TARTU UNIVERSITY
Faculty of Mathematics and Computer Science
Institute of Pure Mathematics
Chair of Algebra

NIKITA SALNIKOV-TARNOVSKI

# ON FLATNESS PROPERTIES OF S-POSETS

Master thesis

Supervisor: prof. Mati Kilp

Author: ...................................  " ..... "  May 2005

Supervisor: ................................  " ..... "  May 2005

Tartu 2005

# Contents

# 1 Introduction

Over the past several decades there was a lot of papers investigating what are usually called *flatness properties of acts over monoids.* These investigations are usually devoted to preservation properties of the functor $A_S \otimes -$ (from the category of left $S$-acts to the category of sets), for a right act $A_S$ over a monoid $S$. For a complete source for these results the reader is referred to monograph [4] by Kilp, Knauer and Mikhalev.

In the 1980s, Fakhruddin published some results (e.g. [3]) devoted to tensor products and flatness properties in the context of ordered monoids acting (monotonically in both arguments) on ordered sets (that is, $S$-posets). But only recently new papers [6], [5], [2], [1] have appeared that continue investigations started by Fakhruddin.

In the present thesis the author is trying to transfer some classical results presented in [4] onto new ground of ordered monoids and $S$-posets. The main aim are results giving the necessary and sufficient conditions for a given $S$-poset to have a given flatness property. While many proofs and results can be carried over more or less *verbatim*, in some cases there are considerable differences betwen classical, unordered, case and ordered one. The results concerning monocyclic congruences are good examples of such differences.

The second section of the thesis gives definitions and results describing our basic tools in the field of ordered $S$-posets which will be used in the rest of the paper. In the third section results concerning torsion freeness and po-torsion freeness are presented. The forth section is devoted to principally weak flatness and principally weak po-flatness. The case of PP pomonoids is described in more detail. In the last section some results about weak flatness and weak po-flatness are given.

## 2 Preliminaries and definitions

A ***partially ordered monoid*** (or ***pomonoid***) is a monoid $S$ together with the partial order $\leq$ on $S$ which is compatible with multiplication in $S$. This means that if $s, s', u \in S$ and $s \leq s'$ then $su \leq s'u$ and $us \leq us'$. A ***right*** $S$***-poset***, denoted as $A_S$, is a poset $A$ with the partial order $\leq$ and a right action $A \times S \to A$, $(a, s) \mapsto as$, that satisfies the following conditions:

(1) $a(ss') = (as)s'$

(2) $a1 = a$

(3) $a \leq a'$ implies $as \leq a's$

(4) $s \leq s'$ implies $as \leq as'$

for all $s, s' \in S$ and $a, a' \in a$.

***Left*** $S$***-posets*** are defined analogously.

An $S$***-morphism*** from $S$-poset $A_S$ to $S$-poset $B_S$ is a monotonic map that preserves $S$-action.

The class of all right $S$-posets together with all $S$-morphisms forms a category, which is denoted by POS-S. Monomorphisms of POS-S are exactly the injective $S$-morphisms.

An $S$-morphism is called an ***embedding*** in category POS-S if it is order embedding (in other words an $S$-morphism $f : A_S \to B_S$ is an embedding if $\forall a, b \in A \ a \leq b \Leftrightarrow f(a) \leq f(b)$).

As a generalization of [3] the notion of factor $S$-posets was well developed in [2]. Let us repeat the essentials. A ***congruence*** on an $S$-poset $A_S$ is an $S$-act congruence $\theta$ that has the further property that the factor act $A/\theta$ can be equipped with a compatible order so that the natural projection $A \to A/\theta$ is an $S$-morphism.

**Definition 1.** Let $A_S$ be an $S$-poset and $\theta$ an $S$-poset congruence on $A$. An order relation $\leq$ on $A_S/\theta$ is called $\theta$***-compatible*** if the natural projection $A_S \to (A_S/\theta, \leq)$ is an $S$-morphism.

A given $S$-act congruence can give rise to many different factor $S$-posets: for example ([1]) if $A_S$ is any $S$-poset with compatible order $\leq$, then the identity relation $\triangle$ is an $S$-poset congruence on $A_S$, and the corresponding factor acts are simply the $S$-posets $(A_S, \leq')$ where the relation $\leq'$ is compatible and finer than the relation $\leq$.

Suppose $A_S$ is an $S$-poset and $\alpha$ is a binary relation on $A$ that is reflexive, transitive and compatible with the $S$-action. We write $a \underset{\alpha}{\leq} a'$ if so-called ***α-chain*** exists from $a$ to $a'$ in $A$:

$$a \leq a_1 \ \alpha \ a_1' \leq a_2 \ \alpha \ a_2' \leq \cdots \ \alpha \ a_m' \leq a',$$

where each $a_i, a_i'$ belongs to $A$. This chain is called ***closed*** if $a = a'$ and ***open*** otherwise.

**Theorem 2.1 ([3] Theorem 1.1).** *Let $A_S$ be an $S$-poset and $\theta$ an $S$-act congruence on $A$. Then $\theta$ is $S$-poset congruence if and only if* ***closed chains condition*** *holds: $a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a$ implies $a\theta a'$ for all $a, a' \in A$.*

**Proposition 2.2.** *Let $A_S$ be an $S$-poset and $\theta$ an $S$-poset congruence on $A$. The order relation $\leq$ on $A_S/\theta$ defined by*

$$[a]_\theta \leq [a']_\theta \text{ if and only if } a \underset{\theta}{\leq} a'$$

*for $a, a' \in A$ is the smallest $\theta$-compatible order on $A_S/\theta$.*

*Proof.* Let $\leq'$ be a $\theta$-compatible order on $A_S/\theta$. Suppose that $a \underset{\theta}{\leq} a'$. This means that there exist $a_1, \ldots, a_m, a_1', \ldots, a_m' \in A$ such that

$$a \leq a_1 \theta a_1' \leq a_2 \theta a_2' \leq \cdots \theta a_m' \leq a'.$$

The natural projection $A \to (A/\theta, \leq')$ is $S$-morphism which means that $b \leq c$ for some $b, c \in A$ implies $[b]_\theta \leq' [c]_\theta$. So we have that

$$[a]_\theta \leq' [a_1]_\theta = [a_1']_\theta \leq' [a_2]_\theta = [a_2']_\theta \leq' \ldots = [a_m']_\theta \leq' [a']_\theta$$

and thus $[a]_\theta \leq' [a']_\theta$. □

In the rest of this thesis if not said otherwise we use the smallest $\theta$-compatible order relation.

The following construction was introduced in [2]. Let $A_S$ be an $S$-poset and let $H \subseteq A \times A$. Define a relation $\alpha(H)$ on $A$ by $a\alpha(H)a'$ if and only if $a = a'$ or

$$\begin{array}{llllll} a = & x_1 s_1 & & y_2 s_2 & = x_3 s_3 \quad \ldots & y_n s_n = a' \\ & y_1 s_1 & = & x_2 s_2 & \quad\quad\quad \ldots & \end{array}, \tag{1}$$

for some $(x_i, y_i) \in H$ and $s_i \in S$. Note that that relation $\alpha(H)$ is transitive, reflexive and compatible with $S$-action.

**Definition 2 ([2] Definition 2.1).** Let $A_S$ be any $S$-poset and let $H \subseteq A \times A$. Then the relation $\nu(H)$ defined by

$$a\nu(H)a' \text{ if and only if } a \underset{\alpha(H)}{\leq} a' \text{ and } a' \underset{\alpha(H)}{\leq} a,$$

(where $\alpha(H)$ and $\underset{\alpha(H)}{\leq}$ are defined as above) is called the ***$S$-poset congruence on $A$ induced by $H$***. The order relation on $A_S/\nu(H)$ given by

$$[a]_{\nu(H)} \leq [a']_{\nu(H)} \text{ if and only if } a \underset{\alpha(H)}{\leq} a'$$

is called the ***order relation on $A_S/\nu(H)$ induced by $H$***.

As was shown in Proposition 2.2 the order relation on the factor $S$-poset $A/\nu(H)$ given by

$$[a]_{\nu(H)} \leq [a']_{\nu(H)} \Longleftrightarrow a \underset{\nu(H)}{\leq} a'$$

is the smallest $\nu(H)$-compatible order. Now the natural question arises whether the order relation induced by $H$ is indeed larger than the smallest one or do they coincide. This question gets a negative answer in the following example.

**Example 1.** *Let $S = (\{1, s\}, \triangle)$ be an idempotent pomonoid with the equality as the trivial order relation. Consider the $S$-poset $S_S$. Let $H = \{(1, s)\} \subset S \times S$. Then in sequence (1) we have that $x_i = 1$ and $y_i = s$ and so $\alpha(H) = \{(1, 1), (s, s), (1, s)\}$. This gives us the $S$-poset congruence $\nu(H) = \{(1, 1), (s, s)\}$ and the factor $S$-poset $S_S/\nu(H)$ along with two order relations: one using $\nu(H)$-chains and the other one using $\alpha(H)$-chains. For $a, a' \in S$ we have that $a \underset{\nu(H)}{\leq} a'$ if and only if $a = a'$. In the same time $1 \underset{\alpha(H)}{\leq} s$.*

The following Homomorphism Theorem for $S$-morphism will be used further in this paper:

**Proposition 2.3 ([2] Proposition 2.3).** *Let $\phi : A_S \to B_S$ be a surjective $S$-poset morphism, and define*

$$H_\phi = \{(a, a') \in A \times A : \phi(a) \leq \phi(a')\}.$$

*Then:*

*(1) The relations $\alpha(H_\phi)$ and $\underset{\alpha(H_\phi)}{\leq}$ both coincide with $H_\phi$ itself.*

*(2)* $\nu(H_\phi) = \ker\phi$, *and in* $A_S/\ker\phi$, $[a]_{\ker\phi} \le [a']_{\ker\phi}$ *if and only if* $\phi(a) \le \phi(a')$.

*(3) The mapping* $\overline{\phi} : A_S/\ker\phi \to B_S$ *defined by* $\overline{\phi}([a]_{\ker\phi}) = \phi(a)$ *for* $a \in A$ *is an* $S$*-poset isomorphism and* $\overline{\phi} \circ \pi = \phi$, *where* $\pi : A_S \to A_S/\ker\phi$ *is the canonical morphism.*

In [3] tensor products of $S$-posets were introduced. The following description of them is due to [6]. Let $A$ be a right and $B$ a left $S$-posets. Cartesian product $A \times B$ of posets $A$ and $B$ is a poset with the Cartesian order $(a, b) \le (c, d) \Leftrightarrow a \le c$ and $b \le d$. Put $H = \{((as, b), (a, sb)) | a \in A, b \in B, s \in S\}$ and let $\nu = \nu(H)$ be the smallest congruence on partially ordered set $A \times B$ which identifies all pairs from $H$. The factor $(A \times B)/\nu$ is a poset, called the ***tensor product of*** $A$ ***and*** $B$ ***over*** $S$, and is denoted by $A \otimes_S B$. As usual, for $a \in A$ and $b \in B$ the equivalence class of $(a, b)$ in $A \otimes_S B$ is denoted by $a \otimes b$.

It was shown in [6] that the order relation on $A \otimes_S B$ is described as follows:

$$a \otimes b \le a' \otimes b'$$

for $a, a' \in A$ and $b, b' \in B$ if and only if, there exist $s_1, t_1, \ldots, s_n, t_n \in S$, $a_1, \ldots, a_n \in A$ and $b_2, \ldots, b_n \in B$ such that

$$\begin{array}{rclcrl} a & \le & a_1 s_1 & & & \\ a_1 t_1 & \le & a_2 s_2 & & s_1 b \le & t_1 b_2 \\ & \cdots & & & \cdots & \\ a_n t_n & \le & a' & & s_n b_n \le & t_n b', \end{array}$$

Then $a \otimes b = a' \otimes b'$ if and only if $a \otimes b \le a' \otimes b'$ and $a \otimes b \ge a' \otimes b'$ both hold.

The following lemma is right-left dual to Corollary 3.3 in [5].

**Lemma 2.4.** *Let* $A$ *be a right* $S$*-poset,* $s, s' \in S, a, a' \in A$. *Then* $a \otimes s \le a' \otimes s'$ *in* $A \otimes S$ *if and only if* $as \le a's'$.

To conclude the section we give explanations of some more notations which will be used in what follows.

For a poset $P$ and its subposet $X$ we denote by $(X]$ the set of all elements of $P$ that are smaller than some element of $X$, that is

$$(X] = \{p \in P \mid \exists\, x \in X,\ p \le x\}.$$

$[X)$ is defined dually.

For an $S$-poset ${}_SA$ and $a \in {}_SA$ we denote by $\rho_a$ the $S$-morphism from ${}_SS$ into ${}_SA$ defined by $\rho_a(s) = sa$ for every $s \in S$.

Finally, recall that a subposet $X$ of poset $P$ is called ***convex***, if for any $x, y \in X$ one has that every element $z \in P$ such that $x \leq z \leq y$ also belongs to $X$.

# 3 Torsion free and po-torsion free S-posets

## 3.1 Definitions and general properties

**Definition 3.** An $S$-poset $A_S$ is called ***po-torsion free*** if the functor $1_{A_S} \otimes -$ preservers all self-embeddings $\iota : {}_SS \to {}_SS$ in S-POS.

**Definition 4.** An $S$-poset $A_S$ is called ***torsion free*** if the induced morphism $A_S \otimes_S S \to A_S \otimes_S S$ is injective whenever ${}_SS \to {}_SS$ is an embedding in S-POS.

Notice that from definitions above it follows that po-torsion freeness implies torsion freeness.

**Definition 5 ([1]).** Let $S$ be a pomonoid. An element $c \in S$ is called ***right po-cancellable*** if $sc \leq s'c$ implies $s \leq s'$ for all $s, s' \in S$.

**Lemma 3.1.** *Let $S$ be a pomonoid. $S$-poset morphism $\iota : {}_SS \to {}_SS$ is embedding if and only if $\iota(1)$ is right po-cancellable element of $S$.*

*Proof.* Let $s, s' \in S$.

*Necessity.* Suppose $\iota$ is an embedding $s\iota(1) \leq s'\iota(1)$. Then we have

$$s\iota(1) \leq s'\iota(1) \Rightarrow \iota(s) \leq \iota(s') \Rightarrow s \leq s'.$$

*Sufficiency.* Suppose $\iota(1)$ is a right po-cancellable element of $S$. Then we have

$$\iota(s) \leq \iota(s') \Rightarrow s\iota(1) \leq s'\iota(1) \Rightarrow s \leq s'.$$

□

**Theorem 3.2.** *Let $S$ be a pomonoid. An $S$-poset $A_S$ is po-torsion free if and only if $ac \leq a'c$ implies $a \leq a'$ whenever $a, a' \in A$, and $c$ is a right po-cancellable elements of $S$.*

*Proof. Necessity.* Let an $S$-poset $A_S$ be po-torsion free, $a, a \in A_S$, $c \in S$ be right po-cancellable and $ac \leq a'c$. Let $\iota = \rho_c : {}_SS \to {}_SS$. From right po-cancellability of $c$ it follows that $\iota$ is an $S$-poset embedding. Then $a\iota(1) \leq a'\iota(1)$ and so by Lemma 2.4 $a \otimes \iota(1) \leq a' \otimes \iota(1)$ in the poset $A_S \otimes {}_SS$. By assumption the induced morphism $A_S \otimes {}_SS \to A_S \otimes {}_SS$ is an embedding which implies that $a \otimes 1 \leq a' \otimes 1$ and thus by Lemma 2.4 we have $a \leq a'$.

*Sufficiency.* Let $\iota : {}_SS \to {}_SS$ be an embedding. Suppose $a \otimes \iota(s) \leq a' \otimes \iota(s')$ in the poset $A_S \otimes {}_SS$. Then $a \otimes s\iota(1) \leq a' \otimes s'\iota(1)$ and thus by Lemma 2.4

$as\iota(1) \leq a's'\iota(1)$. From $\iota$ being embedding it follows that $\iota(1)$ is right po-cancellable element of pomonoid $S$ and thus $as \leq a's'$. The latter implies by Lemma 2.4 $a \otimes s \leq a' \otimes s'$ and so the induced morphism $A_S \otimes {}_S S \to A_S \otimes {}_S S$ is embedding. □

The proof of the following theorem is easily derivable from the previous one by substituting $\leq$ for $=$.

**Theorem 3.3.** *Let $S$ be a pomonoid. An $S$-poset $A_S$ is torsion free if and only if $ac = a'c$ implies $a = a'$ whenever $a, a' \in A$, and $c$ is a right po-cancellable element of $S$.*

Notice that there is no common agreement by different authors so far in definitions of torsion freeness and po-torsion freeness. As Theorem 3.2 has shown the definitions of po-torsion freeness given in the current thesis and in [1] coincide. In [1] torsion freeness is defined as follows:

An $S$-poset $A_S$ is called torsion free if $ac = a'c$ implies $a = a'$ whenever $a, a' \in A$ and $c$ is right cancellable element of $S$.

It follows from Theorem 3.3 that this definition of torsion freeness is stricter than one used in the current thesis.

## 3.2 Torsion free and po-torsion free cyclic S-posets

**Proposition 3.4.** *(Torsion free) Let $\theta$ be an $S$-poset congruence. Then the right $S$-poset $S/\theta$ is torsion free if and only if $(s,t) \notin \theta$ implies $(sc, tc) \notin \theta$ for every right po-cancellable element $c \in S$.*

*Proof. Necessity.* Suppose that $(s,t) \notin \theta$ and $(sc, tc) \in \theta$ for $s, t, c \in S$, $c$ right po-cancellable. This means $[s]_\theta c = [t]_\theta c$. Since $S/\theta$ is torsion free then by Theorem 3.3 this implies $[s]_\theta = [t]_\theta$ or $(s,t) \in \theta$, a contradiction.

*Sufficiency.* Suppose $[s]_\theta c = [t]_\theta c$ for $s, t, c \in S$, $c$ right po-cancellable. This means $sc\ \theta\ tc$ and by assumption it implies $s\ \theta\ t$ or $[s]_\theta = [t]_\theta$. Hence $S/\theta$ is torsion free by Theorem 3.3 □

**Proposition 3.5.** *(Po-torsion free) Let $S$ be a pomonoid $S$ and let $\theta$ be an $S$-poset congruence. Then the right $S$-poset $S/\theta$ is po-torsion free if and only if $s \underset{\theta}{\not\leq} t$ implies $sc \underset{\theta}{\not\leq} tc$ for every right po-cancellable element $c \in S$.*

*Proof. Necessity.* Suppose that $s \underset{\theta}{\not\leq} t$ and $sc \underset{\theta}{\leq} tc$ for $s, t, c \in S$, $c$ right po-cancellable. This means $[s]_\theta c \leq [t]_\theta c$. Since $S/\theta$ is po-torsion free then by Theorem 3.2 this implies $[s]_\theta \leq [t]_\theta$ or $s \underset{\theta}{\leq} t$, a contradiction.

*Sufficiency.* Suppose $[s]_\theta c \leq [t]_\theta c$ for $s, t, c \in S$, $c$ right po-cancellable. This means $sc \underset{\theta}{\leq} tc$ and by assumption it implies $s \underset{\theta}{\leq} t$ or $[s]_\theta \leq [t]_\theta$. Hence $S/\theta$ is po-torsion free by Theorem 3.2. □

## 3.3 Po-torsion free and torsion free Rees factor S-posets

**Lemma 3.6 ([1] Lemma 3).** *Let $K_S$ be a convex, proper right ideal of the pomonoid $S$. Then for $x, y \in S$,*

$$[x] \leq [y] \text{ in } S/K_S \Leftrightarrow (x \leq y) \text{ or } (x \in (K] \text{ and } y \in [K)).$$

*Moreover, $[x] = [y]$ in $S/K_S$ if, and only if, either $x = y$ or else $x, y \in K$.*

**Proposition 3.7.** *(Torsion free) Let $K_S$ be a convex, proper right ideal of the pomonoid $S$. Then $S/K_S$ is torsion free if, and only if, for every $s \in S$ and every right po-cancellable $c \in S$, $sc \in K$ implies $s \in K$.*

*Proof. Necessity.* Suppose $sc \in K_S$ for $s, c \in S$, where $c$ is a right po-cancellable element. Set $\nu = \nu(K \times K)$. Then $[s]_\nu c = [0]_\nu = [0]_\nu c$. Since $S/K_S$ is torsion free then by Proposition 3.4 $[s]_\nu = [0]_\nu$ or $s \in K_S$.

*Sufficiency.* Suppose $[s]_\nu c = [t]_\nu c$, $s, t, c \in S$, $c$ is a right po-cancellable element. If $[s]_\nu c = [t]_\nu c = [0]_\nu$ then $sc, tc \in K_S$ and by assumption $s, t \in K_S$ which means $[s]_\nu = [0]_\nu = [t]_\nu$. If $[s]_\nu c = [t]_\nu c \neq [0]_\nu$ then $sc = tc$. Hence $s = t$ and $[s]_\nu = [t]_\nu$. By Proposition 3.4 this means that $S/K_S$ is torsion free. □

**Proposition 3.8 ([1] Proposition 6).** *(Po-torsion free) Suppose $K_S$ is a proper, convex right ideal of a pomonoid $S$. Then $S/K_S$ is po-torsion free if, and only if, whenever $c$ is a right po-cancellable element of $S$ then $sc \in (K]$ implies $s \in (K]$ and $tc \in [K)$ implies $t \in [K)$.*

It was mentioned above that po-torsion freeness implies torsion freeness and torsion freeness in the sense of [1] implies torsion freeness. Since in [1] there was shown that po-torsion freeness and torsion freeness in the sense of [1] are incomparable properties then it is now clear that torsion freeness cannot imply po-torsion freeness.

# 4 Principally weakly flat and principally weakly po-flat S-posets

## 4.1 Definitions and general properties

**Definition 6 ([5] Definition 3.11).** An $S$-poset $A_S$ is called ***principally weakly po-flat*** if the functor $1_{A_S} \otimes -$ preserves embeddings of principal left ideals into $S$.

In the language of elements this definition means that if inequality $a \otimes s \leq a' \otimes s$ for $a, a' \in A_S, s \in S$ holds in the tensor product $A_S \otimes {}_S S$ then it holds already in the tensor product $A_S \otimes {}_S Ss$.

**Lemma 4.1.** *An $S$-poset $A_S$ is principally weakly po-flat if and only if $as \leq a's$ for $a, a' \in A, s \in S$ implies $a \otimes s \leq a' \otimes s$ in the tensor product $A_S \otimes {}_S Ss$.*

*Proof.* By Corollary 2.4 $a \otimes s \leq a' \otimes s$ for $a, a' \in A, s \in S$ in the tensor product $A_S \otimes {}_S S$ if and only if $as \leq a's$. □

Notice that by definition principally weak po-flatness implies torsion po-freeness.

**Definition 7 ([1]).** An $S$-poset $A_S$ is called ***principally weakly flat*** if the functor $A_S \otimes -$ maps embeddings of principal left ideals to monomorphism.

In the language of elements this definition means that if equality $a \otimes s = a' \otimes s$ for $a, a' \in A_S, s \in S$ holds in the tensor product $A_S \otimes_S S$ then it holds already in the tensor product $A_S \otimes {}_S Ss$.

**Lemma 4.2.** *An $S$-poset $A_S$ is principally weakly flat if and only if $as = a's$ for $a, a' \in A, s \in S$ implies $a \otimes s = a' \otimes s$ in the tensor product $A_S \otimes {}_S Ss$.*

*Proof.* Using Corollary 2.4 we get that $a \otimes s = a' \otimes s$ for $a, a' \in A, s \in S$ in the tensor product $A_S \otimes {}_S S$ if and only if $as = a's$. □

Notice that by definition principally weak po-flatness implies principally weak flatness.

## 4.2 Principally weakly flat and principally weakly po-flat cyclic S-posets

Recall (see [4]) that if $\rho$ and $\lambda$ are equivalence relations on a set $X$ then their join $\rho \vee \lambda$ is the relation defined by:

$x(\rho \vee \lambda)x \Leftrightarrow$ there exist $z_1, z_2, \ldots, z_n \in X$ such that $x \; \rho \; z_1 \; \lambda \; z_2 \; \rho \; z_3 \; \cdots \; z_n \; \lambda \; y$

**Lemma 4.3.** *Let $A_S$ be an $S$-poset and $\rho$ and $\lambda$ $S$-poset congruences on $A_S$. Then $a \underset{\rho\vee\lambda}{\leq} a'$ for some $a, a' \in A$ if and only if there exist $u_0, \dots, u_n \in A$ such that*

$$a = u_0 \underset{\rho}{\leq} u_1 \underset{\lambda}{\leq} u_2 \underset{\rho}{\leq} u_3 \cdots u_{n-1} \underset{\lambda}{\leq} u_n = a'.$$

*Proof. Necessity.* By definition $a \underset{\rho\vee\lambda}{\leq} a'$ if and only if there exist $a_1, \dots, a_n$, $a'_1, \dots, a'_n \in A$ such that

$$a \leq a_1(\rho \vee \lambda)a'_1 \leq a_2(\rho \vee \lambda)a'_2 \leq \cdots (\rho \vee \lambda)a'_n \leq a'.$$

For each $1 \leq i \leq n$ one has $a_i(\rho\vee\lambda)a'_i$ if and only if there exist $u^i_1, \dots, u^i_n \in A$ such that

$$a_i \;\rho\; u^i_1 \;\lambda\; u^i_2 \;\rho\; \cdots\; u^i_n \;\lambda\; a'_i$$

which implies that

$$a_i \underset{\rho}{\leq} u^i_1 \underset{\lambda}{\leq} u^i_2 \underset{\rho}{\leq} \cdots u^i_n \underset{\lambda}{\leq} a'_i$$

and the required follows.

*Sufficiency.* For each $0 \leq i \leq n-2$ one has $u_i \underset{\rho}{\leq} u_{i+1} \underset{\lambda}{\leq} u_{i+2}$ if and only if there exist $v^i_1, \dots, v^i_n, {v^i_1}', \dots, {v^i_n}', w^i_1, \dots, w^i_n, {w^i_1}', \dots, {w^i_n}'$ such that

$$u_i \leq v^i_1 \rho {v^i_1}' \leq \cdots \rho {v^i_n}' \leq u_{i+1} \leq w^i_1 \lambda {w^i_1}' \leq \cdots \lambda {w^i_n}' \leq u_{i+2}.$$

This implies

$$u_i \leq v^i_1(\rho \vee \lambda){v^i_1}' \leq \cdots (\rho \vee \lambda){v^i_n}' \leq w^i_1(\rho \vee \lambda){w^i_1}' \leq \cdots (\rho \vee \lambda){w^i_n}' \leq u_{i+2}$$

and thus $u_i \underset{\rho\vee\lambda}{\leq} u_{i+2}$ and so $a \underset{\rho\vee\lambda}{\leq} a'$. □

**Lemma 4.4.** *Let $\rho$ be a right and $\lambda$ a left $S$-poset congruence on pomonoid $S$. Then*

$$[s]_\rho \otimes [t]_\lambda \leq [s']_\rho \otimes [t']_\lambda$$

*in $S/\rho \otimes S/\lambda$ for $s, s', t, t' \in S$ if and only if $st \underset{\rho\vee\lambda}{\leq} s't'$.*

*Proof. Necessity.* Let $[s]_\rho \otimes [t]_\lambda \leq [s']_\rho \otimes [t']_\lambda$ in $S/\rho \otimes S/\lambda$ for $s, s', t, t' \in S$. This means that we have a tossing

$$\begin{array}{rclrcl}
[s]_\rho & \leq & [u_1]_\rho s_1 & & & \\
[u_1]_\rho t_1 & \leq & [u_2]_\rho s_2 & s_1[t]_\lambda & \leq & t_1[v_2]_\lambda \\
[u_2]_\rho t_2 & \leq & [u_3]_\rho s_3 & s_2[v_2]_\lambda & \leq & t_2[v_3]_\lambda \\
 & \cdots & & & \cdots & \\
[u_n]_\rho t_n & \leq & s' & s_n[v_n]_\lambda & \leq & t_n[t']_\lambda
\end{array}$$

where $s_1, \ldots, s_n, t_1, \ldots, t_n, u_1, \ldots, u_n, v_2, \ldots v_n \in S$. From the first row of this tossing we have $st \underset{\rho}{\leq} u_1 s_1 t$. The second row gives us $s_1 t \underset{\lambda}{\leq} t_1 v_2$ and $u_1 t_1 \underset{\rho}{\leq} u_2 s_2$. Moving in such a manner downwards in the tossing we get that

$$st \underset{\rho}{\leq} u_1 s_1 t \underset{\lambda}{\leq} u_1 t_1 v_2 \underset{\rho}{\leq} u_2 s_2 v_2 \underset{\lambda}{\leq} \cdots \underset{\rho}{\leq} u_n s_n v_n \underset{\lambda}{\leq} u_n t_n t' \underset{\rho}{\leq} s' t'$$

and so $st \underset{\rho \vee \lambda}{\leq} s't'$.

*Sufficiency.* Let $st \underset{\rho \vee \lambda}{\leq} s't'$ for $s, t, s', t' \in S$. Then there exist $u_1, u_2, \ldots, u_n \in S$ such that $st \underset{\rho}{\leq} u_1 \underset{\lambda}{\leq} u_2 \underset{\rho}{\leq} u_3 \underset{\lambda}{\leq} \ldots \underset{\rho}{\leq} u_n \underset{\lambda}{\leq} s't'$ and

$$\begin{aligned}
[s]_\rho \otimes [t]_\lambda & \leq [st]_\rho \otimes [1]_\lambda \leq [u_1]_\rho \otimes [1]_\lambda \leq [1]_\rho \otimes [u_1]_\lambda \leq [1]_\rho \otimes [u_2]_\lambda \leq \cdots \\
& \leq [1]_\rho \otimes [u_n]_\lambda \leq [1]_\rho \otimes [s't']_\lambda \leq [s']_\rho \otimes [t']_\lambda
\end{aligned}$$

in $S/\rho \otimes S/\lambda$. □

**Lemma 4.5.** *Let ${}_SA = Sa$ be a cyclic $S$-poset. Then $S/\ker \rho_a \cong A_S$ with the corresponding $S$-isomorphism $g : S/\ker \rho_a \to {}_SA$ defined by $g([s]_{\rho_a}) = sa$ for every $s \in S$.*

*Proof.* Les ${}_SA = Sa$ for some $a \in {}_SA$. The $S$-morphism $\rho_a : {}_SS \to {}_SA = Sa$ is obviously surjective. By Proposition 2.3 we get that ${}_SA = Sa \cong S/\ker \rho_a$. □

**Lemma 4.6.** *Let $\rho$ be a right $S$-poset congruence on $S$ and $s \in S$. Then $[u]_\rho \otimes s \leq [v]_\rho \otimes s$ in $S/\rho \otimes_S Ss$ for $u, v \in S$ if and only if $u \underset{\rho \vee \ker \rho_s}{\leq} v$.*

*Proof. Necessity.* Let $[u]_\rho \otimes s \leq [v]_\rho \otimes s$ in $(S/\rho) \otimes_S Ss$. Using the $S$-isomorphism $Ss \cong S/\ker\rho_s$ from the Lemma 4.5 we get that $[u]_\rho \otimes [1]_{\ker\rho_s} \leq [v]_\rho \otimes [1]_{\ker\rho_s}$ in $S/\rho \otimes S/\ker\rho_s$. By Lemma 4.4 it means that $u \underset{\rho\vee\ker\rho_s}{\leq} v$.

*Sufficiency.* Let $u \underset{\rho\vee\ker\rho_s}{\leq} v$ for $u, v \in S$. Then there exist $u_1, u_2, \ldots, u_n \in S$ such that

$$u \underset{\rho}{\leq} u_1 \underset{\ker\rho_s}{\leq} u_2 \underset{\rho}{\leq} u_3 \underset{\ker\rho_s}{\leq} \ldots \underset{\rho}{\leq} u_n \underset{\ker\rho_s}{\leq} v$$

and

$$\begin{aligned}[u]_\rho \otimes s &\leq [u_1]_\rho \otimes s \leq [1]_\rho \otimes u_1 s \leq [1]_\rho \otimes u_2 s \leq [u_2]_\rho \otimes s \leq \cdots \\ &\leq [1]_\rho \otimes u_n s \leq [1]_\rho \otimes vs \leq [v]_\rho \otimes s\end{aligned}$$

in $S/\rho \otimes_S Ss$. □

**Proposition 4.7.** *Let $\rho$ be a right $S$-poset congruence on a pomonoid $S$. Then $S/\rho$ is a principally weakly po-flat if and only if $[u]_\rho s \leq [v]_\rho s$, $u, v, s \in S$, implies $u \underset{\rho\vee\ker\rho_s}{\leq} v$.*

*Proof. Necessity.* Let $[u]_\rho s \leq [v]_\rho s$ for $u, v, s \in S$. Since $S/\rho$ is principally weakly po-flat then we have by Proposition 4.1 that $[u]_\rho \otimes s \leq [v]_\rho \otimes s$ in $S/\rho \otimes_S Ss$. Now it follows from Lemma 4.6 that $u \underset{\rho\vee\ker\rho_s}{\leq} v$.

*Sufficiency.* Let $[u]_\rho s \leq [v]_\rho s$ for $u, v, s \in S$. By hypothesis it implies $u \underset{\rho\vee\ker\rho_s}{\leq} v$. By Lemma 4.6 we have $[u]_\rho \otimes s \leq [v]_\rho \otimes s$ in $S/\rho \otimes_S Ss$. Hence $S/\rho$ is principally weakly po-flat. □

**Proposition 4.8.** *Let $\rho$ be a right $S$-poset congruence on a pomonoid $S$. Then $S/\rho$ is a principally weakly flat if and only if $[u]_\rho s = [v]_\rho s$, $u, v, s \in S$, implies $u \underset{\rho\vee\ker\rho_s}{\leq} v \underset{\rho\vee\ker\rho_s}{\leq} u$.*

*Proof. Necessity.* Let $[u]_\rho s = [v]_\rho s$ for $u, v, s \in S$. Since $S/\rho$ is principally weakly flat then we have by Proposition 4.2 that $[u]_\rho \otimes s = [v]_\rho \otimes s$ in $S/\rho \otimes_S Ss$. This means that $[u]_\rho \otimes s \leq [v]_\rho \otimes s$ and $[v]_\rho \otimes s \leq [u]_\rho \otimes s$ in $S/\rho \otimes_S Ss$. Now it follows from Lemma 4.6 that $u \underset{\rho\vee\ker\rho_s}{\leq} v \underset{\rho\vee\ker\rho_s}{\leq} u$.

*Sufficiency.* Let $[u]_\rho s = [v]_\rho s$ for $u, v, s \in S$. By hypothesis it implies $u \underset{\rho\vee\ker\rho_s}{\leq} v$ and $v \underset{\rho\vee\ker\rho_s}{\leq} u$. By Lemma 4.6 we have $[u]_\rho \otimes s \leq [v]_\rho \otimes s$ and $[v]_\rho \otimes s \leq [u]_\rho \otimes s$ in $S/\rho \otimes_S Ss$. So $[u]_\rho \otimes s = [v]_\rho \otimes s$ in $S/\rho \otimes_S Ss$ and hence $S/\rho$ is principally weakly flat. □

### 4.3 On monocyclic S-posets

By analogy with the unordered case a right congruence $\nu$ on pomonoid $S$ is said to be ***monocyclic*** if it is induced by a single pair of elements $(s,t)$, $s,t \in S$, and is then denoted by $\nu(s,t)$. A right factor $S$-poset of a pomonoid $S$ by a monocyclic right congruence is called a ***monocyclic right*** $S$***-poset*** (see Definition 1.4.18 in [4] for the unordered case). Let us describe in more details the construction of monocyclic $S$-posets of the form $S/\nu(wt,t)$ for $w,t \in S$.

In case of $H = \{(wt,t)\}$ in (1) we have that $x_i = wt$ and $y_i = t$ and hence $a\alpha(H)a'$ if and only if $a = w^n a'$, for some $n \geq 0$, and $w^i a' \in tS$ whenever $0 \leq i < n$. Inequality $a \underset{\alpha(H)}{\leq} a'$ means then that there exists an $\alpha(H)$-chain

$$a \leq a_1\alpha(H)a_1' \leq a_2\alpha(H)a_2' \leq \cdots \alpha(H)a_m' \leq a'$$

from $a$ to $a'$ in $S$. Now $a_i\alpha(H)a_i'$ gives us that $a_i = w^{n_i}a_i'$ for some $n_i \in \mathbf{N}$ and $w^j a_i' \in tS$ whenever $0 \leq j < n_i$. We can rewrite $\alpha(H)$-chain as the following sequence of inequalities:

$$\begin{gathered} a \leq w^{n_1}a_1' \\ a_i' \leq w^{n_{i+1}}a_{i+1}' \text{ for every } 1 \leq i \leq m-1 \\ a_m' \leq a'. \end{gathered} \tag{2}$$

**Proposition 4.9.** *Let $S$ be a pomonoid and $w,t \in S$. Then, for any $a,a' \in S$, $a \underset{\alpha(wt,t)}{\leq} a'$ implies $a \leq w^n a'$ for some $n \geq 0$, where $w^i a' \in [tS)$ whenever $0 \leq i < n$.*

*Proof.* As was shown above $a \underset{\alpha(wt,t)}{\leq} a'$ if and only if there exists a sequence of the form (2) for some $n_i \in \mathbf{N}$ and $w^j a_i' \in tS$ whenever $0 \leq j < n_i$. So we have that $a \leq w^{n_1+n_2+\ldots+n_m}a'$.

For any $0 \leq j < n_1+n_2+\ldots+n_m$ we can write $j = l+n_k+n_{k+1}+\ldots+n_m$ for some $1 \leq k \leq m$ and $0 \leq l < n_{k-1}$. Then

$$\begin{aligned} w^j a' = \; & w^{l+n_k+n_{k+1}+\ldots+n_m}a' = \\ & w^{l+n_k+n_{k+1}+\ldots+n_{m-1}}(w^{n_m}a') \geq \\ & w^{l+n_k+n_{k+1}+\ldots+n_{m-1}}a_{m-1}' = \\ & w^{l+n_k+n_{k+1}+\ldots+n_{m-2}}(w^{n_{m-1}}a_{m-1}') \geq \\ & w^{l+n_k+n_{k+1}+\ldots+n_{m-2}}a_{m-2} = \\ & \ldots \\ & w^l a_{k-1}. \end{aligned}$$

But $w^l a_{k-1} \in tS$ and so $w^l a_{k-1} \leq w^j a' \in [tS)$. $\square$

**Proposition 4.10.** *Let $S$ be a pomonoid and $w, t \in S$. If for some $a, a' \in S$, there exist $n \geq 0$, such that $a \leq w^n a'$ and $w^i a' \in tS$ whenever $0 \leq i < n$, then $a \underset{\alpha(wt,t)}{\leq} a'$.*

*Proof.* Let $a, a' \in S$ and suppose that there exist such $n \geq 0$, that $a \leq w^n a'$ and $w^i a' \in tS$ whenever $0 \leq i < n$. Then $w^n a' \; \alpha(wt,t) \; a'$. So we have that $a \leq w^n a' \; \alpha(wt,t) \; a'$ and thus $a \underset{\alpha(wt,t)}{\leq} a'$. $\square$

In case of $H = \{(t^2, t)\}$ in (1) we have that $x_i = t^2$ and $y_i = t$ and hence $a\alpha(H)a'$ if and only if $a = t^n a'$, for some $n \geq 0$, and $a' \in tS$. The inequality $a \underset{\alpha(H)}{\leq} a'$ means then that there exists an $\alpha(H)$-chain

$$a \leq a_1 \alpha(H) a'_1 \leq a_2 \alpha(H) a'_2 \leq \cdots \alpha(H) a'_m \leq a'$$

from $a$ to $a'$ in $S$. Now $a_i \alpha(H) a'_i$ gives us that $a_i = t^{n_i} a'_i$ for some $n_i \in \mathbf{N}$ and $a'_i \in tS$. We can rewrite our $\alpha(H)$-chain as the following sequence of inequalities:

$$\begin{gathered} a \leq t^{n_1} a'_1 \\ a'_i \leq t^{n_{i+1}} a'_{i+1} \text{ for every } 1 \leq i \leq m-1 \\ a'_m \leq a', \end{gathered} \tag{3}$$

where $a'_i \in tS$ for every $1 \leq i \leq m$.

**Proposition 4.11.** *Let $S$ be a pomonoid and $t \in S$. Then, for any $a, a' \in S$, $a \underset{\alpha(t^2,t)}{\leq} a'$ if and only if $a \leq t^n u$ and $tu \leq a'$ for some $n \geq 1$ and $u \in S$.*

*Proof. Necessity.* As was shown above $a \underset{\alpha(t^2,t)}{\leq} a'$ if and only if there exists a sequence of the form (3) for some $n_i \in \mathbf{N}$ and $a'_i \in tS$ whenever $1 \leq i \leq m$. So we have that $a \leq t^{n_1+n_2+\ldots+n_m} a'_m$, $a'_m \in tS$ and $a'_m \leq a'$. Denoting $n' = n_1 + n_2 + \ldots + n_m + 1$ and $a'_m = tu$ we have $a \leq t^{n'} u$ and $tu \leq a'$.

*Sufficiency.* Let $a, a', t, u \in S$ and $a \leq t^n u$ and $tu \leq a'$ for some $1 \leq n \in N$. Then we have that $a \leq t^n u \; \alpha(t^2, t) \; tu \leq a'$ and so $a \underset{\alpha(t^2,t)}{\leq} a'$. $\square$

Notice that from the last proposition it follows, that whenever there exists an $\alpha(t^2, t)$-chain of arbitrary length from $a$ to $a'$, there exists in fact an $\alpha(t^2, t)$-chain of length 1.

Applying the above method to a pair $(t, wt)$ we will have the following two statements.

**Proposition 4.12.** *Let $S$ be a pomonoid and $w, t \in S$. Then, for any $a, a' \in S$, $a \underset{\alpha(t,wt)}{\leq} a'$ implies $w^n a \leq a'$ for some $n \geq 0$, where $w^i a \in (tS]$ whenever $0 \leq i < n$.*

**Proposition 4.13.** *Let $S$ be a pomonoid and $t \in S$. Then, for any $a, a' \in S$, $a \underset{\alpha(t,t^2)}{\leq} a'$ if and only if $t^n u \leq a'$ and $a \leq tu$ for some $n \geq 1$ and $u \in S$.*

## 4.4 Principally weakly flat and principally weakly po-flat monocyclic S-posets

**Proposition 4.14.** *If $w, t \in S$, $wt \neq t$, and if $S/\nu(wt, t)$ is principally weakly flat, then $t \in [tSt)$ and $wt \in [tSt)$.*

*Proof.* Set $\nu = \nu(wt, t)$ and suppose that $S/\nu$ is principally weakly flat. Then $[wt]_\nu = [t]_\nu$ implies $w \underset{\nu \vee \ker \rho_t}{\leq} 1$ and $1 \underset{\nu \vee \ker \rho_t}{\leq} w$ by Proposition 4.8. The first inequality means that there exist $u_1, \ldots, u_n, v_1, \ldots, v_n \in S$ such that

$$w = u_1 \underset{\nu}{\leq} v_1 \underset{\ker \rho_t}{\leq} u_2 \underset{\nu}{\leq} \cdots v_n \underset{\ker \rho_t}{\leq} 1.$$

Assume that this sequence is the shortest among such sequences. We can rewrite it in the following way:

$$\begin{array}{ccccccccc} w = u_1 \underset{\nu}{\leq} & v_1 & & u_2 & \underset{\nu}{\leq} v_2 & \ldots & u_n \underset{\nu}{\leq} & v_n \\ & v_1 t & \leq & u_2 t & & \ldots & & v_n t \leq t. \end{array} \tag{4}$$

For each $1 \leq i \leq n$ the inequality $u_i \underset{\nu}{\leq} v_i$ implies $u_i \underset{\alpha(wt,t)}{\leq} v_i$ and so by Proposition 4.9 there exist $p_i \geq 0$ such that $u_i \leq w^{p_i} v_i$ and $w^{k_i} v_i \in [tS)$ whenever $0 \leq k_i < p_i$. If $p_n = 0$ then we have that $u_n \leq v_n$ and $v_{n-1} t \leq u_n t \leq v_n t \leq t$ and so we can rewrite the sequence (4) as follows:

$$\begin{array}{ccccccccc} w = u_1 \underset{\nu}{\leq} & v_1 & & u_2 & \underset{\nu}{\leq} v_2 & \ldots & u_{n-1} \underset{\nu}{\leq} & v_{n-1} \\ & v_1 t & \leq & u_2 t & & \ldots & & v_{n-1} t \leq t. \end{array}$$

But this contradicts to (4) being the shortest sequence. Hence $p_n > 0$ and $v_n \in [tS)$. This means that there exists such $u \in tS$ that $v_n \geq u$. Then $t \geq v_n t \geq ut$ and $t \in [tSt)$.

Applying the same method to the inequality $1 \underset{\nu \vee \ker \rho_t}{\leq} w$ we get that $wt \in [tSt)$. □

**Corollary 4.14.1.** *If $w, t \in S$, $wt \neq t$, and if $S/\nu(wt, t)$ is principally weakly po-flat, then $t \in [tSt)$ and $wt \in [tSt)$.*

## 4.5 Principally weakly po-flat and principally weakly flat Rees factor S-posets

**Proposition 4.15 ([1] Proposition 10).** *For any pomonoid $S$ and any convex, proper ideal $K_S$ of $S$, $S/K_S$ is principally weakly po-flat if, and only if, for every $s \in S$, $s \in [K)$ implies $s \in [Ks)$ and $s \in (K]$ implies $s \in (Ks]$.*

**Proposition 4.16 ([1] Proposition 9).** *For any pomonoid $S$ and any convex, proper ideal $K_S$ of $S$, $S/K_S$ is principally weakly flat if, and only if, for every $k \in K$ $k \in [Kk) \cap (Kk]$.*

## 4.6 Principally weakly po-flat PP pomonoids

The concept of ***PP pomonoids*** was introduced in [6] as an analogy of PP monoids.

**Definition 8.** An element $a$ of pomonoid $S$ is called ***right po-e-cancellable*** for an idempotent $e \in S$ if $a = ea$ and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$.

**Definition 9 ([6] Proposition 4.8).** A pomonoid $S$ is called ***left PP pomonoid*** if every element $a \in S$ is right po-e-cancellable for some idempotent $e \in S$.

The ***left po-e-cancellable elements*** and ***right PP pomonoids*** are defined dually.

**Lemma 4.17.** *Let $S$ be a left PP pomonoid and let $A_S$ be an $S$-poset. If for some $a, a' \in A_S$ and $s \in S$ the inequality $a \otimes s \leq a' \otimes s$ holds in $A_S \otimes_S Ss$ then there exists $e \in E(S)$ such that $es = s$ and $ae \leq a'e$.*

*Proof.* Suppose inequality $a \otimes s \leq a' \otimes s$ holds in $A_S \otimes_S Ss$. Then we have the scheme

$$\begin{array}{rclrl} a & \leq & a_1 s_1 & & \\ a_1 t_1 & \leq & a_2 s_2 & s_1 s \leq & t_1 u_2 s \\ a_2 t_2 & \leq & a_3 s_3 & s_2 u_2 s \leq & t_2 u_3 s \\ & \cdots & & \cdots & \\ a_n t_n & \leq & a' & s_n u_n s \leq & t_n s, \end{array}$$

where $s_1, \ldots, s_n, u_2, \ldots, u_n \in S$, $a_1, \ldots, a_n \in A$.

Since $S$ is a left PP pomonoid there exists an idempotent $e \in S$ such that $s = es$ and

$$\begin{array}{rl} s_1 e \leq & t_1 u_2 e \\ s_2 u_2 e \leq & t_2 u_3 e \\ \cdots & \\ s_n u_n e \leq & t_n e. \end{array}$$

Then $ae \leq (a_1 s_1)e \leq a_1(t_1 u_2 e) \leq (a_2 s_2)u_2 e \leq a_2(t_2 u_3 e) \leq \ldots \leq (a_n s_n)u_n e \leq a_n(t_n e) \leq a'e$. □

**Theorem 4.18.** *Let $S$ be a left PP pomonoid. An $S$-poset $A_S$ is principally weakly po-flat if and only if, for every $a, a' \in A$ and $s \in S$, $as \leq a's$ implies that there exists $e \in E(S)$ such that $es = s$ and $ae \leq a'e$.*

*Proof. Necessity.* Suppose $A_S$ is principally weakly po-flat and $as \leq a's$, $a, a' \in A$, $s \in S$. Then by Proposition 4.1 $a \otimes s \leq a' \otimes s$ in tensor product $A_S \otimes_S Ss$. Then by Lemma 4.17 there exists $e \in E(S)$ such that $es = s$ and $ae \leq a'e$.

*Sufficiency.* Suppose $as \leq a's$, $a, a' \in A$, $s \in S$. By assumption there exists an idempotent $e \in S$ such that $es = s$ and $ae \leq a'e$. Then we have

$$a \otimes s = a \otimes es = ae \otimes s \leq a'e \otimes s = a' \otimes es = a' \otimes s$$

in the tensor product $A_S \otimes_S Ss$. Hence $A_S$ is principally weakly po-flat. □

**Theorem 4.19.** *Let $S$ be a left PP pomonoid. An $S$-poset $A_S$ is principally weakly flat if and only if, for every $a, a' \in A$ and $s \in S$, $as = a's$ implies that there exists $e \in E(S)$ such that $es = s$, $ae = a'e$.*

*Proof. Necessity.* Suppose $A_S$ is principally weakly flat and $as = a's$, $a, a' \in A$, $s \in S$. Then by Proposition 4.2 $a \otimes s = a' \otimes s$ in tensor product $A_S \otimes_S Ss$. This means that $a \otimes s \leq a' \otimes s$ and $a' \otimes s \leq a \otimes s$ in $A_S \otimes_S Ss$ and so we have the following scheme:

$$\begin{array}{rcl rl}
a & \leq & a_1 s_1 & & \\
a_1 t_1 & \leq & a_2 s_2 & s_1 s \leq & t_1 u_2 s \\
a_2 t_2 & \leq & a_3 s_3 & s_2 u_2 s \leq & t_2 u_3 s \\
 & \cdots & & \cdots & \\
a_n t_n & \leq & a' & s_n u_n s \leq & t_n s, \\
a' & \leq & a'_1 s'_1 & & \\
a'_1 t'_1 & \leq & a'_2 s'_2 & s'_1 s \leq & t'_1 u'_2 s \\
a'_2 t'_2 & \leq & a'_3 s'_3 & s'_2 u'_2 s \leq & t'_2 u'_3 s \\
 & \cdots & & \cdots & \\
a'_n t'_n & \leq & a & s'_n u'_n s \leq & t'_n s,
\end{array}$$

where $s_1, \ldots, s_n, s'_1, \ldots, s'_n, u_2, \ldots, u_n, u'_2, \ldots, u'_n \in S$, $a_1, \ldots, a_n, a'_1, \ldots, a'_n \in A$.

Since $S$ is a left PP pomonoid there exists an idempotent $e \in S$ such that $s = es$ and

$$\begin{aligned} s_1e &\leq t_1u_2e \\ s_2u_2e &\leq t_2u_3e \\ &\cdots \\ s_nu_ne &\leq t_ne \\ s'_1e &\leq t'_1u'_2e \\ s'_2u'_2e &\leq t'_2u'_3e \\ &\cdots \\ s'_nu'_ne &\leq t'_ne. \end{aligned}$$

Then

$$ae \leq (a_1s_1)e \leq a_1(t_1u_2e) \leq (a_2s_2)u_2e \leq \ldots \leq (a_ns_n)u_ne \leq a_n(t_ne) \leq a'e$$

and

$$a'e \leq (a'_1s'_1)e \leq a'_1(t'_1u'_2e) \leq (a'_2s'_2)u'_2e \leq \ldots \leq (a'_ns'_n)u'_ne \leq a'_n(t'_ne) \leq ae.$$

Hence $ae = a'e$.

*Sufficiency.* Suppose $as = a's$, $a, a' \in A$, $s \in S$. By assumption there exist idempotent $e \in S$ such that $es = s$, $ae = a'e$. Then we have

$$a \otimes s = a \otimes es = ae \otimes s = a'e \otimes s = a' \otimes es = a' \otimes s$$

in the tensor product $A_S \otimes_S Ss$ and $A_S$ is principally weakly flat. $\square$

# 5 Weakly flat and weakly po-flat S-posets

## 5.1 Definitions and general properties

**Definition 10 ([5] Definition 3.11).** An $S$-poset $A_S$ is called ***weakly po-flat*** if the functor $1_{A_S} \otimes -$ preserves embeddings of left ideals into $S$

In the language of elements this means that if for $a, a' \in A_S$ and $s, t \in {}_SK$, where ${}_SK$ is a left ideal of $S$, $a \otimes s \leq a' \otimes t$ in the tensor product $A_S \otimes_S S$ then this inequality holds already in the tensor product $A_S \otimes_S K$.

**Lemma 5.1.** *An $S$-poset $A_S$ is weakly po-flat if and only if $as \leq a't$ for $a, a' \in A_S$, $s, t \in S$ implies $a \otimes s \leq a' \otimes t$ in the tensor product $A_S \otimes_S (Ss \cup St)$.*

*Proof.* Note that by Proposition 2.4 $a \otimes s \leq a' \otimes t$ in the tensor product $A_S \otimes_S S$ for $a, a' \in A_S$, $s, t \in S$ if and only if $as \leq a't$, and that $Ss \cup St$ is a left ideal of $S$. □

**Theorem 5.2 ([5] Theorem 3.12).** *A right $S$-poset $A_S$ is weakly po-flat if and only if it is principally weakly po-flat and satisfies Condition*
*(W) If $as \leq a't$ for $a, a' \in A_S$, $s, t \in S$ then there exist $a'' \in A_S$, $p \in Ss$, $q \in St$, such that $p \leq q$, $as \leq a''p$, $a''q \leq a't$.*

## 5.2 Weakly po-flat and weakly flat Rees factor S-posets

**Definition 11 ([1]).** Pomonoid $S$ is called ***weakly right reversible*** if $Ss \cap (St] \neq \emptyset$ for all $s, t \in S$.

**Proposition 5.3 ([1] Proposition 13).** *(Weakly po-flat) For any pomonoid $S$ and any convex, proper right ideal $K_S$, $S/K_S$ is weakly po-flat if, and only if,*

*(1) $S/K_S$ is principally weakly po-flat, and,*

*(2) $S$ is weakly right reversible.*

**Proposition 5.4 ([1] Proposition 14).** *(Weakly flat) For any pomonoid $S$ and any convex, proper right ideal $K_S$, $S/K_S$ is weakly flat if, and only if,*

*(1) $S/K_S$ is principally weakly flat, and,*

*(2) $S$ is weakly right reversible.*

# Osaliselt järjestatud polügoonide lamedusomadustest

## Nikita Salnikov-Tarnovski

## Resümee

Käesolevas töös püüab autor üldistada klassikalisi tulemusi erinevate lamedusomaduste kohta polügoonide jaoks uuele osaliselt järjestatud polügoonide (üle osaliselt järjestatud monoidide) juhule. Peamine eesmärk on esitada tulemused, mis annavad piisavaid ja tarvilikke tingimusi selleks, et antud osaliselt järjestatud polügoonil oleks konkreetne lamedusomadus. Suur osa tõestustest ja tulemustest on väga sarnased klassikalise juhuga, kuid samal ajal leiduvad ka märgatavad erinevused. Oluliselt erinevas situatsioonis on siin näiteks monotsüklilised osaliselt järjestatud polügoonid.

Sissejuhatusele järgnevas teises paragrahvis esitame definitsioonid ja abitulemused, mis on järgnevas vajalikud. Kolmandas paragrahvis käsitletakse väändeta ja po-väändeta osaliselt järjestatud polügoone. Neljas paragrahv on pühendatud nõrgale lamedusele ja nõrgale po-lamedusele. Detailsemalt käsitletakse vasakpoolsete osaliselt järjestatud PP-monoidide juhtu. Viimases paragrahvis on esitatud tulemused nõrgalt lamedate ja nõrgalt po-lamedate osaliselt järjestatud polügoonide kohta. Kõigil neil juhtudel püütakse esitada tsükliliste osaliselt järjestatud polügoonide, mitmesuguste monotsükliliste osaliselt järjestatud polügoonide ja Rees'i faktorpolügoonide jaoks tingimused, mil nad on vastava paragrahvis vaadeldavate omadustega.

Märgime, et lamedusomadusi on osaliselt järjestatud polügoonide situatsioonis võimalik defineerida kas klassikalise juhu definitsioone otseselt üle kandes või siis rangemalt järjestust arvestades (mõistete "po"-versioonid). Läbi kogu töö vaadeldakse neid üldistusi paralleelselt.

## References

[1] Bulman-Fleming, S.; Gutermuth, D.; Gilmour, A.; Kilp, M. *Flatness Properties of S-posets*, (to appear).

[2] Bulman-Fleming, S.; Laan, V. *Lazard's Theorem for S-posets* (to appear).

[3] Fakhruddin, S. M. *Absolute flatness and amalgams in pomonoids*, Semigroup Forum Vol. 33 (1986)

[4] Kilp, M.; Knauer, U.; Mikhalev, A. *Monoids, Acts and Categories.* Walter de Gruyter: Berlin, New York, 2000; 529 pp.

[5] Shi, X. *Strongly flat and po-flat S-posets*, Comm. Alg. (to appear).

[6] Shi, X.; Lui, Z.; Wang, F.; Bulman-Fleming, S. *Indecomposable, Projective and Flat S-Posets*, Comm. Alg. Vol. 33 Issue 1, 2005; p235