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Non-Interactive Shuffle Arguments
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ABSTRACT

A fundamental requirement for all democratic governments is a secure voting system. In recent years some countries, like Estonia and Switzerland, have adopted internet voting (i-voting) and many more have experimented (e.g., Norway and Australia) or have plans to adopt it in the future (e.g., Lithuania and Russia). I-voting has the potential to offer better convenience, lower administrative costs, and higher voter turnout, but this comes with increased security concerns and significant technical challenges.

Consider the simple procedure of shaking the ballot box to mix the order of the ballots. It is far from obvious how to achieve an equivalent result with encrypted digital ballots. Who should perform the mixing procedure? How to guarantee that it was performed correctly? Is it possible that no one can trace the ballots?

This problem can be solved with a distributed system called a mix-network. The idea is to let each peer in the mix-network shuffle (permute and rerandomize) the ciphertexts. This makes it computationally hard to trace the input ciphertexts to the output ciphertexts given that at least one peer is honest. However, each peer should also give a proof that the shuffling was done correctly to avoid substitution attacks. The proof has to be hard to forge (sound) and should leak nothing but the truth of the statement (zero-knowledge). Such proofs are called zero-knowledge shuffle arguments, and in this thesis, we study their constructions. Importantly, we avoid the heuristic security model, used in many of the previous works, which (incorrectly) treats a cryptographic hash function as a truly random function. We show that it possible to construct shuffle arguments, that are efficient enough for large-scale elections, in other security models than the random oracle model.

First, we construct a very efficient non-interactive shuffle argument that avoids the random oracle model and instead uses the generic group model. This is achieved by combining several recent tools like quasi-adaptive zero-knowledge arguments and SNARKs. We implement it and observe practical efficiency for large-scale elections: the proving time for 100,000 ciphertexts is less than a minute, and verification time is less than 1.5 minutes on modest hardware. Unfortunately, security requires that the prover and the verifier have access to a trusted common reference string (CRS).

Secondly, we study how to reduce trust assumptions. There are efficient multi-party computation (MPC) protocols for generating CRSs for a large class of arguments, but they require the random oracle model. We improve upon one such protocol and, among other results, remove the requirement for the random oracle. We prove the security of this protocol in the universal composability setting.

Thirdly, we modify our shuffle argument to be applicable to the above MPC protocol. This guarantees both soundness and zero-knowledge as long as at least one peer in the MPC protocol is honest. We go one step further and show how to get zero knowledge even if all the peers are malicious. Additionally, we simplify the argument construction and prove its security based on weaker assumptions.
CONTENTS

List of original publications 9

1. Introduction 10
   1.1. Contributions and Outline ................................... 11
   1.2. Related Work ................................................. 14

2. Preliminaries 18
   2.1. Pairings ....................................................... 18
   2.2. Assumptions ................................................... 19
   2.3. Cryptosystems ................................................ 22
   2.4. Non-Interactive Zero-Knowledge Arguments ..................... 22
   2.5. Universal Composability ..................................... 25
   2.6. Commitment Schemes ......................................... 26

3. An Efficient Shuffle Argument 29
   3.1. Motivation ..................................................... 29
   3.2. Overview ....................................................... 29
   3.3. Same-Message Argument ..................................... 30
   3.4. Unit Vector Argument ....................................... 31
   3.4.1. Permutation Matrix Argument ............................. 33
   3.5. Consistency Argument ....................................... 34
   3.6. Shuffle Argument ............................................ 36
   3.7. Batching and Implementation ................................ 36

4. UC-secure CRS Generation for NIZK 39
   4.1. Motivation ..................................................... 39
   4.2. Ideal Functionality .......................................... 40
   4.3. Protocol ....................................................... 41
   4.3.1. Class of Circuits .......................................... 41
   4.3.2. Construction ............................................... 42

5. Transparent Shuffle Argument 45
   5.1. Overview ..................................................... 45
   5.2. Modifications in CRS ....................................... 45
   5.3. Unit Vector Argument ..................................... 46
   5.4. Permutation Argument ..................................... 46
   5.5. Shuffle Argument ........................................... 48
   5.6. Subversion Zero-Knowledge .................................. 49

6. Conclusions and Future Work 51

Bibliography 52

6
LIST OF ABBREVIATIONS

AGM : Algebraic Group Model
BDH-KE : Bilinear Diffie-Hellman Knowledge of Exponent
CDH : Computational Diffie-Hellman
CRS : Common Reference String
DDH : Decisional Diffie-Hellman
DL : Discrete Logarithm
GGM : Generic Group Model
IND-CPA : INDistinguishability under Chosen Plain Attack
i-voting : internet voting
KerMDH : Kernel Matrix Diffie-Hellman
NIZK : Non-Interactive Zero-Knowledge
MP : Multi-Pedersen
PDL : Power Discrete Logarithm
PPT : Probabilistic Polynomial Time
pv-RRCCA : publicly verifiable Rerandomizable RCCA
RCCA : Replayable Chosen Ciphertext Attack
RO : Random Oracle
SNARK : Succinct Non-interactive ARgument of Knowledge
SPDL : Symmetric Power Discrete Logarithm
SXDH : Symmetric External Diffie-Hellman
UC : Universal Composability
WI : Witness Indistinguishability
ZK : Zero-Knowledge
Sub-ZK : Subversion ZK
LIST OF ORIGINAL PUBLICATIONS

Publications included in the thesis


Publications not included in the thesis


1. INTRODUCTION

In recent decades, there has been a large shift towards moving all kinds of services online. One exception to this has been voting which has shown strong resistance to digitalization, mainly since it opens up new ways to attack elections. For example, Norway, Australia, France, and Canada have all seen setbacks after the initial trials of internet voting (i-voting). Still, some countries have adopted i-voting. Most successful has been Estonia that allows both municipal and parliamentary elections online, and Switzerland that also has i-voting in several cantons.

Like traditional voting, a secure i-voting system should guarantee anonymity for voters. One way to achieve this is encryption, but it alone is not enough since ballots usually also include some authenticating information, for example, a digital signature. Paper-based voting solves this problem by having voters place ballots into a ballot box (after the voter has been identified) which is then shaken to mix the order of ballots. In i-voting, we should send ciphertexts through an anonymous communication channel before decrypting. Applications for anonymous channels go even far beyond just i-voting and include for example anonymous e-mail [Kat], private information retrieval [IKOS06], and anonymous cryptocurrencies [BNM+14, KRDO17].

Chaum [Cha81] proposed a mix-network protocol for implementing an anonymous channel. Mix-network is a distributed system consisting of $N$ peers. Peers take turns to shuffle the ciphertexts which involves randomizing the appearance of each ciphertext (while keeping the same message) and then randomly permuting their order. Let us assume that at least one peer is honest and the rest are semi-honest, that is, they execute the protocol correctly but study and share all the leaked information. In such a case, it will be computationally hard to match the initial ciphertexts to the final ciphertexts output by the $N$-th peer.

However, the semi-honest model is not sufficient for applications like i-voting considering that any peer could easily substitute some of the ciphertexts while remaining undetected. We can tolerate $N-1$ fully malicious peers if we additionally require each peer to provide proof of correct shuffle execution. Of course, the proof should not leak any information about the permutation or randomness used for blinding the ciphertexts. Otherwise, we would undermine the original goal of the mix-network. This task is well suited for a cryptographic tool called a zero-knowledge argument (or proof)\footnote{Proof systems are unforgeable even against computationally unbounded adversaries whereas argument systems tolerate only efficient adversaries. In this thesis we only consider arguments.} [GMR89]. Zero-knowledge arguments allow proving the validity of a statement while leaking no other information. Arguments can either need interaction between the prover and the verifier or be non-interactive, i.e., prover just outputs a single message which can then be verified by anyone. In the i-voting scenario, we would typically prefer non-interactive shuffle arguments since this allows election auditing even if peers (together with their
private information) are not available anymore.

There has been extensive research into interactive zero-knowledge shuffle arguments [SK95, Abe99, Nef01, FMM’03, GI08, TW10, Gro10, BG12]. More recent of these constructions [Gro10, BG12] rely on standard assumptions (e.g., discrete logarithm assumption) and achieve the efficiency that could be practical for large-scale elections. Moreover, most of them are public coin protocols, meaning that verifier’s messages are chosen uniformly randomly and independently from prover’s messages. This makes it possible to apply the Fiat-Shamir transform [FS87] which makes those arguments non-interactive. Unfortunately, the resulting argument is secure only in the random oracle (RO) model which essentially assumes a black-box implementation of a uniformly random function. In practice, the uniformly random function is substituted with a secure hash function. However, at least on a theoretical level, it is known that this substitution is unwarranted. There are contrived protocols that are secure in the RO model, but trivially insecure when instantiated with a secure hash function [Bar01, GK03, BG’13, BBH’19]. Similar attacks are not known for protocols that have been designed with “honest intentions”, but a distinction between a contrived protocol and a non-contrived protocol is, of course, subjective and thus it is better not to rely on such assumptions whenever possible.

More recently, several non-interactive shuffle arguments have been proposed which do not rely on the RO model [GL07, LZ13, GR16, FL16, FLZ16, FFHR19]. For ease, let us call such protocols RO-free. All of those arguments (actually, even many of the interactive arguments) require a common reference string (CRS) model [BFM88], i.e., it is assumed that a trusted party picks a bitstring $\text{crs}$ from some distribution $D$ and provides it to the prover and the verifier. Due to this reason, it is unclear how to apply those arguments in practice where we usually cannot trust any parties. Moreover, the efficiency of the current RO-free arguments is noticeably worse than the efficiency of the best RO model arguments [Gro10, BG12]. Efficiency and trust requirements of RO-free shuffle arguments are the main issues that we tackle in this thesis.

1.1. Contributions and Outline

Thesis follows three published papers respectively summarized in Chapter 3, Chapter 4, and Chapter 5. Background for these result is given in Chapter 2 and final conclusions and open questions are discussed in Chapter 6.

Since the thesis is based on multi-author papers, then we separately mention some contributions from the author of this thesis. Of course, it should be understood that the distinction of who did what is not so clear-cut in collaborative research. It is rare that one author achieves a result of any significance without being affected by one’s co-authors at all. Thus, this distinction should be viewed such that the noted results are the ones where the author had the biggest impact among all of the co-authors.
Chapter 3: An Efficient Shuffle Argument. We summarize the work of Fauzi, Lipmaa, Siim, and Zajac [FLSZ17a]. More precisely, we follow the full version [FLSZ17b] since the conference version contained a subtle flaw related to zero-knowledge property. The result builds upon the previous work of Fauzi, Lipmaa, Zajac [FLZ16] and proposes so far the most efficient non-interactive zero-knowledge shuffle argument in the CRS model. We denote it by $\delta$ in the following. Our argument uses the standard ElGamal cryptosystem compared to more specialized cryptosystems used in some of the other constructions [GL07, FLZ16, FFHR19]. On a technical level, we mix several previously known but very different arguments like zero-knowledge Succinct Non-interactive ARguments of Knowledge (zk-SNARKs) from Danezis et al. [DFGK14], quasi-adaptive zero-knowledge argument by Kiltz and Wee [KW15], and then some more shuffle specific ideas from Groth and Lu [GL07] and from other previous shuffle arguments [LZ13, FL16, FLZ16]. The final argument has perfect zero-knowledge and standard notion of computational soundness. However, some of the sub.arguments are proved in the generic group model (GGM) which allows an adversary to only make group operations while the group structure is hidden.

We optimize each sub-argument as much as possible and also apply batching techniques [BGR98] which amortize the verifying cost for multiple similar sub-arguments. We provide a well-optimized C++ implementation and see that for 100,000 ciphertexts the proving time is under 1 minute and verification time under 1.5 minutes on a laptop computer. This is close to the running time reported by Bayer and Groth [BG12] (2 minutes for proving and verifying on similar hardware) for one of the most efficient interactive shuffle arguments.

Author’s contributions include optimizing sub-arguments, applying batching techniques, and implementing the argument, but also, for example, reproving zero-knowledge property in the final version of the paper after a subtle flaw was found.

Chapter 4: UC-Secure CRS Generation For NIZK. This chapter is based on the paper by Abdolmaleki, Baghery, Lipmaa, Siim and Zajac [ABL+19b]. The shuffle argument $\delta$ is secure only in the CRS model. Motivated by this, we are interested in reducing the trust requirements of non-interactive zero-knowledge (NIZK) arguments in the CRS model. Common folklore’s suggestion is to generate the CRS with some secure multi-party protocol to distribute trust among multiple peers. Many of the efficient non-interactive zero-knowledge (NIZK) arguments, including our shuffle argument, operate in the pairing-based setting and require a relatively large CRS. Standard secure multi-party protocols are not efficient in this case if we also want to have a large number of peers in a fully malicious setting. Ben-Sasson et al. [BCG+14] designed a special-purpose multi-party protocol for CRS generation. Their protocol works for a large class of pairing-

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2Pairings are bilinear maps $\hat{e} : G_1 \times G_2 \rightarrow G_T$ that allow computing $\hat{e}(g_1^x, g_2^y) = g_T^{xy}$ where $G_i$ is some group with a generator $g_i$ for $i \in \{1, 2, T\}$. We cover them in more detail in Chapter 2.
based NIZK arguments, is much more efficient for CRS generation compared to standard multi-party protocols, and requires only 1 peer to be honest. Unfortunately, it also assumes the RO model and is thus unsuitable for the argument $S$.

In this work, we modified Ben-Sasson et al.’s protocol and make it RO-free. A key tool that we use is a discrete logarithm (DL) extractable commitment by Abdolmaleki et al. [ABL+19a] which opens to $g^x$ but allows extraction of $x$ in the security proof. Essentially, in the first phase of Ben-Sasson et al.’s protocol, each peer has to reveal some shares of the form $g^x$. However, peers must pick those shares independently. The original protocol achieves this by forcing peers to output an RO-based NIZK argument which proves that they know $x$. We instead run a coin-tossing protocol where all peers first commit to their share $x$ with a DL-extractable commitment in the first round and then open it to $g^x$ in the next round. Since the DL-extractable commitment we use is universally composable (UC) [Can01], then it is also non-malleable and we get that shares are chosen independently. Extractability allows us to simulate the protocol transcript and prove that the whole protocol is also UC-secure. We note that the DL-extractable commitment itself might also need a trust assumption but this depends on the concrete instantiation and is hopefully a weaker assumption than a trusted CRS.

The UC-security guarantees that our CRS generation protocol can be safely composed with the NIZK argument without affecting the security properties of the NIZK itself. Moreover, we introduce several efficiency improvements and show that class of arguments $C$ that we can cover includes the most efficient SNARK by Groth [Gro16].

The author’s contributions here include (among others) proposing the ideal functionality for the CRS generation protocol and proving the security of the protocol.

Chapter 5: Transparent Shuffle Argument. The final result is based on the work of Aggelakis, Fauzi, Korfatis, Louridas, Mergoupis-Anagnostou, Siim, and Zajac [AFK+20]. Our main objective with the CRS generation protocol was to distribute the trust of argument $S$. It turns out that the argument $S$ is outside of the class $C$ and thus we cannot directly apply the CRS generation protocol. We have to modify the argument $S$ and construct a new argument $S^+$ that we call a transparent shuffle argument due significantly reduced trust assumptions. Mainly we modify the structure of the CRS, but we also simplify the way that different sub-arguments are combined. After making several non-trivial changes to the argument, we also need to reprove the soundness and zero-knowledge properties. Here, we also manage to make some improvements. We prove the soundness of some of the sub-arguments under weaker assumptions compared to Fauzi et al. [FLSZ17a] and other arguments that had proof in GGM we now prove in a weaker algebraic group model (AGM) [FKL18].

CRS generation protocol guarantees soundness and zero-knowledge if at least one of the peers is honest. Turns out that it is possible to go one step further. Following the works of [BFS16, ABLZ17, Fuc18] we construct a CRS verification
algorithm such that if it accepts the CRS, then the zero-knowledge property will hold. This is known as statistical subversion zero-knowledge in [BFS16, ABLZ17, Fuc18] or equivalently no-auxiliary-string non-black-box zero-knowledge in the bare public-key model as was recently defined in [ALSZ18]. Importantly, it means that zero-knowledge holds even if all the peers in the CRS generation protocol are malicious. Finally, we also implement a complete set of algorithms. That includes the prover’s algorithm, verifier’s algorithm, distributed CRS generation protocol, and the CRS verification algorithm.

The main contributions of the author were modifications of the CRS to make $\delta^+$ suitable for the CRS generation protocol, coming up with the CRS verification algorithm, and proving many of the related results.

Claim of the Thesis. Based on the above results we formulate the main claim of this thesis:

*It is possible to construct a non-interactive zero-knowledge shuffle argument that is RO-free and still practical for large-scale elections.*

The notion of practicality is slightly subjective, but we believe that it is well justified in our case. In particular, it follows from subsequent facts:

1. Shuffle arguments $\delta$ and $\delta^+$ are sufficiently efficient for large numbers of ciphertexts as was shown in [FLSZ17a].
2. Trust assumptions will not be a major obstacle in practice. As we showed in [AFK+20], the argument $\delta^+$ is secure if we only trust 1 out of $N$ peers in the setup phase of the argument.
3. Argument $\delta$ has been integrated to the Zeus i-voting system [TPLT13] which is extensively used in practice [BCD+19]. Integration is in progress for the newer argument $\delta^+$.

1.2. Related Work

We will briefly compare state-of-the-art shuffle arguments to the results of this thesis. Efficiency wise the most interesting parameters are prover’s and verifier’s computational complexity, proof size, and CRS size. Since some shuffle arguments use specialized cryptosystems, occasionally with relatively long ciphertexts, then we will count shuffling complexity as part of the prover’s overall complexity and ciphertext size as part of the proof size. This seems fair to us considering that a mixer will output both the proof and the shuffled ciphertexts.

We also consider several qualitative properties that include the type of cryptosystem, type of CRS, security assumptions, the flavor of soundness, and flavor of zero-knowledge. Since a mix-network is usually part of a larger system (say, part of an i-voting system), then it is preferable to have a relatively standard cryptosystem with short ciphertexts such as Elgamal. Uniformly random CRS is usually preferred over a more specialized distribution since it is easier to implement based on some natural source of randomness (sunspot data, stock market fluctua-
tions, etc.). Perfect or statistical zero-knowledge is usually considered to be better than computational zero-knowledge since for the former we do not need to worry that some computational assumption gets broken.

There are numerous interactive shuffle arguments [SK95, Abe99, Nef01, FS01, FMM+03, GI08, TW10, Gro10, BG12]. However, since our main interest is non-interactive RO-free shuffle arguments, then we will not focus on them too much. State-of-the-art interactive shuffle are [Gro10] by Groth which has linear complexity in all of the previously mentioned efficiency parameters (with quite small coefficients) and [BG12] by Bayer and Groth which has a sublinear proof size (excluding the ciphertext size). Both of them support for example Elgamal ciphertexts and require only discrete logarithm (DL) assumption.

The line of work on non-interactive shuffle arguments was started by Groth and Lu [GL07] that proposed the first construction secure without RO. They proved only culpable soundness and used BBS cryptosystem [BBS04] which has ciphertext size of 3 group elements. As the main tool, they applied Groth-Sahai proofs [GS08] together with some novel assumptions. In their construction, all relevant efficiency parameters are linear in the number of ciphertexts but mostly with relatively large constants.

Lipmaa and Zhang [LZ13] proposed two non-interactive shuffle arguments that achieve the usual notion of soundness, but with a (non-falsifiable, see Section 2.2) knowledge assumption. They relied on an even more complicated variant of BBS cryptosystem which has 6 group elements and is lifted (decrypting requires computing discrete logarithm). Efficiency-wise, one of the construction has CRS size $\Theta(\sqrt{n})$, but other parameters are superlinear and the second construction has linear parameters with smaller constants than [GL07] except for the CRS size.

Fauzi and Lipmaa [FL16] further improved efficiency compared to Lipmaa and Zhang’s result. However, they use both knowledge assumptions and prove only culpable soundness. They give the first construction which works with standard non-lifted Elgamal cryptosystem. The unit vector argument used in our constructions is essentially borrowed from this work.

Fauzi, Lipmaa, and Zajac [FLZ16] achieve significant improvement in verification time compared to previous works. However, they prove the whole argument in the generic group model (GGM) which restricts adversary’s operations to group operations and hides the group structure. They use the non-standard lifted ILin cryptosystem [EHK+13] which contains 3 group elements.

González and Ráfols [GR16] proposed an argument that uses standard Elgamal cryptosystem and soundness is based only on falsifiable assumptions which are considered more safe [Nao03, GK16]. It also achieves relatively good efficiency in all parameters except for CRS size which is $\Theta(n^2)$. Latter is, unfortunately, a big drawback and makes it impractical for large-scale i-voting systems.

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3Elgamal cryptosystem itself, of course, requires decisional Diffie-Hellman (DDH) assumption.
4Culpable soundness restricts the adversary to have knowledge of his cheating. See Section 2.4 for further discussion and the formal definition.
Since the publication of our work [FLSZ17a] in 2017, Faonio et al. [FFHR19] proposed a mix-network which uses a specialized cryptosystem in a more fundamental way. First, they construct a re-randomizable RCCA (Replayable Chosen Ciphertext Attack)\(^5\) [CKN03] secure cryptosystem. They make it publicly verifiable by including a Groth-Sahai proof and then use techniques similar to Faonio and Fiore [FF18] (which itself is RO-based mix-network) to construct a shuffle argument. Interestingly, CRS in their case is a relatively small uniformly random string which only depends on the number of mixers and their construction is UC-secure under DDH assumption in \(\mathbb{G}_1\) and \(\mathbb{G}_2\) (SXDH). On a negative side, their publicly verifiable re-randomizable RCCA (pv-RRCCA) cryptosystem is relatively inefficient. For example the ciphertext size is 12 elements in \(\mathbb{G}_1\), 11 elements in \(\mathbb{G}_2\), and 4 elements in \(\mathbb{G}_T\).

Finally, we mention a series of works [CKLM12, CKLM13b, CKLM13a] that construct controllably malleable shuffle arguments respectively from Groth-Sahai proofs [GS08], Groth and Lu shuffle, and SNARKs. Essentially the idea is to let each mixer malle the argument returned by the previous mixer and include a short proof of participation. The whole mix-network can be verified with a single argument that is returned by the last mixer. Importantly, the argument size grows only additively in the number mixers, i.e., the size is \(\Theta(n + N)\) where \(N\) is the number of mixers and \(n\) is the number of input ciphertexts. All other arguments would have a collective argument size of \(\Theta(nN)\) assuming that a single argument has size \(\Theta(n)\). Their techniques seem to be fundamentally incompatible with Fiat-Shamir based arguments since hashing is non-malleable. However, it seems plausible that the constructions of this paper can also be extended to controlled malleability but with significantly better efficiency than the Groth and Lu argument. We leave this as an open problem.

Efficiency of the best non-interactive arguments is compared in Table 1 and other properties are compared in Table 2. We see that [Gro10, BG12] are still slightly more efficient than RO-free shuffles but the gap is small. Computational security assumptions are still less standard except for Faonio et al.’s construction which only requires SXDH assumption. Of course, we remind that [Gro10, BG12] also requires the RO model in the non-interactive setting which is only a heuristic. Arguably the best RO-free shuffle arguments are the constructions from this thesis and the construction of Faonio et al. Ours has better efficiency and uses a standard cryptosystem, but [FFHR19] has better security properties and a very short and uniform CRS.

\(^5\)This is a slightly weaker version of CCA security which allows malleability if the message of the ciphertext stays the same.
Table 1. Efficiency comparison of state-of-the-art shuffles. Here \( n \) is the number of ciphertexts, \( N \) the number of mixers, \( E \) is the cost of exponentiation, and \( P \) is the cost of pairing. Prover’s complexity includes shuffling time and proof size includes the size of the output ciphertexts. We drop constant terms.

<table>
<thead>
<tr>
<th></th>
<th>Prover efficiency</th>
<th>Verifier efficiency</th>
<th>Proof size</th>
<th>CRS size</th>
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<tbody>
<tr>
<td>[Gro10]</td>
<td>( 8n \times E )</td>
<td>( 6n \times E )</td>
<td>( 2n \times G )</td>
<td>( n \times G )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( 3n \times \mathbb{Z}_p )</td>
<td></td>
</tr>
<tr>
<td>[BG12]</td>
<td>( 2n + 2n \log(\sqrt{n}) \times E )</td>
<td>( 4n \times E )</td>
<td>( 2n + 11\sqrt{n} \times G )</td>
<td>( \sqrt{n} \times G )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \frac{5\sqrt{n}}{n} \times \mathbb{Z}_p )</td>
<td></td>
</tr>
<tr>
<td>[FL16]</td>
<td>( 20n \times E )</td>
<td>( 18n \times P )</td>
<td>( 7n \times G_1 )</td>
<td>( 6n \times G_1 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( 2n \times G_2 )</td>
<td></td>
</tr>
<tr>
<td>[FLZ16]</td>
<td>( 18n \times E )</td>
<td>( 14n \times E )</td>
<td>( 5n \times G_1 )</td>
<td>( 2n \times G_1 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 3n \times P )</td>
<td>( 4n \times G_2 )</td>
<td>( n \times G_2 )</td>
</tr>
<tr>
<td>[GR16]</td>
<td>( 13n \times E )</td>
<td>( 13n \times P )</td>
<td>( 4n \times G_1 )</td>
<td>( 2n \times G_2 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>( n^2 + 24n \times G_1 )</td>
<td></td>
</tr>
<tr>
<td>[FFHR19]</td>
<td>( 72n \times E )</td>
<td>( 22n \times P )</td>
<td>( 12n \times G_1 )</td>
<td>( 2N \times G_1 )</td>
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<tr>
<td></td>
<td>( 5n \times P )</td>
<td></td>
<td>( 11n \times G_2 )</td>
<td>( 2N \times G_2 )</td>
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<td></td>
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<td>( 4n \times G_T )</td>
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<tr>
<td>( \delta ) (Chapter 3)</td>
<td>( 11n \times E )</td>
<td>( 7n \times E )</td>
<td>( 4n \times G_1 )</td>
<td>( 4n \times G_1 )</td>
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<tr>
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<td>( 3n \times P )</td>
<td>( 3n \times G_2 )</td>
<td>( n \times G_2 )</td>
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<tr>
<td>( \delta^+ ) (Chapter 5)</td>
<td>( 11n \times E )</td>
<td>( 7n \times E )</td>
<td>( 4n \times G_1 )</td>
<td>( 5n \times G_1 )</td>
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<td>( 3n \times P )</td>
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Table 2. Qualitative properties of state-of-the-art non-interactive shuffle arguments

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<th></th>
<th>Encryption</th>
<th>Assumption</th>
<th>Soundness</th>
<th>ZK</th>
<th>CRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Gro10]</td>
<td>Elgamal</td>
<td>ROM, DL</td>
<td>standard</td>
<td>statistical</td>
<td>uniform</td>
</tr>
<tr>
<td>[BG12]</td>
<td>Elgamal</td>
<td>ROM, DL</td>
<td>standard</td>
<td>statistical</td>
<td>uniform</td>
</tr>
<tr>
<td>[FL16]</td>
<td>Elgamal</td>
<td>knowledge</td>
<td>culpable</td>
<td>statistical</td>
<td>structured</td>
</tr>
<tr>
<td>[FLZ16]</td>
<td>Elgamal</td>
<td>ILin (lifted)</td>
<td>GGM</td>
<td>standard</td>
<td>structured</td>
</tr>
<tr>
<td>[GR16]</td>
<td>Elgamal</td>
<td>falsifiable</td>
<td>standard</td>
<td>computational</td>
<td>structured</td>
</tr>
<tr>
<td>[FFHR19]</td>
<td>pv-RRCCA</td>
<td>SXDH</td>
<td>computational UC-secure mix-net</td>
<td>uniform</td>
<td></td>
</tr>
<tr>
<td>( \delta ) (Chapter 3)</td>
<td>Elgamal</td>
<td>GGM</td>
<td>standard</td>
<td>statistical</td>
<td>structured</td>
</tr>
<tr>
<td>( \delta^+ ) (Chapter 5)</td>
<td>Elgamal</td>
<td>AGM</td>
<td>standard</td>
<td>subversion statistical</td>
<td>structured</td>
</tr>
</tbody>
</table>
2. PRELIMINARIES

We start by introducing some basic notation used throughout this thesis. We denote the security parameter by $\lambda$. A function $f : \mathbb{N} \to [0,1]$ is called negligible if $f(\lambda) = O(\lambda^{-c})$ for every constant $c \in \mathbb{N}$. We denote this by $f \approx \lambda$.

Integers modulo $p$ are denoted by $\mathbb{Z}_p$ and the ring of polynomials with variables $X_1,\ldots,X_n$ over $\mathbb{Z}_p$ by $\mathbb{Z}_p[X_1,\ldots,X_n]$. Vectors $a := (a_i)_{i=1}^n \in \mathbb{Z}_p^n$ are column vectors by default and are usually denoted by bold lower-case letters. Length $n$ vector of ones is denoted by $\mathbf{1}_n$ and vector of zeros by $\mathbf{0}_n$. Matrices $A \in \mathbb{Z}_p^{m \times n}$ are denoted bold upper-case letters and $i$-th row vector of $A$ is denoted by $\mathbf{A}_i$ and $i$-th column by $A^{(i)}$. Identity matrix is denoted by $\mathbf{I}_n \in \mathbb{Z}_p^{n \times n}$. Set of permutations on $n$ elements is denoted by $\mathcal{S}_n$. Set $\{1,\ldots,n\}$ is denoted by $[n]$.

Sampling $x$ from a distribution $\mathcal{D}$ is written as $x \leftarrow \mathcal{D}$. If $\mathcal{D}$ is a set, then $x \leftarrow \mathcal{D}$ means uniform sampling. We often express probabilities in the form $\Pr[x \leftarrow \mathcal{D} : \text{Pred}(x)]$ where $\mathcal{D}$ is a probability distribution and Pred some predicate. Sometimes we need to explicitly use random coins of an algorithm $A$. We denote sampling sufficiently long randomness by $r \leftarrow \text{RND}_\lambda(A)$ and $A$ outputting $y$ on input $x$ and random coins $r$ is denoted by $y \leftarrow A(x;r)$. Often we leave out the random coins and just write $y \leftarrow A(x)$. Probabilistic polynomial time is abbreviated by PPT. We assume that algorithms are stateful, e.g., we may first execute $A$ on input $x$ and then again on another input $y$ without $A$ losing its state from the first execution.

Lagrange polynomials. Let $\omega_1,\ldots,\omega_{n+1}$ be unique elements in $\mathbb{Z}_p$. Lagrange polynomial $\ell_i(X)$ is the unique $n$-th degree polynomial such that $\ell_i(\omega_i) = 1$ and $\ell_i(\omega_j) = 0$ for $j \neq i$. More explicitly we may express $\ell_i(X) = \prod_{j \neq i}^{n+1} X - \omega_j$. Lagrange polynomials $\ell_1(X),\ldots,\ell_{n+1}(X)$ form a basis for all polynomials in $\mathbb{Z}_p[X]$ which are at most degree $n$. Lagrange polynomials are often used for interpolation. Namely, if $f \in \mathbb{Z}_p[X]$ has a degree at most $n$ and $f(\omega_i) = a_i$ for $i \in [1..n+1]$, then $f(X) = \sum_{i=1}^{n+1} a_i \ell_i(X)$.

2.1. Pairings

All the main constructions of this thesis use cryptographic bilinear pairings. Let $\text{BPgen}$ be a deterministic polynomial time algorithm that on input $1^\lambda$ outputs a tuple $\text{bp} = (p,\hat{\epsilon},G_1,G_2,G_T,\mathcal{P}_1,\mathcal{P}_2)$ where $p$ is a prime of length $\Theta(\lambda)$, $G_1, G_2, G_T$ are additive groups of order $p$ with an efficient operation, $\mathcal{P}_1$ and $\mathcal{P}_2$ are respectively the generators of $G_1$ and $G_2$, and $\hat{\epsilon}$ is a map $G_1 \times G_2 \to G_T$ with the following properties:

Bilinearity: $\hat{\epsilon}(x\mathcal{P}_1, y\mathcal{P}_2) = (xy)\mathcal{P}_T$ where $\mathcal{P}_T := \hat{\epsilon}(\mathcal{P}_1, \mathcal{P}_2)$ for any $x,y \in \mathbb{Z}_p$. 

Non-degeneracy: $\mathcal{P}_T$ as defined above is a non-zero element and thus also a generator of the target group $G_T$.

Efficiency: $\hat{e}$ can be efficiently computed.

Pairings are usually divided into three types: Type 1 pairings have $G_1 = G_2$, Type 2 pairings have $G_1 \neq G_2$, and Type 3 pairings have $G_1 \neq G_2$ even such that there is no efficiently computable isomorphism between $G_1$ and $G_2$. Type 3 pairings are currently the most widely used due to their efficiency and also in this thesis, we will use Type 3 pairings. We will not discuss how pairings are constructed and will use them as a black box. For state-of-the-art pairings, we refer the reader to [BD19].

Bracket Notation. We will be using pairings quite extensively and thus we are going to use convenient bracket notation of Escala et al. [EHK13] which has become widely used in pairing-based research. Firstly, instead of writing $x \mathcal{P} y$, for $i \in \{1, 2, T\}$, we write $[x]_i$. This extends naturally to matrices $[A]_i \in \mathbb{Z}_p^{n \times m}$ where $A \in \mathbb{Z}_p^m$. Secondly, instead of writing the pairing operation as $\hat{e}([x]_1, [y]_1)$, we write $[x]_1 \cdot [y]_2 = [x]_1[y]_2 = [xy]_T$ and similarly we can write $[A]_1 \cdot [B]_2 = [A]_1[B]_2 = [AB]_T$ for matrices $A$ and $B$ of suitable dimensions. Sometimes we also write $[x]_{1,2}$ as a shorthand for $[x]_1$, $[x]_2$.

2.2. Assumptions

Let GPgen be a PPT group generator that on input $1^k$ outputs $(p, G, \mathcal{P})$ where $G$ is an additive group of prime order $p = \Theta(\lambda)$ and $\mathcal{P}$ a random generator of the group. We still use the bracket notation to represent group elements but drop the lower index.

We start by defining some common assumptions in group-based cryptography.

**Definition 1 (DL Assumption).** Discrete Logarithm (DL) assumption holds relative to GPgen, if for any PPT adversary $A$, $\varepsilon \approx_\lambda 0$, where

$$\varepsilon := \Pr\left[gp \leftarrow \text{GPgen}(1^k), x \leftarrow \mathbb{Z}_p : A(gp, [x]) = x \right].$$

**Definition 2 (q-PDL Assumption).** $q$-Power Discrete Logarithm (PDL) assumption holds relative to GPgen, if for any PPT adversary $A$, $\varepsilon \approx_\lambda 0$, where

$$\varepsilon := \Pr\left[gp \leftarrow \text{GPgen}(1^k), x \leftarrow \mathbb{Z}_p : A(gp, [x, \ldots, x^q]) = x \right].$$

We can also have a similar assumption in the pairing-based setting.

**Definition 3 (q-SPDL Assumption).** $q$-Symmetric Power Discrete Logarithm (SPDL) assumption holds relative to BPgen, if for any PPT adversary $A$, $\varepsilon \approx_\lambda 0$, where

$$\varepsilon := \Pr\left[bp \leftarrow \text{BPgen}(1^k), x \leftarrow \mathbb{Z}_p : A(gp, [x, \ldots, x^q]_{1,2}) = x \right].$$

**Definition 4 (CDH Assumption).** Computational Diffie-Hellman (CDH) assumption holds relative to GPgen, if for any PPT adversary $A$, $\varepsilon \approx_\lambda 0$, where

$$\varepsilon := \Pr\left[gp \leftarrow \text{GPgen}(1^k), x, y \leftarrow \mathbb{Z}_p : A(gp, [x], [y]) = [xy] \right].$$
Definition 5 (DDH Assumption). Decisional Diffie-Hellman (DDH) assumption holds relative to GPgen, if for any PPT adversary \( A \), \(|\epsilon_0 - \epsilon_1| \approx \lambda 0\), where

\[
\epsilon_b := \Pr \left[ \text{gp} \leftarrow \text{GPgen}(1^\lambda), x, y \leftarrow \mathbb{Z}_p, \text{if } b = 0 \text{ then } z \leftarrow xy \text{ else } z \leftarrow z \mathbb{Z}_p : A(\text{gp}, [x], [y], [z]) = 1 \right] .
\]

Morillo et al. [MRV16] introduced the Kernel Matrix Diffie-Hellman (KernelMDH) assumption which generalizes many different previously known computational assumptions. We present it here for bilinear groups. Let \( D_{\ell,k} \) be some matrix distribution over \( \mathbb{Z}_p^{\ell \times k} \), where \( \ell > k \).

Definition 6 (KernelMDH Assumption). \( D_{\ell,k} \)-KernelMDH (Kernel Matrix Diffie-Hellman) assumption holds relative to BPgen, if for any PPT adversary \( A \), \( \epsilon \approx \lambda 0 \), where

\[
\epsilon := \Pr \left[ \text{bp} \leftarrow \text{BPgen}(1^\lambda), A \leftarrow D_{\ell,k}, [x]_2 \leftarrow A(\text{bp}, [A]_1) : A^\top x = 0 \land x \neq 0 \right] .
\]

One usually takes \( \ell = k + 1 \) for best efficiency and writes \( D_k := D_{k+1,k} \). Some common families, that are conjectured to be secure, include

\[
L_k := \begin{pmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\vdots & \ldots & \vdots \\
1 & \ldots & 1
\end{pmatrix},
E_k := \begin{pmatrix}
a_1 & 0 & \ldots & 0 \\
0 & a_2 & \ldots & 0 \\
0 & \ldots & \ldots & 0 \\
1 & \ldots & \ldots & 1
\end{pmatrix},
U_k := \begin{pmatrix}
a_{1,1} & \ldots & a_{1,k} \\
a_{2,1} & \ldots & a_{2,k} \\
\vdots & \ldots & \vdots \\
a_{k+1,1} & \ldots & a_{k+1,k}
\end{pmatrix},
\]

where \( a_i, a_{ij} \leftarrow \mathbb{Z}_p \).

We will also be using a very simple knowledge assumption.

Definition 7 (BDH-KE [ABLZ17]). Bilinear Diffie-Hellman Knowledge of Exponent assumption holds relative to BPgen, if for any PPT \( A \) exists a PPT extractor \( \text{Ext} \), such that

\[
\Pr \left[ \text{bp} \leftarrow \text{BPgen}(1^\lambda), r \leftarrow \text{RND}_k(A), ([x]_1, [x]_2) \leftarrow A(\text{bp}; r), x' \leftarrow \text{Ext}(\text{bp}; r) : x \neq x' \right] \approx \lambda 0 .
\]

Generic Group Model. Generic (bilinear) group model (GGM) [Sho97,Mau05] defines a restricted mode of computation in a group-based setting. The idea is to allow an algorithm to only make group-based operations and keep the structure of the group itself hidden. In the context of this thesis, we consider natural operations of bilinear groups which are exponentiation, group addition, pairing, and equality check. One way to formalize GGM is by making operations through oracle calls where oracle only returns a new memory index of group element or a bit in case of equality test. We refrain from giving a full formal description and redirect the reader to [Mau05].

GGM has found two common applications in cryptography. One is to justify novel group-based security assumptions by showing that they hold at least respect to generic adversaries, e.g. [Sho97,BBG05]. Second is by directly proving the security of a cryptographic primitive, e.g [BFF+15,Gro16]. Security proofs in GGM are usually unconditional rather than reductions to some assumption. Like RO model, GGM (at least the Shoup’s version) is known to fail in some contrived protocols [Den02].
Algebraic Group Model. More recently, Fuchsbauer, Kiltz, and Loss [FKL18] proposed a relaxation of GGM that they called algebraic group model (AGM). Here, the group structure is revealed to the adversary and the only restriction is that the adversary has to provide a linear representation together with each group element. More precisely, suppose that an algebraic adversary gets \([x_1, \ldots, x_n]\) and \(gp\) as an input. Then together with each group element \([y]\) that adversary outputs, it also has to output a representation \(a_0, a_1, \ldots, a_n \in \mathbb{Z}_p\) such that \(y = a_0 + \sum_{i=1}^{n} a_i x_i\).

This model is slightly more realistic since attacks can take into account the concrete structure of the group. On the other hand, security proofs in AGM are typically reductions (usually to some flavor of DL assumption) unlike unconditional proofs in GGM. Known weaknesses of GGM like [Den02] do not carry over to AGM as was recently observed in [AHK20].

Falsifiable vs. Non-Falsifiable Assumptions. Naor [Nao03] introduced the classification of security assumptions based on the notion of falsifiability. This was later re-formulated by Gentry and Wichs [GW11] (see also the classification by Pass [Pas13]) to what is currently understood by a falsifiable assumption. Gentry and Wichs called an assumption falsifiable if it can be formulated as an interactive protocol between an efficient challenger and an adversary such that the challenger can verify if the adversary won the game. Most cryptographic assumptions, e.g. DL, DDH assumptions from above, are falsifiable.

However, some assumptions do not fit under this umbrella. Some assumptions require the challenger to have superpolynomial computational power (see for example one-more discrete logarithm assumption [BNPS03]) and some seemingly do not fit into the challenger-adversary framework at all. Of the latter type are all kinds of knowledge assumptions. For example Damgård’s knowledge-of-exponent assumption [Dam92] states that for any PPT adversary \(A\) there exist a PPT extractor \(\text{Ext}\) such that if \(A\) on input \(([1], [x])\) outputs \(([z], [zx])\), then \(\text{Ext}\), given the same random coins and input, can output \(z\) with an overwhelming probability. Idea is that essentially the only way to compute \(([z], [zx])\) is by first computing \(z\) and then multiplying \(([1], [x])\) with \(z\). Then \(\text{Ext}\) can be seen as a slight modification of \(A\)’s code which outputs \(z\) instead of \(([z], [zx])\). BDH-KE assumption is a variation of this assumption.

A common sentiment is that non-falsifiable assumptions are less trustworthy due to difficulty in assessing their validity (see for example [Nao03, GK16]). There is even some concrete evidence that knowledge assumptions do not hold respect to arbitrary (efficiently sampleable) auxiliary input [BP15]. AGM and GGM are even stronger than concrete group-based (knowledge) assumptions. However, AGM and GGM are often used to justify novel group-based assumptions. In our opinion, if the novel assumption is essentially equivalent to the cryptographic primitive, it might be more reasonable to directly prove the primitive itself in AGM or GGM.
2.3. Cryptosystems

A public-key cryptosystem is a triple of PPT algorithms \((\mathsf{Kgen}, \mathsf{Enc}, \mathsf{Dec})\). Key generator \(\mathsf{Kgen}\) takes in a security parameter \(1^\lambda\) and outputs a public key \(\mathsf{pk}\) and a secret key \(\mathsf{sk}\). Encryption algorithm \(\mathsf{Enc}\) takes as input a public key \(\mathsf{pk}\) and a message \(m \in \mathcal{M}\) where \(\mathcal{M}\) is the message space. Decryption algorithm takes as input a secret key \(\mathsf{sk}\) and a ciphertext \(c\) and outputs a message \(m\).

Cryptosystem should be functional in the sense that for any key pair \((\mathsf{pk}, \mathsf{sk})\) and \(m \in \mathcal{M}\) we have \(\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, m)) = m\). A common security requirement is INDistinguishability under Chosen Plaintext Attack (IND-CPA) which requires that ciphertexts are indistinguishable even for known plaintexts. More precisely, for any PPT adversary \(\mathcal{A}\), \(|\epsilon_0 - \epsilon_1| \approx \chi \lambda\) where,

\[
\epsilon_b := \Pr\left[ (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Kgen}(1^\lambda), (m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) : \mathcal{A}(\mathsf{Enc}(\mathsf{pk}, m_b)) = 1 \right]
\]

We call a group based cryptosystem lifted if the message space is \(\mathcal{M} = \mathbb{Z}_p\) and decryption involves computing the discrete logarithm of \([m]\) to recover \(m\). This means that only decryption of short messages, say 30 or 40 bits, is possible.

Elgamal Cryptosystem. In this thesis, we only use the (non-lifted) Elgamal cryptosystem [ElG84] that we define next. Let us sample a group description \((p, G, \mathcal{P}) \leftarrow \mathsf{GPgen}(1^\lambda)\). We set the message space \(\mathcal{M} = G\). Key generator \(\mathsf{Kgen}\) outputs a random secret key \(\mathsf{sk} \leftarrow \{0\}^p\) and sets the public key to \(\mathsf{pk} \leftarrow [1, \mathsf{sk}]\). Encryption algorithm \(\mathsf{Enc}\) on input \((\mathsf{pk}, m = [m]) \in G\) samples \(r \leftarrow \{0\}^p\) and outputs \([0, m] + r \cdot [1, \mathsf{sk}]\). Sometimes we write \(\mathsf{Enc}(\mathsf{pk}, m; r)\) when \(r\) is sampled outside of the algorithm \(\mathsf{Enc}\). Decryption algorithm \(\mathsf{Dec}\) takes as an input \((\mathsf{sk}, c = [c_1, c_2])\) and outputs \(c_2 - \mathsf{sk} c_1\).

It can be verified with a direct substitution that the Elgamal cryptosystem is functional. It also satisfies IND-CPA security under the DDH assumption. Moreover, the Elgamal cryptosystem has the blinding property which allows randomizing ciphertexts. Namely, given an arbitrary ciphertext \(c = [r, m + r \cdot \mathsf{sk}]\), we can pick a new randomness \(r' \leftarrow \{0\}^p\) and compute a new ciphertext \(c' \leftarrow ([r'] + [r], r' [\mathsf{sk}] + [m + r \cdot \mathsf{sk}] = \mathsf{Enc}(\mathsf{pk}, [m]; r + r')\). Ciphertext \(c'\) still decrypts to the same message \([m]\) but is now computationally indistinguishable from a ciphertext with any other message.

2.4. Non-Interactive Zero-Knowledge Arguments

Let \(\mathcal{L}\) be a NP-language and \(\mathcal{R}\) the corresponding binary relation of all pairs \((x, w)\) where \(x\) that we call a statement is in \(\mathcal{L}\) and \(w\) is any of the efficiently verifiable witnesses for \(x\).

A non-interactive zero-knowledge (NIZK) argument \(\mathcal{A}\) for a relation \(\mathcal{R}\) is a tuple of following PPT algorithms:
• Parameter generator $P_{\text{gen}}$ that takes as an input a security parameter $1^\lambda$ and outputs a setup parameters $\text{par}$. In our construction $\text{par}$ is the bilinear group description.

• CRS generation algorithm $C_{\text{gen}}$ that takes as an input $\text{par}$ and outputs a CRS $\text{crs}$ and a trapdoor $\text{td}$.

• Prover algorithm $P$ that takes an input $(\text{par}, \text{crs}, x, w)$ such that $(x, w) \in \mathcal{R}$ and outputs an argument $\pi$.

• Verifier algorithm $V$ that takes an input $(\text{par}, \text{crs}, x, \pi)$ and either rejects the argument by outputting 0 or accepts the argument by outputting 1.

• Simulator algorithm $\text{Sim}$ that takes an input $(\text{par}, \text{crs}, x, \text{td})$ with $x \in \mathcal{L}$ and outputs a forged argument $\pi$.

NIZK arguments must always satisfy some flavor of the properties, completeness, soundness, and zero-knowledge. Completeness establishes that the protocol is functional, i.e., if the CRS generator, the prover, and the verifier are all honest, then the proof will be accepted. Soundness captures a security property that it should be computationally hard to construct an acceptable argument for an invalid statement $x \notin \mathcal{L}$. Zero-knowledge means that the algorithm $\text{Sim}$, given additionally a trapdoor $\text{td}$, can efficiently produce an argument $\pi$ that is indistinguishable from an honest argument. Therefore if one knows the trapdoor, then he does not actually “learn” anything new from the proof besides that $x \in \mathcal{L}$. We define these properties more formally below.

**Definition 8 (Completeness).** An argument $\mathcal{A}$ is perfectly complete if for any $\lambda \in \mathbb{N}$ and $(x, w) \in \mathcal{R}$,

$$\Pr \left[\text{par} \leftarrow P_{\text{gen}}(1^\lambda), (\text{crs}, \text{td}) \leftarrow C_{\text{gen}}(\text{par}), \pi \leftarrow P(\text{par}, \text{crs}, x, w) : V(\text{par}, \text{crs}, x, \pi) = 1\right]\right] = 1.$$

**Definition 9 (Soundness).** An argument $\mathcal{A}$ is computationally sound if for any PPT adversary $A$,

$$\Pr \left[\text{par} \leftarrow P_{\text{gen}}(1^\lambda), (\text{crs}, \text{td}) \leftarrow C_{\text{gen}}(\text{par}), (x, \pi) \leftarrow A(\text{par}, \text{crs}) : V(\text{par}, \text{crs}, x, \pi) = 1 \land x \notin \mathcal{L}\right]\approx_\lambda 0.$$

One can strengthen the above definition and require that prover has knowledge of the witness $w$ corresponding to $x$. By knowledge, we mean that an adversary could efficiently compute $w$. We capture this with an efficient extraction algorithm $\text{Ext}$ that outputs $w$ given the same input and the same random tape as the adversary.

**Definition 10 (Knowledge Soundness).** An argument $\mathcal{A}$ is computationally knowledge sound if for any PPT adversary $A$, there exists a PPT extractor $\text{Ext}$ such that,

$$\Pr \left[\text{par} \leftarrow P_{\text{gen}}(1^\lambda), (\text{crs}, \text{td}) \leftarrow C_{\text{gen}}(\text{par}), r \leftarrow \text{RND}_\lambda(A), (x, \pi) \leftarrow A(\text{par}, \text{crs}; r), w \leftarrow \text{Ext}(\text{par}, \text{crs}; r) : V(\text{par}, \text{crs}, x, \pi) = 1 \land (x, w) \notin \mathcal{R}\right]\approx_\lambda 0.$$
Groth, Ostrovsky, and Sahai [GOS06a, GOS06b] introduced a weaker form of soundness called culpable soundness (originally called co-soundness) which intuitively guarantees security only against adversaries that can provide a witness of their cheating. More formally, we fix an efficiently verifiable relation $R_{gl} = \{(x, w_{gl}) : x \in \overline{L}\}$ where $\overline{L}$ is the complement of the language $L$ and define culpable soundness as follows.

**Definition 11 (Culpable Soundness).** An argument $\mathcal{A}$ is computationally culpably sound respect to $R_{gl}$ if for any PPT adversary $A$,

$$\Pr \left[ \par \leftarrow Pgen(1^\lambda), (\text{crs}, \text{td}) \leftarrow Cgen(\par), (x, w_{gl}, \pi) \leftarrow A(\par, \text{crs}) : \right. \right.$$

$$\left. V(\par, \text{crs}, x, \pi) = 1 \land (x, w_{gl}) \in R_{gl} \right] \approx \lambda^0.$$  

The idea here is that choosing a suitable $R_{gl}$ can allow using $w_{gl}$ to make a reduction to some security assumptions. Of course, in reality, there can be PPT adversaries that do not know $w_{gl}$ but can still provide an acceptable proof $\pi$ for a false statement $x$. Moreover, note that culpable soundness is possible only for languages in $\text{NP} \cap \text{coNP}$.

**Definition 12 (Zero-Knowledge).** An argument $\mathcal{A}$ is perfectly zero-knowledge if for any adversary $A$, $\varepsilon_0 = \varepsilon_1$, where

$$\varepsilon_b := \Pr \left[ \par \leftarrow Pgen(1^\lambda), (\text{crs}, \text{td}) \leftarrow Cgen(\par), (x, w) \leftarrow A(\par, \text{crs}), \right.$$  

$$\left. (x, w_0, w_1) \leftarrow A(\par, \text{crs}), \right.$$  

$$\left. \text{if } b = 0, \text{ then } \pi \leftarrow P(\par, \text{crs}, x, w) \text{ else } \pi \leftarrow \text{Sim}(\par, \text{crs}, x, \text{td}) : A(\pi) = 1 \right].$$  

There is a slightly weaker form of privacy called witness indistinguishability (WI) which requires that adversary cannot distinguish which witness was used to construct the proof. Observe however that if a statement $x$ has only one valid witness $w$, then WI can hold even if argument leaks $w$.

**Definition 13 (Witness-Indistinguishability).** An argument $\mathcal{A}$ is perfectly witness-indistinguishable if for any adversary $A$, $\varepsilon_0 = \varepsilon_1$, where

$$\varepsilon_b := \Pr \left[ \par \leftarrow Pgen(1^\lambda), (\text{crs}, \text{td}) \leftarrow Cgen(\par), (x, w_0, w_1) \leftarrow A(\par, \text{crs}), \right.$$  

$$\left. \pi \leftarrow P(\par, \text{crs}, x, w_b) : A(\pi_b) = 1 \land (x, w_0) \in R \land (x, w_1) \in R \right].$$

**Subversion zero-knowledge.** Finally, we present a very strong notion of zero-knowledge where even the CRS generator can be malicious. We follow the definition from [ABLZ17]. The idea is to additionally have a CRS verification algorithm $CV$ which is used by the prover. Prover reveals the proof only if CRS passes the verification. To simulate, we need to make sure that the CRS subverter knows the trapdoor in case the CRS passes verification. Thus, we also require an extractor algorithm that outputs a trapdoor given subverter’s random coins as an input.

Formally, we say that a NIZK argument $\mathcal{A}$ for relation $R$ is subversion resistant if it additionally contains the following PPT algorithm.

- CRS verification algorithm $CV$ that takes as an input $(\par, \text{crs})$ and either accepts the CRS by outputting 1 or rejects it by outputting 0.
**Definition 14 (Sub-ZK).** A subversion resistant argument is statistically subversion zero-knowledge if for any PPT subverter \( \text{Sub} \), there exists an extractor \( \text{Ext} \), such that for any adversary \( A \),

\[ \varepsilon_0 \approx_{\lambda} \varepsilon_1, \]

where

\[ \varepsilon_b := \Pr \left[ \begin{array}{l}
\text{par} \leftarrow \text{Pgen}(1^\lambda), r \leftarrow \text{RND}_\lambda(\text{Sub}), (\text{crs}, \text{aux}) \leftarrow \text{Sub}(\text{par}; r), \\
\text{td} \leftarrow \text{Ext}(\text{par}; r), (x, w) \leftarrow A(\text{aux}), \\
\text{If } b = 0, \text{ then } \pi \leftarrow P(\text{par}, \text{crs}, x, w) \text{ else } \pi \leftarrow \text{Sim}(\text{par}, \text{crs}, x, \text{td}) : \\
\text{CV}(\text{par}, \text{crs}) = 1 \land A(\pi) = 1
\end{array} \right]. \]  

### 2.5. Universal Composability

Universal composability (UC) [Can01] is a security model where protocols retain their security properties even when arbitrarily composed with other protocols or different instances of the same protocol. Security requirements in the UC model are captured by an ideal functionality. We have two games called the real world (includes an actual protocol) and an ideal world (with an ideal functionality) that should be indistinguishable. Below we give their rough description, but for a detailed explanation, we refer the reader to the original paper by Canetti [Can01].

The real-world game \( \text{Real}_{Z, A, \Pi} \) consists of PPT algorithms: environment \( Z \), adversary \( A \), some number of parties \( P_1, \ldots, P_N \), and a protocol \( \Pi \). Environment \( Z \) models all the other possible protocols running in parallel. The environment is allowed to give inputs and receive outputs from parties and freely communicate with \( A \). Adversary \( A \) can corrupt new parties at any point in the execution and will have full control over them which includes seeing their internal state. Parties in real-world execute the protocol \( \Pi \) and the communication is routed through \( A \). In the end, \( Z \) will output a guess 1 if it believes to be in the real world and 0 otherwise.

In the ideal world game \( \text{Ideal}_{Z, \text{Sim}, F} \) we also have an environment \( Z \) and parties \( P_1, \ldots, P_N \) but instead of the adversary we have a simulator \( \text{Sim} \) and instead of \( \Pi \) we have an ideal functionality \( F \). Ideal functionality is essentially a trusted third party that receives input data (possibly private data) from parties and returns them the correct output. Depending on the primitive there can even be back and forth communication between \( F \) and the parties. (See DL-extractable commitment in Section 2.6 for a relatively simple example.) Parties in the ideal world only transfer inputs to the ideal functionality and outputs from ideal functionality back to the environment. Simulator \( \text{Sim} \) has to be able to simulate the output of \( A \) which might include for example the protocol transcript. Importantly, \( \text{Sim} \) does not see the communication between ideal functionality and parties. Finally, \( Z \) again outputs a guess.

**Definition 15.** We say that a protocol \( \Pi \) UC-realizes a functionality \( F \) if for any PPT \( A \), there exists a PPT simulator \( \text{Sim} \) such for all PPT \( Z \) and \( x \in \{0, 1\}^{\text{poly}(\lambda)} \),

\[ \Pr[\text{Real}_{Z, A, \Pi}(x) = 1] \approx_{\lambda} \Pr[\text{Ideal}_{Z, \text{Sim}, F}(x) = 1] \]  

where \( x \) is an auxiliary input to \( Z \).
If \( \Pi \) uses some ideal functionality \( F^* \) as a subprotocol to UC-realize \( F \), then we say that \( \Pi \) UC-realizes \( F \) in \( F^* \)-hybrid model.

The main importance of the UC model is the universal composability theorem which, roughly speaking, says that \( \Pi \) that UC-realizes a functionality is secure even when running concurrently with other arbitrary protocols. See [Can01] for the formally rigorous statement.

### 2.6. Commitment Schemes

A commitment scheme is a pair of PPT algorithms \( \mathcal{C} = (Kgen, Com) \). Key generation algorithm \( Kgen \) takes in a security parameter \( 1^\lambda \) and outputs a commitment key \( ck \). Commitment algorithm \( Com \) takes in \( ck, m \) from message space \( \mathcal{M} \), randomness \( r \) from randomness space \( \mathcal{R} \) and outputs a commitment \( c \).

We can think of a commitment scheme as a cryptosystem which does not necessarily have a decryption algorithm. Commitment \( c \) should not reveal the message \( m \) (hiding) and it should be computationally difficult to find \( r' \in \mathcal{R} \) and \( m' \in \mathcal{M} \) such that \( c = Com(ck, m'; r) \) but \( m \neq m' \) (binding).

**Definition 16** (hiding). A commitment scheme \( \mathcal{C} \) is computationally hiding if for any PPT adversary \( A \), \( |\varepsilon_0 - \varepsilon_1| \approx_\lambda 0 \) where,

\[
\varepsilon_b := \Pr \left[ ck \leftarrow Kgen(1^\lambda), (m_0, m_1) \leftarrow A(ck) : A(Com(ck, m_b)) = 1 \right] .
\]

\( \mathcal{C} \) is perfectly hiding if \( \varepsilon_0 = \varepsilon_1 \) for even any computationally unbounded \( A \).

**Definition 17** (binding). A commitment scheme \( \mathcal{C} \) is computationally binding if for any PPT adversary \( A \), \( \varepsilon \approx_\lambda 0 \) where,

\[
\varepsilon := \Pr \left[ \begin{array}{l}
ck \leftarrow Kgen(1^\lambda), (m_0, m_1, r_0, r_1) \leftarrow A(ck) : \\
Com(ck, m_0, r_0) = Com(ck, m_1, r_1) \wedge m_0 \neq m_1
\end{array} \right].
\]

\( \mathcal{C} \) is perfectly binding if \( \varepsilon = 0 \) for even any unbounded \( A \).

Every cryptosystem can be used as a perfectly binding and computationally hiding commitment scheme.

**p(X)-Multi-Pedersen Commitment.** We formalize a slight extension of the common (Multi-)Pedersen commitment. Let \( \mathcal{G} \) be a cyclic group of prime order \( p \) as before. We define the message space as \( \mathbb{Z}_p^n \) and randomness space as \( \mathbb{Z}_p \).

Let \( p(X) = (p_1(X), \ldots, p_n(X))\top \) be a vector of linearly independent polynomials of (small) degree at most \( d \). We define the \( p(X) \)-Multi-Pedersen (MP) commitment scheme as follows. Key generator \( Kgen \) samples \( x, \rho \leftarrow \mathbb{Z}_p \) such that \( \rho \neq 0 \) and outputs \( ck \leftarrow [\rho, p_1(x), \ldots, p_n(x)] \). Commitment algorithm \( Com \) takes in \( ck, m \in \mathbb{Z}_p^n, r \leftarrow \mathbb{Z}_p \) and outputs \( c \leftarrow r[\rho] + \sum_{i=1}^n m_i[p_i(x)] \).

**Theorem 1.** \( p(X) \)-MP is perfectly hiding and computationally binding under d-PDL assumption.
Let parties be $P_1, \ldots, P_N$ and the message space is $\mathbb{Z}_p$.

- On receiving input $(\text{commit}, \text{sid}, \text{cid}, P_i, P_j, m)$ from party $P_i$ where $m \in \mathbb{Z}_p$, if no tuple with $(\text{sid}, \text{cid})$ has been recorded, then store $(\text{commit}, \text{sid}, \text{cid}, P_i, P_j, m)$ and send $(\text{received}, \text{sid}, \text{cid}, P_i, P_j)$ to $P_j$ and simulator Sim.
- On receiving $(\text{open}, \text{sid}, \text{cid})$ from $P_i$, if a tuple $(\text{commit}, \text{sid}, \text{cid}, P_i, P_j, m)$ has been recorded then send $(\text{open}, \text{sid}, \text{cid}, P_i, P_j, [m])$ to $P_j$ and Sim.

**Proof.** Hiding. Since $[\rho] \neq [0]$ is a generator of $\mathbb{G}$, then $r[\rho]$ is a uniformly random in $\mathbb{G}$ and thus the commitment itself is uniformly random and independent of the message $m$. It follows that the scheme is perfectly hiding.

**Binding.** Suppose a PPT adversary $A$ outputs $(m, r, m', r')$ with probability $\varepsilon$ such that $r[\rho] + \sum_{i=1}^n m_i p_i(x) = r'[\rho] + \sum_{i=1}^n m'_i p_i(x)$ but $m \neq m'$. We will construct an adversary $B$ that breaks the $d$-PDL assumption with the same probability. First, $B$ receives $[x, \ldots, x^d]$ as a challenge and samples $\rho \leftarrow \mathbb{Z}_p$ which is enough (together with the input) to construct the commitment key $ck$. It runs $A$ on input $ck$ to get $(m, r, m', r')$. Let us define a polynomial $p(X) := \sum_{i=1}^n (m_i - m'_i) p_i(X)$. Observe that if $A$ was successful, then $p(X)$ is a non-zero polynomial since $\{p_i(X)\}_i$ are linearly independent and at least one of the coefficients $m_i - m'_i$ is non-zero. Thus, $B$ can run an efficient root-finding algorithm to obtain $d + 1$ roots of $p(X)$ (assuming that $d$ is small), check which of them is $x$, and then return it.

This commitment scheme is also additively homomorphic which means that $\text{Com}(ck, m_1; r_1) + \text{Com}(ck, m_2; r_2) = \text{Com}(ck, m_1 + m_2; r_1 + r_2)$.

**$DL$-extractable Commitment.** We also use a discrete logarithm (DL) extractable commitment from [ABL+19a]. DL-extractable commitment allows us to commit to a message $m \in \mathbb{Z}_p$, but only $[m]$ needs to be revealed for opening. However, it should be possible to extract $m$ from the commitment when given a secret key.

Abdolmaleki et al. [ABL+19a] gave a UC-functionality of Fig. 1 for this primitive. There are $N$ parties. Different execution sessions are differentiated by unique session IDs sid and different commitments in the same session are differentiated by unique commitment IDs cid. For committing, sending party will send a message $m$ and recipient information to the functionality, upon which the functionality informs simulator and receiver that a commitment was made. Later sender can send an open message to the functionality and functionality will send $[m]$ to the simulator and the receiver. Such a commitment must necessarily be extractable. Namely, the only way to simulate a malicious commitment is for Sim to extract $m$ from the commitment and send it to the ideal functionality.

Abdolmaleki et al. proposed an efficient construction in the registered public
key (RPK) model [BCNP04] where, roughly speaking, each party can have their own trusted authority that stores the secret key. This is a somewhat weaker model than the CRS model. Moreover, it is known that every UC-functionality can be achieved in the uniform random string model [CLOS02] where all parties have access to a common uniformly random string. This construction would not be as efficient but would still serve our concrete application as we only need to use the commitment a small number of times.
3. AN EFFICIENT SHUFFLE ARGUMENT

3.1. Motivation

As we discussed in the introduction, zero-knowledge shuffle arguments allow us to construct verifiable mix-networks. Mix-nets essentially form a hard-to-trace communication channel similar to well-known Tor network [TOR18] but with slightly different properties. Namely, mix-network has provable security if at least one mixer is honest but overall lower latency. This makes it more suitable for applications like ballot shuffling in i-voting and anonymous e-mail [Cha81] and less suitable for anonymous instant messaging. For a longer overview of different types of mix-networks and their applications, we refer the reader to [SP06].

Our goal with this work is to come up with a secure and efficient shuffle argument. We are thinking of i-voting in mind as our main application which can require hundreds of thousands or even millions of ciphertexts to be shuffled. As we mentioned previously, many efficient interactive shuffle arguments are known and they can be transformed into non-interactive arguments in the RO model. However, the RO model is flawed (at least) on a theoretical level and thus we are going to study shuffle arguments that are RO-free.

3.2. Overview

We construct a non-interactive zero-knowledge shuffle argument $S$ in the pairing-based setting. Let $pk = [1, sk]$ be an Elgamal public key\(^1\) and $n$ the number of input ciphertexts. We represent $n$ input and output ciphertexts respectively as matrices $[C]_1, [C']_1 \in \mathbb{G}_2^{n \times 2}$ where each row $[C]_1$ and $[C']_1$ is a ciphertext. Ciphertexts are shuffled correctly if there exists a permutation $\sigma \in S_n$ and randomizers $t \in \mathbb{Z}_p$ such that $[C]_1 = [C_{\sigma(i)}]_1 + \text{Enc}(pk, [0]; t_i)$ for $i = 1, \ldots, n$. If we represent the permutation $\sigma$ as a permutation matrix $A$, i.e., $A = (a_{i,j}) \in \{0, 1\}^{n \times n}$ such that $a_{i,j} = 1$ if and only if $\sigma(i) = j$, then the shuffle relation is as follows

\[
\mathcal{R}_{sh}^{(n)} = \left\{ ([C, C']_1, (A, t)) \in \mathbb{G}_1^{n \times 4} \times (\mathbb{Z}_p^{2n} \times \mathbb{Z}_p^n) \mid \begin{array}{c} C' = A \cdot C + t \cdot pk \land A \text{ is a permutation matrix} \end{array} \right\}. \tag{3.1}
\]

Essential to our construction is a $p(X)$-Multi-Pedersen commitment with certain well-chosen polynomials $p(X)$. The main idea is to commit to each column of the matrix $A$ and to randomness vector $t$, and then (i) prove knowledge of the permutation matrix $A$, (ii) prove that the committed permutation and $t$ were used to shuffle the ciphertexts. The first argument we call a permutation matrix argument and the second we call a consistency argument.

Unfortunately, we are unable to make both arguments work respect to the same $p(X)$-MP commitments. Thus we commit once more to each row of $A$ but this

\(^1\)We present version of the argument where cryptosystem uses $G_1$. Original paper [FLSZ17a] gave a construction where ciphertexts are in $G_2$ which is slightly less efficient.
Cgen\(_{\text{sm}}(\text{par}, [M]_2)\): \(K \leftarrow \mathbb{Z}_p^{2 \times 1}\), \([P]_2 \leftarrow [M]_2^T K \in \mathbb{Z}_p^{n \times 1}\); Return \((\text{crs} = ([K]_1, [P]_2), \text{td} = K)\);

\(P_{\text{sm}}(\text{par, crs}, x = [a, \hat{a}]_2, w = (m_1, \ldots, m_n, r, \hat{r}))\): Return \([d]_2 \leftarrow w[P]_2\);

\(V_{\text{sm}}(\text{par, crs}, x = [a, \hat{a}]_2, \pi = [d]_2)\):

\(1\) \([\pi]_2 \leftarrow \pi = [K]_2^T [a, \hat{a}]_2\);

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Same-message argument}
\end{figure}

time with a different \(q(X)\)-MP commitment scheme (polynomials \(p(X)\) and \(q(X)\) are defined later). Finally, we construct a same-message argument which shows that prover knows how to open both commitments to the same vector. Below we give a more detailed explanation of each sub-argument. We state the main theorems, but for proofs refer the reader to [FLSZ17b].

We note that the conference version [FLSZ17a] contained a subtle flaw. Original construction used the same randomness for \(p(X)\)-MP and \(q(X)\)-MP commitments. This turned out to be insecure and we fixed it later in the full version [FLSZ17b].

### 3.3. Same-Message Argument

Let \(ck = [\rho, p_1(x), \ldots, p_n(x)]_2\) and \(\hat{ck} = [\hat{\rho}, q_1(x), \ldots, q_n(x)]_2\) be respectively commitment keys for \(p(X)\)-MP commitment scheme and \(q(X)\)-MP commitment scheme (we will use specific polynomials in later arguments). We want to prove knowledge of an opening of two commitments \([a]_2\) and \([\hat{a}]_2\) which use the respective commitment schemes. More concretely,

\[
\mathcal{R}^{(n)}_{\text{sm}} = \left\{ ([a, \hat{a}]_2^T, (m_1, \ldots, m_n, r, \hat{r})) \in \mathbb{G}_2^2 \times \mathbb{Z}_p^{n+2} \mid a = r \cdot \rho + \sum_{i=1}^{n} m_i p_i(x) \land \hat{a} = \hat{r} \cdot \hat{\rho} + \sum_{i=1}^{n} m_i q_i(x) \right\}.
\]

Let us first observe that the above language can be restated as a linear language \([a, \hat{a}]_2^T = [M]_2 \cdot w^T\) where

\[
M = \begin{pmatrix}
p(x) & \rho \\
q(x) & \hat{\rho}
\end{pmatrix} \in \mathbb{Z}_p^{2 \times (n+2)}\text{ and } w = (m_1, \ldots, m_n, r, \hat{r})�
\]

Kiltz and Wee [KW15] proposed an efficient quasi-adaptive NIZK argument for linear subspace languages where in the most efficient setting argument size is only 1 group element. Moreover, zero-knowledge is perfect and soundness holds under a standard falsifiable assumption. In Fig. 2 we have (the most efficient version of) Kiltz-Wee linear subspace argument adapted to our concrete matrix \(M\). We

\(^2\)Quasi-adaptive means that CRS can depend on the language parameter, in our case \([M]_2\).
\[ C_{\text{gen}_w}(\text{par}) : \]
1. \( x, \rho \leftarrow \mathbb{Z}_p \)
2. \( u \leftarrow \{ p_i(x) \}_{i=0}^n \cup \{ \rho \}; \)
3. \( v \leftarrow \{ ((p_i(x) + p_0(x))^2 - 1)/\rho \}_{i=1}^n \)
4. Return \((\text{crs} = ([u]_1, [u, v]_2), \text{td} = (x, \rho)) \)

\[ P_{av}(\text{par}, \text{crs}, x = [a]_2, w = (I, r)) : \]
1. \( [b]_1 \leftarrow [p_1(x)]_1 + r[\rho]_1 \)
2. \( [c]_2 \leftarrow 2r[a]_2 + 2r[p_0]_2 - r^2[\rho]_2 + [(p_1(x) + p_0(x))^2 - 1]/\rho]_2 \)
3. Return \( \pi = ([b]_1, [c]_2) \)

\[ V_{av}(\text{par}, \text{crs}, x = [a]_2, \pi = ([b]_1, [c]_2)) : \]
1. \( \alpha \leftarrow s \mathbb{Z}_p \)
2. \( ([b]_1 + [\alpha]_1 + [p_0(x)]_1)([a]_2 - [\alpha]_2 + [p_0(x)]_2) \stackrel{?}{=} [\rho]_1 [c]_2 + [1 - \alpha^2]r \)

**Figure 3.** Unit vector argument

Additionally need the same message argument to be knowledge sound. We prove this in GGM.

**Theorem 2.** The same-message argument is perfectly complete, knowledge sound in GGM, and perfectly zero-knowledge.

### 3.4. Unit Vector Argument

With a unit vector argument, we want to prove that prover knows a unit vector opening to some \( p(X) \)-MP commitment \([a]_2\). More formally, we need a knowledge sound argument for the following relation

\[ \mathcal{R}_{av}^{(n)} = \left\{ ([a]_2, (m, r)) \in \mathbb{G}_2 \times \mathbb{Z}_p^{n+1} \mid a = r \cdot \rho + \sum_{i=1}^n m_i p_i(x) \land m \in \mathbb{Z}_n \right\}, \]

where \( \mathbb{Z}_n \) is the set of all unit vectors of length \( n \), i.e., the set of binary vectors \( \{0, 1\}^n \) where exactly one position is 1. Alternatively, we can represent the witness as \((I, r) \in [1..n] \times \mathbb{Z}_p \) where \( I \) indicates that \( m_I = 1 \). This allows us to express \( a = r \cdot \rho + p_I(x) \) in the previous relation.

Similarly to [FL16, FLZ16], we construct the unit vector argument using square span programs of Danezis et al. [DFGK14]. Vector \( m \) is a unit vector if \( m_I \in \{0, 1\} \) for \( i \in [1..n] \) and \( \sum_{i=1}^n m_i = 1 \). We can restate this as the following matrix equation,

\[ (V m + b - 1_{n+1}) \circ (V m + b - 1_{n+1}) = 1_{n+1}, \quad (3.2) \]

where \( V = \begin{pmatrix} 2I \end{pmatrix}_n, \) \( b = \begin{pmatrix} 0 \end{pmatrix}_1 \), and \( \circ \) denotes point-wise multiplication. First \( n \) rows assert that \((2m_i - 1)^2 = 1\) which is equivalent to \( m_i (m_i - 1) = 0 \) and holds
if and only if \( m_i \in \{0, 1\} \). Last row of the equation shows that \( (\sum_{i=1}^{n} m_i)^2 = 1 \)
which is equivalent to \( (\sum_{i=1}^{n} m_i - 1)(\sum_{i=1}^{n} m_i + 1) = 0 \). Assuming that \( n < p - 1 \) and \( m \in \{0, 1\}^n \), the only solution is \( \sum_{i=1}^{n} m_i = 1 \). Thus, Eq. (3.2) is satisfied when \( m \) is a unit vector.

We will use this equation to construct a zk-SNARK for the unit vector relation. Let us fix polynomials \( p_i(x) \) in a particular way to enable this. We define \( p_i(X) \) as an interpolation polynomial of the \( i \)-th column of \( V \), that is,

\[
p_i(X) := 2\ell_i(X) + \ell_{n+1}(X) \quad \text{for} \quad i \in [1..n],
\]

where \( \ell_i(X) \) is the \( i \)-th Lagrange polynomial. Additionally we define \( p_0(X) \) as an interpolation of \( b - 1_{n+1} \),

\[
p_0(X) := -\sum_{i=1}^{n} \ell_i(X) = 1 - \ell_{n+1}(X).
\]

Last equality holds since interpolation of the constant polynomial 1 is equal to \( \sum_{i=1}^{n+1} \ell_i(X) \).

Using our newly defined polynomials, we can state Eq. (3.2) in a polynomial form which is better suited for a zk-SNARK. Let us define \( s(X) := (\sum_{i=1}^{n} m_ip_i(X) + p_0(X))^2 - 1 \) and \( r(X) := \prod_{i=1}^{n+1} X - \omega_i \), where \( \omega_i \) are interpolation points of Lagrange polynomials (see Chapter 2). It is easy to verify that if \( s(\omega_i) = 0 \) for \( i \in [1..n+1] \) then Eq. (3.2) is satisfied, or alternatively, if \( t(X) \) divides \( s(X) \) then Eq. (3.2) is satisfied.

Now we are ready to give an intuition for the unit vector argument in Fig. 3. The statement of the language is \([a]_2 = [p_1(x)]_2 + r[p]_2\). Prover outputs as part of the proof the same commitment also in the first group \([b]_1 = [p_1(x)]_1 + r[p]_1\). In particular, this means that we need commitment key of \( p(X) \)-MP commitment as part of the CRS and key is given both in \( G_1 \) and \( G_2 \). We verify the equality of commitments with an equation \([b]_1[1]_2 = [1]_1[a]_2\). To verify that \( t(X) \mid s(X) \), we proceed as follows. We include in the CRS elements \([(p_1(x) + p_0(x))^2 - 1)/p]_2\) for \( i \in [1..n] \) where division by \( p \) guarantees linear independence of other CRS elements. Prover sends \( I \)-th of those elements in the proof (element \([c]_2\) in Fig. 3) together with some extra payload to cancel out the randomness of commitments. Verifier will check that \( ([b]_1 + [p_0(x)]_1)[a]_2 + [p_0(x)]_2 - [1]_1r = [p]_1[c]_2 \) where the left-hand side essentially corresponds to \( s(X) \) and the right-hand side must be (due to multiplication by \( p \)) in the span of \( \{(p_1(X) + p_0(X))^2 - 1\} \) which is
Cgen_{pm}(\text{par}): \text{Return } Cgen_{uv}(\text{par});

P_{pm}(\text{par}, crs, x = [a]_2, w = (A, r_i)):
1. For $i \in [1..n]$: $\pi_i \leftarrow P_{uv}(\text{par}, crs, [a_i]_2, (A^{(i)}, r_i));$
2. Return $\pi = (\pi_1, \ldots, \pi_n);

V_{pm}(\text{par}, crs, x = [a]_2, \pi = (\pi_1, \ldots, \pi_n)):
1. $[a_n]_2 = \left(\sum_{i=1}^{n} [p_i(x)]_2\right) - \left(\sum_{i=1}^{n-1} [a_i]_2\right);
2. For $i \in [1..n]$: $V_{uv}(\text{par}, crs, [a_i]_2, \pi_i) = 1;

Figure 4. Permutation matrix argument
divisible by $t(X)$. If the right-hand side is divisible by $t(X)$, then so must be the
left-hand side which corresponds to $s(X)$.

Element $\alpha$ on the verifier side is used to batch those two equations together.
Namely, if the verification equation in Fig. 3 is satisfied, then with an overwhelm-
ing probability also $[b]_1[1]_2 = [1]_1[a]_2$ and $([b]_1 + [p_0(x)]_1)([a]_2 + [p_0(x)]_2) - [1]_T = [\rho]_1[c]_2$ are satified. This allows us to reduce the number of expensive pairing op-
erations.

Unfortunately, the above intuition for our argument is not perfectly correct.
After carefully going through the proof in the generic group model, one finds that $[a]_2$ might contain an additive term $m_0[p_0(x)]_1$ where $m_0 \in \mathbb{Z}_p$ is not necessarily
0. We take a very simple approach here and just assume that the same-message
argument respect to the same $[a]_2$ already accepted. This will essentially guarantee
that $[a]_2$ is in the span of $\{[p_i(x)]_2\}_{i=1}^{n} \cup \{[\rho]_2\}$ and thus $m_0 = 0$. In Chapter 5 we
provide a more elegant solution to this problem which merges the same-message
argument and the unit vector argument.

Theorem 3. The unit vector argument is perfectly complete and perfectly witness-
indistinguishable. Assume that the same-message argument accepts, then the unit
vector argument is knowledge sound in the GGM.

3.4.1. Permutation Matrix Argument

From the unit vector argument, it is relatively easy to construct a permutation
matrix argument. More precisely, we want to show that prover knows how to
open $n$ commitments to a permutation matrix, that is, to a $n \times n$ matrix where each
row and each column is a unit vector.

Let us observe that matrix is a permutation matrix if and only if each of its
columns is a unit vector and the sum of the columns is a vector of 1s. Following
this idea, we will have a $\textbf{p}(X)$-MP commitment $[a_i]_2$ for each column and
prove that it is a unit vector. However, the last commitment $[a_n]_2$ we compute
homomorphically,

\[ [a_n]_2 \leftarrow \text{Com}(ck,1_n;0) - \left( \sum_{i=1}^{n-1} [a_i]_2 \right) = \left( \sum_{i=1}^{n} [p_i(x)]_2 \right) - \left( \sum_{i=1}^{n-1} [a_i]_2 \right). \]

This will guarantee that \( \sum_{i=1}^{n} [a_i]_2 = \text{Com}(ck,1_n;0) \), i.e., that columns sum to an all-one vector. For the sake of completeness, we also give the formal relation and the argument (see Fig. 4).

\[ R_{pm}^{(n)} = \left\{ \left( [a]_2, (A,r) \right) \in \mathbb{G}_2^n \times (\mathbb{S}_n \times \mathbb{Z}_p^n) \mid a = r \cdot \rho + A^\top p(x) \wedge \sum_{i=1}^{n} r_i = 0 \wedge A \text{ is a permutation matrix} \right\}. \]

**Theorem 4.** Permutation matrix argument is perfectly complete, perfectly witness-indistinguishable, and knowledge sound in the GGM (under the same assumption as unit vector argument).

### 3.5. Consistency Argument

The final sub-argument is the consistency argument which shows that \( n \) ciphertexts \( [C']_1 \) are in a linear span of ciphertexts \( [C]_1 \in \mathbb{G}_1^{n \times 2} \) and the public key \( pk \).

More formally we can phrase this as the following relation

\[ R_{con}^{(n)} = \left\{ \left( [C,C']_1, (E,t) \right) \in \mathbb{G}_1^{n \times 4} \times (\mathbb{Z}_p^{n \times n} \times \mathbb{Z}_p^n) \mid C' = E \cdot C + t \cdot pk \right\}. \]

Observe that if we restrict the matrix \( E \) to be a permutation matrix then we actually obtain the shuffle relation in Eq. (3.1). We only prove culpable soundness of the consistency argument and do it respect to the following guilt relation

\[ R_{con-glt}^{(n)} = \left\{ \left( [C,C']_1, ([\hat{a}]_2, E, \hat{r}, sk) \right) \in \mathbb{G}_1^{n \times 4} \times (\mathbb{G}_2^n \times \mathbb{Z}_p^{n \times n} \times \mathbb{Z}_p^n \times \mathbb{Z}_p^p) \mid \text{Dec}(sk,C') \neq E \cdot \text{Dec}(sk,C) \wedge [\hat{a}] = E^\top \cdot p(x) + \hat{r} \cdot \hat{\rho} \right\}. \]

In other words, we will prove soundness against adversaries that know a vector of commitments \( [\hat{a}]_2 \) and their openings \( (E,\hat{r}) \) together with the decryption key. The proof is based on a variant of KerMDH assumption (in Chapter 5 we prove a similar result under a more minimalistic assumption) where the adversary gets some additional information as part of auxiliary input.

Let \( D_{con} \) be the CRS distribution for the whole shuffle argument in Fig. 6 (which includes output of \( \text{Cgen}_{con}(\text{par}) \)) and let us divide the output to a matrix \( [D]_2 = [q(x) \hat{\rho}]_2 \) and the rest we will denote by \( \text{aux} \).
\[ \text{Cgen}_{\text{con}}(\text{par}): x, \hat{\rho} \leftarrow \mathbb{Z}_p^n; \text{crs} \leftarrow [q(x), \hat{\rho}]_2; \text{Return } \langle \text{crs}, \text{td} = (x, \rho) \rangle; \]

\[ \text{P}_{\text{con}}(\text{par}, \text{crs}, x = [C, C']_2, w = (E, t)); \]
1. \( \hat{\rho} \leftarrow \mathbb{Z}_2^n; [\hat{a}]_2 \leftarrow \mathbf{E}^T[q(x)]_2 + \hat{\rho}[\hat{\rho}]_2; \)
2. \( r_t \leftarrow \mathbb{Z}_2^n; [s]_2 \leftarrow t^T[q(x)]_2 + r_t[\hat{\rho}]_2; \)
3. \( [\hat{e}]_1 \leftarrow \hat{r}^T[C]_1 + r_t \cdot pk; \)
4. \( \text{Return } \pi \leftarrow ([\hat{e}]_1, [\hat{a}, s])_2; \)

\[ \text{V}_{\text{con}}(\text{par}, \text{crs}, x = [C, C']_2, \pi = ([\hat{e}]_1, [\hat{a}, s])_2); \]
1. \( [C']_1^T[q(x)]_2 - [C]_1^T[\hat{a}]_2 = \text{pk} \cdot [s]_2 - [\hat{e}]_1[\hat{\rho}]_2; \)

**Figure 5.** Consistency argument

**Definition 18 (KerMDH with aux).** \( \mathcal{D}_{\text{con}} \)-KerMDH assumption with auxiliary input holds relative to BPgen, if for any PPT adversary \( A \), \( \varepsilon \approx \chi_0 \), where

\[
\varepsilon := \Pr \left[ \text{BPgen}(1^k), ([D]_2, \text{aux}) \leftarrow \mathcal{D}_{\text{con}}, [x]_1 \leftarrow A(\text{bp}, [D]_2, \text{aux}) : \begin{array}{c} [x]_1^T[D]_2 = [0]_T \wedge [x]_T \neq [0]_T \\ \end{array} \right].
\]

We remind that so far we have fixed polynomials \( p(X) \) but have not fixed \( q(X) \). Fauzi et al. [FLSZ17b] prove that under a certain restriction on \( q(X) \), \( \mathcal{D}_{\text{con}} \)-KerMDH assumption with auxiliary input holds in GGM.\(^3\)

**Theorem 5.** Assume that \( \{q_i(X) \cdot p_j(X)\}_{i,j=1}^\eta \) is linearly independent. Then \( \mathcal{D}_{\text{con}} \)-KerMDH with auxiliary input holds in GGM.

Let us fix \( q(X) \) for the sake of concreteness:

\[ q_i(X) := X^{(i+1)(n+1)} \text{ for } i \in [1..n]. \]

In particular, \( \{q_i(X) \cdot p_j(X)\}_{i,j=1}^\eta \) is linearly independent.

Our version of the consistency argument, as presented in Fig. 5, is very similar to constructions by Groth and Lu [GL07] and Fauzi et al. [FL16, FLZ16]. To explain the intuition of the argument, let us make some simplifications to \( \mathcal{D}_{\text{con}} \)-glt relation and to the consistency argument in Fig. 5: (i) commitment to \( E \) is without randomization, i.e., \([\hat{a}]_2 = E^T[q(x)]_2 \), (ii) ciphertexts \([C']_1 \) are also not randomized, i.e., \([C']_1 = E \cdot [C]_1 \), and (iii) right hand side of the verification equation is \([0]_T \), that is,

\[ [C']_1^T[q(x)]_2 - [C]_1^T[\hat{a}]_2 = [0]_T. \]

An adversary in the culpable soundness definition outputs (among other elements) \( E \) and a commitment \([\hat{a}]_2 = E^T[q(x)]_2 \) such that \([C']_1 \neq E[C]_1 \). By substituting \([\hat{a}]_2 \) to the verification equation, we can rewrite it as \( ([C']_1^T - [C]_1^T E^T)q(x)]_2 = [0]_T. \) This, however, means that we have found a non-zero vector in the kernel.

\(^3\)Theorem in [FLSZ17b] is actually even more general than the one presented here.
of \([q(x)]_2^T\) and broken some form of KerMDH assumption. Real consistency argument follows the same idea but additionally takes randomization into account. That is why the argument in Fig. 5 also includes a commitment \([s]_2\) to message \(t\) and the ciphertext \([\hat{c}]_1\).

**Theorem 6.** Consistency argument is perfectly complete, perfectly zero-knowledge, and culpably sound respect to relation \(\mathcal{R}^{[n]}_{\text{con-get}}\) under \(\mathcal{D}_{\text{con}}\)-KerMDH assumption with auxiliary input.

### 3.6. Shuffle Argument

Finally, in Fig. 6 we combine the same-message argument, the permutation matrix argument, and the consistency argument to obtain the shuffle argument \(S\). We start by committing to columns of permutation matrix \(A\) with \(p(X)\)-MP commitment and \(q(X)\)-MP commitment and obtain \([a]_2\) and \([\hat{a}]_2\). Randomness is picked such that \(\sum_{i=1}^n [a_i]_2 = \text{Com}(ck_1, l_n, 0)\) as we discussed in Section 3.4.1. We prove with the permutation matrix argument that prover knows an opening \((A, r)\) of commitments \([a]_2\) such that \(A\) is a permutation matrix and with the same-message argument we show that \([\hat{a}]_2\) opens to the same \(A\) but possibly with a different randomness \(\hat{r}\). We give a consistency argument respect to the same \([\hat{a}]_2\). Intuitively, the shuffle argument is sound since otherwise, we can construct an adversary that breaks culpable soundness of the consistency argument. In the reduction we can sample \(sk\) and \(pk\) ourselves, then run the adversary against the soundness of argument \(S\) and extract permutation matrix \(A\) and randomness \(r\) from the permutation argument and same-message argument respectively. Now we can return proof for the consistency argument together with the guilt witness \(([\hat{a}]_2, A, \hat{r}, sk)\). If \(\text{Dec}(sk, C') \neq E : \text{Dec}(sk, C)\), then we have broken culpable soundness and otherwise \([C', C]_1\) is a valid shuffle.

**Theorem 7.** Shuffle argument \(S\) is perfectly complete and perfectly zero-knowledge. If commitments are binding and \(\mathcal{D}_{\text{con}}\)-KerMDH assumption with auxiliary input holds, then \(S\) is computationally sound in GGM.

### 3.7. Batching and Implementation

It is possible to significantly increase the verification speed by using techniques for equation batching [BGR98]. It allows us to substitute most of the costly pairing operations with a lot less costly scalar multiplications in \(G_1\) or \(G_2\). For example, we need to verify that \(n\) same-message arguments \([d_1, \ldots, d_n]_2\) satisfy the equation \([1]_1 [d_i]_2 = [K_1]_1 [a_i]_2 + [K_2]_1 [\hat{a}_i]_2\) for \(i \in [1..n]\) which would take \(3n\) pairing operations. Instead, the verifier can sample \(z_1, \ldots, z_n \leftarrow \mathbb{Z}_p\) and check that

\[
\sum_{i=1}^n z_i ([1]_1 [d_i]_2) = \sum_{i=1}^n z_i ([K_1]_1 [a_i]_2 + [K_2]_1 [\hat{a}_i]_2).
\]
64bit Linux Ubuntu 16.10) and even on such modest hardware achieved quite

improve exponentiation time by sampling

This by itself would give no efficiency gain, but we can use bilinearity properties to write it as

This equation only needs 3 pairings and 3n scalar multiplications. Depending on the pairing construction, scalar multiplication in \( \mathbb{G}_2 \) can be several times faster than a pairing operation (see [FLSZ17b] for concrete numbers). One can further improve exponentiation time by sampling \( z_i \) from a smaller set, say \( \{0,1\}^{40} \subset \mathbb{Z}_p \).

Furthermore, we implement the whole shuffle argument\(^4\) in C++ library Libsnark together with batching and several other optimizations. We tested our argument on a small laptop computer (Intel Core i5-CPU with 4 threads, 8 GB RAM, 64bit Linux Ubuntu 16.10) and even on such modest hardware achieved quite

\[^4\text{Implementation is available at https://bitbucket.org/JannoSiim/hat_shuffle_implementation/}\]

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\[\text{Figure 6. Shuffle argument}\]
Table 3. Running times for $n = 10\,000, 100\,000, 300\,000$ ciphertexts

<table>
<thead>
<tr>
<th></th>
<th>10,000</th>
<th>100,000</th>
<th>300,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>CRS generation</td>
<td>2.6s</td>
<td>20.5s</td>
<td>55.0s</td>
</tr>
<tr>
<td>Prover</td>
<td>9.1s</td>
<td>1m 24s</td>
<td>4m 2.4s</td>
</tr>
<tr>
<td>Prover (online)</td>
<td>0.5s</td>
<td>4.1s</td>
<td>13.5s</td>
</tr>
<tr>
<td>Verifier</td>
<td>8.3s</td>
<td>1m 22.0</td>
<td>4m 49.2s</td>
</tr>
<tr>
<td>Verifier (online)</td>
<td>5.0s</td>
<td>49.9s</td>
<td>2m 55.5s</td>
</tr>
</tbody>
</table>

good running times. In Table 3 we have running times for CRS generation, proof generation, and verification time for $n = 10\,000, 100\,000, 300\,000$ ciphertexts.

Some of the computation can be done offline before the ciphertexts are even known (e.g., permutation matrix argument and the same-message argument). For the i-voting scenario, this computation can be done weeks before the elections. Other computations are online computation, that is, ciphertexts need to be available (shuffling and the consistency argument). We see from the table that online computation, especially for the prover, is very fast for our argument.
4. UC-SECURE CRS GENERATION FOR NIZK

4.1. Motivation

Most of the current NIZK arguments are secure either in the RO model or in the CRS model (sometimes even both, e.g. [BG18]). By folklore reasoning, uniformly random CRS can be obtained from some physical source of randomness (stock market fluctuations, sunspot data, etc.) \(^1\). The same does not seem to be possible with a more structured distribution like \([x, x^2]\) over some elliptic curve group. This raises a major issue with CRS based NIZK arguments: How should CRS be obtained? One option is to include a trusted third party that generates the CRS and does not leak the trapdoor or any other side-information, but this is a very strong trust assumption that might not be warranted in practice.

We can reduce trust by distributing CRS generation among multiple independent parties. This is can be done with a secure multi-party computation protocol that outputs a CRS but the secret information (trapdoor) is shared among all parties. Such an approach was considered by Ben Sasson et al. [BCG + 14]. According to their analysis, if we want to involve many parties (many of them possibly malicious), then the current generic secure multi-party computation protocols are inefficient for CRS generation for SNARKs or our shuffle argument. Thus they proposed their own distributed CRS generation protocol which works for a large class of pairing-based NIZK arguments. It shows good efficiency and requires only 1 out of \(N\) parties to be honest. Unfortunately, it also assumes the RO model and that the CRS is symmetric, i.e., discrete logarithms of \(G_1\) elements are the same as discrete logarithms of \(G_2\) elements. Thus, it does not fit with the structure and the security model of our shuffle argument \(\delta\).

Motivated by this, we propose in [ABL + 19b] an improvement to Ben-Sasson et al.’s protocol and remove both of the restrictions.

**Removing the RO model**: We observe that [BCG + 14] requires the RO model since they use non-interactive sigma protocols to show that each party knows a share of the trapdoor. In our construction, we will substitute sigma protocols with a UC-secure DL-extractable commitment scheme [ABL + 19a]. We have each party commit to their trapdoor share \(t_s\) and once everyone has committed they will open the commitment to \([t_s]_1, [t_s]_2\). On the one hand, it guarantees that trapdoor shares are independently chosen and on the other hand it is possible to extract \(t_s\) from the commitment just as in [BCG + 14]. It turns out that by using a UC-secure DL-extractable commitment we get UC-security for the whole CRS generation protocol. This works well with our objective since it means that elements leaked in the CRS generation protocol cannot affect soundness or zero-knowledge properties.

\(^1\)Even with uniform CRS, the situation is not so simple. Natural distributions are typically non-uniform and thus cannot directly be used as a CRS.
Asymmetric CRS: It turns out to be possible to change the original protocol such that there is only partial symmetry between $G_1$ and $G_2$ elements of the CRS. Namely we require that for each element $\alpha \in td$, CRS contains $[\alpha]_1$ and $[\alpha]_2$. Besides that, CRS does not need to be symmetrical. This change also means that the proof strategy must be slightly different compared to the original protocol. Other restrictions on the CRS in [BCG+14] will stay mostly the same for our protocol. We will cover them in detail in Section 4.3.1.

Overview. We start by explaining the main idea of the protocol with an example. Suppose that parties $P_1, \ldots, P_N$ want to generate a CRS $[\alpha, \alpha^2]_1$ where $\alpha \leftarrow \mathbb{Z}_p^\ast$. We have each party $P_i$ generate a share $\alpha_i \leftarrow \mathbb{Z}_p$ and we implicitly define $\alpha := \prod_{i=1}^N \alpha_i$. We would like each party to reveal $[\alpha]_2$ but such that shares are guaranteed to be picked independently and thus $\alpha$ is uniformly random in $\mathbb{Z}_p^\ast$. This problem is easily solved by starting the protocol with a round where parties broadcast DL-extractable commitments of $[\alpha]_2$ and open it in the next round. Once this is done, parties can start computing the actual CRS. This will proceed in a round-robin fashion: $P_1$ broadcasts $[\alpha_1]_1$, $P_2$ broadcasts $\alpha_2[\alpha_1]_1$, and so on until $P_N$ broadcasts $\alpha_N[\alpha_1 \ldots \alpha_{N-1}]_1 = [\alpha]_1$. Importantly, each computation can be verified by each party. Say if $P_i$-1 outputted $[\alpha_{i-1}]_1 = [\alpha_{1} \ldots \alpha_{i-1}]_1$ and we want to verify output $[\alpha_i]_1$ of party $P_i$, then it can be done with a pairing equation $[\alpha_i]_1[\alpha_i]_2 = [\alpha_{i-1}]_1[\alpha_i]_2$. To get $[\alpha^2]_1$, we add one more round-robin phase: $P_1$ broadcasts $\alpha_1[\alpha]_1$, $P_2$ broadcasts $\alpha_2[\alpha \alpha_1]_1$, until $P_N$ broadcasts $\alpha_N[\alpha \alpha_1 \ldots \alpha_{N-1}]_1 = [\alpha^2]_1$. Verification can be done as before.

One caveat here is that if say parties $P_1, \ldots, P_{N-1}$ are corrupted, then they can compute also $[\alpha]_2$ just from $\alpha_1, \ldots, \alpha_{N-1}$ and $[\alpha_N]_2$. In some NIZK arguments, this might affect soundness or zero-knowledge properties. Therefore, we will explicitly require that for each trapdoor element $x$, CRS must contain both $[x]_1$ and $[x]_2$. Thus in our example, CRS should actually be $[\alpha, \alpha^2]_1$, $[\alpha]_2$. Full protocol in Section 4.3 follows the same ideas, but we additionally allow addition (and division) which makes the protocol more involved.

### 4.2. Ideal Functionality

First, we give a UC ideal functionality for pairing-based CRS generation protocols. We start by formalizing a pairing-based CRS. Let $C_1 : \mathbb{Z}_p^\ast \to \mathbb{Z}_p^{h_1}$ and $C_2 : \mathbb{Z}_p^\ast \to \mathbb{Z}_p^{h_2}$ be two arithmetic circuits such that $\text{crs} = ([C_1(td)]_1, [C_2(td)]_2)$ for $td \leftarrow \mathcal{D}$ where $\mathcal{D}$ is some distribution over $\mathbb{Z}_p^\ast$. Protocol will involve parties $P_1, \ldots, P_N$. Each party $P_i$ will have a share $ts_i$ of trapdoor $td$. Honest parties will have their $ts_i$ sampled from $\mathcal{D}$ by the ideal functionality whereas malicious parties can submit their own $ts_i$ as long as $ts_i \in \mathcal{D}$. Ideal functionality combines shares $ts_1, \ldots, ts_N$ to trapdoor $td$ using some combining function $c : (\mathbb{Z}_p^\ast)^N \to \mathbb{Z}_p^\ast$. Once
Let parties be $P_1, \ldots, P_N$ and let $C_1, C_2, \mathcal{D}$ define the CRS.

**Share collection from honest $P_i$:** On input $(\text{collect}, \text{sid}, P_i)$ from honest $P_i$
check that no message with collect and sid has been stored. If so, then sample $ts_i \leftarrow \mathcal{D}$, store $(\text{collect}, \text{sid}, P_i, ts_i)$, and send $(\text{collect}, \text{sid}, P_i)$ to Sim.

**Share collection from malicious $P_i$:** On input $(\text{collect}, \text{sid}, P_i, ts_i)$ from malicious $P_i$, check that $ts_i \in \mathcal{D}$ and that no message with collect and sid has been stored. If so, then store $(\text{collect}, \text{sid}, P_i, ts_i)$.

**CRS generation:** Once $ts_i$ is stored for each $i \in [1..N]$:
1. Compute $td \leftarrow c(ts_1, \ldots, ts_N)$.
2. Set $\text{crs} \leftarrow ([C_1(td)]_1, [C_2(td)]_2)$ and send $(\text{output}, \text{sid}, \text{crs})$ to Sim.
3. If Sim returns $(\text{output}, \text{sid}, \text{ok})$ then send $(\text{output}, \text{sid}, \text{crs})$ to all $P_i$ for $i \in [1..N]$.

**Figure 7.** Ideal functionality $F_{dcrs}$ for distributed CRS generation

The ideal functionalty has trapdoor shares from all parties, it will compute the CRS from $td$ and return it to all parties. Full details of the ideal functionalty $F_{dcrs}$ can be found in Fig. 7.

This ideal functionalty does what we expect if the adversary cannot distinguish between a CRS generated by an honest party and a CRS generated by $F_{dcrs}$. Firstly, we cannot guarantee it for any non-trivial CRS if all parties are corrupted. Thus, we must allow the adversary to corrupt at most $N-1$ parties. Secondly, if $td \leftarrow \mathcal{D}$ and $td \leftarrow c(ts_1, \ldots, ts_N)$ are indistinguishable then so is the CRS. Hence, we should fix the combining function $c$ such that the adversary would not be able to distinguish between $td \leftarrow \mathcal{D}$ and $td \leftarrow c(ts_1, \ldots, ts_N)$ when there is at least one honest party. There are two natural settings for this.

- If $\mathcal{D}$ is uniform distribution over $\mathbb{Z}_p^*$, then $c$ could be point-wise addition.
- If $\mathcal{D}$ is $(\mathbb{Z}_p^*)^t$, then $c$ could be point-wise multiplication.

Our protocol will use the latter option.

### 4.3. Protocol

#### 4.3.1. Class of Circuits

We describe the class of CRSs that our protocol can securely generate. Our presentation here significantly differs from [ABL+19b] and will be more similar to [BGM17] which is hopefully more digestible.

Firstly, we assume that trapdoors $td$ are distributed uniformly over $(\mathbb{Z}_p^*)^t$ and later our combining function $c$ will be pointwise multiplication. Thus, if we have shares $ts_1, \ldots, ts_N \in (\mathbb{Z}_p^*)^t$ and at least one of them is sampled uniformly and independently, then $c(ts_1, \ldots, ts_N)$ is also distributed uniformly over $(\mathbb{Z}_p^*)^t$.
Let parties be $P_1, \ldots, P_N$.

- On input $(\text{broadcast, sid, } P_i, x)$ from party $P_i$, send $(\text{broadcast, sid, } P_i, x)$ to $P_i$ for $i \in [1..N] \setminus \{i\}$ and Sim.

Figure 8. Ideal functionality $F_{bc}$ for authenticated broadcast

Secondly, we need to describe circuits $C_k : \mathbb{Z}_p^t \rightarrow \mathbb{Z}_p^h$ for $k \in \{1, 2\}$. The circuit will contain (i) multiplication-division gates that have three input wires with values $x, y, z$ and an output wire with value $(x \cdot y)/z$, and (ii) addition gates with some constant number $e$ of input wires with values $x_1, \ldots, x_e$ and an output wire with value $\sum_{i=1}^{e} a_i x_i$ where $a_i$ for $i \in [1..e]$ are some constants. All gates in $C_k$ can be divided into disjoint layers $M_1, A_2, \ldots, M_{d-1}, A_d$ where $M_i$ contains only multiplication-division gates and $A_i$ contains only addition gates. Moreover, input wires of a gate in $i$-th layer are from a gate at the same or from a lower layer or are directly circuit inputs. For efficiency reasons, it is always best to partition gates such that the number of layers is minimum. Additionally, we will have the following restrictions for gates:

1. The output of each gate will also be the output of the whole circuit. This will allow us to publicly verify the computation of each gate.
2. For each input $td_j$ of the circuit we have one trivial multiplication-division gate (as part of $M_1$) which outputs $(1 \cdot td_j)/1 = td_j$. This means that CRS will contain $(\lfloor td_j \rfloor_1, \lfloor td_j \rfloor_2)$ which will be essential to simulate the protocol in the UC-security proof.
3. Multiplication-division gates must always have left input wire (wire with $x$) to be the output of another gate (or constant) and middle and right input wires (respectively wires with $y$ and $z$) as one of the inputs $td_i$ of the circuit (or constant). In the protocol, this will allow us to verify the computation of the gate with simple pairing-based equations.

4.3.2. Construction

We assume to have access to an authenticated broadcast [CLOS02] defined in Fig. 8 and DL-extractable commitment in Fig. 1. Our setting is very similar to multi-party protocol SPDZ [DPSZ12]. Namely, we allow $N - 1$ out $N$ parties to be malicious which means that successful termination cannot be guaranteed. If cheating is detected by an honest party, he will simply abort the protocol. We assume in the following that $C_1$ and $C_2$ follow description from Section 4.3.1.

Share collection phase. Honest party $P_i$ receives an input $(\text{collect, sid, } P_i)$ from the environment which indicates that the share collection phase has started. Party $P_i$ samples trapdoor shares $ts_i = (ts_{i1}, \ldots, ts_{ih})$ uniformly from $(\mathbb{Z}_p^t)^i$. It proceeds by sending a DL-extractable commitment of $ts_i$ to each party and then waits until he has received commitments from all other parties. Once this has hap-
pened, \( P_i \) sends opening (which includes \([ts_j]_1\)) to each party and additionally calls (broadcast, \text{sid}, P_i, [ts_j]_1, [ts_j]_2). This will guarantee that (i) some malicious party \( P_j \) cannot send a commitment of \( ts_j \) to one party and a commitment of different message \( ts_j^* \) to another party, (ii) \([ts_j]_2\) will also be publicly known by all parties. That means \( P_i \) will check that (i) \([ts_j]_1\) received as part of an opening and \([ts_j^*]_1\) received from the broadcast are equal, (ii) shares in both groups are equal, i.e., \([ts_j]_1[1]_2 = [1]_1[ts_j]_2\). He will abort if either of the checks fails. Party will wait until all messages are received from other parties and only then will proceed to the CRS generation phase.

**CRS generation phase.** At this point, each party knows \([ts_j]_1, [ts_j]_2\) for \( j \in [1..N] \) and the rest of the computation will be a deterministic evaluation of circuits \( C_1 \) and \( C_2 \). Parties will evaluate circuits layer-by-layer and \( C_1 \) and \( C_2 \) can be processed in parallel. Multiplication-division layers are evaluated in a round-robin fashion, that is, parties will include their shares one by one to the total result. Therefore, evaluation of one layer \( M_i \) takes exactly \( N \) rounds to complete. Addition layers can be computed locally as we will see.

Let us describe the evaluation of one multiplication-division layer \( M \) of circuit \( C_1 \) (\( C_2 \) layer would be similar).

Evaluation starts with party \( P_1 \) and he will evaluate gates of \( M \) in topological order, i.e., if gate \( G_r \) depends on gate \( G_s \) then \( G_s \) is evaluated before \( G_r \). Suppose that \( P_1 \) is evaluating multiplication-division gate \( G \).

For the left wire there are three options:

- If the left wire of \( G \) is some constant \( c \), then we set \( [x]_1 \leftarrow [c]_1 \).
- If the left wire of \( G \) is from a lower layer, then this gate has been already evaluated and we can set \( [x]_1 \) to this value. Actually, if we minimize the number of layers, then the left wire can only come from the addition gate (or is a constant). Otherwise, \( G \) itself should be from a lower layer.
- If the left wire is from some multiplication-division gate \( G' \) of the same layer \( M \). Then due to evaluation in topological order, \( P_1 \) has already computed output for \( G' \) and we set \( [x]_1 \) to be this value.

The middle wire value and the right wire value are treated as follows.

- If the middle wire (respectively right wire) is a constant \( c \), then we set \( y \leftarrow c \) (respectively \( z \leftarrow c \)).
- If the middle wire (respectively the right wire) corresponds to trapdoor input \( \text{td}_j \), then we set \( y \leftarrow ts_{1,j} \) (respectively \( z \leftarrow ts_{1,j} \)).

Now \( P_1 \) will broadcast \((y/z)\cdot[x]_1 = [(xy)/z]_1\) as his local evaluation of gate \( G \). Importantly all parties will know \([x]_1, [y]_2, \text{and } [z]_2\) which means that they can verify this output with the equation \([xy/z]_1[2] = [x]_1[y]_2\). Party will abort if verification does not pass.

Subsequent parties will proceed slightly differently. Suppose that \( P_{i-1} \) has finished his computation and it is now \( P_i \)'s turn. Evaluation of gates is still done
in the same order, but with a small modification. Namely, $P_i$ should include his shares to the computation of the previous party.

We do it with the following algorithm. To evaluate gate $G$ we first set $[x]_1$ to be $P_{i-1}$'s output for gate $G$ and we set a temporary value $G' \leftarrow G$.

While $G' \in M$ we perform the following:

1. If the middle wire of $G'$ has a constant $c$, then we set $y \leftarrow 1$ ($P_1$ already included constant $c$) and if it corresponds to $td_{ij}$, then we set $y \leftarrow ts_{i,j}$.
2. If the right wire of $G'$ has a constant $c$, then we set $z \leftarrow 1$ ($P_1$ already included constant $c$) and if it corresponds to $td_{ij}$, then we set $z \leftarrow ts_{i,j}$.
3. $P_i$ will broadcast element $[x']_1 \leftarrow (y/z)[x]_1$ and each party will verify it with the equation $[x']_1[z]_2 = [x]_1[y]_2$.
4. We set $[x]_1 \leftarrow [x']_1$ and new $G'$ to be the left gate of the current $G'$ or $\bot$ if there is no left gate.

We consider last broadcasted value to be $P_i$'s evaluation of gate $G$. All the intermediate values are used just to check that this value is computed correctly. When the final party $P_N$ has finished evaluating $G$, then $P_N$'s last output is the actual element of the CRS. In particular, it will include shares of all the parties in the correct way.

Evaluating gates on some addition layer $A$ is much easier and can be done locally by each party. Evaluation proceeds again in topological order. For each input wire of an addition gate, we already know the final value $[x_j]_1$:

1. If a wire comes from another addition gate, then we just use the locally computed evaluation of that.
2. If it comes from a multiplication-division gate then we will use $P_N$’s evaluation of that gate.
3. If the wire is from the input of the circuit for some trapdoor $td_i$, then we use $P_N$’s evaluation of the trivial gate $[1 \cdot td_i/1]_1$.

Therefore each party can locally compute $\sum_{j=1}^{c} a_j [x_j]_1$.

In [ABL+19b], we prove the following theorem.

**Theorem 8.** The protocol above UC-realizes $F_{dcrs}$ functionality in $(F_{dl-com}, F_{bc})$-hybrid model.

We remind that instantiation of $F_{dl-com}$ itself might require a trust assumption and this depends on the concrete construction of the DL-extractable commitment. We refer the reader to Section 2.6 for some possible instantiations, e.g., in the RPK model or in the uniform random string model.
5. TRANSPARENT SHUFFLE ARGUMENT

5.1. Overview

This section follows the results in [AFK+20]. Our goal is to reduce trust in the shuffle argument $\delta$ from Chapter 3 as much as possible. We achieve it in two ways. Firstly, we apply the distributed CRS generation protocol from the previous chapter. This guarantees that soundness and zero-knowledge will hold if at least one out of $N-1$ parties in CRS generation protocol are honest. Unfortunately, CRS of $\delta$ does not fit into restrictions laid out in section Section 4.3.1. This requires us to make several non-trivial modifications to $\delta$.

Secondly, we make our shuffle subversion zero-knowledge by following ideas of [BFS16, ABLZ17, Fuc18]. In particular, we define a CRS verification algorithm CV such that if CRS passes this verification, then zero-knowledge will hold even if CRS came from an untrusted party or equivalently if all $N$ parties in the CRS generation protocol were dishonest.

We also make some other modifications along the way. We will merge the same-message argument with the unit vector argument since their security does not hold in isolation as we discussed in Chapter 3. We also skip the consistency argument since it has only culpable soundness and we directly construct shuffle argument from the permutation argument. We also prove in the paper that the new unit vector argument (merge of unit vector argument and same-message argument) has knowledge soundness in the algebraic group model (AGM) under the SPDL assumption. This is an improvement over GGM proof in [FLSZ17b]. Shuffle argument’s soundness holds if permutation argument is knowledge sound and a novel kernel-type assumption holds that we call GapKerMDH assumption. Although this assumption is new, we believe it to be more natural than KerMDH assumption with auxiliary input in Definition 18. We prove subversion zero-knowledge under the knowledge assumption BDH-KE, similar to Abdolmaleki et al. [ABLZ17].

5.2. Modifications in CRS

CRS of $\delta$ needs several modifications to satisfy structural requirements for our CRS generation protocol. We list the modifications below.

1. First, we observe that setting $\hat{\rho} = 1$ is secure, i.e., we can actually remove this trapdoor element.

2. We add $[x]_1$, $[x]_2$, and $[K]_2 = [K_1, K_2]_2^\top$ to the CRS so that each trapdoor is a discrete logarithm of a $G_1$ element and a $G_2$ element.

3. Since we need to compute $[p_i(x)]_1$, $[p_i(x)]_2$, and $[(p_i(x) + p_0(x))^2 - 1]/\rho$ for $i \in [1..n]$, we include to the CRS $\{[x^i_1]\}_{i=1}^n$ and $\{[x^i_2]\}_{i=1}^{2n}$. This will allow us to compute $[p_i(x)]_1$, $[p_i(x)]_2$, and $[(p_i(x) + p_0(x))^2 - 1]$ with addi-
tion gates and finally \([((p_i(x) + p_0(x))^2 - 1)/\rho]\) with multiplication-division gates.

4. Polynomials \(q_i(X)\) in the original argument have the degree \((i + 1)(n + 1)\). This would imply that we need to include \(\{[x_i^{[n]}]\}_{i=1}^{(n+1)^2}\) to the CRS. We will avoid this quadratic blowup by introducing a new random point \(\theta\) and by redefining the polynomials \(q_i(X)\). Namely, we will have \(q_i^+(X_\theta) := X_\theta^{2i}\) for \(i \in [1..n]\) and thus will use \(q_i^+(X_\theta)\)-MP commitment scheme.\(^1\) Additionally, we will need to include \([\theta]_1\) and \(\{[\theta]^i\}_{i=1}^{2n}\) to the CRS.

5. Finally, in order to compute \([K_1 p_i(x) + K_2 \tilde{q}_i(x)]\), we would need to reveal \([K_1 p_i(x)]\) and \([K_2 \tilde{q}_i(x)]\). This, however, makes the unit vector argument insecure. Instead, we will modify the unit vector argument such that it uses \(K_1[K_1 p_i(x) + K_2 \tilde{q}_i(x)] = [K_1^2 p_i(x) + K_1 K_2 \tilde{q}_i(x)], [K_1 \rho],\) and \([K_1 K_2]\). In such case, we can reveal \([K_1 p_i(x)]\), \([K_2 \tilde{q}_i(x)]\) for \(i \in [1..n]\) and also \([K_1 \rho]\) without affecting the security of the unit vector argument.

It is now straightforward to verify that the CRS satisfies the necessary circuit structure. We refer the reader to [AFK\textsuperscript{+20}, Fig. 7] for an explicit description of the circuit.

### 5.3. Unit Vector Argument

As mentioned before, the unit vector argument in \(S^+\) is a merge of the unit vector argument and the same-message argument in \(S\) and additionally instead of \([K]\) we will use \([K^+]\) for \([K^+_1 K_2]\), a detailed description is given in Fig. 10.

New argument is for the relation

\[
\mathcal{R}_{uv}^{(n)} = \{ \hat{a} | ([a]_2, (i, \hat{r})) \in G_2 \times ([1..n] \times \mathbb{Z}_p) | \hat{a} = q_i^+(x) + \hat{r} \},
\]

and in [AFK\textsuperscript{+20}] we prove the following theorem.

**Theorem 9.** The new unit vector argument is perfectly complete and perfectly zero-knowledge. If \((3n - 1)\)-SPDL assumption holds, then it has computational knowledge soundness.

### 5.4. Permutation Argument

Construction of the permutation argument (see Fig. 10) is essentially identical to the construction of the permutation matrix argument in \(S\). However, for a slightly different relation

\[
\mathcal{R}_{pm}^{(n)} = \left\{ \left( [\hat{a}]_2, (A, \hat{r}) \right) \in G_2^n \times (\mathbb{Z}_p^{n \times n} \times \mathbb{Z}_p^n) | \hat{a} = A^\top q_i^+(x) + \hat{r} \land \sum_{i=1}^n \hat{r}_i = 0 \land A \text{ is a permutation matrix} \right\}.
\]

\(^1\)The more obvious definition of \(q_i^+(X_\theta) = X_\theta^i\) turns out to be insecure as we explain in [AFK\textsuperscript{+20}].
Cgen\textsuperscript{pm}(par):
1. \(x, \rho, \theta \leftarrow \mathbb{Z}_p^n; K = (K_1, K_2)^\top \leftarrow \mathbb{Z}_p^2;\)
2. \(u \leftarrow \{p_i(x)\}_{i=0}^n \cup \{\rho\}; v \leftarrow \{(p_i(x) + p_0(x))^2 - 1\}/\rho\}_{i=1}^n;\)
   \[w \leftarrow \{q_i^+(\theta)\}_{i=1}^n;\]
3. \(U = \left(\begin{array}{cc} p(x) & \rho \\ q^+(\theta) & 0 \end{array}\right); K^+ = (K_1^2, K_2)^\top; P \leftarrow M^\top \cdot K^+ \in \mathbb{Z}_p^{n \times 1};\)
4. Return \((\text{crs} = (\{u, K^+\}_1, \{u, v, w, P\}_2), \text{td} = (x, \rho, \theta, K));\)

\[\begin{align*}
\text{P}_{\text{av}}(\text{par}, \text{crs}, x = [\hat{a}]_2, w = (I, \hat{r})): \\
&1. \ r \leftarrow \mathbb{Z}_p^n; \ [b]_1 \leftarrow [p_1(x)]_1 + r[\rho]_1; \ [\alpha]_2 \leftarrow [q_i^+(\theta)]_2 + r[\rho]_2; \\
&2. \ [c]_2 \leftarrow 2r[a]_2 + 2r[0]_2 - r^2[\rho]_2 + ((p_1(x) + p_0(x))^2 - 1)/[\rho]_2; \\
&3. \ \pi_{\text{sm}} \leftarrow [\hat{P}]_2 \cdot (\hat{e}_t, r, \hat{r})^\top; \ // \ e_t \ is \ the \ I-\text{th \ unit \ vector} \\
&4. \ Return \ \pi = ([b]_1, [\alpha, c]_2, \pi_{\text{sm}}); \\
\end{align*}\]

\[\begin{align*}
\text{V}_{\text{av}}(\text{par}, \text{crs}, x = [\hat{a}]_2, \pi = ([b]_1, [\alpha, c]_2, \pi_{\text{sm}})): \\
&1. \ \alpha \leftarrow \mathbb{Z}_p^n; \\
&2. \ ([b_1 + [\alpha]_1 + [p_0(x)]_1)([a]_2 - [\alpha]_2 + [p_0(x)]_2) = [\rho]_2 + [1 - \alpha^2]_T; \\
&3. \ [K^+]_1 = [1]_1 \cdot \pi_{\text{sm}}; \\
\end{align*}\]

Figure 9. New unit vector argument

\[\begin{align*}
\text{Cgen}_{\text{pm}}(\text{par}): \text{Return Cgen}_{\text{av}}^+(\text{par});
\end{align*}\]

\[\begin{align*}
\text{P}_{\text{pm}}(\text{par}, \text{crs}, x = [\hat{a}]_2, w = (A, \hat{r})): \\
&1. \ For \ i \in [1..n]: \ \pi_i \leftarrow \text{P}_{\text{av}}(\text{par}, \text{crs}, [\hat{a}]_2, (A_i^\top, r_i)); \\
&2. \ Return \ \pi = (\pi_1, \ldots, \pi_n); \\
\end{align*}\]

\[\begin{align*}
\text{V}_{\text{pm}}(\text{par}, \text{crs}, x = [\hat{a}]_2, \pi = (\pi_1, \ldots, \pi_n)): \\
&1. \ [\hat{a}_n]_2 = \left(\sum_{i=1}^n [q_i^+(x)]_2 \right) - \left(\sum_{i=1}^{n-1} [\hat{a}_i]_2 \right); \\
&2. \ For \ i \in [1..n]: \ \text{V}_{\text{av}}(\text{par}, \text{crs}, [\hat{a}_i]_2, \pi_i) \leftarrow 1; \\
\end{align*}\]

Figure 10. New permutation argument

We will also prove a slightly stronger result compared to permutation matrix argument of \(\delta\).

**Theorem 10.** Permutation argument is perfectly complete and perfectly zero-knowledge. If the unit vector argument is knowledge sound and \(q^+\).MP commitment is binding, then the permutation argument is knowledge sound.
Construction of the shuffle argument

The soundness of $\delta^+$ is based on a novel variation of KerMDH assumption that we call GapKerMDH.

**Definition 19** (GapKerMDH Assumption). $n$-GapKerMDH assumption holds relative to BPgen, if for any PPT adversary $A$,

$$\Pr\left[ \begin{array}{l}
bp \leftarrow \text{BPgen}(1^k), \theta \leftarrow \mathbb{Z}_p, [x]_1 \leftarrow A(bp, [\theta]_1, \{(\theta^i)_{j=1}^{2n}\}) : \\
(1, \theta^2, \ldots, \theta^{2i}, \ldots, \theta^{2n}) \cdot x = 0 \land x \neq \theta \end{array}\right] \approx_\lambda 0.
$$

Notice that $(1, \theta^2, \ldots, \theta^{2i}, \ldots, \theta^{2n})$ has only even powers of $\theta$. These gaps
CV(par, crs):

1. Assert that crs can be parsed according to Fig. 11 and elements belong to correct groups.
2. \[ \rho_2 \in \mathbb{G}_2 \setminus [0]_2; [1]_r = [1]_1[1]_2; \]
3. For \( z \in \{x, \theta, \rho, K_1, K_2\} \): \[ [z]_1[1]_2 = [1][z]_2; \]
4. For \( i \in [2..2n] \): \[ [1]_1[\theta^i]_2 = [\theta][\theta^{i-1}]_2; \]
5. \[ [K_1^2]_1[1]_2 = [K_1][K_2]_2; [1]_1[\rho_1^2]_2 = [K_1^2]_1[\rho]_2; \]
6. \[ [1]_1[K_1K_2]_2 = [K_2][K_1]_2; [K_1K_2][1]_1 = [1][K_1K_2]_2; \]
7. \[ [p_0(x)]_1[1]_2 = [1][p_0(x)]_2; \]
8. For \( i \in [1..n] \):
   a. \[ [p_i(x)]_1[1]_2 = [1][p_i(x)]_2; \]
   b. \[ [1]_1[K_1^2 p_i(x)]_2 = [K_1^2]_1[p_i(x)]_2 + [K_1K_2][q_i^2(x)]_2; \]
   c. \[ [\rho]_1([p_i(x) + p_0(x)] - 1)/\rho]_2 = [p_i(x) + p_0(x)][p_i(x) + p_0(x)]_2 - [1]_r; \]
9. Return 0 if any of the equations is not satisfied and 1 otherwise.

**Figure 12.** CRS verification algorithm for \( S^+ \)

between powers are significant. If the vector would contain consecutive powers \( \ldots , \theta^{2i}, \theta^{2i+1}, \ldots \) at any point, then \( A \) can output \([x]_1 = [0, \ldots , 0, \theta, -1, 0, \ldots , 0]_1\) and break the assumption. We prove the following result in [AFK+20].

**Theorem 11.** If \((2n)\)-SPDL assumption holds, then the \( n \)-GapKerMDH assumption holds in AGM.

Now we can also state the security claim for the shuffle argument.

**Theorem 12.** \( S^+ \) is perfectly complete and perfectly zero-knowledge. If the permutation argument is knowledge sound and \( n \)-GapKerMDH assumption holds, then \( S^+ \) has soundness.

### 5.6. Subversion Zero-Knowledge

We modified CRS such that we can use the CRS generation protocol from Chapter 4. This guarantees soundness and zero-knowledge as long as at least 1 party in the protocol is honest. We now show how to achieve zero-knowledge even if all parties are malicious.

We construct a CRS verification algorithm CV that, roughly speaking, checks that all CRS elements used by the prover are well-formed. We do this with pairing-based verification equations described in Fig. 12. For convenience, we write elements such that they are already well-formed. In reality, we, of course, do not know it before executing the CV algorithm.
Let us consider line 4 in Fig. 12 as an example and we are going to denote the element that we verifying by $X$. Thus we have an equation $[1]_1[X]_2 \overset{?}{=} [\theta]_1[\theta^{i-1}]_2$ which is supposed to establish that $X = \theta^i$. We assume here that $[\theta]_1$ and $[\theta^{i-1}]_1$ are already known to be well-formed from previous equations. In such a case, the left-hand side of the equation is $[1]_1[X]_2 = [X]_T$ and the right-hand side of the equation is $[\theta]_1[\theta^{i-1}]_2 = [\theta^i]_T$. Thus clearly $X = \theta^i$. All the other elements can be shown to be well-formed using similar reasoning.

Finally, in [AFK\textsuperscript{+}20] we prove that such a CV algorithm guarantees subversion zero-knowledge under BDH-KE assumption. Knowledge assumption is needed since we need to extract the trapdoor from the subverter and provide it to the simulator.

**Theorem 13.** *If BDH-KE assumption holds, then $\mathcal{S}^+$ is Sub-ZK.*
6. CONCLUSIONS AND FUTURE WORK

In this thesis, we first proposed a NIZK shuffle argument in the CRS model which is secure without the RO model but is close in efficiency even to the best RO-based shuffle arguments. Then we constructed an efficient distributed CRS generation protocol for a special class of pairing-based NIZKs which also does not require the RO model. Finally, we proposed a slightly modified shuffle argument that has much lower trust requirements. In particular, we can apply the CRS generation protocol and this will guarantee security if at least one party in the protocol was honest. Zero-knowledge holds even without any trust assumptions.

We believe that the results of this thesis make RO-free shuffle arguments a viable option in real-life systems. The first argument is already part of the Zeus i-voting system which has been used for elections in many organizations [BCD+ 19]. Our second argument, together with the CRS generation protocol and CRS verification algorithm, is planned to be integrated with Zeus in the future.

We mention one interesting direction for future work. Chase et al. [CKLM13b] proposed a controllably malleable shuffle argument that uses Groth and Lu’s shuffle argument [GL07] as a building block. Essentially the idea is to let each mixer mall the proof of the previous mixer. Verifier only needs to check the final proof and proof size is almost independent of the number mixers. Since our shuffle argument is partly based on ideas of Groth and Lu, it seems plausible that similar malleability techniques apply here as well. However, our argument is several times more efficient in almost every parameter and could make such mix-nets much more efficient. Interestingly, malleability seems much more challenging in RO-based shuffles due to the non-malleability of hashing. Controlled malleability could be another big advantage for pairing-based shuffles, besides avoiding the RO model.

Still many open problems remain in this area. As a short term goal, it would be interesting to obtain as efficient non-interactive shuffle arguments under more standard assumptions. Faonio et al. [FFHR19] made recently good progress in this direction by achieving linear complexity (in all parameters) under SXDH assumption. However, there is still a noticeable gap in concrete efficiency between their work and the fastest shuffle arguments. In the long term, it is also important to obtain efficient post-quantum secure non-interactive shuffle arguments. Only very recently have we seen the first attempts in this area, e.g. [CMM17,Str19,CMM19].
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53


[CKLM13a] Melissa Chase, Markulf Kohlweiss, Anna Lysyanskaya, and Sarah Meiklejohn. Succinct malleable NIZKs and an application to com-


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Mitte-interaktiivsed segamise nullteadmustõestused


Võtame näiteks hääle segamise protseduuri läbi hääletuskasti raputamise, mis peab kaotama seose hääletaja ja tema hääle vahel. Seda on väga lihtne teostada paberhääletega, kuid pole kaugeltki ilmne, kuidas saavutada sama tulemus digitaalse kripteeritud hääleta. Kes peaks teostama segamise? Kuidas me saame kindlad olla, et segamine tehti korrektselt? Kas hääle omanikku on tõesti võimatu tuvastada?


Teiseks uurime, kuidas sarnastes nullteadmustõestustes usaldust vähendada. On olemas turvalisi ühisarvutussüsteeme, mis suudavad arvutada viitesõnast hääletele nullteadmustõestust. Kahjuks kasutavad 100 000 krüptogrammi segamine ja tõestamine aega alla 1,5 minuti ning ka tõestuse verifiteerimiseks kulub umbes sama palju aega. Kahjuks kehtivad turvagarantiid vaid juhul, kui tõestajal ja verifiteerijal on ligipääs usaldatuvi되는.

Kolmandaks modifitseerime eelnevat segamise nullteadmustõestust viisil, et
see toimiks koos ühisarvutuse protokolliga. See kindlustab, et nullteadmistöes-
tus on turvaline seni, kuni vähemasti üks osapool ühisarvutuses oli aus. Lisaks
astume samu vörre kaugemale ja näitame, kuidas säilitada praeva tus ka siis,
kui kõik osapoole ühisarvutuses olid pahatalikud. Samuti lihtsustame esialg-
set konstruktsiooni ja tõestame protokollit turvalisust nõrgematel krüptograafilistel
eeldustel.
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Teadustegevus

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152