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SUBOBJECT CLASSIFICATION AND GEOMETRIC MORPHISMS OF PARTIALLY ORDERED ACTS

Master thesis

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Introduction

One way of viewing a category is to consider it as a very generic typed monoid, ie a monoid where only some elements can be multiplied (when they are of the same “type”) and whose elements form a class instead of merely a set. Conversely a rather important example of a category is the “dot-category” based on a monoid.

The latter view of monoids as categories also provides us with a functorial description of acts over this monoid, these being the presheaves (or set-valued functors) on the “dot-category”. Presheaves themselves are relatively important tools of topos theory and give rise to a wealth of notions and results (some of which would be Grothendieck toposes, sheaves, sheafification and the logical functors arising from truth value objects). Since acts are presheaves (and rather archetypal ones at that), these notions apply to them as well, providing us with a general view of known act-specific results (for instance, that every act is a quotient of free acts, or the Hom-tensor adjunction) and new ones (the sheafification technique and corresponding adjoints, or the logic on subacts).

One may take this a little farther and consider not just sets, but partially ordered sets. In the current work we work with the category of partially ordered acts or posets over a pomonoid (partially ordered monoid). These can be seen as the “dot-category” version of partial-order-valued functors.

The master thesis consists of two primary parts. In the first part we examine how close the category of ordered acts is to being a topos. It turns out that the category is complete and cocomplete, and even cartesian closed, but unfortunately is not a topos, as it lacks a subobject classifier. There are some limited subobject classifiers and generalizations thereof, but none for any of the more common kinds of monomorphisms.

Sydney Bulman-Fleming and Mojgan Mahmoudi have concurrently done a lot of the same work in their recent article [BFM]. There are things in this thesis that they did not consider, namely the regularly extremal morphisms. Also, they study the topos-characteristic notions in much less detail, and subobject classification actually gets no explicit mention in their work.

In the second part we try to generalize the notion of geometric morphisms into one that would be useful for posets. For this we introduce the notions of pofunctors, poadjunctions and universal pococones (generalizations of colimits). We prove a version of the usual Hom-tensor adjunction and find some naturally occurring geometric morphisms arising from pomonoid homomorphisms. Finally, we define the notion of a point in a poset category. In the end we find that points correspond to flat posets over pomonoids, ie posets that induce a tensor multiplication that preserves universal pocones.

1 Preliminaries

For general background in category theory we refer the reader to [Bo] or [CWM]. The basic text for topos theory used in the current work is [MLM]. For a reasonable degree of self-containment we shall review most of the basic notions in the following part of the thesis.

1.1 Posets over a pomonoid

The objects of our study are right S-posets over a pomonoid S.

Definition 1.1 A partially ordered monoid (a pomonoid) is an ordered algebraic structure $(S,\leq,\cdot)$ such that the following hold:

a) $\forall x, y, z \in S \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$,

b) $\exists 1 \in S : \forall x \in S \quad 1 \cdot x = x \cdot 1 = x$,

c) $\forall x \in S \quad x \leq x$,

d) $\forall x, y \in S \quad (x \leq y \wedge y \leq x) \Rightarrow (x = y)$,

e) $\forall x, y, z \in S \quad (x \leq y \wedge y \leq z) \Rightarrow (x \leq z)$,

f) $\forall x, y, z \in S \quad (x \leq y) \Rightarrow (x \cdot z \leq y \cdot z)$,

g) $\forall x, y, z \in S \quad (x \leq y) \Rightarrow (z \cdot x \leq z \cdot y)$.

Definition 1.2 A partially ordered right set over a fixed pomonoid S (a right S-poset) is an ordered algebraic structure $(A,\leq,\cdot s)_{s\in S}$ such that the following hold:

a) $\forall x \in A \, \forall s, t \in S \quad (x \cdot s) \cdot t = x \cdot (s \cdot t)$,

b) $\forall x \in A \quad x \cdot 1 = x$,

c) $\forall x \in A \quad x \leq x$,

d) $\forall x, y \in A \quad (x \leq y \wedge y \leq x) \Rightarrow (x = y)$,

e) $\forall x, y, z \in A \quad (x \leq y \wedge y \leq z) \Rightarrow (x \leq z)$,

f) $\forall x, y \in A \, \forall s \in S \quad (x \leq y) \Rightarrow (x \cdot s \leq y \cdot s)$,

g) $\forall x \in A \, \forall s, t \in S \quad (s \leq t) \Rightarrow (x \cdot s \leq x \cdot t)$.

For better distinction between the two multiplications we write simply st instead of $s \cdot t$ when s and t are elements of a pomonoid.

In the following, we also allow empty posets and empty (one-sided) ideals of pomonoids. All ideals of a pomonoid are taken as purely algebraic (one-sided) ideals, with no order restrictions (as it has been done in some studies).

Naturally, left S-posets can be treated in the same way as right S-posets. Keeping this in mind we deal primarily with right S-posets. Note that proofs for right S-posets can be carried over to left S-posets and vice versa. We refer to this operation as taking the left-(right-)sided version of the proof in question. We write A_S to emphasize that A is a right S-poset and ${}_SA$ to stress that it is a left S-poset.

In the following, let S be a partially ordered monoid (pomonoid) and Pos_S the category of partially ordered right sets over this pomonoid with order-preserving act homomorphisms as morphisms. Also, let Act_S denote the usual category of right S-acts. The categories of left S-posets and S-acts are denoted correspondingly ${}_S\mathrm{Pos}$ and ${}_S\mathrm{Act}$.

A mapping $f : A_S \to B_S$ between two S-posets is therefore an S-poset homomorphism iff the following hold:

a) $\forall x \in A \forall s \in S \quad f(x) \cdot s = f(x \cdot s)$,

b) $\forall x, y \in A \quad (x \leq y) \Rightarrow (f(x) \leq f(y))$.

The morphism sets of the category Pos_S can also be ordered. For this take $f, g : A_S \to B_S$ in Pos_S and define $f \leq g$ iff $f(a) \leq g(a)$ for all $a \in A$. In the following we only consider this pointwise ordering for S-poset morphisms.

The pomonoid S can be made into a right poset S_S over itself with its monoid multiplication and natural order. For any other right S-poset B_S and element $b \in B$, we can define a morphism $\mathbf{b} : S_S \to B_S$ with

$$\mathbf{b}(s) = b \cdot s.$$

As

$$\mathbf{b}(st) = b \cdot (st) = (b \cdot s) \cdot t = \mathbf{b}(s) \cdot t$$

for all $s, t \in S$, this is an act homomorphism. If $s \leq t$ in S, then

$$\mathbf{b}(s) = b \cdot s \leq b \cdot t = \mathbf{b}(t)$$

and $\mathbf{b}$ is order-preserving as well. So this definition does give us an S-poset morphism.

Lemma 1.1 *For $b, c \in B, b \leq c$, $s \in S$*

1) $\mathbf{b} \circ \mathbf{s} = \mathbf{b} \cdot \mathbf{s}$*;*

2) $\mathbf{b} \leq \mathbf{c}$.

Proof. Obviously

$$(\mathbf{b} \circ \mathbf{s})(t) = b \cdot (s \cdot t) = (b \cdot s) \cdot t = (\mathbf{b} \cdot \mathbf{s})(t)$$

for all $t \in S$. Likewise,

$$\mathbf{b}(t) = b \cdot t \leq c \cdot t = \mathbf{c}$$

for all $t \in S$. ■

Definition 1.3 A right T-poset A_T that is also a left S-poset ${}_SA$ is called an (S, T)-biposet if

$$s \cdot (a \cdot t) = (s \cdot a) \cdot t$$

for all $a \in A, s \in S, t \in T$.

For a category $\mathcal{C}$, we denote the class of its objects as $\mathrm{Ob}(\mathcal{C})$.

In the same way, if $A, B \in \mathrm{Ob}(\mathcal{C})$, then for the set of all morphisms from A to B we write $\mathrm{Mor}_{\mathcal{C}}(A, B)$.

The category of all sets and functions will be denoted by **Sets**.

For the terminal object of a category $\mathcal{C}$ we write $\mathbf{1}_{\mathcal{C}}$ or simply $\mathbf{1}$, and for any $C \in \mathrm{Ob}(\mathcal{C})$ we will denote the unique morphism to $\mathbf{1}$ as $!_C : C \to \mathbf{1}$.

If $\leq$ is a partial order, then we define $<$ as the relation $< := \leq \setminus =$.

1.2 Monomorphisms

In category theory there are several different types of monomorphisms. Let us also recall these definitions.

Definition 1.4 A morphism $\iota : B \to C$ in a category $\mathcal{C}$ is called

- a *coretraction* (or a *section*), if it is left invertible, i.e.

$$(\exists f : C \to B)(f \circ \iota = 1_B);$$

- a *regular monomorphism*, if it is an equalizer, i.e.

$$(\exists D \in \mathrm{Ob}(\mathcal{C}))(\exists f, g : C \to D)((B, \iota) \approx Equ(f, g));$$

- a *strict monomorphism*, if

$$(\forall H \in \mathrm{Ob}(\mathcal{C}))(\forall h : H \to C)[(\forall D \in \mathrm{Ob}(\mathcal{C}))(\forall f, g : C \to D)$$
$$(f \circ \iota = g \circ \iota \Rightarrow f \circ h = g \circ h) \Rightarrow (\exists! k : H \to B)(\iota \circ k = h)];$$

- a *strong monomorphism*, if

$$(\forall U, V \in \mathrm{Ob}(\mathcal{C}))(\forall f : U \to B)(\forall g : V \to C)(\forall \pi : U \to V)$$
$$(\iota \circ f = g \circ \pi \wedge \pi \text{ is epimorphism } \Rightarrow$$
$$(\exists h : V \to B)(f = h \circ \pi \wedge g = \iota \circ h));$$

- an *extremal monomorphism*, if

$$(\forall D \in \mathrm{Ob}(\mathcal{C}))(\forall \pi : B \to D)(\forall f : D \to C)$$
$$(\iota = f \circ \pi \wedge \pi \text{ is epimorphism } \Rightarrow \pi \text{ is isomorphism}),$$

- a *monomorphism*, if it is left cancellable, i.e.

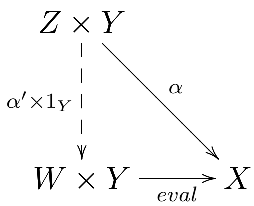
$$(\forall D \in \mathrm{Ob}(\mathcal{C}))(\forall f, g : D \to B)(\iota \circ f = \iota \circ g \Rightarrow f = g).$$

We have the following implications (see [HS], pages 103-104, 265-266 and 110 for example):

coretraction $\Rightarrow$ regular monomorphism $\Rightarrow$ strict monomorphism $\Rightarrow$
strong monomorphism $\Rightarrow$ extremal monomorphism $\Rightarrow$ monomorphism

1.3 Toposes

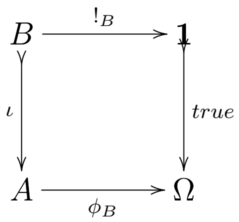
Definition 1.5 In a finitely complete category we say that an object W is the *exponential object* of objects Y and X, if there exists such a morphism $eval : W \times Y \to X$ that for any other morphism $\alpha : Z \times Y \to X$ there is a unique morphism $\alpha' : Z \to W$ such that the diagram



commutes.

The usual notation for the exponential object of Y and X is X^Y. The above definition is of course a writeout of the (alternate) definition that exponentiation is right adjoint to multiplication (ie $- \times Y \dashv -^Y$), with adjunction expressed in terms of a universal morphism.

Definition 1.6 In a finitely complete category the *subobject classifier* is a monomorphism $true : \mathbf{1} \to \Omega$ such that for any other monomorphism $\iota : B \to A$ there is a unique morphism ϕ_B such that the square



turns out to be a pullback. The object Ω is usually called the *truth value object* of this category, and it is unique up to isomorphism. Actually, the morphism *true* is unique up to isomorphism (in the category of morphisms) as well.

Note that monomorphisms can be replaced with subobjects in the above definition since equivalent monics are isomorphic. In the following, when dealing with partially ordered acts, if we have a subobject, we identify it with the representative monomorphism that injects the image corresponding to all the monomorphisms belonging to the equivalence class that makes up this subobject. So ι will always be an injection of a subset.

Definition 1.7 A category $\mathcal{C}$ is called an *(elementary) topos*, if

(i) there exist all finite limits and colimits;

(ii) for each pair of objects $E, F \in \mathrm{Ob}(\mathcal{C})$ there exists their exponential object E^F;

(iii) there is a (the) subobject classifier Ω.

Definition 1.8 A category $\mathcal{C}$ is called a *cartesian closed category*, if it is finitely complete and satisfies (ii).

2 Subobject classification

Our aim in this part is to see whether the category of S-posets is a topos and if not, whether there exist some similar constructions for replacing those in a topos.

2.1 Cartesian closedness

The following fact is also proved at the beginning of section 2.3 of [BFM]

Lemma 2.1 *Monomorphisms in the category* Pos_S *are precisely the injective order-preserving* S*-act homomorphisms.*

Proof. Since Pos_S is a concrete category, every injective order-preserving S-act homomorphism is a monomorphism. To prove the converse, we only need to find a free S-poset with one generator. The S-poset S_S with its natural order is just that. Therefore no non-injective morphism can be a monomorphism and our proof is complete. ■

Lemma 2.2 *The terminal object in* Pos_S *is the one-element poset* $\mathbf{1}$ *with its only possible action.*

Proof. For an arbitrary S-poset A, define a mapping $!_A : A \to \mathbf{1}$ as $!_A(x) = *$ for all $x \in A$, where $*$ is the only element of $\mathbf{1}$. We actually cannot define any other maps from A to $\mathbf{1}$. We have $!_A(x) \cdot s = * \cdot s = * = !_A(x \cdot s)$ for any $x \in A, s \in S$ and $!_A(x) = * \leq * = !_A(y)$ for any $x \leq y$ in A. Hence $!_A$ is a morphism and since we cannot get any other maps $A \to \mathbf{1}$, we cannot get any other morphisms either. So $\mathbf{1}$ is the terminal object in Pos_S. ■

The following proposition is effectively the same as Theorem 18 of [BFM].

Proposition 2.1 *The category* Pos_S *is cartesian closed.*

Proof. For the existence of limits and colimits see [Fa]. For the sake of ease of understanding later, we should mention that products are taken with componentwise order and action. Also, the equalizer of $f : A \to B$ and $g : A \to B$ is the embedding of $\{a \in A | f(a) = g(a)\}$ into A.

In the following we ignore the cases when one of the S-posets is empty, since these cases can be trivially verified.

Take arbitrary objects $X, Y \in \mathrm{Ob}(\mathrm{Pos}_S)$, both of these are of course S-posets. As the exponential object X^Y take the morphism set

$$X^Y := \mathrm{Mor}_{\mathrm{Pos}_S}(S_S \times Y, X),$$

i.e. the set of all order-preserving S-act homomorphisms from the product S-poset $S_S \times X$ to the poset Y. Here, S_S is just the pomonoid S considered as a poset over itself. The order on X^Y is the pointwise one. Define the S-action exactly in the same way as it is done in the case of S-acts (see [MLM] page 62): for $f : S_S \times Y \to X$ and $s \in S$, define $f \cdot s : S_S \times Y \to X$ with the equation

$$(f \cdot s)(t, y) = f(st, y)$$

for all $t \in S, y \in Y$. We have

$$(f \cdot s)((t, y) \cdot u) = (f \cdot s)(tu, y \cdot u) = f(stu, y \cdot u) = (f(st, y)) \cdot u = ((f \cdot s)(t, y)) \cdot u$$

for all $t, u \in S, y \in Y$ and if $(t_1, y_1) \leq (t_2, y_2)$, then

$$(f \cdot s)(t_1, y_1) = f(st_1, y_1) \leq f(st_2, y_2) = (f \cdot s)(t_2, y_2)$$

since f itself preserves the order and so does multiplication in S. So indeed $f \cdot s \in \mathrm{Mor}_{\mathrm{Pos}_S}(S_S \times Y, X)$.

Moreover, $(f \cdot 1)(s, y) = f(1s, y) = f(s, y)$ for all $s \in S, y \in Y$, so $f \cdot 1 = f$. Similarly,

$$((f \cdot s) \cdot t)(u, y) = (f \cdot s)(tu, y) = f(s(tu), y) = f((st)u, y) = (f \cdot (st))(u, y)$$

for all $s, t, u \in S, y \in Y$ and therefore $(f \cdot s) \cdot t = f \cdot (st)$. For the order, if $f \leq g$, then for any $s, t \in S, y \in Y$

$$(f \cdot s)(t, y) = f(st, y) \leq g(st, y) = (g \cdot s)(t, y),$$

that is, $f \cdot s \leq g \cdot s$. In the same line of thought, if $s_1 \leq s_2$, then

$$(f \cdot s_1)(t, y) = f(s_1 t, y) \leq f(s_2 t, y) = (f \cdot s_2)(t, y) \quad \text{for all } t \in S, y \in Y$$

since f is order-preserving and $s_1 t \leq s_2 t$. This concludes our verification that X^Y with the S-action defined above is an S-poset.

Take the evaluation morphism also precisely the same as in the case of Act_S: for $eval : X^Y \times Y \to X$, have

$$eval(f, y) = f(1, y) \quad \text{for all } f \in X^Y, y \in Y.$$

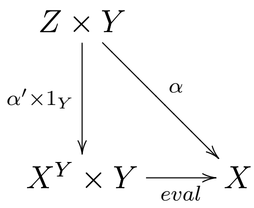
Then

$$\begin{aligned} eval((f, y) \cdot s) &= eval(f \cdot s, y \cdot s) = (f \cdot s)(1, y \cdot s) \\ &= f(s \cdot 1, y \cdot s) = (f(1, y)) \cdot s = (eval(f, y)) \cdot s \end{aligned}$$

for all $f \in X^Y, y \in Y, s \in S$. If $(f_1, y_1) \leq (f_2, y_2)$, then

$$eval(f_1, y_1) = f_1(1, y_1) \leq f_1(1, y_2) \leq f_2(1, y_2) = eval(f_2, y_2).$$

Thus $eval$ is indeed a morphism in Pos_S.



To show that this morphism set X^Y is indeed the exponential object, we have to show that for any object $Z \in \mathrm{Ob}(\mathrm{Pos}_S)$ and any S-poset morphism $\alpha : Z \times Y \to X$ there is a unique morphism $\alpha' : Z \to X^Y$ such that $eval \circ (\alpha' \times 1_Y) = \alpha$. For that we define α' as follows: for any $z \in Z, t \in S, y \in Y$

$$\alpha'(z)(t, y) = \alpha(z \cdot t, y).$$

Then we have

$$\alpha'(z)((t, y) \cdot s) = \alpha(z \cdot (ts), y \cdot s) = (\alpha(z \cdot t, y)) \cdot s = (\alpha'(z)(t, y)) \cdot s$$

for all $s, t \in S, z \in Z, y \in Y$. Likewise, if $(t_1, y_1) \leq (t_2, y_2)$, then

$$\alpha'(z)(t_1, y_1) = \alpha(z \cdot t_1, y_1) \leq \alpha(z \cdot t_2, y_2) = \alpha'(z)(t_2, y_2).$$

Therefore $\alpha'(z)$ is in $X^Y = \mathrm{Mor}_{\mathrm{Pos}_S}(S_S \times Y, X)$.

Additionally, if $z \in Z$ and $s \in S$, then

$$\alpha'(z \cdot s)((t, y)) = \alpha((z \cdot s) \cdot t, y) = \alpha(z \cdot (st), y) = \alpha'(z)(st, y) = (\alpha'(z) \cdot s)(t, y).$$

So $\alpha'(z \cdot s) = \alpha'(z) \cdot s$. Also, for $z_1 \leq z_2$

$$\alpha'(z_1)(t, y) = \alpha(z_1 \cdot t, y) \leq \alpha(z_2 \cdot t, y) = \alpha'(z_2)(t, y)$$

for all $t \in S, y \in Y$ as $z_1 \cdot t \leq z_2 \cdot t$ and α is order-preserving. This shows that α' is a morphism of Pos_S.

Consider the equation

$$eval \circ (\alpha' \times 1_Y) = \alpha.$$

This is equivalent to

$$(eval \circ (\alpha' \times 1_Y))(z, y) = \alpha(z, y)$$

holding for arbitrary $z \in Z$ and $y \in Y$. Some calculation yields

$$\begin{aligned}
\alpha(z \cdot s, y) &= (eval \circ (\alpha' \times 1_Y))(z \cdot s, y) = eval(\alpha'(z \cdot s), y) \\
&= (\alpha'(z) \cdot s)(1, y) = \alpha'(z)(s1, y) = \alpha'(z)(s, y).
\end{aligned}$$

This shows that the α' defined earlier does make this equation true, and is also the only Pos_S morphism to do so. Proof completed. ■

2.2 Subobject classifiers

In [BFM] the nonexistence of a subobject classifier and the fact that Pos_S is not a topos is shown through the fact that not all monomorphisms are regular (Remark 19, point (2) of [BFM]). Here, we will see why exactly the subobject classifier cannot exist and we will also seek to remedy that in some manner.

Proposition 2.2 *For any pomonoid S, the category Pos_S does not have a subobject classifier. Furthermore, it does not have one for even only the embeddings, ie order-reflecting monomorphisms.*

Proof. Take $A = \{a < b < c\}$ and $x \cdot s = x$ for all $s \in S, x \in A$. Then A turns out to be an S-poset. We have the subobject $B = \{a < c\}$ of A, which is the equivalence class of the monomorphism $\iota : B \to A$, with $\iota = 1_A|_B$. If we had a subobject classifier $true : \mathbf{1} \to \Omega$, where Ω is the truth value object, we would have to have the unique map $\phi_B : A \to \Omega$ which makes the diagram

$$\begin{array}{ccc}
B & \xrightarrow{\ !_B\ } & \mathbf{1} \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle true} \\
A & \xrightarrow[\ \phi_B\ ]{} & \Omega
\end{array}$$

into a pullback. Obviously $true$ fixes one element $true(*)$ in Ω. If the aforementioned diagram is a pullback, then $B \cong \{(x, *) \in A \times \mathbf{1} | \phi_B(x) = true(*)\}$ (for more details see [BFM] page 5).

So $\phi_B(a) = \phi_B(c) = true(*)$. Since $a < b < c$ and ϕ_B is order-preserving, we have $\phi_B(b) = true(*)$. But then $\{(x, *) \in A \times \mathbf{1} | \phi_B(x) = true(*)\} = A$. We have thus found out that regardless of the nature of Ω, B cannot even have a characteristic map ϕ_B making the above diagram into a pullback, not to mention its uniqueness. Therefore, no object is fit to be the truth value object. Since ι was also an embedding, we do not have a subobject classifier for embeddings instead of monomorphisms either. ■

Theorem 2.1 *The category* Pos_S *has the subobject classifier for subobjects that are downwards closed.*

The truth value object is

$$\Omega_d = \{X | X \text{ is a right ideal of } S \text{ and } X \text{ is downwards closed}\}$$

with $true(*) = S$. *The order on* Ω_d *is that of reverse inclusion, i.e.*

$$X \leq Y \Leftrightarrow Y \subseteq X.$$

The S*-action is*

$$X \cdot s = \{t \in S | st \in X\}.$$

Proof. Since the S-action is taken directly from the topos of S-acts (see [MLM] page 35), we only have to see that $X \cdot s$ is downwards closed. Take $t \in X \cdot s$ and $t' \leq t$. Since S is a pomonoid, $st' \leq st$. As X is downwards closed, $st' \in X$ and consequently $t' \in X \cdot s$.

We have to verify that Ω_d is an S-poset. As it is done in the case of S-acts, it can be shown that Ω_d is an S-act. If $Y \subseteq X$ and $s \in S$, then

$$Y \cdot s = \{t \in S | st \in Y\} \subseteq \{t \in S | st \in X\} = X \cdot s.$$

Also, if $s \leq s'$, then $st \leq s't$ for any $t \in S$ and

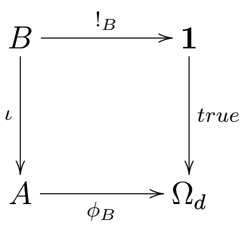
$$X \cdot s' = \{t \in S | s't \in X\} \subseteq \{t \in S | st \in X\} = X \cdot s$$

since X is downwards closed. This concludes the proof that Ω_d is an S-poset.

Take a monomorphism $\iota : B \to A$ where B is downwards closed and define

$$\phi_B(x) = \{s \in S | x \cdot s \in B\}$$

for any $x \in A$. Take $x \cdot s \in B$ and $t \leq s$. Then $x \cdot t \leq x \cdot s$ and consequently $x \cdot t \in B$. So ϕ_B is a well-defined homomorphism of S-acts. If $x \leq y$, then $x \cdot s \leq y \cdot s$ and we have $\phi_B(y) = \{s \in S | y \cdot s \in B\} \subseteq \{s \in S | x \cdot s \in B\} = \phi_B(x)$ as B is downwards closed. This shows that ϕ_B is also order-preserving and hence a morphism in the category Pos_S.



Obviously $\phi_B \circ \iota = true \circ !_B$ as B is a subact. Also $\{(a,*) \in A \times \mathbf{1} | \phi_B(a) = true(*) = S\} = B$, because if $a \notin B$, then $1 \notin \{s \in S | a \cdot s \in B\}$, or equivalently $S \neq \{s \in S | a \cdot s \in B\} = \phi_B(a)$. So we have that B is the pullback of ϕ_B and *true*.

Let $\psi_B : A \to \Omega_d$ be a morphism such that B is the pullback of ψ_B and *true*. Then $B \cong \{(a,*) \in A \times \mathbf{1} | \psi_B(a) = true(*) = S\}$. If $x \cdot s \in B$, then $\{t \in S | st \in \psi_B(x)\} = \psi_B(x) \cdot s = \psi_B(x \cdot s) = S \ni 1$ and so $s \cdot 1 \in \psi_B(x)$. Hence $\{s \in S | x \cdot s \in B\} \subseteq \psi_B(x)$. If $x \cdot s \notin B$ (then $\psi_B(x \cdot s) \neq S$), we get $S \neq \psi_B(x \cdot s) = \psi_B(x) \cdot s = \{t \in S | st \in \psi_B(x)\}$ and so $1 \notin \{t \in S | st \in \psi_B(x)\}$. So $s \notin \psi_B(x)$ and $\psi_B(x) = \{s \in S | x \cdot s \in B\} = \phi_B(x)$. We have verified, that ϕ_B is the only morphism that gives us the desired pullback. This completes our proof. ■

By dualizing the order on S, using Theorem 2.1 and dualizing back to S, we get

Corollary 2.1 *The category* Pos_S *has the subobject classifier for subobjects that are upwards closed.*

The truth value object is

$$\Omega_u = \{X | X \text{ is a right ideal of } S \text{ and } X \text{ is upwards closed}\}$$

with $true(*) = S$. *The order on* Ω_u *is that of inclusion, i.e.*

$$X \leq Y \Leftrightarrow X \subseteq Y.$$

The S*-action is*

$$X \cdot s = \{t \in S | st \in X\}.$$

Proof. ■

2.3 Monomorphism types

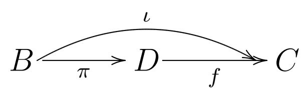
Now that we have seen that monomorphisms and embeddings fail to give us a subobject classifier, we shall examine whether any other types of monomorphism might do better.

In the following we first examine which notions of monomorphisms are different in the category of S-posets over a pomonoid S. Recall that an embedding is an injective homomorphism of S-acts that both preserves and reflects the order.

Proposition 3 of [BFM] shows a part of the following proof (epimorphisms are surjective homomorphisms) and Theorem 7 of [BFM] describes the extremal monomorphisms.

Lemma 2.3 *In* Pos_S *extremal monomorphisms are embeddings.*

Proof. Take an extremal morphism $\iota : B \to C$ and suppose it is not an embedding. Then we have $x, y \in B$ such that $\iota(x) \leq \iota(y)$, but $x \nleq y$. We can thus define $\pi : B \to D$ and $f : D \to C$ with $D = \mathrm{Im}(\iota)$, $\pi(z) = \iota(z)$ for all $z \in B$ and f inserting $D = \mathrm{Im}(\iota)$ into C. As π is a surjective S-act homomorphism, it must also be an S-act epimorphism.



To see that, take $g \circ \pi = f \circ \pi$, with f and g being morphisms in Pos_S. We get $f = g$ in S-Act which means they coincide as mappings and consequently as Pos_S homomorphisms.

It is trivial to see that π preserves the order. So π is an epimorphism of S-posets. Obviously $\iota = f \circ \pi$. But π is not left invertible, because if we had $i \circ \pi = 1_B$, then $x = (i \circ \pi)(x) = i(\iota(x)) \leq i(\iota(y)) = (i \circ \pi)(y) = y$, as i must preserve the order. Therefore we have a factorization $\iota = f \circ \pi$, with π an epimorphism but not an isomorphism. This cannot happen as ι is extremal and therefore our assumption that ι was not an embedding does not hold. ■

Lemma 2.4 *In* Pos_S*, every embedding is a regular monomorphism.*

Proof. Let $\iota : B \to C$ be an embedding and $B' = \mathrm{Im}\iota$. Take

$$D = C \amalg (C \setminus B').$$

For a clearer notation, have

$$D = C_1 \amalg C_2 \amalg B'$$

with $C_1 = C_2 = C \setminus B'$.

We need an order on D and for that define $x \leq y$ in D iff one of the following holds

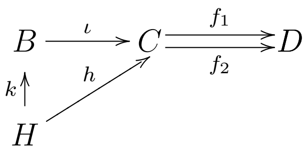
- $x \in B', y \in C_i, i = 1, 2$ and $x \leq y$ in C;
- $x \in C_i, i = 1, 2, y \in B'$ and $x \leq y$ in C;
- $x, y \in B'$ and $x \leq y$ in C;
- $x, y \in C_i, i = 1, 2$ and $x \leq y$ in C;
- $x \in C_i, y \in C_j, i \neq j$ and $\exists z \in B'$ such that $x \leq z \leq y$ in C.

Essentially this is the transitive closure of the amalgamated coproduct of C with itself over B' in the sense of Act_S. Obviously we get a reflexive relation. By taking transitive closure we ensure transitivity, since the only failures to remain transitive have to compare elements from different copies of C. Antisymmetry within the two copies of C carries over, and if we take $x \in C_i, y \in C_j, i \neq j$ from different copies, then $x \leq y$ and $y \leq x$ holds only if there are $u, v \in B'$ such that $x \leq u \leq y \leq v \leq x$, whence $x = u \in B'$, an impossibility. So we have a partial order on D.

The action on C can clearly be transferred to D, as B' is a subact and so both partial actions (of S on C) are confined to respective copies of C.

Now take from D two elements $d \leq d'$ and $s \in S$. Then $ds \leq d's$. The only place where that might not hold (as C is an S-poset) is when $d \in C_i$ and $d' \in C_j$, $i \neq j$. Then $d \leq d'' \leq d'$, with $d'' \in B'$. But in that case $ds \leq d''s \leq d's$, and consequently $ds \leq d's$.

Likewise, for $d \in D$ and $s, s' \in S$, with $s \leq s'$, we get $ds \leq ds'$ as this holds within both copies of C without problems. So D is indeed an S-poset.



Define $f_1, f_2 : C \longrightarrow D$, with $f_i(x) = x$ if $x \in B'$ and $f_i(x) = g_i(x)$ if $x \notin B'$, where $g_i : C \to C_i \amalg B'$ are the isomorphisms. Trivially $f_1 \circ \iota = f_2 \circ \iota$. Since f_i are both essentially identities, they preserve the order and the S-action on D (which was unchanged within the separate copies of C). So they are both morphisms in Pos_S.

Moreover, if $h : H \to C$ is an order-preserving S-act homomorphism and $f_1 \circ h = f_2 \circ h$ then clearly $\mathrm{Im}\, h \subseteq B'$. Let $x \in H$. Then there exists a unique $b \in B$ such that $\iota(b) = h(x)$. Define

$$k(x) = b.$$

Because ι is an embedding, k is order-preserving, and obviously also a homomorphism. We have

$$(\iota \circ k)(x) = \iota(b) = h(x)$$

for all $x \in H$ and hence $\iota \circ k = h$. Since ι is a monomorphism, this means that we have shown (B, ι) to be the equalizer of f_1 and f_2. Therefore, ι is a regular monomorphism. ■

So far, we have not managed to get anything different from embeddings and usual monomorphisms (injective homomorphisms). One might try to get something different and generalize the extremal monomorphisms as follows.

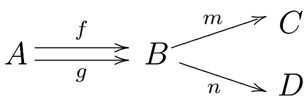
Definition 2.1 A morphism $\iota : B \to C$ in a category $\mathcal{C}$ is a *regularly extremal morphism*, if

$$(\forall D \in \mathrm{Ob}(\mathcal{C}))(\forall \pi : B \to D)(\forall f : D \to C)$$
$$(\iota = f \circ \pi \wedge \pi \text{ is regular epimorphism } \Rightarrow \pi \text{ is isomorphism}).$$

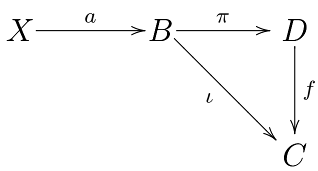
The following result shows that this definition is much weaker and is actually not even a generalization, but more of an overgeneralization.

Lemma 2.5 *Monomorphisms are always regularly extremal morphisms, but regularly extremal morphisms do not have to be monomorphisms.*

Proof. First, let us see that a regularly extremal morphism does not have to be a monomorphism. Take a category with four objects A, B, C and D. Let $f, g : A \to B$, $m : B \to C$, $n : C \to D$ and their composites be the only non-trivial morphisms in this category, with the added restrictions $m \circ f = m \circ g$ and $n \circ f = n \circ g$. The morphism m is obviously not a monomorphism. The only possible factorizations of m are either $m \circ 1_B$ or $1_C \circ m$. 1_B is a regular epimorphism, but also an isomorphism. The morphism m itself is not an isomorphism, not even a coretraction. Therefore it can only be the coequalizer of f and g. But there are no morphisms from C to D and thus m is not a regular epimorphism and therefore is a regularly extremal morphism.



Now, let $\iota : B \to C$ be a monomorphism in an arbitrary category and let $\iota = f \circ \pi$, where $\pi : B \to D$ is a regular epimorphism. Then π must be the coequalizer of a single morphism a (with a copy of this morphism to coequalize with), since whatever π coequalizes, $f \circ \pi = \iota$ makes equal and m is a monomorphism.



From this we establish that π has a left inverse, since $1_B \circ a = 1_B \circ a$ and thus there must be a unique $b : D \to B$ such that $b \circ \pi = 1_B$. In the same line of reasoning, $\pi \circ a = \pi \circ a$ and so there is a unique $c : D \to D$ such that $c \circ \pi = \pi$. But $(\pi \circ b) \circ \pi = \pi = 1_D \circ \pi$, therefore $c = \pi \circ b = 1_D$ and π is an isomorphism. This proves that ι is regularly extremal. ■

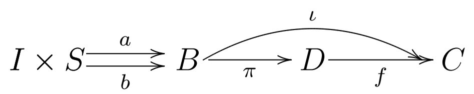
It turns out that in the case of Pos_S, the converse holds as well.

Lemma 2.6 *In* Pos_S *all monomorphisms are precisely the regularly extremal morphisms.*

Proof. We have seen that all monomorphisms are regularly extremal. Suppose $\iota : B \to C$ is not a monomorphism, then

$$\emptyset \neq \{(x, y) \in B^2 | \iota(x) = \iota(y), x \neq y\} = \{(x_i, y_i) \in B^2 | i \in I\}$$

for some index set I. We can take $D = \mathrm{Im}(\iota)$, with $\pi : B \to D$ defined as $\pi(z) = \iota(z)$ for all $z \in B$. Define $f : D \to C$ with $f(z) = z$ for all $z \in D$. Both π and f are S-poset morphisms and clearly $f \circ \pi = \iota$.



Take a new poset, denoted $I \times S$, such that it consists of all (i, s), where $i \in I, s \in S$ and $(i, s) \cdot t = (i, st)$ for all $t \in S$. With S-induced order (that is, the only order relations are $i \cdot s \leq i \cdot s'$ with $s \leq s'$) and the obvious action, it is evidently an S-poset. The epimorphism π is the coequalizer of the pair $a, b : I \times S \to B$ of S-poset morphisms with $a(i, s) = x_i \cdot s, b(i, s) = y_i \cdot s$. To see this, have $g \circ a = g \circ b$ for some morphism $g : B \to G$, which means $g(x_i) = g(y_i)$ for each $i \in I$. Then we define a new morphism $k : D \to G$ as follows. Take $z \in D$, in which case there is at least one $u \in B$ such that $\pi(u) = z$. Define

$$k(z) = g(u).$$

For any other such u' that $\pi(u') = z$ we have $\iota(u') = \iota(u) = z$, so $g(u) = g(u')$ as well and the map is well-defined. It is also easy to see that k preserves both the order and the S-action. Of course $k \circ \pi = g$. As π is an epimorphism, k must be unique. Therefore ι is not regularly extremal because π is not an isomorphism, but is a regular epimorphism. ■

2.4 Category of posets as a topos

By Lemmas 2.3, 2.4 and 2.6 we obtain the following:

Proposition 2.3 *In* Pos_S *the notions of regular, strict, strong and extremal monomorphism coincide, being precisely all the embeddings of* Pos_S*. Regularly extremal morphisms are the same as monomorphisms.*

As we can see, most notions of monomorphism lead us to embeddings, with (actual) monomorphisms (the same as regularly extremal morphisms) and coretractions being separate. In Pos_S we cannot classify neither embeddings nor monomorphisms. Therefore we cannot use any of those notions to get Pos_S to have a partial (i.e. valid only for a certain class of monomorphisms) subobject classifier.

In the end, Pos_S is not a topos, but it is rather close. It has arbitrary limits and colimits, exponentials and a restricted subobject classifier. Our hope was to get a result like that of usual S-acts, which form a Grothendieck topos (see [MLM], third chapter). This was motivated by S-posets being very similar to S-acts in the functorial sense, the former as **2**-functors to the category of posets and the latter as functors to the category of sets. Unfortunately, we can not get a Grothendieck topos out of it (since these have to be toposes), and one can only hope that there is a suitable **2**-categorical notion of Grothendieck topology that might be used on S-posets as some sort of **2**-sheaves.

2.5 Submonomorphisms and subclassifiers

In the study of S-posets, in some situations it has proved to be useful to substitute the equalities of composites with inequalities of composites in the definitions of (co)limits. Subpullbacks, subequalizers etc thus derived have been introduced in [BFM] and [BFL]. Such an approach motivates our definition of submonomorphisms in the same way. From [BFM] we obtain that subpullbacks of $f : A \to C$ and $g : B \to C$ can be canonically constructed as

$$\{(a, b)|a \in A, b \in B, f(a) \leq g(b)\}$$

and subpullbacks of g and f as

$$\{(b, a)|a \in A, b \in B, g(b) \leq f(a)\}.$$

Definition 2.2 We call an S-poset morphism $m : A_S \to B_S$ a *submonomorphism* iff for all S-poset morphisms $f, g : C_S \to A_S$ whenever $m \circ f \leq m \circ g$, $f \leq g$ as well.

Lemma 2.7 *Submonomorphisms in* Pos_S *are precisely embeddings.*

Proof. If m is an embedding and $m \circ f \leq m \circ g$ for $f, g : C_S \to A_S$, then $m(f(c)) \leq m(g(c))$ for all $c \in C$. Therefore $f(c) \leq g(c)$ and $f \leq g$.

If m is a submonomorphism and $m(x) \leq m(y)$, then we can consider morphisms $\mathbf{x} : S_S \to A_S$ and $\mathbf{y} : S_S \to A_S$. Then

$$(m \circ \mathbf{x})(s) = m(x \cdot s) = m(x) \cdot s \leq m(y) \cdot s = m(y \cdot s) = (m \circ \mathbf{y})(s)$$

for all $s \in S$. Thus $m \circ \mathbf{x} \leq m \circ \mathbf{y}$, whence $\mathbf{x} \leq \mathbf{y}$. In particular, $x = x \cdot 1 = \mathbf{x}(1) \leq \mathbf{y}(1) = y \cdot 1 = y$. So $x \leq y$ and m reflects order. Every submonomorphism is a monomorphism, since equality implies both inequalities and vice versa. Therefore m is indeed an embedding. ■

While we could define subregular, substrict, substrong etc morphisms, by [BFM] subequalizers are embeddings, subcoequalizers are surjections and we would not get any new classes of morphisms.

Instead, we define subobject subclassifiers as follows. Note that since terminal objects are not defined with equalities of composites, subterminal objects are the same as terminal objects.

Definition 2.3 In a category with finite sublimits the *subobject subclassifier (subobject supclassifier)* is a monomorphism $true : \mathbf{1} \to \Omega$ such that for any other monomorphism $\iota : B \to A$ there is a unique morphism ϕ_B such that the square

$$\begin{array}{ccc} B & \xrightarrow{!_B} & \mathbf{1} \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle true} \\ A & \xrightarrow[\phi_B]{} & \Omega \end{array}$$

turns out to be a subpullback of ϕ_B and $true$ (of $true$ and ϕ_B).

These objects are also unique up to isomorphism, which is proved in the same way as in the case of subobject classifiers.

Once again, it suffices to consider only injections of subsets.

In the case of Pos_S we have the following result.

Theorem 2.2 *The category* Pos_S *has the subobject subclassifier for subobjects that are downwards closed and the subobject supclassifier for subobjects that are upwards closed. Moreover, any other subobjects can not be subclassified or supclassified.*

Proof. Take precisely the same Ω_d, $true$ and ϕ_B for downwards closed subobjects as we did in Theorem 2.1. Take a monomorphism $\iota : B \to A$. Since $true(*) = S$,

$$\begin{aligned} \phi_B(a) \leq true(*) \quad &\Leftrightarrow \quad S \subseteq \phi_B(a) \Leftrightarrow S = \phi_B(a) \\ &\Leftrightarrow \quad \forall s \in S \quad a \cdot s \in B \Leftrightarrow a \in B \end{aligned}$$

for every $a \in A$. Then

$$\{(a, *) \in A \times \mathbf{1} | \phi_B(a) \leq true(*)\} = B \times \mathbf{1} \cong B.$$

Therefore we do obtain a subpullback.

Let $\psi_B : A \to \Omega_d$ be a morphism such that B is the subpullback of ψ_B and $true$. Then $B \cong \{(a, *) \in A \times \mathbf{1} | \psi_B(a) \leq true(*) = S\}$. We are trying to prove $\phi_B = \psi_B$. If $x \cdot s \in B, x \in A, s \in S$, then $\{t \in S | st \in \psi_B(x)\} = \psi_B(x) \cdot s = \psi_B(x \cdot s) \supseteq S \ni 1$ and so $s \cdot 1 \in \psi_B(x)$. Hence $\{s \in S | x \cdot s \in B\} \subseteq \psi_B(x)$. If $x \cdot s \notin B$ (then $\psi_B(x \cdot s) \neq S$ because otherwise $\psi_B(x \cdot s) \supseteq S$ and $x \cdot s \in B$ from the subpullback), we get $S \neq \psi_B(x \cdot s) = \psi_B(x) \cdot s = \{t \in S | st \in \psi_B(x)\}$ and so $1 \notin \{t \in S | st \in \psi_B(x)\}$. So $s \notin \psi_B(x)$ and $\psi_B(x) = \{s \in S | x \cdot s \in B\} = \phi_B(x)$. We have verified, that ϕ_B is the only morphism that gives us the desired subpullback.

Upwards closed subobjects are supclassified by using Ω_u, $true$ and ϕ_B implied by Corollary 2.1.

Suppose a subobject $\iota : B \to A$ can be subclassified and there is an $a \in A$ such that $a \leq b$ for some $b \in B$, but $a \notin B$. Then $\phi_B(a) \supseteq \phi_B(b) = S$, so $a \in B$, a contradiction. Similarly only upwards closed subobjects can be supclassified. ■

3 Geometric morphisms

As the previous part demonstrated, Pos_S is not a topos and therefore the usual notion of a geometric morphism (for more details, see [MLM], chapter VII) does not apply there. In the following we show that by adding a few order-related restrictions, we can obtain similar morphisms (pogeometric morphisms) between S-poset categories.

3.1 Pofunctors and poadjunctions

Definition 3.1 We say that a functor $F : \mathrm{Pos}_S \to \mathrm{Pos}_T$ is a *pofunctor* if for any pair of S-poset morphisms $f_1, f_2 : B_S \to B'_S$ with $f_1 \leq f_2$ also $F(f_1) \leq F(f_2)$.

Definition 3.2 We say that two adjoint pofunctors $L : \mathrm{Pos}_S \to \mathrm{Pos}_T$ and $R : \mathrm{Pos}_T \to \mathrm{Pos}_S$, $L \dashv R$, form a *poadjunction* (L is *left poadjoint* to R), if the corresponding binatural isomorphism

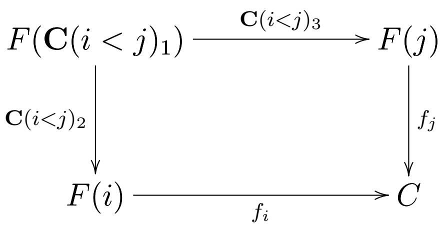
$$\alpha : \mathrm{Mor}_{\mathrm{Pos}_T}(L(-), -) \to \mathrm{Mor}_{\mathrm{Pos}_S}(-, R(-))$$

also preserves and reflects the pointwise order of S- and T-poset morphisms. This means that for all S-posets B_S, T-posets C_T and $f_1, f_2 : L(B_S) \to C_T$, $f_1 \leq f_2$ we have $\alpha_{B,C}(f_1) \leq \alpha_{B,C}(f_2)$. Also, for all $g_1, g_2 : B_S \to R(C_T)$, $g_1 \leq g_2$ it must hold that $\alpha^{-1}_{B,C}(g_1) \leq \alpha^{-1}_{B,C}(g_2)$.

Definition 3.3 Let $F : I \to \mathrm{Pos}_S$ be a functor on an index category I, where the objects form a poset $(\mathrm{Ob}(I), \leq)$. Let $\mathbf{C} :<\to \mathrm{Ob}(I) \times \mathrm{Mor}_{\mathrm{Pos}_S} \times \mathrm{Mor}_{\mathrm{Pos}_S}$ be such a mapping that for each $i < j$, where $i, j \in \mathrm{Ob}(I)$, there is an object $\mathbf{C}(i < j)_1 \in \mathrm{Ob}(I)$ and a pair of morphisms $\mathbf{C}(i < j)_2 : F(\mathbf{C}(i < j)_1) \to F(i)$ and $\mathbf{C}(i < j)_3 : F(\mathbf{C}(i < j)_1) \to F(j)$ in Pos_S. We say that a cocone $(C, (f_i)_{i \in \mathrm{Ob}(I)})$ on a diagram $(F(i))_{i \in \mathrm{Ob}(I)}$ of shape F is a *pococone* of F with respect to mapping $\mathbf{C}$ if

$$f_i \circ \mathbf{C}(i < j)_2 \leq f_j \circ \mathbf{C}(i < j)_3$$

for all $i < j, i, j \in \mathrm{Ob}(I)$.



If we have another functor $G : \text{Pos}_S \to \text{Pos}_T$, then we write $G \circ \mathbf{C}$ instead of the longer $(1 \times G \times G) \circ \mathbf{C}$ to denote the corresponding mapping for the functor $G \circ F$, ie $(G \circ \mathbf{C})(i < j) = (\mathbf{C}(i < j)_1, G(\mathbf{C}(i < j)_2), G(\mathbf{C}(i < j)_3))$.

A *universal pococone* of a functor $F : I \to \text{Pos}_S$ with respect to mapping $\mathbf{C}$ is a pococone $(U, (u_i)_{i \in \text{Ob}(I)})$ with the property that for every other pococone $(V, (v_i)_{i \in \text{Ob}(I)})$ of the same functor and with respect to the same mapping there is a unique morphism $f : U \to V$ such that $f \circ u_i = v_i$ for all $i \in \text{Ob}(I)$.

Note that universal pococones of the same functor with respect to the same mapping $\mathbf{C}$ are unique up to isomorphism, this is proved in the usual way. Universal pococones of trivial orders are ordinary colimits.

Lemma 3.1 *If $(U, (u_i)_{i \in \text{Ob}(I)})$ is the universal pococone of $F : I \to \text{Pos}_S$ with respect to $\mathbf{U}$, $(V, (v_i)_{i \in \text{Ob}(I)})$ is the universal pococone of $G : I \to \text{Pos}_S$ with respect to $\mathbf{V}$, functors F and G are naturally isomorphic with natural isomorphisms $\alpha_i : F(i) \to G(i)$ for all $i \in \text{Ob}(I)$, $\mathbf{U}(i < j)_1 = \mathbf{V}(i < j)_1$ and*

$$\mathbf{V}(i < j)_2 \circ \alpha_{\mathbf{U}(i<j)_1} = \alpha_i \circ \mathbf{U}(i < j)_2,$$

$$\mathbf{V}(i < j)_3 \circ \alpha_{\mathbf{U}(i<j)_1} = \alpha_j \circ \mathbf{U}(i < j)_3$$

for all $i < j$ in $\text{Ob}(I)$, then there is a unique morphism $\alpha_U : U \to V$ such that

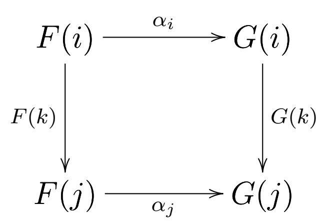
$$\alpha_U \circ u_i = v_i \circ \alpha_i$$

for all $i \in \text{Ob}(I)$, which is also an isomorphism.

Proof. Due to naturality of α,

$$G(k) \circ \alpha_i = \alpha_j \circ F(k)$$

for all morphisms $k : i \to j$ in I.



This implies

$$v_j \circ \alpha_j \circ F(k) = v_j \circ G(k) \circ \alpha_i = v_i \circ \alpha_i$$

for $k : i \to j$. Therefore

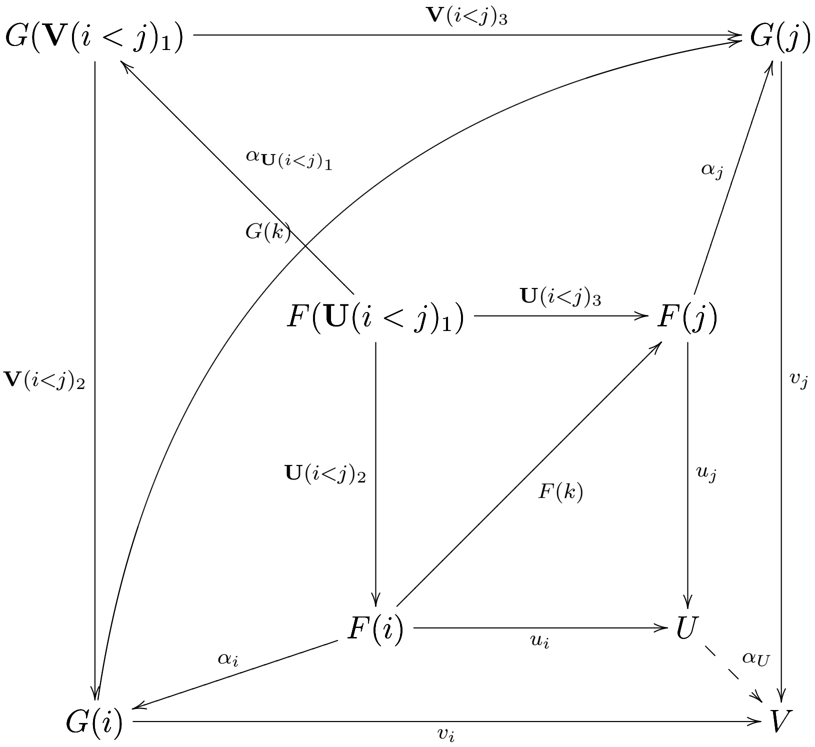
$$(V, (v_i \circ \alpha_i)_{i \in \text{Ob}(I)})$$

is a cocone of F. Also,

$$\begin{aligned} v_i \circ \alpha_i \circ \mathbf{U}(i<j)_2 &= v_i \circ \mathbf{V}(i<j)_2 \circ \alpha_{\mathbf{U}(i<j)_1} \\ &\leq v_j \circ \mathbf{V}(i<j)_3 \circ \alpha_{\mathbf{U}(i<j)_1} \\ &= v_j \circ \alpha_j \circ \mathbf{U}(i<j)_3. \end{aligned}$$

So $(V, (v_i \circ \alpha_i)_{i \in \mathrm{Ob}(I)})$ is a cocone of F with respect to $\mathbf{U}$. Therefore we have a unique $\alpha_U : U \to V$ such that

$$\alpha_U \circ u_i = v_i \circ \alpha_i.$$



Reversing the roles of U and V in the previous argument, we obtain that there is a unique $\alpha_V : V \to U$ such that

$$\alpha_V \circ v_i = u_i \circ \alpha_i^{-1}.$$

Therefore

$$\alpha_V \circ \alpha_U \circ u_i = \alpha_V \circ v_i \circ \alpha_i = u_i \circ \alpha_i^{-1} \circ \alpha_i = u_i.$$

Since also

$$1_U \circ u_i = u_i,$$

the uniqueness property of universal cocone U gives us $\alpha_V \circ \alpha_U = 1_U$. Additionally

$$\alpha_U \circ \alpha_V \circ v_i = \alpha_U \circ u_i \circ \alpha_i^{-1} = v_i \circ \alpha_i \circ \alpha_i^{-1} = v_i$$

and the universal property of V provides $\alpha_U \circ \alpha_V = 1_V$. Thus α_U is the isomorphism we wanted. ■

Proposition 3.1 *Left poadjoints preserve universal pococones. More precisely, if $(C, (f_i)_{i\in \mathrm{Ob}(I)})$ is the universal pococone of $F : I \to \mathrm{Pos}_S$ with respect to mapping $\mathbf{C}$ and the functor $L : \mathrm{Pos}_S \to \mathrm{Pos}_T$ is a left poadjoint, then $(L(C), (L(f_i))_{i\in \mathrm{Ob}(I)})$ is the universal pococone of $L \circ F$ with respect to mapping $L \circ \mathbf{C}$.*

Proof. Let $L : \mathrm{Pos}_S \to \mathrm{Pos}_T$ be left poadjoint to $R : \mathrm{Pos}_T \to \mathrm{Pos}_S$ with the corresponding natural isomorphisms

$$\alpha_{A,B} : \mathrm{Mor}_{\mathrm{Pos}_T}(L(A), B) \to \mathrm{Mor}_{\mathrm{Pos}_S}(A, R(B)),$$

$A \in \mathrm{Ob}(\mathrm{Pos}_S), B \in \mathrm{Ob}(\mathrm{Pos}_T)$. Let I be an index category the object set of which is also ordered and consider a functor $F : I \to \mathrm{Pos}_S$. Let $(C, (f_i)_{i\in \mathrm{Ob}(I)})$ be the universal pococone of F with respect to $\mathbf{C}$, which means

$$f_i \circ \mathbf{C}(i < j)_2 \le f_j \circ \mathbf{C}(i < j)_3$$

for all $i < j, i, j \in \mathrm{Ob}(I)$. We are trying to prove that $(L(C), (L(f_i))_{i\in \mathrm{Ob}(I)})$ is a universal pococone of $L \circ F$ with regard to $L \circ \mathbf{C}$.

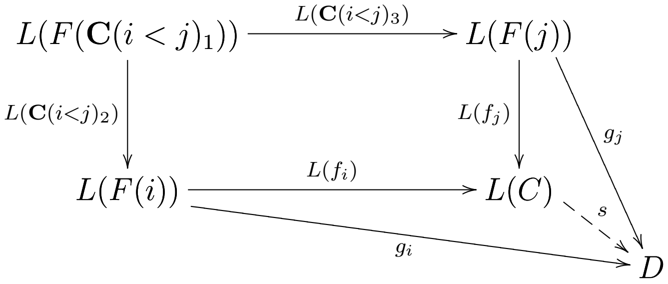
The functorial image of a cocone is a cocone. Since L is a pofunctor, we have

$$L(f_i) \circ L(\mathbf{C}(i < j)_2) \le L(f_j) \circ L(\mathbf{C}(i < j)_3)$$

for all $i < j, i, j \in \mathrm{Ob}(I)$. Therefore $(L(C), (L(f_i))_{i\in \mathrm{Ob}(I)})$ is a pococone of $L \circ F$ with respect to $L \circ \mathbf{C}$.

Now consider a pococone $(D, (g_i)_{i\in \mathrm{Ob}(I)})$ of $L \circ F$ with respect to $L \circ \mathbf{C}$, implying

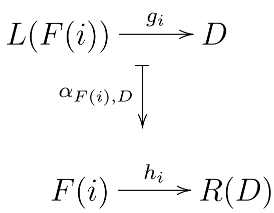
$$g_i \circ L(\mathbf{C}(i < j)_2) \le g_j \circ L(\mathbf{C}(i < j)_3).$$



Due to adjointness we have morphisms

$$h_i = \alpha_{F(i),D}(g_i) : F(i) \to R(D)$$

for all $i \in \mathrm{Ob}(I)$.



For a morphism $k : i \to j$ the naturality of the isomorphisms yields

$$\begin{aligned} h_i &= \alpha_{F(i),D}(g_i) = \alpha_{F(i),D}(g_j \circ L(F(k))) \\ &= (\alpha_{F(i),D} \circ \mathrm{Mor}_{\mathrm{Pos}_T}(L(F(k)), D))(g_j) \\ &= (\mathrm{Mor}_{\mathrm{Pos}_S}(F(k), R(D)) \circ \alpha_{F(j),D})(g_j) \\ &= \mathrm{Mor}_{\mathrm{Pos}_S}(F(k), R(D))(h_j) = h_j \circ F(k). \end{aligned}$$

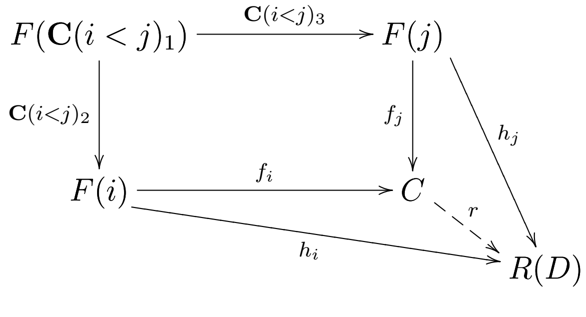
Additionally, the poadjunction implies

$$\alpha_{F(\mathbf{C}(i<j)_1),D}(g_i \circ L(\mathbf{C}(i<j)_2)) \leq \alpha_{F(\mathbf{C}(i<j)_1),D}(g_j \circ L(\mathbf{C}(i<j)_3))$$

for $i < j, i, j \in \mathrm{Ob}(I)$, whence

$$\begin{aligned} h_i \circ \mathbf{C}(i<j)_2 &= \alpha_{F(i),D}(g_i) \circ \mathbf{C}(i<j)_2 \\ &= [\mathrm{Mor}_{\mathrm{Pos}_S}(\mathbf{C}(i<j)_2, R(D)) \circ \alpha_{F(i),D}](g_i) \\ &= [\alpha_{F(\mathbf{C}(i<j)_1),D} \circ \mathrm{Mor}_{\mathrm{Pos}_T}(L(\mathbf{C}(i<j)_2), D)](g_i) \\ &= \alpha_{F(\mathbf{C}(i<j)_1),D}(g_i \circ L(\mathbf{C}(i<j)_2)) \\ &\leq \alpha_{F(\mathbf{C}(i<j)_1),D}(g_j \circ L(\mathbf{C}(i<j)_3)) \\ &= [\alpha_{F(\mathbf{C}(i<j)_1),D} \circ \mathrm{Mor}_{\mathrm{Pos}_T}(L(\mathbf{C}(i<j)_3), D)](g_j) \\ &= [\mathrm{Mor}_{\mathrm{Pos}_S}(\mathbf{C}(i<j)_3, R(D)) \circ \alpha_{F(j),D}](g_j) \\ &= \alpha_{F(j),D}(g_j) \circ \mathbf{C}(i<j)_3 = h_j \circ \mathbf{C}(i<j)_3. \end{aligned}$$

This makes $(R(D), (h_i)_{i \in \mathrm{Ob}(I)})$ into a pococone on F with respect to $\mathbf{C}$.



From this we get the unique factorization $h_i = r \circ f_i$ with $r : C \to R(D)$, for all $i \in \mathrm{Ob}(I)$. By adjointness we have $s = \alpha^{-1}_{C,D}(r) : L(C) \to D$, and by naturality of $\alpha^{-1}_{C,D}$ also

$$\begin{aligned} s \circ L(f_i) &= \mathrm{Mor}_{\mathrm{Pos}_T}(L(f_i), D)(\alpha^{-1}_{C,D}(r)) = \alpha^{-1}_{F(i),D}(\mathrm{Mor}_{\mathrm{Pos}_S}(f_i, R(D))(r)) \\ &= \alpha^{-1}_{F(i),D}(r \circ f_i) = \alpha^{-1}_{F(i),D}(h_i) = g_i. \end{aligned}$$

$$\begin{array}{ccc} \mathrm{Mor}_{\mathrm{Pos}_S}(C, R(D)) & \xrightarrow{\alpha^{-1}_{C,D}} & \mathrm{Mor}_{\mathrm{Pos}_T}(L(C), D) \\ \downarrow {\scriptstyle \mathrm{Mor}_{\mathrm{Pos}_S}(f_i, R(D))} & & \downarrow {\scriptstyle \mathrm{Mor}_{\mathrm{Pos}_T}(L(f_i), D)} \\ \mathrm{Mor}_{\mathrm{Pos}_S}(F(i), R(D)) & \xrightarrow[\alpha^{-1}_{F(i),D}]{} & \mathrm{Mor}_{\mathrm{Pos}_T}(L(F(i)), D) \end{array}$$

For any other $s' = \alpha^{-1}_{C,D}(r') : L(C) \to D$ such that $s' \circ L(f_i) = g_i$ for every $i \in \mathrm{Ob}(I)$, the naturality of $\alpha_{C,D}$ gives

$$\begin{aligned} r' \circ f_i &= \alpha_{C,D}(s') \circ f_i = \mathrm{Mor}_{\mathrm{Pos}_S}(f_i, R(D)(\alpha_{C,D}(s')) \\ &= \alpha_{F(i),D}(\mathrm{Mor}_{\mathrm{Pos}_T}(L(f_i), D)(s')) \\ &= \alpha_{F(i),D}(s' \circ L(f_i)) = \alpha_{F(i),D}(g_i) = h_i. \end{aligned}$$

As $(C, (f_i)_{i \in \mathrm{Ob}(I)})$ is universal, $r = r'$ and s is unique. ■

Lemma 3.2 *Every S-poset is a universal pococone of free S-posets S_S.*

Proof. Let B_S be an S-poset. Construct a new category $\mathrm{El}(B)$, where $\mathrm{Ob}(\mathrm{El}(B)) = B$ and if $b, c \in B$, then $\mathrm{Mor}_{\mathrm{El}(B)}(b, c) = \{s \in S | c \cdot s = b\}$ and the composition is the multiplication of S. Note that B is a partial order.

Define a functor $F : \mathrm{El}(B) \to \mathrm{Pos}_S$ by

$$F(b) = S_S$$

and

$$F(s) = \mathbf{s}$$

for all $b, c \in B, s \in \mathrm{Mor}_{\mathrm{El}(B)}(b, c)$. From Lemma 1.1

$$F(ss') = \mathbf{ss'} = \mathbf{s} \circ \mathbf{s'} = F(s) \circ F(s'),$$

$F(1) = \mathbf{1} = 1_S$ and we do obtain a functor.

We have the S-poset morphisms $f_b : F(b) \to B_S$ defined as

$$f_b = \mathbf{b}$$

for all $b \in B$.

If $b < b', b, b' \in B$, we take

$$\mathbf{B}(b < b')_1 = b$$

and

$$\mathbf{B}(b < b')_2 = \mathbf{B}(b < b')_3 = \mathbf{1} = 1_S : S_S \to S_S.$$

We want to show that $(B_S, (f_b)_{b\in B})$ is the universal pococone of F with respect to the order of B and mapping $\mathbf{B}$.

First of all, take $b, c \in B$, $c \cdot s = b$. Then by Lemma 1.1

$$f_c \circ F(s) = \mathbf{c} \circ \mathbf{s} = \mathbf{c} \cdot \mathbf{s} = \mathbf{b} = f_b$$

for any $s : b \to c$ in $\mathrm{El}(B)$ and we have a cocone.

Secondly, take $b < c, b, c \in B$. Then

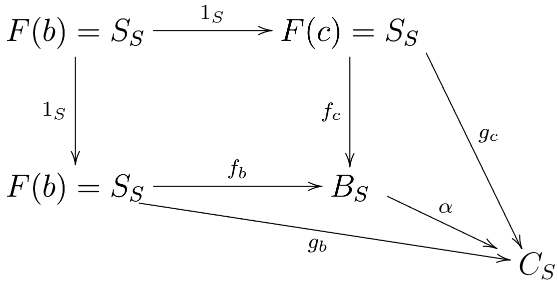
$$f_b \circ \mathbf{B}(b < c)_2 = \mathbf{b} \circ \mathbf{1} = \mathbf{b} \leq \mathbf{c} = \mathbf{b} \circ \mathbf{1} = f_c \circ \mathbf{B}(b < c)_3$$

and $(B_S, (f_b)_{b\in B})$ turns out to be a pococone.

Take another pococone $(C_S, (g_b)_{b\in B})$ with respect to $\mathbf{B}$, which means

$$g_b = g_b \circ \mathbf{B}(b < c)_2 \leq g_c \circ \mathbf{B}(b < c)_3 = g_c$$

when $b < c, b, c \in B$.



To prove the universal property, we define $\alpha : B_S \to C_S$ with

$$\alpha(b) = g_b(1).$$

If $b < c$, then

$$\alpha(b) = g_b(1) \leq g_c(1) = \alpha(c)$$

and evidently $\alpha(b) \leq \alpha(b)$. Also,

$$\alpha(b \cdot s) = g_{b \cdot s}(1) = (g_b \circ F(s))(1) = g_b(s1) = g_b(1) \cdot s = \alpha(b) \cdot s$$

by naturality of the cocone $(C_S, (g_b)_{b \in B})$. Hence α is an S-poset morphism.

Moreover,

$$(\alpha \circ f_b)(s) = \alpha(bs) = g_{bs}(1) = g_b(s)$$

for all $s \in S$.

Suppose there is also an S-poset morphism $\alpha' : B_S \to C_S$ such that $\alpha' \circ f_b = g_b$. Then

$$\alpha'(b) = \alpha'(f_b(1)) = (\alpha' \circ f_b)(1) = g_b(1) = \alpha(b)$$

for all $b \in B$. Therefore $\alpha = \alpha'$ and the proof is finished. ■

3.2 Tensor products

The study of tensor products and flatness properties of S-posets was initiated by S.M. Fakhruddin in the 1980s so the following facts are nothing new, but there is no standard reference for them either. So let us recall a few basic facts about tensor products.

Definition 3.4 The *tensor product* $B \otimes_S A$ of S-posets B_S and ${}_SA$ is the coequalizer of the pair of poset morphisms

$$B \times S \times A \overset{f}{\underset{g}{\rightrightarrows}} B \times A$$

where $f(b, s, a) = (b \cdot s, a)$ and $g(b, s, a) = (b, s \cdot a)$.

Recall that coequalizers in the category of posets have a canonical construction as quotients by the smallest congruences (and corresponding smallest orders) that identify the images at every point.

If we take a right S-poset B_S and an (S, T)-biposet ${}_SA_T$, the tensor product $B \otimes_S A$ can be made into a T-poset by taking

$$(b \otimes a) \cdot t = b \otimes (a \cdot t)$$

for all $a \in A, b \in B, t \in T$.

Similarly, a product $B_S \times {}_SA$ in Pos (with componentwise order) can be made into a T-poset if A is an (S, T)-biposet, by defining

$$(b, a) \cdot t = (b, a \cdot t)$$

for all $a \in A, b \in B, t \in T$.

Lemma 3.3 *In the tensor product $B \otimes_S A$ we have $b \otimes a \leq b' \otimes a'$ iff there exist $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$, $b_1, \ldots, b_n \in B$ and $a_2, \ldots, a_n \in A$ such that*

$$\begin{array}{rclcrcl} b & \leq & b_1 s_1 & & & & \\ b_1 t_1 & \leq & b_2 s_2 & & s_1 a & \leq & t_1 a_2 \\ & \vdots & & & & \vdots & \\ b_n t_n & \leq & b' & & s_n a_n & \leq & t_n a' \end{array} \tag{1}$$

Moreover, if $a \leq a'$, then $b \otimes a \leq b \otimes a'$ and if $b \leq b'$, then $b \otimes a \leq b' \otimes a'$ for any $b, b' \in B, a, a' \in A$.

Proof. The first result is due to [BFL], page 6.

By taking $b = b_1, s_1 = t_1 = 1$ in the system of inequalities (1), we obtain both desired inequalities. ■

Lemma 3.4 *A T-act morphism $f : B \times A_T \to C_T$ which preserves the order in both arguments and satisfies the condition*

$$f(b \cdot s, a) = f(b, s \cdot a) \tag{2}$$

yields a well-defined T-poset morphism $f' : (B \otimes_S A)_T \to C_T$ by taking

$$f'(b \otimes a) = f(b, a).$$

Proof. Taking $b \otimes a \leq b' \otimes a'$, we then obtain that

$$\begin{aligned} f'(b \otimes a) &= f(b, a) \leq f(b_1 s_1, a) = f(b_1, s_1 a) \leq f(b_1, t_1 a_2) = f(b_1 t_1, a_2) \\ &\leq f(b_2 s_2, a_2) = f(b_2, s_2 a_2) \leq f(b_2, t_2 a_3) = \cdots = f(b_n, s_n a_n) \\ &\leq f(b_n, t_n a') = f(b_n t_n, a') \leq f(b', a') = f'(b' \otimes a'). \end{aligned}$$

Given $b \otimes a = b' \otimes a'$, we also have $b \otimes a \leq b' \otimes a'$ and $b \otimes a \geq b' \otimes a'$ and can simply execute the previous argument twice to obtain $f'(b \otimes a) = f'(b' \otimes a')$.

Finally,

$$f'((b \otimes a) \cdot t) = f'(b \otimes (a \cdot t)) = f(b, a \cdot t) = f((b, a) \cdot t) = f(b, a) \cdot t = f'(b \otimes a) \cdot t$$

for any $a \in A, b \in B, t \in T$. ■

Morphisms that satisfy the equation (2) are generally called *balanced* morphisms.

Take an (S,T)-biposet ${}_S A_T$. Recall that the S-action on $\mathrm{Mor}_{\mathrm{Pos_T}}(A,B)$ is defined by

$$(f \cdot s)(a) = f(s \cdot a)$$

for $a \in A, f \in \mathrm{Mor}_{\mathrm{Pos_T}}(A,B)$. The order is simply the pointwise order.

For an (S,T)-biposet ${}_S A_T$, consider the usual two functors

$$- \otimes_S A : \mathrm{Pos}_S \to \mathrm{Pos}_T$$

and

$$\mathrm{Mor}_{\mathrm{Pos_T}}(A,-) : \mathrm{Pos}_T \to \mathrm{Pos}_S,$$

where for an S-poset morphism $f : B_S \to B'_S$ we have

$$(f \otimes_S A)(b \otimes a) = f(b) \otimes a$$

and for a right T-poset morphism $f : C_T \to C'_T$

$$\mathrm{Mor}_{\mathrm{Pos_T}}(A,f)(g) = f \circ g,$$

for every $a \in A, b \in B$ and $g \in \mathrm{Mor}_{\mathrm{Pos}_T}(A,C)$.

Lemma 3.5 *The functors* $- \otimes_S A : \mathrm{Pos}_S \to \mathrm{Pos}_T$ *and* $\mathrm{Mor}_{\mathrm{Pos_T}}(A,-) : \mathrm{Pos}_T \to \mathrm{Pos}_S$ *are pofunctors.*

Proof. Note that since $f(b \cdot s) \otimes a = [f(b) \cdot s] \otimes a = f(b) \otimes (s \cdot a)$, the morphism $f \otimes_S A$ is balanced and the functor $- \otimes_S A$ is well-defined by Lemma 3.4 (as both f and tensor multiplication are order-preserving).

Consider a pair of S-poset morphisms $f_1, f_2 : B_S \to B'_S$ with $f_1 \leq f_2$. Then by Lemma 3.3

$$(f_1 \otimes_S A)(b \otimes a) = f_1(b) \otimes a \leq f_2(b) \otimes a = (f_2 \otimes_S A)(b \otimes a)$$

for all $a \in A, b \in B$.

Similarly for a pair of T-poset morphisms $f_1, f_2 : C_T \to C'_T$ where $f_1 \leq f_2$ we have

$$\begin{aligned} \mathrm{Mor}_{\mathrm{Pos_T}}(A,f_1)(g)(a) &= (f_1 \circ g)(a) = f_1(g(a)) \leq f_2(g(a)) \\ &= (f_2 \circ g)(a) = \mathrm{Mor}_{\mathrm{Pos_T}}(A,f_2)(g)(a) \end{aligned}$$

for any $a \in A$. Therefore $\mathrm{Mor}_{\mathrm{Pos_T}}(A,f_1)(g) \leq \mathrm{Mor}_{\mathrm{Pos_T}}(A,f_2)(g)$ for all $g \in \mathrm{Mor}_{\mathrm{Pos}_T}(A,C)$. ■

Proposition 3.2 *If ${}_SA_T$ is an (S,T)-biposet, then the pofunctor* $-\otimes_S A$: $\mathrm{Pos}_S \to \mathrm{Pos}_T$ *is left poadjoint to the pofunctor* $\mathrm{Mor}_{\mathrm{Pos_T}}(A,-) : \mathrm{Pos}_T \to \mathrm{Pos}_S$.

Proof. In terms of Mor-sets the adjunction can be expressed as a binatural isomorphism between the functors

$$\mathrm{Mor}_{\mathrm{Pos}_T}(-\otimes_S A,-)$$

and

$$\mathrm{Mor}_{\mathrm{Pos}_S}(-,\mathrm{Mor}_{\mathrm{Pos_T}}(A,-))$$

([MLM] pp 17-18). This means that in order to prove the proposition, we have to find a binatural collection of bijections

$$\alpha_{B,C} : \mathrm{Mor}_{\mathrm{Pos}_T}(B\otimes_S A, C) \to \mathrm{Mor}_{\mathrm{Pos}_S}(B, \mathrm{Mor}_{\mathrm{Pos_T}}(A,C))$$

over all $B_S \in \mathrm{Ob}(\mathrm{Pos}_S)$ and $C_T \in \mathrm{Ob}(\mathrm{Pos}_T)$ that also satisfy the poadjunction condition.

To this end we define

$$\alpha_{B,C}(f)(b)(a) = f(b\otimes a)$$

for $f : (B\otimes_S A)_T \to C_T, b\in B, a\in A$.

As long as f is well-defined, α is as well. First we need to verify that $\alpha_{B,C}(f)(b) : A \to C$ is indeed a T-poset morphism. For that take $a \le a'$, then

$$\alpha_{B,C}(f)(b)(a) = f(b\otimes a) \le f(b\otimes a') = \alpha_{B,C}(f)(b)(a')$$

since f is order-preserving. That $b\otimes a \le b\otimes a'$ follows from Lemma 3.3. Also,

$$\alpha_{B,C}(f)(b)(a\cdot t) = f(b\otimes(a\cdot t)) = f((b\otimes a)\cdot t) = f(b\otimes a)\cdot t = \alpha_{B,C}(f)(b)(a)\cdot t.$$

Thus indeed $\alpha_{B,C}(f)(b) \in \mathrm{Mor}_{\mathrm{Pos_T}}(A,C)_S$. We do not yet know whether $\alpha_{B,C}(f) : B_S \to \mathrm{Mor}_{\mathrm{Pos_T}}(A,C)$ is actually an S-poset morphism. It is clearly order-preserving, which is verified by noticing that if $b\le b'$, then $b\otimes a \le b'\otimes a$ for every $a\in A$ and therefore $f(b\otimes a) \le f(b'\otimes a)$. Also,

$$\begin{aligned}\alpha_{B,C}(f)(b\cdot s)(a) &= f(b\cdot s\otimes a) = f(b\otimes s\cdot a)\\ &= \alpha_{B,C}(f)(b)(s\cdot a) = [(\alpha_{B,C}(f)(b))\cdot s](a)\end{aligned}$$

for all $a\in A$. This concludes the verification that $\alpha_{B,C}$ is well-defined.

We notice that if $g \in \mathrm{Mor}_{\mathrm{Pos}_S}(B, \mathrm{Mor}_{\mathrm{Pos_T}}(A,C))$, then

$$g(b\cdot s)(a) = (g(b)\cdot s)(a) = g(b)(s\cdot a)$$

and

$$g(b)(a \cdot t) = g(b)(a) \cdot t$$

for every $b \in B, a \in A$ and $s \in S, t \in T$. Define

$$\widehat{\beta_{B,C}(g)}(b,a) = g(b)(a)$$

for any $g \in \mathrm{Mor}_{\mathrm{Pos}_S}(B, \mathrm{Mor}_{\mathrm{Pos_T}}(A,C))$, $b \in B, a \in A$. The mapping $\widehat{\beta_{B,C}(g)} : (B \times A)_T \to C_T$ is clearly order-preserving in both arguments and the previous remarks show that it is also a balanced T-poset morphism. By this and Lemma 3.4 the inverse of α, defined by

$$\beta_{B,C}(g)(b \otimes a) = g(b)(a)$$

is also a well-defined T-poset morphism (applying the lemma to $\widehat{\beta_{B,C}(g)}$). Since clearly

$$[\alpha_{B,C}(\beta_{B,C}(g))](b)(a) = [\beta_{B,C}(g)](b \otimes a) = g(b)(a)$$

and

$$[\beta_{B,C}(\alpha_{B,C}(f))](b \otimes a) = \alpha_{B,C}(f)(b)(a) = f(b \otimes a)$$

for any $a \in A, b \in B$, the components $\alpha_{B,C}$ are bijections.

To check the naturality in B, take an S-poset morphism $h : B_S \to B'_S$. Then

$$\begin{aligned} & [[\alpha_{B,C} \circ \mathrm{Mor}_{\mathrm{Pos}_T}(h \otimes_S A, C)](f)](b)(a) = [\alpha_{B,C}(f \circ (h \otimes_S A))](b)(a) \\ = \; & f(h(b) \otimes a) = \alpha_{B',C}(f)(h(b))(a) = ((\alpha_{B',C}(f) \circ h)(b))(a) \\ = \; & [[\mathrm{Mor}_{\mathrm{Pos}_S}(h, \mathrm{Mor}_{\mathrm{Pos_T}}(A,C)) \circ \alpha_{B',C}](f)](b)(a), \end{aligned}$$

for all $f \in \mathrm{Mor}_{\mathrm{Pos}_T}(B' \otimes_S A, C), b \in B, a \in A$.

$$\begin{array}{ccc} \mathrm{Mor}_{\mathrm{Pos}_T}(B' \otimes_S A, C) & \xrightarrow{\alpha_{B',C}} & \mathrm{Mor}_{\mathrm{Pos}_S}(B', \mathrm{Mor}_{\mathrm{Pos_T}}(A,C)) \\ {\scriptstyle \mathrm{Mor}_{\mathrm{Pos}_T}(h \otimes_S A, C)} \downarrow & & \downarrow {\scriptstyle \mathrm{Mor}_{\mathrm{Pos}_S}(h, \mathrm{Mor}_{\mathrm{Pos_T}}(A,C))} \\ \mathrm{Mor}_{\mathrm{Pos}_T}(B \otimes_S A, C) & \xrightarrow[\alpha_{B,C}]{} & \mathrm{Mor}_{\mathrm{Pos}_S}(B, \mathrm{Mor}_{\mathrm{Pos_T}}(A,C)) \end{array}$$

Likewise for naturality in C, fix a T-poset morphism $h : C_T \to C'_T$. Similar calculation yields

$$\begin{aligned} & [[\alpha_{B,C'} \circ \mathrm{Mor}_{\mathrm{Pos}_T}(B \otimes_S A, h)](f)](b)(a) = [\alpha_{B,C'}(h \circ f)](b)(a) \\ = \; & h(f(b \otimes a)) = (h \circ (\alpha_{B,C}(f)(b))(a) \\ = \; & [[\mathrm{Mor}_{\mathrm{Pos}_S}(B, \mathrm{Mor}_{\mathrm{Pos_T}}(A,h)) \circ \alpha_{B,C}](f)](b)(a), \end{aligned}$$

for all $f \in \mathrm{Mor}_{\mathbf{Pos}_T}(B \otimes_S A, C), b \in B, a \in A$.

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathbf{Pos}_T}(B \otimes_S A, C) & \xrightarrow{\alpha_{B,C}} & \mathrm{Mor}_{\mathbf{Pos}_S}(B, \mathrm{Mor}_{\mathbf{Pos}_T}(A, C)) \\
\Big\downarrow {\scriptstyle \mathrm{Mor}_{\mathbf{Pos}_T}(B \otimes_S A, h)} & & \Big\downarrow {\scriptstyle \mathrm{Mor}_{\mathbf{Pos}_S}(B, \mathrm{Mor}_{\mathbf{Pos}_T}(A, h))} \\
\mathrm{Mor}_{\mathbf{Pos}_T}(B \otimes_S A, C') & \xrightarrow[\alpha_{B,C'}]{} & \mathrm{Mor}_{\mathbf{Pos}_S}(B, \mathrm{Mor}_{\mathbf{Pos}_T}(A, C'))
\end{array}$$

For the last part, take $f_1, f_2 \in \mathrm{Mor}_{\mathbf{Pos}_T}(B \otimes_S A, C)$ with $f_1 \leq f_2$. Then

$$\alpha_{B,C}(f_1)(b)(a) = f_1(b \otimes a) \leq f_2(b \otimes a) = \alpha_{B,C}(f_2)(b)(a)$$

for all $b \in B, a \in A$. So $\alpha_{B,C}$ indeed preserves order.

For the converse take $g_1, g_2 \in \mathrm{Mor}_{\mathbf{Pos}_S}(B, \mathrm{Mor}_{\mathbf{Pos}_T}(A, C))$, $g_1 \leq g_2$. Then

$$\beta_{B,C}(g_1)(b \otimes a) = g_1(b)(a) \leq g_2(b)(a) = \beta_{B,C}(g_2)(b)(a)$$

and the proof is complete. ■

There is a special case of turning a tensor product into a T-poset where the second poset ${}_SA_T$ is simply ${}_SS_S$ with its natural order. In this case we have the following result.

Lemma 3.6 *For any S-poset B_S, there are canonical isomorphisms*

$$(B \otimes_S S)_S \cong B_S \cong \mathrm{Mor}_{\mathbf{Pos}_S}(S_S, B_S)_S.$$

Proof. We define $\alpha : (B \otimes_S S)_S \to B_S$ and $\beta : B_S \to \mathrm{Mor}_{\mathbf{Pos}_S}(S_S, B_S)_S$ with the equations

$$\alpha(b \otimes s) = b \cdot s$$

and

$$\beta(b)(s) = b \cdot s,$$

that is, $\beta(b) = \mathbf{b}$. Since

$$\alpha(b \cdot s \otimes t) = (b \cdot s) \cdot t = b \cdot (st) = \alpha(b \otimes st)$$

and by definition the multiplication in an S-poset is order-preserving in both arguments, α is a well-defined poset morphism according to Lemma 3.4. Additionally,

$$\alpha((b \otimes s) \cdot t) = \alpha(b \otimes (st)) = b \cdot (st) = (b \cdot s) \cdot t = \alpha(b \otimes s) \cdot t$$

and therefore α is an S-poset morphism as well. The morphism α is clearly surjective, as $\alpha(b \otimes 1) = b \cdot 1 = b$ for every $b \in B$. Suppose $b \cdot s = b' \cdot s'$. Then

$$b \otimes s = (b \cdot s) \otimes 1 = (b' \cdot s') \otimes 1 = b' \otimes s'$$

and α proves to be injective as well. Also, if $b \leq b'$, then $b \otimes 1 \leq b' \otimes 1$ and α reflects order. We have thus shown that the first morphism α is an isomorphism.

In the preliminaries we saw that $\mathbf{b}$ is an S-poset morphism. Since

$$(\beta(b) \cdot t)(s) = \beta(b)(t \cdot s) = b \cdot (ts) = (b \cdot t) \cdot s = \beta(b \cdot t)(s)$$

for all $s, t \in S$ and $b \in B$, and multiplication is order-preserving in both arguments, β is a right S-poset morphism. If we take $g \in \mathrm{Mor}_{\mathrm{Pos}_S}(S_S, B_S)$, we can see that

$$\beta(g(1))(s) = g(1) \cdot s = g(s)$$

for all $s \in S$ and so $\beta(g(1)) = g$. Moreover, if $\beta(b) = \beta(b')$ (ie $\beta(b)(s) = \beta(b')(s)$ for all $s \in S$) then

$$b = b \cdot 1 = \beta(b)(1) = \beta(b')(1) = b' \cdot 1 = b'$$

and β is both injective and surjective. Finally, if $g_1 \leq g_2$ in $\mathrm{Pos}_{\mathrm{S}}(S_S, B_S)$, then $g_1(1) \leq g_2(1)$ which means that β reflects order. Thus it is also an S-poset isomorphism. ■

Let $f : T \to S$ be a morphism of pomonoids, ie an order-preserving monoid homomorphism. Any right S-poset A_S can be considered as a right T-poset A_T^f by taking

$$a \cdot t = a \cdot f(t),$$

$a \in A, t \in T$. The same applies to left posets, taking ${}_SA$ to ${}^f_TA$.

In this manner we obtain a functor $(-)_T^f : \mathrm{Pos}_S \to \mathrm{Pos}_T$. If we take $f_1, f_2 : A_S \to A'_S, f_1 \leq f_2$, then

$$(f_1)_T^f(x) = f_1(x) \leq f_2(x) = (f_2)_T^f(x)$$

for all $x \in (A)_T^f$ and this functor is a pofunctor.

It is clear that the (S, S) biposet ${}_SS_S$ can be considered as an (S, T)-biposet in this way.

Lemma 3.7 *For any pomonoid morphism $f : T \to S$, there is a natural isomorphism of pofunctors*

$$- \otimes_S S_T^f \cong (-)_T^f \cong \mathrm{Mor}_{\mathrm{Pos}_S}({}^f_TS_S, -).$$

Proof. The result follows from the isomorphisms of Lemma 3.6, where the S-posets are taken as T-posets via the morphism f. More precisely, the isomorphisms α and β are the same, but are taken as right T-poset morphisms:

$$\begin{aligned}\alpha_B((b\otimes s)\cdot_T t) &= \alpha_B(b\otimes(s\cdot_T t)) = \alpha_B(b\otimes(sf(t)))\\ &= b\cdot_S(sf(t)) = (b\cdot_S s)\cdot_S f(t) = \alpha_B(b\otimes s)\cdot_T t,\end{aligned}$$

$$\begin{aligned}\beta_B(b)(s\cdot_T t) &= \beta_B(b)(sf(t)) = b\cdot_S(sf(t))\\ &= (b\cdot_S s)\cdot_S f(t) = (\beta_B(b)(s))\cdot_S f(t) = (\beta_B(b)(s))\cdot_T t.\end{aligned}$$

Also, the isomorphisms are natural in B, ie for an S-poset morphism $h : B \to B'$

$$\begin{aligned}(\alpha_{B'}\circ(h\otimes_S S_T^f))(b\otimes s) &= \alpha_{B'}(h(b)\otimes s) = h(b)\cdot s = h(b\cdot s)\\ &= h(\alpha_B(b\otimes s)) = (h\circ\alpha_B)(b\otimes s)\end{aligned}$$

where $b\in B, s\in S$.

$$\begin{array}{ccc} B\otimes_S S_T^f & \xrightarrow{\alpha_B} & B_T^f \\ \downarrow{\scriptstyle h\otimes_S S_T^f} & & \downarrow{\scriptstyle h} \\ B'\otimes_S S_T^f & \xrightarrow[\alpha_{B'}]{} & B'^f_T \end{array}$$

Similarly

$$\begin{aligned}[(\beta_{B'}\circ h)(b)](s) &= [\beta_{B'}(h(b))](s) = h(b)\cdot s = h(b\cdot s) = h(\beta_B(b)(s))\\ &= (h\circ\beta_B(b))(s) = [(\mathrm{Mor}_{\mathbf{Pos}_S}({}_T^fS_S, h)\circ\beta_B)(b)](s)\end{aligned}$$

for any $b\in B, s\in S$.

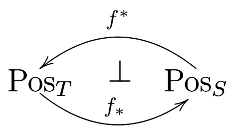
$$\begin{array}{ccc} B_T^f & \xrightarrow{\beta_B} & \mathrm{Mor}_{\mathbf{Pos}_S}({}_T^fS_S, B) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{Mor}_{\mathbf{Pos}_S}({}_T^fS_S, h)} \\ B'^f_T & \xrightarrow[\beta_{B'}]{} & \mathrm{Mor}_{\mathbf{Pos}_S}({}_T^fS_S, B') \end{array}$$

■

3.3 Pogeometric morphisms

Now we can finally define geometric morphisms for the categories of S-posets.

Definition 3.5 A *pogeometric morphism* $f : \mathrm{Pos}_T \to \mathrm{Pos}_S$ is a pair of pofunctors $f^* : \mathrm{Pos}_S \to \mathrm{Pos}_T$ and $f_* : \mathrm{Pos}_T \to \mathrm{Pos}_S$ such that f^* is left poadjoint to f_* and f^* preserves finite universal pocones (that is, f^* is *left poexact*).



Proposition 3.3 *A pomonoid morphism $f : T \to S$ induces a pogeometric morphism*

$$f : \mathrm{Pos}_T \to \mathrm{Pos}_S$$

with $f^*(B_S) = B_T^f$. The pofunctor f^* also has a left poadjoint $f_!$.

Proof. By Lemma 3.7 we have

$$\mathrm{Mor}_{\mathrm{Pos}_S}({}^f_T S_S, -) \cong f^* \cong - \otimes_S S^f_T.$$

Proposition 3.2 gives us the poadjunctions

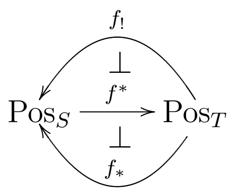
$$- \otimes_S S^f_T \dashv \mathrm{Mor}_{\mathrm{Pos}_T}(S^f_T, -) =: f_*$$

and

$$f_! := - \otimes_T {}^f_T S \dashv \mathrm{Mor}_{\mathrm{Pos}_S}({}^f_T S_S, -).$$

Hence

$$f_! = - \otimes_T {}^f_T S \dashv \mathrm{Mor}_{\mathrm{Pos}_S}({}^f_T S_S, -) \cong f^* \cong - \otimes_S S^f_T \dashv \mathrm{Mor}_{\mathrm{Pos}_T}(S^f_T, -) = f_*.$$



Note that these functors are pofunctors. As a right poadjoint, f^* preserves finite universal pocones and is left poexact. ■

3.4 Points

Note that Pos, the category of partially ordered sets, is the same as $\mathrm{Pos}_{\mathbf{1}}$, the category of posets over the trivial monoid.

Definition 3.6 A *point* of a right S-poset category Pos_S is a pogeometric morphism $f : \mathrm{Pos} = \mathrm{Pos}_{\mathbf{1}} \to \mathrm{Pos}_S$.

Let Points_S be the category of points of Pos_S, where the morphisms are the natural transformations between the left poadjoints, ie $\alpha : f \to g$ in Points_S iff $\alpha : f^* \to g^*$ is a natural transformation.

The morphism sets in this category can be partially ordered by taking $\alpha \leq \beta : f^* \to g^*$ iff

$$\alpha_B \leq \beta_B$$

in the poset $\mathrm{Mor}_{\mathrm{Pos}}(f^*(B), g^*(B))$ for every right S-poset B_S.

Let $f : \mathrm{Pos} \to \mathrm{Pos}_S$ be a point of Pos_S. In this case we have a special poset $A = f^*(S_S)$, which can be turned into a left S-poset ${}_SA$ by defining

$$s \cdot a = f^*(\mathbf{s})(a) \tag{3}$$

for all $a \in A$ and $s \in S$. Since $f^*(\mathbf{s}) : f^*(S_S) \to f^*(S_S)$ must be a poset morphism and f^* is a pofunctor, multiplication is indeed associative and order-preserving.

Lemma 3.8 *There is a natural isomorphism of functors*

$$- \otimes_S A \cong f^* : \mathrm{Pos}_S \to \mathrm{Pos},$$

where f is a point of Pos_S and ${}_SA = {}_Sf^*(S_S)$ is defined as above.

Proof. Take a right S-poset B_S and define a mapping $\alpha_B : B \otimes_S A \to f^*(B)$ by

$$\alpha_B(b \otimes a) = f^*(\mathbf{b})(a).$$

Define $u : B \times A \to \mathrm{Mor}_{\mathrm{Pos}_S}(S_S, B_S) \times A$ as $u(b, a) = (\mathbf{b}, a)$. The mapping u is clearly order-preserving. Since $f^* : \mathrm{Pos}_S \to \mathrm{Pos}$ is a pofunctor, the mapping $f^*(-)(-) \circ u : B \times A \to f^*(B)$ is order-preserving in both arguments. Note that by Lemma 1.1 $\mathbf{b} \cdot \mathbf{s} = \mathbf{b} \circ \mathbf{s} : S_S \to S_S \to B_S$ for all $s \in S, b \in B$. Additionally,

$$f^*(\mathbf{b} \cdot s)(a) = f^*(\mathbf{b} \cdot \mathbf{s})(a) = f^*(\mathbf{b} \circ \mathbf{s})(a) = f^*(\mathbf{b})(f^*(\mathbf{s})(a)) = f^*(\mathbf{b})(s \cdot a).$$

Thus $f^*(-)(-) \circ u$ is balanced and by Lemma 3.4 α_B is a well-defined poset morphism.

If $B_S = S_S$, then

$$\alpha_S(s \otimes a) = f^*(\mathbf{s})(a) = s \cdot a$$

which means that $\alpha_S : S \otimes_S A \to f^*(S)$ is the left-sided version of the isomorphism from Lemma 3.7 restricted to posets. Thus

$$S \otimes_S A \cong f^*(S_S)$$

in Pos.

Every right S-poset B_S is a universal pococone of free S-posets S_S with identity morphisms (see Lemma 3.2). Since left poadjoints preserve universal pococones and both the tensor product $- \otimes_S A$ and f^* are left poadjoints, $(B \otimes_S A, (\mathbf{b} \otimes_S A)_{b \in B})$ is the universal pococone of objects $S \otimes_S A$ (with respect to $(- \otimes_S A) \circ \mathbf{B}$, and $\mathbf{B}$ from Lemma 3.2) and $(f^*(B), (f^*(\mathbf{b}))_{b \in B})$ is the universal pococone of objects $f^*(S)$ (with respect to $f^* \circ \mathbf{B}$).

Observe that the diagram for B_S as a cocone consists of morphisms $\mathbf{s} : S_S \to S_S$. For $a \in A = f^*(S_S)$ clearly

$$f^*(\mathbf{s})(a) = s \cdot a = \alpha_S(s \otimes a) = \alpha_S((\mathbf{s} \otimes A)(1 \otimes a)) = (\alpha_S \circ (\mathbf{s} \otimes_S A))(\alpha_S^{-1}(a))$$

and the functors $(- \otimes_S A) \circ F$ and $f^* \circ F : \mathrm{El}(B) \to \mathrm{Pos}$ are isomorphic with natural isomorphisms $\alpha_b = \alpha_S$ for all $b \in B$. Here, the functor F is also from Lemma 3.2.

Note that

$$(((- \otimes_S A) \circ \mathbf{B})(b < c))_1 = ((f^* \circ \mathbf{B})(b < c))_1 = \mathbf{B}(b < c)_1.$$

Also, if $b < c$ in B and $s \otimes a \in S \otimes_S A$, then

$$\begin{aligned} & f^*(\mathbf{B}(b < c)_2) \circ \alpha_{\mathbf{B}(b<c)_1}(s \otimes a) = (f^*(1_S) \circ \alpha_S)(s \otimes a) = f^*(1_S)(s \cdot a) \\ = \; & s \cdot a = \alpha_S(s \otimes a) = (\alpha_S \circ (1_S \otimes_S A))(s \otimes a) \\ = \; & (\alpha_S \circ (\mathbf{B}(b < c)_2 \otimes_S A))(s \otimes a) = (\alpha_b \circ ((- \otimes_S A)(\mathbf{B}(b < c)_2)))(s \otimes a) \end{aligned}$$

and

$$\begin{aligned} & f^*(\mathbf{B}(b < c)_3) \circ \alpha_{\mathbf{B}(b<c)_1}(s \otimes a) = (f^*(1_S) \circ \alpha_S)(s \otimes a) = f^*(1_S)(s \cdot a) \\ = \; & s \cdot a = \alpha_S(s \otimes a) = (\alpha_S \circ (1_S \otimes_S A))(s \otimes a) \\ = \; & (\alpha_S \circ (\mathbf{B}(b < c)_3 \otimes_S A))(s \otimes a) = (\alpha_c \circ ((- \otimes_S A)(\mathbf{B}(b < c)_3)))(s \otimes a). \end{aligned}$$

So

$$f^*(\mathbf{B}(b < c)_2) \circ \alpha_{\mathbf{B}(b<c)_1} = \alpha_b \circ (- \otimes_S A)(\mathbf{B}(b < c)_2)$$

and

$$f^*(\mathbf{B}(b<c)_3)\circ\alpha_{\mathbf{B}(b<c)_1}=\alpha_c\circ(-\otimes_S A)(\mathbf{B}(b<c)_3).$$

By Lemma 3.1 there is a unique morphism $u : B\otimes_S A\to f^*(B)$ such that

$$u\circ(\mathbf{b}\otimes_S A)=f^*(\mathbf{b})\circ\alpha_b$$

for all $b\in B$. But

$$\begin{aligned}(\alpha_B\circ(\mathbf{b}\otimes_S A))(s\otimes a) &= \alpha_B((b\cdot s)\otimes a)=f^*(\mathbf{b}\cdot\mathbf{s})(a)=f^*(\mathbf{b}\circ\mathbf{s})(a)\\ &= (f^*(\mathbf{b})\circ f^*(\mathbf{s}))(a)=f^*(\mathbf{b})(f^*(\mathbf{s})(a))\\ &= f^*(\mathbf{b})(s\cdot a)=(f^*(\mathbf{b})\circ\alpha_b)(s\otimes a)\end{aligned}$$

for all $s\otimes a\in S\otimes_S A$. So α_B must be this unique morphism, and according to Lemma 3.1, it is also an isomorphism.

Finally, we check the naturality of these isomorphisms. Take a right S-poset morphism $h : B_S\to B'_S$. Note that $\mathbf{h}(\mathbf{b})=h\circ\mathbf{b}$ for all $b\in B$. Then

$$\begin{aligned}(\alpha_{B'}\circ(h\otimes_S A))(b\otimes a) &= \alpha_{B'}(h(b)\otimes a)=f^*(\mathbf{h}(\mathbf{b}))(a)\\ &= f^*(h\circ\mathbf{b})(a)=f^*(h)(f^*(\mathbf{b})(a))\\ &= (f^*(h)\circ\alpha_B)(b\otimes a)\end{aligned}$$

for $b\in B, a\in A$. So these isomorphisms are natural in B. ■

Definition 3.7 We call an S-poset ${}_SA$ *flat* when the induced tensor multiplication functor $-\otimes_S A : \mathrm{Pos}_S\to\mathrm{Pos}$ is left poexact.

Definition 3.8 We define the category of flat left S-posets ${}_S\mathrm{Flat}$ to be the full subcategory of ${}_S\mathrm{Pos}$ of flat left S-posets.

Theorem 3.1 *There is an equivalence of categories*

$${}_S\mathrm{Flat}\underset{\rho}{\overset{\tau}{\rightleftarrows}}\mathrm{Points}_S$$

where τ and ρ are order-preserving functors.

Proof. We define

$$\tau(A)^*=-\otimes_S A,\qquad \tau(A)_*=\mathrm{Mor}_{\mathrm{Pos}}(A,-),$$

when $A\in\mathrm{Ob}({}_S\mathrm{Flat})$ and

$$\rho(f)={}_SA$$

for $f \in \mathrm{Ob}(\mathrm{Points}_S)$ and ${}_SA$ is defined with equation (3). Since A_S is flat, $\tau(A)$ is a point of Pos_S by Proposition 3.2. Conversely, for a point f, the functor f^* is left poexact. From Lemma 3.8

$$- \otimes_S A \cong f^*$$

and if f^* is left poexact, the isomorphic functor is as well.

Clearly if we have a left S-poset morphism $f : {}_SA \to {}_SA'$, then $\tau(f) : - \otimes_S A \to - \otimes_S A'$ should be defined with

$$\tau(f)_B = B \otimes_S f$$

for all right S-posets B_S. These components are morphisms in Pos by the left-sided version of Lemma 3.5. When $h : B \to B'$ is a right S-poset morphism, we get

$$\begin{aligned} (\tau(A')^*(h) \circ \tau(f)_B)(b \otimes a) &= (h \otimes_S A')(b \otimes f(a)) = h(b) \otimes f(a) \\ &= \tau(f)_{B'}(h(b) \otimes a) \\ &= (\tau(f)_{B'} \circ \tau(A)^*(h))(b \otimes a) \end{aligned}$$

for $b \in B, a \in A$. So $\tau(f)$ is indeed a natural transformation.

$$\begin{array}{ccc} B \otimes_S A & \xrightarrow{\tau(f)_B} & B \otimes_S A' \\ {\scriptstyle \tau(A)^*(h) = h \otimes_S A} \downarrow & & \downarrow {\scriptstyle h \otimes_S A' = \tau(A')^*(h)} \\ B' \otimes_S A & \xrightarrow[\tau(f)_{B'}]{} & B' \otimes_S A' \end{array}$$

Due to the left-sided version of Lemma 3.5 we have

$$\begin{aligned} (\tau(g) \circ \tau(f))_B &= \tau(g)_B \circ \tau(f)_B = (B \otimes_S g) \circ (B \otimes_S f) \\ &= B \otimes_S (g \circ f) = \tau(g \circ f)_B \end{aligned}$$

and

$$\tau(1_A)_B = B \otimes_S 1_A = 1_{B \otimes_S A}$$

for all $f : {}_SA \to {}_SA'$, $g : {}_SA' \to {}_SA''$, $B_S \in \mathrm{Ob}(\mathrm{Pos}_S)$. Therefore $\tau(g \circ f) = \tau(g) \circ \tau(f)$, $\tau(1_A) = 1_{- \otimes_S A}$ and τ is a functor.

Even better, for $f, g : {}_SA \to {}_SA'$ with $f \leq g$ in ${}_S\mathrm{Flat}$ we have

$$\tau(f)_B = B \otimes_S f \leq B \otimes_S g = \tau(g)_B$$

for all right S-posets B_S, again by the left-sided version of Lemma 3.5. So the natural transformations have the same order objectwise and τ is an order-preserving functor.

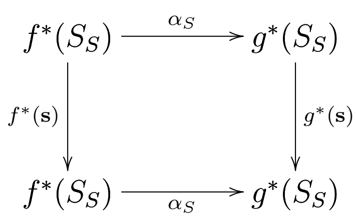
For all morphisms of points $\alpha : f \to g$, ie natural transformations $\alpha : f^* \to g^*$, we define a mapping $\rho(\alpha) : f^*(S_S) \to g^*(S_S)$ as

$$\rho(\alpha) = \alpha_S.$$

Clearly α_S is order-preserving. As α must be natural, then

$$\rho(\alpha)(s \cdot a) = \rho(\alpha)(f^*(\mathbf{s}(a))) = (\alpha_S \circ f^*(\mathbf{s}))(a) = (g^*(\mathbf{s}) \circ \alpha_S)(a) = s \cdot \rho(\alpha)(a)$$

for $a \in f^*(S_S)$, $s \in S$.



So $\rho(\alpha)$ is indeed a left S-poset morphism. Moreover, for natural transformations $\alpha : f^* \to g^*$ and $\beta : g^* \to h^*$ it holds that

$$\rho(\beta \circ \alpha) = (\beta \circ \alpha)_S = \beta_S \circ \alpha_S = \rho(\beta) \circ \rho(\alpha)$$

and

$$\rho(1_{f^*}) = (1_{f^*})_S = 1_{f^*(S_S)}.$$

So ρ is also a functor.

Take two natural transformations $\alpha, \beta : f^* \to g^*$ with $\alpha_B \leq \beta_B$ for all right S-posets B. Then

$$\rho(\alpha)(a) = \alpha_S(a) \leq \beta_S(a) = \rho(\beta)(a)$$

for $a \in A = f^*(S_S)$. So $\rho(\alpha) \leq \rho(\beta)$ and ρ turns out to be order-preserving as well.

We want to find natural isomorphisms

$$\rho \circ \tau \cong 1_{S\mathrm{Flat}}.$$

For every flat left S-poset ${}_SA$ we have

$$\rho(\tau(A)) = S \otimes_S A \cong A$$

in ${}_S\mathrm{Flat}$ by the left-sided version of Lemma 3.6. These isomorphisms are natural in A, the verification of this is done in the same way as the right-sided version in Lemma 3.7.

We also want to find natural isomorphisms

$$\tau \circ \rho \cong 1_{\mathrm{Points}_S}.$$

For a pogeometric morphism $f : \mathrm{Pos} \to \mathrm{Pos}_S$ there are natural isomorphisms

$$\tau(\rho(f))^* = - \otimes_S \rho(f) = - \otimes_S f^*(S_S) = - \otimes_S A \cong f^*$$

by Lemma 3.8. Both of these also have their right poadjoints (unique, since these are right adjoints). We shall denote these natural isomorphisms by $\alpha^f : - \otimes_S f^*(S_S) \to f^*$, with

$$\alpha^f_B(b \otimes a) = f^*(\mathbf{b}(a))$$

for all $b \in B, a \in f^*(S_S)$ as in Lemma 3.8. Now take a natural transformation $\beta : f^* \to g^*$, whence

$$g^*(h) \circ \beta_B = \beta_{B'} \circ f^*(h)$$

for all right S-poset morphisms $h : B \to B'$. From the above α^f are morphisms in Points_S. Additionally,

$$\begin{aligned} (\alpha^g \circ (\tau(\rho(\beta))))_B(b \otimes a) &= (\alpha^g_B \circ (B \otimes_S \rho(\beta)))(b \otimes a) \\ &= \alpha^g_B((B \otimes_S \beta_S)(b \otimes a)) \\ &= \alpha^g_B(b \otimes \beta_S(a)) = g^*(\mathbf{b})(\beta_S(a)) \\ &= \beta_B(f^*(\mathbf{b})(a)) = (\beta \circ \alpha^f)_B(b \otimes a) \end{aligned}$$

for a right S-poset B_S, $a \in f^*(S_S), b \in B$.

$$\begin{array}{ccc} \tau(\rho(f))^* & \xrightarrow{\alpha^f} & f^* \\ {\scriptstyle \tau(\rho(\beta))}\downarrow & & \downarrow{\scriptstyle \beta} \\ \tau(\rho(g))^* & \xrightarrow[\alpha^g]{} & g^* \end{array}$$

Thus the natural isomorphisms (morphisms between points) α^f are natural in f.

In conclusion, we have shown that

$$\rho \circ \tau \cong 1_{{}_S\mathrm{Flat}}$$

and

$$\tau \circ \rho \cong 1_{\mathrm{Points}_S}.$$

So τ (or ρ) is indeed an equivalence of categories. It is even an order equivalence. ∎

Alamobjektide eristamine ja geomeetrilised morfismid osaliselt järjestatud polügoonides

Lauri Tart

Resümee

Käesolevas magistritöös vaadeldakse osaliselt järjestatud polügoonide kategooria toposeteoreetilisi omadusi. Selline lähenemine on motiveeritud sellest, et järjestatud polügoonid on küllaltki sarnased polügoonidega, ja polügoonide kategooria on topos (isegi Grothendiecki topos).

Töö koosneb kahest peamisest osast. Esimeses osas uuritakse universaalseid konstruktsioone osaliselt järjestatud polügoonidel. Osutub, et tegu on täieliku, kotäieliku ja eksponentsiaalobjektidega kategooriaga. Seevastu alamobjektide eristajaid üldjuhul selles kategoorias ei ole. Teatud erijuhul alamobjektide eristajad siiski eksisteerivad, kuid kahjuks mitte tuntumate monomorfismide klasside puhul. Seega ei ole osaliselt järjestatud polügoonide kategooria topos.

Töö teises osas üldistatakse topostevahelise geomeetrilise morfismi mõistet. Selle tulemusena saadakse nn pogeomeetriline morfism osaliselt järjestatud polügoonide kategooriate vahel. Lisaks vaadeldakse tensorkorrutisi ja viimaste abil saadavaid pogeomeetrilisi morfisme. Viimase ja olulisima tulemusena defineeritakse punktid osaliselt järjestatud polügoonide kategooria jaoks ja leitakse, et need on ekvivalentsed lamedate (universaalseid pokoonuseid säilitavate) osaliselt järjestatud polügoonidega.

References

[Bo] F. Borceux, *Handbook of Categorical Algebra 1: Basic Category Theory*, Cambridge University Press, Cambridge, 1994.

[BFL] S. Bulman-Fleming, V. Laan, *Lazard's theorem for S-posets*, Math. Nachr., to appear.

[BFM] S. Bulman-Fleming, M. Mahmoudi, *The category of S-posets*, submitted.

[Fa] S. M. Fakhruddin, *On the category of S-posets*, Acta Sci. Math., 52 (1988), pp 85-92.

[HS] H. Herrlich, and G.E. Strecker: *Category Theory,* Heldermann Verlag Berlin, 1979.

[CWM] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1971.

[MLM] S. Mac Lane, I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer, New York, 1992.