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Daugavet- and Delta-points in Lipschitz-free Banach Spaces

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DAUGAVET- AND DELTA-POINTS IN LIPSCHITZ-FREE BANACH SPACES

Master's thesis

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Abstract: In this thesis, we characterize Daugavet- and Delta-points in Lipschitz-free Banach spaces. Our results complement the ones in the recent preprint *"Daugavet points and Δ-points in Lipschitz-free Banach spaces"* by M. Jung and A. Rueda Zoca.

CERCS research specialisation: P140 Series, Fourier analysis, functional analysis.

Key Words: Banach space, Lipschitz-free space, slice, Daugavet property, Daugavet-point, Δ-point.

DAUGAVETI- JA DELTA-PUNKTID LIPSCHITZI-VABADES BANACHI RUUMIDES

Magistritöö

Triinu Veeorg

Lühikokkuvõte: Magistritöös esitatakse kriteeriumid Daugaveti- ja Delta-punktide kirjeldamiseks Lipschitzi-vabades Banachi ruumides. Saadud tulemused täiendavad M. Jung ja A. Rueda Zoca hiljutises eeltrükis *"Daugavet points and Δ-points in Lipschitz-free Banach spaces"* avaldatud tulemusi.

CERCS teaduseriala: P140 Jadad, Fourier analüüs, funktsionaalanalüüs.

Märksõnad: Banachi ruum, Lipschitzi-vaba ruum, viil, Daugaveti omadus, Daugaveti-punkt, Δ-punkt.

Contents

Introduction

This Master's thesis is in the field of functional analysis, more specifically, in Banach space geometry. In the thesis we study Daugavet- and Δ-points in Lipschitz-free Banach spaces. This is a fairly new direction in the investigation of the famous Daugavet property and diameter-2 properties; Daugavet- and Δ-points were first introduced in 2018 [1] by T. A. Abrahamsen, R. Haller, V. Lima and K. Pirk.

This new line of study has received the interest of several researchers, in particular interest to us is the very recent preprint [11] by M. Jung and A. Rueda Zoca. They obtained several results both on Daugavet-points and on Δ-points in Lipschitz-free spaces. Most importantly, the following characterization were provided in [11]:

- If M is a compact metric space, then $\mu \in S_{\mathcal{F}(M)}$ is a Daugavet-point if and only if $\|\mu - \nu\| = 2$ for every denting point μ;
- If M is a metric space and $x \neq y \in M$, then $m_{xy} \in S_{\mathcal{F}(M)}$ is a Δ-point if and only if for every $\varepsilon > 0$ and slice S with $m_{xy} \in S$ there exist $u \neq v \in M$ such that $m_{uv} \in S$ and $d(u,v) < \varepsilon$.

This left open two important questions:

- How to characterize Daugavet-points in Lipschitz-free spaces when M is not compact?
- How to characterize Δ-points that are not molecules in Lipschitz-free spaces?

The goal of the thesis was to provide answers to these questions.

The thesis are divided into three chapters. In the first chapter we will introduce main concepts as well as some useful auxiliary results.

The second chapter is dedicated to Daugavet-points. We shall present two results from [11] and then add some new results. Our main purpose here is to provide a characterization for Daugavet-points in Lipschitz-free spaces that works for any metric space M. Two such characterizations are provided in Theorem 2.6. Subsequently we will apply this theorem to construct an example of a metric space M such that the corresponding Lipschitz-free space $\mathcal{F}(M)$ has the Radon–Nikodým property and also contains a Daugavet-point.

In the third chapter we examine Δ-points. First we will present several results from [11] and then add a few original results to the existing ones. Our main purpose of here is to provide a characterization for Δ-points among convex combinations of molecules, which is archived by Theorem 3.7. We shall apply our new result to construct an example showing that a convex combination of molecules that are not Δ-points can be a Δ-point.

In the thesis, we consider only real Banach spaces. We use common notation. For Banach space X we will denote the closed unit ball by B_X, the unit sphere by S_X and the dual space by X^*.

1 Preliminaries

In this chapter we will introduce some concepts and results that will be used throughout the thesis.

First let us recall the definition of closed convex hull.

Definition 1.1. Let A be a subset of Banach space X. The **convex hull** of subset A is the set

$$\operatorname{conv} A = \{\sum_{i=1}^{n} \lambda_i x_i \colon n \in \mathbb{N}, x_1, \ldots, x_n \in A, \lambda_1, \ldots, \lambda_n > 0, \lambda_1 + \cdots + \lambda_n = 1\}.$$

The **closed convex hull** $\overline{\operatorname{conv}}\, A$ of subset A is the closure of the convex hull.

1.1 Daugavet- and Delta-points

There are many different ways to define Daugavet- and Δ-points. In this thesis, we will define these points using slices.

Definition 1.2. Let X be a Banach space. A **slice** of the unit ball B_X is a set

$$S(x^*, \alpha) = \{y \in B_X \colon x^*(y) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

Definition 1.3 (see [1, Lemmas 2.1,2.2])**.** We say that a norm-1 element x of a Banach space X is a **Daugavet-point**, if for every slice S of B_X and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.

We say that a norm-1 element x of a Banach space X is a **Δ-point**, if for every slice S of B_X with $x \in S$ and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.

The concepts of Daugavet-point and Δ-point were first introduced in 2018 by the authors of the article [1], where they defined these points by using convex combinations and also presented the equivalent definition used above. The concepts are closely related to the Daugavet property and the diametral local diameter two property.

Definition 1.4 (see [14, Lemma 2.2])**.** We say that a Banach space X has the **Daugavet property**, if for every $x \in S_X$, for every $\varepsilon > 0$ and for every slice S of B_X there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.

Definition 1.5 (see [5, Page 2])**.** We say that a Banach space X has the **diametral local diameter two property**, if for every slice S of B_X, for every $\varepsilon > 0$ and for every $x \in S_X \cap S$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.

It is easy to see that a Banach space X has the Daugavet property if and only if every $x \in S_X$ is a Daugavet-point, and X has the diametral local diameter two property if and only if every $x \in S_X$ is a Δ-point.

We will provide another equivalent criterion for Δ-points that will be used in the thesis.

Lemma 1.6 (see [11, Remark 2.4])**.** *Let X be a Banach space, $x \in S_X$ and let $A \subset B_X$ be such that $\overline{\text{conv}}\, A = B_X$. Then x is a Δ-point if and only if for every slice S with $x \in S$ and $\varepsilon > 0$, there exists $y \in A \cap S$ such that $\|x - y\| \geq 2 - \varepsilon$.*

The last lemma of this section allows us to choose however small α when proving an element is a Δ-point.

Lemma 1.7 (see [10, Lemma 2.1])**.** *Let X be a Banach space, $x^* \in S_{X^*}$ and $\alpha > 0$. Then for every $x \in S(x^*, \alpha)$ and every $\delta > 0$ there exists $y* \in S_{X^*}$ such that $x \in S(y^*, \delta)$ and $S(y^*, \delta) \subset S(x^*, \alpha)$.*

1.2 Lipschitz-free Banach spaces

Definition 1.8. Let M and N be metric spaces and $f\colon M \to N$. We say that a mapping f is **Lipschitz** if there exists a constant $L \geq 0$ such that

$$d\big(f(p), f(q)\big) \leq L \cdot d(p, q)$$

for all $p, q \in M$. The least such number L is called the **Lipschitz constant** of f and is denoted by $\text{Lip}(f)$.

Definition 1.9. A metric space M together with a fixed point $0 \in M$ is called a **pointed metric space**.

From here on we assume that all metric spaces M are in fact pointed. We will denote by $\text{Lip}_0(M)$ the vector space of all such Lipschitz functions $f\colon M \to \mathbb{R}$ that $f(0) = 0$. It is a known fact (see, e.g., [13, Proposition 1.29]) that $\text{Lip}_0(M)$ with the norm

$$\|f\| = \text{Lip}(f)$$

is a Banach space.

Let $\delta\colon M \to \text{Lip}_0(M)^*$ be the canonical isometric embedding of M into $\text{Lip}_0(M)^*$, which is given by $x \longmapsto \delta_x$, where $\delta_x(f) = f(x)$. The norm closed linear span of $\delta(M)$ is called the **Lipschitz-free space** and denoted by $\mathcal{F}(M)$. An element in $\text{Lip}_0(M)^*$ of the form

$$m_{xy} = \frac{\delta_x - \delta_y}{d(x, y)}$$

for $x \neq y \in M$ is called a **molecule**. We denote the set of all molecules by $\mathrm{Mol}(M)$. It is well-known and easy to prove that $\|m_{xy}\| = 1$ for every $x \neq y \in M$. Also, note that

$$\overline{\mathrm{conv}}\big(\mathrm{Mol}(M)\big) = B_{\mathcal{F}(M)},$$

therefore we can use Lemma 1.6 when deriving results about Δ-points.

Theorem 1.10 (see, e.g., [13, Theorem 3.3])**.** *Let M be a pointed metric space. Then $\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M)$.*

By Theorem 1.10, in Banach space $\mathcal{F}(M)$ we can choose the defining functional of the slice from $S_{\mathrm{Lip}_0(M)}$. Note that for convex combinations of molecules, f and $f + a$ give the same value for every $f \in \mathrm{Lip}_0(M)$ and constant a. Therefore, we do not need to ensure that $f(0) = 0$, when creating slices.

Theorem 1.11 (see, e.g., [13, Theorem 1.33])**.** *Let M be a metric space, let M_0 be a nonempty subset of M and let f_0 be a Lipschitz function from M_0 into $\mathbb{R}$. Then there exists an extension $f\colon M \to \mathbb{R}$ of f_0 which has the same Lipschitz constant as f_0.*

1.3 Auxiliary lemmas

In this section, we will introduce some lemmas that will be used throughout the thesis. The results of this chapter hold for all metric spaces M.

Lemma 1.12 (see [13, Proposition 1.32])**.** *Let $f, g \in \mathrm{Lip}_0(M)$. Then we have $\min\{f, g\}, \max\{f, g\} \in \mathrm{Lip}_0(M)$. Furthermore,*

$$\big\| \min\{f, g\} \big\| \leq \max\big\{\|f\|, \|g\|\big\}$$

and

$$\big\| \max\{f, g\} \big\| \leq \max\big\{\|f\|, \|g\|\big\}.$$

When proving that a unit sphere element is a Daugavet-point or a Δ-point we usually examine elements that are almost at distance two from each other. We shall provide two lemmas that assist us with this.

Lemma 1.13. *Let $x \neq y, u \neq v \in M$ and $\varepsilon > 0$. The following statements are equivalent:*

(i) $\|m_{xy} + m_{uv}\| \geq 2 - \varepsilon$;

(ii) $d(x, v) + d(u, y) \geq d(x, y) + d(u, v) - \varepsilon \max\big\{d(x, y), d(u, v)\big\}$.

Proof. Without loss of generality, let us assume that $d(x,y) \geq d(u,v)$.
$(i) \Rightarrow (ii)$. Assume that $\|m_{xy} + m_{uv}\| \geq 2 - \varepsilon$. Then

$$\begin{aligned} 2 - \varepsilon &\leq \|m_{xy} + m_{uv}\| \\ &= \frac{\big\|m_{xv} d(x,v) + m_{uy} d(u,y) + m_{uv}\big(d(x,y) - d(u,v)\big)\big\|}{d(x,y)} \\ &\leq \frac{d(x,v) + d(u,y) + d(x,y) - d(u,v)}{d(x,y)}. \end{aligned}$$

Therefore

$$d(x,v) + d(u,y) \geq (1-\varepsilon)d(x,y) + d(u,v) = d(x,y) + d(u,v) - \varepsilon \max\big\{d(x,y), d(u,v)\big\}.$$

$(ii) \Rightarrow (i)$. Assume that

$$d(x,v) + d(y,u) \geq d(x,y) + d(u,v) - \varepsilon \max\big\{d(x,y), d(u,v)\big\},$$

i.e.,

$$d(v,x) + d(y,u) - d(u,v) \geq (1 - \varepsilon)d(x,y).$$

Let us examine the function $f \colon M \to \mathbb{R}$,

$$f(p) = \min\big\{d(y,p), d(v,p) + d(y,u) - d(u,v)\big\}.$$

According to Lemma 1.12, $f \in \mathrm{Lip}_0(M)$ and $\|f\| \leq 1$. Let us note that

$$\begin{aligned} f(x) &= \min\big\{d(y,x), d(v,x) + d(y,u) - d(u,v)\big\} \geq (1-\varepsilon)d(x,y), \\ f(y) &= \min\big\{0, d(v,y) + d(y,u) - d(u,v)\big\} = 0, \\ f(u) &= \min\big\{d(y,u), d(v,u) + d(y,u) - d(u,v)\big\} = d(y,u), \\ f(v) &= \min\big\{d(y,v), 0 + d(y,u) - d(u,v)\big\} = d(y,u) - d(u,v). \end{aligned}$$

Therefore

$$\|m_{xy} + m_{uv}\| \geq \frac{f(x) - f(y)}{d(x,y)} + \frac{f(u) - f(v)}{d(u,v)} \geq 1 - \varepsilon + 1 = 2 - \varepsilon.$$

□

We will derive our next lemma from the following theorem.

Theorem 1.14 (see [11, Theorem 2.6])**.** *Let (u_n), (v_n) be two sequences in M such that $u_n \neq v_n$ for every $n \in \mathbb{N}$ and $d(u_n, v_n) \to 0$. Then for every $\mu \in S_{\mathcal{F}(M)}$ we get that*

$$\|\mu + m_{u_n v_n}\| \to 2.$$

Lemma 1.15. *Let $\mu \in S_{\mathcal{F}(M)}$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $u \neq v \in M$ satisfy $d(u,v) < \delta$, then $\|\mu - m_{uv}\| \geq 2 - \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Assume for the sake of contradiction that for every $n \in \mathbb{N}$ there exist $u_n \neq v_n \in M$ such that $d(u_n, v_n) < 1/n$ and $\|\mu - m_{u_n v_n}\| < 2 - \varepsilon$. Then $d(u_n, v_n) \to 0$ and $\|\mu - m_{u_n v_n}\| \not\to 2$, which contradicts Theorem 1.14.

Therefore there must exists $n \in \mathbb{N}$ such that $\|\mu - m_{uv}\| \geq 2 - \varepsilon$ for every $u \neq v \in M$ with $d(u, v) < 1/n$. By choosing $\delta = 1/n$, we conclude the proof. $\square$

Some results about Δ-points in $\mathrm{Lip}_0(M)$ are achieved by constructing a special Lipschitz function. The last two lemmas we introduce in this chapter will assist us with that.

Lemma 1.16. *Let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n > 0$, $x_1 \neq y_1, \ldots, x_n \neq y_n \in M$ and $f \in S_{\mathrm{Lip}_0(M)}$ be such that*

$$f\Big(\sum_{i=1}^{n} \lambda_i m_{x_i y_i}\Big) = \Big\| \sum_{i=1}^{n} \lambda_i m_{x_i y_i} \Big\| = \sum_{i=1}^{n} \lambda_i.$$

If $m \in \{1, \ldots, n\}$ and $k_1, \ldots, k_{m+1} \in \{1, \ldots, n\}$ with $k_1, \ldots, k_m$ pairwise distinct and $k_{m+1} = k_1$ are such that

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}),$$

then $f(x_{k_1}) - f(y_{k_2}) = d(x_{k_1}, y_{k_2})$. In particular, $f(x_i) - f(y_i) = d(x_i, y_i)$ for all $i \in \{1, \ldots, n\}$.

Proof. Let $m \in \{1, \ldots, n\}$ and $k_1, \ldots, k_{m+1} \in \{1, \ldots, n\}$ be such that $k_1, \ldots, k_m$ are pairwise distinct, $k_{m+1} = k_1$ and

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}).$$

Set

$$\lambda_0 = \min_{i \in \{1, \ldots, n\}} \frac{\lambda_i}{d(x_i, y_i)}$$

and

$$l_i = \begin{cases} \lambda_i - \lambda_0 d(x_i, y_i), & \text{if } i \in \{k_1, \ldots, k_m\}, \\ \lambda_i, & \text{if } i \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_m\}. \end{cases}$$

Clearly $l_i \geq 0$ for every $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
\sum_{i=1}^{n} l_i + \lambda_0 \sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) &= \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i \frac{f(x_i) - f(y_i)}{d(x_i, y_i)} \\
&= \sum_{i=1}^{n} l_i \frac{f(x_i) - f(y_i)}{d(x_i, y_i)} + \lambda_0 \sum_{i=1}^{m} \big(f(x_{k_i}) - f(y_{k_i})\big) \\
&= \sum_{i=1}^{n} l_i \frac{f(x_i) - f(y_i)}{d(x_i, y_i)} + \lambda_0 \sum_{i=1}^{m} \big(f(x_{k_i}) - f(y_{k_{i+1}})\big) \\
&\leq \sum_{i=1}^{n} l_i + \lambda_0 \sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}).
\end{aligned}$$

In particular, $f(x_{k_1}) - f(y_{k_2}) = d(x_{k_1}, y_{k_2})$. $\square$

Lemma 1.17. *Let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n > 0$ and $x_1 \neq y_1, \ldots, x_n \neq y_n \in M$ be such that $\sum_{i=1}^{n} \lambda_i = \big\| \sum_{i=1}^{n} \lambda_i m_{x_i y_i} \big\|$. There exists $f \in S_{\mathrm{Lip}_0(M)}$ such that $f\big(\sum_{i=1}^{n} \lambda_i m_{x_i y_i}\big) = \big\| \sum_{i=1}^{n} \lambda_i m_{x_i y_i} \big\|$ and for all $k_1, k_2 \in \{1, \ldots, n\}$ the following conditions are equivalent:*

- *(i)* $f(x_{k_1}) - f(y_{k_2}) = d(x_{k_1}, y_{k_2})$;
- *(ii)* *Then either $k_1 = k_2$ or there exist $m \in \{1, \ldots, n\}$ and $k_3, \ldots, k_{m+1} \in \{1, \ldots, n\}$ with $k_1, \ldots, k_m$ pairwise distinct and $k_{m+1} = k_1$ such that*

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}).$$

Proof. Write $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i y_i}$. From Lemma 1.16 we see that if for $f \in S_{\mathrm{Lip}_0(M)}$ we have $f(\mu) = \|\mu\|$, then $(ii) \Rightarrow (i)$ holds true. Therefore our goal is to find such $f \in S_{\mathrm{Lip}_0(M)}$ that $f(\mu) = \|\mu\|$ and $(i) \Rightarrow (ii)$ holds true. We will start with a function $g \in S_{\mathrm{Lip}_0(M)}$ such that $g(\mu) = \|\mu\|$ and use g to define new functions. Note that in case $n = 1$ we can choose $f = g$, therefore from here on we will assume $n > 1$.

Let $A \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ be such that $(k_1, k_2) \in A$ if and only if $k_1 \neq k_2$ and for every $m \in \{1, \ldots, n\}$ and $k_3, \ldots, k_{m+1} \in \{1, \ldots, n\}$ with $k_1, \ldots, k_m$ pairwise distinct and $k_{m+1} = k_1$ we have

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) \neq \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}).$$

For each $k = (k_1, k_2) \in A$ we will define a function $h_k \in S_{\mathrm{Lip}_0(M)}$ such that $h_k(\mu) = \|\mu\|$ and $h_k(x_{k_1}) - h_k(y_{k_2}) < d(x_{k_1}, y_{k_2})$.

Fix $k = (k_1, k_2) \in A$. If $g(x_{k_1}) - g(y_{k_2}) < d(x_{k_1}, y_{k_2})$, then let $h_k = g$.

Now we consider the case where $g(x_{k_1}) - g(y_{k_2}) = d(x_{k_1}, y_{k_2})$. Set

$$M_0 = \{x_1, \dots, x_n, y_1, \dots, y_n\} \subseteq M.$$

Let us first define a function $\tilde{h}_k$ on M_0. To do so, we will first define a set of indexes. Let $B \subseteq \{1, \dots, n\}$ be such that $i \in B$ if and only if there exist $m \in \{2, \dots, n\}$ and $k_3, \dots, k_{m+1} \in \{1, \dots, n\}$ with $k_1, \dots, k_m$ pairwise distinct and $k_{m+1} = i$ such that

$$g(x_{k_j}) - g(y_{k_{j+1}}) = d(x_{k_j}, y_{k_{j+1}})$$

for every $j \in \{1, \dots, m\}$.

From Lemma 1.16 we get that $g(x_{k_2}) - g(y_{k_2}) = d(x_{k_2}, y_{k_2})$. Therefore we can take $m = 2$ and $k_3 = k_2$, giving us $k_2 \in B$.

Let us assume that $k_1 \in B$. Then there exist $m \in \{2, \dots, n\}$ and $k_3, \dots, k_{m+1} \in \{1, \dots, n\}$ with $k_1, \dots, k_m$ pairwise distinct and $k_{m+1} = k_1$ such that

$$g(x_{k_j}) - g(y_{k_{j+1}}) = d(x_{k_j}, y_{k_{j+1}})$$

for every $j \in \{1, \dots, m\}$. By reshuffling we get

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} \big(g(x_{k_j}) - g(y_{k_{j+1}})\big) = \sum_{j=1}^{m} \big(g(x_{k_j}) - g(y_{k_j})\big) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}),$$

contradicting $(k_1, k_2) \in A$. Therefore $k_1 \notin B$.

Let us show that if $g(x_i) - g(y_{i'}) = d(x_i, y_{i'})$ for $i \in B$ and $i' \in \{1, \dots, n\}$, then $i' \in B$. There exist $m \in \{2, \dots, n\}$ and $k_3, \dots, k_{m+1} \in \{1, \dots, n\}$ with $k_1, \dots, k_m$ pairwise distinct and $k_{m+1} = i$ such that

$$g(x_{k_j}) - g(y_{k_{j+1}}) = d(x_{k_j}, y_{k_{j+1}})$$

for every $j \in \{1, \dots, m\}$. If $i' = k_j$ for some $j \in \{3, \dots, m+1\}$, then $k_3, \dots, k_j$ are suitable indexes, giving us $i' \in B$. Otherwise by the pigeonhole principle we must have $m < n$ and therefore $k_3, \dots, k_{m+1}, i'$ are suitable indexes, giving us $i' \in B$.

Now let $C = \{x_i \colon i \in B\} \cup \{y_i \colon i \in B\}$. Assume that $g(p) - g(q) = d(p, q)$ for some $p \in C$ and $q \in M_0$. Let us show that then $q \in C$. Let $i \in B$, $j \in \{1, \dots, n\}$ be such that $p \in \{x_i, y_i\}$ and $q \in \{x_j, y_j\}$. Then

$$\begin{aligned} g(x_i) - g(y_j) &= g(x_i) - g(p) + g(p) - g(q) + g(q) - g(y_j) \\ &= d(x_i, p) + d(p, q) + d(q, y_j) \\ &\geq d(x_i, y_j). \end{aligned}$$

This gives us that $j \in B$, i.e., $q \in C$. From this we deduce that if $p \in C$, $q \in M_0 \setminus C$, then $g(p) - g(q) < d(p, q)$.

Let $\delta > 0$ be such that for every $p, q \in M_0$, if $g(p) - g(q) < d(p,q)$, then $g(p) - g(p) + \delta < d(p,q)$. Finally we are ready to define the function $\tilde{h}_k$. Let

$$\tilde{h}_k(p) = \begin{cases} g(p) + \delta, & \text{if } p \in C, \\ g(p), & \text{if } p \in M_0 \setminus C. \end{cases}$$

First we will show that $\|\tilde{h}_k\| \leq 1$. Let $p \in C$ and $q \in \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \setminus C$. We showed previously that in this case $g(p) - g(q) < d(p,q)$ and therefore $g(p) - g(q) + \delta < d(p,q)$. Now we see that

$$\tilde{h}_k(p) - \tilde{h}_k(q) = g(p) - g(q) + \delta < d(p,q)$$

and

$$\tilde{h}_k(q) - \tilde{h}_k(p) = g(q) - g(p) - \delta < d(p,q)$$

giving us $|\tilde{h}_k(p) - \tilde{h}_k(q)| < d(p,q)$. Therefore $\mathrm{Lip}(\tilde{h}_k) \leq 1$.

Let us also note that for every $i \in \{1, \ldots, n\}$ we have either $x_i, y_i \in C$ or $x_i, y_i \notin C$ giving us that $\tilde{h}_k(x_i) - \tilde{h}_k(y_i) = g(x_i) - g(y_i)$. Therefore

$$\tilde{h}_k(\mu) = g(\mu) = \|\mu\|.$$

Last we point out that

$$\tilde{h}_k(x_{k_1}) - \tilde{h}_k(y_{k_2}) = g(x_{k_1}) - g(y_{k_2}) - \delta < d(x_{k_1}, y_{k_2}).$$

We will extend $\tilde{h}_k$ by Theorem 1.11 to a Lipschitz map h_k on M, with $\mathrm{Lip}_0(h_k) = \mathrm{Lip}_0(\tilde{h}_k) = 1$

Now we have defined h_k for every $k \in A$. Let

$$f := \frac{1}{|A|} \sum_{k \in A} h_k.$$

By the triangle inequality $\|f\| \leq 1$ and

$$f(\mu) = \frac{1}{|A|} \sum_{k \in A} \sum_{i=1}^{n} \lambda_i \frac{h_k(x_i) - h_k(y_i)}{d(x_i, y_i)} = \|\mu\|.$$

Furthermore, for every $k = (k_1, k_2) \in A$ we have

$$\begin{aligned} f(x_{k_1}) - f(y_{k_2}) &= \frac{1}{|A|} \sum_{l \in A} \big(h_l(x_{k_1}) - h_l(y_{k_2})\big) \\ &\leq \frac{|A| - 1}{|A|} d(x_{k_1}, y_{k_2}) + h_k(x_{k_1}) - h_k(y_{k_2}) < d(x_{k_1}, y_{k_2}). \end{aligned}$$

□

2 Daugavet-points in Lipschitz-free Banach spaces

In this section, we study Daugavet-points in Lipschitz-free spaces. We present previously known results and generalize some of them.

In order to present these results we first need to introduce the concept of denting points, extreme points and preserved extreme points.

Definition 2.1 (see, e.g., [7, Definition 3.59], [6, Page 119])**.** Let K be a bounded closed convex subset of a Banach space X. We say that $x \in K$ is an **extreme point** of K if $x_1 = x_2 = x$ whenever $x_1, x_2 \in K$ and $x = 1/2(x_1 + x_2)$.

We say that $x \in K$ is a **denting point** of K if $x \notin \overline{\text{conv}}\big(K \setminus B(x, \varepsilon)\big)$ for every $\varepsilon > 0$.

We denote the set of all extreme points of K by $\text{ext}(K)$ and the set of all denting points of K by $\text{dent}(K)$.

We say that an extreme point $x \in \text{ext}(B_X)$ is a **preserved extreme point** of B_X if $x \in \text{ext}(B_{X^{**}})$. Note that in Lipschitz-free spaces the concepts of denting point and preserved extreme point of $B_{\mathcal{F}(M)}$ are equivalent (see [8, Theorem 2.4]). Therefore any result attained for preserved extreme point holds for denting points as well.

Lemma 2.2 (see [13, Corollary 3.44], [8, Theorem 2.4])**.** *Let M be a complete metric space and let $\mu \in \text{dent}(B_{\mathcal{F}(M)})$. Then $\mu = m_{uv}$ for some $u \neq v \in M$.*

Let us introduce some more notation. For every $u, v \in M$ and $\delta > 0$ we denote

$$[u, v] = \big\{p \in M : d(u, p) + d(v, p) = d(u, v)\big\}$$

and

$$\text{Line}(u, v, \delta) = \big\{p \in M : d(u, p) + d(v, p) < (1 + \delta)d(u, v)\big\}.$$

This first set contains all points that are on the segment between u and v and the second set contains all points that are close to the segment between u and v.

Theorem 2.3 (see [8, Theorem 2.6])**.** *Let M be a metric space and $u \neq v \in M$. The following are equivalent:*

(i) The molecule m_{uv} is a denting point of $B_{\mathcal{F}(M)}$.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{Line}(u, v, \delta) \subseteq B(u, \varepsilon) \cup B(v, \varepsilon).$$

2.1 Characterization of Daugavet-point in Lipschitz-free Banach spaces

The following necessary condition for a norm-1 element to be a Daugavet-point was presented in [11].

Proposition 2.4 (see, e.g. [11, Proposition 3.1])**.** *Let X be a Banach space and $x \in S_X$ be a Daugavet-point. Then for every $y \in \mathrm{dent}(B_X)$ we have $\|x - y\| = 2$.*

It was shown in [11] that in Lipschitz-free spaces over compact metric spaces the previous condition is, in fact, an equivalent condition.

Theorem 2.5 (see [11, Theorem 3.2])**.** *Let M be a compact metric space and $\mu \in S_{\mathcal{F}(M)}$. The following statements are equivalent:*

(i) μ is a Daugavet-point.

(ii) For every $\nu \in \mathrm{dent}(B_{\mathcal{F}(M)})$ we have $\|\mu - \nu\| = 2$.

Moreover, if μ is of the form m_{xy} for $x \neq y \in M$, then the previous two statements are equivalent to:

(iii) If $u, v \in M$ satisfy $[u, v] = \{u, v\}$ then

$$d(x,y) + d(u,v) \leq \min\{d(x,u) + d(y,v), d(x,v) + d(y,u)\}. \tag{2.1}$$

Here we would like to point out that according to Lemma 1.13 the condition 2.1 is equivalent to $\|m_{xy} \pm m_{uv}\| = 2$.

This theorem was used to provide an example of a Lipschitz-free space that contains a Daugavet-point, but does not have the Daugavet property (see [11, Example 3.3]).

In this thesis we will not present the original proof of Theorem 2.5 (for the original proof please see [11]). Instead we will provide a characterization of Daugavet-point for any metric space M and use that to prove Theorem 2.5.

Theorem 2.6. *Let M be a metric space and $\mu \in S_{\mathcal{F}(M)}$. The following statements are equivalent:*

(i) μ is a Daugavet-point;

(ii) For every $\nu \in \mathrm{dent}(B_{\mathcal{F}(M)})$ we have $\|\mu - \nu\| = 2$;

(iii) For every $u \neq v \in M$ and $\varepsilon > 0$, if there exists $\delta > 0$ such that

$$B\big(u, (1-\varepsilon)d(u,v)\big) \cap B\big(v, (1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u, v, \delta) = \emptyset,$$

then $\|\mu - m_{uv}\| \geq 2 - 4\varepsilon$.

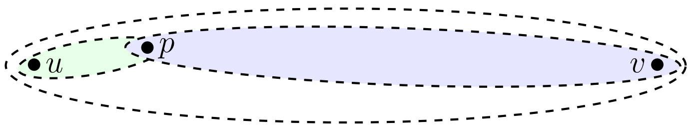

Figure 1: Illustration of Lemma 2.7

Our proof of Theorem 2.6 is long, therefore we shall divide the proof into parts and first introduce two lemmas used in the proof.

Lemma 2.7. *Let $u \neq v \in M$, $\delta > 0$ and let $p \in \mathrm{Line}(u, v, \delta)$. Then there exists $\delta' > 0$ such that*

$$\mathrm{Line}(p, v, \delta') \subseteq \mathrm{Line}(u, v, \delta)$$

and

$$\mathrm{Line}(u, p, \delta') \subseteq \mathrm{Line}(u, v, \delta).$$

Proof. Since $p \in \mathrm{Line}(u, v, \delta)$ then there exists $\delta' > 0$ such that

$$(1 + \delta')\big(d(u, p) + d(v, p)\big) < (1 + \delta)d(u, v).$$

If $q \in \mathrm{Line}(p, v, \delta')$, then

$$\begin{aligned} d(u, q) + d(v, q) &< d(u, q) + (1 + \delta')d(v, p) - d(p, q) \\ &\leq d(u, p) + (1 + \delta')d(v, p) \\ &< (1 + \delta)d(u, v) \end{aligned}$$

giving us $\mathrm{Line}(p, v, \delta') \subseteq \mathrm{Line}(u, v, \delta)$. The inclusion $\mathrm{Line}(u, p, \delta') \subseteq \mathrm{Line}(u, v, \delta)$ can be proved analogously. $\square$

Lemma 2.8. *Let $u \neq v \in M$, $\varepsilon \in (0, 1/2)$ and $\delta > 0$ be such that*

$$B\big(u, (1 - \varepsilon)d(u, v)\big) \cap B\big(v, (1 - \varepsilon)d(u, v)\big) \cap \mathrm{Line}(u, v, \delta) = \emptyset.$$

For every $\alpha > 0$, there exist $\beta > 0$, $x \in B\big(u, (1 - \varepsilon)d(u, v)\big) \cap \mathrm{Line}(u, v, \delta)$ and $y \in B\big(v, (1 - \varepsilon)d(u, v)\big) \cap \mathrm{Line}(u, v, \delta)$ such that the following holds

(1) $B\big(x, (1 - \alpha)d(x, y)\big) \cap B\big(y, (1 - \alpha)d(x, y)\big) \cap \mathrm{Line}(x, y, 4\beta) = \emptyset$,

(2) $\mathrm{Line}(x, y, \beta) \subseteq \mathrm{Line}(u, v, \delta)$,

(3) $d(x, y) \leq d(u, v)$,

(4) $d(u, x) + d(v, y) + d(x, y) < (1 + \delta)d(u, v)$.

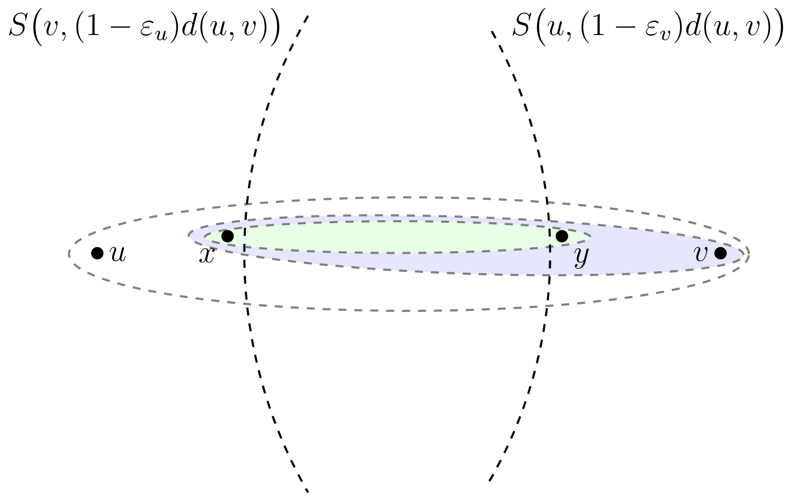

Figure 2: Illustration of Lemma 2.8

Proof. Fix $\alpha > 0$. Let $\delta_u > 0$ be such that $\delta_u < \delta$ and $3\delta_u < \alpha(1-2\varepsilon)$. Then

$$B\big(u,(1-\varepsilon)d(u,v)\big) \cap B\big(v,(1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta_u) = \emptyset.$$

Set

$$\varepsilon_u = \sup\big\{\varepsilon_0 : B\big(u,(1-\varepsilon)d(u,v)\big) \cap B\big(v,(1-\varepsilon_0)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta_u) \neq \emptyset\big\}.$$

We see that $0 \le \varepsilon_u \le \varepsilon$. If $\varepsilon_u = 0$, then let $\gamma_u = 0$ and $x = u$, otherwise let $\gamma_u > 0$ be such that $\gamma_u < \varepsilon_u$ and $\gamma_u < \delta_u/2$ and let

$$x \in B\big(u,(1-\varepsilon)d(u,v)\big) \cap B\big(v,(1-\varepsilon_u+\gamma_u)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta_u).$$

Note that $\gamma_u - \varepsilon_u \le 0$. Furthermore, $d(x,v) \ge (1-\varepsilon_u)d(u,v)$ since

$$B\big(u,(1-\varepsilon)d(u,v)\big) \cap B\big(v,(1-\varepsilon_0)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta_u) = \emptyset$$

for every $\varepsilon_0 > \varepsilon_u$. According to Lemma 2.7 there exists $\delta_v > 0$ such that

$$\mathrm{Line}(x,v,\delta_v) \subseteq \mathrm{Line}(u,v,\delta_u).$$

We may assume that δ_v is small enough to satisfy $\delta_v(1-\varepsilon_u+\gamma_u) < \gamma_u/2$,

$$d(u,x) + (1+\delta_v)d(x,v) < (1+\delta_u)d(u,v),$$

and if $\varepsilon_u > 0$, then also $(1+\delta_v)(1-\varepsilon_u+\gamma_u) < 1$. Note that from $\mathrm{Line}(x,v,\delta_v) \subseteq \mathrm{Line}(u,v,\delta_u)$ we get

$$B\big(u,(1-\varepsilon)d(u,v)\big) \cap B\big(v,(1-\varepsilon_u-\gamma_u)d(u,v)\big) \cap \mathrm{Line}(x,v,\delta_v) = \emptyset.$$

Now set

$$\varepsilon_v = \sup\bigl\{\varepsilon_0 : B\bigl(u,(1-\varepsilon_0)d(u,v)\bigr)\cap B\bigl(v,(1-\varepsilon_u-\gamma_u)d(u,v)\bigr)\cap \mathrm{Line}(x,v,\delta_v)\neq\emptyset\bigr\}.$$

We see that $0\le\varepsilon_v\le\varepsilon$. If $\varepsilon_v=0$, then let $\gamma_v=0$ and $y=v$, otherwise let $\gamma_v>0$ be such that $\gamma_v<\varepsilon_v$ and $\gamma_v<\gamma_u/2$ and let

$$y\in B\bigl(u,(1-\varepsilon_v+\gamma_v)d(u,v)\bigr)\cap B\bigl(v,(1-\varepsilon_u-\gamma_u)d(u,v)\bigr)\cap \mathrm{Line}(x,v,\delta_v).$$

Note that $\gamma_v-\varepsilon_v\le 0$ and $d(y,u)\ge(1-\varepsilon_v)d(u,v)$. According to Lemma 2.7 there exists $\beta>0$ such that

$$\mathrm{Line}(x,y,\beta)\subseteq \mathrm{Line}(x,y,4\beta)\subseteq \mathrm{Line}(x,v,\delta_v)\subseteq \mathrm{Line}(u,v,\delta_u)\subseteq \mathrm{Line}(u,v,\delta).$$

Hence (2) holds. Note that

$$d(u,x)+d(v,y)+d(x,y)<d(u,x)+(1+\delta_v)d(x,v)<(1+\delta_u)d(u,v)$$

and therefore condition (4) is also true.

If $\varepsilon_u=0$, then $x=u$ and

$$d(x,y)=d(u,y)\le(1-\varepsilon_v+\gamma_v)d(u,v)\le d(u,v).$$

If $\varepsilon_u>0$, then

$$\begin{aligned}
d(x,y)&<(1+\delta_v)d(x,v)-d(v,y)\\
&\le(1+\delta_v)(1-\varepsilon_u+\gamma_u)d(u,v)-d(u,v)+d(u,y)\\
&<d(u,v)-d(u,v)+(1-\varepsilon_v+\gamma_v)d(u,v)\\
&\le d(u,v).
\end{aligned}$$

Therefore $d(x,y)\le d(u,v)$, i.e., (3) is true. Also,

$$\begin{aligned}
d(x,y)&<(1+\delta_v)d(x,v)-d(v,y)\\
&\le(1+\delta_v)(1-\varepsilon_u+\gamma_u)d(u,v)-d(u,v)+d(u,y)\\
&\le\delta_v(1-\varepsilon_u+\gamma_u)d(u,v)+(\gamma_u-\varepsilon_u)d(u,v)+(1-\varepsilon_v+\gamma_v)d(u,v)\\
&<(1-\varepsilon_u-\varepsilon_v+2\gamma_u)d(u,v)\\
&<(1-\varepsilon_u-\varepsilon_v+\delta_u)d(u,v).
\end{aligned}$$

Assume that there exists

$$z\in B\bigl(x,(1-\alpha)d(x,y)\bigr)\cap B\bigl(y,(1-\alpha)d(x,y)\bigr)\cap \mathrm{Line}(x,y,4\beta).$$

Recall that $d(x,v)\ge(1-\varepsilon_u)d(u,v)$ and therefore

$$\begin{aligned}
d(u,z)&\le d(u,x)+d(x,z)\\
&<(1+\delta_u)d(u,v)-d(v,x)+(1-\alpha)d(x,y)\\
&\le(1+\delta_u)d(u,v)-(1-\varepsilon_u)d(u,v)+(1-\alpha)(1-\varepsilon_v-\varepsilon_u+\delta_u)d(u,v)\\
&=(1-\varepsilon_v)d(u,v)+2\delta_u d(u,v)-\alpha(1-\varepsilon_v-\varepsilon_u+\delta_u)d(u,v)\\
&<(1-\varepsilon_v)d(u,v)+\bigl(2\delta_u-\alpha(1-2\varepsilon)\bigr)d(u,v)\\
&<(1-\varepsilon_v-\delta_u)d(u,v).
\end{aligned}$$

Analogously, $d(v,z) < (1-\varepsilon_u-\delta_u)d(u,v)$.

Notice that $z \in \mathrm{Line}(x,y,4\beta) \subseteq \mathrm{Line}(x,v,\delta_v)$ and therefore

$$z \in B\big(u,(1-\varepsilon_v-\delta_u)d(u,v)\big) \cap B\big(v,(1-\varepsilon_u-\delta_u)d(u,v)\big) \cap \mathrm{Line}(x,v,\delta_v),$$

which is a contradiction since this intersection is empty. This gives us

$$B\big(x,(1-\alpha)d(x,y)\big) \cap B\big(y,(1-\alpha)d(x,y)\big) \cap \mathrm{Line}(x,y,4\beta) = \emptyset.$$

Therefore (1) holds. $\square$

Now we are ready to prove Theorem 2.6

Proof of Theorem 2.6. $(i) \Rightarrow (ii)$. Is a consequence of Proposition 2.4.

$(ii) \Rightarrow (iii)$. Assume that $\|\mu-\nu\| = 2$ for every $\nu \in \mathrm{dent}(B_{\mathcal{F}(M)})$. First we will prove $(ii) \Rightarrow (iii)$ assuming M is complete.

Note that the case $\varepsilon \geq 1/2$ is trivial. Fix $u \neq v \in M$ and $\varepsilon \in (0,1/2)$, such that there exists $\delta > 0$ with

$$B\big(u,(1-\varepsilon)d(u,v)\big) \cap B\big(v,(1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta) = \emptyset.$$

Note that we can choose δ to be however small. Therefore, if we show $\|\mu - m_{uv}\| \geq 2-4\varepsilon-4\delta$, then from that we can derive $\|\mu - m_{uv}\| \geq 2-4\varepsilon$. Additionally, we assume that $\delta < 1/2-\varepsilon$.

Let us find $x \in B\big(u,(1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta)$ and $y \in B\big(v,(1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta)$ such that m_{xy} is a denting point. To do so, we shall inductively construct two Cauchy sequences (u_n) and (v_n) of elements in the sets

$$B\big(u,(1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta)$$

and

$$B\big(v,(1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u,v,\delta),$$

respectively, and also find positive numbers δ_n and ε_n for every $n \in \mathbb{N}$. We will define these in such a way that for every $n \in \mathbb{N}$ we have

(1) $B\big(u_n,(1-\varepsilon_n)d(u_n,v_n)\big) \cap B\big(v_n,(1-\varepsilon_n)d(u_n,v_n)\big) \cap \mathrm{Line}(u_n,y_n,4\delta_n) = \emptyset$;

(2) $\mathrm{Line}(u_n,v_n,\delta_n) \subseteq \mathrm{Line}(u,v,\delta/4)$;

(3) $d(u_{n+1},v_{n+1}) \leq d(u_n,v_n)$;

(4) $d(u_n,u_{n+1}) + d(v_n,v_{n+1}) + d(u_{n+1},v_{n+1}) < (1+\delta_n)d(u_n,v_n)$;

(5) $\max\big\{d(u,u_n),d(v,v_n)\big\} < (1-\varepsilon-\delta/2^{n-1})d(u,v)$;

(6) $\delta_{n+1}, \varepsilon_{n+1} < \delta_n/2$;

starting by $\delta_1 = \delta/4$, $\varepsilon_1 = \varepsilon$, $u_1 = u$ and $v_1 = v$. Then (u_n) and (v_n) converge to the elements x and y we are looking for.

It is easy to see that conditions (1), (2), (5) and (6) hold for $\delta_1 = \delta/4$, $\varepsilon_1 = \varepsilon$, $u_1 = u$ and $v_1 = v$. Assume that we have found u_n, v_n, δ_n and ε_n for $n \in \mathbb{N}$.

Let $\varepsilon_{n+1} \in (0, \delta_n/2)$. By Lemma 2.8 (taking $u = u_n$, $v = v_n$, $\delta = \delta_n$, $\varepsilon = \varepsilon_n$ and $\alpha = \varepsilon_{n+1}$) choose δ_{n+1} $(= \beta)$, u_{n+1} $(= x)$ and v_{n+1} $(= y)$. Additionally assume that $\delta_{n+1} < \delta_n/2$.

Note that $u_{n+1} \in B\big(u_n, (1-\varepsilon_n)d(u_n, v_n)\big) \cap \operatorname{Line}(u_n, v_n, \delta_n)$. Therefore

$$d(v_n, u_{n+1}) \geq (1-\varepsilon_n)d(u_n, v_n)$$

and we get

$$\begin{aligned} d(u_n, u_{n+1}) &< (1+\delta_n)d(u_n, v_n) - d(v_n, u_{n+1}) \\ &\leq (1+\delta_n)d(u_n, v_n) - (1-\varepsilon_n)d(u_n, v_n) \\ &= (\delta_n + \varepsilon_n)d(u, v). \end{aligned}$$

If $n = 1$, then recall that $\delta + \varepsilon < 1/2$ and therefore

$$d(u, u_2) = d(u_1, u_2) < (\delta_1 + \varepsilon_1)d(u, v) < (\delta + \varepsilon)d(u, v) < (1 - \varepsilon - \delta)d(u, v).$$

If $n \neq 1$, then

$$\begin{aligned} d(u, u_{n+1}) &\leq d(u, u_n) + d(u_n, u_{n+1}) \\ &< \Big(1 - \varepsilon - \frac{\delta}{2^{n-1}}\Big)d(u, v) + (\varepsilon_n + \delta_n)d(u, v) \\ &< \Big(1 - \varepsilon - \frac{\delta}{2^{n-1}}\Big)d(u, v) + \frac{\delta}{2^n}d(u, v) \\ &= \Big(1 - \varepsilon - \frac{\delta}{2^n}\Big)d(u, v). \end{aligned}$$

Therefore

$$d(u, u_{n+1}) < (1 - \varepsilon - \delta/2^n)d(u, v).$$

Analogously,

$$d(v, v_{n+1}) < (1 - \varepsilon - \delta/2^n)d(u, v).$$

Now we have two sequences (u_n) and (v_n). Let us show that these are Cauchy sequences. Note that for every $m > n \in \mathbb{N} \setminus \{1\}$ we have

$$d(u_n, u_m) \leq \sum_{i=n}^{m-1} d(u_i, u_{i+1}) < \sum_{i=n}^{m-1} (\delta_i + \varepsilon_i)d(u, v) < \sum_{i=n}^{m-1} \frac{\delta}{2^i}d(u, v) < \frac{\delta}{2^{n-1}}d(u, v).$$

Therefore $d(u_n, u_m) \to 0$ and there exists $x \in M$ such that $u_n \to x$. Analogously we see that (v_n) is a Cauchy sequence and there exists $y \in M$ such that $v_n \to y$. Note that

$$u_n \in B\big(u, (1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u, v, \delta/4)$$

and

$$v_n \in B\big(v, (1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u, v, \delta/4),$$

for every $n \in \mathbb{N}$. Therefore

$$x \in B\big(u, (1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u, v, \delta)$$

and

$$y \in B\big(v, (1-\varepsilon)d(u,v)\big) \cap \mathrm{Line}(u, v, \delta).$$

Furthermore by (3) we have $d(x, y) \leq d(u_n, v_n)$ for every $n \in \mathbb{N}$.

Now we show that m_{xy} is a denting point. Fix $\varepsilon' > 0$. Let $n \in \mathbb{N}$ be such that $d(u_n, x) < \varepsilon'/2$, $d(v_n, y) < \varepsilon'/2$ and $(\varepsilon_n + 4\delta_n)d(u_n, v_n) < \varepsilon'/2$. Note that

$$B\big(u_n, (1-\varepsilon_n)d(u_n,v_n)\big) \cap B\big(v_n, (1-\varepsilon_n)d(u_n,v_n)\big) \cap \mathrm{Line}(u_n, v_n, \delta_n) = \emptyset.$$

There exists $m > n$ such that $d(u_m, x) + d(v_m, y) < \delta_n d(u_n, v_n)$. By conditions (4) and (6) we get

$$\begin{aligned}
d(u_n, u_m) + d(v_n, v_m) &\leq \sum_{i=n}^{m-1} \big(d(u_i, u_{i+1}) + d(v_i, v_{i+1})\big) \\
&= \sum_{i=n}^{m-1} \big(d(u_i, u_{i+1}) + d(v_i, v_{i+1}) + d(u_{i+1}, v_{i+1})\big) - \sum_{i=n}^{m-1} d(u_{i+1}, v_{i+1}) \\
&< \sum_{i=n}^{m-1} (1+\delta_i) d(u_i, v_i) - \sum_{i=n}^{m-1} d(u_{i+1}, v_{i+1}) \\
&= \sum_{i=n}^{m-1} \delta_i d(u_i, v_i) + d(u_n, v_n) - d(u_m, v_m) \\
&\leq \sum_{i=n}^{m-1} \frac{\delta_n}{2^{i-n}} d(u_n, v_n) + d(u_n, v_n) - d(u_m, v_m) \\
&< 2\delta_n d(u_n, v_n) + d(u_n, v_n) - d(u_m, v_m).
\end{aligned}$$

If $p \in \mathrm{Line}(x, y, \delta_n)$, then

$$\begin{aligned}
d(u_n, p) + d(v_n, p) &\leq d(u_n, u_m) + d(v_n, v_m) \\
&\quad + d(u_m, x) + d(v_m, y) + d(x, p) + d(y, p) \\
&< 2\delta_n d(u_n, v_n) + d(u_n, v_n) - d(u_m, v_m) \\
&\quad + \delta_n d(u_n, v_n) + (1+\delta_n) d(x, y) \\
&\leq (1 + 4\delta_n) d(u_n, v_n).
\end{aligned}$$

Therefore

$$\mathrm{Line}(x, y, \delta_n) \subseteq \mathrm{Line}(u_n, v_n, 4\delta_n).$$

Let $p \in \mathrm{Line}(x, y, \delta_n)$. Our aim is to show that $p \in B(x, \varepsilon') \cup B(y, \varepsilon')$. Note that $\delta_n + \varepsilon_n < 1/2$. Hence

$$d(u_n, p) + d(v_n, p) < (1 + 4\delta_n)d(u_n, v_n) < 2(1 - \varepsilon_n)d(u_n, v_n)$$

and either $p \in B\big(u_n, (1 - \varepsilon_n)d(u_n, v_n)\big)$ or $p \in B\big(v_n, (1 - \varepsilon_n)d(u_n, v_n)\big)$. We only consider the case $p \in B\big(u_n, (1-\varepsilon_n)d(u_n, v_n)\big)$, the case $p \in B\big(v_n, (1-\varepsilon_n)d(u_n, v_n)\big)$ is analogous. Then $p \notin B\big(v_n, (1 - \varepsilon_n)d(u_n, v_n)\big)$. Now we get

$$\begin{aligned} d(x, p) &\leq d(x, u_n) + d(u_n, p) < \frac{\varepsilon'}{2} + (1 + 4\delta_n)d(u_n, v_n) - d(v_n, p) \\ &\leq \frac{\varepsilon'}{2} + (1 + 4\delta_n)d(u_n, v_n) - (1 - \varepsilon_n)d(u_n, v_n) < \varepsilon'. \end{aligned}$$

Therefore $\mathrm{Line}(x, y, \delta_n) \subseteq B(x, \varepsilon') \cup B(y, \varepsilon')$. According to Theorem 2.3, m_{xy} is a denting point and by assumption (ii) we have $\|\mu - m_{xy}\| = 2$.

Since $x \in B\big(u, (1 - \varepsilon)d(u, v)\big) \cap \mathrm{Line}(u, v, \delta)$ and

$$B\big(u, (1 - \varepsilon)d(u, v)\big) \cap B\big(v, (1 - \varepsilon)d(u, v)\big) \cap \mathrm{Line}(u, v, \delta) = \emptyset,$$

we have

$$d(x, u) \leq (1 + \delta)d(u, v) - d(x, v) \leq (1 + \delta)d(u, v) - (1 - \varepsilon)d(u, v) = (\varepsilon + \delta)d(u, v).$$

Analogously, $d(y, v) \leq (\varepsilon + \delta)d(u, v)$. Therefore

$$\begin{aligned} \|m_{xy} - m_{uv}\| &= \frac{\big\|m_{xu}d(x, u) + m_{vy}d(v, y) - m_{uv}\big(d(u, v) - d(x, y)\big)\big\|}{d(u, v)} \\ &\leq \frac{d(x, u) + d(y, v) + \big|d(u, v) - d(x, y)\big|}{d(u, v)} \\ &\leq \frac{2(\varepsilon + \delta)d(x, y) + d(x, u) + d(y, v)}{d(u, v)} \\ &\leq 4(\varepsilon + \delta). \end{aligned}$$

Consequently,

$$\|\mu - m_{uv}\| \geq \|\mu - m_{xy}\| - \|m_{xy} - m_{uv}\| \geq 2 - 4(\varepsilon + \delta).$$

Hence $\|\mu - m_{uv}\| \geq 2 - 4\varepsilon$. This closes the case when M is complete.

Now assume M is not complete and let M' be its completion. Then $\mathcal{F}(M) = \mathcal{F}(M')$ and therefore for every $\nu \in \mathrm{dent}(B_{\mathcal{F}(M')})$ we have $\|\mu - \nu\| = 2$.

Let us note that, if for $u \neq v \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that in M we have

$$B\big(u, (1-\varepsilon)d(u,v)\big) \cap B\big(v, (1-\varepsilon)d(u,v)\big) \cap \operatorname{Line}(u,v,\delta) = \emptyset,$$

then in M' we have

$$B\big(u, (1-\varepsilon')d(u,v)\big) \cap B\big(v, (1-\varepsilon')d(u,v)\big) \cap \operatorname{Line}(u,v,\delta') = \emptyset$$

for every $\varepsilon' \in (0,\varepsilon)$ and $\delta' \in (0,\delta)$. Then by the first case we get $\|\mu - m_{uv}\| \geq 2 - 4\varepsilon'$ for every $\varepsilon' \in (0,\varepsilon)$ and therefore $\|\mu - m_{uv}\| \geq 2 - 4\varepsilon$.

This is what we wanted to prove.

$(iii) \Rightarrow (i)$. Assume that (iii) holds. We will show that μ is a Daugavet-point. Fix $f \in S_{\operatorname{Lip}_0(M)}$ and $\alpha, \varepsilon > 0$. We will prove that there exist $u \neq v \in M$ such that $m_{uv} \in S(f,\alpha)$ and $\|\mu - m_{uv}\| \geq 2 - \varepsilon$.

Let $u_0 \neq v_0 \in M$ be such that $f(u_0) - f(v_0) > (1-\alpha)d(u_0,v_0)$. According to Lemma 1.15 there exists $\gamma > 0$ such that if $x \neq y \in M$ satisfy $d(x,y) < \gamma$, then $\|\mu - m_{xy}\| \geq 2 - \varepsilon$. Let $n \in \mathbb{N}$ and $\delta > 0$ be such that

$$\Big(1 - \frac{\varepsilon}{4}\Big)^n d(u_0,v_0) < \gamma$$

and $f(u_0) - f(v_0) > (1-\alpha)(1+\delta)^n d(u_0,v_0)$.

If $\|\mu - m_{u_0v_0}\| \geq 2 - \varepsilon$, then we have found suitable points u and v.

Consider the case where $\|\mu - m_{u_0v_0}\| < 2 - \varepsilon$. By (iii) there exists

$$p \in B\Big(u_0, \big(1 - \frac{\varepsilon}{4}\big)d(u_0,v_0)\Big) \cap B\Big(v_0, \big(1 - \frac{\varepsilon}{4}\big)d(u_0,v_0)\Big) \cap \operatorname{Line}(u_0,v_0,\delta).$$

Therefore

$$\begin{aligned} f(u_0) - f(p) + f(p) - f(v_0) &> (1-\alpha)(1+\delta)^n d(u_0,v_0) \\ &> (1-\alpha)(1+\delta)^{n-1}\big(d(u_0,p) + d(v_0,p)\big). \end{aligned}$$

Then either

$$f(u_0) - f(p) > (1-\alpha)(1+\delta)^{n-1} d(u_0,p)$$

or

$$f(p) - f(v_0) > (1-\alpha)(1+\delta)^{n-1} d(v_0,p).$$

Additionally we have

$$\max\big\{d(u_0,p), d(v_0,p)\big\} \leq \Big(1 - \frac{\varepsilon}{4}\Big)d(u_0,v_0).$$

Therefore there exist $u_1 \neq v_1 \in M$ such that

$$f(u_1) - f(v_1) > (1-\alpha)(1+\delta)^{n-1} d(u_1,v_1)$$

and $d(u_1, v_1) < (1 - \varepsilon/4)d(u_0, v_0)$.

Now we will repeat this step as many times as needed, but no more than n times. Assume for $k \in \{1, \dots, n-1\}$ that

$$f(u_k) - f(v_k) > (1 - \alpha)(1 + \delta)^{n-k} d(u_k, v_k)$$

and $d(u_k, v_k) < (1 - \varepsilon/4)^k d(u_0, v_0)$.

If $\|\mu - m_{u_k v_k}\| \geq 2 - \varepsilon$, then u_k and v_k are suitable points, since $f(u_k) - f(v_k) > (1 - \alpha)d(u_k, v_k)$.

If $\|\mu - m_{u_k v_k}\| < 2 - \varepsilon$, then as we did before, we can find $u_{k+1} \neq v_{k+1} \in M$ such that

$$f(u_{k+1}) - f(v_{k+1}) > (1 - \alpha)(1 + \delta)^{n-k-1} d(u_{k+1}, v_{k+1})$$

and $d(u_{k+1}, v_{k+1}) < (1 - \varepsilon/4)^{k+1} d(u_0, v_0)$.

If by the n-th step we have not found suitable points, then we have $f(u_n) - f(v_n) > (1 - \alpha)d(u_n, v_n)$ and

$$d(u_n, v_n) < \left(1 - \frac{\varepsilon}{4}\right)^n d(u_0, v_0) < \gamma,$$

which gives us $\|\mu - m_{u_n v_n}\| \geq 2 - \varepsilon$. Now we have found suitable points u and v, therefore μ is a Daugavet-point. $\square$

The following corollary can be directly derived from Theorem 2.6 and Lemma 1.13 in case μ is a molecule.

Corollary 2.9. *Let M be a metric space and $x \neq y \in M$. The following statements are equivalent:*

(i) m_{xy} in a Daugavet-point;

(ii) For every $u \neq v \in M$ and $\varepsilon > 0$, if there exists $\delta > 0$ such that

$$B\big(u, (1 - \varepsilon)d(u, v)\big) \cap B\big(v, (1 - \varepsilon)d(u, v)\big) \cap \mathrm{Line}(u, v, \delta) = \emptyset,$$

then

$$d(x, u) + d(y, v) \geq d(x, y) + d(u, v) - 4\varepsilon \max\big\{d(x, y), d(u, v)\big\}.$$

Another natural question to consider is whether ε and δ are indeed needed in condition (iii) of Theorem 2.6. From Theorem 2.5 we see that in case of compact spaces the characterization can be presented without using ε and δ. In Theorem 2.6 we examine only bounded sets, which indicates that ε and δ can be left out in case of proper metric spaces, i.e., metric spaces in which all closed bounded sets are compact. This is indeed so, as can be seen by following lemma.

Lemma 2.10 (see [4, Proposition 2.3], [8, Theorem 2.4]). *Let M be a proper metric space. Then the following are equivalent for all $u \neq v \in M$:*

(i) $m_{uv} \in \operatorname{dent} B_{\mathcal{F}(M)}$*;*

(ii) $[u, v] = \{u, v\}$.

Note that if M is proper then it is complete and by Lemma 2.2 all denting points are molecules.

Corollary 2.11. *Let M be a proper metric space and $\mu \in S_{\mathcal{F}(M)}$. Then the following statements are equivalent:*

(i) μ *is a Daugavet-point;*

(ii) For every $u \neq v \in M$ *if* $[u, v] = \{u, v\}$*, then* $\|\mu - m_{uv}\| = 2$.

Let us notice by using Lemma 1.13 we can also simplify Corollary 2.11 when μ is a molecule.

Corollary 2.12. *Let M be a proper metric space and $x \neq y \in M$. The following statements are equivalent:*

(i) m_{xy} *in a Daugavet-point;*

(ii) For every $u \neq v \in M$*, if* $[u, v] = \{u, v\}$*, then*

$$d(x, u) + d(y, v) \geq d(x, y) + d(u, v).$$

Now we see that Theorem 2.5 can be derived from Theorem 2.6 and Corollary 2.12.

As a last part of this section, we ask if positive ε and δ are really needed in Theorem 2.6 in case M is not proper. Our aim is to present an example of complete metric space M such that there exists $x \neq y \in M$ such that m_{xy} is a Daugavet-point, but condition (ii) from Corollary 2.11 does not hold true. This example is far from trivial and it also serves as an example of a metric space M such that $\mathcal{F}(M)$ has the Radon–Nikodým property and a Daugavet-point. This example is provided in the following chapter.

2.2 Example of Lipschitz-free Banach space with Radon–Nikodým property and with Daugavet-point

In this chapter, we will provide an example of a Lipschitz-free space with the Radon–Nikodým property and a Daugavet-point. We will first show that the Lipschitz-free space in our example has the Schur property and then apply Theorem 2.15 to conclude that this space also has the Radon–Nikodým property. First, let us introduce these properties.

Definition 2.13 (see [12])**.** We say that a Banach space X has the **Radon–Nikodým property** if every nonempty bounded closed convex set has a denting point.

Definition 2.14 (see [7, Page 253])**.** We say that a Banach space X has the **Schur property** if every weakly convergent sequence in X is norm convergent.

Theorem 2.15 (see [2, Theorem 4.6])**.** *For Lipschitz-free Banach spaces these properties are equivalent:*

(i) $\mathcal{F}(M)$ *has the Radon–Nikodým property;*

(ii) $\mathcal{F}(M)$ *has the Schur property.*

To show that $\mathcal{F}(M)$ in our example has the Schur property we will use following result.

Lemma 2.16 (see [3, Corollary 2.7])**.** *Let M be a countable complete metric space. Then $\mathcal{F}(M)$ has the Schur property.*

Now we are ready to present our example.

Example 2.17. Let $x = (0, 0)$, $y = (1, 0)$ and $S_0 = \{x, y\}$. For every $n \in \mathbb{N}$ let

$$S_n = \left\{ \left(\frac{k}{2^n}, \frac{1}{2^n} \right) : k \in \{0, 1, \ldots, 2^n\} \right\}.$$

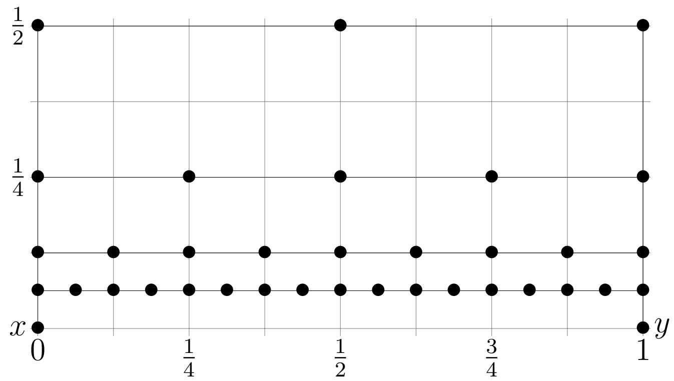

Figure 3: The sets $S_0, \ldots, S_4$

Let

$$M = \bigcup_{n=0}^{\infty} S_n$$

be a metric space with metric

$$d\big((a_1,a_2),(b_1,b_2)\big) = \begin{cases} |a_1-b_1|, & \text{if } a_2 = b_2 \\ \min\{a_1+b_1, 2-a_1-b_1\} + |a_2-b_2|, & \text{if } a_2 \neq b_2. \end{cases}$$

Let us convince that d is indeed a metric. Let $a = (a_1, a_2)$ and $b = (b_1, b_2) \in M$. Clearly $d(a,b) \geq 0$ and $d(a,b) = d(b,a)$. If $d(a,b) = 0$, then we must have $a_2 = b_2$, which implies also $a_1 = b_1$.

Let us show that $d(a,b) \leq d(a,c) + d(b,c)$ for $c = (c_1, c_2) \in M$ by examining three cases.

1. If $a_2 = b_2 = c_2$, then
$$d(a,b) = |a_1 - b_1| \leq |a_1 - c_1| + |b_1 - c_1| = d(a,c) + d(b,c).$$

2. If $a_2 \neq c_2$ and $b_2 = c_2$ (the case where $a_2 = c_2$ and $b_2 \neq c_2$ is analogous), then let us notice
$$\begin{aligned} \min\{a_1+b_1, 2-a_1-b_1\} &= \min\{a_1+c_1+b_1-c_1, 2-a_1-c_1+c_1-b_1\} \\ &\leq \min\{a_1+c_1, 2-a_1-c_1\} + |b_1-c_1| \end{aligned}$$
and therefore $d(a,b) \leq d(a,c) + d(b,c)$.

3. Consider the case where $a_2 \neq c_2$ and $b_2 \neq c_2$. Clearly $|a_1 - b_1| \leq a_1 + b_1$, and since $a_1, b_1 \leq 1$ we conclude that $|a_1 - b_1| \leq 2 - a_1 - b_1$. This gives us
$$|a_1 - b_1| \leq \min\{a_1+b_1, 2-a_1-b_1\}.$$
Therefore, it is enough, if we examine only the case when $a_2 \neq b_2$.
If $2 - a_1 - c_1 \leq a_1 + c_1$ and $2 - b_1 - c_1 \leq b_1 + c_1$, then
$$2 - a_1 - b_1 \leq 2 - a_1 - c_1 + 2 - b_1 - c_1$$
by using $c_1 \leq 1$. Otherwise
$$a_1 + b_1 \leq \min\{a_1+c_1, 2-a_1-c_1\} + \min\{b_1+c_1, 2-b_1-c_1\}$$
by using $a_1, b_1 \leq 1$. This gives us $d(a,b) \leq d(a,c) + d(b,c)$.

We conclude that $d(a,b) \leq d(a,c) + d(b,c)$ and therefore the triangle inequality holds.

Next we show that M is complete. Let (a_n) be a Cauchy sequence in M. To show that (a_n) converges to an element of M we consider two cases.

1. First assume that there exists $m \in \mathbb{N}$ such that (a_n) is in $\cup_{n=0}^{m} S_n$. Then there exists $\varepsilon > 0$ such that for every $u \neq v \in \cup_{n=0}^{m} S_n$ we have $d(u,v) > \varepsilon$. Therefore, (a_n) is eventually constant.

2. Assume that for every $m \in \mathbb{N}$, there exist $k > m$ and $n \in \mathbb{N}$ such that $a_n \in S_k$. Choose a subsequence (a_{n_k}) such that $a_{n_k} \in S_{m_k}$, where $m_1 < m_2 < m_3 \cdots$. The distance between any two different elements $a_{n_k} = (b_{n_k}, c_{n_k}) \neq a_{n_l} = (b_{n_l}, c_{n_l})$ is defined as
$$\min\{b_{n_k} + b_{n_l}, 2 - b_{n_k} - b_{n_l}\} + |c_{n_k} - c_{n_l}|.$$
Since (a_{n_k}) is a Cauchy sequence, either $a_{n_k} \to (0,0)$ or $a_{n_k} \to (1,0)$.

According to Lemma 2.16, $\mathcal{F}(M)$ has the Schur property, since M is countable and complete. By Theorem 2.15, $\mathcal{F}(M)$ also has the Radon–Nikodým property.

We will now show that m_{xy} is a Daugavet-point; recall that $x = (0,0)$ and $y = (1,0)$.

By Theorem 2.6, it suffices to show that $\|m_{xy} - \nu\| = 2$ for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(M)})$. According to Lemma 2.2 ν is a denting point in $B_{\mathcal{F}(M)}$ only if $\nu = m_{uv}$ for some $u \neq v \in M$.

Fix $m_{uv} \in \operatorname{dent}(B_{\mathcal{F}(M)})$. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then according to Theorem 2.3, for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\operatorname{Line}(u, v, \delta) \subseteq B(u, \varepsilon) \cup B(v, \varepsilon).$$

If there exists $p \in [u,v] \setminus \{u,v\}$, then $p \in \operatorname{Line}(u,v,\delta)$ for every $\delta > 0$ and there exists $\varepsilon > 0$ such that $\min\big\{d(u,p), d(v,p)\big\} > \varepsilon$. This implies that $\operatorname{Line}(u,v,\delta) \not\subseteq B(u,\varepsilon) \cup B(v,\varepsilon)$ for every $\delta > 0$, which is a contradiction. Therefore $[u,v] = \{u,v\}$.

We will now show that $u, v \notin \{x,y\}$. Assume that $u \in \{x,y\}$ (case $v \in \{x,y\}$ is analogous). Then $u_2 = 0$. If $v_2 \neq 0$, then
$$z := (u_1, v_2/2) \in [u,v] \setminus \{u,v\},$$
because $d(u,z) = v_2/2$ and $d(v,z) = \min\{u_1 + v_1, 2 - u_1 - v_1\} + v_2/2$. This is in contradiction with $[u,v] = \{u,v\}$ and therefore $v_2 = 0$, i.e., $v \in \{x,y\}$. Since the conditions for u and v are symmetrical then let us assume $u = x$ and $v = y$. However, m_{xy} is not a denting point because
$$z := \left(\frac{1}{2}, \frac{1}{2^{n+1}}\right) \in \operatorname{Line}\left(x, y, \frac{1}{2^n}\right)$$
and $\min\{d(u,z), (v,z)\} > 1/2$ for every $n \in \mathbb{N}$, since $d(x,z) = d(y,z) = 1/2 + 1/2^n$.

Now we know that $[u,v] = \{u,v\}$ and $u, v \notin \{x,y\}$. Let us show that $\|m_{xy} - m_{uv}\| = 2$.

If $u_2 = v_2$, then $|u_1 - v_1| = u_2$, because otherwise either

$$(u_1 + u_2, u_2) \in [u, v] \setminus \{u, v\}$$

or

$$(u_1 - u_2, u_2) \in [u, v] \setminus \{u, v\}.$$

Hence

$$d(x, u) + d(y, v) = u_1 + u_2 + 1 - v_1 + v_2 \geq 1 + |u_1 - v_1| = d(x, y) + d(u, v)$$

and according to Lemma 1.13 we get $\|m_{xy} - m_{uv}\| = 2$.

If $u_2 \neq v_2$, then either $u_1 = v_1 = 0$ or $u_1 = v_1 = 1$, because otherwise one of the four points $(0, u_2), (1, u_2), (0, v_2), (1, v_2)$ is in $[u, v] \setminus \{u, v\}$. Hence

$$d(x, u) + d(y, v) = u_1 + u_2 + 1 - v_1 + v_2 = 1 + u_2 + v_2 \geq 1 + |u_2 - v_2| = d(x, y) + d(u, v)$$

and according to Lemma 1.13 we get $\|m_{xy} - m_{uv}\| = 2$.

Now we have shown that for every $\nu \in \operatorname{dent}(B_{\mathcal{F}(M)})$ we have $\|m_{xy} - \nu\| = 2$ and therefore m_{xy} is a Daugavet-point.

Furthermore, we see that condition (ii) from Corollary 2.11 does not hold for m_{xy} since $[x, y] = \{x, y\}$ and $\|m_{xy} - m_{xy}\| = 0$.

3 Delta-points in Lipschitz-free Banach spaces

In this chapter, we will examine Δ-points in Lipschitz-free spaces. We will present some results and examples from [11] and add some new related results.

This chapter is mostly dedicated to examining convex combinations of molecules. For simplicity, when writing $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i y_i} \in \operatorname{conv}\big(\operatorname{Mol}(M)\big) \cap S_{\mathcal{F}(M)}$, we assume by default that $\lambda_i > 0$ for every $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} \lambda_i = 1$.

3.1 Characterization of Delta-point in Lipschitz-free Banach spaces

In the article [11], the question of when a molecule is a Δ-point was studied. They provided one equivalence condition for molecule to be a Δ-point as well one sufficient condition. An example of a Δ-point that was not a Daugavet-point was also presented, thus proving Δ- and Daugavet-points are not the same in Lipschitz-free spaces.

Definition 3.1 (see [11, Definition 4.1]). Let $x \neq y \in M$. We say that points x and y are **connectable** if for every $\varepsilon > 0$ there exists 1-Lipschitz mapping $\alpha\colon [0, d(x,y)+\varepsilon] \to M$ with $\alpha(0) = y$ and $\alpha(d(x,y)+\varepsilon) = x$, and say that α **connects** x and y.

Proposition 3.2 (see [11, Proposition 4.2]). *Let M be a metric space and let $x \neq y \in M$ be connectable. Then m_{xy} is a Δ-point.*

This result is a good tool for proving that a certain point is a Δ-point. In [11] it was also used to show that Δ-points and Daugavet-points are not the same in Lipschitz-free spaces.

Example 3.3 (see [11, Example 4.3]). Let $0 < r < 1$ and define $M := [0,1] \times \{0\} \cup \{(0,r),(1,r)\} \subseteq (\mathbb{R}^2, \|.\|_2)$ and consider $x := (1,0)$ and $y := (0,0)$. Note that m_{xy} is a Δ-point because there exists a 1-Lipschitz mapping $\alpha\colon [0,1] \to M$ connecting x and y, namely $\alpha(t) := (t,0)$ for every $t \in [0,1]$. However, m_{xy} is not a Daugavet-point. For $u := (1,r)$ and $v := (1,r)$ we get $[u,v] = \{u,v\}$, however

$$d(x,u) + d(y,v) = 2r < 2 = d(x,y) + d(u,v),$$

so by Theorem 2.5, m_{xy} is not a Daugavet-point.

Theorem 3.4 (see [11, Theorem 4.6]). *Let $x \neq y \in M$. Then m_{xy} is a Δ-point if and only if for every $\varepsilon > 0$ and slice S with $m_{xy} \in S$ there exists $u \neq v \in M$ such that $m_{uv} \in S$ and $d(u,v) < \varepsilon$.*

We will not present the original proof in this thesis. Instead we will prove that a generalization of Theorem 3.4 holds true for arbitrary $\mu \in \operatorname{conv}\big(\operatorname{Mol}(M)\big) \cap S_{\mathcal{F}(M)}$, not just for $m_{xy} \in \operatorname{conv}\big(\operatorname{Mol}(M)\big) \cap S_{\mathcal{F}(M)}$. To do so we will find a function similar to f_{xy} that was used in the proof of Theorem 3.4.

Lemma 3.5 (see [9, Lemma 3.6])**.** *Let $x \neq y \in M$ and let*

$$f_{xy}(p) = \frac{d(x,y)}{2} \cdot \frac{d(y,p) - d(x,p)}{d(x,p) + d(y,p)}.$$

We have

(1) $\|f_{xy}\| = 1$*;*

(2) If $m_{uv} \in S(f_{xy}, \alpha)$ for some $u \neq v \in M$, then

$$(1-\alpha)\max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} < d(x,y).$$

Note that the main benefit of f_{xy} is condition (2), which tells us that if a molecule m_{uv} is in the slice defined by f_{xy}, then both u and v are close to the segment $[x,y]$. Our aim is to find for $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i y_i} \in \operatorname{conv}\big(\operatorname{Mol}(M)\big) \cap S_{\mathcal{F}(M)}$ a norm-1 function f_μ such that if a molecule m_{uv} is in the slice defined by f_μ then it is close to segment $[x_i, y_i]$ for some $i \in \{1, \dots, n\}$. Note that if for some $m \in \{1, \dots, n\}$ and $k_1, \dots, k_{m+1} \in \{1, \dots, n\}$ with $k_1 = k_{m+1}$ we have

$$\sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) = \sum_{i=1}^{m} d(x_{k_i}, y_{k_i})$$

then there exists a presentation of μ as convex combination of molecules, where all the molecules $m_{k_1 k_2}, \dots, m_{k_m k_{m+1}}$ are included. With that in mind we will construct the function.

Lemma 3.6. *Let $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i y_i} \in \operatorname{conv}\big(\operatorname{Mol}(M)\big) \cap S_{\mathcal{F}(M)}$. There exists $f_\mu \in S_{\operatorname{Lip}_0(M)}$ and $\delta > 0$ such that the following holds:*

(1) $f_\mu(\mu) = 1$*;*

(2) For every $u \neq v \in M$ and $\alpha \in (0, \delta)$ with $m_{uv} \in S(f_\mu, \alpha)$ there exists $m \in \{1, \dots, n\}$ and $k_1, \dots, k_{m+1} \in \{1, \dots, n\}$ with $k_1, \dots, k_m$ pairwise distinct and $k_1 = k_{m+1}$, such that

$$\sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) = \sum_{i=1}^{m} d(x_{k_i}, y_{k_i})$$

and

$$(1-\alpha)\max\{d(x_{k_1}, v) + d(y_{k_2}, v), d(x_{k_1}, u) + d(y_{k_2}, u)\} < d(x_{k_1}, y_{k_2}).$$

Proof. According to Lemma 1.17 there exists $g \in S_{\mathrm{Lip}_0(M)}$ such that $g(\mu) = 1$ and for $k_1 \neq k_2 \in \{1, \dots, n\}$ we have

$$g(x_{k_1}) - g(y_{k_2}) = d(x_{k_1}, y_{k_2})$$

if and only if either $k_1 = k_2$ or there exists $m \in \{1, \dots, n\}$ and $k_3, \dots, k_{m+1} \in \{1, \dots, n\}$ with $k_1, \dots, k_m$ pairwise distinct and $k_{m+1} = k_1$ such that

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}).$$

Let $\delta > 0$ be such that if $g(x_i) - g(y_j) < d(x_i, y_j)$ for some $i, j \in \{1, \dots, n\}$, then $g(x_i) - g(y_j) < (1-\delta)d(x_i, y_j)$. Set

$$h_i(p) = \max\left\{\frac{d(x_i, y_j)d(x_i, p)}{d(x_i, p) + d(y_j, p)} : j \in \{1, \dots, n\}, g(x_i) - g(y_j) = d(x_i, y_j)\right\}$$

for every $i \in \{1, \dots, n\}$. Note that $g(x_i) - g(y_i) = d(x_i, y_i)$ for every $i \in \{1, \dots, n\}$ by Lemma 1.16, therefore the set we use in definition of h_i is not empty. Furthermore,

$$\begin{aligned} \frac{d(x_i, y_j)}{2} - f_{x_i y_j}(p) &= \frac{d(x_i, y_j)}{2} \frac{d(x_i, p) + d(y_j, p) - \big(d(y_j, p) - d(x_i, p)\big)}{d(x_i, p) + d(y_j, p)} \\ &= \frac{d(x_i, y_j)d(x_i, p)}{d(x_i, p) + d(y_j, p)}, \end{aligned} \tag{3.1}$$

then from Lemmas 1.12 and 3.5 we get $\|h_i\| \leq 1$. Set

$$f_\mu(p) = \max_{i \in \{1, \dots, n\}} \Big\{g(x_i) - \max\big\{h_i(p), (1-\delta)d(x_i, p)\big\}\Big\}.$$

Note that that from Lemma 1.12 we get $\|f_\mu\| \leq 1$. We will show that f_μ and δ satisfy the conditions of the lemma.

For every $i \in \{1, \dots, n\}$ we have

$$f_\mu(x_i) \geq g(x_i) - \max\big\{h_i(x_i), (1-\delta)d(x_i, x_i)\big\} = g(x_i).$$

Next we show that $f_\mu(y_i) \leq g(y_i)$ for every $i \in \{1, \dots, n\}$. For fixed $i \in \{1, \dots, n\}$ let $j \in \{1, \dots, n\}$ be such that

$$f_\mu(y_i) = g(x_j) - \max\big\{h_j(y_i), (1-\delta)d(x_j, y_i)\big\}.$$

If $g(x_j) - g(y_i) = d(x_j, y_i)$, then

$$f_\mu(y_i) = g(x_j) - \max\big\{h_j(y_i), (1-\delta)d(x_j, y_i)\big\} = g(x_j) - d(x_j, y_i) = g(y_i).$$

If $g(x_j) - g(y_i) < d(x_j, y_i)$, then by the choice of δ we have $g(x_j) - g(y_i) < (1-\delta)d(x_j, y_i)$ and therefore

$$\begin{aligned} f_\mu(y_i) &= g(x_j) - \max\big\{h_j(y_i), (1-\delta)d(x_j, y_i)\big\} \\ &\leq g(y_i) + g(x_j) - g(y_i) - (1-\delta)d(x_j, y_i) \\ &< g(y_i). \end{aligned}$$

This gives us $f_\mu(x_i) - f_\mu(y_i) \geq g(x_i) - g(y_i)$ for every $i \in \{1, \dots, n\}$ and therefore $f_\mu(\mu) \geq g(\mu) = 1$. Note that $\|f_\mu\| \leq 1$, hence $f_\mu \in S_{\mathrm{Lip}_0(M)}$ and $f_\mu(\mu) = 1$.

Now let us show that condition (2) holds. Fix $u \neq v \in M$ and $\alpha \in (0, \delta)$ such that $m_{uv} \in (f_\mu, \alpha)$.

Let $k_1 \in \{1, \dots, n\}$ be such that

$$f_\mu(u) = g(x_{k_1}) - \max\big\{h_{k_1}(u), (1-\delta)d(x_{k_1}, u)\big\}.$$

Then

$$\begin{aligned} (1-\alpha)d(u,v) &< f_\mu(u) - f_\mu(v) \\ &\leq g(x_{k_1}) - \max\big\{h_{k_1}(u), (1-\delta)d(x_{k_1}, u)\big\} \\ &\qquad - g(x_{k_1}) + \max\big\{h_{k_1}(v), (1-\delta)d(x_{k_1}, v)\big\} \\ &= \max\big\{h_{k_1}(v), (1-\delta)d(x_{k_1}, v)\big\} - \max\big\{h_{k_1}(u), (1-\delta)d(x_{k_1}, u)\big\}. \end{aligned}$$

If $\max\big\{h_{k_1}(v), (1-\delta)d(x_{k_1}, v)\big\} = (1-\delta)d(x_{k_1}, v)$, then

$$\begin{aligned} (1-\alpha)d(u,v) &< \max\big\{h_{k_1}(v), (1-\delta)d(x_{k_1}, v)\big\} - \max\big\{h_{k_1}(u), (1-\delta)d(x_{k_1}, u)\big\} \\ &\leq (1-\delta)\big(d(x_{k_1}, v) - d(x_{k_1}, u)\big) \leq (1-\delta)d(u,v), \end{aligned}$$

which contradicts with $\alpha \in (0, \delta)$. Hence

$$\max\big\{h_{k_1}(v), (1-\delta)d(x_{k_1}, v)\big\} = h_{k_1}(v).$$

There exists $k_2 \in \{1, \dots, n\}$ with $g(x_{k_1}) - g(y_{k_2}) = d(x_{k_1}, y_{k_2})$ such that

$$h_{k_1}(v) = \frac{d(x_{k_1}, y_{k_2})d(x_{k_1}, v)}{d(x_{k_1}, v) + d(y_{k_2}, v)}.$$

By 3.1 we get that

$$\begin{aligned} (1-\alpha)d(u,v) &< \max\big\{h_{k_1}(v), (1-\delta)d(x_{k_1}, v)\big\} - \max\big\{h_{k_1}(u), (1-\delta)d(x_{k_1}, u)\big\} \\ &\leq \frac{d(x_{k_1}, y_{k_2})d(x_{k_1}, v)}{d(x_{k_1}, v) + d(y_{k_2}, v)} - \frac{d(x_{k_1}, y_{k_2})d(x_{k_1}, u)}{d(x_{k_1}, u) + d(y_{k_2}, u)} \\ &= f_{x_{k_1}y_{k_2}}(u) - f_{x_{k_1}y_{k_2}}(v). \end{aligned}$$

From Lemma 3.5 we get

$$d(x_{k_1}, y_{k_2}) > (1-\alpha)\max\big\{d(x_{k_1}, u) + d(y_{k_2}, u), d(x_{k_1}, v) + d(y_{k_2}, v)\big\}.$$

Note that $g(x_{k_1}) - g(y_{k_2}) = d(x_{k_1}, y_{k_2})$, therefore either $k_1 = k_2$ or there exists $m \in \{1, \ldots, n\}$ and $k_3, \ldots, k_{m+1} \in \{1, \ldots, n\}$ with $k_1, \ldots, k_m$ pairwise distinct and $k_{m+1} = k_1$ such that

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}).$$

Since in case $k_1 = k_2$ we can choose $m = 1$ and k_1, k_2 as suitable indexes, then we conclude the proof. $\square$

Now we present our generalization of Theorem 3.4.

Theorem 3.7. *Let $\mu \in \operatorname{conv}\big(\operatorname{Mol}(M)\big) \cap S_{\mathcal{F}(M)}$. Then μ is a Δ-point if and only if for every $\varepsilon > 0$ and a slice S with $\mu \in S$ there exist $u \neq v \in M$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$.*

Proof. ($\Leftarrow$) This is a direct consequence of Lemma 1.15.

($\Rightarrow$) Assume that μ is a Δ-point. Let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $x_1 \neq y_1 \ldots, x_n \neq y_n \in M$ be such that $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i y_i}$.

According to Lemma 3.6 there exists $f_\mu \in S_{\operatorname{Lip}_0(M)}$ and $\delta > 0$ such that $f_\mu(\mu) = 1$ and for every $u \neq v \in M$ and $\delta' \in (0, \delta)$ with $f_\mu(u) - f_\mu(v) > (1 - \delta')d(u, v)$ there exist $m \in \{1, \ldots, n\}$ and $k_1, \ldots, k_{m+1} \in \{1, \ldots, n\}$ with $k_1, \ldots, k_m$ pairwise distinct and $k_1 = k_{m+1}$ such that

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j})$$

and

$$(1 - \delta') \max\big\{d(x_{k_1}, v) + d(y_{k_2}, v), d(x_{k_1}, u) + d(y_{k_2}, u)\big\} < d(x_{k_1}, y_{k_2}).$$

Fix $\varepsilon > 0$ and a slice $S = S(f, \alpha)$ such that $\mu \in S$. According to Lemma 1.7 we can assume that $\alpha < \delta$ and

$$\Big(\frac{1}{(1 - \alpha)^2} - 1\Big) \max_{i,j \in \{1, \ldots, n\}} d(x_i, y_j) < \varepsilon.$$

Our aim is to show there exists $u \neq v \in M$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$. Set $g = f + f_\mu$ and

$$\lambda_0 = \min_{i \in \{1, \ldots, n\}} \frac{\lambda_i}{d(x_i, y_i)}.$$

It is easy to see that $g(\mu) = f_\mu(\mu) + f(\mu) > 2 - \alpha$. Then

$$\mu \in S\Big(\frac{g}{\|g\|}, 1 - \frac{2 - \alpha}{\|g\|}\Big).$$

Since μ is a Δ-point, by Lemma 1.6 there exist $u \neq v \in M$ such that

$$m_{uv} \in S\Big(\frac{g}{\|g\|}, 1 - \frac{2-\alpha}{\|g\|}\Big)$$

and

$$\|\mu - m_{uv}\| \geq 2 - \lambda_0 \alpha \min\big\{d(x_i, y_j) \colon i, j \in \{1, \dots, n\}, d(x_i, y_j) \neq 0\big\}.$$

It is easy to see that $f_\mu(u) - f_\mu(v) > (1-\alpha)d(u,v)$ and $f(u) - f(v) > (1-\alpha)d(u,v)$, which means $m_{uv} \in S(f, \alpha)$. Now we will show that $d(u,v) < \varepsilon$.

Since $f_\mu(u) - f_\mu(v) > (1-\alpha)d(u,v)$ and $\alpha \in (0, \delta)$, there exist $m \in \{1, \dots, n\}$ and $k_1, \dots, k_{m+1} \in \{1, \dots, n\}$ with $k_1, \dots, k_m$ pairwise distinct and $k_1 = k_{m+1}$ such that

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}) \tag{3.2}$$

and

$$(1-\alpha) \max\big\{d(x_{k_1}, v) + d(y_{k_2}, v), d(x_{k_1}, u) + d(y_{k_2}, u)\big\} < d(x_{k_1}, y_{k_2}).$$

Clearly $d(x_{k_1}, y_{k_2}) > 0$ and therefore

$$\begin{aligned}\|\mu - m_{uv}\| &\geq 2 - \lambda_0 \alpha \min\big\{d(x_i, y_j) \colon i, j \in \{1, \dots, n\}, d(x_i, y_j) \neq 0\big\} \\ &\geq 2 - \lambda_0 \alpha d(x_{k_1}, y_{k_2}).\end{aligned}$$

Now we will show $\|m_{x_{k_1} y_{k_2}} - m_{uv}\| \geq 2 - \varepsilon$. Note that $\lambda_i \geq \lambda_0 d(x_i, y_i)$ for every $i \in \{1, \dots, n\}$. Let

$$l_i = \begin{cases} \lambda_i - \lambda_0 d(x_i, y_i), & \text{if } i \in \{k_1, \dots, k_m\}, \\ \lambda_i, & \text{if } i \in \{1, \dots, n\} \setminus \{k_1, \dots, k_m\}. \end{cases}$$

By 3.2 we have

$$\mu = \sum_{i=1}^{n} \lambda_i m_{x_i y_i} = \sum_{i=1}^{n} l_i m_{x_i y_i} + \lambda_0 \sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) m_{x_{k_i} y_{k_{i+1}}}.$$

We see that

$$\begin{aligned} 2 - \alpha \lambda_0 d(x_{k_1}, y_{k_2}) &\leq \Big\| \sum_{i=1}^{n} \lambda_i m_{x_i y_i} - m_{uv} \Big\| \\ &= \Big\| \sum_{i=1}^{n} l_i m_{x_i y_i} + \lambda_0 \sum_{i=1}^{m} d(x_{k_i}, y_{k_{i+1}}) m_{x_{k_i} y_{k_{i+1}}} - m_{uv} \Big\| \\ &\leq \sum_{i=1}^{n} l_i \|m_{x_i y_i}\| + \lambda_0 \sum_{i=2}^{m} d(x_{k_i}, y_{k_{i+1}}) \|m_{x_{k_i} y_{k_{i+1}}}\| \\ &+ \big(1 - \lambda_0 d(x_{k_1}, y_{k_2})\big) \|m_{uv}\| + \lambda_0 d(x_{k_1}, y_{k_2}) \|m_{x_{k_1} y_{k_2}} - m_{uv}\| \\ &= 2 - 2\lambda_0 d(x_{k_1}, y_{k_2}) + \lambda_0 d(x_{k_1}, y_{k_2}) \|m_{x_{k_1} y_{k_2}} - m_{uv}\|. \end{aligned}$$

Therefore $\|m_{x_{k_1}y_{k_2}} - m_{uv}\| \geq 2-\alpha$. From Lemma 1.16 we get

$$\|m_{x_{k_1}y_{k_2}} + m_{uv}\| \geq \frac{f_\mu(x_{k_1}) - f_\mu(y_{k_2})}{d(x_{k_1}, y_{k_2})} + \frac{f_\mu(u) - f_\mu(v)}{d(u,v)} > 2-\alpha.$$

By Lemma 1.13 we get that

$$\begin{aligned}\min &\big\{d(x_{k_1}, v) + d(y_{k_2}, u), d(x_{k_1}, u) + d(y_{k_2}, v)\big\} \\ &\geq d(x_{k_1}, y_{k_2}) + d(u,v) - \alpha \max\big\{d(x_{k_1}, y_{k_2}), d(u,v)\big\} \\ &> (1-\alpha)\big(d(x_{k_1}, y_{k_2}) + d(u,v)\big).\end{aligned}$$

We also know that

$$(1-\alpha)\max\big\{d(x_{k_1}, v) + d(y_{k_2}, v), d(x_{k_1}, u) + d(y_{k_2}, u)\big\} < d(x_{k_1}, y_{k_2})$$

and therefore

$$\begin{aligned}d(u,v) &< \frac{d(x_{k_1}, v) + d(y_{k_2}, u) + d(x_{k_1}, u) + d(y_{k_2}, v)}{2(1-\alpha)} - d(x_{k_1}, y_{k_2}) \\ &< \frac{2d(x_{k_1}, y_{k_2})}{2(1-\alpha)^2} - d(x_{k_1}, y_{k_2}) \\ &\leq \Big(\frac{1}{(1-\alpha)^2} - 1\Big) \max_{i,j\in\{1,\dots,n\}} d(x_i, y_j) < \varepsilon.\end{aligned}$$

Therefore we have found $u \neq v \in M$ such that $m_{uv} \in S$ and $d(u,v) < \varepsilon$. □

It was also shown in [11] that Theorem 3.4 implies the following corollary

Corollary 3.8 (see [11, Corollary 4.7]). *Let $x \neq y \in M$ and let m_{xy} be a Δ-point. Then for every $r \in \big(0, d(x,y)\big)$ and $\varepsilon > 0$ we get that*

$$B(x, r+\varepsilon) \cap B\big(y, d(x,y) - r + \varepsilon\big) \neq \emptyset.$$

Moreover, if M is proper then $S(x,r) \cap S\big(y, d(x,y) - r\big) \neq \emptyset$.

Proof. The following proof is from [11], small modifications have been made to fit the style of the thesis.

Fix $r \in \big(0, d(x,y)\big)$ and $\varepsilon > 0$. Let $\delta \in (0, \varepsilon/2)$ be such that $r + \delta < d(x,y)$ and $d(x,y) - r + \delta < d(x,y)$. Let

$$\begin{aligned}f_1(t) &:= \max\big\{r + \delta - d(x,t), 0\big\} \\ f_2(t) &:= \min\big\{-(d(x,y) - r) - \delta + d(y,t), 0\big\}\end{aligned}$$

and let $f := f_1 + f_2$. Then $f_1(y) = f_2(x) = 0$ and therefore

$$f(x) - f(y) = f_1(x) - f_2(y) = r + \delta + \big(d(x,y) - r\big) + \delta = d(x,y) + 2\delta > d(x,y).$$

Then $m_{xy} \in S\big(f/\|f\|, 1-1/\|f\|\big)$ and from Theorem 3.7 we get that there exist $u \neq v \in M$ such that $m_{uv} \in S\big(f/\|f\|, 1-1/\|f\|\big)$ and $d(u,v) < \delta$. Hence

$$f(u) - f(v) > \|f\|\Big(1 - 1 + \frac{1}{\|f\|}\Big)d(u,v) = d(u,v).$$

Notice that according to Lemma 1.12 we have $\|f_1\| \leq 1$ and $\|f_2\| \leq 1$. Therefore $f_1(u) - f_1(v) > 0$, which gives us $u \in B(x, r+\delta)$. Analogously, $v \in B\big(y, d(x,y) - r + \delta\big)$. Then

$$d(y,u) \leq d(y,v) + d(u,v) < d(x,y) - r + \delta + \delta < d(x,y) - r + \varepsilon.$$

Thus

$$u \in B(x, r+\varepsilon) \cap B\big(y, d(x,y) - r + \varepsilon\big),$$

i.e.,

$$B(x, r+\varepsilon) \cap B\big(y, d(x,y) - r + \varepsilon\big) \neq \emptyset.$$

Now assume that M is proper. Fix $r \in \big(0, d(x,y)\big)$. We can construct a sequence

$$u_n \in B\Big(x, r + \frac{1}{n}\Big) \cap B\Big(y, d(x,y) - r + \frac{1}{n}\Big).$$

Let (u_{n_k}) be a convergent subsequence of (u_n) and let $u_{n_k} \to u$. It is easy to see that

$$u \in S(x,r) \cap S\big(y, d(x,y) - r\big),$$

i.e.,

$$S(x,r) \cap S\big(y, d(x,y) - r\big) \neq \emptyset.$$

□

A natural question to consider is whether the reverse of Corollary 3.8 holds. In [11] it was shown that in general it does not (see [11, Example 4.10]), although for some compact metric spaces the reverse of Corollary 3.8 is in fact true. Note that the metric space M from example provided in [11] was non-compact, therefore leaving open the question of whether the reverse of Corollary 3.8 holds for all compact metric spaces. The following example concludes that the reverse does not hold for all compact metric spaces.

Example 3.9. Let $M = [0,1]$ be a metric space with distance

$$d(a,b) = |a-b| \min\{1+a, 1+b, 2-a, 2-b\}.$$

First, we will show that d is indeed a metric. Let $a, b \in M$. Clearly $d(a,b) \geq 0$ and $d(a,b) = d(b,a)$. If $d(a,b) = 0$, then we must have $a = b$.

Now we will show $d(a,b) \leq d(a,c) + d(b,c)$ for any $c \in M$. Without loss of generality we will assume that $\min\{1+a, 2-a\} \leq \min\{1+b, 2-b\}$.

If $\min\{1+a, 2-a\} \le \min\{1+c, 2-c\}$, then

$$\begin{aligned} d(a,b) &= |a-b| \min\{1+a, 1+b, 2-a, 2-b\} \\ &\le (|a-c| + |b-c|) \min\{1+a, 1+b, 2-a, 2-b, 1+c, 2-c\} \\ &\le |a-c| \min\{1+a, 1+c, 2-a, 2-c\} \\ &+ |b-c| \min\{1+b, 1+c, 2-b, 2-c\} \\ &= d(a,c) + d(b,c). \end{aligned}$$

If $\min\{1+a, 2-a\} > \min\{1+c, 2-c\} = 1+c$, then $a > c$ and $b > c$. Therefore

$$\begin{aligned} d(a,b) &= |a-b| \min\{1+a, 1+b, 2-a, 2-b\} \\ &\le |a-b|(1+\min\{a,b\}) \\ &= |a-b|(1+c) + |a-b|(\min\{a,b\} - c) \\ &\le |a-b|(1+c) + 2(\min\{a,b\} - c)(1+c) \\ &= (a-c+b-c)(1+c) \\ &= |a-c| \min\{1+a, 1+c, 2-a, 2-c\} \\ &+ |b-c| \min\{1+b, 1+c, 2-b, 2-c\} \\ &= d(a,c) + d(b,c). \end{aligned}$$

If $\min\{1+a, 2-a\} > \min\{1+c, 2-c\} = 2-c$, then $a < c$ and $b < c$

$$\begin{aligned} d(a,b) &= |a-b| \min\{1+a, 1+b, 2-a, 2-b\} \\ &\le |a-b|(2-\max\{a,b\}) \\ &= |a-b|(2-c) + |a-b|(c - \max\{a,b\}) \\ &\le |a-b|(2-c) + 2(c - \max\{a,b\})(2-c) \\ &= (c-a+c-b)(2-c) \\ &= |a-c| \min\{1+a, 1+c, 2-a, 2-c\} \\ &+ |b-c| \min\{1+b, 1+c, 2-b, 2-c\} \\ &= d(a,c) + d(b,c). \end{aligned}$$

Note that M is compact since $|a-b| \le d(a,b) \le 2|a-b|$ for all $a, b \in M$.

Denote $x = 0$ and $y = 1$. Then

$$r \in S(x,r) \cap S(y, 1-r)$$

for every $r \in (0,1)$. However, m_{xy} is not a Δ-point. Let $\lambda \in (0, 1/8)$ and let us define two functions $f, g \colon M \to \mathbb{R}$ by

$$f(p) = -p$$

and

$$g(p) = \min\Big\{ \max\big\{1-p, \frac{1}{2} - \lambda\big\}, \frac{1}{2} + \lambda\Big\}.$$

Clearly $\|f\| \leq 1$. Note that by Lemma 1.12 we also have $\|g\| \leq 1$. Let

$$h = (1-\lambda)f + \lambda g.$$

Then

$$h(x) - h(y) = 1 - \lambda + \lambda\Big(\frac{1}{2} + \lambda - \Big(\frac{1}{2} - \lambda\Big)\Big) = 1 - \lambda + 2\lambda^2.$$

Therefore

$$m_{xy} \in S\big(h/\|h\|, 1 - (1-\lambda+\lambda^2)/\|h\|\big).$$

Assume that $d(u,v) < \lambda$ and $m_{uv} \in S\big(h/\|h\|, 1 - (1-\lambda+\lambda^2)/\|h\|\big)$. Then

$$\begin{aligned}(1-\lambda+\lambda^2)d(u,v) &< h(u) - h(v)\\ &= (1-\lambda)\big(f(u) - f(v)\big) + \lambda\big(g(u) - g(v)\big)\\ &\leq (1-\lambda)d(u,v) + \lambda\big(g(u) - g(v)\big).\end{aligned}$$

Therefore $g(u) - g(v) > 0$ and then from $d(u,v) < \lambda$ we see that either $u \in (1/2-\lambda, 1/2+\lambda)$ or $v \in (1/2-\lambda, 1/2+\lambda)$. Furthermore,

$$\begin{aligned}(1-\lambda)d(u,v) &< (1-\lambda)\big(f(u) - f(v)\big) + \lambda\big(g(u) - g(v)\big) - \lambda^2 d(u,v)\\ &< (1-\lambda)|u-v| + \lambda d(u,v),\end{aligned}$$

giving us $(1-2\lambda)d(u,v) < (1-\lambda)|u-v|$. Hence

$$\min\{u, v, 1-u, 1-v\} = \frac{d(u,v)}{|u-v|} - 1 < \frac{1-\lambda}{1-2\lambda} - 1 = \frac{\lambda}{1-2\lambda}.$$

Now we see that either

$$u \in \big[0, \lambda/(1-2\lambda)\big) \cap \big(1 - \lambda/(1-2\lambda), 1\big]$$

or

$$v \in \big[0, \lambda/(1-2\lambda)\big) \cap \big(1 - \lambda/(1-2\lambda), 1\big].$$

Therefore

$$\lambda > d(u,v) \geq |u-v| > \frac{1}{2} - \lambda - \frac{\lambda}{1-2\lambda} > \frac{1}{2} - \frac{1}{8} - \frac{1}{6} > \frac{1}{8} > \lambda,$$

which is absurd. Therefore, by Theorem 3.7, m_{xy} is not a Δ-point.

From Theorem 3.7 we can easily derive the following result.

Corollary 3.10. *Let $\mu = \sum_{i=1}^n \lambda_i m_{x_i y_i} \in \operatorname{conv}\big(\operatorname{Mol}(M)\big) \cap S_{\mathcal{F}(M)}$. If $m_{x_i y_i}$ is a Δ-point for every $i \in \{1, \ldots, n\}$, then μ is a Δ-point.*

Proof. Let $m_{x_i y_i}$ be a Δ-point for every $i \in \{1, ..., n\}$. Fix $f \in S_{Lip(M)}$ and $\alpha, \varepsilon > 0$ such that $\mu \in S(f, \alpha)$. There exists $i \in \{1, \ldots, n\}$ such that $m_{x_i y_i} \in S(f, \alpha)$. Since $m_{x_i y_i}$ is a Δ-point, then according to Theorem 3.7 there exist $u \neq v \in M$ such that $m_{uv} \in S(f, \alpha)$ and $d(u,v) < \varepsilon$. That means that μ is a Δ-point. $\square$

It is natural to ask whether the converse of this corollary holds. The following example shows that in general it does not.

EXAMPLE 3.11. Let $M = \big\{(a,b)\colon a \in \{0,1\}, b \in [0,1]\big\}$ be a metric space with distance

$$d\big((a_1,b_1),(a_2,b_2)\big) = \max\big\{|a_1-a_2|, |b_1-b_2|\big\}.$$

Let $x_1 = (0,0)$, $y_1 = (1,0)$, $x_2 = (1,1)$ and $y_2 = (0,1)$.

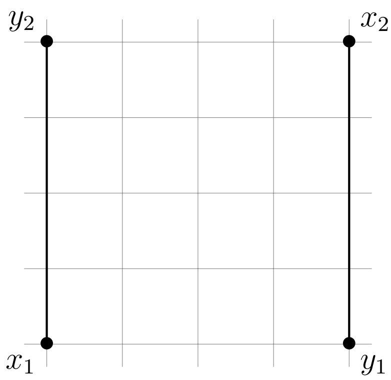

Figure 4: Metric space M from Example 3.11

We have $d(x_1,y_1) = d(x_1,y_2) = d(x_2,y_1) = d(x_2,y_2) = 1$ and from Lemma 1.13 we derive that

$$\|m_{x_1y_1} + m_{x_2y_2}\| = \|m_{x_1y_2} + m_{x_2y_1}\| = 2.$$

By Corollary 3.8, $m_{x_1y_1}$ and $m_{x_2y_2}$ are not Δ-points. However, from Proposition 3.2 we see that $m_{x_1y_2}$ and $m_{x_2y_1}$ are Δ-points. Therefore

$$\frac{m_{x_1y_1} + m_{x_2y_2}}{2} = \frac{m_{x_1y_2} + m_{x_2y_1}}{2}$$

is a Δ-point according to Corollary 3.10.

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