University of Tartu
Faculty of Science and Technology
Institute of Mathematics and Statistics

Nikita Leo

Plasticity of the unit ball of a Banach space

Mathematics

Bachelor's thesis (9 ECTS)

Supervisors:
Rainis Haller (University of Tartu)
Olesia Zavarzina (V. N. Karazin Kharkiv National University)

Tartu 2021

BANACHI RUUMI ÜHIKKERA PLASTILISUS

Bakalaureusetöö

Nikita Leo

Lühikokkuvõte

Selles bakalaureusetöös vaadeldakse väljakutsuvat lahtist küsimust Banachi ruumi ühikkera plastilisusest. Töös antakse ülevaade probleemist ja laiendatakse osaliste positiivsete tulemuste nimekirja, tõestades ühikkera plastilisuse ruumis c. Samuti saadakse üks nõrgem omadus ruumi c_0 ühikkera jaoks – tõestatakse, et mittelaiendav bijektsioon on isomeetria, kui selle pöördkujutus on pidev.

CERCS teaduseriala: P140 Jadad, Fourier analüüs, funktsionaalanalüüs.

Märksõnad: Funktsionaalanalüüs, Banachi ruumid, meetrilised ruumid, mittelineaarsed operaatorid, plastilisus.

PLASTICITY OF THE UNIT BALL OF A BANACH SPACE

Bachelor's thesis

Nikita Leo

Abstract

In this bachelor's thesis, we consider a challenging open problem of whether the unit ball of every Banach space is a plastic metric space. We give an overview of the problem and extend the list of partial positive results by proving the plasticity of the unit ball of c. We also obtain a slightly weaker property for the unit ball of c_0 – we prove that a non-expansive bijection is an isometry, provided that it has a continuous inverse.

CERCS research specialisation: P140 Series, Fourier analysis, functional analysis.

Keywords: Functional analysis, Banach spaces, metric spaces, non-linear operators, plasticity.

Contents

Introduction

A function $f\colon X \to Y$ between two metric spaces is called *non-expansive*, if $d\big(f(a), f(b)\big) \le d(a,b)$ for every a and b in X. If $d\big(f(a), f(b)\big) = d(a,b)$ for every a and b in X, then a function f is called an *isometry*. We call a metric space *plastic* if every non-expansive bijection from the space onto itself is an isometry. The notion was introduced in 2006 by S. A. Naimpally, Z. Piotrowski and E. J. Wingler [1]. The only general result about the class of plastic metric spaces is that every totally bounded (and hence every compact) space is plastic. Conversely, it is known that a plastic metric space need not be totally bounded nor bounded. It can also be shown that a bounded space need not be plastic.

It is an open question whether the unit ball of every Banach space is a plastic metric space. The unit ball of a finite-dimensional space is compact and therefore is plastic, since compactness implies plasticity. So the question is really just about the infinite-dimensional spaces. So far, the plasticity of the unit ball has been proved for the following classes of Banach spaces:

- strictly convex spaces
  (B. Cascales, V. Kadets, J. Orihuela and E. J. Wingler; 2016 [2]);
- ℓ_1-sums of strictly convex spaces
  (V. Kadets and O. Zavarzina; 2018 [3]);
- spaces whose unit sphere is the union of all its finite-dimensional polyhedral extreme subsets
  (C. Angosto, V. Kadets and O. Zavarzina; 2018 [4]).

In this thesis, we are going to extend this list by proving the plasticity of the unit ball of c. We are also going to establish a slightly weaker property for the unit ball of c_0 – we prove that a non-expansive bijection with a continuous inverse is an isometry.

The aim of the first section is to provide some intuition for the notion of plasticity. The second section is devoted to giving an overview of the problem and introducing the main tools used in its study. The last two sections are meant for presenting the two original results of this thesis.

1 Plastic metric spaces

In this section we are going to define a plastic metric space and provide some examples of plastic and non-plastic metric spaces.

Definition 1.1. Let X and Y be metric spaces. A function $f\colon X \to Y$ is called

- *non-expansive*, if for each a and b in X we have $d\big(f(a), f(b)\big) \leq d(a, b)$;
- *non-contractive*, if for each a and b in X we have $d\big(f(a), f(b)\big) \geq d(a, b)$;
- an *isometry*, if for each a and b in X we have $d\big(f(a), f(b)\big) = d(a, b)$.

Being non-expansive or non-contractive is a weaker condition than being an isometry. The question is, under which conditions does non-expansiveness or non-contractiveness imply being an isometry? We may try adding some conditions to the left-hand side and one possible choice for this is to consider injectivity, surjectivity and bijectivity. It gives rise to some properties of metric spaces like the property that every non-expansive bijection from the space onto itself is an isometry. There is a total of eight properties to consider. These properties are listed in the following table, where nE and nC stand for non-expansive and non-contractive:

1a)	nE $\implies$ isometry;	1b)	nC $\implies$ isometry;
2a)	nE, injection $\implies$ isometry;	2b)	nC, injection $\implies$ isometry;
3a)	nE, surjection $\implies$ isometry;	3b)	nC, surjection $\implies$ isometry;
4a)	nE, bijection $\implies$ isometry;	4b)	nC, bijection $\implies$ isometry.

The spaces with property 1a are exactly one-element spaces, so this property is not of any interest. Some of these properties are equivalent. Note that every non-contractive map is injective, so 1b is equivalent to 2b and 3b is equivalent to 4b. Also note that the inverse of a non-expansive bijection is a non-contractive bijection and vice versa. This makes 4a equivalent to 4b. If we exclude 1a, we are left with four properties:

$$\text{A – 2a, B – 3a, C – 1b/2b, D – 3b/4a/4b.}$$

Obviously, properties A, B and C each imply property D. It is not hard to find counterexamples for all the remaining conditionals, so the properties A, B, C and D are all different. We will focus on property D.

Definition 1.2. A metric space X is called *plastic* if every non-expansive bijection from X onto itself is an isometry.

The notion was introduced by S. A. Naimpally, Z. Piotrowski and E. J. Wingler in [1], where they study the properties of the class of plastic metric spaces. In this paper, the term *EC-plastic* was used, but we prefer a shorter version given above. One may also define plasticity using one of the two equivalent conditions mentioned before. It is known that every totally bounded metric space is plastic. Conversely, a plastic metric space need not be totally bounded nor bounded. There are also examples of bounded spaces that are not plastic. Overall, it seems that the class of plastic metric spaces does not have a good simple characterization.

We are going to start with proving the theorem establishing the plasticity of all totally bounded metric spaces. Recall that a metric space is called *totally bounded* if its every sequence has a Cauchy subsequence. A related notion is compactness. A metric space is called *compact* if its every sequence has a convergent subsequence. Compact spaces constitute a subclass of totally bounded spaces – a metric space is compact if and only if it is complete and totally bounded. The next theorem implies the plasticity of all totally bounded spaces. The theorem does not assume injectivity or surjectivity, so it actually asserts more than just plasticity. The theorem and its proof are extracted from [1].

Theorem 1.3 ([1], Theorem 1.1)**.** *Let X be a totally bounded metric space and $f\colon X \to X$ be a non-contractive map. Then f is an isometry.*

Proof. Let a and b be two arbitrary points of X. Our goal is to show that $d\big(f(a), f(b)\big) = d(a, b)$. Since f is non-contractive, we have $d\big(f(a), f(b)\big) \geq d(a, b)$, so it suffices to show that $d\big(f(a), f(b)\big) \leq d(a, b)$. Given a point $x \in X$ we can consider a sequence (x_n) where $x_1 = x$ and $x_{n+1} = f(x_n)$. Let us consider sequences (a_n) and (b_n) defined as above. It is time to make use of the condition that X is a totally bounded space. So it follows that both our sequences have Cauchy subsequences. But this is not enough, because we want these subsequences to have the same set of indices. This can be accomplished as follows. First, extract a Cauchy subsequnce from (a_n), then use the same indices to extract the corresponding subsequence of (b_n). The latter doesn't have to be Cauchy, but we know that it contains a Cauchy subsequence. Now, consider the corresponding subsequence of (a_n). This is a subsequence of the initial Cauchy subsequence extracted from (a_n). Therefore, the resulting subsequence of (a_n) is also Cauchy. It also has the same set of indices as the previously obtained Cauchy subsequence of (b_n).

So now we have an increasing sequence of indices (n_k) such that (a_{n_k}) and (b_{n_k}) are both Cauchy sequences. Let $\varepsilon > 0$ be arbitrary. The definition of Cauchy sequence implies that there exists an index K such that $d(a_{n_i}, a_{n_j}) < \varepsilon$ for all indices $i \geq K$ and $j \geq K$ and some another index K such that $d(b_{n_i}, b_{n_j}) < \varepsilon$ for all indices $i \geq K$ and $j \geq K$. Taking the maximum of these two, we obtain an index K such that the both conditions hold simultaneously. Now we see that there exists an index n and a positive integer p such that both $d(a_n, a_{n+p})$ and $d(b_n, b_{n+p})$ are smaller than ε. One may choose $n = n_i$ and $n + p = n_j$, where i and j are some indices with $K \leq i$ and $i < j$.

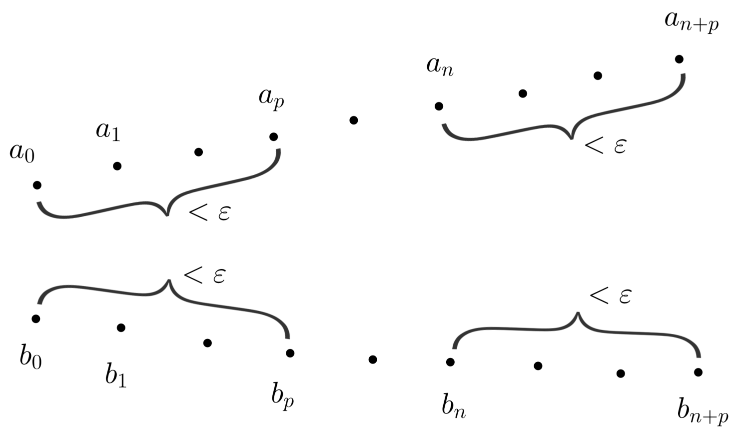


Figure 1: Illustration of the proof of Theorem 1.3

It is time to use the fact that f is non-contractive. First, we obtain the inequalities $d(a_0, a_p) \leq d(a_n, a_{n+p})$ and $d(b_0, b_p) \leq d(b_n, b_{n+p})$. As $d(a_n, a_{n+p})$ and $d(b_n, b_{n+p})$ are smaller than ε, it follows that $d(a_0, a_p)$ and $d(b_0, b_p)$ are too (see Figure 1 above). We also obtain the inequality $d(a_1, b_1) \leq d(a_p, b_p)$. Now we can give an upper estimate to $d\big(f(a), f(b)\big)$ by using the triangle inequality:

$$\begin{aligned} d\big(f(a), f(b)\big) = d(a_1, b_1) \leq d(a_p, b_p) &\leq d(a_p, a_0) + d(a_0, b_0) + d(b_0, b_p) \\ &< d(a_0, b_0) + 2\varepsilon = d(a, b) + 2\varepsilon. \end{aligned}$$

Since ε was chosen arbitrarily, it follows that $d\big(f(a), f(b)\big) \leq d(a, b)$, which completes the proof. □

We finish this section with a couple of examples. Plastic spaces tend to be "small", which seems to be supported by the last theorem, but there are exceptions – we are going to see that a plastic metric space need not be totally bounded nor bounded. One may also wonder whether all "small" spaces are plastic, but there are also exceptions to this – we will see an example of a space that is bounded, but not plastic.

The first example is of a plastic space that is not bounded. This example is taken from [1], but a slightly more intuitive proof is provided.

Example 1.4. The set of integers with the standard absolute value metric is a plastic metric space, that is not bounded. Obviously, the space is not bounded. Let us show that the space is plastic. Consider an arbitrary non-expansive injection $f\colon \mathbb{Z} \to \mathbb{Z}$. Let us show that f is an isometry. Denote $f(0)$ by a. Note that $d(0,1) = 1$, so $d\big(f(0), f(1)\big) \leq 1$, as f is non-expansive. This means that $f(1)$ should be $a-1$, a or $a+1$. However, the injectivity of f excludes the case $f(1) = a$, so we should have either $f(1) = a-1$ or $f(1) = a+1$. If $f(1) = a+1$, then we can use induction to prove that $f(x) = x+a$ for each $x \in \mathbb{Z}$. Similarly, if $f(1) = a-1$, then we can use induction to show that $f(x) = -x+a$ for every $x \in \mathbb{Z}$. In both cases, f turns out to be an isometry. This proves that every non-expansive injection is an isometry, hence $\mathbb{Z}$ is plastic.

Note that in the last proof, we do not assume the surjectivity of f, so here we again have a property stronger than plasticity. The next example is of a space that is bounded, but not plastic. We are going to omit the details, because we only want to present the main idea. This example is a generalization of [2], Example 2.7.

Example 1.5. Consider a Banach space $H = \ell_p(\mathbb{Z})$, where $p \in (1,\infty)$. Let $\alpha\colon \mathbb{Z} \to \mathbb{R}$ be a non-increasing sequence of positive numbers and suppose that there exist positive numbers M and N such that $M \leq \alpha_k \leq N$ for every $k \in \mathbb{Z}$. Let us also assume that there exists at least one index $k \in \mathbb{Z}$ with $\alpha_{k+1} < \alpha_k$. Consider a subset of H defined by

$$A = \Big\{ x \in H \colon \sum_{k\in\mathbb{Z}} \Big| \frac{x_k}{\alpha_k} \Big|^p \leq 1 \Big\}.$$

The set A is bounded, convex, closed and symmetric. Consider an operator $T\colon H \to H$ defined by $T(x)_k = x_{k-1}\alpha_k/\alpha_{k-1}$. The restriction of T to A is a non-expansive bijection from A onto itself, that is not an isometry. Therefore, A is a bounded metric space that is not plastic.

2 Plasticity of the unit ball

It is an open question whether the unit ball of every Banach space is a plastic metric space. The unit ball of a finite-dimensional space is compact and therefore is also plastic by the theorem proved in the previous section. So the question is really just about the infinite-dimensional spaces. So far, the plasticity of the unit ball has been proved for the following classes of Banach spaces:

- strictly convex spaces;
- ℓ_1-sums of strictly convex spaces;
- spaces whose unit sphere is the union of all its finite-dimensional polyhedral extreme subsets.

The first positive result was for strictly convex spaces and it was presented by B. Cascales, V. Kadets, J. Orihuela and E. J. Wingler in 2016 [2]. In the same article, the question about plasticity of the unit ball was first posed. The second positive result was initially for just ℓ_1 and it was presented by V. Kadets and O. Zavarzina in the same year [5]. This result was generalized to ℓ_1-sums of strictly convex spaces by V. Kadets and O. Zavarzina in 2018 [3]. Finally, the same year a positive result for the third class was obtained by C. Angosto, V. Kadets and O. Zavarzina [4]. The class of strictly convex spaces is a subclass of the other two classes, so the second and the third result might be seen as generalizations of the first.

The unit ball seems to be a set that is likely to be plastic. However, a Banach space can contain subsets that are not plastic, but yet very similar to the unit ball. One example of such a subset is given in Example 1.5 of the previous section. The subset considered in this example can be thought of as an infinite-dimensional ellipsoid. This is indeed very similar to the unit ball. In particular, both subsets are convex, closed, bounded and symmetric.

Extreme points play a significant role in the problem of plasticity of the unit ball. Conversely, there is not much we can do without extreme points. As a consequence, all the positive results obtained so far are for spaces that have many extreme points. The first positive result was for strictly convex spaces and these are spaces where all the points of the unit sphere are extreme. The later extensions allow for some non-extreme points, but they still require the space to have many extreme points. On the other hand, nothing is known about spaces with little or no extreme points at all. It is natural to suppose some of these spaces may actually not have a plastic unit ball. This motivates us to study plasticity of the unit ball in spaces with little or no extreme points. A positive result for any such space would be also significant – it would show that plasticity of the unit ball does not require the space to have many extreme points. The space c_0 is one such space, as it has no extreme points. This is also one of the two spaces considered in this thesis.

The plasticity of the unit ball is known for all ℓ_p sequence spaces except ℓ_∞ – there is a positive result for ℓ_1, while ℓ_p with p in $(1, \infty)$ is strictly convex and there is a positive result for strictly convex spaces. The difficulty of ℓ_∞ seems to be lying in the fact that this space is too big. In particular, it is not separable, while the spaces ℓ_p with $p < \infty$ are. However, ℓ_∞ contains some common separable spaces like c and c_0. While the plasticity of the unit ball of ℓ_∞ seems to be a hard problem to tackle, spaces c and c_0 may be worth a try. Spaces ℓ_∞ and c can be said to have many extreme points. In contrast, the space c_0 is special – it has no extreme points at all.

There is also a more general problem related to the problem at hand. Given two Banach spaces X and Y and a non-expansive bijection $F\colon B_X \to B_Y$, is it true that F is an isometry? As one can see, this is the same as plasticity of the unit ball, except that now we have two spaces. The problem was posed by O. Zavarzina in 2018 [6]. This article contains a positive result for pairs of spaces where Y is finite-dimensional, strictly convex or ℓ_1. The same year, two another positive results were obtained: [3] gives a positive answer for pairs of spaces where Y is ℓ_1-sum of strictly convex spaces and [4] provides a positive result for pairs where Y is a space whose unit sphere is the union of all its finite-dimensional polyhedral extreme subsets. Additionally, it is easy to see that if X is finite-dimensional, then Y is also finite-dimensional. It is also easy to see that if X is strictly convex, then Y is also strictly convex (see [4], Theorem 4.2). This yields a positive answer for the case where X is finite-dimensional or strictly convex.

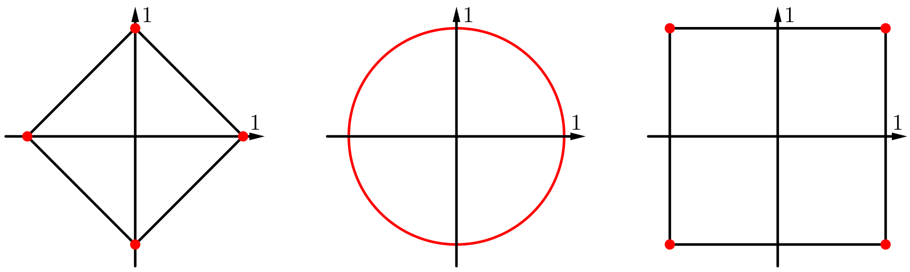
Usually, we are not interested in non-complete spaces. However, it is hard to say whether completeness plays a role in the problem at hand, so it might be reasonable to consider non-complete spaces as well. To the best of our knowledge, there is no known example of a non-complete space that doesn't have a plastic unit ball. Nonetheless, it might occur that Banach spaces have a plastic unit ball, but there is a non-complete space, the unit ball of which is not plastic. In connection to this, there are two questions to consider: if X is a non-complete normed space and $\overline{X}$ is its completion, then does X having a plastic unit ball imply that $\overline{X}$ has a plastic unit ball and does $\overline{X}$ having a plastic unit ball imply that X has a plastic unit ball?

In this section we are going to introduce the main tools used in the study of plasticity of the unit ball. The next two sections will be devoted to proving the two main results of this thesis. The two spaces considered in this thesis are c_0 and c. We prove the plasticity of the unit ball of c. For c_0, we will not be able to prove the plasticity of the unit ball, but we show that a non-expansive bijection having a continuous inverse is an isometry.

As mentioned above, extreme points play an important role in the context of plasticity of the unit ball. So, we are going to start with introducing the concept of an extreme point. If u and v are points of a normed space, the line segment connecting these two points is a set of all points of form $(1-\lambda)u+\lambda v$, where λ is a real number in $[0, 1]$. Sometimes, the word "non-trivial" may be added to stress that u and v should be distinct. The line segment connecting points u and v is denoted by $[u, v]$. By the interior of a line segment we mean the line segment with the endpoints excluded (this is different from the usual notion of the interior of a set). The notion of an extreme point can be defined for an arbitrary convex subset of a normed space, but we will focus on extreme points of the unit ball, referring to them as simply "extreme".

Definition 2.1. A point of the unit sphere of a normed space is called *extreme* if it does not belong to the interior of any non-trivial line segment contained in the unit sphere.

It is obvious that the set of extreme points is symmetric (x is an extreme point if and only if $-x$ is an extreme point). Below is a diagram showing the unit balls of $\mathbb{R}^2$ with respect to three different norms. The extreme points are highlighted.



$\|(x, y)\| = |x| + |y|$ $\|(x, y)\| = \sqrt{x^2 + y^2}$ $\|(x, y)\| = \max\{|x|, |y|\}$

Figure 2: Extreme points of the unit ball

By definition, extreme points are part of the unit sphere, but a point of the unit sphere need not be an extreme point. A normed space is called *strictly convex* if every point of its unit sphere is an extreme point. Equivalently, a strictly convex space is a space the unit ball of which does not contain any non-trivial line segments.

An extreme point may be equivalently defined as a point of the unit sphere, that is not a middle point of any non-trivial line segment contained in the unit sphere. Indeed, a middle point of a non-trivial line segment is obviously an element of its interior. Conversely, if a point belongs to the interior of some non-trivial line segment, then it is also a middle point of some another non-trivial line segment, which is contained in the first segment. It is also easy to see that if a point of the unit sphere belongs to the interior of a line segment connecting two distinct points of the unit sphere, then the whole segment should be actually contained in the unit sphere. That is, to show that a point of the unit sphere is not extreme, it suffices to show that this point is contained in the interior of a line segment connecting two distinct points of the unit sphere.

There is one useful property of extreme points which is important to emphasize. Let X be a normed space and a be a point of X. Let δ be a positive real number and e be an element of S_X. Consider points a and $b = a + \delta e$. Let α and β be positive numbers such that $\alpha + \beta = \delta$. Consider the set D of all points of X that are distance α from a and distance β from b. Obviously, the line segment connecting the points a and b contains a point with this property – this is the point $\frac{\beta}{\delta}a + \frac{\alpha}{\delta}b$. It is also clear that it does not contain any other points with this property. The question is, are there any other points with this property outside of the line segment connecting a and b? In other words, when is the set D one-element? The symmetry of a normed space should make it obvious that the situation depends exclusively on the direction of e and does not depend on a and δ. For $x \in S_X$ and $\lambda \in (0, 1)$ denote by $D(x, \lambda)$ the set of all points that are distance λ from 0 and distance $1 - \lambda$ from x. It is easy to see that $D = a + \delta D(e, \frac{\alpha}{\delta})$, so D is one-element if and only if $D(e, \frac{\alpha}{\delta})$ is one-element. This means that it should suffice to consider the case where $a = 0$ and $\delta = 1$. It turns out that the following equivalence is true – the set D is a one-element set if and only if e is an extreme point. To prove this equivalence, it suffices to prove that for each $\lambda \in (0, 1)$ and each $x \in S_X$ the set $D(x, \lambda)$ is one-element if and only if x is an extreme point. Let us state this as a proposition. This proposition is a slightly modified version of Lemma 2.1 of [2].

Proposition 2.2 ([2], Lemma 2.1)**.** *Let X be a normed space, $x \in S_X$ and $\lambda \in (0,1)$. Denote by $D(x,\lambda)$ the set $\{z \in X : \|z\| = \lambda,\ \|x - z\| = 1 - \lambda\}$. The set $D(x,\lambda)$ is one-element if and only if x is an extreme point.*

Proof. Since $\lambda x \in D(x,\lambda)$, then the condition that $D(x,\lambda)$ is a one-element set is equivalent to the condition that the set $D(x,\lambda)$ does not contain a point distinct from λx. We prove the statement by proving the two following claims:

- if $D(x,\lambda)$ contains an element distinct from λx, then x is not extreme;
- if x is not extreme, then $D(x,\lambda)$ contains an element distinct from λx.

We begin with giving the set $D(x,\lambda)$ the following description:

$$D(x,\lambda) = \left\{ z \in X : x + \frac{z - \lambda x}{\lambda} \in S_X,\ x - \frac{z - \lambda x}{1 - \lambda} \in S_X \right\}.$$

This is easy to verify. The first condition says that $\|x + \frac{z-\lambda x}{\lambda}\| = 1$. Multiplying both sides by λ, we get $\|z\| = \lambda$. The second condition says that $\|x - \frac{z-\lambda x}{1-\lambda}\| = 1$. Multiplying both sides by $1-\lambda$, we get $\|x - z\| = 1 - \lambda$. This means that the first condition is equivalent to $\|z\| = \lambda$ and the second condition is equivalent to $\|x - z\| = 1 - \lambda$. This proves the equality. Now it is easy to see that if z is a point of $D(x,\lambda)$ which is different from λx, then x belongs to the interior of a line segment connecting two distinct points $x + \frac{z-\lambda x}{\lambda}$ and $x - \frac{z-\lambda x}{1-\lambda}$, which both belong to the unit sphere, so x is not an extreme point. That is, if the set $D(x,\lambda)$ contains any element distinct from λx, then x is not extreme. This proves one of the two directions. For the second direction, suppose that x is not an extreme point. The description of $D(x,\lambda)$ given above can be rewritten the following way:

$$D(x,\lambda) = \left\{ \lambda x + w : x + \frac{w}{\lambda} \in S_X,\ x - \frac{w}{1 - \lambda} \in S_X \right\}.$$

The idea is to find a non-zero w which satisfies the two conditions given above. As x is not an extreme point, there exists $a \in X \setminus \{0\}$ such that the line segment $[x - a, x + a]$ is contained in S_X. Define $w = \min\{\lambda, 1 - \lambda\}a$, then $x + \frac{w}{\lambda}$ and $x - \frac{w}{1-\lambda}$ belong to $[x - a, x + a]$ and hence to S_X. Now we see that $z = \lambda x + w$ is an element of $D(x,\lambda)$ which is distinct from λx. This proves the second direction. □

Let us illustrate the last proposition by the following diagrams (see Figure 3 below). When drawing a picture, it is useful to think of the set $D(x,\lambda)$ as the intersection of the balls $B(0,\lambda)$ and $B(x, 1-\lambda)$. The following three diagrams depict the space $\mathbb{R}^2$ equipped with the maximum-norm. The bigger square is the unit ball. On the unit sphere, a point x is chosen. The two smaller squares are the balls $B(0, 3/5)$ and $B(x, 2/5)$. Their intersection is the set $D(x, 3/5)$.

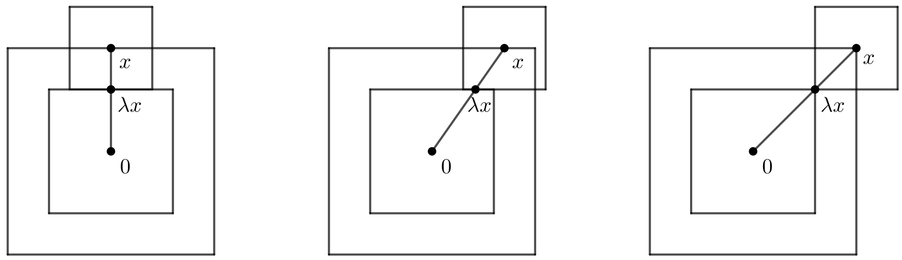


Figure 3: Illustration of Proposition 2.2

We can make some conclusions from the proposition that we have just proved. First, as we mentioned above, the situation depends on the direction, but not the position, so the latter proposition can be generalized by the following corollary.

Corollary 2.3. *Let X be a normed space. Let a be a point of X and e an element of S_X. Let δ be a positive real number. Consider points a and $b = a + \delta e$. Let α and β be two positive real numbers such that $\alpha + \beta = \delta$. Denote by D the set of all points which are distance α from a and distance β from b. The set D is one-element if and only if e is an extreme point.*

Proof. It is straightforward to verify that $D = a + \delta D(e, \frac{\alpha}{\delta})$. This shows that D is one-element if and only if $D(e, \frac{\alpha}{\delta})$ is one-element. By the last proposition, the latter is equivalent to the fact that e is an extreme point. $\square$

Note that if X is a strictly convex space, then the set D consists always of just one element. The next corollary is just an application of Corollary 2.3 to a specific situation.

Corollary 2.4. *Let X be a normed space and let x be a point of S_X. Consider the set D of all points that are distance one from x and distance one from $-x$. The set D is one-element if and only if x is an extreme point.*

The main aim of this section is to introduce the tools used in the study of plasticity of the unit ball. Let us start with the following easy observations.

Proposition 2.5 ([2], Theorem 2.3)**.** *Let X and Y be normed spaces and let $F\colon B_X \to B_Y$ be a non-expansive bijection. The following claims hold true:*

1) $F(0_X) = 0_Y$*;*

2) $\|F(x)\| \leq \|x\|$*;*

3) if $y \in S_Y$*, then* $F^{-1}(y) \in S_X$*.*

Proof. Let us start with proving the first item. The idea of the proof is that zero is a unique point of the unit ball such that all the other points of the unit ball are within distance one from it. We are going to use a proof by contradiction. Denote $F(0_X)$ by y and suppose by contrary that $y \neq 0_Y$. In this case, there exists a point $y' \in B_Y$ such that $\|y' - y\| > 1$. An obvious choice for such a point is $-\frac{y}{\|y\|}$. For this point, the distance $\|y' - y\|$ is equal to $1 + \|y\|$, which is clearly greater than one. Denote $F^{-1}(y')$ by x'. Note that the distance from x' to 0_X is at most one, but the distance from $F(x')$ to $F(0_X)$ is greater than one. This contradicts the fact that F is non-expansive, which proves the first item. The second and the third item follow easily from the first one. The norm of an element is the distance of this element from zero, so item 2) follows directly from item 1) and the non-expansiveness of F. Item 3) is a straightforward consequence of item 2). □

When dealing with plasticity of the unit ball, we are normally considering a non-expansive bijection from the unit ball of some space onto itself. However, the last proposition was stated in a more general setting of a non-expansive bijection between unit balls of two normed spaces. This is motivated by a more general problem which was mentioned at the beginning of this section. We are going to follow this line in the next propositions.

When we consider a non-expansive map between two metric spaces and it is known to preserve the distance between some two points, then we can make some useful conclusions on the behaviour of nearby points. Let us illustrate this. Consider a non-expansive map $F\colon M \to N$ between two metric spaces M and N. Let x and y be two distinct elements of M and let α and β be positive real numbers such that $\alpha+\beta = d(x, y)$. Let z be a point of M which is distance α from x and distance β from y. Non-expansiveness implies $d\big(F(x), F(z)\big) \leq \alpha$ and $d\big(F(z), F(y)\big) \leq \beta$. However, if $d\big(F(x), F(y)\big) = d(x, y)$, then $F(z)$ will be also distance α from $F(x)$ and distance β from $F(y)$, because otherwise we would have $d\big(F(x), F(z)\big)+d\big(F(z), F(y)\big) < d\big(F(x), F(y)\big)$, which contradicts the triangle inequality. To sum up, if F preserves the distance between points x and y and a point z is such that $d(x, z) + d(z, y) = d(x, y)$, then F preserves the distance between points x and z and points z and y. This simple fact is often used in the context of plasticity. This is also used in the proof of the following proposition.

We have mentioned that extreme points play a significant role in the problem of plasticity of the unit ball. The following proposition is the reason for this.

Proposition 2.6 ([2], Theorem 2.3)**.** *Let X and Y be normed spaces and let $F\colon B_X \to B_Y$ be a non-expansive bijection. If y is an extreme point of B_Y, then $F^{-1}(y)$ is an extreme point of B_X and $F^{-1}(\alpha y) = \alpha F^{-1}(y)$ for each $\alpha \in [-1, 1]$.*

Proof. Denote $F^{-1}(y)$ by x. Let us start with proving that x is an extreme point. We know that $y \in S_Y$, so $x \in S_X$ by Proposition 2.5. Note that F preserves the distance between 0_X and x. This allows us to use the idea presented just above this proposition. Take an arbitrary α from $(0, 1)$. Consider the set $D(x, \alpha)$ of all points of X that are distance α from zero and distance $1-\alpha$ from x. Regardless of x, this set contains at least one point, which is αx. Analogously, consider the set $D(y, \alpha)$ of all points of Y that are distance α from zero and distance $1 - \alpha$ from y. Again, regardless of y, this set contains at least one point, which is αy. The fact that F is non-expansive implies that $F(D(x, \alpha)) \subset D(y, \alpha)$. Now, as y is an extreme point, then Proposition 2.2 implies that $D(y, \alpha)$ is actually one-element – its only element is αy. The injectivity of F together with the last inclusion imply that $D(x, \alpha)$ is also one-element – its only element should be αx. Applying Proposition 2.2 again, we get that x is an extreme point.

Now, let us show that $F^{-1}(\alpha y) = \alpha x$ for every $\alpha \in (0, 1)$. Let $\alpha \in (0, 1)$ be arbitrary. Note that the last equation is equivalent to $F(\alpha x) = \alpha y$. The point αx belongs to the set $D(x, \alpha)$. We know that $F(D(x, \alpha)) \subset D(y, \alpha)$, so $F(\alpha x)$ belongs to $D(y, \alpha)$. As y is an extreme point, then the only element of $D(y, \alpha)$ is αy. Therefore, $F(\alpha x)$ should be equal to αy.

Let us show that $F^{-1}(-y) = -x$. Denote $F^{-1}(-y)$ by x'. Note that the distance between y and $-y$ is equal to two, so $x = F^{-1}(y)$ and $x' = F^{-1}(-y)$ should be also distance two apart. Consider a point $m = \frac{x+x'}{2}$. This point is distance one from x and distance one from x'. Note that F preserves the distance between x and x'. It follows that $F(m)$ is distance one from y and distance one from $-y$. As y is an extreme point, then Corollary 2.4 implies that $F(m) = 0_Y$. As $F(0_X) = 0_Y$ by Proposition 2.5, then the injectivity of F implies that $m = 0_X$. The latter yields $x' = -x$, so we are done.

The equation $F^{-1}(\alpha y) = \alpha F^{-1}(y)$ is now proved for $\alpha \in (0, 1)$ and $\alpha = -1$. This is also obvious for $\alpha = 1$. For $\alpha = 0$, the equation follows from the first item of Proposition 2.5. It is left to show that $F^{-1}(\alpha y) = \alpha F^{-1}(y)$ for every $\alpha \in (-1, 0)$. This is equivalent to showing that $F^{-1}(-\alpha y) = -\alpha F^{-1}(y)$ for every $\alpha \in (0, 1)$. Using the fact that $F^{-1}(-y)$ is equal to $-F^{-1}(y)$, the last equation can be rewritten as $F^{-1}(\alpha(-y)) = \alpha F^{-1}(-y)$. Since y is an extreme point, then $-y$ is also an extreme point, so the latter equation can be proved by applying the above argument to $-y$ instead of y. □

The properties described in the two last propositions were observed in [2], the first article on plasticity of the unit ball, where the positive result for strictly convex spaces was obtained. These properties were initially described for the case of a non-expansive bijection from the unit ball of a normed space to itself, but the generalization to two spaces is straightforward.

When dealing with plasticity of the unit ball, it is also useful to know the following result by P. Mankiewicz. In 1932, S. Mazur and S. Ulam proved that every isometric bijection from a normed space X onto a normed space Y is affine. Recall that a map $f\colon X \to Y$ is called *affine* if there exists a linear operator $A\colon X \to Y$ and an element $y_0 \in Y$ such that f can be given by $f(x) = A(x) + y_0$. In a sense, affine map is a "shifted" linear map.

One can consider a more general problem. Let U be a subset of X and V be a subset of Y and let $T\colon U \to V$ be an isometric bijection. Is it true, that T extends to an affine isometric bijection $\widetilde{T}\colon X \to Y$? It is easy to verify that this is not true in general. Obviously, one should add some assumptions on U and V.

This problem was considered by P. Mankiewicz. In 1972, he was able to give two sufficient conditions [7]. One of them is given by the following theorem.

Theorem 2.7 ([7], Theorem 5)**.** *Let X and Y be normed spaces. Let $U \subset X$ and $V \subset Y$ be convex with non-empty interior and let $T\colon U \to V$ be an isometric bijection. Then T extends to an affine isometric bijection $\widetilde{T}\colon X \to Y$.*

This theorem is useful in the context at hand. Let U be the unit ball of X and V be the unit ball of Y. Note that the unit ball is convex and has a non-empty interior. So the theorem implies that every isometric bijection T from U onto V extends to an affine isometric bijection $\widetilde{T}$ from X onto Y. The first item of Proposition 2.5 implies that T sends zero to zero, so the same can be said for the extension of T. This means that $\widetilde{T}$ is actually linear. That is, the theorem implies the following: if $F\colon B_X \to B_Y$ is an isometric bijection, then F extends to an isometric isomorphism between two spaces – a linear isometric bijection from X onto Y. In case $X = Y$, the result will be an isometric automorphism of X – a linear isometric bijection from X onto itself.

This last fact can be useful when proving plasticity of the unit ball, because at some point of the proof one might discover that the unit ball of some finite-dimensional subspace is mapped bijectively onto itself or a copy of itself. In this situation one can apply the fact that the unit ball of a finite-dimensional space is plastic and then apply the mentioned result to conclude the linearity, which can be useful in the future.

The last theorem is also useful in one other way. If we want to prove that every non-expansive bijection from the unit ball of some space onto itself is an isometry, then it might be useful to know what bijective isometries from the unit ball of this space onto itself are there. The last result says that these are precisely the restrictions of isometric automorphisms of this space. One possible approach to proving plasticity of the unit ball in some specific space X consists of considering a non-expansive bijection $F\colon B_X \to B_X$, extracting some information about this function to choose an isometric automorphism that this function seems to resemble and then proving that the two functions actually coincide. Of course, this approach requires that we know a characterization of the isometric automorphisms of X.

There is also a sufficient condition for a non-expansive bijection $F\colon B_X \to B_Y$ to be an isometry. This is given by the following theorem, which is taken from [2]. In this article, the case $X = Y$ is considered, but the generalization to two spaces is straightforward. The proof can be found in [2] and is based on smooth points and the differentiation of the norm.

Theorem 2.8 ([2], Lemma 2.5)**.** *Let X and Y be normed spaces and let $F\colon B_X \to B_Y$ be a non-expansive bijection. If $F(S_X) = S_Y$ and $F(\alpha x) = \alpha F(x)$ for every $x \in S_X$ and every $\alpha \in [-1, 1]$, then F is an isometry.*

So, the sufficient condition is that F is homogeneous and maps S_X bijectively onto S_Y. Below is a slightly more general result, which is actually a straightforward generalization of the last theorem. This is extracted from [6].

Theorem 2.9 ([6], Lemma 2.3)**.** *Let X and Y be normed spaces and let $F\colon B_X \to B_Y$ be a non-expansive bijection such that $F(S_X) = S_Y$. Let V be a subset of S_X which consists of all those $x \in S_X$ such that $F(\alpha x) = \alpha F(x)$ for every $\alpha \in [-1, 1]$. Then F keeps distances between elements of the set $A = \{\alpha x\colon \alpha \in [-1, 1],\ x \in V\}$. That is, the restriction $F|_A$ is an isometry.*

The sufficient condition given by Theorem 2.8 may be also equivalently restated as follows. Let us state this as a corollary (extracted from [3]).

Corollary 2.10 ([3], Lemma 2.9)**.** *Let X and Y be normed spaces and let $F\colon B_X \to B_Y$ be a non-expansive bijection. If $F^{-1}(\alpha y) = \alpha F^{-1}(y)$ for every $y \in S_Y$ and every $\alpha \in [-1, 1]$, then F is an isometry.*

It is easy to see how the last corollary implies the plasticity of the unit ball for strictly convex spaces. Indeed, if X is a strictly convex space and $F\colon B_X \to B_X$ is a non-expansive bijection, then Proposition 2.6 implies that $F^{-1}(\alpha x) = \alpha F^{-1}(x)$ for every $x \in S_X$, because the condition that X is strictly convex means that every point of S_X is extreme. More generally, the last corollary implies that if X and Y are normed spaces and Y is strictly convex, then every non-expansive bijection from B_X onto B_Y is an isometry.

3 Plasticity of the unit ball of c_0

Let us fix the notation for the following two sections. Given a sequence $x \in c$, denote the n-th element by x_n. Given a sequence $\xi \in c^{\mathbb{N}}$, denote the k-th sequence by ξ^k and the n-th element of the k-th sequence by ξ^k_n. For $n \in \mathbb{N}$ denote by e^n a sequence such that $e^n_n = 1$ and $e^n_i = 0$ for every $i \in \mathbb{N} \setminus \{n\}$. Denote the unit ball of c by B. For $h \in [-1, 1]$ denote by B_h the subset of B which consists of all sequences converging to h. In particular, B_0 is going to stand for the unit ball of c_0. For a subset $S \subset \mathbb{N}$ and $h \in [-1, 1]$ denote by B^S_h the subset of B_h defined by

$$B^S_h = \{x \in B_h : x_n = h \text{ for all } n \in \mathbb{N} \setminus S\}.$$

For $h \in [-1, 1]$ denote by B^*_h the subset of B_h defined by

$$B^*_h = \{x \in B_h \colon x_n \neq h \text{ for finitely many } n \in \mathbb{N}\}.$$

For $x \in c$ and $r > 0$ denote by $B(x, r)$ the corresponding closed ball of the space c.

Let us consider an arbitrary non-expansive bijection $F \colon B_0 \to B_0$ from the unit ball of c_0 onto itself. We are going to try to infer as much as possible about the behaviour of this function. The maximum goal is to show that F is an isometry, but we will not be able to achieve this. However, we will show that F is an isometry, provided that F^{-1} is continuous.

One possible approach to proving plasticity of the unit ball in some specific space X consists of considering a non-expansive bijection $F \colon B_X \to B_X$, extracting some information about this function to choose an isometric automorphism that this function seems to resemble and then proving that the two functions actually coincide. Of course, this approach requires that we know a characterization of the isometric automorphisms of X. We are going to apply this approach in the proof at hand. The isometric automorphisms of c_0 have the form $\mathcal{A}(x)_{\sigma_n} = \alpha_n x_n$, where $\sigma \colon \mathbb{N} \to \mathbb{N}$ is a bijection and α is a sequence of ones and minus ones.

So, the first step is to extract some information about F to choose an isometric automorphism of c_0 that the function F seems to resemble. First, we need the following lemma, which is concerned with covering the unit ball by two closed balls of radius one.

Lemma 3.1. *Let x and y be two non-zero elements of B_0. The balls $B(x, 1)$ and $B(y, 1)$ cover the ball B_0 if and only if there exists an index n such that $x_i = y_i = 0$ for all $i \neq n$ and either x_n is positive and y_n is negative or x_n is negative and y_n is positive.*

Proof. Suppose that there exists an index n such that $x_i = y_i = 0$ for all $i \neq n$ and either x_n is positive and y_n is negative or x_n is negative and y_n is positive. Consider the case where x_n is positive and y_n is negative. Let z be an arbitrary element of B_0. We see that if $z_n \geq 0$, then $z \in B(x,1)$, and if $z_n \leq 0$, then $z \in B(y,1)$. That is, z is always contained in at least one of the balls $B(x,1)$ and $B(y,1)$. This means that the balls $B(x,1)$ and $B(y,1)$ cover the ball B_0. The second case is analogous. This proves one of the two directions.

For the second direction, assume that B_0 is covered by $B(x,1)$ and $B(y,1)$. Since x is non-zero, there exists an index n such that $x_n \neq 0$. Suppose that there exists an index $i \neq n$ such that $y_i \neq 0$. Consider a sequence $z = -\operatorname{sgn}(x_n)e^n - \operatorname{sgn}(y_i)e^i$. Note that $z \in B_0$, but z is not covered by $B(x,1)$ and $B(y,1)$. This contradicts our assumption, so we can conclude that $y_i = 0$ for every $i \neq n$. If y_n was also equal to zero, then y would be equal to zero, therefore $y_n \neq 0$. Now we can repeat the above argument swapping the roles of x and y. As a result, we get that $x_i = 0$ for every $i \neq n$. Finally, we have to exclude the possibility that x_n and y_n are both positive or both negative. This is easy to see, because if x_n and y_n are both positive, then $B(x,1)$ and $B(y,1)$ do not cover $-e^n \in B_0$, and if x_n and y_n are both negative, then $B(x,1)$ and $B(y,1)$ do not cover $e^n \in B_0$. This proves the second direction. □

Now we can extract some information about F to choose a corresponding isometric automorphism of c_0.

Lemma 3.2. *There exists a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ and a sequence $\alpha\colon \mathbb{N} \to \{-1,1\}$ such that for every $x \in B_0$ and $n \in \mathbb{N}$ we have the following:*

1) if $x_n = 0$, then $F(x)_{\sigma_n} = 0$;

2) if $x_n < 0$ and $\alpha_n = 1$ ($\alpha_n = -1$), then $F(x)_{\sigma_n} \leq 0$ ($F(x)_{\sigma_n} \geq 0$);

3) if $x_n > 0$ and $\alpha_n = 1$ ($\alpha_n = -1$), then $F(x)_{\sigma_n} \geq 0$ ($F(x)_{\sigma_n} \leq 0$).

Proof. Let $n \in \mathbb{N}$ be arbitrary. Consider elements e^n and $-e^n$. Note that B_0 is covered by the balls $B(e^n,1)$ and $B(-e^n,1)$. As F is non-expansive and surjective, then B_0 should be also covered by the balls $B\big(F(e^n),1\big)$ and $B\big(F(-e^n),1\big)$. Moreover, item 1) of Theorem 2.5 implies that $F(e^n)$ and $F(-e^n)$ are non-zero. The previous lemma implies that there exists an index k such that $F(e^n)_i = F(-e^n)_i = 0$ for all $i \neq k$ and either $F(e^n)_k$ is positive and $F(-e^n)_k$ is negative or $F(e^n)_k$ is negative and $F(-e^n)_k$ is positive. Define $\sigma_n = k$. Define $\alpha_n = 1$ if $F(e^n)_k$ is positive and $\alpha_n = -1$ otherwise.

Now, we can show that if a sequence $x \in B_0$ is such that $x_i = 0$ for every $i \neq n$, then $F(x)_i = 0$ for all $i \neq k$. Let $x \in B_0$ be such that $x_i = 0$ for every $i \neq n$. If $x_n = 0$, then x is zero, so $F(x)$ is also zero by item 1) of Theorem 2.5 and it is true that $F(x)_i = 0$ for all $i \neq k$. So let us consider the case where $x_n \neq 0$. Suppose that $x_n > 0$. Note that B_0 is covered by the balls $B(-e^n, 1)$ and $B(x, 1)$. As F is non-expansive and surjective, then B_0 should be also covered by the balls $B\big(F(-e^n), 1\big)$ and $B\big(F(x), 1\big)$. Item 1) of Theorem 2.5 implies that $F(-e^n)$ and $F(x)$ are non-zero, so the previous lemma implies that $F(x)_i = 0$ for every $i \neq k$. The case $x_n < 0$ is analogous.

Consider a function $f_n\colon [-1, 1] \to [-1, 1]$ defined by $f_n(t) = F(te^n)_{\sigma_n}$. The non-expansiveness and the injectivity of F imply the non-expansiveness and the injectivity of f_n. Since f_n is continuous and injective, then it is either strictly increasing or strictly decreasing. If $\alpha_n = 1$, then f_n is increasing, $f_n(-1) < 0$, $f_n(0) = 0$ and $f_n(1) > 0$. If $\alpha_n = -1$, then f_n is decreasing, $f_n(-1) > 0$, $f_n(0) = 0$ and $f_n(1) < 0$. Since f_n is continuous, then the image of f_n is a segment. This segment should have the form $[a, b]$, where $a < 0$ and $b > 0$. Now it is easy to see that the injectivity of F requires the injectivity of σ. Indeed, suppose by contrary that there exist distinct indices m and n such that $\sigma_m = \sigma_n = k$. Then the line segments $[-e^n, e^n]$ and $[-e^m, e^m]$ have just one common point, but their images are line segments $[ae^k, be^k]$ and $[a'e^k, b'e^k]$, where $a, a' < 0$ and $b, b' > 0$. These segments have more than one common point, which contradicts the injectivity of F.

We want to show that for each $x \in B_0$ and $n \in \mathbb{N}$ the items 1)–3) are true. As F is continuous and B_0^* is dense in B_0, then it suffices to consider the case where $x \in B_0^*$. Denote by S the set $\{n \in \mathbb{N}\colon x_n \neq 0\}$. Note that S is finite. For every $n \in S$ let $s^n = -\operatorname{sgn}(x_n)e^n$. Note that B_0 is covered by the ball $B(x, 1)$ together with the balls $B(s^n, 1)$, $n \in S$. Indeed, for arbitrary $z \in B_0$, if there exists an index $n \in S$ such that either $x_n > 0$ and $z_n < 0$ or $x_n < 0$ and $z_n > 0$, then z belongs to $B(s^n, 1)$. Otherwise, z belongs to $B(x, 1)$. It follows that B_0 should be also covered by the balls $B\big(F(x), 1\big)$ and $B\big(F(s^n), 1\big)$, $n \in S$.

Now, let us construct a sequence y as follows:

- for $k \notin \sigma(S)$ let $y_k = 0$;
- for $n \in S$ let $y_{\sigma_n} = \operatorname{sgn}(x_n)\alpha_n$.

Note that $y_k = 0$ for $k \notin \sigma(S)$ and $y_k = 1$ or $y_k = -1$ for $k \in \sigma(S)$. As $\sigma(S)$ is finite, then y belongs to B_0. Note that y does not belong to any of the balls $B\big(F(s^n), 1\big)$, $n \in S$. On the other hand, y is an element of B_0 and B_0 should be covered by the balls $B\big(F(x), 1\big)$ and $B\big(F(s^n), 1\big)$, $n \in S$. It follows that y should belong to $B\big(F(x), 1\big)$. This implies items 2) and 3).

To prove item 1), consider an index $k \notin \sigma(S)$. Alter the definition of y such that $y_k = 1$. Note that y still belongs to B_0 and does not belong to any of the balls $B\big(F(s^n), 1\big)$, $n \in S$, so it should belong to $B\big(F(x), 1\big)$. This yields $F(x)_k \geq 0$. Similarly, if we take $y_k = -1$, then we get $F(x)_k \leq 0$. So we can conclude that $F(x)_k = 0$. This proves item 1).

Continuity of F, together with the fact that B_0^* is dense in B_0, allows us to extend the result to all elements of B_0. Finally, we have to make sure σ is a surjection. Suppose by contrary that there exists an index $k \notin \sigma(\mathbb{N})$. From the above argument it follows that $F(x)_k = 0$ for every $x \in B_0^*$. Continuity of F and the fact that B_0^* is dense in B_0 imply that the same holds for every $x \in B_0$. This contradicts the surjectivity of F. Therefore, σ should be surjective. □

The previous lemma fixes a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ and a sequence $\alpha\colon \mathbb{N} \to \{-1, 1\}$. Let $\mathcal{A}$ be the corresponding isometric automorphism of c_0 defined by $\mathcal{A}(x)_{\sigma_n} = \alpha_n x_n$. Denote by G the restriction of $\mathcal{A}$ to B_0, which is an isometric bijection from B_0 onto itself. Our goal is to show that $F = G$. This is equivalent to showing that $G^{-1} \circ F$ is the identity map of B_0. Denote the map $G^{-1} \circ F$ by $\widetilde{F}$. Note that the definition of $\widetilde{F}$ implies that $\widetilde{F}$ is a non-expansive bijection from B_0 onto itself. The previous lemma implies that $\widetilde{F}$ has the following properties.

Lemma 3.3. *For each $x \in B_0$ and $n \in \mathbb{N}$,*

1) if $x_n = 0$, then $\widetilde{F}(x)_n = 0$;

2) if $x_n < 0$, then $\widetilde{F}(x)_n \leq 0$;

3) if $x_n > 0$, then $\widetilde{F}(x)_n \geq 0$.

The three properties listed above imply the following property of $\widetilde{F}^{-1}$.

Lemma 3.4. *For each $y \in B_0$ and $n \in \mathbb{N}$, if $y_n < 0$, then $\widetilde{F}^{-1}(y)_n < 0$, and if $y_n > 0$, then $\widetilde{F}^{-1}(y)_n > 0$.*

Now we are going to continue collecting some properties that describe the behaviour of $\widetilde{F}$.

Lemma 3.5. *For each $y \in B_0$ and $n \in \mathbb{N}$, if $y_n = 1$ and $y_i \neq 0$ for all $i \neq n$, then $\widetilde{F}^{-1}(y)_n = 1$, and if $y_n = -1$ and $y_i \neq 0$ for all $i \neq n$, then $\widetilde{F}^{-1}(y)_n = -1$.*

Proof. Let $y \in B_0$ be such that $y_n = 1$ and $y_i \neq 0$ for all $i \neq n$. Consider a sequence $z \in B_0$ defined by $z_n = -1$ and $z_i = y_i$ for all $i \neq n$. Denote $\widetilde{F}^{-1}(y)$ and $\widetilde{F}^{-1}(z)$ by y' and z'. The distance between y and z equals two, so the distance between y' and z' should be also equal to two. Consider an index $i \neq n$. We know that $y_i \neq 0$, so y_i is either positive or negative. If $y_i > 0$, then Lemma 3.4 implies that y'_i and z'_i should be positive too. Similarly, if $y_i < 0$, then Lemma 3.4 implies that y'_i and z'_i should be also negative. In either case we have $|y'_i - z'_i| < 1$. It follows, that for the distance between y' and z' to be equal to two, we need to have $|y'_n - z'_n| = 2$. This means that either $y'_n = 1$ and $z'_n = -1$ or $y'_n = -1$ and $z'_n = 1$. However, Lemma 3.4 excludes the second case. Therefore, we have $y'_n = 1$ as wanted. The case $y_n = -1$ is similar. □

Lemma 3.6. *For each $x \in B_0$ and $n \in \mathbb{N}$, if $x_n < 0$, then $\widetilde{F}(x)_n \in [x_n, 0]$, and if $x_n > 0$, then $\widetilde{F}(x)_n \in [0, x_n]$.*

Proof. Let us consider the case $x_n > 0$. Lemma 3.3 implies that $\widetilde{F}(x)_n \geq 0$, so it remains to show that $\widetilde{F}(x)_n \leq x_n$. Let us construct a sequence $y \in B_0$ as follows. First, let $y_n = -1$. Then, for every $i \neq n$

- choose y_i from $[-1, 0)$ if $x_i < 0$;
- choose y_i from $(0, 1]$ if $x_i > 0$;
- choose y_i from $[-1, 1] \setminus \{0\}$ if $x_i = 0$.

For y to belong to B_0, it is important to choose y_i such that the sequence converges to zero. Clearly, such choice is possible. One possible choice is to define $y_i = 1/i$ for $x_i \geq 0$ and $y_i = -1/i$ for $x_i < 0$.

Denote $\widetilde{F}^{-1}(y)$ by y'. Note that $y_n = -1$ and $y_i \neq 0$ for every $i \neq n$. Lemma 3.5 implies that $y'_n = -1$. Let us show that $|x_i - y'_i| \leq 1$ for every $i \neq n$. Consider an index i distinct from n. If $x_i < 0$, then $y_i < 0$, so Lemma 3.4 implies that $y'_i < 0$. As x_i and y'_i are both negative, then $|x_i - y'_i| < 1$. Similarly, if $x_i > 0$, then $y_i > 0$, so Lemma 3.4 implies that $y'_i > 0$. As x_i and y'_i are both positive, then $|x_i - y'_i| < 1$. Finally, if $x_i = 0$, then the inequality $|x_i - y'_i| \leq 1$ is obvious. Now we see that the distance between x and y' is equal to $1 + x_n$. Indeed, $|x_n - y'_n| = 1 + x_n$ and $|x_i - y'_i| \leq 1$ for $i \neq n$. This implies that the distance between $\widetilde{F}(x)$ and y is at most $1 + x_n$, therefore $|\widetilde{F}(x)_n - y_n| \leq 1 + x_n$, which yields $\widetilde{F}(x)_n \leq x_n$. This concludes the proof for the case $x_n > 0$. The proof for the case $x_n < 0$ is similar. $\square$

The last lemma implies the following properties of the inverse function.

Lemma 3.7. *For each $y \in B_0$ and $n \in \mathbb{N}$, if $y_n < 0$, then $\widetilde{F}^{-1}(y)_n \in [-1, y_n]$, and if $y_n > 0$, then $\widetilde{F}^{-1}(y)_n \in [y_n, 1]$.*

The important feature of c_0 is that every element attains its norm. That is, for every $x \in c_0$ there is an index n such that $|x_n| = \|x\|$. In other words, the supremum used to define the norm of x is actually a maximum. We are going to make use of this feature in the proof of the next proposition.

Lemma 3.8. *Let $y \in B_0$ be such that the set $S = \{i \in \mathbb{N}\colon y_i = 0\}$ is finite. Then $\widetilde{F}^{-1}(y)_i = 0$ for each $i \in S$.*

Proof. Let us proceed by induction on the number of elements of S. If S is empty, then there is nothing to prove. Now, let N be a non-negative integer and suppose that the claim holds whenever S has up to N elements. Let us show that the claim also holds when S has $N + 1$ elements. So let $y \in B_0$ be a sequence such that the set S has $N + 1$ elements. Denote $\widetilde{F}^{-1}(y)$ by y'. We are going to use a proof by contradiction. Suppose by contrary that there exists $n \in S$ such that $y'_n \neq 0$. Construct a sequence $z \in B_0$ as follows. First, set $z_n = 1$ if $y'_n > 0$ and $z_n = -1$ if $y'_n < 0$. Then, for each $i \neq n$

- choose z_i from $[-1, 0)$ if $y'_i < 0$;
- choose z_i from $(0, 1]$ if $y'_i > 0$;
- set $z_i = 0$ if $y'_i = 0$.

For z to belong to B_0, it is important to choose z_i such that the sequence converges to zero. Clearly, such choice is possible. One possible choice is to define $z_i = 1/i$ for $y'_i > 0$ and $z_i = -1/i$ for $y'_i < 0$.

By definition, $z_n \neq 0$. If $i \in \mathbb{N} \setminus S$, then $y_i \neq 0$, so Lemma 3.4 implies that $y'_i \neq 0$ and the definition of z implies that $z_i \neq 0$. It follows that the set $\{i \in \mathbb{N}\colon z_i = 0\}$ is contained in the set $S \setminus \{n\}$. Therefore, the set $\{i \in \mathbb{N}\colon z_i = 0\}$ has at most N elements. This means that the induction hypothesis can be applied to z, so we know that $\widetilde{F}(z)_i = 0$ whenever $z_i = 0$.

Denote $\widetilde{F}^{-1}(z)$ by z'. Let us show that the distance between y' and z' is smaller than one. Since every element of c_0 attains its norm, it suffices to show that $|y'_i - z'_i| < 1$ for each $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be arbitrary. Consider the case $y'_i > 0$. The definition of z implies that $z_i > 0$. As $z_i > 0$, then Lemma 3.4 implies that $z'_i > 0$. Since z'_i and y'_i are both positive, then $|y'_i - z'_i| < 1$. Similarly, if $y'_i < 0$, then the definition of z implies that $z_i < 0$. As $z_i < 0$, then Lemma 3.4 implies that $z'_i < 0$. Since z'_i and y'_i are both negative, then $|y'_i - z'_i| < 1$. Finally, if $y'_i = 0$, then the definition of z implies that $z_i = 0$. Since $z_i = 0$, then the application of the induction hypothesis to z gives $z'_i = 0$, which makes the inequality $|y'_i - z'_i| < 1$ obvious. This shows that the distance between y' and z' is smaller than one.

If the distance between y' and z' is smaller than one, then the distance between y and z should be also smaller than one, but we have $|y_n - z_n| = 1$, which is a contradiction. □

Lemma 3.9. *Let $x \in B_0$ be such that the set $S = \{i \in \mathbb{N} \colon x_i = 0\}$ is finite. Then $\widetilde{F}(x)_i \neq 0$ for each $i \in \mathbb{N} \setminus S$.*

Proof. Let $n \in \mathbb{N} \setminus S$ be arbitrary. Construct a sequence $y \in B_0$ as follows. First, set $y_n = \operatorname{sgn}(x_n)$. Then, for each $i \neq n$

- choose y_i from $[-1, 0)$ if $x_i < 0$;
- choose y_i from $(0, 1]$ if $x_i > 0$;
- set $y_i = 0$ if $x_i = 0$.

For y to belong to B_0, it is important to choose y_i such that the sequence converges to zero. Clearly, such choice is possible. Denote $\widetilde{F}^{-1}(y)$ by y'. Note that the sequence y has only finitely many zeros. Therefore, Lemma 3.8 can be applied to y. This means that $y'_i = 0$ for every $i \in S$. Now, we can show that the distance between x and y' is smaller than one – the argument is identical to the one that appeared in the proof of the previous lemma. Since the distance between x and y' is smaller than one, then the distance between $\widetilde{F}(x)$ and y should be also smaller than one. In particular, we should have $|\widetilde{F}(x)_n - y_n| < 1$, which yields $\widetilde{F}(x)_n \neq 0$. □

Lemma 3.8 says that if we have $y_n = 0$, then we should also have $\widetilde{F}^{-1}(y)_n = 0$, provided that the sequence y has only finitely many zeros. If we assume the continuity of $\widetilde{F}^{-1}$, then we can get rid of that additional assumption.

Lemma 3.10. *Suppose that $\widetilde{F}^{-1}$ is continuous. Let $y \in B_0$ and $n \in \mathbb{N}$ be such that $y_n = 0$. Then $\widetilde{F}^{-1}(y)_n = 0$.*

Proof. It is possible to construct a sequence $\psi\colon \mathbb{N} \to B_0$ such that $\psi^k_n = 0$, $\psi^k_i \neq 0$ for $i \neq n$ and $\psi^k \to y$. One possible choice is to define

$$\psi^k_i = \begin{cases} 0, & i = n, \\ y_i, & i \neq n,\ y_i \neq 0, \\ 1/(ik), & i \neq n,\ y_i = 0. \end{cases}$$

For each $k \in \mathbb{N}$, the sequence $\psi^k \in B_0$ has exactly one zero at index n, so Lemma 3.8 implies that $\widetilde{F}^{-1}(\psi^k)_n = 0$. Since $\psi^k \to y$ and $\widetilde{F}^{-1}$ is continuous, then $\widetilde{F}^{-1}(\psi^k) \to \widetilde{F}^{-1}(y)$. This implies the convergence $\widetilde{F}^{-1}(\psi^k)_n \to \widetilde{F}^{-1}(y)_n$. As $\widetilde{F}^{-1}(\psi^k)_n = 0$ for every $k \in \mathbb{N}$, then it follows that $\widetilde{F}^{-1}(y)_n = 0$. □

Now we can prove the main result.

Theorem 3.11. *If F^{-1} is continuous, then F is an isometry.*

Proof. Let S be a finite subset of $\mathbb{N}$. Consider the subset

$$B^S_0 = \{x \in B_0 : x_n = 0 \text{ for all } n \in \mathbb{N} \setminus S\}.$$

Lemma 3.3 says that $\widetilde{F}(x)_n = 0$ whenever $x_n = 0$. This yileds the inclusion $\widetilde{F}(B^S_0) \subset B^S_0$. If F^{-1} is continuous, then $\widetilde{F}^{-1}$ is also continuous, so we can apply Lemma 3.10, which yields the inclusion $\widetilde{F}^{-1}(B^S_0) \subset B^S_0$. Combining these two together, we get that the set B^S_0 is mapped bijectively onto itself. It follows that the restriction of $\widetilde{F}$ to B^S_0 is a non-expansive bijection from the unit ball of a finite-dimensional space onto itself. Since the unit ball of a finite-dimensional space is plastic, then it follows that the restriction of $\widetilde{F}$ to B^S_0 is an isometry. Moreover, Theorem 2.7 says that the latter is actually a restriction of an isometric automorphism of the underlying finite-dimensional space. Combining this with some previously acquired information, we can conclude that the restriction of $\widetilde{F}$ to B^S_0 is an identity map. Indeed, Lemma 3.3 says that for each $n \in S$ the element e^n is mapped to te^n, where $t > 0$. However, for the norm to be preserved, we need to have $t = 1$. Therefore, the elements e^n, $n \in S$ are mapped to itself. Since every element of B^S_0 is a linear combination of these, then the linearity implies that $\widetilde{F}$ should keep all elements of B^S_0 in place.

If $\widetilde{F}$ restricted to B_0^S is an identity map, then the restriction of $\widetilde{F}$ to B_0^* is also an identity map, because the latter is the union of all the subsets B_0^S, where S is a finite subset of $\mathbb{N}$. Since $\widetilde{F}$ restricted to B_0^* is an identity map, $\widetilde{F}$ is continuous and B_0^* is dense in B_0, then it follows that $\widetilde{F}$ is an identity map of B_0. This means that F is a restriction of an isometric automorphism of c_0 defined by $\mathcal{A}(x)_{\sigma_n} = \alpha_n x_n$. In particular, F is an isometry. $\square$

4 Plasticity of the unit ball of c

The notation for this section is fixed in the beginning of the previous section. Also, some parts of the proof will be identical to the corresponding parts of the proof for c_0 and we are going to omit these parts. Therefore, it is advisable to take a look at the previous section before reading the proof at hand.

Let us consider an arbitrary non-expansive bijection $F\colon B \to B$ from the unit ball of c onto itself. Our goal is to show that F is an isometry. As with the space c_0, the first step is to extract some information about F to choose a corresponding isometric automorphism of c. The isometric automorphisms of c have the form $\mathcal{A}(x)_{\sigma_n} = \alpha_n x_n$, where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a bijection and α is a sequence of ones and minus ones that is constant from some point.

When dealing with plasticity of the unit ball, it is important to know the extreme points. While the space c_0 has no extreme points, the space c has quite many. The extreme points of c are the sequences that consist of just ones and minus ones and are constant from some point.

The next lemma is an analog of Lemma 3.1. The only difference is that now we have B instead of B_0. The proof is identical to the one of Lemma 3.1.

Lemma 4.1. *Let x and y be two non-zero elements of B. The balls $B(x, 1)$ and $B(y, 1)$ cover the ball B if and only if there exists an index n such that $x_i = y_i = 0$ for all $i \neq n$ and either x_n is positive and y_n is negative or x_n is negative and y_n is positive.*

Let us retrieve some information about F to fix an isometric automorphism of c that the function F seems to resemble. The next lemma is an analog of Lemma 3.2. The difference from the space c_0 is that now we have to ensure that the sequence α is constant from some point.

Lemma 4.2. *There exists a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ and a sequence $\alpha\colon \mathbb{N} \to \{-1, 1\}$, which is constant starting from some index, such that for every $x \in B$ and $n \in \mathbb{N}$ we have the following:*

1) if $x_n = 0$, then $F(x)_{\sigma_n} = 0$;

2) if $x_n < 0$ and $\alpha_n = 1$ ($\alpha_n = -1$), then $F(x)_{\sigma_n} \leq 0$ ($F(x)_{\sigma_n} \geq 0$);

3) if $x_n > 0$ and $\alpha_n = 1$ ($\alpha_n = -1$), then $F(x)_{\sigma_n} \geq 0$ ($F(x)_{\sigma_n} \leq 0$).

Proof. The first part of the proof repeats the first three paragraphs of the proof of Lemma 3.2. We only have to replace B_0 by B. We fix a function $\sigma\colon \mathbb{N} \to \mathbb{N}$ and a sequence $\alpha\colon \mathbb{N} \to \{-1, 1\}$. We show that for each $n \in \mathbb{N}$ there exists a continuous function $f_n\colon [-1,1] \to [-1,1]$ such that $F(te^n) = f_n(t)e^{\sigma_n}$ for each $t \in [-1,1]$. We also know that if $\alpha_n = 1$, then f_n is strictly increasing, $f_n(-1) < 0$, $f_n(0) = 0$ and $f_n(1) > 0$, and if $\alpha_n = -1$, then f_n is strictly decreasing, $f_n(-1) > 0$, $f_n(0) = 0$ and $f_n(1) < 0$. We also show that the function σ is injective.

The next step is to ensure that the sequence α is constant from some point. Let y be an arbitrary extreme point of B. The sequence y consists of ones and minus ones and is constant from some point. Denote $F^{-1}(y)$ by x. By Theorem 2.6 we know that x is also an extreme point. Therefore, the sequence x consists of ones and minus ones and is constant from some point. Let us show that for each $n \in \mathbb{N}$ we have $y_{\sigma_n} = \alpha_n x_n$. Let $n \in \mathbb{N}$ be arbitrary. We know that x_n is either 1 or -1. Let us consider the case $x_n = 1$. We need to show that $y_{\sigma_n} = \alpha_n$. Consider elements e^n and x. The distance between e^n and x is equal to one. It follows that the distance between $F(e^n)$ and y should be at most one. We know that $F(e^n)$ is te^{σ_n}, where $t > 0$ if $\alpha_n = 1$ and $t < 0$ if $\alpha_n = -1$. We know that y_{σ_n} is either 1 or -1. If $y_{\sigma_n} \neq \alpha_n$, then $|F(e^n)_{\sigma_n} - y_{\sigma_n}| > 1$, which implies that the distance between $F(e^n)$ and y is greater than one, but this can not be the case. Therefore, we must have $y_{\sigma_n} = \alpha_n$. The case $x_n = -1$ is analogous. Now, the fact that $y_{\sigma_n} = \alpha_n x_n$ for each $n \in \mathbb{N}$ and the fact that the sequences x and y are constant from some point imply that the sequence α should be also constant from some point.

Now we want to show that for each $x \in B$ and $n \in \mathbb{N}$ the items 1)–3) are true. As F is continuous and $B \setminus B_0$ is dense in B, then it suffices to consider the case where $\lim x_n \neq 0$. Note that we can not apply the approach used in the proof for c_0, because the subset B_0^* is not dense in B. Denote by S the set $\{n \in \mathbb{N}\colon x_n \neq 0\}$. For every $n \in S$ let $s^n = -\operatorname{sgn}(x_n)e^n$. Note that B is covered by the ball $B(x,1)$ together with the balls $B(s^n,1)$, $n \in S$. It follows that B should be also covered by the balls $B\big(F(x),1\big)$ and $B\big(F(s^n),1\big)$, $n \in S$.

As with the case of c_0, the next step is to find an element $y \in B$, which does not belong to any of the balls $B\big(F(s^n),1\big)$, $n \in S$. The difficult part is to ensure that the sequence y belongs to B. Let us construct a sequence y as follows:

- for $k \notin \sigma(S)$ let $y_k = \lim(\operatorname{sgn}(x_n)\alpha_n)$;
- for $n \in S$ let $y_{\sigma_n} = \operatorname{sgn}(x_n)\alpha_n$.

Let us ensure that the sequence y belongs to B. Since $\lim x_n \neq 0$, then either $\lim x_n > 0$ or $\lim x_n < 0$. If $\lim x_n > 0$, then there exists an index N such that $\operatorname{sgn}(x_n) = 1$ for each $n \geq N$. If $\lim x_n < 0$, then there exists an index N

such that $\operatorname{sgn}(x_n) = -1$ for each $n \geq N$. Either way, the sequence $(\operatorname{sgn} x_n)$ is constant starting from index N. The sequence (α_n) is also constant from some point, as shown above. This means that the sequence $(\operatorname{sgn}(x_n)\alpha_n)$ is also constant from some point, so the limit $\lim(\operatorname{sgn}(x_n)\alpha_n)$ exists and is equal to 1 or -1. Now we see that the sequence y consists of ones and minus ones and is constant from some point. Therefore, y belongs to B.

Note that y does not belong to any of the balls $B\big(F(s^n), 1\big)$, $n \in S$. On the other hand, y is an element of B and B should be covered by the balls $B\big(F(x), 1\big)$ and $B\big(F(s^n), 1\big)$, $n \in S$. It follows that y should belong to $B\big(F(x), 1\big)$. This implies items 2) and 3). The item 1) can be proved by the same argument as in the proof for c_0.

Continuity of F and the fact that $B \setminus B_0$ is dense in B allow us to extend the result to all elements of B. Finally, we have to make sure σ is a surjection. This can be proved by the same argument as in the proof for c_0. $\square$

The previous lemma fixes a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ and a sequence $\alpha\colon \mathbb{N} \to \{-1, 1\}$, that is constant from some point. Let $\mathcal{A}$ be the corresponding isometric automorphism of c defined by $\mathcal{A}(x)_{\sigma_n} = \alpha_n x_n$. Define $\widetilde{F}$ as in the proof for c_0. Our goal is to show that $\widetilde{F}$ is an identity map. The previous lemma implies the following properties of $\widetilde{F}$.

Lemma 4.3. *For each $x \in B$ and $n \in \mathbb{N}$,*

- *if $x_n = 0$, then $\widetilde{F}(x)_n = 0$;*
- *if $x_n < 0$, then $\widetilde{F}(x)_n \leq 0$;*
- *if $x_n > 0$, then $\widetilde{F}(x)_n \geq 0$.*

The three properties listed above imply the following properties of $\widetilde{F}^{-1}$.

Lemma 4.4. *For each $y \in B_0$ and $n \in \mathbb{N}$, if $y_n < 0$, then $\widetilde{F}^{-1}(y)_n < 0$, and if $y_n > 0$, then $\widetilde{F}^{-1}(y)_n > 0$.*

In the proof for c_0, the next step was to prove Lemmas 3.5 and 3.6. The proofs of these propositions work for the space c as well. We only need to substitute B for B_0. Therefore, we obtain the following.

Lemma 4.5. *For each $y \in B$ and $n \in \mathbb{N}$, if $y_n = 1$ and $y_i \neq 0$ for all $i \neq n$, then $\widetilde{F}^{-1}(y)_n = 1$, and if $y_n = -1$ and $y_i \neq 0$ for all $i \neq n$, then $\widetilde{F}^{-1}(y)_n = -1$.*

Lemma 4.6. *For each $x \in B$ and $n \in \mathbb{N}$, if $x_n < 0$, then $\widetilde{F}(x)_n \in [x_n, 0]$, and if $x_n > 0$, then $\widetilde{F}(x)_n \in [0, x_n]$.*

As for the case of c_0, the previous lemma implies the following properties of the inverse function.

Lemma 4.7. *For each $y \in B_0$ and $n \in \mathbb{N}$, if $y_n < 0$, then $\widetilde{F}^{-1}(y)_n \in [-1, y_n]$, and if $y_n > 0$, then $\widetilde{F}^{-1}(y)_n \in [y_n, 1]$.*

In the case of c_0, the next step was Lemma 3.8. The proof of this lemma relies on the fact that every element of c_0 attains its norm, but this is not true in c. Therefore, we are forced to use some alternative approach.

Lemma 4.8. *Let $x \in B$ be such that the set $S = \{i \in \mathbb{N} \colon x_i \notin \{-1, 1\}\}$ is finite. Then $\widetilde{F}(x) = x$.*

Proof. Let us proceed by induction on the number of elements of S. For the base of induction, consider the case where the set S is empty. If the set S is empty, then $x_i \in \{-1, 1\}$ for every $i \in \mathbb{N}$ and the application of Lemma 4.7 gives $\widetilde{F}^{-1}(x) = x$, which is equivalent to $\widetilde{F}(x) = x$. This proves the base of induction.

Now, let N be an arbitrary non-negative integer. Suppose that the claim holds whenever the set S has at most N elements. Let us prove that the claim also holds when the set S has $N + 1$ elements. Suppose that the set S has $N + 1$ elements. We need to show $\widetilde{F}(x) = x$, which is equivalent to showing $\widetilde{F}^{-1}(x) = x$. Applying Lemma 4.7, we obtain that $\widetilde{F}^{-1}(x)_n = x_n$ for every $n \notin S$. It remains to show that $\widetilde{F}^{-1}(x)_n = x_n$ is true for every $n \in S$. Let n be an arbitrary element of S. Consider sequences y and z defined by $y_n = 1$, $z_n = -1$ and $y_i = z_i = \widetilde{F}^{-1}(x)_i$ for every $i \neq n$. Note that the sets $S_y = \{i \in \mathbb{N} \colon y_i \notin \{-1, 1\}\}$ and $S_z = \{i \in \mathbb{N} \colon z_i \notin \{-1, 1\}\}$ are contained in the set $S \setminus \{n\}$. This implies that the sets S_y and S_z have at most N elements. Therefore, we can apply the induction hypothesis to obtain $\widetilde{F}(y) = y$ and $\widetilde{F}(z) = z$. Since the sequences y, z and $\widetilde{F}^{-1}(x)$ coincide for all indices distinct from n, then the distance between elements $\widetilde{F}^{-1}(x)$ and y is equal to $|\widetilde{F}^{-1}(x)_n - y_n|$ and the distance between elements $\widetilde{F}^{-1}(x)$ and z is equal to $|\widetilde{F}^{-1}(x)_n - z_n|$. It follows that the distance between elements x and y is at most $|\widetilde{F}^{-1}(x)_n - y_n|$ and the distance between elements x and z is at most $|\widetilde{F}^{-1}(x)_n - z_n|$. Combining these two facts, we obtain $\widetilde{F}^{-1}(x)_n = x_n$. $\square$

From the last lemma it follows that $\widetilde{F}$ is an identity map on $B_1^* \cup B_{-1}^*$. Since $\widetilde{F}$ is continuous and $B_1^* \cup B_{-1}^*$ is dense in $B_1 \cup B_{-1}$, then $\widetilde{F}$ is also an identity map on $B_1 \cup B_{-1}$. It turns out that we can say something about other levels too.

Lemma 4.9. *Let $x \in B$, $h = \lim x_k$ and $n \in \mathbb{N}$.*

1) If $|x_n| < |h|$, then $\widetilde{F}(x)_n = x_n$.

2) If $x_n \geq |h|$, then $\widetilde{F}(x)_n \in [|h|, x_n]$.

3) If $x_n \leq -|h|$, then $\widetilde{F}(x)_n \in [x_n, -|h|]$.

Proof. For the case $h = 0$, the three items follow from Lemma 4.6 and the item 1) of Lemma 4.3, so it remains to consider the case $h \neq 0$.

Let us start with proving the first item. For the case $x_n = 0$, the claim follows from Lemma 4.3, so it remains to consider the case $x_n \neq 0$. Let $\varepsilon = |h| - |x_n|$. Note that $\varepsilon > 0$. Since the sequence x converges to h, then there exists an index $N \in \mathbb{N}$ such that $|x_k - h| < \varepsilon$ for each $k \geq N$. Note that $n < N$. Define a sequence $y \in B$ as

$$y_k = \begin{cases} \operatorname{sgn}(x_n), & k = n, \\ x_k, & k < N,\ k \neq n, \\ \operatorname{sgn}(h), & k \geq N. \end{cases}$$

Note that the sequence y satisfies the conditions of Lemma 4.8, hence we have $\widetilde{F}(y) = y$. Compare sequences x and y. For $k = n$ we have $|x_k - y_k| = 1 - |x_n|$. For $k < N$, $k \neq n$ we have $|x_k - y_k| = 0$. For $k \geq N$ we have $|x_k - y_k| < 1 - |x_n|$. It follows that the distance between x and y is equal to $1 - |x_n|$. Therefore, the distance between $\widetilde{F}(x)$ and y is at most $1 - |x_n|$ (recall that $\widetilde{F}(y) = y$). If $x_n < 0$, then the latter fact implies $\widetilde{F}(x)_n \leq x_n$, while Lemma 4.6 implies $\widetilde{F}(x)_n \geq x_n$. If $x_n > 0$, then the latter fact implies $\widetilde{F}(x)_n \geq x_n$, while Lemma 4.6 implies $\widetilde{F}(x)_n \leq x_n$. In either case we have $\widetilde{F}(x)_n = x_n$ as wanted.

Now, let us consider the second item. By Lemma 4.6 we know $\widetilde{F}(x)_n \leq x_n$, so it remains to show $\widetilde{F}(x)_n \geq |h|$. Let ε be an arbitrary positive number. Since the sequence x converges to h, then there exists an index $N \in \mathbb{N}$ such that $|x_k - h| < \varepsilon$ for each $k \geq N$. If it happens that $N \leq n$, then choose N to be any index greater than n. Define a sequence $y \in B$ as before. Note that the sequence y satisfies the conditions of Lemma 4.8, hence we have $\widetilde{F}(y) = y$. Compare sequences x and y. For $k = n$ we have $|x_k - y_k| = 1 - |x_n| \leq 1 - |h|$. For

$k < N$, $k \neq n$ we have $|x_k - y_k| = 0$. For $k \geq N$ we have $|x_k - y_k| < 1 - |h| + \varepsilon$. It follows that the distance between x and y is at most $1 - |h| + \varepsilon$. Therefore, the distance between $\widetilde{F}(x)$ and y is also at most $1 - |h| + \varepsilon$ (recall that $\widetilde{F}(y) = y$). This yields $\widetilde{F}(x)_n \geq |h| - \varepsilon$. Since ε was arbitrary, then it follows that $\widetilde{F}(x)_n \geq |h|$. This concludes the proof of the second item. The proof of the third item is analogous. □

We can make some conclusions from the properties obtained in the last lemma. First, we see that $\widetilde{F}$ preserves the limit – for every $x \in B$ we have $\lim \widetilde{F}(x)_k = \lim x_k$. Moreover, we see that the inverse function has the following property.

Lemma 4.10. *Let $y \in B$, $h = \lim y_k$ and $n \in \mathbb{N}$. If $|y_n| < |h|$, then $\widetilde{F}^{-1}(y)_n = y_n$.*

Proof. Denote $\widetilde{F}^{-1}(y)$ by x. As mentioned above, $\widetilde{F}$ preserves the limit. Therefore, we have $\lim x_k = \lim y_k = h$. We can have three cases: $|x_n| < |h|$, $x_n \geq |h|$ and $x_n \leq -|h|$. If $x_n \geq |h|$, then Lemma 4.9 implies that $y_n \geq |h|$, which contradicts our assumption. If $x_n \leq -|h|$, then Lemma 4.9 implies that $y_n \leq -|h|$, which contradicts our assumption. Therefore, we are left with the case $|x_n| < |h|$, so Lemma 4.9 implies that $y_n = x_n$. □

We are almost done. Recall that in the case of c_0 the last step was to show that the set B_0^S, where S is a finite subset of $\mathbb{N}$, is mapped bijectively onto itself. To finish the proof at hand, it will suffice to show the same for the set B_h^S, where S is a finite subset of $\mathbb{N}$ and $h \in [-1, 1]$. Lemma 4.9 implies the inclusion $\widetilde{F}(B_h^S) \subset B_h^S$, so it remains to show that the same is true for the inverse function. This is exactly what the next lemma asserts. It will be more convenient to limit ourselves to the case $h \in (-1, 1) \setminus \{0\}$. Fortunately, this will be sufficient.

Lemma 4.11. *Let $h \in (-1, 1) \setminus \{0\}$ and let S be a finite subset of $\mathbb{N}$. Then $\widetilde{F}^{-1}(B_h^S) \subset B_h^S$.*

Proof. Let us consider the case $h > 0$. Let y be an arbitrary element of the set B_h^S. Our goal is to show that $\widetilde{F}^{-1}(y) \in B_h^S$. To prove this, we need to show that $\widetilde{F}^{-1}(y)_n = h$ for every $n \in \mathbb{N} \setminus S$. Let n be an arbitrary element of $n \in \mathbb{N} \setminus S$. Since $y_n = h$ and $h > 0$, then Lemma 4.7 says $\widetilde{F}^{-1}(y)_n \geq h$, so we only need to show that the reverse inequality is also true. For the sake of contradiction, suppose that $\widetilde{F}^{-1}(y)_n > h$. Denote $1 - h$ by d. Define a sequence $z \in B$ as

$$z_k = \begin{cases} 1, & k = n, \\ y_k, & |y_k| < h, \\ h + d/2(1 - 1/2^k), & y_k \geq h, \ k \neq n, \\ -h - d/4, & y_k \leq -h. \end{cases}$$

It is straightforward to check that $z_k \in [-1, 1]$ for every $k \in \mathbb{N}$. We see that for every $k \notin S \cup \{n\}$ we have the third case. Since the set $S \cup \{n\}$ is finite and $h + d/2(1 - 1/2^k) \to h + d/2$, then we also have $z_k \to h + d/2$. Therefore, we see that z is indeed an element of B. See Figure 4 below for an illustration. Denote $\widetilde{F}^{-1}(y)$ and $\widetilde{F}^{-1}(z)$ by y' and z'. To obtain a contradiction, let us show that the distance between y' and z' is smaller than d. Since $z_n = 1$, then Lemma 4.7 says $z'_n = 1$. Consider an index $k \neq n$. Recall that $\lim z_k = h + d/2$. Since we have $|z_k| < |h + d/2|$, then Lemma 4.10 implies $z'_k = z_k$. It follows that $z' = z$. Therefore, we need to show that the distance between y' and z is smaller than d. According to our assumption, we have $y'_n > h$, hence $|y'_n - z_n| < d$. To show that the distance between y' and z is smaller than d, it will suffice to show that $|y'_k - z_k| \leq 3d/4$ for every $k \neq n$. Consider the case $|y_k| < h$. Lemma 4.10 implies $y'_k = y_k$ and the definition of z implies $z_k = y_k$, so $y'_k = z_k$ and $|y'_k - z_k| = 0$. Let us consider the cases $y_k \geq h$ and $y_k \leq -h$. If $y_k \geq h$, then Lemma 4.7 implies that $y'_k \in [h, 1]$, and if $y_k \leq -h$, then Lemma 4.7 implies that $y'_k \in [-1, -h]$. In either case, the greatest possible distance between y'_k and z_k is $3d/4$. This shows that the distance between y' and z is smaller than d. Since the distance between y' and z is smaller than d, then the distance between y and z should be also smaller than d, but we have $|y_n - z_n| = d$, which is a contradiction. The case $h < 0$ is analogous. □

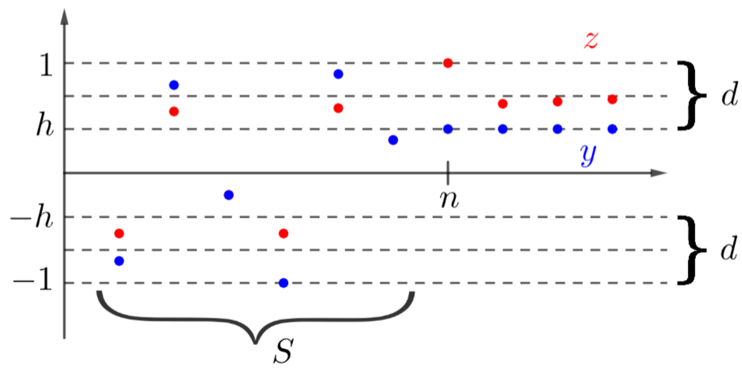


Figure 4: Illustration of the proof of Lemma 4.11

Now, we can finish the proof. The remaining part is very similar to the way we finished the proof for c_0.

Theorem 4.12. *F is an isometry.*

Proof. Let $h \in (-1, 1) \setminus \{0\}$ and let S be a finite subset of $\mathbb{N}$. If $x \in B_h^S$, then $\lim x_k = h$ and Lemma 4.9 implies that $\widetilde{F}(x)_n = h$ whenever $x_n = h$. This yields the inclusion $\widetilde{F}(B_h^S) \subset B_h^S$. Applying Lemma 4.11, we obtain the inclusion $\widetilde{F}^{-1}(B_h^S) \subset B_h^S$. Combining these two together, we see that the set B_h^S is mapped bijectively onto itself. It follows that the restriction of $\widetilde{F}$ to B_h^S is a non-expansive bijection from the unit ball of a finite-dimensional space onto itself (the set B_h^S, as a metric space, can be identified with B_0^S). Since the unit ball of a finite-dimensional space is plastic, then it follows that the restriction of $\widetilde{F}$ to B_h^S is an isometry. Theorem 2.7 says that the latter is actually a restriction of an isometric automorphism of the underlying finite-dimensional space. Combining this with some previously acquired information, we can conclude that the restriction of $\widetilde{F}$ to B_h^S is an identity map.

Since $\widetilde{F}$ is an identity map on B_h^S for every finite subset S, then the restriction of $\widetilde{F}$ to B_h^* is also an identity map, because the latter is the union of all the subsets B_h^S, where S is a finite subset of $\mathbb{N}$. Since $\widetilde{F}$ is an identity map on B_h^*, $\widetilde{F}$ is continuous and B_h^* is dense in B_h, then it follows that the restriction of $\widetilde{F}$ to B_h is also an identity map.

We have seen that $\widetilde{F}$ is an identity map on B_h for every $h \in (-1, 1) \setminus \{0\}$. Previously, we have also seen that $\widetilde{F}$ is an identity map on B_{-1} and B_1. It follows that the restriction of $\widetilde{F}$ to $B \setminus B_0$ is an identity map. Since $\widetilde{F}$ is continuous and $B \setminus B_0$ is dense in B, then it follows that $\widetilde{F}$ is an identity map. This means that F is a restriction of an isometric automorphism of c defined by $\mathcal{A}(x)_{\sigma_n} = \alpha_n x_n$. In particular, F is an isometry. □

References

[1] S. A. Naimpally, Z. Piotrowski, and E. J. Wingler. “Plasticity in metric spaces”. In: *J. Math. Anal. Appl* 313 (2006), pp. 38–48.

[2] B. Cascales, V. Kadets, J. Orihuela, and E. J. Wingler. “Plasticity of the unit ball of a strictly convex Banach space”. In: *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* 110.2 (2016), pp. 723–727.

[3] V. Kadets and O. Zavarzina. “Nonexpansive bijections to the unit ball of the ℓ_1-sum of strictly convex Banach spaces”. In: *Bulletin of the Australian Mathematical Society* 97 (2018), pp. 285–292.

[4] C. Angosto, V. Kadets, and O. Zavarzina. “Non-expansive bijections, uniformities and polyhedral faces”. In: *Journal of Mathematical Analysis and Applications* 471.1 (2019), pp. 38–52.

[5] V. Kadets and O. Zavarzina. “Plasticity of the unit ball of ℓ_1”. In: *Visnyk of V. N. Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics* 83 (2016), pp. 4–9.

[6] O. Zavarzina. “Nonexpansive bijections between unit balls of Banach spaces”. In: *Ann. Funct. Anal* 9.2 (2018), pp. 271–281.

[7] P. Mankiewicz. “On extension of isometries in normed linear spaces”. In: *Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys.* 20 (1972), pp. 367–371.

Lihtlitsents lõputöö reprodutseerimiseks ja üldsusele kättesaadavaks tegemiseks

Mina, Nikita Leo,

1. annan Tartu Ülikoolile tasuta loa (lihtlitsentsi) minu loodud teose

   "Plasticity of the unit ball of a Banach space",

   mille juhendajad on Rainis Haller ja Olesia Zavarzina, reprodutseerimiseks eesmärgiga seda säilitada, sealhulgas lisada digitaalarhiivi DSpace kuni autoriõiguse kehtivuse lõppemiseni.

2. Annan Tartu Ülikoolile loa teha punktis 1 nimetatud teos üldsusele kättesaadavaks Tartu Ülikooli veebikeskkonna, sealhulgas digitaalarhiivi DSpace kaudu Creative Commonsi litsentsiga CC BY NC ND 3.0, mis lubab autorile viidates teost reprodutseerida, levitada ja üldsusele suunata ning keelab luua tuletatud teost ja kasutada teost ärieesmärgil, alates 10.06.2022 kuni autoriõiguse kehtivuse lõppemiseni.

3. Olen teadlik, et punktides 1 ja 2 nimetatud õigused jäävad alles ka autorile.

4. Kinnitan, et lihtlitsentsi andmisega ei riku ma teiste isikute intellektuaalomandi ega isikuandmete kaitse õigusaktidest tulenevaid õigusi.

Nikita Leo
14.06.2021