number domain is also very important as the solution set could differ in case of, for example, the real and complex domain. It should be noted that the complex domain is not distinguished only by an imaginary unit in the answer, but it could also be present in case of real (or even integer) solutions. Is $-1$ the solution to the equation $x + \sqrt{x} = \sqrt{x} - 1$? Yes, in case of the complex domain. Furthermore, hidden complex domain can also be seen in case of the answer with $\arcsin 2$. We distinguish between two subtypes:

(3a) if the provided answer is not suitable even in the complex domain, and

(3b) if the provided answer is a correct answer in the complex domain.

The subtype (3a) is related to the question of division by zero.

(4) Fewer solutions than expected.

(4a) If we search for a general solution to a trigonometric equation, the particular solutions that several CAS provide could be noted as fewer solutions than expected.

(4b) There are also other examples where a solution is missing (for example, in case of $\sqrt[3]{x+45} - \sqrt[3]{x-16} = 1$, some CAS miss $-109$). In case of literal equations the issue could be about presenting different branches. Almost all CAS provide only the main branch (in case of equation $ax = 1$ only branch where $a \neq 0$) as mentioned also by (Bernardin, 1999). The only exception is the use of the *Reduce* command in Mathematica.

The last types are the most unsuccessful and, fortunately, rare.

(5) Did not solve or only some transformations were completed. Error message.

Unlike in case of type 2, where an error message was rather expected, here the error message indicates incapacity for solving. Figure 9 presents the response in case of $2^{2x} = 2^{x-1}$ by Sage where only some transformations are completed.

```
[2^x == -2^(1/2*x - 1/2), 2^x == 2^(1/2*x - 1/2)]
```

**Figure 9.** Response with only some transformations completed

(6) Very complicated answer.

Figure 10 shows a fragment of response in case of $2(\sin x + \cos x) + \sin 2x + 1 = 0$ by Maxima is presented.

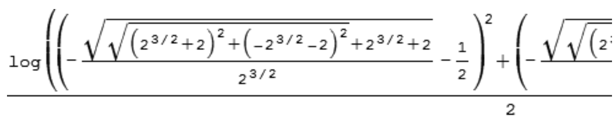

**Figure 10.** A fragment of complicated answer

Examples of equations are presented in the following tables:

- linear equations in Table 1;
- quadratic equations in Table 2;
- fractional equations in Table 3;
- irrational equations in Table 4;
- equations that contain an absolute value of an expression in Table 5;
- exponential equations in Table 6;
- logarithmic equations in Table 7;
- trigonometric equations in Table 8 and Table 9;
- literal equations in Table 10.

The idea was not to discuss, which CAS is better or worse, or even more or less suitable for school. A deviation from school expectations can even be an advantage in terms of promoting discussion.

This section classified the differences between CAS answers and school answers into six types. Some remarks on using the differences are presented in the next section.

| Equation $\to$ school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $2x-4=0 \to 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $2x-3=0 \to 3/2$ ($1\frac{1}{2}$ is discussed in Section 5.2.2) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $2x=0 \to 0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x+1=x \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x-(x-1)=0 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x-1=1-(2-x) \to \mathbb{R}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 1.** Linear equations

| Equation $\to$ school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $x^2-1=0 \to \pm 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x^2+x=0 \to 0,-1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x^2-3x+2=0 \to 1,2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $2x^2-4x-5=0 \to 1 \pm \frac{\sqrt{14}}{2}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x^2-2x+1=0 \to 1,1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x^2+x+1=0 \to \emptyset$ | 1 | 1 | 1 | 3b | 3b | 1 | 1 | 1 |
| $x^2=-1 \to \emptyset$ | 1 | 1 | 1 | 3b | 3b | 1 | 1 | 1 |
| $2(x-1)(x-2)=0 \to 1,2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $(x-1)x=(2x+1)(x-1) \to 1,-1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $(x+1)x-x^2=1 \to 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x^3=x \to 0,1,-1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x^4-5x^2+4=0 \to -1,1,-2,2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 2.** Quadratic equations

| Equation $\to$ school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{x} = 1 \to 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{1}{x} = 0 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x + \frac{1}{x} = \frac{1}{x} \to \emptyset$ | 1 | 3a | 3a | 3a | 3a | 3a | 1 | 1 |
| $x + \frac{1}{x-1} = 1 + \frac{1}{x-1} \to \emptyset$ | 1 | 3a | 3a | 3a | 3a | 3a | 1 | 1 |
| $\frac{x^2-x+1}{x-1} = \frac{x}{x-1} \to \emptyset$ | 1 | 1 | 3a | 3a | 3a | 3a | 1 | 1 |
| $x^2 + 2 + \frac{1}{x-1} = 3x + \frac{1}{x-1} \to 2$ | 1 | 3a | 3a | 3a | 3a | 3a | 1 | 1 |
| $\frac{x}{x-1} = \frac{1}{x-1} \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 3a | 1 | 1 |
| $\frac{x(x-1)}{x-1} = 1 \to \emptyset$ | 1 | 3a | 3a | 3a | 3a | 3a | 1 | 1 |
| $\frac{x^2}{x-1} = \frac{3x-2}{x-1} \to 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{1}{x-1} = \frac{x}{2} \to 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{x}{2x-2} - \frac{1}{x^2-1} = \frac{2x}{2x+2} \to 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{x^2+1}{x^2} = 2 - \frac{1}{x} + x \to \pm 1, \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{x^2}{x} = 0 \to \emptyset$ | 1 | 3a | 3a | 3a | 3a | 3a | 1 | 1 |

**Table 3.** Fractional equations

| Equation $\to$ school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $\sqrt{x}=1 \to 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt{x}=-1 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt{x}+\sqrt{x-1}=-2 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt{2x}=\sqrt{x-1} \to \emptyset$ | 3b | 1 | 1 | 3b | 3b | 1 | 3b | 3b |
| $\sqrt{x^2+2}=\sqrt{3x} \to 1,2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x-\sqrt{1-x^2}=1 \to 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x+\sqrt{1-x^2}=1 \to 1,0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt{x+4}+\sqrt{x+1}=3 \to 0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt{x+5}+\sqrt{20-x}=7 \to 4,11$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt{2x+1}+\sqrt{x-3}=4 \to 4$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt[3]{x}=2 \to 8$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt[3]{2x}=\sqrt[3]{x-1} \to -1$ | 1 | 1 | 1 | 1 | 4b | 1 | 1 | 1 |
| $\sqrt[3]{x^2+2}=\sqrt[3]{3x} \to 1,2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt[3]{x^2}-3\sqrt[3]{x}+2=0 \to 1,8$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $2\sqrt{x^2-2x+4}-\sqrt{x^2-2x+9}=1$ $\to 0,2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sqrt[3]{x+45}-\sqrt[3]{x-16}=1 \to 80,-109$ | 1 | 4b | 4b | 4b | 4b | 4b | 4b | 4b |
| $\sqrt{x^2+1}=x-2 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x+\sqrt{x}=2 \to 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $2\sqrt{x}+3x^{1/4}-2=0 \to \frac{1}{16}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{x}{\sqrt{x}}=0 \to \emptyset$ | 1 | 3a | 3a | 3a | 3a | 1 | 1 | 1 |
| $x\sqrt{x}=\sqrt{x} \to 0,1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x+\sqrt{x}=\sqrt{x}-1 \to \emptyset$ | 3b | 3b | 3b | 3b | 3b | 3b | 3b | 3b |
| $\frac{x}{\sqrt{x-1}}=\frac{1}{\sqrt{x-1}} \to \emptyset$ | 1 | 1 | 1 | 3a | 3a | 1 | 1 | 1 |
| $\sqrt{x^2}=1 \to -1,1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 4.** Irrational equations

| Equation → school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $\|x\| = 1 \to \pm 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\|x-1\| = 2 \to -1, 3$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\|2x-1\| = -3x \to -1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\|x\| = -1 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\|2x-1\| = -3 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\|x-1\| + \|x+1\| = 2 \to [-1, 1]$ | 1 | 1 | 1 | 2 | 5 | 4b | 1 | 1 |
| $\|2x+5\| = \|x-2\| \to -1, -7$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\|x-1\| + \|2x+1\| = 3 \to \pm 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x + \frac{1}{\|x\|} = \frac{1}{\|x\|} \to \emptyset$ | 1 | 1 | 3a | 3a | 3a | 3a | 1 | 1 |
| $\frac{x}{\|x\|} = 0 \to \emptyset$ | 1 | 3a | 1 | 1 | 3a | 1 | 1 | 1 |

**Table 5.** Equations with absolute value

| Equation → school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $2^x = 8 \to 3$ | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 |
| $2^x = 7 \to \log_2 7$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $2^x = -1 \to \emptyset$ | 1 | 1 | 1 | 3b | 3b | 1 | 3b | 1 |
| $2^{2x} = 2^{x-1} \to -1$ | 1 | 1 | 1 | 5 | 5 | 1 | 1 | 1 |
| $2^{x+1} + 2^x = 3 \to 0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $4^x - 3 \cdot 2^x + 2 = 0 \to 1, 0$ | 1 | 1 | 1 | 5 | 5 | 1 | 1 | 1 |
| $2^{x^2-3x} = \frac{1}{4} \to 1, 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $(x-6)^x = 2^x \to 8, 4, 0$ | 4b | 4b | 1 | 5 | 5 | 5 | 4b | 4b |
| $x^x = x \to 1, -1$ | 5 | 4 | 1 | 5 | 5 | 5 | 4 | 5 |

**Table 6.** Exponential equations

| Equation → school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $x^{\log_x(x^2+3)} = 4 \to \emptyset$ | 3a | 3a | 3a | 5 | 5 | 3a | 3a | 3a |
| $\log_{10} x = 2 \to 100$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\log_x 100 = 2 \to 10$ | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |
| $\log_x 90 = 2 \to 3\sqrt{10}$ | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 |
| $\log_{10}^3 x = \log_{10} x \to 1, 10, \frac{1}{10}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{\log_{10}^2 x}{\log_{10} x} = 0 \to \emptyset$ | 1 | 3a | 3a | 3a | 3a | 3a | 1 | 1 |
| $\log_{10}(2x) = \log_{10}(x-1) \to \emptyset$ | 3b | 3b | 1 | 3b | 3b | 1 | 3b | 3b |
| $\log_{x+1} 4 = 2 \to 1$ | 1 | 1 | 1 | 3b | 2 | 1 | 1 | 1 |
| $\log_3(\log_2 x) = 0 \to 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\log_2(x-2) + \log_2(x-3) = 1 \to 4$ | 1 | 1 | 1 | 3b | 3b | 1 | 1 | 1 |
| $\log_{10} x^2 + \log_{10} x = 3 \to 10$ | 1 | 1 | 1 | 1 | 3b | 1 | 1 | 1 |
| $\log_{10}^2 x - 3\log_{10} x + 2 = 0 \to 10, 100$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $x + \log x = \log x - 1 \to \emptyset$ | 3b | 3b | 3b | 3b | 3b | 3b | 3b | 3b |
| $\frac{x}{\log x} = \frac{1}{\log x} \to \emptyset$ | 1 | 1 | 1 | 3a | 3a | 1 | 1 | 1 |
| $x^{\log_{10} x} = 100 \to 10^{\sqrt{2}}, 10^{-\sqrt{2}}$ | 2 | 2 | 2 | 5 | 5 | 1 | 2 | 2 |
| $\sqrt{\log x} = \log \sqrt{x} \to 1, e^4$ | 5 | 1 | 1 | 1 | 1 | 5 | 1 | 5 |

**Table 7.** Logarithmic equations

| Equation $\to$ school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $\sin x = 0 \to n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin x = \frac{1}{2} \to (-1)^n \frac{\pi}{6} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin(3x) = -1 \to (-1)^{n+1} \frac{\pi}{6} + \frac{n\pi}{3}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin(x - \frac{\pi}{3}) = \frac{\sqrt{2}}{2} \to \frac{\pi}{3} - (-1)^n \frac{\pi}{4} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin(4x+2) = \frac{\sqrt{3}}{2} \to$ $-\frac{1}{2} + (-1)^n \frac{\pi}{12} + n \cdot \frac{\pi}{4}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin x = \frac{1}{10} \to (-1)^n \arcsin \frac{1}{10} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin x = 2 \to \emptyset$ | 1 | 1 | 1 | 3b | 3b | 1 | 3b | 4a |
| $\cos x = 0 \to \pm \frac{\pi}{2} + 2n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\cos(x - \frac{\pi}{6}) = 0.5 \to \frac{\pi}{6} \pm \frac{\pi}{3} + 2n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\cos(x - \frac{\pi}{6}) = 1/2 \to \frac{\pi}{6} \pm \frac{\pi}{3} + 2n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\cos x = 2 \to \emptyset$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\tan x = 0 \to n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\tan x = -\frac{\sqrt{3}}{3} \to \frac{\pi}{6} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\tan x = 2 \to \arctan 2 + n\pi$ | 1 | 1 | 2 | 1 | 4a | 4a | 1 | 4a |
| $\tan x = \tan \frac{\pi}{4} \to \frac{\pi}{4} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\tan x = \tan \frac{\pi}{2} \to \emptyset$ | 1 | 1 | 3a | 3a | 1 | 1 | 3a | 1 |
| $\cot x = 0 \to \frac{\pi}{2} + n\pi$ | 1 | 1 | 1 | 1 | 4a | 5 | 1 | 4a |
| $\cot x = \sqrt{3} \to \frac{\pi}{6} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\cot x = \cot 0 \to \emptyset$ | 1 | 1 | 3a | 3a | 1 | 1 | 3a | 1 |
| $\sin x = \cos x \to \frac{\pi}{4} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 1 |
| $\tan x = 1 \to \frac{\pi}{4} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin x = \tan x \to 0 + n\pi, 0 + n2\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 8.** Trigonometric equations (1)

| Equation → school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $\tan(x+\frac{\pi}{4}) = 2\cot x - 1 \to$ $\frac{\pi}{2} + n\pi$, $\arctan\frac{1}{2} + n\pi$ | 1 | 1 | 1 | 6 | 4b | 4 | 1 | 2 |
| $\tan^3 x = \tan x \to n\pi, \pm\frac{\pi}{4} + n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{\tan^2 x}{\tan x} = 0 \to \emptyset$ | 3a | 1 | 3a | 3a | 3a | 3a | 3a | 1 |
| $2\sin x \cos 2x - 1 + 2\cos 2x - \sin x = 0$ $\to \pm\frac{\pi}{6} + \pi n, -\frac{\pi}{2} + 2n\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $2(\sin x + \cos x) + \sin 2x + 1 = 0 \to$ $-\frac{\pi}{4} + n\pi$ | 2 | 1 | 1 | 6 | 6 | 4a | 3b | 4a |
| $\cos(x^2 - 2) = \frac{1}{2} \to$ $\pm\sqrt{2 \pm \frac{\pi}{3} + 2\pi n}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\sin^2 x - 3\sin x + 2 = 0 \to \frac{\pi}{2} + 2\pi n$ | 1 | 1 | 1 | 6 | 3b | 4a | 3b | 4a |
| $3\sin^2 x \cos x - \cos^3 x = 0 \to$ $(2n+1)\frac{\pi}{2}$, $(6n+1)\frac{\pi}{6}$, $(6n-1)\frac{\pi}{6}$ | 2 | 4a | 1 | 2 | 4a | 4a | 1 | 4a |

**Table 9.** Trigonometric equations (2)

| Equation → school answer | GGB | MPL | MTH | MXM | SAGE | WRS | WA | XCS |
|---|---|---|---|---|---|---|---|---|
| $ax = 0 \to$ if $a \neq 0$ then 0, else $\mathbb{R}$ | 4b | 4b | 4b | 4b | 4b | 4b | 4b | 4b |
| $ax = 1 \to \frac{1}{a}$ | 4b | 4b | 4b | 4b | 4b | 4b | 4b | 4b |
| $ax^2 - 3x + 2 = 0 \to$ if $a = 0$ then $\frac{2}{3}$ else $\frac{3 \pm \sqrt{9-8a}}{2a}$ | 4b | 4b | 4b | 4b | 4b | 4b | 4b | 4b |
| $ax^2 + bx + c = 0$ | 4b | 4b | 4b | 4b | 4b | 4b | 4b | 4b |
| $ax + b = 0$ | 4b | 4b | 4b | 4b | 4b | 4b | 4b | 4b |

**Table 10.** Literal equations

## 4.4. Some remarks on using the differences between CAS answers and school answers

Nice concrete examples of educational use of the differences of CAS answers from expectations can be found in the works referred in Section 2.3 of this dissertation. The ideas presented here, in Section 4.4, are not well-developed and they are provided mainly as an evidence of the ongoing search for an applicable idea.

We primarily discuss the topics that could be highlighted. The keywords are certainly already there — equivalence, domain, extraneous solution, properties of a particular function or operation, particular and general solutions.

The difficulty of the task of checking equivalence depends on the particular equation. It could be very easy if the answers are obviously different. However, it could be complicated in other cases. Of course, a teacher should consider students' level of learning, the aim of the lesson, etc. For example, checking equivalence of the answers to a quadratic equation could be instructive if manipulation of such expressions is the current topic of study but it could become boring later. In addition, a CAS answer can sometimes be ahead of the school program. For example, an imaginary unit can be disturbing if presented too early. On the other hand, it could be very interesting for some students even at the early stages.

Solutions to trigonometric equations are a good source for tasks, as even the correct solutions can look externally very different. Chapter 8 (based on Tonisson, 2013 and Tonisson & Lepp, 2015) presents the lessons in which university students solved trigonometric equations at first by themselves manually and after that with a CAS.

The comparison of a CAS answer to the students' (possibly incorrect) answer opens new and interesting perspectives. We suggest that this method could stimulate discussion and provide a possibility to activate students.

Domain issues (both a domain of the function and number domain general) are also very important and the above equations could provide ground for discussion. For example, $x+\frac{1}{x}=\frac{1}{x}$ leads to the domain of a function $f(x)=\frac{1}{x}$ and also to real and complex domains. Many examples in the previous table are (deliberately) quite simple — the critical issue is quite transparent. It is possible to decrease the transparency somewhat; for example, one could use $\frac{4x}{2}-x+\frac{(x-1)^2-x(x-2)}{x}=\frac{1}{3(x-1)-2x+3}$ instead of $x+\frac{1}{x}=\frac{1}{x}$. The observable function could be present already in the equation but it could also appear in the solution. For example, in $2^x=8$ the answer is $\frac{\ln(8)}{\ln(2)}$ and the logarithm appears only in the solution (which is unfinished). The domain issue is directly related to extraneous solutions. The common strategy for excluding extraneous solutions is checking the provisional solution in the initial equation. It could be done by a CAS. Sometimes the extraneous solution is caught; for example, $0+\frac{1}{0}$ causes an error message. Sometimes an extraneous solution can pass the test. For example, in case of equation $x+\sqrt{x}=\sqrt{x}-1$ the substitution $-1+\sqrt{-1}=\sqrt{-1}-1$ could be simplified to True (which is correct in the complex domain).

Imaginary numbers appear in different cases — the most famous case is square root. It is common that $i$ is introduced by $\sqrt{-1}$ from $x^2=-1$. Actually, imaginary numbers can be part of solutions of exponential, logarithmic and trigonometric equations. An exhaustive discussion of such solutions requires knowledge about complex analyses, branch cuts, etc., which is definitely beyond school mathematics and probably not even covered in math teacher training. Sometimes answers with complex number solutions can be (perhaps frighteningly) large, occupying dozens of lines. The questions of uncertainty and infinity also belong to an area that almost immediately crosses the border of school mathematics. Some CAS still divide by zero.

The ideas that are presented in this section could potentially be developed for actual use in classes. However, only one approach was actually developed and tested in the frames of this dissertation.

## 4.5. Conclusion

Section 4.5 concludes Chapter 4.

A study, in which 127 equations of different types of school equations were solved by 8 computer algebra systems, was presented in the chapter. The tables in this chapter provide an answer to RQ1 *Where differences between CAS and school answers could be detected in equations within the school curriculum?* The answers to linear and quadratic equations were practically same as school answers in case of all the CAS. There were some differences in other types of equations. In case of the literal equation, all CAS provided only the main branch which would not be a complete answer in the school context. In general, GeoGebra answers had the least differences from school answers.

The differences between CAS answers and expected school answers were classified into six types. This classification could be accounted as one answer to RQ2 *How can the detected differences between CAS and school answers of equations in the school curriculum be described and classified?* Other classifications are possible and a different one is provided in Chapter 5. Chapter 5 also includes more ideas about instructive use of the differences.

A teacher should decide if solving a particular equation with a particular CAS is suitable for stimulating (hopefully rich) discussion. It could be reasonable to choose another CAS or avoid the equation at all. The possible topics of classroom discussion are related to equivalence, domains and ranges of different functions, extraneous solutions, real and complex domain.

Many examples from the test suite are 'explicit' and could be 'disguised' for the purposes of actual use. Some cases (especially answers with complex numbers and infinity) are definitely out of students' and probably teachers' scope of learning. It is possible to examine some issues in more detail. One of them is the number domain that is discussed in Chapter 6. As quite many solutions include branching in some ways the issues of branches are in the focus of Chapter 7.

Some trigonometric equations from the test suite were used in the experiment described in Chapter 8.

# 5. CONTENT-ORIENTED CLASSIFICATION OF DIFFERENCES

## 5.1. Introduction

This chapter focuses mainly on the research question RQ2: *How can the detected differences between CAS and school answers of equations in the school curriculum be described and classified?* A classification of the differences between school answers and CAS answers as an answer to RQ2 was described in Chapter 4. It mirrored mainly the possible set-theoretic relations between answers as solution sets. Another, more content-oriented, classification will be provided in this chapter. The types of differences between CAS answers and school answers are described and examples are chosen to illustrate each particular type. Chapter 5 is based on Tonisson, 2011. The CAS Maxima, OpenAxiom, Sage, WIRIS, WolframAlpha are used and the chapter reflects the situation in the summer 2010.

It was mentioned several times before that most answers offered by CAS to equations are customary for school, but there are some answers that would be somewhat unexpected (or incompatible with the teaching practice) in school. For our purposes, an answer is considered to be unexpected if it differs in some way from the answer that the student/teacher/textbook expects/waits for/presents. In reality, it depends on many circumstances. The expectations could vary depending on the curriculum, the teacher, the textbook, etc.

The unexpected CAS answers can be divided between mistakes and reasonable unexpected answers. Reasonable unexpected CAS answers are correct (or sometimes partially correct), but conform to a different standard. As we try to discuss didactically useful answers, the terms 'didactical answers' or 'instructive answers' can be appropriate as well. Some of the reasonable answers are equivalent to the expected answers but some are not. A further breakdown, as presented in Figure 11, is possible. The types are based on the form, completion, dependence on number domain, and branching of answers and automatic simplification of equation. The types were inspired by different papers (e.g., Aslaksen, 1999; Bradford et al., 2002; Kadijevich, 2009; Stoutemyer, 1991; Wester, 1999b) but this particular set as a whole has not been used before. In fact, the types *Form*, *Unfinished*, *Domain*, *Branches*, *Automatic* can be broken down even further. Hence we use the notation *Type(subtype)*, e.g., *Domain(R/C)*, *Unfinished(log)*. It should be noted that the classification presented in Figure 11 is about answers. However, we also consider input issues and present type *Input*.

The test suite includes linear, quadratic and fractional equations, equations that contain an absolute value of an expression, irrational, exponential, logarithmic, trigonometric and literal equations. More than 120 equations (partially the same as in Chapter 4) were used as test examples but only distinguishing ones are presented in the chapter. Most of them are school-like equations but some are rather artificial equations, designed to elicit specific behavior. As CAS are constantly

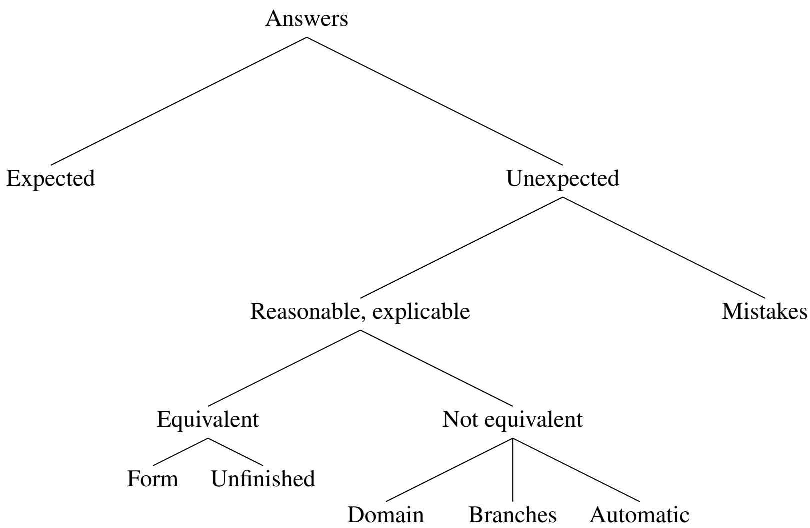


**Figure 11.** Types of answers

improved and new versions and new CAS appear, the situation with a particular version of a particular CAS is not overly important. Rather, what could be useful is an understanding of the possible range and nature of the phenomena, especially when the didactic value of an unexpected answer can be capitalized on in the presentation or accentuation of some topics. Such topics (e.g., equivalence, domain) can be concealed at first, but are actually important. The scope of this chapter is restricted to mapping and (preliminary) classification. The ascertainment of the actual importance requires further experiments, involving teachers and students as well.

The phenomena are identified in the Section 5.2 by types of equations. The types of phenomena are discussed separately in Section 5.3.

## 5.2. Equations

### 5.2.1. Introduction

The main types of school equations are observed in this chapter to identify the various kinds of differences between CAS and school answers. However, other equations (e.g., Diophantine equations) can be included in the curriculum in certain regions. The curricula of different countries and school types include different amounts of equations. The order of study and names of equation types can vary as well, e.g., radical equation — irrational equation, literal equation — equation with parameter, absolute-value equation — equation with absolute value — equation containing absolute value.

The following subsections all have a similar structure. The subsections include a brief introduction of every type (and subtype) of equations. This is followed by a discussion of the interesting phenomena. The test examples were selected with the intention to find simple non-trivial examples, characteristic of the particular (sub)type. Some were taken directly from a textbook; others were simplified from textbook examples. There are also examples from previous chapters.

### 5.2.2. Linear equations

The first equations we study are linear equations. Linear equations are the first equations learned in school and may seem quite trivial to both students and CAS. Still, some important phenomena appear here. It is possible to identify three subtypes of linear equations: equation that has one solution, equation that is true for all numbers, and equation that has no solutions.

The solution could be (at least at the school level where linear equations are introduced) an integer or a fractional number. There are no problems with the integer, but two points can be observed in case of a fractional answer. In case of $3x+2=0$, all CAS give $-2/3$ (or $-\frac{2}{3}$). In case of the equation $2x=3$, only Sage gives 1.5, the others $3/2$ (or $\frac{3}{2}$). Should the solution be a common fraction or a decimal fraction? Although students should be familiar with both forms, the particular form selected by the program could be slightly unexpected. We denote this issue as *Form(fraction)*. The second issue is more country-dependent. Namely, mixed numbers are used in some countries and, instead of $3/2$, the answer $1\frac{1}{2}$ is expected. CAS do not support mixed numbers. We denote the issue as *Form(mixed)*.

The cases where there are no solutions or the solution set consists of all numbers lead to the notation issue denoted as *Form(all/empty)*. CAS use quite different approaches in this case:

- $[x=x]$, $[x==r1]$, $0=0$, 'All values of x are solutions';
- [ ], 'No solutions exist', 'Warning, difficulty. It is not possible to find a result or solution'.

Some examples are listed in Table 11.

| **Equation** | **Answer, Remark** | **CAS** | **Phenomenon** |
|---|---|---|---|
| $2x=3$ | 1.5 (vs 3/2) | Sage | *Form(fraction)* |
| $2x=3$ | 3/2 or 1.5 (vs $1\frac{1}{2}$) | All | *Form(mixed)* |
| $2(x+1)=2x+2$ | $[x=r1]$ | Sage | *Form(all/empty)* |
| $2(x+1)=2x+1$ | [ ] | Maxima | *Form(all/empty)* |

**Table 11.** Linear equations

### 5.2.3. Quadratic equations

Quadratic equations can be classified by the (manual) solution process and by the number of real solutions. It should be noted that different countries seem to prefer slightly different solutions. For example, solving of quadratic equation by completing the square is popular in several countries, but almost not used at all in others.

Incomplete quadratic equations in the form $ax^2 + bx = 0$ (like $10x^2 + 17x = 0$) are solved by all CAS without new problems. (The issue of presentation of fractions is still present.) The equations in the form $ax^2 + c = 0$ (like $2x^2 - 8 = 0$) raise the issue of sign $\pm$ (*Form($\pm$)*), WolframAlpha presents $\pm 2$ as the answer to the equation $2x^2 - 8 = 0$, others present the solutions separately.

A quadratic equation can have 2, 1 (actually two equal) or 0 real solutions. All of them can provide unexpected answers. In case of 2 solutions, there can be differences between CAS in the order of terms, etc. For example, one of the answers to the equation $2x^2 - 4x - 5 = 0$ could be presented by different CAS as $-\frac{\sqrt{14}}{2} + 1$ (WIRIS, see Figure 12), $-\frac{\sqrt{14} - 2}{2}$ (Maxima, see Figure 13), $\frac{-\sqrt{14} + 2}{2}$ (Axiom, see Figure 14), $-\frac{1}{2}\sqrt{14} + 1$ (Sage, see Figure 15) or $1 - \sqrt{\frac{7}{2}}$ (WolframAlpha, see Figure 16). We denote the issue as *Form(radical)*. In case of two equal solutions, there is a question about presentation of multiple solutions (*Branches(mult)*); all CAS present them one at a time — the equation $x^2 - 2x + 1 = 0$ has the answer $x = 1$ (not $x_1 = 1, x_2 = 1$).

The most essential issue related to the quadratic equation is presentation of the answer when there are no real solutions (but there are complex solutions, of course). an empty set { }; Maxima, Sage and WolframAlpha give the complex solutions that include an imaginary unit; the answer in Axiom has a negative number under square root in solutions. We denote the issue as *Domain(C)*.

Some examples are listed in Table 12.

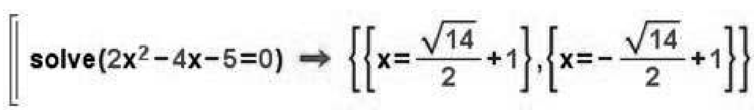

**Figure 12.** Solutions of quadratic equation (WIRIS)

### 5.2.4. Fractional equations

Fractional equations introduce extraneous solutions. One can perform all steps (excl. checking) of the solution algorithm correctly, but the solution can still be wrong. There are two possible strategies for managing this case (both for human

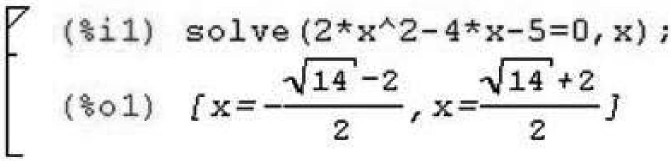


**Figure 13.** Solutions of quadratic equation (Maxima)

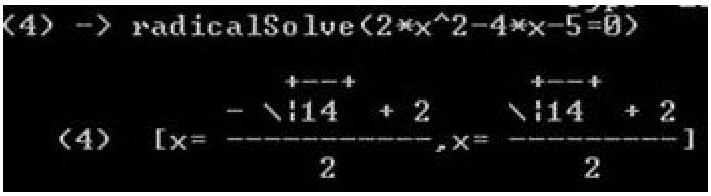


**Figure 14.** Solutions of quadratic equation (Axiom)

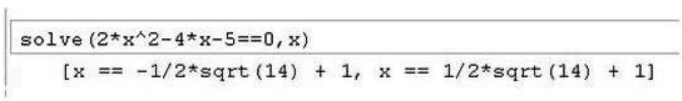


**Figure 15.** Solutions of quadratic equation (Sage)

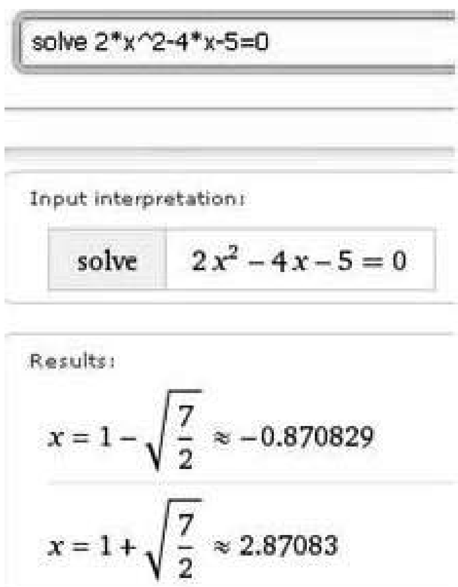


**Figure 16.** Solutions of quadratic equation (WolframAlpha)

| Equation | Answer, Remark | CAS | Phenomenon |
|---|---|---|---|
| $2x^2-8=0$ | $\pm$ in anwswer | WolframAlpha | *Form($\pm$)* |
| $2x^2-4x-5=0$ | Different forms | All | *Form(radical)* |
| $2x^2-2x+1=0$ | $x=1$ vs $x_1=1, x_2=1$ | All | *Branches(mult)* |
| $x^2+x+2=0$ | $i$ in answer | Maxima, Sage, WolframAlpha | *Domain(C)* |

**Table 12.** Quadratic equations

and CAS operations). The 'forbidden' values can be determined in advance before solving or the potential answer can be checked when solving is finished. CAS give correct answers in case of ordinary fractional equations, like $\frac{1}{x}=0$, $\frac{x-1}{x}=1$, $\frac{3x-1}{x}-2=0$, $\frac{x-2}{x-1}+1=\frac{x-3}{2x-2}$, $\frac{2}{x}-x=1$ or $\frac{x+1}{x-1}=\frac{2}{x^2-x}$.

One could lure CAS with the somewhat artificial equations $\frac{x\cdot x}{x}=0$ and $\frac{1}{x}=\frac{1}{x}$. All CAS give disputable answers: 0 to the first equation and 'all values' to the other equation. This is probably related to automatic simplification — the equation is automatically transformed to the (standard) form and indeterminacy is cancelled. We denote this issue as *Automatic(indet)*.

Some examples are listed in Table 13.

| Equation | Answer, Remark | CAS | Phenomenon |
|---|---|---|---|
| $\frac{x\cdot x}{x}=0$ | Division by 0 | All | *Automatic(indet)* |
| $\frac{1}{x}=\frac{1}{x}$ | Division by 0 | All | *Automatic(indet)* |

**Table 13.** Fractional equations

### 5.2.5. Equations that contain an absolute value

Equations that contain an absolute value could be presented to students after linear equations or much later or not at all. Solving such equations is complicated and some textbooks deal only with simple examples, solvable by definition.

The first question related to CAS is how to input an absolute value. WolframAlpha understands the mark | | (typed from the keyboard); WIRIS has a special button in the palette, but the most common approach is to use the *abs( )* function. We denote the issue as *Input(abs)*.

Unlike with the previous types, CAS have quite different efficiencies in case of equations that contain absolute values. It should be noted that absolute value is also a theoretically complicated topic in computer algebra. WIRIS and WolframAlpha cope well with all examples: $|x|=3$, $|x|=-3$, $|x+2|=1$, $|x-3|=|x+2|$, $|x-3|=|3-x|$, $|x+2|-|x|=x-3$, $|x^2-1|=-2x$, $|x^2-x|+3x=5$,

$||x^3 - \sqrt{x+1}| - 3| = x^3 + \sqrt{x+1} - 7$.

Axiom gives answers when $\sqrt{(f(x))^2}$ is used instead of $|f(x)|$ in the input, e.g., $\sqrt{(x+2)^2} = 1$. We denote this issue as *Input(sqrt(ˆ2))*. However, there will be the extraneous solutions in case of $|x^2 - 1| = -2x$, $|x^2 - x| + 3x = 5$; and in case of $||x^3 - \sqrt{x+1}| - 3| = x^3 + \sqrt{x+1} - 7$ the answer did not appear in reasonable time. Maxima and Sage provide the answer in response to numerical command *find_root*. We denote this issue as *Input(numerical)*.

Some examples are listed in Table 14.

| Equation | Answer, Remark | CAS | Phenomenon |
|---|---|---|---|
| All | *abs()* vs $\mid\,\mid$ | | *Input(abs)* |
| All | $\sqrt{(f(x))^2}$ vs $\mid\,\mid$ | Axiom | *Input(sqrt(ˆ2))* |
| All | *find_root* vs *solve* | Maxima, Sage | *Input(numerical)* |

**Table 14.** Equations that contain an absolute value

### 5.2.6. Irrational equations

There are several subtypes of irrational equations in school mathematics. Some of them are solvable simply by raising to power (perhaps more than once) and then solving a linear or a quadratic equation. Sometimes, the equation is algebraic equation with respect to the radical expression. The checking of solutions is essential part of solving.

In case of CAS, the first question is again about input. While square roots are easily to enter (with a special button or function *sqrt()*), the other roots (e.g., $\sqrt[3]{\ }$, $\sqrt[6]{\ }$) can be problematic. We denote the issue as *Input(radical)*.

All CAS provide correct solutions to simpler irrational equations with only one radical on the one side and a number on the other, like $\sqrt{x} = 0$, $\sqrt{x} = 2$, $\sqrt{x+1} = 2$ and $\sqrt{x^2 - 3x} = 2$. Additionally, all CAS can solve $\sqrt{7 - \sqrt{x-3}} = 2$ and $\frac{1}{\sqrt{x}} = \frac{1}{2}$.

In case of most irrational equations (like $\sqrt{2x} = \sqrt{x+1}$, $\sqrt{x} = x$, $x - 2 = \sqrt{x}$, $x - \sqrt{25 - x^2} = 1$, $\sqrt{x+6} - \sqrt{x-1} = 1$, $\sqrt{2x-1} - \sqrt{x-4} = 2$ and $\sqrt{x+2} - \sqrt{x-2} = \sqrt{3x+2}$), Axiom, WIRIS and WolframAlpha offer correct answers, while Maxima and Sage do not solve symbolically but solve numerically with the *find_root* command (*Input(numerical)*). Axiom, Sage, WIRIS and WolframAlpha solve $\sqrt{x^2} = 2$ correctly, Maxima gives $|x| = 2$ as a reaction (but solves numerically with *find_root*). In case of an empty set of solutions ($\sqrt{x} = -2$, $\sqrt{x+3} = -2$ and $\sqrt{3x+1} = \sqrt{x} - 1$), Axiom, WIRIS and WolframAlpha give correct reaction. Maxima, Sage, WIRIS and WolframAlpha are correct with $\sqrt{3x+4} + \sqrt{x} = -3$, but Axiom gives a faulty answer.

Axiom and WolframAlpha present a real solution that is considered as extraneous in school: $-1$ in case of $\sqrt{2x} = \sqrt{x-1}$ and $-\frac{27}{7}$ (in addition to 3) in case of $\sqrt{2x+6} + \sqrt{x-3} = 2\sqrt{x}$. We denote this issue as *Domain(R/C)*.

In case of $\sqrt[3]{x+45} - \sqrt[3]{x-16} = 1$, WIRIS gives both solutions (80 and $-109$), Axiom and WolframAlpha give only 80. When $-109$ is substituted to the left side of the equation, Axiom and WolframAlpha give $-\sqrt[3]{-1}$ as a result. We denote this issue as *Form(cbrt)*. It is related to definition of multivalued functions (see Jeffrey & Norman, 2004). Maxima and Sage do not solve it symbolically but solve numerically with *find_root*. WIRIS and WolframAlpha solve $\sqrt[3]{x+1} + 2\sqrt[6]{x+1} = 3$ correctly. Axiom gives, in addition to 0, the strange faulty solutions which include $\sqrt{-3\%x72^2 + (-2\%x71 - 760)\%x72 - 3\%x71^2 - 760\%x71 - 11232}$, for example. Again, Maxima and Sage do not solve it symbolically but solve numerically with *find_root*.

Some examples are listed in Table 15.

| **Equation** | **Answer, Remark** | **CAS** | **Phenomenon** |
|---|---|---|---|
| $\sqrt[3]{\ }$, etc | What function | All | *Input(radical)* |
| All | *find_root* vs *solve* | Maxima, Sage | *Input(numerical)* |
| $\sqrt{2x} = \sqrt{x-1}$ | $-1 \to \sqrt{-2}$ | Axiom, WolframAlpha | *Domain(R/C)* |
| $\sqrt[3]{x+45} - \sqrt[3]{x-16} = 1$ | Multivalued $-\sqrt[3]{-1}$ | Axiom, WolframAlpha | *Form(cbrt)* |

**Table 15.** Irrational equations

### 5.2.7. Exponential equations

Some useful ideas for solving exponential equations are introduced in school. One of them is to observe whether bases are equal (or transformable to equal). Another idea is to use logarithms. There are equations that are linear or quadratic equation with respect to the term $a^x$ and are also more complicated.

The first exponential equation observed is $2^x = 2^3$ (same as $2^x = 8$). WolframAlpha gives a correct answer 3. WIRIS gives 3.0 which is correct, but seems to be numerical. We denote this as *Form(1.)*. Axiom, Maxima and Sage give $\frac{\log(8)}{\log(2)}$ as a solution. It seems to be somewhat unfinished and we denote this as *Unfinished(log)*. The same situation occurs in case of $\sqrt{3^x} = 9$, where the left side of the equation is a little bit trickier, but the right side is still a number. When there is an exponential term on the right side as well, e.g., $2^{x+1} = 2^{2x}$, Axiom and WolframAlpha give the exact answer, WIRIS gives 1.0 but Maxima and Sage do not solve symbolically but give the solution numerically with the help of *find_root* (*Input(numerical)*).

If the same base is somewhat hidden ($5^{1-x}+\left(\frac{1}{5}\right)^{x-2}+25^{-\frac{x}{2}}=155$ or $3^{2x-1}=27^x$), Axiom does not solve, but if the first step is made manually ($3^{2x-1}=3^{3x}$ and $\frac{5}{5^x}+\frac{25}{5^x}+\frac{1}{5^x}=155$), the CAS can solve it. In case of equations that are a linear or quadratic equation with respect to the term $a^x$ ($3^{x+2}-3^{x+1}+3^x=21$, $2^{2x}-8\cdot 2^x+16=0$ and $3^{2x-1}-3^{x-1}-2=0$), WolframAlpha gives a correct answer, WIRIS gives a *Form(1.)* answer, Axiom gives an *Unfinished(log)* answer, and Maxima and Sage give a numerical answer with the command *find_root*.

WolframAlpha and WIRIS give expected answers to equation $\frac{3^x+3^{-x}}{3^x-3^{-x}}=2$, 1/2 and 0.5, respectively. Maxima and Sage give 1/2 and $\frac{\log(-\sqrt{3})}{\log(3)}$ (*Form(log(−1))*), Axiom gives $\frac{\log(\sqrt{3})}{\log(3)}$ and $\frac{\log(-\sqrt{3})}{\log(3)}$ (*Unfinished(log)*, *Unfinished(log(−1))*). It should be noted that WolframAlpha also offers the complex answer (see Figure 17, denoted as *Branches(Cn)*, *Domain(C)*). In case of 'unlikeable' answers (like $3^x=5$, $3^{2x+1}=15$ and $e^x=30$), WIRIS gives *Form(1.)* answer (like 0.73249) and the four other CAS give an *Unfinished(log)* answer; WolframAlpha also gives a numerical value ($\frac{\log(5)}{\log(9)}\approx 0.732487$ ).

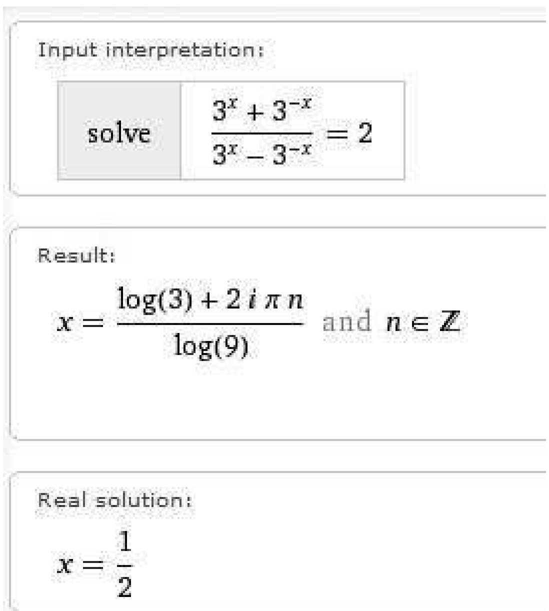


**Figure 17.** Periodic solution to exponential equation (WolframAlpha)

When the number is in decimal form in the equation $10^{-x} = 0.0347$, manual transformation to $10^{-x} = \frac{347}{10000}$ is needed in Axiom (*Input(fraction)*).

WIRIS gives a warning message in case of equations $2^x = -4$, $e^x + 1 = 0$ that have no real solutions. WolframAlpha gives all sets of complex solutions (like $\frac{\log(4) + i\pi(2n+1)}{\log(2)}$ or $i(2\pi n + \pi)$ ) (denoted as *Branches(Cn)*, *Domain(C)*); Sage gives one complex solution (like $i\pi + \frac{\log(4)}{\log(2)}$) (denoted as *Branches(C1)*, *Domain(C)*). Axiom and Maxima give $\frac{\log(-4)}{\log(2)}$ (*Unfinished(log(*$-1$*))*).

In case of unequal bases (like $2^{2-x} = 3^{2-x}$ and $4^{x+1} - 3^x = 3^{x+2} - 4^x$), WolframAlpha gives the expected answers; Maxima and Sage give numerical answers with the help of *find_root*. WIRIS gives a warning message, and Axiom gives empty brackets [ ]. In case of $2 \cdot 4^x - 5 \cdot 6^x + 3 \cdot 9^x = 0$, WolframAlpha gives only a set of complex answers (*Branches(Cn)*, *Domain(C)*) where case $n = 0$ gives the needed real answers.

Some examples are listed in Table 16.

| **Equation** | **Answer, Remark** | **CAS** | **Phenomenon** |
|---|---|---|---|
| $2^x = 2^3$ | 3.0 | WIRIS | *Form(1.)* |
| $2^x = 2^3$ | $\frac{\log(8)}{\log(2)}$ | Axiom, Maxima, Sage | *Unfinished(log)* |
| $2^{x+1} = 2^{2x}$ | *find_root* vs *solve* | Maxima, Sage | *Input(numerical)* |
| $\frac{3^x + 3^{-x}}{3^x - 3^{-x}} = 2$ | $\frac{\log(-\sqrt{3})}{\log(3)}$ | Axiom, Maxima, Sage | *Form(log(*$-1$*))* |
| $10^{-x} = 0.0347$ | manually $0.0347 \to 347/10000$ | Axiom | *Input(fraction)* |
| $2^x = -4$, $e^x + 1 = 0$ | All sets of complex solutions | WolframAlpha | *Branches(Cn)*, *Domain(C)* |
| $2^x = -4$, $e^x + 1 = 0$ | One complex solution | Sage | *Branches(C1)*, *Domain(C)* |

**Table 16.** Exponential equations

### 5.2.8. Logarithmic equations

Before discussing logarithmic equations, we should consider the function log(). It means $\log_{10}$ in WIRIS, but $\log_e$ in the other CAS used. There are log(b, a) function for the $\log_a b$ in WIRIS, but log(a, b) (or log_a(b)) in WolframAlpha. In other CAS, the necessary function should be defined (e.g., `logv(b,a):=log(b)/log(a)`). We denote the issue as *Input(log)*.

The main idea in solving logarithmic equations is to transform the equation to the form $\log_a f(x) = c$ or $\log_a f(x) = \log_a g(x)$. We start from equations that already have the required form. All CAS solve the equations $\log_2 x = 4$, $\log_2 x = \log_2 14$; WIRIS gives a decimal answer (e.g., 16.).

When the number is in decimal form in the equation $\log_2 x = 4.5$, manual transformation to $\log_2 x = 9/2$ is needed in Axiom (*Input(fraction)*).

Then, Axiom gives $e^{\frac{9\log(2)}{2}}$ as a solution. Also, Maxima and Sage give $e^{\frac{9\log(2)}{2}}$ as a solution in case of $\log_2 x = 4.5$. It seems to be unfinished and we denote this as *Unfinished(log/e)*. Similarly, Axiom, Maxima and Sage give $e^{\frac{\log(4)}{2}} - 1$ in case of $\log_{x+1} 4 = 2$ $e^{\frac{\log(4)}{2}} - 1$. Axiom gives $e^{2\log(2)}$ in case of $\log_2(\log_2 x) = 1$ and $e^{2\log(10)}$ in case of $\log_{10}^2(x) - 3\log_{10}(x) + 2 = 0$.

In case of more complicated equations, like

$$\log_{2x}(x^2+3) = 2,$$

$$\log_{10}(x+1) + \log_{10}(x-1) - \log_{10}(2x+5) = \log_{10} 3,$$
$$\log_{10}^2(100x) - 3\log_{10}^2(10x) - \log_{10}(x) = 14,$$

WolframAlpha gives the expected answers. Sage gives numerical answers with the help of *find_root* (*Input(numerical)*). Maxima does not solve (with the help of *find_root* either) and Axiom gives empty brackets [ ]. WIRIS gives the expected answer in case of the first equation and *Form(1.)* answers (like 10.0) in case of others.

The exponential logarithmic equation $x^{\log_{10} x} = 100x$ is solved only by WolframAlpha.

WIRIS and Axiom present a real solution that is considered as extraneous in school: $-1$ in case of $\ln(2x) = \ln(x-1)$. We denote this issue as *Domain(R/C)*.

The equation $\log_b(x^2) = \log_b(2x-1)$ includes a parameter. WolframAlpha gives the answer $x = 1$ and $\log(b) \neq 0$, while Axiom only gives 1. This issue will be discussed in the section on literal equations. Maxima, Sage and WIRIS do not solve it.

Some examples are listed in Table 17.

### 5.2.9. Trigonometric equations

Trigonometric equations can be included in school curricula and textbooks in very different ways. Sometimes only the sine, cosine and tangent are covered. The cotangent could also be included, but secant and cosecant are virtually unknown. The complexity of equations is very variable. The variety also extends to the requirements for solution — sometimes a general solution, sometimes one solution or solutions in specified interval. A question of unit — radian or degree — is also possible.

Trigonometric equations can be classified in different ways. In this chapter, we distinguish the following subtypes:

- Basic equations

| Equation | Answer, Remark | CAS | Phenomenon |
|---|---|---|---|
| $\log_a b$ | Should be defined | Axiom, Sage, Maxima | *Input(log)* |
| $\log_2 x = 4.5$ | manually $4.5 \to 9/2$ | Axiom | *Input(fraction)* |
| $\log_2 x = 4.5$ | $e^{\frac{9\log(2)}{2}}$ | Maxima, Sage | *Unfinished(log/e)* |
| $\log_2(\log_2 x) = 1$ | $e^{2\log(2)}$ | Axiom | *Unfinished(log/e)* |
| $\log_{x+1} 4 = 2$ | $e^{\frac{\log(4)}{2}} - 1$ | Axiom, Sage, Maxima | *Unfinished(log/e)* |
| $\log_{2x}(x^2+3) = 2$ | *find_root* vs *solve* | Sage | *Input(numerical)* |
| $\log_{10}^2 x - 3\log_{10} x + 2 = 0$ | $e^{2\log(10)}$ | Axiom | *Unfinished(log/e)* |
| $\ln(2x) = \ln(x-1)$ | $-1 \to \sqrt{-2}$ | Axiom, WolframAlpha | *Domain(R/C)* |

**Table 17.** Logarithmic equations

- 'likeable' answer, like $\sin x = 0$, $\cos x = \frac{\sqrt{3}}{2}$ or $\cot x = -1$
- 'less likeable' answer, like $\sin x = \frac{1}{10}$, $\sin x = 0.1$
- unsolvable (in $\mathbb{R}$), like $\sin x = 2$, $\cos x = 2$

- Advanced equations ('one-function')
  - more complicated argument, like $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$
  - factorization, like $\sin x(1 - \sin x) = 0$
  - quadratic equations, like $\sin^2 x - 2\sin x - 3 = 0$
  - biquadratic equations, like $2\tan^4 3x - 3\tan^2 3x + 1 = 0$
- More advanced ('function-change')
  - change function, like $\tan x + 3\cot x = 4$
  - homogeneous, like $2\sin x - 3\cos x = 0$
  - more complicated, like $2 + \cos^2(2x) = (2 - \sin^2 x)^2$

The following phenomena can be notice in case of basic trigonometric equations. The number of solutions can be one (Axiom, Sage) or two (WIRIS), or a general solution can be given (WolframAlpha). Maxima gives one solution but also a warning: 'Some solutions may be lost'. We denote the issue as *Branches(number of solutions)*. The single solution can be different in different CAS, for example, in case of $\cot x = -1$, Axiom gives $\frac{3\pi}{4}$ but Maxima and Sage give $\frac{-\pi}{4}$ (*Branches(choice of solution)*). WIRIS does not solve this equation.

Decimal answers are expected in case of 'less likeable' answers, but they can sometimes also appear in case of an expected 'likeable' answer (e.g., $\sin x = \frac{1}{2}$ in WIRIS) (*Form(fraction)*). It is possible that an inverse function (*arcsin*, *arccos*, *arctan*, *arccot*) is included in the solution (*Form(invtrig)*). The inverse functions are presented differently in different CAS (see Figures 18, 19 and 20). In case of Axiom, the inverse function appears already in the 'likeable' answer (e.g., Axiom, $\sin x = \frac{1}{2}$ ). In case of Maxima, Sage and WolframAlpha, the inverse function appears in case of 'less likeable' answers (e.g., $\sin x = \frac{1}{10}$). In case of 'unsolvable' equations (e.g., $\cos x = 2$), WIRIS does not give solutions (that is reasonable). Other CAS give inverse functions in the solution.

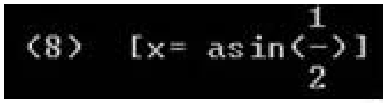


**Figure 18.** Inverse trigonometric functions in answers (Axiom)


**Figure 19.** Inverse trigonometric functions in answers (Sage)

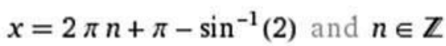


**Figure 20.** Inverse trigonometric functions in answers (WolframAlpha)

The advanced ('one-function') equations add more complicated answers and sometimes the checking of equivalence can be quite intriguing. Are $x = n\pi$ and $x = (-1)^n \frac{\pi}{2} + n\pi$ as found in (Estonian) textbooks equivalent to $x = 2\pi n$, $x = 2\pi n + \pi$ and $x = \frac{1}{2}(4\pi n + \pi)$ as given by WolframAlpha (*Form(periodic)*)? Biquadratic trigonometric equation $2\tan^4 3x - 3\tan^2 3x + 1 = 0$ is too complicated for Maxima and Sage, but $2t^4 - 3t^2 + 1 = 0$ is solvable. Therefore, we denote the issue as *Input(substitute)*.

From the more advanced ('function-change') equations, we mention here the equation $2 + \cos^2(2x) = (2 - \sin^2 x)^2$ , introduced by Abramovich (2005) as the equation that has at least three different reasonable ways of solving and each way produces a different-looking answer. Similarly, CAS give different answers (Maxima and Sage do not solve the equation).

Some examples are listed in Table 18.

| Equation | Answer, Remark | CAS | Phenomenon |
|---|---|---|---|
| All | Particular vs general solutions | All | *Branches(number of solutions)* |
| $\cot x = -1$ | $\frac{3\pi}{4}$ vs $\frac{-\pi}{4}$ | Axiom, Sage, Maxima | *Branches(choice of solution)* |
| $\sin x = \frac{1}{2}$ | Decimal answers | WIRIS | *Form(fraction)* |
| $\sin x = \frac{1}{2}$ | Inverse function | Axiom | *Form(invtrig)* |
| $\sin x = \frac{1}{10}$ | Inverse function | Maxima, Sage, WolframAlpha | *Form(invtrig)* |
| $\sin x(1-\sin x) = 0$ | Different forms of solutions | WolframAlpha | *Form(periodic)* |
| $2\tan^4 3x - 3\tan^2 3x + 1 = 0$ | Substitute $t = \tan 3x$ | Maxima, Sage | *Input(substitute)* |

**Table 18.** Trigonometric equations

### 5.2.10. Literal equations

In principle, parameters can be included in every type of equations; one such example was presented in the section on logarithmic equations. Here we focus on linear and quadratic equations. Drijvers (2002) discussed parameter very thoroughly. Drijvers emphasizes that literal equation is basically a different type of equation, because the answer is algebraic expression with letters. The issue is related to obstacle 4 in Drivers' list: *The tendency to accept only numerical solutions and not algebraic solutions.* It is explained as: *Students often are not satisfied with answers such as* $x = \frac{1}{2}s - \frac{1}{2}v$*. In the end they want to know what value x stands for. This is called the 'expected answer obstacle'.*

The test set included, e.g., $ax = 1$, $ax = b$, $ax + b = 5$, $x^2 = a$, $V = \pi r^2 h$. One can notice the different forms of the answer, e.g., in case of $ax^2 + bx + c = 0$,

- WIRIS gives $-\frac{\sqrt{-4ac+b^2}}{2a} + \frac{-b}{2a}$ and $\frac{\sqrt{-4ac+b^2}}{2a} + \frac{-b}{2a}$,
- Maxima $-\frac{\sqrt{b^2-4ac}+b}{2a}$ and $\frac{\sqrt{b^2-4ac}-b}{2a}$,
- Sage $-\frac{1}{2} \cdot \frac{b+\sqrt{-4ac+b^2}}{a}$ and $-\frac{1}{2} \cdot \frac{b-\sqrt{-4ac+b^2}}{a}$.

The answer of WolframAlpha is shown in Figure 21. The issue of different forms of radicals is denoted as *Form(radical)*. Axiom does not solve this equation.

When we look at the WolframAlpha answer, we see different branches. It is very common that a complete answer to a literal equation has branches but CAS present branches differently (*Branches(literal)*).

Sometimes a branch can be the source of a new problem to be solved by students. For example, WolframAlpha gives to the equa-

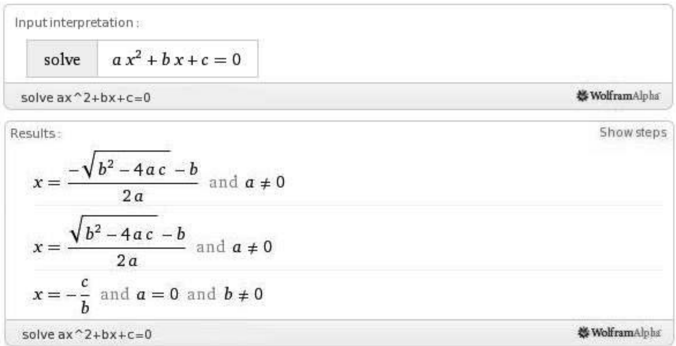


**Figure 21.** Branches in the answer of a literal equation (WolframAlpha)

tion $\dfrac{3mx-5}{(m+2)(x^2-9)} = \dfrac{2m+1}{(m+2)(x-3)} - \dfrac{5}{x+3}$ the answer $x = \dfrac{21m+38}{6m+9}$ and $3m+3 \neq 0$ and $9m^3 + 66m^2 + 151m + 110 \neq 0$.

| **Equation** | **Answer, Remark** | **CAS** | **Phenomenon** |
|---|---|---|---|
| $ax^2+bx+c=0$ | Different forms of radicals | All but Axiom | *Form(radical)* |
| $ax^2+bx+c=0$ | Different branches | All but Axiom | *Branches(literal)* |

**Table 19.** Literal equations

Some examples are listed in Table 19.

All basic types of school equation were observed to highlight the various types of differences between CAS answers and school answers. The next section discusses the detected phenomena individually by type.

## 5.3. Identified phenomena

### 5.3.1. Introduction

Section 5.3 discusses the phenomena highlighted in Section 5.2. The individual phenomena are examined separately in Sections 5.3.2–5.3.7.

In addition to a general description of each type, the following sections present a few brief ideas on the potential didactical use of the types. In the explanation of didactical ideas, the answers from different sources are labeled as follows: $answer_{textbook}$, $answer_{CAS}$, $answer_{CAS(command)}$, $answer_{CAS(specification)}$, $answer_{student}$, etc.

The connections between type and Drijvers' (2002) list of obstacles are presented as well. Drijvers (2002) presented the obstacles that students can encounter while working in a computer algebra environment. The paper was based on his experiments. As this chapter focuses on a greater number of school equations and a greater number of CAS, the obstacles listed by Drijvers are partially also relevant to the phenomena discussed here. The following obstacles were listed in his paper:

1. The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'.
2. The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.
3. The flexible conception of variables and parameters that using a CAS requires.
4. The tendency to accept only numerical solutions and no algebraic solutions.
5. The limitations of the CAS, and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.
6. The inability to decide when and how computer algebra can be useful.
7. The black box character of the CAS.
8. The limited conception of algebraic substitution.
9. The limited conception of algebraic solution.
10. The conception of an expression as a process.
11. The difficult transfer between CAS technique and paper-and-pencil.
12. The difficulty in interpreting the CAS output.

### 5.3.2. Phenomena related to input

We assumed in Chapter 4 that input does not pose particular problems, or at least we did not discuss them in greater depth. However, we start with the issues that occur before an answer is produced. In these cases, it is necessary to enhance the input line; the command *solve(equation)* is not powerful enough.

7 phenomena were identified: *Input(abs)*, *Input(sqrt(ˆ2))*, *Input(radical)*, *Input(log)*, *Input(fraction)*, *Input(numerical)*, *Input(substitute)*. The relevant equation types are listed in Table 20.

| Phenomenon | Equations |
|---|---|
| *Input(abs)* | Absolute (Table 14) |
| *Input(sqrt(ˆ2))* | Absolute (Table 14) |
| *Input(radical)* | Irrational (Table 15) |
| *Input(log)* | Logarithmic (Table 17) |
| *Input(fraction)* | Exponential, logarithmic (Tables 16, 17) |
| *Input(numerical)* | Absolute, irrational, logarithmic (Tables 14, 15, 17) |
| *Input(substitute)* | Trigonometric (Table 18) |

**Table 20.** Input

There are several cases where simply using the solve-command is not sufficient and some adjustments are needed. They could be related to entering a specific function or operation, for example, absolute value, root, logarithm (*Input(abs)*, *Input(radical)*, *Input(log)*). At first, the different notations and ways of expression may seem obstructive, but they can also be instructive.

Sometimes formulae (like $|a| = \sqrt{a^2}$ (*Input(sqrt(^2))*) or $\log_a b = \dfrac{\log b}{\log a}$ (*Input(log)*)) are required. The use of a formula helps to understand the properties of the operation or function.

The issue is related to obstacle 5 on Drijvers' list: *The limitations of the CAS, and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.* It is explained as: *Sometimes, ..., there is no direct command to perform a task, or the CAS is unable to carry it out without any help from the user. In such cases, cooperation between users' expertise and CAS capacities is needed to find a result.*

The next issue concerns the form of the enterable number (fraction). The use of 4.5 or $9/2$ in equations can give different responses in solving (*Input(fraction)*). For example, when the number is in decimal form in the equation $\log_2 x = 4.5$, manual transformation to $\log_2 x = 9/2$ is needed in Axiom (*Input(fraction)*).

The question of decimal approximation is also important if a CAS does not solve the equation symbolically and numerical solving methods (for example, with the command *find_root*) are invoked (*Input(numerical)*). Although decimal approximation and numerical methods are very interesting and important topics, they are not discussed in this dissertation. Here, it is classified under the *Input* phenomena as the choice of a command can be treated as a part of input. This leads to questions about initial values, precision, etc. The importance of numerical calculations and decimal answers can be quite different in different countries. The issue is closely related to obstacle 2 on Drijvers' list: *The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.* It is explained as: *For many students $\sqrt{2}$ is not a real answer: they consider 1.41 as the ultimate result. They do not really understand the difference in status of the two answers: 'still has some algebra in it', whereas 1.41 is purely numerical. The CAS is not always clear about this difference in status.*

The expected form of the answer seems to depend on the country. In several countries, $\sqrt{2}$ is preferred as the ultimate answer. It could also be classified as *Form(numerical)* instead of *Input(numerical)*.

Sometimes, partial manual solving of the equations helps to produce the answer. For example, instead of $2\tan^4 3x - 3\tan^2 3x + 1 = 0$ (which could be unsolvable for a particular CAS), we will solve $2t^4 - 3t^2 + 1 = 0$. After that, the corresponding trigonometric equation will be solved (*Input(substitute)*). The issue is related to obstacle 8 on Drijvers' list: *The limited conception of algebraic substitution.* It is explained as: *Students often think that substitution is limited to 'filling in numerical values'. That conception has to be extended to algebraic substitution of expressions.*

### 5.3.3. Phenomena related to form of answer

In Chapter 4, we regarded many slightly different answers as equivalent (Type (2) *Equivalent but different*). In this subsection we are able to increase the degree of 'sensitivity' and focus on the differences. 10 phenomena were identified: *Form(fraction)*, *Form(mixed)*, *Form(all/empty)*, *Form(±)*, *Form(radical)*, *Form(cbrt)*, *Form(1.)*, *Form(log(−1))*, *Form(invtrig)*, *Form(periodic)*. The relevant equation types are listed in Table 21.

| **Phenomenon** | **Equations** |
|---|---|
| *Form(fraction)* | Linear (Table 11) |
| *Form(mixed)* | Linear (Table 11) |
| *Form(all/empty)* | Linear (Table 11) |
| *Form(±)* | Quadratic (Table 12) |
| *Form(radical)* | Quadratic, literal (Tables 12, 19) |
| *Form(cbrt)* | Irrational (Table 15) |
| *Form(1.)* | Exponential (Table 16) |
| *Form(log(−1))* | Exponential (Table 16) |
| *Form(invtrig)* | Trigonometric (Table 18) |
| *Form(periodic)* | Trigonometric (Table 18) |

**Table 21.** Form

This subsection groups together the phenomena where the answer is equivalent to the expected one but different in some way. However, the answers in this subsection seem to be in the ultimate form; the seemingly unfinished answers are treated in the next subsection. (The boundary between these two classes is quite ambiguous.)

We start with the fractional answers. It depends on the approach adopted in a country which form, $\frac{3}{2}$, $1\frac{1}{2}$ or 1.5 is preferred (*Form(mixed)*, *Form(fraction)*). The situation is slightly different if a decimal answer appears in case of an integer, 1. instead of 1 (*Form(1.)*). The answer could also be expressed by more complicated expressions, like $-\frac{\sqrt{14}}{2}+1$, $-\frac{\sqrt{-4ac+b^2}}{2a}+\frac{-b}{2a}$ (*Form(radical)*) or $x=2\pi n$, $x=2\pi n+\pi$ and $x=\frac{1}{2}(4\pi n+\pi)$ (*Form(periodic)*), and the answer could differ from the expected one. The handling of $-\sqrt[3]{-1}$ (*Form(cbrt)*) can also be placed in this group of phenomena.

The solution can be in a different but still correct form. The checking of equivalence can be easy or not so easy but instructive. The issue is related to obstacle 1 on Drijvers' list: *The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'.* Recognizing equivalent expressions is a central issue in algebra and this is still the case when working in a computer algebra environment.

Sometimes the response of a CAS can contain symbols that are unfamiliar to a student. If a symbol (like $\pm$) is included in the curriculum (but is presented at a later stage) the introduction via CAS can be appropriate (*Form(±)*). (Also, one can understand $\pm$ as interval.) If the symbol is not in the curriculum of the school, the introduction could be problematic but still instructive. There are also some CAS-dependent notation questions (like $[x = x]$ or $[\,]$ in case of equations where the solution set includes all numbers or is empty) (*Form(all/empty)*). Understanding the different error or warning messages is also a part of understanding the notation.

A somewhat transitional issue between this and the next subsection is the appearance of inverse trigonometric functions in the solution (*Form(invtrig)*).

Now we look at solutions of quadratic equation, which are seemingly different but actually equivalent.

The didactical example includes quadratic equations that have square roots in the solutions. Different CAS can use different rules for presentation of an expression. For example, one of the answers to the equation $2x^2 - 4x - 5 = 0$ can be presented by different CAS as $-\frac{\sqrt{14}}{2} + 1$ (Wiris), $-\frac{\sqrt{14}-2}{2}$ (Maxima), $\frac{-\sqrt{14}+2}{2}$ (Axiom), $-\frac{1}{2}\sqrt{14}+1$ (Sage) or $1 - \sqrt{\frac{7}{2}}$ (WolframAlpha).

When do answers with a square root appear? If coefficients $a$, $b$, and $c$ are rational and $b^2 - 4ac$ is not a perfect square, then the roots are irrational and not equal. Furthermore, the roots are conjugate surds (it is possible to construct the equation with irrational coefficients so that the discriminant is not a perfect square but the roots are rational). The issues of perfect square, conjugate surds, etc., could also be interesting and instructive, but probably do not pertain to normal school mathematics.

Another (probably simpler) way is to solve the equation with CAS and see whether square root appears or not. It is not very difficult to generate an equation to represent this phenomenon: using arbitrary integer coefficients, we usually get an equation with solutions that include square root.

There are different possibilities to use these equations and answers in learning and teaching, as the issue of equivalence of such expressions is relevant for school. Some possible tasks include:

- Check $answer_{textbook} = answer_{CAS}$ or $answer_{student} = answer_{CAS}$ or $answer_{CAS1} = answer_{CAS2}$
  - With a CAS. It is possible to check equivalence of the answers by simplification of difference of the two expressions;
  - Manually.
- Transform $answer_{CAS} \longrightarrow answer_{textbook}$.
- When does the solution include square roots?

### 5.3.4. Phenomena related to unfinished answers

3 phenomena were identified: *Unfinished(log)*, *Unfinished(log(−1))*, *Unfinished(log/e)*. The relevant equation types are listed in Table 22.

| **Phenomenon** | **Equations** |
|---|---|
| *Unfinished(log)* | Exponential (Table 16) |
| *Unfinished(log(−1))* | Exponential (Table 16) |
| *Unfinished(log/e)* | Logarithmic (Table 17) |

**Table 22.** Unfinished

This subsection discusses the solutions that seem to be unfinished, so that almost every user would like to simplify them further. The subsection is somewhat parallel to the subsection on *Input* which referred to (manual) steps before solving. This type only contains logarithm-related examples. There are cases (like $\frac{\log(8)}{\log(2)}$ (*Unfinished(log)*) and $e^{2\log(2)}$ (*Unfinished(log/e)*)) where only the final step seems to be needed. The negative argument of logarithm ($\frac{\log(-\sqrt{3})}{\log(3)}$ (*Unfinished(log(-1))*)) leads to the question of domains.

These CAS answers could also be used in teaching and learning, as students could be tasked with performing the final simplification.

- Simplify $answer_{CAS}$
  - With a CAS;
  - Manually.
- Transform $answer_{CAS} \longrightarrow answer_{textbook}$.

### 5.3.5. Phenomena related to domain issues

2 phenomena were identified: *Domain(C)*, *Domain(R/C)*. The relevant equation types are listed in Table 23.

| **Phenomenon** | **Equations** |
|---|---|
| *Domain(C)* | Quadratic (Table 12) |
| *Domain(R/C)* | Irrational, logarithmic (Tables 15, 17) |

**Table 23.** Domain

Complex numbers are included in school curricula in some countries but not in others. If complex numbers are not included, complex solutions would be unexpected. The solution can explicitly include an imaginary unit or a negative number under square root or as an argument of logarithm (*Domain(C)*). Nevertheless, the introduction of a complex solution could be instructive, at least by indicating that there are more numbers than we use in school.

It is somewhat disputable what would be the correct answer for equations like $\sqrt{2x} = \sqrt{x-1}$ or $\ln(2x) = \ln(x-1)$. On the one hand, $-1$ is, of course, a real solution. On the other hand, it is not appropriate when operating with real numbers only, since a negative number appears under the square root signs and as an argument for ln (*Domain(R/C)*). The topic is more thoroughly discussed in Chapter 6.

The example takes us back to the quadratic equation. The quadratic equation does not have real solutions if the discriminant is negative, but there are complex solutions.

The bounds could be described through a 'semi-solution' (by finding the discriminant) but it could be easier just to solve the equation with a CAS and check for $i$.

It is important that not all CAS produce $i$. Wiris gives real solutions by default (it is possible to ask for complex solutions). WolframAlpha gives real solutions if the word 'real' is added. Axiom gives a negative number under the square root in response to the command `radicalSolve` and an approximate answer (precision 0.1) with $i$ in response to the command `complexSolve`.

$Equation$: $x^2 + 2 = 0$
$Answer_{Axiom(radicalSolve)}$: $\sqrt{-2}$, $-\sqrt{-2}$
$Answer_{Axiom(complexSolve)}$: $-1.40625i$, $1.40625i$
$Answer_{Maxima}$: $\sqrt{2}i$, $-\sqrt{2}i$
$Answer_{WolframAlpha(SolveEquation)}$: $\pm(i\sqrt{2})$

Is it reasonable to introduce the imaginary unit at all if it is not mentioned in the school mathematics curriculum? It could be instructive for many students, while it could be excessive noise for others. However, the statement 'no real solutions' could lead to the question 'what about other solutions'. An imaginary unit in a CAS answer can provide a good basis for introduction and further discussion. A possible task is:

- Solve the equation and try to explain the answer. Can you guess the meaning of $i$?
  - Use the Internet.

The imaginary solutions will also appear in case of exponential and trigonometric equations (see Chapter 6).

The next issue is about checking the solution of an irrational equation. Are $\sqrt{-1}$ and $\sqrt{-1}$ equal?

On the one hand, the question about the equality of two expressions that are the same may seem strange. On the other hand, the expression seems to be 'illegal': a negative number cannot be under square root (in 'real' life). The question is based on equations where a solution (or solutions) is a real number but changes the expression under square root to negative. The topic is more thoroughly discussed in Chapter 5 .

$Equation$: $\sqrt{2x+1} = \sqrt{x}$
$Answer_{Axiom}$, $Answer_{Maxima(to\_poly\_solver)}$, $Answer_{Sage(to\_poly\_solve)}$,
$Answer_{WolframAlpha(SolveEquation)}$: $-1$

(Wiris and WolframAlpha give no solutions in the real domain.)

Number $-1$ is, of course, a real number, but substitution to the equation gives $\sqrt{-1}$.

What happens when we substitute the solution for the equation in CAS?

$Equation$: $\sqrt{2\cdot(-1)+1} = \sqrt{(-1)}$
$Answer_{Axiom}$: $\sqrt{-1} = \sqrt{-1}$
$Answer_{Maxima}$, $Answer_{Sage}$: $i = i$
$Answer_{WolframAlpha}$: $True$

To check whether an equation causes such 'illegal' real solution, one can solve the equation. It seems, however, that instructions for constructing an example are more useful. An equation that has a square root on both sides could be based on linear or quadratic equation, for example. It is easy to compose an equation with the solution $-1$, for instance. Then, we can distribute the terms so that one side contains the expression, which is negative in this particular solution $(-1)$. Finally, square roots are added.

$(x-3)(x+1) = x^2 - 2x - 3$
$x^2 - 3 = 2x$
$\sqrt{x^2-3} = \sqrt{2x}$

Such 'illegal' real solutions create a basis for discussion in a mathematics class. Possible initial tasks include:

- Solve the equation and check the answer
    - with a CAS;
    - manually.
- Compare the answers of the different CAS.

The phenomenon is also possible in case of logarithmic equations (and rarely used inverse trigonometric equations).

The issues of domain are more thoroughly discussed in Chapter 5.

### 5.3.6. Phenomena related to branches of solutions

6 phenomena were identified: *Branches(mult)*, *Branches(Cn)*, *Branches(C1)*, *Branches(number of solutions)*, *Branches(choice of solution)*, *Branches(literal)*. The relevant equation types are listed in Table 24.

| Phenomenon | Equations |
|---|---|
| *Branches(mult)* | Quadratic (Table 12) |
| *Branches(Cn)* | Exponential (Table 16) |
| *Branches(C1)* | Exponential (Table 16) |
| *Branches(number of solutions)* | Trigonometric (Table 18) |
| *Branches(choice of solution)* | Trigonometric (Table 18) |
| *Branches(literal)* | Literal (Table 19) |

**Table 24.** Branches

In many cases when solving an equation, the solution is separable into branches in some manner.

Different textbooks can treat some issues a little differently. For example, in case of two equal solutions, some textbooks say that there are two equal roots, some say one real root (a double root), and some say just one real solution. All CAS present them only one at a time (at least by default).

There are different sources for branching in case of trigonometric equations — periodicity and the families of solutions. Textbooks may or may not provide general solutions, and the same applies to CAS.

The choice of between different solutions (like $\frac{3\pi}{4}$ or $\frac{-\pi}{4}$) from the solution set is denoted as *Branches(choice of solution)*.

The topic of literal equation is a classic branching topic (*Branches(literal)*). Bernardin (1999) criticized the behavior of CAS. In case of $ax = b$ Bernardin says: *When asked to solve with respect to x, all the systems returned the solution* $x = \frac{b}{a}$ *even when this answer is obviously not correct for* $a = 0$. He notes that there may be different commands (e.g., *Reduce* in Mathematica) that work better.

In addition, the answers in case of complex numbers can include branch issues (*Branches(Cn)*, *Branches(C1)*). However, it is not an issue of the school-level and we will skip it here.

In case of trigonometric equation, the solution can include more numbers in a specific systematic way. For example, the general solution to a trigonometric equation consists of one or several 'series'. There are many possible types of trigonometric equations in textbooks. In addition, the expected answer could be different: sometimes the general solution is expected but in many cases only particular solutions fit the specified range. We focus mainly on the general solution, although particular solutions are also considered. Actually, it is not surprising that trigonometric equations (especially in more complicated cases) may give different answers depending on choices made in the solving process (see Abramovich, 2005).

Different CAS work quite differently in case of trigonometric equations. WolframAlpha gives the general solution. Maxima and Sage give particular solutions by default but also have possibilities for general solutions. Axiom and Wiris give particular solutions. We focus on two issues here: equivalence of two periodic solutions and relation between general and particular solutions.

We need at least two different general solutions for analysis of equivalence. Interestingly, in many cases, WolframAlpha gives different answers, depending on whether one enters only an equation ($Answer_{WolframAlpha(Equation)}$) or an equation with the command `solve` ($Answer_{WolframAlpha(SolveEquation)}$). In many cases the difference is quite superficial, only with different terms removed from parentheses.

$Equation$: $\sin x = \frac{1}{2}$
$Answer_{WolframAlpha(Equation)}$: $\frac{1}{6}(12\pi n + \pi)$, $n \in \mathbb{Z}$ and $\frac{1}{6}(12\pi n + 5\pi)$, $n \in \mathbb{Z}$
$Answer_{WolframAlpha(SolveEquation)}$: $\pi(2n + \frac{1}{6})$, $n \in \mathbb{Z}$ and $\pi(2n + \frac{5}{6})$, $n \in \mathbb{Z}$

Identification of the exact boundaries is not an easy task and it is better to examine it on the basis of CAS answers.

The cases where different answers (textbooks vs. CAS or CAS1 vs. CAS2) treat the 'series' of solutions differently (for example, two series in one answer are merged in the other) are more instructive.

$Equation$: $\sin x = 0$
$Answer_{WolframAlpha(Equation)}$: $\pi n$, $n \in \mathbb{Z}$
$Answer_{WolframAlpha(SolveEquation)}$: $2\pi n$, $n \in \mathbb{Z}$ and $2\pi n + \pi$, $n \in \mathbb{Z}$

When does this phenomenon appear? If there are different numbers of series (given by textbook, student or CAS), it is an indication that the phenomenon appears. In (some) textbooks, $(-1)^n$ is usual in case of sine and $\pm$ in case of cosine, while the a CAS (WolframAlpha) gives an equivalent answer with two series.

$Equation$: $\cos x = \frac{1}{2}$
$Answer_{Textbook}$: $\pm\frac{\pi}{3} + 2n\pi$, $n \in \mathbb{Z}$
$Answer_{WolframAlpha(Equation)}$: $\frac{1}{3}\pi(6n - 1)$, $n \in \mathbb{Z}$ and $\frac{1}{3}(6\pi n + \pi)$, $n \in \mathbb{Z}$

Therefore, appearance of the phenomenon can be predicted from the textbook answer: As the textbook uses $(-1)^n$ and $\pm$ but a CAS does not use them, we have different but equivalent answers. For a detailed overview we need a description of the solving and presenting strategy of the CAS.

Another significant (but not consistent) indicator of the phenomenon is $tan^{-1}$

(inverse tangent) in WolframAlpha.

$Equation$: $\sin x = \cos x$
$Answer_{Textbook}$: $\frac{\pi}{4} + n\pi$, $n \in \mathbb{Z}$
$Answer_{WolframAlpha(Equation)}$: $2(\pi n - tan^{-1}(1 - \sqrt{2})), n \in \mathbb{Z}$ and $2(\pi n - 2tan^{-1}(1 + \sqrt{2})), n \in \mathbb{Z}$
$Answer_{Maxima(to\_poly\_solver)}$: $\frac{4\pi\%z1 + \pi}{4}$, $n \in \mathbb{Z}$

It is explainable that $\arctan(1 - \sqrt{2})$ is equal to $-\frac{\pi}{8}$ and $\arctan(1 + \sqrt{2})$ to $\frac{3\pi}{8}$.

The second issue (which is not purely a question of equivalence) is the relation between particular and general solutions. If a CAS gives particular solutions it is instructive to place them in the context of the general solution. Maxima and Sage give only one solution per series. Similarly, Axiom gives one solution, but quite often simply with inverse trigonometric function. The question when the inverse trigonometric function justified as the exact answer, is interesting but not discussed here. The appearance of inverse trigonometric functions in different CAS is not discussed here either.

$Equation$: $\sin x(1 - \sin x) = 0$
$Answer_{Maxima}, Answer_{Sage}, Answer_{Axiom}$: 0, $\frac{\pi}{2}$
$Answer_{Wiris}$: 0, $\pi$, $\frac{\pi}{2}$

(For example, the 'Wiris' strategy is interesting and consists of two substrategies.)

Branches seem to be suitable for use in a mathematics class. Possible tasks include:

- Check $answer_{textbook} = answer_{CAS}$ or $answer_{student} = answer_{CAS}$ or $answer_{CAS1} = answer_{CAS2}$
  - With a CAS;
  - Manually.
- Transform $answer_{CAS} \longrightarrow answer_{textbook}$.
- What about the relation between the particular and general solution? What is the value of n?
- Graphical presentation of solutions is also a possible basis of tasks.

The experiment described in Chapter 8 is partially based on this type of differences.

The issues of branching are more thoroughly discussed in Chapter 6.

### 5.3.7. Phenomena related to automatic simplification of equation

1 phenomenon was identified: *Automatic(indet)*.

In case of the solving the somewhat artificial equations $\frac{x \cdot x}{x} = 0$ and $\frac{1}{x} = \frac{1}{x}$, the equations are automatically transformed to the standard form and indeterminacy is cancelled (*Automatic(indet)*). All CAS give disputable answers: 0 in case of the first equation and 'all values' in case of the other. This is probably related to automatic simplification (the topic is discussed in Tonisson, 2004).

It seems that CAS simplify equation $\frac{x \cdot x}{x} = 0$ to equation $x = 0$ before actually solving it. Kadijevich (2009) also discussed the question of automatic simplification.

It is easy to construct an example if one adds indeterminacy to a linear or quadratic equation by multiplication-division to one side or adding to both sides. Indeterminacy 0/0 is added by multiplication/division in the following example.

$x^2 + 3x - 4 = 0$, solutions are $x = 1$ and $x = -4$
$x^2 = 4 - 3x$, solutions are $x = 1$ and $x = -4$
$\frac{x^2(x-1)}{x-1} = 4 - 3x$, solution is $x = -4$, extraneous solution is $x = 1$
$\frac{x^3 - x^2}{x-1} = 4 - 3x$, solution is $x = -4$, extraneous solution is $x = 1$

The phenomenon provides a chance to discuss the topics of division by 0, indeterminacy, etc., which are not very thoroughly covered in school mathematics (see Tonisson, 2006). The following task could be used:

- Solve the equation and check the answer
  - with a CAS;
  - manually.

The types of phenomena were discussed in this section. The next section concludes the chapter.

## 5.4. Conclusion

In a large number of cases when solving school equations (from linear and quadratic to trigonometric and literal) with CAS, the system gives the answer that is expected by the student or teacher. However, occasionally, this situation also reveals certain phenomena, in which the answer is somewhat different (unexpected). The chapter focused on reasonable unexpected answers — answers that are not mistakes, but are formulated according to standards differing from school standards. The goal was to classify and map such answers, not to criticize or rank any particular CAS.

More than 120 different equations were solved in five computer algebra systems (Axiom, Maxima, Sage, Wiris and WolframAlpha). The chapter identifies 29 types of phenomena with unexpected answers (and input issues), grouping them into categories and supplying each with a brief introduction. There were separate categories for entering the command and the equation, form of the answer, unfinished solutions, domain issues, branching and automatic simplification. Their relations with the items on Drijvers' (2002) list were expressed. Other ways of grouping are certainly possible, but the classification proposed in this chapter is also one answer to RQ2 *How can the detected differences between CAS and school answers of equations in the school curriculum be described and classified?*

It should be noted that the current study was based only on a limited number of CAS and textbooks. The real instructive value of the phenomena could be determined in a study that involves teachers and students. It seems that the most promising topics for study are the equivalence of answers, the domain issues and branching.

# 6. ISSUES OF NUMBER DOMAIN IN SCHOOL MATHEMATICS AND IN COMPUTER ALGEBRA SYSTEMS

## 6.1. Introduction

As different number domains are major source of differences in CAS answers such issues are discussed severally in this chapter. The chapter is based on Tonisson, 2008. Although equations are the main area of the dissertation, simplification of expressions is examined here in addition to solving equations. Simplification of expressions and solving of equations are closely related, both in school context (Usiskin, 1988) and in computer algebra.

The chapter is devoted to the research question RQ3: *When do CAS outputs offer correct and incorrect answers for domain-sensitive examples, specifically for expression simplification and equation solving?*

The solution sets of equations and the validity of transformation rules in manipulation of expressions depend on the domain. For example, equation $x^2+1=0$ has solutions in the complex domain but not in the real (or rational) domain. It is also possible that a real number is the correct solution only in case of the complex domain. The purpose of the chapter is to identify expressions and equations that are domain-sensitive, analyze techniques for selecting domains for school use in different CAS, test CAS with domain-sensitive examples and make some suggestions for improving teaching and CAS. Testing of CAS is inspired by various larger (Wester, 1999b) or smaller (Bernardin, 1996) reviews that are not school-oriented in principle. As our goal is a school-oriented review it is necessary to describe (or at least sketch) what answers are expected in schools. We have presumed that the curriculum does not include complex numbers (in fact, imaginary numbers) even though this may not be the case in the curricula of several countries and schools.

The area of domain-sensitive examples is very multifarious and, therefore, some constraints are needed. We will focus on square root, natural logarithm and arc sine because they are domain-sensitive and taught quite thoroughly at schools. (The topics related to infinities and indeterminates (e.g., division by zero, tangent) are not discussed in this chapter.) The chapter brings together information from different areas, such as works on using CAS in learning and teaching mathematics (in general or specifically), treatment of number domains in mathematics education, comparative reviews of CAS, etc. Some of these works will be cited in the following text. The school textbooks and works on the multi-valued nature of functions are used for describing treatments in school and higher mathematics.

Section 6.2 of this chapter recalls different levels of handling complex numbers. Section 6.3 provides a brief overview of the handling of domain-sensitive functions (mainly $\sqrt{\,}$, $\ln$ and $\arcsin$) in school mathematics in relation to expressions and equations. A list of (didactically) important examples is presented as a result. Section 6.4 reviews various possibilities in CAS for determination of the domain of a calculation result, variable value or equation solution. Different aspects of such tools in different CAS (Derive, Maple, Mathematica, Maxima, MuPAD, TI-92+, TI-nspire and WIRIS) are analyzed. What is the default domain? How could the user select a domain (packages, commands, buttons, etc.)? The goal is to identify a school-friendly (imaginary-free) set of options. Section 6.5 presents the results of testing with the examples listed in Section 6.3, using the possibilities overviewed in Section 6.4. It is important that the CAS versions contemporary in 2008 are used. Some problematic issues are explained with the help of experts of particular CAS. Section 6.6 draws some conclusions and makes suggestions.

## 6.2. In school mathematics and beyond

Section 6.2 discusses different levels of handling the topics of square root, natural logarithm and arc sine. In school mathematics, the domains are restricted (accordingly, to $x \geq 0$, $x > 0$ and $-1 \leq x \leq 1$) and the functions are mainly treated as single-valued. (However, there are some doubtful cases, e.g., is $\sqrt{4}$ equal to 2 or $\pm 2$, even attempts to use special notation $\underset{+}{\sqrt{1}} = 1$ could be found in Novosjelov, 1955.)

Treatment of the topic of complex numbers varies in different (intermediate or college algebra) textbooks (e.g., Barnett & Kearns, 1990) but is generally quite superficial. It is only a rough (and unverified) guess, but it seems likely that in very many cases the school knowledge about complex numbers is almost exclusively restricted to $\sqrt{-1} = i$ and solving quadratic equations with negative discriminant. Therefore, it may be quite a surprise for some users when a CAS produces results like $\ln(-1) = \pi i$ or $\arcsin(5/4) = \frac{\pi}{2} - i \ln 2$.

In fact, the field is even more complex and the topics of multi-valued functions, branch cuts, signed zero, etc., enable various interesting discussions (at advanced study levels), for example, in the works of Corless, Davenport, Jeffrey and others (Bradford et al., 2002; Corless et al., 2000; Bradford & Davenport, 2002; Jeffrey & Norman, 2004; Rich & Jeffrey, 1996). For example, Bradford et al. (2002); Bradford and Davenport (2002) discussed an identity $\sqrt{1-z}\sqrt{1+z} = \sqrt{1-z^2}$. Aslaksen (1999) discussed the issue of identities (for example, $\ln e^x = x$ and $\ln xy = \ln x + \ln y$). There are 8 tests after theoretical treatment; some of them are used in this chapter. The question of equations (like $-3 = z^{1/2}$) is discussed in Fateman (1996).

This section highlighted some differences between answers in complex domain and real domain. They are also useful for the next section where set of examples is described.

## 6.3. Didactically important domain-sensitive examples

The aim of this section is to describe a set of examples that will be tested in different CAS. The examples should be (at least in principle) usable in schools. Some of them may occur accidentally while using a CAS while others can be used on purpose. Some examples are derived from the materials cited in the previous section and some are new.

When we look at the topics related to square root, natural logarithm and arc sine in textbooks we can find some similarities. The new operation ($\sqrt{}$, ln and arcsin) is introduced with the help of the familiar inverse operation ($x^2$, $e^x$ (or $a^x$), sin). The subtopics are usually presented in the following order: calculation, transformation-simplification, solving of equations. The pace can vary — the processes from learning square root to learning irrational equation could take years while logarithm topics could be concentrated in a short span of time. We try to find examples from the different phases.

We start with calculation of functions in case of arguments that are impossible in (imaginary-free) school: $\sqrt{-1}$, $\ln(-1)$ and $\arcsin(5/4)$ (used also in Rich & Jeffrey, 1996).

Then we introduce 'cancellation expressions' $\sqrt{x^2}$ (is $x$, when $x \geq 0$ , and $|x|$, when $x$ is real), $\ln e^x$ (is $x$, when $x$ is real), $\arcsin(\sin(x))$ (is $x$, when $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$).

The third group of examples checks some well-known identities by simplifying the difference of two expressions: $\sqrt{xy} - \sqrt{x}\sqrt{y}$ (is 0, when $x$ and $y$ are both non-negative); $\sqrt{\frac{1}{x}} - \frac{1}{\sqrt{x}}$ (is 0, when $x$ is positive); $\sqrt{e^x} - e^{x/2}$ (is 0, when $x$ is real); $\ln xy - (\ln x + \ln y)$ (is 0, when $x$ and $y$ are positive); $\ln x^2 - 2\ln x$ (is 0, when $x$ is positive) and also $\sqrt{1-x}\sqrt{1+x} - \sqrt{1-x^2}$ (is 0, when $-1 \leq x \leq 1$) which need somewhat specific explanation. The expected result 0 relieves us of the question, which expression is simpler? It should be noted that the restrictions for domain are somewhat hidden in school textbooks, for example, it could be said: *Unless otherwise stated, all variables are assumed to represent positive number* (Barnett & Kearns, 1990). In general, it is necessary to express the assumptions explicitly in CAS.

The equations $x^2 + 1 = 0$ , $e^x + 1 = 0$ and $\sin x = 5/4$ (the fourth group) are related to the above calculations and do not have solutions in school.

The fifth group includes the equations $\sqrt{x} = \sqrt{2x+1}$, $\ln x = \ln(2x+1)$ and $\arcsin x = \arcsin(2x - 5/4)$. It is somewhat disputable what is the correct answer for the school (without imaginary numbers). On the one hand, $-1$ (and $5/4$ in case of arcsin) are, of course, real solutions. On the other hand, they are not appropriate when operating with real numbers only, since a negative number appears under the square root signs and as an argument for ln (and $5/4 > 1$ for arcsin).

All examples are domain-sensitive in the sense that if we change the domain (e.g., to complex numbers) the results may change.

## 6.4. Domain possibilities of CAS

Several features that are needed to create a school-like situation: provision that the calculation result is a real number; declare the domain of variable to be a real, positive (or non-negative) or in a certain interval; provision that the solution of an equation is a real number are described in Section 6.4. The features that we use in the following tests are listed in Table 25. Hopefully, the users of specific CAS can understand the very short titles.

It should also be noted that the usual commands, like *Simplify* and *Solve*, are used. Notably, that Maxima has, in addition to default simplification, special commands like *rootscontract* that converts products of roots into roots of products or *logcontract* that 'collects' logarithms, for example in the following way

```
(%i1) 2*(a*log(x) + 2*a*log(y))$
(%i2) logcontract(%);
                                   2  4
(%o2)                       a log(x  y )
```

Foremost, we are looking for the symbolic (exact) answers (vs numeric (approximate)). As *realroots* gives approximate solutions in case of polynomial equations in Maxima, we use parentheses in the table — (Realroots). It is possible that CAS have different commands for similar activities (e.g., *Reduce* instead of *Solve* (Mathematica)). Some of them are mentioned in the next section.

| | calculation result in real domain | variable is a real number | variable in a certain interval | solution of an equation is a real number |
|---|---|---|---|---|
| Derive | Not applicable | Author Variable Domain | Author Variable Domain | Solve(..., Real) |
| Maple | RealDomain | Assume RealDomain | Assume | RealDomain |
| Mathematica | RealOnly | Assuming | Assuming | RealOnly |
| Maxima | Not applicable | Default | Assume | (Realroots) |
| MuPAD | Dom::Real | Assume | Assume | Solve(..., Real) |
| TI-92+ | Complex Format Real | Default | \| "with" | Solve (vs cSolve) |
| TI-nspire | Complex Format Real | Default | \| "with" | Solve (vs cSolve) |
| WIRIS | Default | Default | Not applicable | Default |

**Table 25.** Techniques for establishing the domain

Different CAS use different styles. There may be

- special commands (like `assume(x, Type::Real)` (MuPAD)),
- special parameters (like `SOLVE(x^2+1=1,x,Real)` (Derive)),
- menu options (like *Document Settings* $\rightarrow$ *Complex Format* $\rightarrow$ *Real* (TI)), or
- special packages (like `with(RealDomain)` (Maple)), etc.

Some restrictions work just for one line (like `simplify(sqrt(x^2)) assuming x>=0`), some determine the entire subsequent process in that particular worksheet or session. Some commands have different forms in one CAS (like *assume* and *assuming*). Some options are visible on the screen, while others are not.

In general, it seems that all features are simple enough to be used in schools but some may need certain explanations by the teacher. It should be noted that some CAS work, at least partially, in the real domain by default. Maxima generally treats variables as real-valued. WIRIS works in principle in the real domain by default.

## 6.5. Test results

This section presents the results of testing with the examples mentioned in Section 6.3 by using the possibilities overviewed in Section 6.4. At first, Table 26 presents the examples, domains, prospective results and evaluations of the answers offered by CAS. The problematic issues are discussed afterwards. The table includes various reference codes. The code *No* in case of a prospective result indicates that the proper reaction of a CAS should be no solutions or a message: *Nonreal result* or *undefined*, etc.

The answers can receive the following potential evaluations.

- Code *OK* is shown in the cell if the CAS gives the prospective result. In case of $\sqrt{x} = \sqrt{2x-1}$ and $\ln x = \ln(2x+1)$ both *No real solution* and $-1$ are acceptable answers; similarly, in case of $\arcsin x = \arcsin(2x - 5/4)$ *No real solution* and $5/4$ are acceptable. Therefore, codes *OK1* and *OK2* are used.
- Code *OK*$\approx$ indicates that the answer is numerical (approximate) (this feature is used only when a symbolic one is not found).
- Code *RNA* (Restriction Not Applicable) means that it is not currently possible to restrict the domain but an answer is found on larger scale, for example, $\sqrt{-1}$ is simplified to $i$.
- Code *NA* (Not Applicable) indicates that the operation is not applicable (in case of current arguments) at all.
- Code *NS* (Not Simplified) means that in case of simplifications, it is possible that expression is not simplified.

| Example/ Domain | Pros- pective result | DRV | MAP | MTM | MXM | MUP | 92+ | NSP | WRS |
|---|---|---|---|---|---|---|---|---|---|
| $\sqrt{-1}$ Result is real | No | RNA | OK | OK | RNA | OK | OK | OK | OK |
| $\ln(-1)$ Result is real | No | RNA | OK | OK | NS | OK | OK | OK | OK |
| $\arcsin(5/4)$ Result is real | No | RNA | OK | OK≈ | NS | OK | OK | OK | OK |
| $\sqrt{x^2}$ $x \geq 0$ | x | OK | OK | OK | OK | OK | OK | OK | RNA |
| $\sqrt{x^2}$ x is real | \|x\| | OK | OK | OK | OK | OK | OK | OK | OK |
| $\ln(e^x)$ x is real | x | OK | OK | OK | OK | OK | OK | OK | OK |
| $\arcsin(\sin(x))$ $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ | x | OK | OK | OK | OK | OK | OK | OK | NS |
| $\sqrt{x \cdot y} - \sqrt{x}\sqrt{y}$ $x \geq 0, y \geq 0$ | 0 | OK | OK | OK | OK | OK | OK | OK | NS |
| $\sqrt{\frac{1}{x} - \frac{1}{\sqrt{x}}}$ $x > 0$ | 0 | OK | OK | OK | OK | OK | OK | OK | RNA |
| $\sqrt{e^x} - e^{x/2}$ x is real | 0 | OK | OK | OK | OK | OK | OK | OK | NS |
| $\ln(x \cdot y) - (\ln x + \ln y)$ $x > 0, y > 0$ | 0 | OK | OK | OK | OK | OK | OK | OK | RNA |
| $\ln x^2 - 2\ln x$ $x > 0$ | 0 | OK | OK | OK | OK | OK | OK | OK | RNA |
| $x^2 + 1 = 0$ Real solutions | No | OK | OK | OK | OK≈ | OK | OK | OK | OK |
| $e^x + 1 = 0$ Real solutions | No | OK | OK | OK | RNA | OK | OK | OK | OK |
| $\sin x = 5/4$ Real solutions | No | OK | OK | OK | RNA | OK | OK | OK | OK |
| $\sqrt{x} = \sqrt{2x+1}$ Real solutions | No or -1 | OK2 | OK1/2 | OK2 | NA | OK2 | OK2 | OK2 | OK1 |
| $\ln x = \ln(2x+1)$ Real solutions | No or -1 | OK2 | OK1/2 | OK2 | NA | OK2 | OK1 | OK1 | OK1 |
| $\arcsin x = \arcsin(2x - 5/4)$ Real solutions | No or 5/4 | OK2 | OK2 | OK2 | NA | OK2 | OK≈ | OK≈ | OK1 |

**Table 26.** Results of tests

It is important to note the versions of CAS used in the tests. The table includes the results of Derive 6, Maple 8, Mathematica 5.2, Maxima 5.13, MuPAD 4.0, TI-92+, TI-nspire and WIRIS. Sometimes the differences between versions are significant, for example, Maple 8 (with *RealDomain*) gives no solution in case of $\sqrt{x} = \sqrt{2x+1}$ and $\ln x = \ln(2x+1)$ while Maple 12 gives $-1$. It is marked in the table as *OK1/2*. It should be noted that in case of $\ln xy - (\ln x + \ln y)$ the default simplifier of Maxima does not simplify but *logcontract* does.

We discuss the cases that are different from *OK* (or *OK1* and *OK2*) and *NA*. Derive has no possibilities to restrict the results of calculations to the real domain but the answers are correct in the complex domain (code RNA). The situation is the same with Maxima in case of $\sqrt{-1}$ (*RNA*). In case of $\ln(-1)$ and $\arcsin(5/4)$ Maxima does not return imaginary answers but just the input. In principle, it is possible to customize Maxima (as an open-source system) by writing a pattern that will give an error message if they are generated. Code *OK*$\approx$ in case of $\arcsin(5/4)$ and Mathematica means that $\arcsin(5/4)$ does not give the proper result while $\arcsin(1.25)$ does (approximately).

The command *solve* could not be restricted to real solutions in Maxima. However, there is the command *realroots* that works only for polynomial equations and gives approximate (floating-point) solutions. Thus, the code used for $x^2+1=0$ is *OK*$\approx$. There is no such command for other equations. The command *find_root* that finds a root over the closed interval $[a,b]$ is not very suitable in our situation.

TI-92+ and TI-nspire do not solve $\arcsin x = \arcsin(2x-5/4)$ symbolically but *nSolve* gives answer *No solutions found* (code *OK*$\approx$).

WIRIS works in the real domain by default. Unfortunately, the assumptions (like $x>0$) are not possible. If we test the examples without assumptions, we use the code *RNA* to refer to the results that are the same as presumed with the assumptions. Otherwise, when the expression is not simplified we use the code *NS*.

Some problems notwithstanding, the general situation seems to be quite good. Some more problematic issues may occur with more complex examples. Let us look at the expression $\sqrt{1-x}\sqrt{1+x}-\sqrt{1-x^2}$. We restrict the domain to $-1 \le x \le 1$. The prospective answer is 0 in this range (and actually wider (Bradford & Davenport, 2002)). If we use the same commands as previously, only Mathematica gives 0. We may need other commands for this expression. Thanks to experts of current CAS the following variants are listed. The variant `simplify(combine(sqrt(1-x)*sqrt(1+x)-sqrt(1-x^2),symbolic));` works in Maple. The command *rootscontract* works in Maxima. In MuPAD, it is possible to determine maximal number of steps that will be done in simplification (the default is 50): `Simplify(sqrt(1-x)*sqrt(1+x)-sqrt(1-x^2), Steps = 10000)`. Under the above assumptions, this needs 1352 steps (version 4.0.6).

## 6.6. Conclusions and suggestions

The examples from the test suite (18 calculations, simplifications, equations) were solved by different CAS with usage of the features for creation of school-like situation, particularly establishing the domain. Different CAS have different possibilities (commands, packages, buttons, etc) for determination of the domain of a calculation result, variable value or equation solution. Correctness of the CAS answers were evaluated and the answer to RQ3 was presented as a table.

We could say that CAS have quite good features for confining the work in school-like (imaginary-free) domains. Different CAS have different commands and other features and Table 25 could serve as a kind of dictionary in this respect. It would be wonderful if we could create a school-like approach in one swoop in any given CAS. Just as Wester (1999b) hoped: *One could invoke* `mindset(elementary_math_student)` *to initially declare all variables to be real, make $\sqrt{-1}$ undefined, etc., for example.* Of course, various problems should be overcome to create such a feature — differences between different schools, technical problems, etc. The packages *RealDomain* (in Maple) and *RealOnly* (in Mathematica) are already a big step in the right direction.

The current features work quite well as we see in Table 26. However, the user has to be informed and careful. If the teacher knows the problematic issues it is possible to avoid or forestall negative surprises. On the other hand, some issues could be used for disputes with students in mathematics classes. For example, the equations $\sqrt{x} = \sqrt{2x+1}$, $\ln x = \ln(2x+1)$ and $\arcsin x = \arcsin(2x-5/4)$ could be used for introduction of imaginary numbers. Knowledge about complex variable functions would help teachers in this case.

This review could be expanded with more complex examples. A study of different commands (like *testeq* for testing equivalence) could be useful. The testing with other CAS and versions is important when a particular CAS or version is actually used in class. A challenge could be to compose a review where the examples are tested in the complex domain. It would cross the boundary of (imaginary-free) school mathematics and would create powerful possibilities for comparing domains, identities, (multivalued) functions, etc., in schools where the topics of complex numbers are included in the curriculum as well as in universities. Nevertheless, even this chapter can hopefully contribute to the use of different ideas (e.g., searching counterexamples).

This chapter was devoted to the issues of number domain. The next chapter discusses on the issues of branching.

# 7. BRANCH COMPLETENESS IN SCHOOL MATHEMATICS AND IN COMPUTER ALGEBRA SYSTEMS

## 7.1. Introduction

Chapter 7 is devoted to the solutions that are separable into branches in some manner. Here, like in the previous chapter, simplification of expressions is relevant in addition to solving equations. This chapter is based on Tonisson, 2007 and focuses on the research question RQ4: *How can branching be described for answers provided by different CAS software; by different school solutions and textbooks; by the possibilities of mathematical approaches for expressions simplifications and equations solving?*

The cases where separation into branches takes place are different. For example, an expression may be undefined in case of some values of the variables (e.g., $1/x$, $\sqrt{x}$) or an equation may have several roots or root groups. In some cases, these branches are explicitly introduced in school mathematics, in other cases the branches may be hidden. This chapter examines separable branches of solutions of different problems (simplifications, equations) and the completeness of branch sets. We could say that a solution is mathematically branch-complete if all branches are presented. The chapter describes some approaches to branches that are used in school mathematics and CAS. It tries to identify possible reasons behind different approaches and also indicate some ideas how such differences could be explained to students.

The school mathematics side of the chapter is based both on different textbooks (English (e.g., Barnett & Kearns, 1990; Barnett & Ziegler, 1989), Russian (e.g., Govorov, Dybov, Miroshin, & Smirnova, 1983), Norwegian (e.g., Oldervoll, Orskaug, & Vaaje, 1995), Estonian (e.g., Lepik, Nurk, Telgmaa, & Undusk, 2000; Lepmann, Lepmann, & Velsker, 2000)) and on computer algebra systems (Derive 6 (*Webpage of Derive*, 2007), Maple 8 (*Webpage of Maple*, 2007), Mathematica 5.2 (*Webpage of Mathematica*, 2007), Maxima 5.13 (*Webpage of Maxima*, 2007), MuPAD 4.0 (*Webpage of MuPAD*, 2007), TI-92+ (*Webpage of TI-92+*, 2007), TI-*n*spire (*Webpage of TI-nspire*, 2007) and WIRIS (*Webpage of Wiris*, 2007)). It should be noted that Tonisson (2007) was published in 2007 and the versions of CAS were contemporaneous.

Admittedly, these sources do not cover all possible approaches. Different textbooks, other CAS, versions, commands or even some special expressions or equations could work in a different way. However, this chapter is hopefully adequate enough for most cases. The main aim of the chapter is not criticism of a particular CAS or textbook but rather providing a description of a variety of approaches. A special notation ($CAS < SCH = MATH$) is introduced in Section 7.3 for better overview. Some presented topics are fundamental school topics that are discussed in almost every textbook. Some topics are discussed only in particular textbooks and are not discussed at all or only touched upon briefly in others. At the same time these topics could have recognizable educational potential, particularly when using a CAS. Similarly, some presented examples may occur only in a single CAS, being fairly marginal but still having educational value.

There is an overview of previous related works in Section 7.2. An approach to evaluation of branching diversities is introduced in Section 7.3. Section 7.4 discusses simplification questions and solving of equations is examined in Section 7.5. Even though other commands may work better in some cases, the usual *Simplify* and *Solve* commands are used first of all. There are some general comments on branching diversities in Section 7.6 and a conclusion in Section 7.7.

## 7.2. Related works

The author has not found any works that would thoroughly discuss branching in CAS from the viewpoint of school mathematics. However, there is a great deal of material related to the topic. Several works were already introduced in Chapter 2. Some of the related works are discussed in Section 7.2.

Wester (1999b) conducted probably the largest experiment aimed at discovering how different CAS solve problems can be found. However, there were not very many examples from the school in that paper and not many branching examples either. Bernardin (1999) provided some interesting examples and comments. Bernardin said after example $ax = b$: *Often, there seems to be a philosophy among computer algebra systems to return answers even if they do not hold on a finite subset of the parameter space.* Stoutemyer (1991) also discussed the same issues. Stoutemyer listed several theoretical and practical limitations of CAS. Some of them are closely related to branching. His sentence, *It is important for users to be aware of some of the limitations of such systems to use them wisely*, is suitable as a slogan for this research.

Kahan (1987) provided a theoretical overview of branch cuts for complex elementary functions, Aslaksen (1999) discussed complex analysis for CAS. Complex analysis is closely related to branching. It is useful to find parallels from further fields of mathematics, for example the idea of using sequents like in Gentzen-type calculi (Chuaqui & Suppes, 1990). It seems that branching in school mathematics is generally not very often explicitly discussed in papers. The importance of branches tends to be emphasized in more specific papers (for instance, on teaching and learning the absolute value (Wilhelmi, Godino, & Lacasta, 2007)).

There are also some papers by the author of the dissertation that are related to the current topic. Some preliminary work for Tonisson (2007) could be found in Tonisson (2004) that tried to classify CAS answers in relation to correctness, completeness and compactness of the answers.

## 7.3. Structure of overview and notation

Before proceeding to the main sections, the structure of the description and specific notation are explained in this section. There are three matters under consideration — we observe comparatively how branches are treated in school textbooks, in CAS, and what is a mathematically complete branch set. The variations in the first two elements of the above list are the object of this chapter. The mathematical branch completeness is a gauge that is explained in every subsection. Special attention in studying the textbooks is paid to the model solutions and also to the answers, because the expectations for students are mainly presented in these parts. The classic commands (e.g., *Solve* and *Simplify*) are used in CAS at a first approximation.

We evaluate the branching completeness in case of

- CAS answer;
- school answer;
- mathematically complete answer.

Then we compare them and determine the *evaluations of branching diversities* (EBD) for each problem type, e.g.,

$$CAS < SCH = MATH.$$

CAS refers to the treatment of branches in computer algebra systems, SCH refers to the treatment of branches in school textbooks, and MATH refers to a mathematically branch-complete solution. The equality sign (=) indicates that branches are similarly presented, the sign $<$ shows that the second treatment is more complete. As different textbooks and CAS may have differing branch sets, it is possible that there is more than one EBD in a particular problem type. The specification is added in parentheses in these cases (e.g., $CAS(1)$ may mean that a multiple root is presented in one time). As the specification is context dependent, there is an explanation for the corresponding EBD. Section 7.6 includes comments on all found EBDs.

There are more areas where we can find branching, but in this chapter we focus on two important areas of school mathematics — simplification of expressions (Section 7.4) and solving of equations (Section 7.5). The sections are structured as follows: a general introduction to branching in this area, followed by a discussion of a number of more colorful topics. The treatments of branches in textbooks and CAS, and mathematical branch-completeness are explained for each problem type. The evaluations of branching diversities (EBD) are also introduced.

We do not discuss the notation of answers (incl. branches); it is assumed that the notation is understandable for the students. If the notation is complicated or confused a teacher should explain it.

Now, it is possible to move to the main part of the chapter. Firstly, simplification is under consideration.

## 7.4. Simplification

### 7.4.1. Introduction

Although the main topic of the dissertation is solving equations, it is justified to discuss also issues of simplification of expressions. Different simplification exercises can be found in textbooks at many places, usually after introduction of a new operation or function. The great majority of the simplification exercises in textbooks are without (or at least without explicitly presented) branching. Three areas are discussed in greater depth in this section. The topic of 'forbidden branches' (Section 7.4.2) is a good example of hidden branches while in the case of expressions with absolute value (Section 7.4.3) the branches are (sometimes) explicitly presented. The topic of $\sqrt{a^2}$ (Section 7.4.4) is closely related to the absolute value but has independent importance as well. Finally, one more topic is listed (Section 7.4.5).

### 7.4.2. Forbidden branches

The student learns for a number of operations and functions that operating is impossible (at least in school) in case of some values of arguments. The first contact with these problems occurs in early grades during subtracting when $2-3$ is problematic, because negative numbers are (at least 'officially') not yet known to students. The matter is not discussed in depth in these grades and such expressions are simply avoided. The first commented contact with the 'forbidden' operands appears in case of division by zero. The fact that division by zero is undefined is explained by means of multiplication. An important argument is the fact that multiplying by 0 always results in 0.

Explanations are also given for further problematic operations. Thus, the student knows (after more or less explanation) the following restrictions:

- division by zero is undefined;
- there is no square root for a negative number;
- the domain of a logarithmic function is the set of all positive real numbers;
- the base of logarithm is positive and differs from 1;
- tangent function is not defined if $x=(2n+1)\pi/2$, $n \in \mathbb{Z}$;
- argument of arc sine and arc cosine is from the interval $[-1;1]$ (not expressly discussed in school textbooks).

The limits of what is allowed and what is not are fairly clear while calculating with numbers. The situation changes when variables appear in denominators or under square roots or in arguments of other problematic functions. In order to follow these restrictions correctly in mathematical terms one should demonstrate all 'forbidden branches' separately in expression transformation exercises. In reality, the distinguishing of forbidden branches is discarded, and the practice is even legalized. For example, $\frac{x^2}{x}$ is transformed to $x$ without comments.

A textbook (Lepmann et al., 2000) said:

- *Even though not always explicitly stated, we always assume that variables are restricted so that division by 0 is excluded*;
- *Unless stated to the contrary all variables are restricted so that all quantities involved are real numbers*;
- *The equality is valid only at such variable values where the value of either side of the equality is calculable. For instance, the equality*

  $$\frac{x}{x-1} = \frac{x \cdot x}{(x-1)x}$$

  *is valid only where $x \neq 0$ and $x \neq 1$. As mentioned above, such restrictions are henceforth not explicitly stated in the equalities.*

Such conventions allow students to remorselessly reduce, expand, isolate variables from the radical, etc., without giving a thought to division by zero or extracting the square root of a negative number, etc.

The CAS do not show 'forbidden branches' either. We look at the expressions where in simplification some parts are cancelled, for example

$$\frac{97x}{x}.$$

All computer algebra systems solve it by giving the answer 97, without recording the peculiarity of $x = 0$. (It is noteworthy that TI-*n*spire adds a warning message: *Domain of the result may be larger*.) In addition to the general style of disregard for the special cases, it could also be attributed to automatic simplification that some CAS use. We claim that a result is mathematically branch-complete if 'forbidden branches' are also explained. Therefore, EBD for cases where CAS and school mathematics forget 'forbidden branches' is $CAS = SCH < MATH$.

It is another matter whether a computer algebra system (or some other software application) could behave in a more precise manner and separately record special cases. This issue has been examined in Chuaqui and Suppes (1990), who propose to write the results of solution steps in the form of sequents (as in Gentzen-type calculi in mathematical logic). It is possible to present only the main branch with the condition(s), for example,

$$x \neq 0 \Rightarrow \frac{97x}{x} = 97.$$

Furthermore, it is possible to present all branches separately, for example,

$$x \neq 0 \Rightarrow \frac{97x}{x} = 97 \text{ and } x = 0 \Rightarrow \frac{97x}{x} = \frac{0}{0}.$$

A case of $\frac{0}{0}$ leads to another area that is discussed in Beeson and Wiedijk (2005). This question is also discussed in Stoutemyer (1991) as Sets of Measure Zero.

Stoutemyer suggests including in CAS the option of simplifying that distinguishes the branches in the conditional form of *if ... then ... else ...*.

In fact, there are some textbooks containing some specific exercises that emphasize the domain of the expression, for example, *Find the domain of the expression*

$$\sqrt{\frac{4}{x-1}}$$

(Lepmann et al., 2000).

There are also books (e.g. Govorov et al., 1983 — the collection of problems used in entrance examinations in the former Soviet Union) where (at least in some exercises) the branches are presented. One such exercise is presented in Section 7.4.4. In these cases, where school textbooks refer to the domain, EBD is coded as $CAS < SCH = MATH$.

The topic of 'forbidden branches' is tied to the topic of infinity-indeterminate and number domains because there may be different restrictions in case of different domains. For example, $\sqrt{e^z} - e^{z/2}$ should not be simplified when $z$ is complex but should be simplified to 0 when $z$ is real (Aslaksen, 1999).

### 7.4.3. Absolute value

The topic of absolute value — a classic branched topic and very educative, as such — is discussed in Section 7.4.3. The branches are introduced already in the definition:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}.$$

There may be exercises that avoid branching by appropriate additional assumption in the text of the exercise: *Simplify expression* $|x+1| - |x-1|$, *where* $-1 \leq x < 1$. Likewise, CAS (except WIRIS) have possibilities for using such assumptions (see Table 25 in Chapter 5). For example, in Maple:

```
simplify(abs(x+1)-abs(x-1)) assuming -1<=x, x<1;
```

As all explanations of a branch are equal in all these 'parties', EBD is $CAS = SCH = MATH$.

In fact, school textbooks do not include too many simplifications that contain the absolute value. Abel, Jõgi, and Mitt (1984) was not a regular school textbook but rather a textbook for teacher training. In case of expression without additional assumptions (e.g., $2 - |x-3|$ ) the branches spring and Abel et al. (1984) presented them.

$$2 - |x-3| = \begin{cases} 2-(x-3) & \text{if } x \geq 3 \\ 2-(-x+3) & \text{if } x < 3 \end{cases} = \begin{cases} 5-x & \text{if } x \geq 3 \\ x-1 & \text{if } x < 3 \end{cases}$$

A separate question would be if this exercise is simplification at all, and the answer to that is more complicated in some sense. CAS do not present branches (at least not in association with the *Simplify* command). EBD is $CAS < SCH = MATH$ if CAS does not present branches.

It is possible to compose miscellaneous expressions that include several absolute values. For example, if absolute values are cancelable, CAS give the correct answer, for example, $|x-3|-|3-x|$ is simplified to 0.

The question of $\sqrt{a^2}$, that is discussed in the next section, is very closely related to the absolute value.

### 7.4.4. Square root of the square ($\sqrt{a^2}$)

There are quite a few simplification exercises that include expressions in the form of $\sqrt{a^2}$, $(a^2)^{1/2}$, $\sqrt[4]{a^4}$, etc., where $a$ is an expression. Such examples are discussed in Section 7.4.4. On the one hand, textbooks say that $\sqrt{a^2} = |a|$. On the other hand in simplification exercises some books (for example Lepmann et al., 2000) may somewhat retract and say *If not required separately we do not write* $\sqrt{a^2y} = |a|\sqrt{y}$ *and* $\sqrt[4]{(x-2)^4} = |x-2|$ *but* $\sqrt{a^2y} = a\sqrt{y}$ *and* $\sqrt[4]{(x-2)^4} = x-2$. It is a hidden assumption $a \geq 0$ similar to the one of 'forbidden branches'.

Derive, Maxima, TI-92+ and TI-*n*spire give $|a|\sqrt{y}$ as an answer in the simplification of $\sqrt{a^2y}$. The other systems do not simplify. This question is also discussed in Aslaksen (1999):

- $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{csgn}(z)z$ *when* $z$ *is complex.*
- $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{sgn}(z)z = |z|$ *when* $z$ *is real.*
- $\sqrt{z^2}$ *should simplify to* $z$ *when* $z$ *is positive.*

The complex sign function csgn(z) is defined (in Aslaksen, 1996)

$$\operatorname{csgn}(z) = \begin{cases} 1 & \text{if } Re(z) > 0 \text{ or } (Re(z) = 0 \text{ and } Im(z) > 0) \\ 0 & \text{if } z = 0 \\ -1 & \text{if } Re(z) < 0 \text{ or } (Re(z) = 0 \text{ and } Im(z) < 0) \end{cases}$$

When the CAS gives the absolute value (the branches are 'compressed') we could (perhaps questionably) say $SCH < CAS = MATH$. It is hard to determine EBD when the CAS does not simplify.

There are still books that require presenting branches; for example, in case of

$$2(a+b)^{-1}(ab)^{1/2}\left(1+\frac{1}{4}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2\right)^{1/2}$$

Govorov et al. (1983) gave branching answer:

$$\begin{cases} 1 & \text{if } a > 0 \text{ and } b > 0 \\ -1 & \text{if } a < 0 \text{ and } b < 0 \end{cases}$$

Forbidden branches are not mentioned. It is related to the question of $\sqrt{a^2}$ but also to several issues listed in the section on 'forbidden branches'. We could say (perhaps questionably again) that EBD is $CAS < SCH = MATH$ as CAS do not present branches.

### 7.4.5. More topics

There are more topics in school mathematics where we could anticipate branching. For example in trigonometry Half-Angle Formulas include $\pm$, e.g.,

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

However, these expressions are very rare in simplification exercises. Sometimes a half-angle may be in the exercise but it is squared and $\pm$ will be eliminated immediately.

## 7.5. Equations

### 7.5.1. Introduction

Equations have a central position in school algebra, and branching is essential in case of some equations. Four areas are discussed in greater depth — multiplicity of roots (Section 7.5.2), extraneous roots (Section 7.5.3), literal equations (Section 7.5.4) and trigonometric equations (Section 7.5.5). It should be noted that here we skip forms of branching that are related to complex numbers although they are important, especially in case of CAS answers (e.g., solutions of exponent equations).

### 7.5.2. Multiplicity of roots

Although multiplicity of roots is a wider topic we concentrate here on quadratic equations. In solving a quadratic equation, one can get two different real roots in case of a positive discriminant; such branching is clearly presented both in textbooks and in CAS answers (EBD is $CAS = SCH = MATH$). A negative discriminant leads to complex numbers and we skip it here. There is a problem of the multiple root if the discriminant is zero (e.g. in case of $x^2 + 2x + 1 = 0$). Some textbooks say that there are two equal roots, some say one real root (a repeated root) or some say just one real solution. The CAS have applied different approaches as well. Derive, Maxima, TI-92+, TI-*n*spire, MuPAD and WIRIS give a single root while Maple and Mathematica give it twice. It is important to note with respect to the notation that the same signs may have different meanings in different CAS. For example, mark {a, b, c} means a set in MuPAD but a list in Mathematica. We consider that presenting roots twice is a complete answer, even though it is not particularly important to emphasize this in school because there are no polynomials and equations of the *n*th degree in school.

As textbooks and CAS can both present roots once (marked by 1 in EBDs) or twice (marked by 2 in EBDs) the possible EBDs are

$CAS(1) = SCH(1) < MATH(2)$,
$CAS(2) = SCH(2) = MATH(2)$,
$SCH(1) < CAS(2) = MATH(2)$ or
$CAS(1) < SCH(2) = MATH(2)$.

### 7.5.3. Extraneous roots

It is necessary to consider the branches in the case of the equations (fractional, radical, logarithmic, etc.) that correspond to the expressions with 'forbidden branches'. Disregarding them may result in extraneous roots. Different variants are used in the textbooks to obtain correct final answers. Mainly, checking of potential roots in the initial equation is used but detection of the domain of the equation can be used as well. Generally, textbooks and CAS give a correct set of solutions (EBD is $SCH = CAS = MATH$) but CAS may be surprisingly 'more complete' than would be correct in case of some equations (EBD is $SCH = MATH < CAS$). The CAS (except Maxima and WIRIS) can present by default a real solution that is considered in school, in the real domain, as extraneous (e.g., $x = -1$ in case of $\sqrt{2x} = \sqrt{x-1}$ (or $\ln(2x) = \ln(x-1)$)).

The other case can be illustrated by the example where all the systems offer 0 as the answer to the equation

$$\frac{x \cdot x}{x} = 0.$$

TI-*n*spire adds a warning-message *Domain of the result may be larger*. It can be explained by the fact that the original equation is automatically simplified before solving. EBD is $SCH = MATH < CAS$.

### 7.5.4. Literal equation

The topic of literal equation is a classic branching topic. Literal equations offer different levels for treatment of branches. The minimal case considered correct in some way would be the one where it is assumed by default that the parameter has no 'suspicious' values, and only the main branch is calculated. This is often assumed in applied problems (e.g., in physics: $A = P + Prt$; *please express r*). At the next level, the parameter values that result in the branch are recorded with the main branch. The level where all cases are shown separately is the most complete. For example, the answer of equation $(a^2-1)x - (2a^2+a-3) = 0$ in Lepmann et al. (2000) is

$$\begin{cases} \mathbb{R} & \text{if } a = 1 \\ \emptyset & \text{if } a = -1 \\ x = \dfrac{2a+3}{a+1} & \text{if } a \neq \pm 1 \end{cases}$$

We consider the level where all cases are shown separately as complete.

Bernardin (1999) criticized the behavior of CAS. In case of $ax = b$ Bernardin sayd: *When asked to solve with respect to x, all the systems returned the solution*

$x = \frac{b}{a}$ *even when this answer is obviously not correct for* $a = 0$. He notes that there may be different commands (e.g., *Reduce* in Mathematica) that work better.

Using somewhat later versions of CAS we could say that the CAS apply different approaches. MuPAD records all branches; Derive, Maple, Maxima, TI-92+, TI-*n*spire and WIRIS present only the main branch. In Mathematica, it depends on what command (*Solve*, *Reduce* or *InequalitySolve*) is run. The command *Solve* gives only the main branch, the command *Reduce* gives the main branch with the corresponding condition and command *InequalitySolve* gives the complete set of branches. We mark the 'main-branch-approach' as *main* and the 'all-branch-approach' as *all*. As both textbooks and CAS can use both approaches we get many possible EBDs:

$CAS(main) < SCH(all) = MATH(all)$,
$CAS(all) = SCH(all) = MATH(all)$,
$SCH(main) < CAS(all) = MATH(all)$ or
$CAS(main) = SCH(main) < MATH(all)$.

### 7.5.5. Trigonometric equation

There are different sources of branching in case of the trigonometric equations — periodicity and the families of solutions. Textbooks provide general solutions. For example, in case of the equation

$$\sin x + \cos 2x = 0$$

textbooks give the answer

$$x_1 = \frac{\pi}{2} + 2n\pi \text{ and } x_2 = \frac{\pi}{6} \pm \frac{\pi}{3} + \frac{4}{3}n\pi$$

or

$$x_1 = (-1)^n \frac{\pi}{2} + n\pi \text{ and } x_2 = (-1)^{n+1} \frac{\pi}{6} + n\pi.$$

However, the CAS work differently. MuPAD gives general solutions; Derive, Maple, Mathematica, TI-92+, TI-*n*spire give (at least by default) only the particular solutions. They have different standards for the choice. This equation is too complicated for Maxima and WIRIS. In case of simpler equations they present particular solutions. It is noteworthy that Mathematica, Maxima, TI-92+, TI-*n*spire add warning messages when presenting particular solutions, e.g., *Some solutions will be lost* or *Some solutions may not be found*.

As we consider general solution as complete, EBDs according to the CAS approach would be:

$CAS < SCH = MATH$ or
$CAS = SCH = MATH$.

| EBD | Problem type |
| --- | --- |
| $CAS = SCH < MATH$ | Forbidden branches are not recorded (CAS, SCH) |
| $CAS < SCH = MATH$ | Forbidden branches are recorded (SCH), not recorded (CAS)<br>Absolute value, all branches (SCH) |
| $SCH < CAS = MATH$ | $\sqrt{a^2} \to a$ (SCH) |

**Table 27.** EBDs (Simplification)

| EBD | Diversity type |
| --- | --- |
| $CAS = SCH < MATH$ | Multiplicity of roots 1 (CAS, SCH)<br>Literal equation 1 branch (CAS, SCH) |
| $CAS < SCH = MATH$ | Multiplicity of roots 1 (CAS), 2 (SCH)<br>Literal equation 1 branch (CAS), all branches (SCH)<br>Particular solution of trigonometric equation (CAS) |
| $SCH < CAS = MATH$ | Multiplicity of roots 1 (SCH), 2 (CAS)<br>Literal equation 1 branch (SCH), all branches (CAS) |
| $SCH = MATH < CAS$ | Extraneous roots |

**Table 28.** EBDs (Equations)

## 7.6. Comments on branching diversities

Theoretically we could compose quite a few different EBDs while only some of them are present in practice. The situations where a textbook and CAS give the same amount of branches ($CAS = SCH = MATH$) are not very interesting from the perspective of this chapter, as the student (and also the teacher) gets the answer that accords with the school presentation and mathematics. Tables 27 and 28 contain all the other aforementioned EBDs with references. This section provides brief comments on the different types of evaluations of branching diversities that we found.

The cases of $SCH = MATH < CAS$ are deficiencies of CAS where CAS provides extraneous solutions. The situation $CAS = SCH < MATH$ needs some comments. The explanation-justification related to hidden forbidden branches is presented (maybe too modestly) in textbooks. It can be assumed that branch completeness is rejected for the sake of compactness. Apparently, it is more complicated as well as more time- and space-consuming to (repeatedly) record several branches and special cases. Furthermore, repeated recording entails the danger of oversights, etc. In addition, it is more difficult to grasp the answer where it contains many special cases and branches, which distract attention from the main line.

In other cases, a CAS may present a complete set of branches while a textbook presents an incomplete set or vice versa. It could be happen, for instance, in case of literal equations where some branches could be 'forgotten'. If a CAS presents a complete and a textbook an incomplete set ($SCH < CAS = MATH$), there may be the explanation-justification in textbook ($\sqrt{a^2} \to a$) or it is possible to use the same explanation as in the textbooks that present the complete set of branches (multiple root, literal equation).

If a textbook is more complete ($CAS < SCH = MATH$), there are three possible approaches: avoid using CAS (simplification of an expression involving absolute value), try to find a complete set manually (or with CAS)(trigonometric equation), or explain that a (more) complete set of branches is not necessary for school (multiple root, literal equation).

## 7.7. Conclusion

It could be said to answer RQ4 that branching can be described by evaluations of branching diversities (EBD).

We described (perhaps questionably sometimes) the mathematically correct and complete branch sets and compared the approaches of textbooks and CAS and the mathematically complete sets. We classified the situations and found four types ($SCH = MATH < CAS$, $CAS = SCH < MATH$, $SCH < CAS = MATH$ and $CAS < SCH = MATH$) that were briefly commented on. In all cases the answers were comparable and the branches in the school answer were similarly presented as the CAS answer or the mathematically complete answer as there are only few reasonable variants of branching.

It seems reasonable to use CAS in teaching and learning the branch-related topics. Almost all diversities can be explained. However, further studies are needed to develop a detailed framework. Hopefully, teachers (and others) could then place their own examples into this framework and get useful information to improve their work.

There are several open research 'branches' that could ensue from this chapter. For example, a study of teachers' attitudes to branching would be very interesting. What approaches are more suitable for teaching and learning? The usual commands *Simplify* and *Solve* with default settings are used in the above examples. The other possibilities (e.g., *assuming* in Maple, or *Reduce* in Mathematica) are only briefly touched on. Actually, the proper use of different commands, assumptions and settings could make CAS more suitable for school mathematics. For example, there are special tools for determination of the real or complex domain. Some of them are listed in Chapter 5.

# 8. STUDENTS' COMPARISON OF THEIR TRIGONOMETRIC ANSWERS WITH THE ANSWERS OF A COMPUTER ALGEBRA SYSTEM

## 8.1. Introduction

The previous chapters included some abstract ideas about potential educational use of the differences between the researched answers. This chapter will describe an experiment where differences between CAS answers and students answers were used in a teaching and learning situation.

Chapter 8 is based on the papers:

- *Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System* (Tonisson, 2013) and
- *Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System in Terms of Equivalence and Correctness* (Tonisson & Lepp, 2015).

In Section 8.2 a lesson scenario is provided as an answer to the research question RQ5:

*What pedagogical approaches could be proposed to utilise the teaching opportunities offered by the differences between CAS and students answers?*

Sections 8.3 and 8.4 are based on two experiments using this lesson scenario to seek answers to RQ6: *How can students identify 1) the equivalence and non-equivalence between CAS and their own answers; 2) correctness of CAS and their own answers of trigonometric equations during lessons based on comparative discussions on students' answers and CAS answers in pairs of students?*

Section 8.3 (and (Tonisson, 2013)) is focused on the adequacy of students' identification of equivalence. Lessons were held in the autumn of 2012 (also referred to as 'Lessons of 2012'). Section 8.4 (and (Tonisson & Lepp, 2015)) is focused on identification of equivalence and correctness by students. The lessons were held in the autumn of 2013 (also referred to as 'Lessons of 2013'). Section 8.5 includes conclusion of Sections 8.2 and 8.4, overviews of problematic issues for students, and also some ideas for future work.

## 8.2. Using comparison of students' and CAS answers

Section 8.2 proposes one pedagogical approach for utilising teaching opportunities offered by the differences between CAS and student answers. The proposed approach is described in Section 8.2.1. The lessons where this approach was used are described in Section 8.2.2. The approach is evaluated in Section 8.2.3.

### 8.2.1. Description of the proposed approach

The aim was to propose a pedagogical approach, in which the differences between CAS and students' answers can be utilized. The proposed approach and its background are described in this section. A lesson scenario is presented, followed by a description of data collection.

In general, it was kept in mind that an approach should be relatively easy to execute in practice and attractive for students. The use of student answers and their own solution processes was designed to encourage deeper engagement with the task. Moreover, solving by pen-and-paper supports the development of students' procedural skills — an aspect, which some believe is jeopardized by the use of CAS. Keeping a balance between procedural skills and conceptual understanding was one of the underlying ideas.

The envisaged approach had to support students' conceptual understanding. As discussion can be a useful tool for developing conceptual understanding, work in pairs was chosen as the preferred approach. As discussion and work in pairs are usually more effective when they are guided, it was decided to provide students with a worksheet. According to these ideas, a lesson scenario, based on a comparative discussion on students' answers and computer algebra system answers in pairs of students is proposed.

As this dissertation focuses on equations, the proposed approach was used on issues that are important in the context of equation solving — solution of equation, equivalence of solutions. The students were paired up and supplied with given worksheets with equations and questions. Initially, the students solve an equation (correctly or not) without a CAS and then with a particular CAS. Solving of trigonometric equations was chosen as the main topic because of the natural variety of possible presentations of solutions and units of measurement. In addition, unlike most of the other school equation types, the solution of trigonometric equation includes no separate numbers but rather series, which reveals some special complications, like understanding the meaning of parameter $n$. Students' difficulties in linking the periodicity of trigonometric functions and integers in solutions of trigonometric equations is highlighted in Chigonga (2016). It should be noted that in Estonian textbooks the solution for $\sin x = m$, is often expressed as

$$x = (-1)^n \arcsin m + n\pi,\ n \in \mathbb{Z}$$

and not as in many other countries

$$\begin{aligned} x &= \arcsin m + 2n\pi,\ n \in \mathbb{Z} \\ x &= \pi - \arcsin m + 2n\pi,\ n \in \mathbb{Z}\ . \end{aligned}$$

In the context of this dissertation, it is important that the proposed approach provides an opportunity to collect various data, for example, to identify students' understandings and misunderstandings. The filled-out worksheets are an important source of data. Moreover, students' pair discussions could be recorded for later analysis. In addition, it is possible to use questionnaires. (Tonisson, 2013) and (Tonisson & Lepp, 2015) and Sections 8.3 and 8.4 are based on the data collected during the experiment with the lesson scenario carried out in the course "Elementary mathematics I" at the university. The analyzed data included the work of 26 pairs in 2012 and 38 pairs in 2013.

The lessons where the proposed scenario was used are described in the next section.

### 8.2.2. Two rounds of lessons. Similarities and differences

Section 8.2.2 describes the lessons. Firstly, the course and the participants are introduced, followed by a description of the lessons. The differences between the lessons in 2012 and 2013 are also highlighted.

The lessons were a part of a course in elementary mathematics for first-year university students. The course "Elementary mathematics" is a course for the first-year university students and it involves some repetition of the elements from school mathematics. The participants were mainly computer science students but also several mathematics students and a few students from other curricula. The students had quite diverging skill levels in solving trigonometric equations. As the advanced students were dismissed from the course (on the basis of a preliminary test), the proportion of wrong answers probably increased. The students had very few experiences with CAS but they were experienced in using a computer (according to pre-questionnaires). CAS were not used in other lessons of the course.

The students were paired up and then worksheets with trigonometric equations and questions were distributed. Initially, the students solved an equation (correctly or not) without a CAS and then with a particular CAS. As different systems can present answers in different ways, a particular CAS was arranged to initiate an 'intrigue' in order to obtain information about the effect of different representations. The systems used were Maxima, Wiris, and WolframAlpha in the lessons of 2012 and only WolframAlpha in the lessons of 2013. The worksheets guided the students to analyze the differences, equivalence and correctness of their own answers and CAS answers (see Appendix C). Their discussions were audio-taped in order to obtain a deeper overview beyond the notes on paper. The audio-tapes were actually used in this dissertation only for understanding questionable places of the worksheets.

The lesson in question was taught by the author (who was not the regular teacher of the course). The lesson lasted for 90 minutes and consisted of an introduction (pre-questionnaire, formation of pairs), a period of equation solving (ca 70 minutes), and closing (saving data, post-questionnaire (Appendix B)). (See also Table 29.) The introduction gave an overview of the lesson, the aims of the study, etc. The computer algebra systems were not specially introduced but the students were warned that the answers of a CAS could differ from human answers

| Minutes | Periods | Activities |
|---|---|---|
| 10 | Introduction | pre-questionnaire, formation of pairs |
| 70 | Solving | distribution of worksheets,<br>for each equation:<br>solve without CAS,<br>solve with CAS,<br>analyze differences, equivalence, correctness |
| 10 | Closing | Saving data, post-questionnaire |

**Table 29.** Lesson scenario

| | **Lessons of 2012 & Lessons of 2013** |
|---|---|
| Topic of the lesson | Solving of trigonometric equations |
| Course | Elementary mathematics I |
| Students | Advanced students were dismissed |
| Teacher | Author (was not the regular teacher) |
| Data | Worksheet, audiotapes, questionnaires |

**Table 30.** Similarities between the lessons of 2012 and 2013

and could also be incorrect. The types of possible differences were not explained. The topic of trigonometric equations was not repeated in the current lesson but the students were equipped with a paper that included the most important trigonometric formulae. The common aspects of the lessons are also presented in Table 30.

In the autumn of 2012 data were collected from 29 pairs. Each pair had a different order of equations and there were 10 equations in total on the worksheets. The students solved fewer equations than the author had hoped and some of the equations had only a few solvers. Three equations were selected for the study presented in Tonisson, 2013 and in Section 8.3. The analysis covered 47 instances of equation-solving from 26 pairs.

The lessons in 2012 seemed to be successful in my opinion, and also in the students' opinions, as shown by a brief post-questionnaire. Moreover, the organization of the lesson seemed to be appropriate for data collection for the most part and it was decided to repeat the experiment again in the autumn of 2013. The differences between the rounds and corresponding studies are also presented in Table 31. In order to get more responses to particular equations, the order of equations was fixed. There were fewer equations (7 from the 10 used in 2012). Seven was enough, only a couple of pairs solved all of them. It was also decided to use only one CAS (WolframAlpha) instead of three in 70 minutes. These changes were made on purpose. The difference between the numbers of students (and therefore pairs) depended on the actual number of students in corresponding years. The study presented in Tonisson & Lepp, 2015 and in Section 8.4 focuses on three equations (112 instances of equation-solving, 38 pairs of students). The latter study also focuses on correctness of CAS and student answers in addition to equivalence/non-equivalence.

| | Lessons 2012 | Lessons 2013 |
|---|---|---|
| Order of equations | prescribed, different for different pairs | prescribed, same for all |
| Total number of equations | 10 | 7 |
| CAS | Maxima, Wiris, WolframAlpha | WolframAlpha |
| Focus of study | equivalence | equivalence, correctness |
| Analyzed equations | 3 | 3 |
| Analyzed instances | 47 | 112 |
| Number of pairs whose work was analyzed | 26 | 38 |
| Paper | (Tonisson, 2013) | (Tonisson & Lepp, 2015) |

**Table 31.** Differences between the lessons of 2012 and 2013

There were no changes in lesson organisation between the lessons in 2012 and 2013. The post-questionnaire was improved and 13 questions were asked (see Table 32).

The filled-out worksheets of student pairs were used as the main source of data. The worksheets were analysed after the lesson.

The worksheets included students' solutions which were evaluated as usually teachers do in case of written solutions. There were also some questions with multiple-choice answers, for example, "How confident are you in the correctness of your answer" and "How unexpected is the CAS answer at the first sight?". The answers to the question "How are your answer and the CAS answer related? (analyse equivalence/non-equivalence, particular solutions/general solutions)" again needed evaluation by the researcher.

In the first study, the data were analysed mainly by the author of the dissertation. Some audio records were used to clarify questionable places on the worksheets. The problematic cases were discussed with colleagues. In the second study, two researchers were involved in analysing. The first researcher, Marina Lepp, the co-author of (Tonisson & Lepp, 2015), analysed the worksheets. After that the second researcher, the author of the dissertation, reviewed the codes using the worksheets and the audio records in some questionable places. A joint decision was made whenever there were problematic cases.

This section described the lessons where the proposed approach — a lesson scenario based on comparative discussion on students' answers and CAS answers in pairs of students — was used. The next section discusses the success of the scenario.

### 8.2.3. Evaluation of the scenario

In Section 8.2.3, an evaluation of the scenario is presented. The evaluation is mainly based on the students' feedback that was collected with a post-questionnaire in the lessons of 2013. A less informative post-questionnaire was used in the lessons of 2012.

The task of comparing their own answers and CAS answers was new for the students. Usually, only the solution of an equation is needed and nothing more. However, the format seemed to be interesting and attractive. Generally, they became accustomed to the style of the lesson and actively discussed the topic of trigonometry throughout the lesson.

After the lesson the students individually filled out the post-questionnaire (see Appendix B). There were 13 statements and the students were asked to circle the numbers that indicated whether they strongly disagreed (1), rather disagreed (2), partially agreed / partially not (3), rather agreed (4) or strongly agreed (5) with the statement.

The statements and mean values are presented in Table 32.

| **No** | **Statement** | **Mean** |
|---|---|---|
| Q1. | I liked solving these tasks in pairs | 3.8 |
| Q2. | The work in pairs went well in our pair | 4.1 |
| Q3. | The personal qualities of my peer were suitable for cooperative work with me | 4.1 |
| Q4. | Discussion with my peer was instructive | 4.0 |
| Q5. | Working in pairs took more time than working alone | 2.7 |
| Q6. | Results of work in pairs were better than they would be working alone | 3.8 |
| Q7. | I got more knowledge and skills thanks to work in pairs than I would get working alone | 3.5 |
| Q8. | I received advice and explanations about the exercises from my peer | 3.9 |
| Q9. | My peer had more knowledge and skills than I | 3.1 |
| Q10. | I played the leading role in the discussion | 2.8 |
| Q11. | The worksheet encouraged discussion | 3.6 |
| Q12. | The tasks on worksheet were suitable for work in pairs | 3.5 |
| Q13. | Both partners should have their own worksheets | 2.5 |

**Table 32.** Post-questionnaire

One of the objectives was to have an approach, which is attractive for students. The feedback from students generally confirmed that the scenario was attractive for them. They rather liked solving these tasks in pairs (Q1), and felt that the work in pairs went well in their pair (Q2). They also felt that the results of work in pairs were better than they would have been working alone (Q6).

The author was the teacher in these lessons and can confirm that the lessons were successful at least in the sense that the students were involved in discussion about mathematics (here trigonometry) for the whole lesson. As the role of the teacher was to go around and observe (and help if necessary), it was possible to follow whether the students stayed on topic. This was also confirmed by a spot check of audio recordings.

Another objective was to support students' conceptual understanding by maintaining a balance between procedural skill and conceptual understanding. A specific measurement of the improvement in students' conceptual understanding was not performed, but at least the students themselves generally felt that the discussion with peers was instructive (Q4), they attained more knowledge and skills thanks to the work in pairs (Q7), and they received advice and explanations about the exercises from their peers (Q8). The balance between procedural skills and conceptual understanding was not measured either, but both procedural skills (such as equation solving with pen and paper) and conceptual understanding (such as the concepts of equation solution and equivalence of trigonometric expressions) were addressed.

The objective *The approach is relatively easy to execute in practice* is somewhat relative. If computers with a CAS are available, only worksheets are necessary. The teacher's role is to walk around in the classroom and offer help if needed. A similar scenario was also used in case of irrational equation with school teachers and it was easy to execute (Hõim, Jukk, Lepp, Pihlap, & Tõnisson, 2015).

One could say conclusively that this scenario promotes student discussion about mathematics throughout the lesson. The students rather approve of such lessons. The actual teachers of the groups agreed that the lessons were successful.

Furthermore, data on students' understandings and misunderstandings were collected, providing the basis for the next sections.

## 8.3. Adequacy of identification of equivalence/non-equivalence

The research question RQ6 is *How can students identify 1) the equivalence and non-equivalence between CAS and their own answers; 2) correctness of CAS and their own answers of trigonometric equations during lessons based on comparative discussions on students' answers and CAS answers in pairs of students?*

Section 8.3 is devoted to the first part of the question about the equivalence and non-equivalence between CAS and students' answers.

The section is based on the lessons of 2012 and Tonisson (2013). The selection of equations is presented in Section 8.3.1. Sections 8.3.2, 8.3.3 and 8.3.4 are devoted to analysing the adequacy of identification of equivalence/nonequivalence in case of the three respective equations.

### 8.3.1. Choice of equations and CAS. Worksheet

The selection of equations is described in Section 8.3.1.

The worksheets with 10 equations (but fewer solutions) of 26 pairs of students were analyzed. The order of solvable equations was prescribed and was different for different pairs. The students first solved an equation (correctly or not) without and then with a particular CAS. The systems used were Maxima, Wiris and WolframAlpha. A specific CAS was prescribed for the equation to attain the expected difference between student answers and the CAS answer. Actually, as students solved the equations themselves, they also made mistakes and the comparison was made between their actual answers and the CAS answers.

Some of the equations were from regular school textbooks, others from books where trigonometry is handled at a somewhat advanced level. Data of more than 100 instances of equation-solving were collected. Three equations

$$\sin(4x+2) = \frac{\sqrt{3}}{2},$$

$$\tan^3 x = \tan x,$$

$$\cos\left(x - \frac{\pi}{6}\right) = 0.5$$

from ten were chosen for deeper analyses in this study (47 instances of equation-solving, 26 pairs of students).

These equations seemed to be more suitable for the topic of *Representations of mathematical knowledge* in the track Mathematical Knowledge Management in the Conferences on Intelligent Computer Mathematics (CICM 2013). The paper (Tonisson, 2013) was a part of the Proceedings of the CICM 2013. The focus of these equations is primarily on different representations of the answers and not so much on extraneous roots, complex domain, etc., (like in case of some other equations).

The first example is the equation $\sin(4x+2) = \frac{\sqrt{3}}{2}$, where the students use the formula

$$x = (-1)^n \arcsin m + n\pi,\ n \in \mathbb{Z}$$

(as taught in Estonian schools) and get the answer like

$$x = (-1)^n \frac{\pi}{12} - \frac{1}{2} + \frac{n\pi}{4},\ n \in \mathbb{Z}.$$

WolframAlpha expresses series separately (see Figure 22). It was relevant in 2012 and 2013. It should be noted that a new version of a CAS can behave differently from a previous version in the case of a particular equation. At the beginning of 2015, WolframAlpha shows only the exact solution by default. The approximate form can be seen by clicking the "Approximate forms" button.

The second example is $\tan^3 x = \tan x$, where students give general solutions, but Wiris gives particular solutions (see Figure 23).

The third example is $\cos\left(x - \frac{\pi}{6}\right) = 0.5$, where Maxima uses its own notation with `union` and `%z` (see Figure 24). The other equations with more specific nuances (extraneous roots, issues of domain, indeterminacy, etc.) are not discussed in this section but are listed for the sake of completeness in Table 33.

The students first had to solve the trigonometric equations by themselves and then with a particular CAS. They were encouraged to analyze differences, equivalence and correctness of their own answers and CAS answers. The worksheet included the following tasks (in the case of the first example):

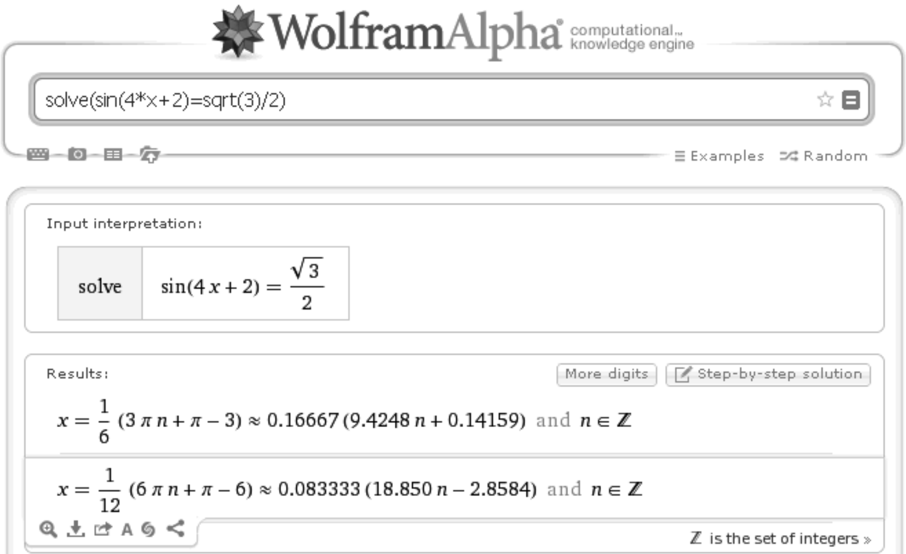


**Figure 22.** $\sin(4x+2)=\dfrac{\sqrt{3}}{2}$ (WolframAlpha)

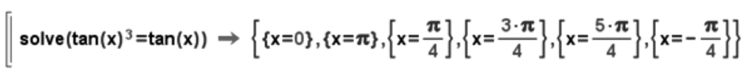


**Figure 23.** $\tan^3 x=\tan x$ (Wiris)

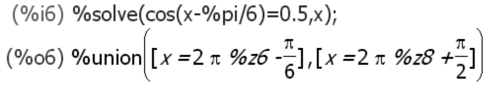


**Figure 24.** $\cos\left(x-\dfrac{\pi}{6}\right)=0.5$ (Maxima)

| No | Equation. CAS | Place of difference |
|---|---|---|
| 1. | $\sin(4x+2)=\frac{\sqrt{3}}{2}$<br>WolframAlpha | Different forms of general solution |
| 2. | $\tan^3 x = \tan x$<br>Wiris | CAS gives<br>particular solutions |
| 3. | $\cos\left(x-\frac{\pi}{6}\right)=0.5$<br>Maxima | Unusual form of arbitrary integer |
| 4. | $2\sin 2x\cos 2x+\cos 2x=0$<br>in $[-30°;0°]$ WolframAlpha | CAS gives general solution |
| 5. | $\frac{\tan^2 x}{\tan x}=0$<br>WolframAlpha | Exceptional solution. Expected mistake<br>also by CAS |
| 6. | $\tan(x+\frac{\pi}{4})=2\cot x-1$<br>WolframAlpha | Possible miss of solution,<br>correct by CAS |
| 7. | $2\cos^2 x+4\cos x=3\sin^2 x$<br>Wiris | CAS gives particular solutions<br>complicated answer |
| 8. | $\sin x-\sin^2 x=1+\cos^2 x$<br>Maxima | Complex numbers in solution |
| 9. | $\frac{1-\cos x}{\sin x}=0$<br>Maxima, Wiris,<br>WolframAlpha | Transfer to $\tan(x/2)$ leads to<br>exceptional solution.<br>Differently by different CAS |
| 10. | $1-\cos x=\sqrt{3}\sin x$<br>WolframAlpha | Squaring leads to exceptional<br>solution |

**Table 33.** Equations, place of difference

1. Solve the equation $\sin(4x+2) = \frac{\sqrt{3}}{2}$ (without the computer at first).
2. How confident are you in the correctness of your answer?
3. Solve the equation with the CAS WolframAlpha using the solve command.
4. How unexpected is the CAS answer at first view?
5. Analyze the accordance of your answer with the CAS answer! If you want to complement/correct your solution, please use the green pen.
6. What are the differences between your answer and the CAS answer?
7. How are your answer and the CAS answer related (analyze equivalence/non-equivalence, particular solutions/general solutions)?
8. Rate the correctness of your (possibly corrected) answer.
9. Rate the correctness of the CAS answer.

The student worksheets were evaluated and analyzed and the results from the three examples are presented in the following subsections.

The analysis of each equation begins with a brief introduction of the example, including reasons for selecting the example, a possible school answer, and a snapshot of the CAS answer. Next, the equivalence/non-equivalence of the students' answers with the CAS answers is discussed. It is based on mathematical reasoning by the author (denoted by the word *Mathematically* in the tables). The second dimension is the students' opinion about the equivalence/non-equivalence that is based on an analysis of paper and audio data (denoted by the words *In students' opinion* in the tables). The discussion concludes with some pedagogical comments.

### 8.3.2. The first equation. Different forms of general solution

Section 8.3.2 focuses on the equations where the CAS answer is particularly unexpected for those who use the $(-1)^n$ formula for solutions of $\sin x = m$ (as is common for Estonian students). The expected Estonian school answer for the equation

$$\sin(4x+2) = \frac{\sqrt{3}}{2}$$

is

$$x = (-1)^n \frac{\pi}{12} - \frac{1}{2} + \frac{n\pi}{4},\ n \in \mathbb{Z}.$$

WolframAlpha gives two series of solutions (see Figure 22). The answers are actually equivalent. The students did not receive any specific information about the CAS answer.

As our textbooks and teachers use mainly the $(-1)^n$ form, the students' answers and the CAS answer seemed quite different at least for this reason. (Twelve pairs (of 17) used $(-1)^n$ form and 4 gave the particular solution. One pair initially gave the particular solution and then corrected it to the $(-1)^n$ form.)

As several pairs made mistakes, the count included 11 cases (of 17) of equivalence with the CAS answer and 6 cases of non-equivalence. Four pairs (of the equivalent cases) used both degrees and radians in the same answer, for example:

$$x = (-1)^n 15^\circ - \frac{1}{2} + 45^\circ \cdot n,\ n \in \mathbb{Z}.$$

| | Equivalent in students' opinion | Non-equivalent in students' opinion | Abstruse | |
|---|---|---|---|---|
| Mathematically equivalent | 4 | 5 | 2 | 11 |
| Mathematically non-equivalent | 3 | 3 | | 6 |
| | 7 | 8 | 2 | |

**Table 34.** $\sin(4x+2) = \frac{\sqrt{3}}{2}$. Equivalence/non-equivalence

Our main focus in the study is to observe how students compare their own and CAS answers. In many cases, their opinion about the equivalence is ascertainable, sometimes not. The results are presented in Table 34.

The depth of discussions about the comparison varied between the student pairs. For example, 3 pairs identified actual equivalence through reasonable discussion, while one pair simply presumed it. There were also 3 pairs whose answer was not equivalent with the CAS answer, but they counted them as equivalent without any real discussion. Seven pairs did not recognize that the answers were equivalent (5 pairs considered as non-equivalent and 2 opinions were abstruse). Mainly, they did not grasp that $n$ in their answer and $n$ in the CAS answer (see Figure 22) was not the same. This points to an automated (and correct) habit of solving the algorithm of trigonometric equation without exhaustive understanding of the solution. Three pairs identified the non-equivalence of their answer and the CAS answer. Their answers were remarkably different from the CAS answer.

It seems that the different representations of the same answer, like in this example, could initiate instructive discussion. It could also point to a possible superficial treatment of the fairly important issue of the meaning of $n$. A simpler equation, like $\sin 4x = \frac{\sqrt{3}}{2}$, could probably be a more straightforward means for clarifying the phenomenon. The example is suitable if the students use the $(-1)^n$ formula. This is also an issue of different traditions. For example, it is usual to find solutions, such as

$$x = (-1)^n \frac{\pi}{6} + n\pi,\ n \in \mathbb{Z}$$

(being the solution of $\sin x = \frac{1}{2}$), in the textbooks of some countries, like in Estonia, but this is not the case in many others.

### 8.3.3. The second equation. CAS gives only particular solutions

The situation where a CAS gives only particular solutions but students are asked to present general solutions is discussed in Section 8.3.3.

The students should frame the CAS solutions up to their own general solutions. In case of the equation

$$\tan^3 x = \tan x$$

the human answer could be

$$x = n\pi, n \in \mathbb{Z}$$
$$x = \pm\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$

or

$$x = n\pi, n \in \mathbb{Z}$$
$$x = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$
$$x = -\frac{\pi}{4} + n\pi, n \in \mathbb{Z} \quad .$$

Wiris gives the particular solutions 0, $\pi$, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $-\frac{\pi}{4}$ (see Figure 23).

Certainly, $\{n\pi, n \in \mathbb{Z}\}$ and $\{0; \pi\}$ are not equivalent in the usual mathematical sense. These answers are counted as equivalent in the sense that all series are represented by 2 instances. The order of solutions in the output of Wiris is quite confusing as the instances of the series of solutions are not always side by side (for example, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are not from same 'club'). The students did not receive any specific information about the CAS answer. Many student pairs (9 of 14) gave the right answer and they also figured out (after more or less effort and discussion) the relationship between their and CAS answer (see Table 35). One pair could not frame $\pi$, $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$ up to their right answer. Again, the meaning of $n$ in the formula seemed to be incoherent for them. The cases where students omitted some solutions were very interesting. One such pair corrected their mistake and finally found the right answer. They added to

$$\pi + \pi n$$
$$\frac{\pi}{4} + \pi n$$

missing

$$-\frac{\pi}{4} + \pi n \quad .$$

Emotions are not focused in this study but their joy after the correction was remarkable. The other pair (initially only $n\pi$ solution) had a member who had already diagnosed their mistake. The third pair did not analyze the CAS solutions thoroughly enough and did not notice that their answer was incomplete. It is impossible to give a thorough overview of the discussion of the pair that got an incomplete answer and also considered it as non-equivalent with the CAS answer, as their discussion was very laconic. It seems that the representation of the answer

| | Equivalent in students' opinion | Non-equivalent in students' opinion | |
|---|---|---|---|
| Mathematically equivalent | 9 | 1 | 10 |
| Mathematically non-equivalent | 2 | 1 | 3 |
| Non-equivalent → Equivalent | 1 | | 1 |
| | 12 | 2 | |

**Table 35.** $\tan^3 x = \tan x$. Equivalence/non-equivalence

is generally accomplishable in this case. The possible corrective virtue is also notable. The standard of representation of answers to trigonometric equations could provide more instructive examples, as the choice of a particular solutions is not always as transparent.

### 8.3.4. The third equation. Unusual form of arbitrary integer

The third example, which is related to CAS notation, is analysed in Section 8.3.4.

The CAS answer is actually very similar to a normal human answer but with some CAS-specific peculiarity. The human answer to the equation

$$\cos\left(x - \frac{\pi}{6}\right) = 0.5$$

could be

$$x = -\frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}$$
$$x = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z} \quad .$$

Maxima gives the same answer in a somewhat distinctive way: $2\pi\%z6 - \frac{\pi}{6}$, $2\pi\%z8 + \frac{\pi}{2}$ (see Figure 24). The package `to_poly_solve` is used for solving trigonometric equations as suggested in the Maxima manual. The Maxima manual is cited for clarity: *Especially for trigonometric equations, the solver sometimes needs to introduce an arbitrary integer. These arbitrary integers have the form* `%zXXX`*, where* `XXX` *is an integer* (*Maxima manual*, 2015). Use of separate arbitrary integers is even better way.

The meaning of `%z` was also an important issue for solving the equation with Maxima. The students did not receive any specific information about the CAS answer, but they had additional brief paper manuals (3 pages) on using different CAS where `%z` was explained. Only two pairs found the info about `%z` in this manual. Almost all pairs mentioned `%z` as a significant difference from their own answer. An explanation was given if the students asked for it. Nevertheless, two pairs remained confused and could not understand the CAS answer. The meaning of such a notation could be more clearly indicated in the CAS user-interface. For example, tooltips could be used. Eight pairs (of 16) got the right answer (see Table 36). Five of these pairs quite easily found the CAS answer to be equivalent. Three pairs had an answer equivalent with the CAS answer but their opinion about equivalence was abstruse. One of these pairs could not understand the CAS answer because of `%z`. The second pair did not observe the CAS answer sufficiently and did not notice the relation between the CAS answer and their own (not fully simplified) answer. The third pair's discussion was too laconic. One pair corrected their mistake and finally found the right answer, from

$$\begin{gathered} \dots \\ x - 30^\circ = \arccos\frac{1}{2} + 2\pi n \\ \dots \end{gathered}$$

to

$$\begin{gathered} \dots \\ x - 30^\circ = \pm\arccos\frac{1}{2} + 2\pi n \\ \dots\ . \end{gathered}$$

Three pairs saw equivalence that really did not exist. There were also four pairs who considered their wrong answers as non-equivalent with the CAS answer. One of these pairs could not understand the meaning of `%z` correctly. Two pairs tried to find their mistakes; one pair had evidently a different answer.

| | Equivalent in students' opinion | Non-equivalent in students' opinion | Abstruse | |
|---|---|---|---|---|
| Mathematically equivalent | 5 | | 3 | 8 |
| Mathematically non-equivalent | 3 | 4 | | 7 |
| Non-equivalent → Equivalent | 1 | | | 1 |
| | 9 | 4 | 3 | |

**Table 36.** $\cos\left(x - \frac{\pi}{6}\right) = 0.5$. Equivalence/non-equivalence

It seems that the different notation can cause major trouble for some people, while it can be easily acceptable for others. It should be mentioned that the students used Maxima for the first time and many issues would probably be resolved by further use.

The conclusions are presented in Section 8.5 after the section based on the lessons of 2013.

## 8.4. Equivalence and correctness

Section 8.4 is devoted to both parts of RQ6: *How can students identify 1) the equivalence and non-equivalence between CAS and their own answers; 2) correctness of CAS and their own answers of trigonometric equations during lessons based on comparative discussions on students' answers and CAS answers in pairs of students?*

The section is based on the lessons (data collected from 38 student pairs) of 2013 described in the paper of Tonisson and Lepp (2015). Like in the experiment described in Section 8.3, the lessons were based on comparative discussion on students' answers and computer algebra system answers in pairs of students (see Section 8.2, Table 29). Unlike the lessons of 2012, the order of equations was fixed and same for all pairs, and all equations were solved with WolframAlpha.

This section includes three parts. Section 8.4.1 explains the selection of equations and Section 8.4.2 describes data collection and analysis. The results are presented in Section 8.4.3.

### 8.4.1. Choice of equations

The selection of equations is described in Section 8.4.1. Three selected equations are presented in detail.

Data of more than 200 instances of equation-solving were collected in the lessons of 2013. Three equations ($\sin(4x+2) = \frac{\sqrt{3}}{2}$, $2\sin 2x \cos 2x + \cos 2x = 0$, $\frac{\tan^2 x}{\tan x} = 0$) from seven were chosen for deeper analyses in this study (112 instances of equation-solving, 38 pairs of students), as they were solved by almost all pairs. (These seven equations are equations 1, 2, 4, 5, 6, 7, 10 in Table 33, which lists the equations used in the lessons of 2012. As it was decided to use only one CAS, the Maxima related examples were excluded.)

The first equation is

$$\sin(4x+2) = \frac{\sqrt{3}}{2}.$$

The expected school answer in our region is

$$x = (-1)^n \frac{\pi}{12} - \frac{1}{2} + \frac{n\pi}{4},\ n \in \mathbb{Z}$$

$$x = \pi\left(n - \frac{1}{4}\right) \approx 3.1416\,(n - 0.25000) \text{ and } n \in \mathbb{Z}$$

$$x = \pi\left(n + \frac{1}{4}\right) \approx 3.1416\,(n + 0.25000) \text{ and } n \in \mathbb{Z}$$

$$x = \pi\left(n - \frac{1}{12}\right) \approx 3.1416\,(n - 0.083333) \text{ and } n \in \mathbb{Z}$$

$$x = \pi\left(n + \frac{7}{12}\right) \approx 3.1416\,(n + 0.58333) \text{ and } n \in \mathbb{Z}$$

**Figure 25.** The answer to the second equation $2\sin 2x\cos 2x + \cos 2x = 0$ (WolframAlpha)

or

$$x = (-1)^n 15^\circ - \frac{1}{2} + n\cdot 45^\circ,\ n \in \mathbb{Z}.$$

It is usual in Estonian schools that, if not stated otherwise, the general solution is needed. Solution provided by WolframAlpha is $\frac{1}{6}(3\pi n + \pi - 3)$, $\frac{1}{12}(6\pi n + \pi - 6)$, $n \in \mathbb{Z}$ (shown in Figure 22).

The second task was to find the solutions of the equation $2\sin 2x\cos 2x + \cos 2x = 0$ in the interval $[-30^\circ; 0]$. The school answer is $-15^\circ$ or $-\pi/12$. The students' worksheets included an example of solving the equation without particular marking of the interval in the students' worksheet. The solution to the equation $2\sin 2x\cos 2x + \cos 2x = 0$ provided by WolframAlpha is shown in Figure 25.

This highlights the question of particular and general solution. WolframAlpha gives a general solution, which is again quite different than probably expected by the students. In case of our usual formulae the general solution is

$$x = \pm\frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$$

and

$$x = (-1)^{n+1}\frac{\pi}{12} + \frac{n\pi}{2},\ n \in \mathbb{Z}.$$

The third equation $\frac{\tan^2 x}{\tan x} = 0$ is interesting because WolframAlpha gives principally different answer than could be expected in the school context. The normal school answer is that there are no solutions. Nevertheless, it is quite natural to propose that $n\pi$ is the solution. And the CAS confirms this answer.

The selection of equations was described in this section. The next section is devoted to data collection and analysis.

### 8.4.2. Data collection and analysis

Section 8.4.2 describes data collection and analysis by worksheets.

Figure 26 shows the part of the worksheet with the students' answers. The assigned task is "Solve the equation $\sin(4x+2) = \frac{\sqrt{3}}{2}$ (without the computer at first!)." The first part of the solution is written before solving with WolframAlpha. Question 1.1 is "How confident are you in the correctness of your answer?" (very confident / quite confident / do not know / quite unsure / very uncertain). These students marked "very confident". The next task is "Solve the equation with the CAS WolframAlpha using the solve command". Question 1.2 is "How unexpected is the CAS answer at the first sight?" (very unexpected / quite unexpected / do not know / quite expected / very expected). These students chose "quite unexpected".

The second part of the solution is written with a green pen after solving the equation with WolframAlpha.

On the next page, the students had to answer the question: "How are your answer and the CAS answer related? (analyze equivalence/non-equivalence, particular solutions/general solutions)". These students wrote: "After we had seen the system's answer, we understood that our solution was not correct and we solved the task again".

After the lesson the worksheets and questionnaires were analysed. Two researchers were involved in this. First, one researcher analysed the worksheets. After that the second researcher reviewed using worksheets and audio records in some questionable places. A joint decision was made in all problematic cases.

Although the equations were different and prevalent mistakes were quite variable as well, a single classification is used. At first, the students' manual solutions were classified on the basis of the relation to the correct school answer. It is quite similar to the process that teachers apply when they check students' solutions. However, our emphasis was on finding different types of solutions instead of giving points or marks. Although a quite detailed classification is possible, a more concise version is used here and the particular mistakes are highlighted as examples. Three main categories are distinguish

- the answer is correct,
- the answer has a particular/general solution inaccuracy,
- the answer is incorrect.

The comments for all categories help us to understand the ranges. Actually, even answers with minor shortcomings are accounted as correct, for example, missing $n \in Z$ or using degrees and radians together in the answer. However, accounting answers containing both radians and degrees as correct answers is problematic because, in the next step, some students tend to add them incorrectly. The particular/general solution inaccuracy means that particular solutions are expressed when actually the general solution is expected or vice versa. It is important that all series are represented, e.g., in case of $\sin x \cdot \cos x = 0$, answers of both $\sin x = 0$ and $\cos x = 0$ are necessary.

Incorrect answers are the largest category of students' manual solutions. Different subtypes can be distinguished in this case. For example, degrees and radians are calculated as same units or solution process is unfinished. (However, a solution where $x$ is already expressed and only a small step remains to be done was

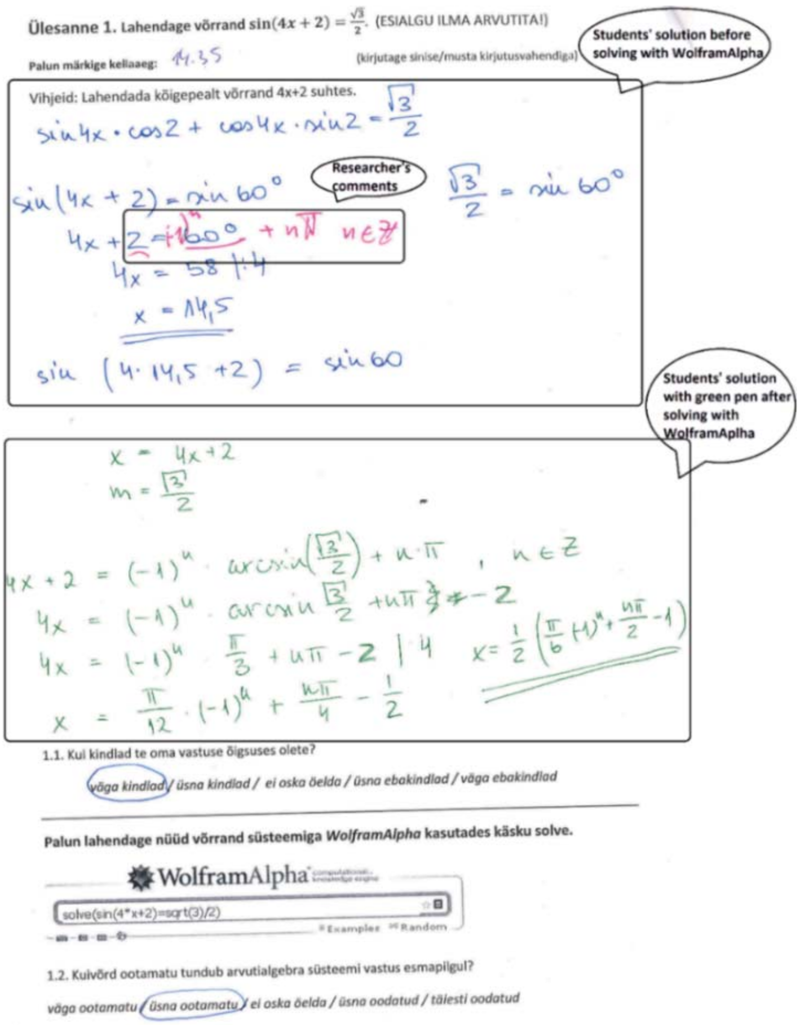


**Figure 26.** The part of the worksheet with the students' solution

still counted as correct.) The following types were used in coding:

- degrees and radians are incorrectly added,
- discontinued,
- mathematical detail,
- mathematical principle (e.g., permission of 0/0),
- trigonometric detail (e.g., missed arcsin),
- trigonometric principle (e.g., problems in application of formula),
- multiple mistakes.

It should be noted that the answer is focused rather than the use of all necessary solution steps. There are some cases (in the second equation, $2\sin 2x\cos 2x + \cos 2x = 0$) where the right answer is expressed but the solution process is not completely correct.

As the CAS answer is not necessarily the same (or even equivalent) as the school answer, it is also reasonable to classify the students' answers in relation to the CAS answer. Here, a similar schema is used. Three main equivalence categories are used:

- the students' answer is equivalent to the CAS answer,
- the students' answer and the CAS answer have a particular/general solution difference,
- the students' answer is not equivalent to the CAS answer.

Again, minor differences in nuances were allowed when counting the answers as equivalent. We did not care about missing $n \in \mathbb{Z}$. Degrees and radians in the same expression certainly maintain equivalence. The answer belongs to the second category if there are only particular solutions in the students' answer but general solution in the CAS answer or vice versa. There are different reasons to count the students' answer and the CAS answer as non-equivalent. For example, a series is missing or is incorrect.

The students had to specify the differences and relations between their answer and the CAS answer. As the equations were quite different and the differences and relations varied as well, we used different schemas for analyzing the differences and relations. The most important trends are expressed in the dissertation. The students were encouraged to complement/correct their solution after obtaining the CAS answer. The possibility was not used very often but there were some cases. Nevertheless, the cases where the CAS answer gives students ideas for changing their original solution are particularly interesting.

This section described how data were collected and analysed. The result is presented in the next section.

### 8.4.3. Results

This section describes the results of the experiment of 2013. The first part looks at how well the students solved the equations, followed by an overview of how

| | Correct | Particular/General | Incorrect |
|---|---|---|---|
| $\sin(4x+2)=\frac{\sqrt{3}}{2}$ | 12 | 5 | 21 |
| $2\sin 2x\cos 2x+\cos 2x=0$ | 19 | 3 | 15 |
| $\frac{\tan^2 x}{\tan x}=0$ | 19 | | 18 |

**Table 37.** How well the students solved

the students perceived the expected CAS answers. The main part of the section is presented with the help of the following questions: *What differences do students notice foremost? How do the students understand correctness of the answers? Are students able to ascertain equivalence/non-equivalence? How do the students explain equivalence/non-equivalence? Are there any differences in this regard between different types of equations and answers?*

First, we consider how well the students solved the equations. The first equation $\sin(4x+2)=\frac{\sqrt{3}}{2}$ was quite complicated for the students. Only 12 pairs from 38 solved it correctly while 21 pairs solved it incorrectly. Five pairs provided a particular solution while a general solution was needed. The most common mistakes were connected to measure units radians and degrees. For example, the students got the correct answer $x=15^\circ+45^\circ n-1/2$, where 15 and 45 are in degrees and $1/2$ is in radians and then they solved further and got $x=14.5^\circ+45^\circ\cdot n$.

The second equation $2\sin 2x\cos 2x+\cos 2x=0$, was solved with a little more success than the first one. The correct answer was find by 19 pairs from 38. Fifteen pairs solved the equation incorrectly (the solution was produced with mistakes or only a half-solution was presented). Three pairs provided the general solution while the particular solution was required. Eleven pairs made some changes (with green pen) after solving the same equation with WolframAlpha and one of them found the correct answer (replacing the previous incorrect answer).

The third equation $\frac{\tan^2 x}{\tan x}=0$ was very interesting as it has no solutions, but it is possible to make a mistake by reducing it and get a solution. This is what WolframAlpha does. The students produced both correct and wrong answers. Nineteen pairs gave the right answer that there is no solutions and 18 pairs solved the equation incorrectly and arrived at a solution. They did not take into account the contradiction and simplified the equation $\frac{\tan^2 x}{\tan x}=0$ in a similar way to WolframAlpha. They got the equation $\tan x=0$ and solved it and got the solution $x=n\pi,\ n\in\mathbb{Z}$. The respective figures are also presented in Table 37.

The WolframAlpha answer to the first equation $\sin(4x+2)=\frac{\sqrt{3}}{2}$ was quite unexpected. The WolframAlpha answer was found to be very (8 pairs) or quite (14 pairs) unexpected by 22 pairs in their worksheets. After using WolframAlpha for solving the first equation the students could guess that the CAS produces solutions according to different formulae. The answer of the second equation

| | Very or quite unexpected | Very or quite expected | Do not know or not answered |
|---|---|---|---|
| $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 22 | 6 | 10 |
| $2\sin 2x\cos 2x + \cos 2x = 0$ | 12 | 18 | 7 |
| $\frac{\tan^2 x}{\tan x} = 0$ | 18 | 15 | 4 |

**Table 38.** Expectness

$2\sin 2x\cos 2x + \cos 2x = 0$ was not as unexpected as the first one. While, in relation to the first equation, 12 pairs marked that the CAS answer was very (2 pairs) or quite (10 pairs) unexpected, this had now risen to 18 pairs for whom this answer was quite (12 pairs) or very (6 pairs) unexpected. The WolframAlpha answer to the third equation $\frac{\tan^2 x}{\tan x} = 0$ was unexpected for 18 pairs and 15 of them had the correct answer in the worksheet. 15 pairs marked the CAS answer as expected and 13 of them had the wrong answer. (See also Table 38.)

The following part of the section is structured with the help of the questions listed in the beginning of Section 8.4.3.

*What differences do the students notice foremost?* The worksheet included two questions about the differences and relations between the students' answer and the CAS answer (What are the differences between your answer and the CAS answer? How are your answer and the CAS answer related (analyze equivalence/non-equivalence, particular solutions/general solutions)?). As the answers to the first equation are quite different in appearance, it would be natural to investigate the CAS answer and try to derive one from the other. However, it seems that the students were not keen on deeper exploration. The students (27 pairs) noticed the important differences between answers (like radians/degrees, one/two answers, particular/general solution), but they did not provide any analysis on the relations between the answers. An analysis of the relation between the students' answer and the CAS answer was almost missing in case of 23 pairs and completely missing in case of 8 pairs. Only 2 pairs provided a sufficient analysis of the answers. Five pairs presented an insufficient description of the relation between the answers.

The descriptions of the differences and relations in case of the second equation, $2\sin 2x\cos 2x + \cos 2x = 0$, were not sufficient and we combine them here. Only 2 pairs checked whether their answer really matched the CAS answer. Many pairs (18) indicated that their solution is in concrete interval but the CAS solution is not. Several pairs (3) mentioned that CAS has more solutions. One pair noticed the relation between the general and particular solution. The other 14 pairs did not write any reasonable descriptions.

In case of the third equation $\frac{\tan^2 x}{\tan x} = 0$ the differences and relations between the students' and the CAS answers were divided mainly into two groups (with some minor exceptions). If the answer was the same as the CAS answer, the

| | Students' opinion about correctness | | | | |
|---|---|---|---|---|---|
| | correct | partially correct | incorrect | do not know | |
| Math. correct | 6 | 4 | 2 | | 12 |
| Math. incorrect | 6 | 1 | 5 | 4 | 16 |
| Math. particular/general | 2 | 3 | | | 5 |
| Math. incorrect → correct | 3 | | 1 | | 4 |
| | 17 | 8 | 8 | 4 | |

**Table 39.** $\sin(4x+2) = \sqrt{3}/2$. Correctness of students' answer / students' opinion about correctness of their answer (after possible correction)

| | Students' opinion about correctness | | | | |
|---|---|---|---|---|---|
| | correct | partially correct | incorrect | do not know | |
| Math. correct | 17 | 1 | 1 | | 19 |
| Math. incorrect | 2 | 9 | 2 | 1 | 14 |
| Math. particular/general | 1 | 1 | | 1 | 3 |
| Math. incorrect → correct | 1 | | | | 1 |
| | 21 | 11 | 3 | 2 | |

**Table 40.** $2\sin 2x\cos 2x + \cos 2x = 0$. Correctness of students' answer / students' opinion about correctness of their answer

differences were not detected (12 pairs). If the students answered that the equation does not have solutions, they (19 pairs) also noticed important differences (no solutions / some solution). The same holds for the description of relations. The pairs with the same answers as the CAS answer (14 pairs) did not have to find the relations between answers, they just marked that the answers are the same. It was not difficult to describe the relations between the answers (CAS simplifies and does not take into account the contradiction) for those pairs who had the correct answer (17 pairs).

*How do the students understand correctness of the answers?* It was somewhat complicated, as they had not been given the correct answer according to school math. Many students seemed to trust CAS while others trusted themselves. It seems that it depends on students' confidence in their answer whether they trust the system or themselves. Furthermore, correctness can sometimes be complicated to evaluate, for example, in case of particular and general solution.

The students' opinions about correctness of their answer were adequate, very roughly, in half of cases. It is interesting to note that, even after seeing the correct answer to the first equation, $\sin(4x+2) = \frac{\sqrt{3}}{2}$, produced by WolframAlpha, six pairs marked their wrong answer as correct.

It is possible that the CAS answer was so different that it did not give any clue about possible mistakes. The WolframAlpha answer was so confusing for some students that they marked their right answer as wrong (2 pairs). Twelve pairs made

| | Students' opinion about correctness | | | | |
|---|---|---|---|---|---|
| | correct | partially correct | incorrect | do not know | |
| Math. correct | 15 | 1 | | 1 | 17 |
| Math. incorrect | 12 | 3 | | 1 | 16 |
| Math. correct → incorrect | | 1 | 1 | | 2 |
| Math. incorrect → CAS like incorrect | 1 | | | | 1 |
| | 28 | 5 | 1 | 2 | |

**Table 41.** $\frac{\tan^2 x}{\tan x} = 0$. Correctness of students' answer / students' opinion about correctness of their answer

some changes with the green pen and 4 of them got the correct answer. (See also Table 39.)

As the second equation, $2\sin 2x\cos 2x + \cos 2x = 0$, required a particular solution and WolframAlpha provided only a general solution, most pairs marked their answers as correct or partially correct. Even the pairs with wrong answers chose the option "correct answer" (2 pairs) or "partially correct" (9 pairs).

These students probably did not bother to investigate the CAS answer deeply enough to discover that they had an incorrect solution. The WolframAlpha answer was quite different (4 series) and this may have been the reason why 1 pair marked their correct answer as incorrect. (See also Table 40.)

The larger majority of both groups (groups with the correct answer and groups with a wrong answer to the third equation $\frac{\tan^2 x}{\tan x} = 0$) had marked their answer as correct on the worksheet after seeing the WolframAlpha answer. Those pairs who had the correct answer understood how WolframAlpha got the wrong answer and that their answer is correct and the CAS answer is not. Only one pair marked their right answer as wrong after seeing the WolframAlpha answer. They probably believed in CAS more than in themselves. The pairs with an incorrect answer saw the WolframAlpha (wrong) answer and were very happy, as the answers were exactly the same. Of course, they marked their answer as correct. Four pairs made some changes with the green pen after solving the equation with WolframAlpha. Two of them changed their correct answer to incorrect answer (they got the same answer as the WolframAlpha). One pair had a completely wrong answer, which was not equivalent to WolframAlpha answer. After seeing the CAS answer they corrected the incorrect answer to another incorrect answer, but this one was equivalent to the CAS answer. (See also Table 41.)

The opinions about correctness of the WolframAlpha answer were quite different. The students had to make different choices for different equations. The CAS answer of the first equation, $\sin(4x+2) = \frac{\sqrt{3}}{2}$, was correct. The correctness of the solutions in case of this equation was identified with the highest degree of accuracy (71%). However, as the form of the WolframAlpha answer of the first

| | Students' opinion about correctness of CAS | | | | |
|---|---|---|---|---|---|
| | correct | partially correct | incorrect | do not know | |
| Math. correct | 8 | | | 4 | 12 |
| Math. incorrect | 10 | | | 7 | 17 |
| Math. particular/general | 5 | | | | 5 |
| Math. incorrect → correct | 4 | | | | 4 |
| | 27 | | | 11 | |

**Table 42.** $\sin(4x+2) = \sqrt{3}/2$. Correctness of students' answer / students' opinion about correctness of CAS answer

| | Students' opinion about correctness of CAS | | | | |
|---|---|---|---|---|---|
| | correct | partially correct | incorrect | do not know | |
| Math. correct | 11 | 5 | | 3 | 19 |
| Math. incorrect | 10 | 1 | 1 | 2 | 14 |
| Math. particular/general | 1 | 1 | | 1 | 3 |
| Math. incorrect → correct | 1 | | | | 1 |
| | 23 | 7 | 1 | 6 | |

**Table 43.** $2\sin 2x \cos 2x + \cos 2x = 0$. Correctness of students' answer / students' opinion about correctness of CAS answer

| | Students' opinion about correctness of CAS | | | | |
|---|---|---|---|---|---|
| | correct | partially correct | incorrect | do not know | |
| Math. correct | 3 | | 12 | 4 | 19 |
| Math. incorrect | 13 | 1 | | | 14 |
| Math. correct → incorrect | 1 | | | 1 | 2 |
| Math. incorrect → CAS like incorrect | 1 | | | | 1 |
| | 18 | 1 | 12 | 5 | |

**Table 44.** $\frac{\tan^2 x}{\tan x} = 0$. Correctness of students' answer / students' opinion about correctness of CAS answer

equation is quite complicated, it is likely that most of the students only believed that the CAS answer was correct (perhaps it shows that students always trust the system even if they do not understand the answer). (See also Table 42.)

We count the answer of the second equation, $2\sin 2x\cos 2x + \cos 2x = 0$, produced by WolframAlpha as partially correct (because a partial solution was needed, but CAS gave a general solution). The quite low rate (19%) of adequate identifications of correctness in the case of the second equation could be explained by the fact that many pairs (62%) counted the general solution as correct. It is another question why they thought that the CAS general solution, which differs from their general solution, is correct. (See also Table 43.) It is likely that, again, they trusted the CAS.

The WolframAlpha answer of the third equation, $\frac{\tan^2 x}{\tan x} = 0$, did not coincide with the correct answer as taught in school. The correct option would have been to select that the WolframAlpha answer is incorrect. It was quite confusing for the students. Twelve pairs (32%) said that the CAS answer is incorrect and all of them got the correct answer in manual calculations. Three pairs with the right answer had marked that the CAS answer as correct. Of course, those students who had the incorrect solution (same as CAS) thought that the CAS answer was correct. (See also Table 44.)

*Are students able to ascertain equivalence/non-equivalence?* There seem to be different 'hindrances' to identification of equivalence/non-equivalence in case of different equations. The form of the WolframAlpha answer of the first equation, $\sin(4x+2) = \frac{\sqrt{3}}{2}$, was so different and unexpected that the students possibly made simply a guess about equivalence (only 5 pairs found that their answer was equivalent with the CAS answer (when it actually was equivalent)). Students with an equivalent answer to the CAS answer said that the answers are not equivalent and vice versa (4 pairs marked that answers are not equivalent and 3 pairs selected the option "do not know" when the answers were equivalent; 4 pairs decided that the answers are equivalent while the students' answer and the CAS answer were not equivalent). The issue of equivalence was not explored in great depth on the worksheets. (See also Table 45.)

The small percentage (13%) of adequate identifications in case of the second equation was caused by the task (find the particular solution). The choices offered to the students were only that answers are equivalent or not equivalent. Maybe if the worksheet had the choice "general solutions are equivalent", the number of adequate identifications would have been higher. As it was, adequate identification was possible only if the students presented a general solution which was equivalent to the WolframAlpha answer (and they said that the answers are equivalent, 1 pair) or the students gave a wrong general solution and marked the answers as non-equivalent (4 pairs). Most of the students chose the equivalence (16 pairs). This is not a wrong choice because general solutions were equivalent in 12 cases. 10 pairs marked the non-equivalence. Again this is not a wrong choice

| | Students' opinion about equivalence | | | |
|---|---|---|---|---|
| | equivalent | non-equivalent | do not know | |
| Math. equivalent | 5 | 4 | 3 | 12 |
| Math. non-equivalent | 4 | 9 | 4 | 17 |
| Math. particular/general | 2 | 2 | 1 | 5 |
| Math. non-equivalent $\rightarrow$ equivalent | 1 | 2 | 1 | 4 |
| | 12 | 17 | 9 | |

**Table 45.** $\sin(4x+2) = \sqrt{3}/2$. Equivalence of students' and CAS answer / students' opinion about equivalence of their and CAS answer

| | Students' opinion about equivalence | | | |
|---|---|---|---|---|
| | equivalent | non-equivalent | do not know | |
| Math. equivalent | 1 | | 2 | 3 |
| Math. non-equivalent | 4 | 4 | 6 | 14 |
| Math. particular/general | 10 | 6 | 3 | 19 |
| Math. non-equivalent $\rightarrow$ particular/general | 1 | | | 1 |
| | 16 | 10 | 11 | |

**Table 46.** $2\sin 2x\cos 2x + \cos 2x = 0$. Equivalence of students' and CAS answer / students' opinion about equivalence of their and CAS answer

as the particular answer ($-\pi/12$) and the general answer ($x = \pm\frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$ and $x = (-1)^{n+1}\frac{\pi}{12} + \frac{n\pi}{2},\ n \in \mathbb{Z}$) are not equivalent. One pair revised their wrong solution and improved it (finding the correct answer), but this answer is not equivalent to the CAS answer (because it is a particular solution). (See also Table 46.)

The high percentage (84%) of adequate identifications in case of the third equation, $\frac{\tan^2 x}{\tan x} = 0$, was caused by the task as well. Fifteen pairs had exactly the same wrong answer as WolframAlpha (13 pairs had wrong solution and 2 pairs had the

| | Students' opinion about equivalence | | | |
|---|---|---|---|---|
| | equivalent | non-equivalent | do not know | |
| Math. equivalent | 13 | | | 13 |
| Math. non-equivalent | 1 | 17 | 1 | 19 |
| Math. particular/general | | 1 | | 1 |
| Math. non-equivalent $\rightarrow$ equivalent | 2 | 1 | | 3 |
| | 16 | 19 | 1 | |

**Table 47.** $\frac{\tan^2 x}{\tan x} = 0$. Equivalence of students' and CAS answer / students' opinion about equivalence of their and CAS answer

right solution and corrected it to incorrect with the green pen). Of course, all of them marked the answers as equivalent. Nineteen pairs had the correct answer. The identification of equivalence/non-equivalence of their answer and the CAS answer was quite easy but still some pairs did not identify it properly (2 pairs). (See also Table 47.)

To conclude, the identification of equivalence/non-equivalence depends very much on the equation and the task. If the students' answer and the CAS answer looked rather different, then the students did not know how to determine equivalence/non-equivalence and quite often did it incorrectly. Equivalence/non-equivalence of particular and general solutions is very questionable. However, in case of some equations, where the CAS answer is the same as a possible student answer or very different (like some solution/ no solutions), it can be done very well.

*How do the students explain equivalence/non-equivalence?* If the students' answer and the CAS answer looked quite different, then the students usually did not explain equivalence/non-equivalence at all. It is maybe one of the important messages for improving the worksheet. A thorough comparison of different answers would be very instructive, for example, for understanding general solutions properly, including the role of the $n$. It is likely that more detailed subtasks would be useful. It could also be a good idea to use equations where the answer is more similar to the school (and hopefully the students') answer.

*Are there any differences in this regard between different types of equations and answers?* There were not enough equations for making extensive conclusions but one could say that the choice of equation (particularly by answer) is very important. It would be useful to evaluate the 'distance' between the CAS answer and school answer (or probable students' answer). For example, the 'distance' seems to be too large in case of the first equation.

As Sections 8.3 and 8.4 are closely connected, the common conclusions are presented in the next section.

## 8.5. Conclusions and suggestions

Section 8.5 consists of three parts. Firstly, common conclusive part of Sections 8.3 and 8.4 is provided in Section 8.5.1. In Section 8.5.2, the issues problematic to students are highlighted. Finally, with conclusion ideas for future work are presented in Section 8.5.3.

### 8.5.1. Adequacy of identification of equivalence/non-equivalence and correctness

Section 8.5.1 continues the search for answers to the research question RQ6: *How can students identify 1) the equivalence and non-equivalence between CAS and their own answers; 2) correctness of CAS and their own answers of trigonometric* *equations during lessons based on comparative discussions on students' answers and CAS answers in pairs of students?*

This section is based on the conclusive parts of the papers Tonisson (2013) and Tonisson and Lepp (2015) but also includes details that were not in the papers. The results on adequacy of identification of equivalence/nonequivalence and correctness were presented mainly by individual equations in Sections 8.3 and 8.4. Here we try to look at the equations together.

However, we start with an overview of how the students took the chance to correct their solution (with the green pen) after seeing the CAS answer. Table 48 includes data about all equations that were under observation in the papers. The table includes columns for: year, equation, the number of solutions, the number of solutions that were not the same as the CAS answer, the number of changed solutions (with the green pen), and the number of successful changes (incorrect to correct). The possible change is natural if the students' answer and CAS answer are different. If we account for all these cases, changes were made in 29% of the solutions (43 of 146).

It should also be noted that, in case of $\frac{\tan^2 x}{\tan x} = 0$, the CAS answer was incorrect. Two pairs changed their correct answer to incorrect answer.

| **Year** | **Equation** | **Solutions** | **Not same** | **Changes** | **Success-ful** |
|---|---|---|---|---|---|
| 2012 | $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 17 | 17 | 6 | 0 |
| 2012 | $\tan^3 x = \tan x$ | 14 | 14 | 5 | 1 |
| 2012 | $\cos\left(x - \frac{\pi}{6}\right) = 0.5$ | 16 | 13 | 4 | 1 |
| 2013 | $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 38 | 38 | 13 | 4 |
| 2013 | $2\sin 2x \cos 2x + \cos 2x = 0$ | 37 | 37 | 11 | 1 |
| 2013 | $\frac{\tan^2 x}{\tan x} = 0$ | 37 | 27 | 4 | -2 |

**Table 48.** Change solution after seeing CAS answer

In many cases, the students did not try to correct their answer. Furthermore, it seems that the students did not make enough use of the provided opportunity to analyze differences, equivalence and correctness of their own answers and the CAS answers in writing. In case of multiple-choice questions they selected something but thorough, judicious explanations were rather rare in questions with a free-text answer.

It is not clear why quite so many students ignored the possibility to discuss and change the different answer. Some ideas for the reasons could be the following:

- Such tasks are unfamiliar for students. Equivalence issues are not usual at school.

| Year | Equation | Percentage of correct solutions |
|---|---|---|
| 2012, 2013 | $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 42% |
| 2012 | $\tan^3 x = \tan x$ | 71% |
| 2012 | $\cos\left(x - \frac{\pi}{6}\right) = 0.5$ | 50% |
| 2013 | $2\sin 2x \cos 2x + \cos 2x = 0$ | 51% |
| 2013 | $\frac{\tan^2 x}{\tan x} = 0$ | 51% |

**Table 49.** Percentage of correct solutions

- The students' answer was too 'far' from the CAS answer. The large gap reduced the motivation to even try to fill it.
- The students' motivation was not geared towards complete understanding. Rather, they preferred to move to the next equation.

Even without a thorough discussion, it was mostly clear what the students thought about equivalence/non-equivalence and correctness of their answers and CAS answers. The adequacy of the opinions was the main topic of the papers Tonisson (2013) and Tonisson and Lepp (2015) and also Sections 8.3 and 8.4.

It should be noted that the topic of solving trigonometric equations is quite complicated for students in general and this was the case in our lessons as well. In spite of working in pairs and having formula sheets available, the percentage of correct solutions was not very high in the group of analyzed equations (see Table 49).

Solving of equations was an important part of the scenario in our experiment, but still only one part. While solving equations is a common task in schools, ascertaining and explaining equivalence/non-equivalence and correctness of answers is not so common.

When we look at the findings (from the lessons of 2012 and 2013) in Sections 8.3 and 8.4, it is possible to single out the cases where the students adequately identified the equivalence/non-equivalence of their answer and the CAS answer. The percentages of these cases are presented in Table 50. The cases where a non-equivalent answer was changed to equivalent in the light of the CAS answer are also included.

There seem to be different 'hindrances' to identification of equivalence/non-equivalence in case of different equations. It is likely that the percentage of adequate identifications of equivalence/non-equivalence can be increased by drawing special attention to the problematic issues before solving or on the worksheets. It is important to decide what issues are relevant and useful for students.

It is probably possible to increase the percentage of student answers that are equivalent to CAS answers. For instance, we could use simpler equations or give more hints about the solution. (For example, $\sin(x) = \frac{\sqrt{3}}{2}$ instead of $\sin(4x+$

| Year | Equation | Percentage of adequate identification |
|---|---|---|
| 2012 | $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 41% |
| 2012 | $\tan^3 x = \tan x$ | 79% |
| 2012 | $\cos\left(x - \frac{\pi}{6}\right) = 0.5$ | 62% |
| 2013 | $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 39% |
| 2013 | $2\sin 2x \cos 2x + \cos 2x = 0$ | 13% |
| 2013 | $\frac{\tan^2 x}{\tan x} = 0$ | 89% |

**Table 50.** Adequate identification of equivalence/non-equivalence

| Year | Equation | Percentage of adequate identification |
|---|---|---|
| 2013 | $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 45% |
| 2013 | $2\sin 2x \cos 2x + \cos 2x = 0$ | 57% |
| 2013 | $\frac{\tan^2 x}{\tan x} = 0$ | 44% |

**Table 51.** Adequate identification of correctness of students' answer

$2) = \frac{\sqrt{3}}{2}$.) Principally, it is possible to compare the pre-developed solutions (also correct or incorrect) with CAS answers but then the students would have a weaker personal connection with the exercise.

Table 51 shows the percentage of adequate identifications of correctness of the students' own answers and Table 52 shows the percentage of adequate identification of correctness of CAS answers. These data was collected only in the lessons of 2013 (not in 2012).

It is possible to draw following conclusions.

- *Students quite rarely take the opportunity to change their answer after see-*

| Year | Equation | Percentage of adequate identification |
|---|---|---|
| 2013 | $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | 71% |
| 2013 | $2\sin 2x \cos 2x + \cos 2x = 0$ | 19% |
| 2013 | $\frac{\tan^2 x}{\tan x} = 0$ | 33% |

**Table 52.** Adequate identification of correctness of CAS answer

*ing the CAS answer.*

- *The discussion is not very thorough and students often prefer moving to the next equation.*
- *Ascertaining and explaining equivalence/non-equivalence and correctness of answers of trigonometric equations is complicated for many students.* There are differences between different equations.

It is possible to formulate several suggestions for different levels — the mathematics curriculum in general, the selection of concrete equations, and the organization of the lesson.

- *The mathematics curriculum should include a more extensive explanation of the topic of equivalence/non-equivalence.*
- *The equations for the lesson should be carefully selected. A too large gap between possible students' answers and CAS answers decreases motivation for discussion. CAS answers that are the same as school answers could be used at first.*
- *The scenario should be used with simpler topics at first. Trigonometric equation is probably one of the most complicated topics in school mathematics.*
- *A better explanation of the scenario, and motivation to discuss thoroughly and to correct their answers after seeing CAS answer should be provided in the introduction (and also during the lesson).*

### 8.5.2. Problematic trigonometric issues revealed. Parameter $n$

The problematic issues that only emerge in the context of the used scenario are discussed in Section 8.5.2.

As students initially solved the equations manually on the worksheet, their mistakes were the same as they usually make in tests. These issues are not discussed here. We are interested in problematic issues that emerge in the context of comparison of student answers and CAS answers.

Solving of trigonometric equations entails quite specific issues, for example, n (or another parameter) in the answers of general solutions. If one solves the trigonometric equation and finally uses formulae like

$$x = \arcsin m + 2n\pi,\ n \in \mathbb{Z}$$

$$x = \pi - \arcsin m + 2n\pi,\ n \in \mathbb{Z}$$

then $n$ will 'appear' in the solution. If there are no mistakes, everything seems fine and the solution is accounted as correct. Indeed, it is procedurally correct solution. Nevertheless, students can have conceptual gaps in their knowledge. The study revealed quite a lot of cases where students had trouble comparing two answers where $n$ had different meanings, even if they solved the equation correctly.

It should be noted that in Estonia mainly formula

$$x = (-1)^n \arcsin m + n\pi,\ n \in \mathbb{Z}$$

is used in case of $\sin x = m$ instead of expressing two series separately. Therefore, the students' answer and the CAS answer were different in all cases of solving $\sin(4x+2) = \frac{\sqrt{3}}{2}$. Attempts to explain equivalence by substitution of concrete numbers to the general solution could be identified only in a very few cases.

The issue of parameter $n$ is also connected to a peculiarity of Maxima. It should be mentioned that the *%zXXX* form can be confusing for some students, but it seems to be easily explainable. However, changing the *%zXXX* form could be considered as a possible suggestion to CAS developers.

The following conclusions could be formulated.

- *Even if a solution looks to be correct, students can have misunderstandings and knowledge gaps that can remain hidden in the usual solving process but could be uncovered through this scenario.*
- *The issue of parameter n is incoherent for many students. The comparison of different series in answers was missing or inadequate in many cases.*
- *The differences between the CAS standard and the school standard can cause additional obstacles.*

The suggestions are also related to parameter $n$.

- *The meaning of parameter n should be explained more thoroughly to students.*
- *There should be a discussion about the preference of formulae* $x = \arcsin m + 2n\pi$ *and* $x = \pi - \arcsin m + 2n\pi$ *instead of* $x = (-1)^n \arcsin m + n\pi$ *in case of* $\sin x = m$ *in (Estonian) school mathematics*.
- *It could be helpful if %zXXX is replaced in Maxima.*

### 8.5.3. Conclusion. Future work

In this section, after a brief conclusion, some ideas for further work are presented.

We can conclude that the method of asking students to compare their own answers with CAS answers seems to have potential in the context of learning as well as research, but further work is certainly needed. This type of scenario is also usable with other topics. An experiment where irrational equation was introduced in a school mathematics class is described in the article of Hõim et al. (2015).

This style of comparison could contribute to the usage of computer-based tools for doing mathematics in different ways. On the one hand, students can see a faster and easier way to perform calculations. On the other hand, one should understand that evaluating a CAS answer may not be a fast and easy exercise for students. It is likely that the exercises facilitate the development of critical thinking ability (particularly, with respect to computer algebra systems).

One could even say that having different answers compared to school solutions is a part of the charm of CAS. It is possible to propose various lesson scenarios other than those used in the lessons described here. A discussion where all students would participate could be very useful. The discussion could take place during the same lesson, after solving and comparing, but it is also possible to arrange the concluding discussion during the next lesson. In any case, the concluding part is desirable, as students need feedback.

It is also possible to direct students to use CAS tools in the comparison of answers. For example, they could try to substitute a solution into the equation, simplify the difference of answers with the help of the CAS. Of course, it is possible that students compare their own answers with CAS answers as they did in these lessons. Another possible task for students could be a comparison of the answers of different computer algebra systems. In addition, one and the same CAS could offer different answers with different commands or assumptions and these answers could be also compared.

On the one hand, it is good when CAS answers are very school-like. On the other hand, a moderate difference between the student and CAS answers can also be positively challenging and useful. Specifying the degree of such moderation is one of the most challenging tasks of further work as it leads to more questions. For example, how would such a specification look like? Would it be possible to work out indicators that qualify the type of answers?

# 9. CONCLUSION

Firstly, the aims of the dissertations are repeated in the conclusion. Secondly, an overview of the results by research questions is presented. Thirdly, the contribution to the work of different user groups is outlined, followed by ideas on possible future work at the end of the chapter.

The dissertation focused on the differences between CAS answers and answers expected from students in schools (*school answers*). A number of such differences have been described in literature, but the exact locations and categories have not been mapped. The first aim of the dissertation was to provide a systematic review of the differences between CAS answers and school answers and the reasons for their occurrence in equation solution tasks.

Although differences between CAS answers and school answers are often seen as confusing and undesirable obstacles they can also be useful for teaching and learning. The second aim of the dissertation was to propose a pedagogical approach for utilising the teaching opportunities offered by the differences between CAS and students' answers. The proposed approach was used in lessons on trigonometric equations and data about students' understandings and misunderstandings was collected.

Some answers to the six research questions stated in the introduction of the dissertation are summarized here.

*RQ1. Where differences between CAS and school answers could be detected in equations within the school curriculum?*

The answer to RQ1 was presented in Chapter 4 in tables 1–10, which included 127 equations. The equations were selected to test all important types of school equations and potentially critical examples. All the equations of the test suite were solved with 8 different CAS (GeoGebra, Maple, Mathematica, Maxima, Sage, Wiris, WolframAlpha, and Xcas) and the differences between CAS answers and school answers were analyzed.

The identified differences were broken down into 6 types, with one key criterion being the comparison of whether a CAS answer includes more or fewer solutions than the expected answer. The types of difference were: *No difference*, *Equivalent but different*, *More solutions than expected*, *Fewer solutions than expected*, *Did not solve or only some transformations are completed*, *Very complicated answer*.

Answers to linear and quadratic equations in case of all CAS were practically the same as school answers. There were some differences in case of other types of equations. In case of the literal equation, all CAS provided only the main branch, which would not be a complete answer in the school context. In general, GeoGebra answers had the least differences from school answers.

*RQ2. How can the detected differences between CAS and school answers to equations in the school curriculum be described and classified?*

The above-described set-theoretic relations between answers — whether the

CAS answer includes more or fewer solutions than the expected answer — resulted in one possible classification. As a second option, a content-oriented classification is presented in Chapter 5. The answers were analyzed in order to describe and classify the differences between CAS and school answers. The classification was based on the form, completion, dependence on the number domain, and branching of answers and automatic simplification of equations. A type based on the form of equation input, which was not just about the answer, was also added. Each type was divided into subtypes (phenomena) and 29 such subtypes were identified.

The differences caused by the number domain and branching were also discussed separately in the light of RQ3 and RQ4.

*RQ3. When do CAS outputs offer correct and incorrect answers for domain-sensitive examples, specifically for expression simplification and equation solving?*

The examples from the test suite (18 examples — calculations, simplifications, equations) were solved by different CAS while using the features for creation of a school-like situation, particularly establishing the domain. Different CAS have different possibilities (commands, packages, buttons, etc.) for determination of the domain of a calculation result, variable value, or equation solution. The correctness of CAS answers was evaluated and the answer to RQ3 was presented as a table in Chapter 6.

*RQ4. How can branching be described for answers provided by different CAS software; by different school solutions and textbooks; by the possibilities of mathematical approaches for expressions simplifications and equations solving?*

Comparison of school, CAS and mathematically branch-complete answers was presented by encoding different evaluations of branching diversities (EBD) in Chapter 7. For example, the evaluation $CAS < SCH = MATH$ expresses that CAS answer is less complete ($<$) than school and mathematically branch complete answers, which present branches similarly ($=$). The answer to RQ4 was presented as a set evaluations (EBD) with comments added for each topic where branching takes place. Four types of EBD were detected ($SCH = MATH < CAS$, $CAS = SCH < MATH$, $SCH < CAS = MATH$ and $CAS < SCH = MATH$).

In a long-term perspective, it could be said that there were some bugs that I found in the beginning of my research before the year 2000. Later, the differences between CAS and school answers have been mainly related to the choices made by CAS authors. The issues (such as domain and branches) have been similar over the years.

*RQ5. What pedagogical approaches could be proposed to utilize the teaching opportunities offered by the differences between CAS and students' answers?*

Chapter 8 was dedicated to finding a suitable pedagogical approach. In addition to other criteria, it was required that the approach should support students' conceptual understanding with keeping the balance between procedural skill and conceptual understanding. Furthermore, the approach should provide data about students' understanding and misunderstanding.

A lesson scenario, based on comparative discussion on students' answers and computer algebra system answers in pairs of students, was proposed and applied. Students were charged with the task of comparing the answers offered by a CAS with their own answers. The topic of trigonometric equations, which has several useful properties, was chosen for testing in the mathematics class. One could say conclusively that this scenario promotes student discussion about mathematics throughout the lesson. The students perceived this lesson format as rather beneficial.

Furthermore, data was collected to answer RQ6.

*RQ6. How can students identify 1) the equivalence and non-equivalence between CAS and their own answers; 2) correctness of CAS and their own answers of trigonometric equations in lessons based on comparative discussions on students' answers and CAS answers in pairs of students?*

It could be mentioned that the students identified the equivalence and non-equivalence and correctness of CAS answers and their own answers of trigonometric equations relatively poorly. It was found that even if a student's solution looks to be correct, students can have misunderstandings and knowledge gaps. The issue of parameter $n$ in periodic solutions of trigonometric equations was confusing for many students.

The dissertation offers broadly two results:

- A review (and classification) of the differences between CAS answers and school answers in case of equations;
- A pedagogical approach based on comparative discussion on students' answers and computer algebra system answers in pairs of students, which is also used as a data collection tool to obtain insights into students' understandings and misunderstandings.

The results provide a contribution to the work of different user groups: CAS developers, teachers, curriculum designers, textbook authors, and researchers.

For CAS developers, the review gives information for improvement of CAS so that they would be better at meeting the needs of the school context. It is possible to improve the options for selecting a school-friendly output. The warning about possible differences can be improved as well.

For teachers, the review may provide encouragement for increased use of CAS, as the possible differences are known and can be either used for learning or avoided. It provides information for explaining the differences between CAS answers and school answers to students. The review also supports teachers in choosing CAS and tasks. The proposed approach complements the repertoire for stimulation of mathematical discussion among students. It is also applicable as a tool for collecting data in order to obtain insights into students' understandings and misunderstandings. The outlined issues of students' misunderstandings in case of trigonometric equations can be used to prevent them.

Curriculum designers, textbook authors and others who are responsible for school mathematics at a more general level could, with the help of the review, fine-tune the use of CAS in the development of mathematics curricula, textbooks, etc., by either avoiding or highlighting the problematic examples. Furthermore, the proposed pedagogical approach may inspire them to include such methods in curricula or textbooks.

Researchers may be inspired to study the differences between CAS answers and school answer and propose and test new ways of utilization of such differences. The proposed approach can be used as a research tool for studying students' understandings and misunderstandings. It could also be improved and used for other topics.

The conclusion ends with some ideas for future work related to this dissertation. It is possible to create a similar review of the differences between CAS answers and school answers for other CAS or CAS versions. Certain issues seemed persistent and the same examples could be used for testing any new CAS or versions. Such review could be done periodically. Similar school-oriented reviews could also be created for topics other than equations. Besides symbolic solving methods, numeric solving methods could also be examined. It should be mentioned that no general large-scale comparisons of CAS have been published after Wester, 1999b. One reason could be that the users of contemporary CAS have different options for customising a CAS and it is not easy to put the answers of a CAS in one row. A description of possible options of different CAS could be interesting.

There amount of data collected during the experimental lessons was larger than has been used in this dissertation. For example, there is room for a more thorough study of the discussions among students.

The pedagogical approach that was proposed and tested in this dissertation could be used for other topics beyond trigonometric equations. It has already been attempted in case of irrational equations (Hõim et al., 2015). Similarly, understandings and misunderstandings could be studied in other topics. The approach could be improved, for example, by adding other activities before and after the work in pairs. Moreover, the approach could be used systematically in case of different topics, starting with simpler ones.

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# Appendix A. CHOICE OF EQUATIONS

The following tables present why particular equations are included to the test suite used in Chapter 4. The source is marked as follows:

- *common* — the equation is or could be in textbooks, for example, $2x^2 - 4x - 5 = 0$;
- *developed* — the equation is developed from trivial equations by attaching an 'intriguing behaviour', for example, $x + \frac{1}{x-1} = 1 + \frac{1}{x-1}$ instead of $x = 1$. In some cases, triviality in disguised, for example, $x + 1 = x$ or $x - (x-1) = 0$ instead of $1 = 0$;
- by reference, for example, (Kolyagin et al., 1977).

| Equation | Answer | Why | Source |
|---|---|---|---|
| $2x - 4 = 0$ | 2 | $ax + b = 0$ | common |
| $2x - 3 = 0$ | $1\frac{1}{2}$ or $\frac{3}{2}$ or 1.5 | mixed number | common |
| $2x = 0$ | 0 | $b = 0$ | common |
| $x + 1 = x$ | $\emptyset$ | pseudolinear $b \neq 0$ | developed |
| $x - (x-1) = 0$ | $\emptyset$ | pseudolinear $b \neq 0$ | developed |
| $x - 1 = 1 - (2 - x)$ | $\mathbb{R}$ | pseudolinear $b = 0$ | developed |

**Table 53.** Linear equation

| Equation | Answer | Why | Source |
|---|---|---|---|
| $x^2 - 1 = 0$ | $\pm 1$ | $ax^2 + c = 0$ | common |
| $x^2 + x = 0$ | $0, -1$ | $ax^2 + bx = 0$ | common |
| $x^2 - 3x + 2 = 0$ | $1, 2$ | with 2 real roots, $b^2 - 4ac > 0$ | common |
| $2x^2 - 4x - 5 = 0$ | $1 \pm \frac{\sqrt{14}}{2}$ | with 2 real roots, $b^2 - 4ac > 0$, "ugly" | common |
| $x^2 - 2x + 1 = 0$ | $1, 1$ | with 1 double root, $b^2 - 4ac = 0$ | common |
| $x^2 + x + 1 = 0$ | $\emptyset$ | no real roots, $b^2 - 4ac < 0$, out of range | common |
| $x^2 = -1$ | $\emptyset$ | out of range | developed |
| $2(x-1)(x-2) = 0$ | $1, 2$ | product = 0 | common |
| $(x-1)x = (2x+1)(x-1)$ | $\pm 1$ | same factor in both sides, danger in division both sides | developed |
| $(x+1)x - x^2 = 1$ | $1$ | pseudoquadratic | common |
| $x^3 = x$ | $0, 1, -1$ | $y^3 = y$ same factor in both sides, danger in division both sides | common |
| $x^4 - 5x^2 + 4 = 0$ | $-1, 1, -2, 2$ | biquadratic | common |

**Table 54.** Quadratic equation

| Equation | Answer | Why | Source |
|---|---|---|---|
| $\frac{1}{x}=1$ | 1 | basic | common |
| $\frac{1}{x}=0$ | $\emptyset$ | out of range | developed |
| $x+\frac{1}{x}=\frac{1}{x}$ | $\emptyset$ | extraneous +=+, to quadratic | developed |
| $x+\frac{1}{x-1}=1+\frac{1}{x-1}$ | $\emptyset$ | extraneous +=+, to quadratic | developed |
| $\frac{x^2-x+1}{x-1}=\frac{x}{x-1}$ | $\emptyset$ | extraneous +=+, already common denominator | developed |
| $x^2+2+\frac{1}{x-1}=3x+\frac{1}{x-1}$ | 2, extran. 1 | extraneous +=+, to quadratic | based on (Boltyanskii et al., 1972) |
| $\frac{x}{x-1}=\frac{1}{x-1}$ | $\emptyset$ | extraneous /=/ 1/0=1/0? | developed |
| $\frac{x(x-1)}{x-1}=1$ | $\emptyset$ | extraneous */ | developed |
| $\frac{x^2}{x-1}=\frac{3x-2}{x-1}$ | 2, extran. 1 | to quadratic, extranous | developed |
| $\frac{1}{x-1}=\frac{x}{2}$ | $-1,2$ | to quadratic, proportion | common |
| $\frac{x}{2x-2}-\frac{1}{x^2-1}=\frac{2x}{2x+2}$ | 2, extran. 1 | to quadratic, quadratic in denominator, not yet common denominator | common |
| $x^2+\frac{1}{x^2}=2-\frac{1}{x}+x$ | $\pm1,\frac{1}{2}\pm\frac{\sqrt{5}}{2}$ | reciprocal equation, substitution | common |
| $\frac{x^2}{x}=0$ | $\emptyset$ | uncertainty $y^2/y$ | developed |

**Table 55.** Fractional equation

| Equation | Answer | Why | Source |
|---|---|---|---|
| $\sqrt{x}=1$ | 1 | basic | common |
| $\sqrt{x}=-1$ | $\emptyset$ | out of range | developed |
| $\sqrt{x}+\sqrt{x-1}=-2$ | $\emptyset$ | out of range | common |
| $\sqrt{2x}=\sqrt{x-1}$ | $\emptyset$ | R–>C–>R $\sqrt{-2}=\sqrt{-2}$ | developed |
| $\sqrt{x^2+2}=\sqrt{3x}$ | $-1$ | sqrt = sqrt | common |
| $x-\sqrt{1-x^2}=1$ | 1, extran. 0 | 1 involution, 1 radical, to quadratic, extraneous | common |
| $x+\sqrt{1-x^2}=1$ | $1,0$ | 1 involution, 1 radical, to quadratic | common |
| $\sqrt{x+4}+\sqrt{x+1}=3$ | 0 | 2 involutions, 2 radicals, to linear | common |
| $\sqrt{x+5}+\sqrt{20-x}=7$ | $4,11$ | 2 involutions, 2 radicals, to quadratic, 2 solutions | common |
| $\sqrt{2x+1}+\sqrt{x-3}=4$ | 4, extran. 84 | 2 involutions, 2 radicals, to quadratic, extraneous | common |
| $\sqrt[3]{x}=2$ | 8 | cbrt | common |
| $\sqrt[3]{2x}=\sqrt[3]{x-1}$ | $1,2$ | 1 involution, cbrt=cbrt, to linear | developed |
| $\sqrt[3]{x^2+2}=\sqrt[3]{3x}$ | $1,2$ | 1 involution, 2 radicals,cbrt=cbrt, to quadratic | common |
| $\sqrt[3]{x^2}-3\sqrt[3]{x}+2=0$ | $1,8$ | quadratic(cbrt) | common |
| $2\sqrt{x^2-2x+4}-\sqrt{x^2-2x+9}=1$ | $0,2$ | subst | common |
| $\sqrt[3]{x+45}-\sqrt[3]{x-16}=1$ | $80,-109$ | using initial equation in solving | common |
| $\sqrt{x^2+1}=x-2$ | $\emptyset$ | repeat Wester | (Wester, 1999b) |
| $x+\sqrt{x}=2$ | 1 | repeat Wester | (Wester, 1999b) |
| $2\sqrt{x}+3x^{1/4}-2=0$ | $\frac{1}{16}$ | repeat Wester | (Wester, 1999b) |
| $\frac{x}{\sqrt{x}}=0$ | $\emptyset$ | $y^2/y$ | developed |
| $x\sqrt{x}=\sqrt{x}$ | $0,1$ | $y^3=y$ | developed |
| $x+\sqrt{x}=\sqrt{x}-1$ | $\emptyset$ | extraneous +=+ | developed |
| $\frac{x}{\sqrt{x-1}}=\frac{1}{\sqrt{x-1}}$ | $\emptyset$ | extraneous /=/ 1/0=1/0? | developed |
| $\sqrt{x^2}=1$ | $\pm 1$ | absolute value | developed |

**Table 56.** Irrational equation

| Equation | Answer | Why | Source |
|---|---|---|---|
| $2^x = 8$ | 3 | basic | common |
| $2^x = 7$ | $\log_2 7$ | basic, "ugly" | common |
| $2^x = -1$ | $\emptyset$ | out or range | developed |
| $2^{2x} = 2^{x-1}$ | $-1$ | $a^{f(x)} = a^{g(x)}$ | common |
| $2^{x+1} + 2^x = 3$ | 0 | linear(f(x)) | common |
| $4^x - 3 \cdot 2^x + 2 = 0$ | $1, 0$ | quadratic(f(x)) | common |
| $2^{x^2-3x} = \frac{1}{4}$ | $1, 2$ | take logarithm, to quadratic | common |
| $(x-6)^x = 2^x$ | $8, 4, 0$ | equal exponents | literature |
| $x^x = x$ | $1, -1$ | repeat Wester | (Wester, 1999b) |

**Table 57.** Exponential equation

| Equation | Answer | Why | Source |
|---|---|---|---|
| $x^{\log_x(x^2+3)} = 4$ | extran. $1, -1$ | exponent | common |
| $\log_{10} x = 2$ | 100 | basic | common |
| $\log_x 100 = 2$ | 10 | basic | common |
| $\log_x 90 = 2$ | $3\sqrt{10}$ | "ugly" | common |
| $\log_{10}^3 x = \log_{10} x$ | $1, 10, \frac{1}{10}$ | $y^3 = y$ | common |
| $\frac{\log_{10}^2 x}{\log_{10} x} = 0$ | extran. 1 | $y^2/y$ | developed |
| $\log_{10}(2x) = \log_{10}(x-1)$ | $\emptyset$ | R–>C–>R | developed |
| $\log_{x+1} 4 = 2$ | 1, extran. $-3$ | definition | common |
| $\log_3(\log_2 x) = 0$ | 2 | $f_1(f_2(x))$ | common |
| $\log_2(x-2) + \log_2(x-3) = 1$ | 4, extran. 1 | 2 logarithms | common |
| $\log_{10} x^2 + \log_{10} x = 3$ | 10 | linear(f(x)) | common |
| $\log_{10}^2 x - 3\log_{10} x + 2 = 0$ | $10, 100$ | quadratic(f(x)) | common |
| $x + \log x = \log x - 1$ | $\emptyset$ | extraneous +=+ | developed |
| $\frac{x}{\log x} = \frac{1}{\log x}$ | $\emptyset$ | extraneous /=/ 1/0=1/0? | developed |
| $x^{\log x} = 100$ | $10^{\sqrt{2}}, 10^{-\sqrt{2}}$ | complex | common |
| $\sqrt{\log x} = \log\sqrt{x}$ | $1, e^4$ | repeat Wester | (Wester, 1999b) |

**Table 58.** Logarithmic equation

| Equation | Answer | Why | Source |
|---|---|---|---|
| $\sin x = 0$ | $n\pi$ | basic | common |
| $\sin x = \frac{1}{2}$ | $(-1)^n \frac{\pi}{6} + n\pi$ | basic | common |
| $\sin(3x) = -1$ | $(-1)^{n+1} \frac{\pi}{6} + \frac{n\pi}{3}$ | lin.arg | common |
| $\sin(x - \frac{\pi}{3}) = \frac{\sqrt{2}}{2}$ | $\frac{\pi}{3} - (-1)^n \frac{\pi}{4} + n\pi$ | lin.arg | common |
| $\sin(4x+2) = \frac{\sqrt{3}}{2}$ | $-\frac{1}{2} + (-1)^n \frac{\pi}{12} + n \cdot \frac{\pi}{4}$ | lin.arg | common |
| $\sin x = \frac{1}{10}$ | $(-1)^n \arcsin \frac{1}{10} + n\pi$ | "ugly" | developed |
| $\sin x = 2$ | $\emptyset$ | out of range | common |
| $\cos x = 0$ | $\pm \frac{\pi}{2} + 2n\pi$ | basic | common |
| $\cos(x - \frac{\pi}{6}) = 0.5$ | $\frac{\pi}{6} \pm \frac{\pi}{3} + 2n\pi$ | lin.arg, decimal | common |
| $\cos(x - \frac{\pi}{6}) = 1/2$ | $\frac{\pi}{6} \pm \frac{\pi}{3} + 2n\pi$ | lin.arg | common |
| $\cos x = 2$ | $\emptyset$ | out of range | developed |
| $\tan x = 0$ | $n\pi$ | basic | common |
| $\tan x = -\frac{\sqrt{3}}{3}$ | $\frac{\pi}{6} + n\pi$ | basic | common |
| $\tan x = 2$ | $\arctan 2 + n\pi$ | "ugly" | common |
| $\tan x = \tan \frac{\pi}{4}$ | $\frac{\pi}{4} + n\pi$ | tan = tan | developed |
| $\tan x = \tan \frac{\pi}{2}$ | $\emptyset$ | out of domain | developed |
| $\cot x = 0$ | $\frac{\pi}{2} + n\pi$ | basic | common |
| $\cot x = \sqrt{3}$ | $\frac{\pi}{6} + n\pi$ | basic | common |
| $\cot x = \cot 0$ | $\emptyset$ | out of domain | developed |

**Table 59.** Trigonometric equation (1)

| Equation | Answer | Why | Source |
|---|---|---|---|
| $\sin x = \cos x$ | $\frac{\pi}{4} + n\pi$ | basic | (Wester, 1999b) |
| $\tan x = 1$ | $\frac{\pi}{4} + n\pi$ | basic | (Wester, 1999b) |
| $\sin x = \tan x$ | $0 + n\pi, 0 + n2\pi$ | basic | (Wester, 1999b) |
| $\tan(x + \frac{\pi}{4}) = 2\cot x - 1$ | $\frac{\pi}{2} + n\pi$, $\arctan\frac{1}{2} + n\pi$ | miss solution $\frac{\pi}{2}$ | (Kolyagin et al., 1977) |
| $\tan^3 x = \tan x$ | $n\pi, \pm\frac{\pi}{4} + n\pi$ | $y^3 = y$ | developed |
| $\frac{\tan^2 x}{\tan x} = 0$ | $\emptyset$ | $y^2/y$ | developed |
| $2\sin x\cos 2x - 1 + 2\cos 2x - \sin x = 0$ | $\pm\frac{\pi}{6} + \pi n$, $-\frac{\pi}{2} + 2n\pi$ | factors | common |
| $2(\sin x + \cos x) + \sin 2x + 1 = 0$ | $-\frac{\pi}{4} + n\pi$ | subst. $\sin x + \cos x = t$ | common |
| $\cos(x^2 - 2) = \frac{1}{2}$ | $\pm\sqrt{2 \pm \frac{\pi}{3} + 2\pi n}$ | trig(quad(x)) | common |
| $\sin^2 x - 3\sin x + 2 = 0$ | $\frac{\pi}{2} + 2\pi n$ | quad(trig(x)) | common |
| $3\sin^2 x\cos^2 x - \cos^3 x = 0$ | $(2n+1)\frac{\pi}{2}$, $(6n+1)\frac{\pi}{6}$, $(6n-1)\frac{\pi}{6}$ | homogeneous | common |

**Table 60.** Trigonometric equation (2)

| Equation | Answer | Why | Source |
|---|---|---|---|
| $\|x\| = 1$ | $\pm 1$ | basic | common |
| $\|x - 1\| = 2$ | $-1, 3$ | abs(lin), 2 solutions | (Wester, 1999b) |
| $\|2x - 1\| = -3x$ | $-1$ | abs(lin), 1 solution | (Wester, 1999b) |
| $\|x\| = -1$ | $\emptyset$ | out of range | common |
| $\|x - 1\| + \|2x + 1\| = 3$ | $\pm 1$ | 2 absolute, 2 solutions | common |
| $\|2x - 1\| = -3$ | $-1$ | 2 absolute, 1 solution | common |
| $\|x - 1\| + \|x + 1\| = 2$ | $[-1, 1]$ | interval | common |
| $\|2x + 5\| = \|x - 2\|$ | $-1, -7$ | repeat Wester | (Wester, 1999b) |
| $x + \frac{1}{\|x\|} = \frac{1}{\|x\|}$ | $\emptyset$ | extraneous +=+ | developed |
| $\frac{x}{\|x\|} = 0$ | $\emptyset$ | extraneous /=/ 1/0=1/0? | developed |

**Table 61.** Equation that contains an absolute value of an expression

| Equation | Answer | Why | Source |
|---|---|---|---|
| $ax = 0$ | if $a \neq 0$ then 0, else $\mathbb{R}$ | basic | (Stoutemyer, 1991) |
| $ax = 1$ | $\frac{1}{c}$ | basic | (Stoutemyer, 1991) |
| $ax^2 - 3x + 2 = 0$ | if $a = 0$ then $\frac{2}{3}$, else $\frac{3 \pm \sqrt{9-8a}}{2a}$ | usual | common |
| $ax^2 + bx + c = 0$ | branches | formula | developed |
| $ax + b = 0$ | branches | formula | developed |

**Table 62.** Literal equation

# Appendix B. QUESTIONNAIRES

**Trigonomeetrilised võrrandid. Paaristöö**

*Eelankeet*

Arvutialgebra süsteemidega saab lahendada väga paljusid matemaatikaülesandeid. See, millisel moel nende süsteemide võimsust matemaatika õppimisel ja õpetamisel saab kasutada, on jätkuvalt arutelu all. Täna proovime ühte lähenemist, mille sobivust ja kasulikkust püüame ka hinnata. Analüüsimiseks on vajalik võimalikult korralikult andmeid koguda ja selle jaoks on Teiega koostöö väga oluline. Teie personaalseid andmeid ei avaldata.

Ülesannete lahendamine toimub paarides. Samas on igal üliõpilasel võimalus jääda eriarvamusele, mida siis palun ka lahenduses märkida. Samuti vastatakse selle ankeedi küsimustele eraldi.

Olge Te tänatud katsetusel kaasalöömise eest!

Eno Tõnisson

**Palun vastake neile küsimustele enne võrrandite lahendamisele asumist**

Nimi:

Hinnake viiepallisüsteemis (võib lisada ka + ja -) oma koolimatemaatika taset:

Kui sageli olete kasutanud järgmisi arvutialgebra süsteeme?

Maxima *ei olegi / üksikutel juhtudel / mõnikord / sageli*

Wiris *ei olegi / üksikutel juhtudel / mõnikord / sageli*

WolframAlpha *ei olegi / üksikutel juhtudel / mõnikord / sageli*

Mõni muu arvutialgebra süsteem

……. *ei olegi / üksikutel juhtudel / mõnikord / sageli*

……. *ei olegi / üksikutel juhtudel / mõnikord / sageli*

Kui palju olete varem koolis tundides paaristööd teinud?

*mitte kunagi / üksikutel juhtudel / mõnikord / sageli*

Kui palju olete varem ülikoolis tundides paaristööd teinud?

*mitte kunagi / üksikutel juhtudel / mõnikord / sageli*

Kuivõrd teile tundides paaristöö meeldib?

*üldse ei meeldi / pigem ei meeldi / raske hinnata / pigem meeldib / väga meeldib*

Kommentaarid:

**Trigonomeetrilised võrrandid. Paaristöö**

*Järelankeet*

Kui suur oli umbes teie panus ülesannete lahendamisel? (50 %, kui mõlema paarilise panus oli umbes sama suur)

Palun hinnake, kuivõrd olete nõus järgmiste väidetega.

| | Üldse pole nõus | Pigem pole nõus | Osaliselt nõus, osaliselt mitte | Pigem nõus | Täiesti nõus |
|---|---|---|---|---|---|
| Mulle meeldis neid ülesandeid lahendada paaristööna. | 1 | 2 | 3 | 4 | 5 |
| Paaristöö sujus meie paaril hästi. | 1 | 2 | 3 | 4 | 5 |
| Paarilise isikuomadused olid minuga koostööks sobivad. | 1 | 2 | 3 | 4 | 5 |
| Paarilisega arutamine oli õpetlik. | 1 | 2 | 3 | 4 | 5 |
| Paaristööga läks kauem aega, kui oleks läinud üksi töötades. | 1 | 2 | 3 | 4 | 5 |
| Paaristöö tulemusena sai töö tulemus parem, kui oleks saanud üksi töötades. | 1 | 2 | 3 | 4 | 5 |
| Sain tänu paaristööle enam teadmisi ja oskusi, kui oleksin saanud üksi töötades. | 1 | 2 | 3 | 4 | 5 |
| Paariliselt sain nõuandeid ja selgitusi ülesannete kohta. | 1 | 2 | 3 | 4 | 5 |
| Paarilisel oli enam teadmisi ja oskusi kui minul. | 1 | 2 | 3 | 4 | 5 |
| Olin arutelus juhtivam pool. | 1 | 2 | 3 | 4 | 5 |
| Tööleht soodustas arutelu. | 1 | 2 | 3 | 4 | 5 |
| Töölehe ülesanded olid paaristööks sobivad. | 1 | 2 | 3 | 4 | 5 |
| Mõlemal paarilisel peaks olema eraldi tööleht. | 1 | 2 | 3 | 4 | 5 |

Üldised kommentaarid eksperimendi kohta, märkused materjali kohta jms.

# Appendix C. FRAGMENT OF WORKSHEET

**Ülesanne 1.** **Lahendage võrrand** $\mathbf{sin}(\mathbf{4}x+\mathbf{2})=\frac{\sqrt{3}}{2}$. (ESIALGU ILMA ARVUTITA!)

**Palun märkige kellaaeg:** (kirjutage sinise/musta kirjutusvahendiga)

Vihjeid: Lahendada kõigepealt võrrand 4x+2 suhtes.

1.1. Kui kindlad te oma vastuse õigsuses olete?

*väga kindlad / üsna kindlad / ei oska öelda / üsna ebakindlad / väga ebakindlad*

---

**Palun lahendage nüüd võrrand süsteemiga *WolframAlpha* kasutades käsku solve.**


1.2. Kuivõrd ootamatu tundub arvutialgebra süsteemi vastus esmapilgul?

*väga ootamatu / üsna ootamatu / ei oska öelda / üsna oodatud / täiesti oodatud*

**Palun analüüsige enda vastuse ja arvutialgebra süsteemi vastuse kooskõla!**

- Kui on vaja oma eelmise lehekülje lahendust **täiendada-parandada**, siis palun tehke seda **rohelise kirjutusvahendiga**.

1.3. Milles seisnevad teie vastuse ja arvutialgebra süsteemi vastuse erinevused?

1.4. Kuidas on teie vastus ja arvutialgebra süsteemi vastus omavahel seotud? (Analüüsige samaväärsust / mittesamaväärsust, erilahendite / üldlahendite seost jms.)

1.5. Palun hinnake nüüd enda (vajadusel parandatud) vastuse õigsust

*õige / osaliselt õige / vale / ei oska öelda*

1.6. Palun hinnake arvutialgebra süsteemi vastuse õigsust

*õige / osaliselt õige / vale / ei oska öelda*

1.7. Palun hinnake enda (vajadusel parandatud) vastuse ja arvutialgebra süsteemi vastuse samaväärsust

*on samaväärsed / ei ole samaväärsed / ei oska öelda*

Kommentaarid!

# ACKNOWLEDGEMENT

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# SUMMARY IN ESTONIAN

## Oodatavate vastuste ja arvutialgebra süsteemide vastuste erinevused koolimatemaatika võrrandite puhul

Arvutialgebra süsteemidega saab lahendada erinevat tüüpi matemaatika-ülesandeid, sealhulgas koolimatemaatika võrrandeid. Sageli langevad arvutialgebra süsteemide vastused kokku koolikontekstis oodatavate vastustega (koolivastustega), vahel aga mitte. Sellised ootamatud arvutialgebra süsteemide vastused on tihti küll matemaatiliselt korrektsed, aga mõne teise standardi järgi, näiteks kompleksarvude vallas.

Käesoleva dissertatsiooni üheks eesmärgiks oli anda ülevaade arvutialgebra süsteemide vastuste ja koolivastuste erinevustest ja nende põhjustest võrrandite puhul. Selleks püstitati uurimisküsimused, mille kaupa järgnev kokkuvõte on üles ehitatud.

1. uurimisküsimus. *Kust võib leida erinevusi arvutialgebra süsteemide vastuste ja koolivastuste vahel koolimatemaatika võrrandite puhul?*

Vastusena sellele küsimusele on 4. peatükis tabelid, mis sisaldavad 127 võrrandit. Need võrrandid on valitud kõigi oluliste koolimatemaatika võrranditüüpide testimiseks. Kõik võrrandid on lahendatud 8 erineva arvutialgebra süsteemiga (GeoGebra, Maple, Mathematica, Maxima, Sage, Wiris, WolframAlpha, and Xcas). Leitud erinevused on jaotatud 6 tüüpi, kusjuures põhikriteeriumiks oli, kas arvutialgebra süsteemi vastus sisaldab rohkem või vähem lahendeid kui koolivastus.

Lineaarvõrrandite ja ruutvõrrandite puhul olid arvutialgebra süsteemide vastused praktiliselt samad kui koolivastused. Teiste võrranditüüpide puhul leidus ka erinevusi. Parameetrit sisaldavate võrrandite puhul andsid arvutialgebra süsteemid ainult peaharu, mis ei ole koolikontekstis täielik vastus. Süsteemide lõikes oli üldiselt GeoGebra vastuste puhul kõige vähem erinevusi koolivastustest.

2. uurimisküsimus. *Kuidas saab koolimatemaatika võrrandite puhul leitud arvutialgebra süsteemide vastuste ja koolivastuste erinevusi kirjeldada ja liigitada?*

Kui eelkirjeldatud liigitus põhines pigem lahendite arvude võrdlemisel, siis 5. peatükis toodud liigitus on sisupõhisem. See põhineb vastuste kujul, täielikkusel, arvuvallast sõltuvusel ja harunemisel ning võrrandite automaatsel lihtsustamisel. Lisaks baseerub üks tüüp võrrandi sisestamisele. Tüübid on veel omakorda jaotatud alamtüüpideks, mida on kokku 29.

Arvuvallast ja harunemisest tulenevaid erinevusi käsitleti eraldi 3. ja 4. uurimisküsimuse abil.

3. uurimisküsimus. *Millal on arvutialgebra süsteemi vastused korrektsed ja millal mittekorrektsed arvuvallast sõltuvate avaldise lihtsustamise ja võrrandi lahendamise näidete puhul?*

Testikomplekti 18 ülesannet (arvutamisi, lihtsustamisi, võrrandeid) lahendati arvutialgebra süsteemidega Derive, Maple, Mathematica, Maxima, MuPAD, TI-

92+, TI-nspire ja WIRIS.

Kasutati arvutialgebra süsteemide võimalusi koolilaadse olukorra (eriti just arvuvalla mõttes) tagamiseks. Erinevates arvutialgebra süsteemides on selleks erinevad võimalused (käsud, paketid, nupud vms). Arvutialgebra süsteemide vastuste korrektsuse hinnangud on esitatud tabelina 6. peatükis.

4. uurimisküsimus. *Kuidas saab kirjeldada erinevate arvutialgebra süsteemide vastuste, erinevate kooli (õpiku) vastuste ja matemaatiliselt täielike vastuste harunemist avaldiste lihtsustamise ja võrrandite lahendamise ülesannete korral?*

Koolivastuse, arvutialgebra süsteemi vastuse ja matemaatiliselt täieliku vastuse võrdlus esitati 7. peatükis vastavate võrratuste abil. Näiteks CAS < SCH = MATH näitab, et arvutialgebra süsteemi vastus ei ole nii täielik (<) kui koolivastus ja matemaatiliselt täielik vastus, mis esitavad harusid sarnaselt (=). Analüüsiti kõiki avaldiste lihtsustamise ja võrrandite teemasid, kus harunemist esineb. Leiti nelja tüüpi variante (SCH = MATH < CAS, CAS = SCH < MATH, SCH < CAS = MATH ja CAS < SCH = MATH).

Koolivastuste ja arvutialgebra süsteemide vastuste erinevusi saab kasutada õpetamisel ja õppimisel. Sellega seoses püstitati 5. uurimisküsimus.

5. uurimisküsimus. *Milliseid pedagoogilisi lähenemisi saab välja pakkuda arvutialgebra süsteemide vastuste ja õppijate vastuste erinevuste kasutamiseks?*

8. peatükis otsiti lähenemist, mis toetaks õppijate kontseptuaalset arusaamist, kusjuures säilitades tasakaalu protseduuriliste oskuste ja kontseptuaalse arusaamise vahel. Samuti pidi lähenemine andma võimaluse koguda andmeid õppijate arusaamise kohta.

Dissertatsioonis on pakutud välja tunnistsenaarium, mis põhineb arvutialgebra süsteemide vastuste ja õppijate endi vastuste võrdlemisel paaristööna. õppijad pidid kõigepealt lahendama ülesande ise pabertöölehel ja seejärel arvutialgebra süsteemiga ning seejärel vastuseid võrdlema. Seda lähenemist testiti trigonomeetrilisi võrrandeid käsitlevates tundides esimese aasta üliõpilastega. Trigonomeetriliste võrrandite teema valiti, sest nende puhul on vastuste võimalik erinev esitus loomulik. Pakutud lähenemine soodustas õpilastevahelist matemaatilist arutelu. õppijad pidasid seda formaati kasulikuks.

Lisaks õpetamisele ja õppimisele saab selle formaadiga koguda andmeid õppijate teemamõistmise kohta. Antud juhul püstitati järgmine uurimisküsimus.

6. uurimisküsimus. *Kuidas suudavad õppijad tuvastada 1) arvutialgebra süsteemide ja endi vastuste ekvivalentsust või mitteekvivalentsust, 2) arvutialgebra süsteemide ja endi vastuste korrektsust trigonomeetrilisi võrrandeid käsitlevad tunnis, mis põhineb arvutialgebra süsteemide vastuste ja õppijate endi vastuste võrdlemisel paaristööna?*

Võib öelda, et õppijad tuvastasid arvutialgebra süsteemi vastuse ja enda vastuse ekvivalentsust ja mitteekvivalentsust ning korrektsust suhteliselt halvasti. Arvestada tuleb, et antud tunniformaat oli õppijatele uus ja ülesanded suhteliselt keerulised. Tulemusena leiti ka, et isegi kui õppijate lahendus paistab korrektne, võib siiski olla lünki arusaamises. Näiteks parameetri n tähendus trigonomeetrilise võrrandi perioodilises lahendis oli paljude õppijate jaoks segane.

Dissertatsioonis toodud ülevaade arvutialgebra süsteemide vastuste ja koolivastuste erinevustest ja nende põhjustest võib aidata arvutialgebra süsteemide arendajaid täiustada süsteeme kooli jaoks sobivamaks.

Õpetajad saavad selle ülevaate abil julgemalt arvutialgebra süsteeme kasutada, kuna nad teavad võimalikke erinevusi ja saavad neid rakendada või vältida. Samuti aitab ülevaade arvuitialgebra süsteeme ja ülesandeid valida. Väljapakutud pedagoogiline lähenemine (tunnistsenaarium) täiendab õpetajate vahendite loetelu õppijatevahelise matemaatilise arutelu initsieerimiseks. Samuti saab seda lähenemist kasutada õppijate arusaamiste kindlakstegemisel.

Õppekava arendajad, õpikute autorid ja teised, kes vastutavad koolimatemaatika eest üldisemalt, saavad pakutud ülevaate abil arvutialgebra süsteeme õppekava ja õpikutega seotult täpsemalt kasutada problemaatilisi näiteid rõhutades või vältides. Väljapakutud pedagoogiline lähenemine võib inspireerida selliseid viise soovitama õppekavas või õpikutes.

Uurijad võivad saada ideid arvutialgebra süsteemide ja koolivastuste erinevuste uurimiseks ja uute kasutamisvõimaluste väljapakkumiseks. Samuti saab pakutud lähenemist kasutada õppijate arusaamise uurimisel. Seda lähenemist saab täiustada ja kasutada ka teiste teemade puhul.

# CURRICULUM VITAE

## Personal data

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## Education

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| 2017 | University of Tartu, computer science PhD student (external) |
| 1997–2002 | University of Tartu, mathematics PhD student |
| 1992–1995 | University of Tartu, MSc in mathematics |
| 1987–1992 | University of Tartu, diploma, mathematics, teacher of mathematics |
| 1976–1987 | Tartu Secondary School No 5 |

## Employment

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| 2001– | University of Tartu, Institute of Computer Science, lecturer |
| 1994–2000 | University of Tartu, Chair of Didactics of Mathematics, assistant |
| 1990–1998 | Tartu Tamme Gymnasium, teacher |

## Scientific work

Main fields of interest:

- computers (specially Computer Algebra Systems) in teaching and learning school mathematics
- carrier choices of ICT students
- didactics of programming

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## Haridus

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| 2017 | Tartu Ülikool, informaatika doktorantuur (ekstern) |
| 1997–2002 | Tartu Ülikool, matemaatika doktorantuur |
| 1992–1995 | Tartu Ülikool, MSc matemaatika |
| 1987–1992 | Tartu Ülikool, diplom matemaatika ja matemaatikaõpetaja erialal |
| 1976–1987 | Tartu 5. Keskkool |

## Teenistuskäik

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| 2001– | Tartu Ülikool, arvutiteaduse instituut, lektor |
| 1994–2000 | Tartu Ülikool, matemaatika didaktika õppetool, assistent |
| 1990–1998 | Tartu Tamme Gümnaasium, õpetaja |

## Teadustegevus

Peamised uurimisvaldkonnad:

- arvutid (eriti arvutialgebra süsteemid) koolimatemaatika õpetamisel ja õppimisel
- infotehnoloogia valdkonna üliõpilaste karjäärivalikud
- programmeerimise didaktika

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