

## CHAPTER X

## FUNDAMENTAL MEASUREMENT

*Summary.* Measurement is the assignment of numerals to represent properties.

Why is the process important and why is it applicable to some properties (e.g. weight) and not to others (e.g. colour)?

The answer must lie in some relation between numerals and measurable properties which does not apply to non-measurable properties. This relation is found in the common possession of order. The conception of order is analysed, as well as the relation between numerals and numbers. All measurable properties are capable of being placed in a natural order by means of definite physical laws which are true of them.

But the possession of order alone will not enable a property to be measured, except possibly by the use of previously established systems of measurement for other properties. In order that a property should be measured as a fundamental magnitude, involving the measurement of no other property, it is necessary that a physical process of addition should be found for it. By a physical process of addition is meant an operation which is similar in a certain manner to the mathematical operation of addition.

This similarity is analysed, the property of weight being taken as an example. It is shown that, if there is to be a satisfactory process of physical addition, two laws, the first and second laws of addition, must be fulfilled. Both these are definite physical laws, so that it is experiment and experiment only that can determine whether a property is fundamentally measurable. The two laws, though closely connected, are independent and one of them may be true without the other.

The difference between properties that are and those that are not capable of satisfactory addition is roughly that between quantities and qualities.

If a fundamental process of measurement can be found at all, the assignment of numerals to represent properties is perfectly definite, except for one arbitrary element, the unit.

It was suggested in Chapter I that physics could be distinguished from other sciences by the part played in it by measurement. Other sciences measure some of the properties which they investigate but it is generally recognised that when they make such measurements they are always depending, directly or indirectly, on the results of physics. All fundamental measurements belong to physics, which might almost be described as the science of measurement. Accordingly before we can enter upon any discussion of the actual results of physics we must have a clear idea of what measurement is and what are the conditions necessary for its application; to this inquiry, and questions arising directly out of it, this part of the volume will be directed.

**Measurement is the assignment of numbers.** Measurement is the process of assigning numbers to represent qualities; the object of measurement is to enable the powerful weapon of mathematical analysis to be applied

to the subject matter of science. The questions that have to be answered are What is the nature of the process, and How does it serve to attain that object? The answers to both of them may seem very obvious, and this obviousness itself may seem the chief reason for the difficulty which I believe that some people will find in giving a perfectly clear and unambiguous answer to them. It will therefore be well to base our discussion on the consideration of a question which will probably appear more capable of a definite answer, Why can and do we measure some properties of bodies while we do not measure others?

I have before me on the table a tray containing several similar crystals. These crystals possess many properties among which may be included the following: Number, weight, density, hardness, colour, beauty. The first three of these qualities are undoubtedly capable of measurement—unless it be judged that number is to be excluded as being more fundamental than any measurement; concerning hardness it is difficult to say whether or no it can be measured, for though various systems of measuring hardness are in common use it is generally felt that none of them are wholly satisfactory. Colour cannot be measured as the others can, that is to say it is impossible to denote the colour of an object by a single number which can be determined with the same freedom from arbitrariness which characterises the assigning of a number to represent weight or density<sup>1</sup>. The last property, beauty, can certainly not be measured, unless we accept the view which is so widely current that beauty is determined by the market value. What is the difference between the properties which determine the possibility or impossibility of measuring them?

We can dispose very easily of the case of “beauty”, which it was perhaps rather fantastic to introduce. Beauty is not a property with which science can have anything to do, because no agreement can be obtained for judgements concerning it; it could only become a matter for scientific investigation if a relation can be established between it and some property, such as price, concerning which agreement could be obtained.

The other properties, however, are all such that agreement concerning them can be obtained (for we will suppose that colour is to be judged only by normal persons—see p. 26), and the reason for the difference must be sought further afield. Since measurement is the process of assigning numbers to represent properties, it is obvious to inquire whether that difference may not be due to some greater resemblance between number and measurable properties than between numbers and immeasurable properties. Let us ask therefore what are the properties of numbers.

**Numerals and Numbers.** At the outset we must note that “number” is often used to denote two (or perhaps three) entirely different things; when I say that the number of my room in a hotel is 187 I am not speaking of the

<sup>1</sup> Of course it might be possible to measure colour by means of the wave-lengths and amplitudes of the components of the reflected light. But the discussion below will show that such a method is irrelevant to our present purpose.

same kind of thing as when I say that two and two are four; or when I say that I have five fingers. The fact that my room is 187 and yours is 58 does not mean that either or both of us have any nearer relation to the occupant of room 245 than to any other person; nor does it imply necessarily that there are 187 rooms in the hotel. "Number" in the first sentence should be replaced by "numeral"; all that I mean by the statement is that there is on the door of my room a piece of brass cut in the shape which is conventionally used to represent the number 187 on a printed page; the proper name for anything which thus represents a number is a "numeral" and it will be used throughout what follows. A numeral is a material or quasi-material symbol, a black mark on a piece of paper or certain sounds which I utter. The numeral 2 is thus clearly distinguished from the number which is involved in the statement that two and two make four, a pure arithmetical proposition; according to Mr Russell this number two is a class of classes, but we do not need to know for our purposes what it is; we only want to distinguish it from a numeral and, later, to know some of its properties. What is the exact meaning of number in the last sense (when I say I have five fingers) and whether it is identical with numeral or mathematical number is one of the problems we have to solve. But in case we have cause later to make a distinction I shall henceforth always write the mathematical concept, the class of classes, with a capital letter, Number.

**Numerals and Order.** The fact that numerals are used to represent Numbers shows that there must be some close relation between them; part at least of this relation is that they have a common property; this property is the possession of a definite order. Numerals are usually and normally found always in the same order; and I shall be seriously annoyed if I find that my room in the hotel is not between Nos. 186 and 188—though I may pardon the landlord if it lies between 185 and 189, the even numerals being on the opposite side of the passage. Now things may come to possess a definite order in one of two ways; the order may either be assigned arbitrarily as the result of a mere convention which everyone accepts, or it may arise from special relations between the things which possess the order. It is order of the first kind which is possessed by numerals. If we forget for the moment the relation between numerals and Numbers, there is no more reason why we should write 1, 2, 3, 4, 5, 6, 7, 8, 9 rather than 2, 6, 3, 8, 1, 4, 7, 9, 5; the usual order of the numerals arises from pure convention; it is as conventional as the order of the letters of the alphabet or the order of ranks in the Lord Chamberlain's table of precedence; it is merely something that civilised people have agreed to accept and which might be entirely changed if we began civilisation over again. But the convention which establishes the order of the numerals has a great advantage over that which establishes the order of the alphabet or of ranks, an advantage so great that the ingenuity which suggested the convention has often been described as one of the most remarkable achievements of the human mind. The advantages are (1) that the convention enables the order to be remembered very easily; we have only to carry in our minds an

arbitrary order of 10 symbols and a few very simple rules for the order of their combinations; (2) that the list of things ordered can be indefinitely extended and yet the order of them remains perfectly definite; (3) that interpolation to any desired extent can be effected without changing the order already established. Thus, if we have already arranged five symbols in the order 1, 2, 3, 4, 5 and we want to add a sixth which is to come between 2 and 3, we can leave the five symbols as they are and add a new symbol 2·1 which is known by the convention to come between 2 and 3; if we find later we want a symbol between 2·1 and 2, we can use 2·11 and so on. It is these advantages which lead to the wide-spread use of numerals, rather than any other arbitrarily ordered symbols, to denote objects of which the order is important, such as houses in a street<sup>1</sup>. It is only very foolish people who put their friends to the trouble of remembering that Chatsworth comes between Seaview and Chez Nous instead of that 16 comes between 15 and 17; and it is almost equally foolish and even more common to call 32 *a* instead of 32·1 the house which has been built between 32 and 33.

**The relation generating order.** But order may also arise, not from an arbitrary convention, but from real properties of the things ordered; and it is of course the existence of this real order which has led to the invention of arbitrary orders to denote the things characterised by it. Such real order is possessed by the houses in a street and by Numbers, and by many of the other objects which are usually denoted by numerals. Now this real order, as pure mathematicians inform us, arises from certain relations obtaining between the things which are ordered, namely, such relations as are technically called "transitive and asymmetrical". A transitive relation is one such that if *A* has it to *B* and *B* has it to *C*, then *A* has it to *C*; a symmetrical relation is one such that if *A* has it to *B* then *B* has it to *A*. Thus, in a family, the relation of father to son is intransitive and asymmetrical; the relation of children of the same parents to each other is symmetrical and intransitive, but would be transitive if we regard a person as bearing that relation to himself; the relation of ancestor to descendant is transitive and asymmetrical. Similarly, of the relations used in science, "of the same colour as" is transitive and symmetrical, "different in colour from" is symmetrical, but may be intransitive (i.e. if *C* is of the same colour as *A*), "heavier than" is transitive and asymmetrical, "one gramme heavier than" is intransitive and asymmetrical. Now a number of terms will form a series and will have a real order if there is a transitive asymmetrical relation such that every term has either this relation or its converse to every other term. (By the converse of a relation *R* is

<sup>1</sup> It is perhaps worth while to note that objects which possess no natural order are often denoted by numerals, for instance soldiers or telephones. The advantage of numerals over any other kind of name for such purposes arises only from the possibility of indefinite extension; however many objects there are we can never run short of names for them if we use numerals, for there are definite rules according to which new names can always be invented. Here again few people have the sense to be consistent and use only numerals; in England (but not in France) we still call our telephone exchanges by pet names, as if they were dogs.

meant the relation which  $B$  has to  $A$  when  $A$  has the relation  $R$  to  $B$ ; for example, "lighter than" is the converse of the relation "heavier than".) The condition that such a relation exists is sufficient to establish a definite order and to place the terms in a series<sup>1</sup>.

**Measurement of hardness.** We are now in a position to decide why measurement is applicable to "hardness" in a way that it is not applicable to colour. In respect of hardness all bodies to which the concept applies at all are related by a transitive asymmetrical relation "harder than" or its converse "softer than"; every body the hardness of which we want to measure at all is either harder than or softer than every other body in that class. (For the moment we omit the possibility that two bodies are equally hard, and shall return to the consideration of it later.) Now "harder than" is a transitive asymmetrical relation. We will leave out of account modern methods of measuring hardness (the Brinell test, the impact test and so on) and suppose that by hardness we mean the property which led to the establishment of Mohs' scale used by mineralogists. Then one body is harder than another if it will scratch it. "Harder than" is transitive because, if  $A$  will scratch  $B$ , and  $B$  will scratch  $C$ , then  $A$  will scratch  $C$ ; it is asymmetrical because if  $A$  will not scratch  $B$ ,  $B$  will not scratch  $A$ <sup>2</sup>. Accordingly since every body with which we are concerned (omitting again those of equal hardness) will either scratch or be scratched by every other, the condition for the formation of a series is fulfilled and it is possible to arrange all these bodies in a series having a definite order of hardness, such that each body is harder (or softer) than the body which follows it and softer (or harder) than that which precedes it in the series.

When we have arranged the bodies in this series, they are suitable for denotation by numerals. We call the softest (or hardest) of them 1, the next in the series 2 and so on. This is precisely the procedure adopted by Mohs in his scale of hardness; it depends simply and solely on the fact that hardness is a transitive and asymmetrical relation. If it had turned out that the relation was symmetrical and that, while  $A$  would scratch  $B$ ,  $B$  would also scratch  $A$ , there would have been no reason for putting  $A$  before  $B$  or  $B$  before  $A$  in the series; if on the other hand it had turned out that the relation was intransitive and that, though  $A$  would scratch  $B$  and  $B$  scratch  $C$ , yet  $A$  would not scratch  $C$ , but  $C$  would scratch  $A$ , then we should not have known whether to place  $C$  after  $B$  or before  $A$ .

Now we must introduce the bodies which are such that they will neither

<sup>1</sup> For a discussion how far the condition is necessary as well as sufficient and for the consideration of certain complications connected with "closed series", which will not concern us, reference should be made to Russell and Whitehead's *Principia Mathematica*. Of course no proof has been offered here that a transitive asymmetrical relation can generate order; perhaps such a proof can hardly be given, for what is "order" except something generated by such a relation?

<sup>2</sup> Neither of these propositions is strictly true. See Winkelmann's *Handbuch*, 2nd ed., *Allgemeine Phys.*, II. p. 860; but Mohs' scale is based on the assumption that they are true—as is pointed out by Auerbach in the passage referred to.

scratch nor be scratched by some body in the series. Their introduction into the scheme, in spite of the fact that the condition of a series is not fulfilled, is possible for this reason. We find that a body *B* which will not scratch *A* or be scratched by *A* behaves towards other bodies exactly as *A* behaves; that is to say, it will scratch all the bodies which *A* scratches and be scratched by all which scratch *A*<sup>1</sup>. It is this fact, which could not have been foretold *a priori*, which makes us call *B* of the "same hardness as *A*"; if *B*, though neither scratching nor being scratched by *A*, had not behaved in the same way as *A* towards any other body *C*, we should not have known where to put it in the series; but as it behaves exactly like *A*, we put it in the same place as *A* and denote it by the same numeral. It will then still be true that any body denoted by a larger numeral will scratch any body denoted by a smaller numeral; we have only to introduce a new idea to deal with bodies denoted by the same numeral (a feature not contemplated in the original formation of the series), namely that bodies denoted by the same numeral will neither scratch nor be scratched by each other.

**Colour cannot be measured.** We cannot apply the same process to colour because we cannot find a similar transitive and asymmetrical relation which expresses differences of colour and covers the whole range of coloured bodies. "Different in colour" is, as we have noted, symmetrical but not always intransitive; "redder than", "darker than" are intransitive and symmetrical relations, but they do not cover the whole field. Some colours are neither redder nor less red than others, two shades of the same blue, for example; and yet they are not "equal in colour", as the two bodies which would neither scratch nor be scratched by each other were equal in hardness. For they are not similarly related to all other coloured bodies; one may be darker and the other lighter than some third colour. Accordingly we cannot range them in a single series characterised by a definite order, even if we have recourse to the device of allowing the same place in the series to be occupied by several terms. There is no natural order of the colours which enables us to denote them by the series of numerals except by the most arbitrary convention<sup>2</sup>. Dyers often issue patterns of wool marked with numerals, but the assignment of the numerals to the colours is as purely arbitrary as the assignment of numerals to soldiers; it is dictated simply by the fact that

<sup>1</sup> Again this proposition is untrue, but again the assumption that it is true is essential to the scale.

<sup>2</sup> It may be noted that if the three-colour theory of vision is true it would be possible to represent colours by numerals in just the same way as minerals are represented in respect of hardness. For then it would be true that every colour is either redder or less red than any other, while at the same time it is either bluer or less blue and greener or less green. Accordingly we might have three scales of redness, blueness, and greenness and denote a colour by three numerals, such as 4, 7, 11, representing respectively its position in the three scales. Colours which were neither redder nor less red, neither bluer nor less blue, neither greener nor less green than some other colour *B* could then be given the place the same as *B* because it would differ from all other colour *C* in the same manner as *B* differs from *C*. The kind of series which would be generated in this manner would be a three-dimensional series, concerning which we shall have more to say in Part III.

numerals are numerous and not by their possession of an order. The assignment of numerals to colours is arbitrary, because it is not dictated by experiment, that is by the results of judgements which form part of the subject matter of science and for which universal and impersonal assent can be obtained. Once we have decided on the scheme of our scale of hardness, the numeral which we are to allot to any body in respect of hardness is fixed by experiment; it is experiment which determines whether  $A$  will scratch  $B$  or be scratched by it and, therefore, whether  $A$  is to be represented by a greater or a less numeral than  $B$ <sup>1</sup>. On the other hand, in allotting numerals, as the dyer does, to represent colours, the particular numeral which is to be allotted to a particular colour is not determined by experiment; no experimental information is conveyed by representing it by one numeral rather than by another. The representation of hardness by numerals does convey certain experimental information; the representation of colours by numerals does not; and that is why we say that hardness is a measurable property in a sense in which colour is not.

**The first conditions for measurement.** This then is our first conclusion. In order that a property shall be measurable at all and in order that its representation by numerals shall convey any information of importance to science, it is necessary that systems which differ in respect of that property shall be related by some transitive asymmetrical relation,  $R$ . When we say that  $A$  differs from  $B$  in respect of this property (and the statement that  $A$  "has" a property means nothing else than that it differs in some respect from other bodies), we must mean that it can be shown by experiment that  $A$  and  $B$  are related by such a relation. •

Further, the class of systems which includes all systems  $A$  and  $B$  such that  $A$  is related to  $B$  by this relation  $R$  (i.e. the class of systems "having the property" in question) must possess the following features. (This class is called "the field" of  $R$ .) If  $X$  is any system in the class,  $X$  must bear either the relation  $R$  or the converse relation  $R'$  to every other system in the class; or, if there is in the class some other system  $X'$  to which  $X$  bears neither of these relations, then  $X$  must bear the same relation as  $X'$  to every other member of the class; that is to say, if  $X'$  bears to any other member  $Y$  the relation  $R$ , the relation  $R'$  or neither of them, then  $X$  must also bear to  $Y$  in each case the same relation. Systems which are related as  $X$  to  $X'$  are said to have the same property or to be equal in respect of it; while if  $X_1$  bears to  $X_2$  the characteristic relation  $R$ , it will be said generally to be "greater than"  $R$ . It should be observed that equality is a transitive symmetrical relation; for if  $X$  has the same relation to  $Y$  as  $X'$ , then  $X'$  has the same relation to  $Y$  as  $X$ , and if  $X$  has the same relation to  $Y$  as  $X'$  and  $X'$  has the same relation to  $Y$  as  $X''$ , then  $X$  has the same relation to  $Y$  as  $X''$ ; that is a necessary consequence of the meaning of the word "same". Equality in

<sup>1</sup> Numerals are not really greater or less; strictly I mean a numeral which comes earlier or later in the arbitrary order and is used conventionally to represent a greater or less Number.

respect of a property is the transitive symmetrical relation which is thus developed out of the transitive asymmetrical relation which is characteristic of the property; we usually regard it as the limit which is reached when the differences expressed by the relation  $R$  and its converse are reduced (e.g. bodies of equal density are regarded as being reached by reducing indefinitely the relations of "just denser than" and "just less dense than"); but the definition given here is capable of being stated more briefly and precisely. Systems which are equal in respect of a property are assigned the same numeral.

**But such measurement is not satisfactory.** This condition is fulfilled by the other three examples which were given of measurable properties, number, weight and density. "More numerous than", "heavier than", "denser than" are all transitive asymmetrical relations; and the further condition is fulfilled that (e.g.), if a body  $B$  is neither heavier nor lighter than  $A$ , then  $B$  is heavier than  $C$  if  $A$  is heavier than  $C$ , and lighter than  $C$  if  $A$  is lighter than  $C$ . These three properties are therefore measurable. Indeed it is obvious that they are not only measurable, but measurable in some higher and more important sense than is hardness (according to Mohs' scale); there is still an arbitrariness about the assignment of numerals on Mohs' scale which is absent from the assignment of numerals to represent weight or density. The unsatisfactoriness of the scale of hardness was thus expressed recently in an engineering journal: "According to Mohs' scale, the hardness of diamond is represented by 10, of ruby by 9 ..., of talc by 1. But these figures are not proportional; and it is probable that the difference in hardness between diamond and ruby is at least as great as between ruby and talc". Nobody would make a similar statement about the scale of weight. What precisely is the distinction between the two scales?

We have seen that, in the scale of hardness, experiment determines uniquely and certainly the order in which bodies are to be placed, and it imposes certain limitations if numerals are to be used at all to represent hardness. Thus, if  $A$  is harder than  $B$ , the numeral which represents the hardness of  $A$  must be greater than that which represents the hardness of  $B$  (according to Mohs' convention, which might equally well have been reversed); if  $A$  is 8,  $B$  cannot be 7. On the other hand there is nothing whatever to determine how much larger the numeral of  $B$  must be; for all experiment can show,  $B$  might be equally well represented by 9, or 50, or a billion; whichever we chose we could still find numerals to represent hardness intermediate between that of  $A$  and  $B$ , and no difficulty or contradiction would arise. In fact experiment, while determining the order of the numerals, leaves the difference between them (or, more accurately, the difference between the Numbers conventionally represented by the numerals) quite undetermined; the choice of the difference is as arbitrary as the dyer's choice of numerals to represent his colours, and the engineer is perfectly right in implying that the differences between the assigned numerals have no physical significance whatever; it does not represent the physical difference<sup>1</sup>. On

<sup>1</sup> Of course the engineer's statement is somewhat confused. He had in his mind

the other hand, when we represent one weight by 8 and another by 3, we feel that the difference in the properties of the two weights is in some manner represented by the difference between the Numbers 8 and 3 which are represented by the same numerals.

Here we encounter entirely new ideas. So far we have been concerned with numerals only as representing order and possessing a quality not essentially different from that of the letters of the alphabet. Instead of representing hardness by numerals 1-10, we should have lost little if we had represented it by letters *A-J* or by ranks, dukes, marquises, earls ...; the only difference would have been that we should have had to have invented some new convention of order when we wanted to interpolate. But when the idea of difference is introduced the distinction between the methods of notation immediately appears; there is a significance in the difference between 1 and 10 which there is not in the difference between *A* and *J*, and this significance is connected, not with the actual symbols used as numerals, but with the Numbers which they are always used to represent. It seems indicated now that measurement is a process of establishing a relation not between properties and numerals, but properties and Numbers. Further these new ideas are associated with a more definite determination of the numerals which are to represent properties; those properties for which the difference between the representative numerals is significant are also those for which these numerals are definitely fixed. When the scale of hardness was found, the choice of a numeral to represent the hardness of one body did not limit the choice of numerals to represent any other body to one and one only; but in our scale of weight, when we have fixed the numeral which is to represent one weight we are not longer left with any arbitrary choice of the remainder; we are forced to represent every other weight by one numeral and one only.

**Density. Derived magnitudes.** Before we proceed to consider what is the distinction between those properties for which numerals are thus fixed, and for which differences are significant, and those which resemble hardness in this matter, we ought to decide to which class those properties which have been enumerated as examples belong. At first sight it will doubtless appear that all the three remaining, number, weight and density, belong to the first class, but a little consideration will show that they are not entirely similar. It is true that the numeral which has to be assigned to represent the density of any substance is fixed quite as definitely as that which represents its weight, but on the other hand the significance of differences of density is not quite the same as that of differences of weight. It will be recognised that the relation between a pair of substances of density 1 and 2 and another pair of density

a definition of hardness by some property other than simple scratching, namely the amount of material of a given kind which can be worn away in given conditions with the use of the substance as an abrasive. This amount can be measured; and what he really meant to assert is that the amount worn away is not proportional to the Number indicated by Mohs' scale. That is to say, he is assuming that a satisfactory method of measuring hardness as a derived quantity (see later) has been found, and stating that Mohs' scale does not agree with the result of that method.

2 and 3 is not so simple as that between the pair of weights 1 and 2 and the pair 2 and 3; we can think at once of a very simple experiment to show that the difference between the weights of one pair is equal to the difference between the weights of the other pair, but there is no experiment of equal simplicity which shows that the difference in density of the two pairs is the same.

Further, if we inquire how we are able to fix with perfect definiteness the numeral to be assigned to represent density, we realise at once that the method employed involves the measurement of other properties; the definite measurement is possible only because we identify density with the ratio of mass to volume and because we can measure mass and volume. It is rather difficult to say whether, in the present stage of development of physics, we actually mean by density this ratio, or whether we merely employ that ratio as an indication of some other property, which is what we really mean by the term; but at any rate we can describe the property which we call density without any reference to mass or volume. Thus we may say that *A* is denser than *B* if a liquid can be found such that *B* will float in it and *A* sink<sup>1</sup>. This definition is not, of course, applicable to all substances which have density, but it would not be difficult to elaborate it so that it would cover the whole range. Now since we are inquiring how properties are measured, it is obviously no answer to our questions to describe a process which involves the measurement of other properties, unless the process by which those properties are measured has already been determined. It has not yet been determined in the case of mass and volume, and hence it is of no use for our present purpose to refer to the measurement of density as a "derived magnitude"; our question is whether density would be measurable if it were defined without any reference to other measurable properties.

The property of sinking in a liquid in which another property will float can be shown by experiment to be transitive and asymmetrical, and to cover the whole range with the exception of bodies which fulfil the condition for equality. Hence density, defined without reference to mass or volume, is at least as measurable as hardness; but is it more so? No; a little inquiry will show that if we define density by means of this property we can proceed exactly as far as and no further than we could with hardness. We can arrange bodies in order of density and assign a series of numerals to represent their density; but the fixing of the numeral to represent the density of one body places no limitation on that to be assigned to any other except that it must be larger or smaller. The only difference between hardness and density is that

<sup>1</sup> I am inclined to think that this is historically the ultimate meaning of density, that to Archimedes, for example, density was simply the property in virtue of which some bodies floated while others sank, and that the discovery that this property was represented by the ratio of the mass (or rather weight) to the volume was a later and independent discovery. It is questions of this kind which ought to be, and never are, discussed in histories of science.

It is worth while here to ask exactly what is meant when it is said that the ratio "represents" the density. It simply means that the order in which bodies are placed in respect of density, or the order generated by the transitive asymmetrical relation "denser than", is the same as the order of the Numbers of the ratios mass/volume.

in the latter and not in the former the generating relation really is transitive and asymmetrical, so that the most minute inquiry would not disclose any discrepancies in our series. The difference which appears to exist between the power of measuring density and the power of measuring hardness arises only from the fact that we can define density adequately as a derived magnitude (see Chapter XIII) and that we cannot at present so define hardness. If the researches that have been undertaken in recent years to define hardness as a derived magnitude, e.g. by the area of the impression caused by the blow of a definite weight falling through a definite distance, are successful, hardness will be measurable in exactly the same degree as density. But the process involved presupposes the measurement of area, just as the measurement of density presupposes the measurement of mass or volume; neither process is a fundamental process of measurement and it is only such processes which interest us for the moment.

**Significance of addition.** Density, therefore, must be classed with hardness, and the break in the series of properties which has been enumerated occurs between density and weight. For weight undoubtedly is a property which is definitely measurable; the fixing of the weight of one body fixes uniquely the weight of all others, and yet the process of measuring weight does not involve the measurement of any other magnitude<sup>1</sup>. For this reason we shall term weight a "fundamental magnitude" in distinction to density which is a derived magnitude. Now the distinction between density and weight in this factor is, it has been suggested, connected with the physical significance of the difference between the Numbers represented by the numerals which represent also the property; in the matter of weight the difference between the weights 2 and 3 is equal to the difference between the weights 3 and 4 in some way in which the difference between the densities 2 and 3 is not equal to the difference between the densities 3 and 4. It is indicated that in the measurement of weight we make use in some manner of the conceptions of addition and subtraction, conceptions which are applicable to Numbers but not to numerals. In the measurement of density or hardness we make use of a similarity between these properties and numerals; in the measurement of weight, we make use of a similarity between that property and Numbers.

And of course it is obvious how we make use of addition to fix definitely the numeral to be assigned to represent weight. We choose some body to which we assign some numeral, say 2; we take another body which has the same weight; we combine them in a manner which we call addition, and so obtain a body of which the weight is definitely and uniquely fixed to be 4. That is to say, the definite fixing of the numeral to be assigned is associated with the finding of some clear physical significance for the process of addition.

The difference between those properties which can be measured perfectly definitely, like weight, and those which cannot arises then from the possibility or impossibility of finding in connection with these properties a physical

<sup>1</sup> With the possible exception of number—a matter to be discussed presently.

significance for the process of addition. Our next inquiry, then, must be what is the process of addition and how physical significance can be attributed to it.

**Nature of addition.** Addition is a process which is peculiarly characteristic of Numbers. What precisely the addition of Numbers is and what are the special properties of Numbers which make addition so particularly applicable to them, these are questions which lie far outside our province; they are discussed in such treatises as that of Russell and Whitehead, and if I attempted to summarise their conclusions I should probably blunder. But there are certain propositions about addition which are undoubtedly true, although the exact nature of their foundation may be doubtful; moreover they are the propositions on which the application of the conception of addition to the further development of arithmetic is based. These propositions are known by the names of the associative and distributive laws; they may be stated thus: If by  $a + b$  we mean the Number which results from the process of addition applied to the Numbers  $a$  and  $b$ , and by  $(a + b) + c$  the Number which results from the addition of  $c$  to the Number which results from the addition of  $a$  and  $b$ , then

$$a + b = b + a,$$

and

$$(a + b) + c = a + (b + c).$$

There is a third proposition which is not so often stated and may possibly have a rather different foundation, though it is not less certainly true, namely, that, if  $a$  is any Number,  $(a + 1)$  is greater than  $a$ , and more particularly that  $(1 + 1)$  is greater than  $1$ . These three propositions contain, I believe, all that is necessary to the development of that part of arithmetic which is concerned only with addition, and not with multiplication or division or any other operation. It should be insisted again that it is not necessary for us to decide whether these statements are mere definitions or significant propositions; it is possible that all are propositions or all definitions, or that some are one and some the other; all that matters to us is that they truly represent certain properties of Numbers and of the process of addition.

It will be observed that the propositions involve, beside the symbol of addition  $+$ , the symbol of equality  $=$ . To state fully the significance of this symbol and to use it properly in developing arithmetic from the propositions, some further knowledge about it is required. We require to know that it represents a transitive symmetrical relation, so that if  $a = b$ ,  $b = a$ , and if  $a = b$  and  $b = c$ , then  $c = a$  or  $a = c$ .

Further, we need for the development of arithmetic certain propositions about individual Numbers and not merely about general relations between them. We need for example to know that  $2 = 1 + 1$ ,  $3 = 2 + 1$ ,  $4 = 3 + 1$  and so on. Again, we need not decide whether these are definitions of the Numbers or propositions about them derived from other propositions; but it is in virtue of such propositions, together with the distributive law, that we arrive at such propositions as that  $4 + 3 = 7 = 5 + 2$ .

Lastly, since our discussion arose from the consideration of the meaning of "difference" it will be convenient, though not strictly necessary, to state the meaning of this term and the associated process of subtraction. For our purpose subtraction may be regarded simply as the inverse of addition; if adding  $b$  to  $a$  is the process which turns  $a$  into  $c$ , then subtraction of  $b$  is the process which turns  $c$  back to  $a$ . And difference will be sufficiently defined for our purpose by stating that, if  $c$  is the sum of  $a$  and  $b$ , then  $b$  is the difference between  $c$  and  $a$ ,  $a$  the difference between  $c$  and  $b$ ; alternatively we may say that the difference of  $p$  and  $q$  is what results by performing on  $p$  the process of subtraction of  $q$ . These definitions are actually equivalent in many, if not all, cases; but it is quite unnecessary for us to inquire why or when they are equivalent.

**Physical addition.** So much for the properties of Numbers in virtue of which addition and subtraction are applicable to them. What is the similarity between these properties and the properties of bodies in respect of weight which enable us to apply to weight the process of addition? The similarity is between the relation denoted by the sign of addition and a relation which can be established experimentally between bodies in virtue of the fact that they have weight; the propositions which are true of one relation are true of the other.

The example of weight will make the meaning clear. We measure weight by means of a balance. We state that the weights of two bodies  $A$  and  $B$  are "equal" when, if  $A$  is placed in one pan of the balance and  $B$  in the other, the final position of the pointer of the balance is unchanged. We say that the body  $C$  is "added to" the body  $A$ , when  $A$  and  $C$  are placed in the same pan of the balance; and that it is "subtracted from" the body composed of  $A$  and  $C$  in the same pan by removing  $C$  from the pan. When we have thus defined "equal" and "added to" in the use of the balance, we can state, corresponding to arithmetical propositions which involve addition and equality, propositions about what will happen to the balance when we place bodies in the pans. Thus suppose that the symbols  $a, b, c$  are taken to represent the bodies  $A, B, C$  ( $a, b, c$  are not numerals necessarily; " $a$ " means simply that we are referring to  $A$  rather than to  $B$  or  $C$ ); then the statement that  $a + b = c$  will simply mean that  $A$  and  $B$  in the same pan balance  $C$ . Now let us state in this manner propositions corresponding to the laws of addition. Then, corresponding to the arithmetical proposition that, if  $a = b$  and  $b = c$ , then  $a = c$ , we shall state that, if a certain body  $A$  balances another body  $B$  and if  $B$  balances another body  $C$ , then  $A$  must balance  $C$ ; corresponding to the distributive law,  $a + (b + c) = (a + b) + c$ , we shall state that if  $P$  is a body which balances  $B$  and  $C$  on the same pan and  $Q$  a body which balances  $A$  and  $B$  on the same pan, then  $A$  and  $P$  on the same pan must balance  $C$  and  $Q$  on the same pan; and so on for the other laws.

Now these statements concern experimental facts; they assert that, in certain circumstances, we shall observe something. The statements may be true or false; and, as with all statements of experimental fact, experiment

only can determine whether they are true or false. If they are true, there will be a certain similarity between the arithmetical process of addition and the arithmetical relation of equality on the one hand and the physical process of addition and the physical relation of equality on the other; if they are false, there will not be this similarity. It is similarity of this sort which was meant at the opening of this section.

**The principle of measurement.** If these experimental propositions, corresponding to the laws of addition, are true (and in the case of weight they are) we can proceed at once to measurement. We know that the Number 2 is that which results from the addition of the Number 1 to the Number 1; that the Number 3 results from the addition of the Number 2 to the Number 1, and so on for all other Numbers. Accordingly if we can find a body corresponding to the Number 1 we can discover in a unique and perfectly determinate fashion what bodies will correspond to other Numbers; and if we agree that the weights of bodies are to be represented by the numerals which represent also the Numbers to which they correspond we can assign numerals to represent their weights. For the present we shall not inquire on what principle we choose the body which corresponds to the Number 1, but suppose it is selected by an act of purely arbitrary choice. Having selected it we proceed thus. We have first to introduce again a definition used in establishing order, namely that, if two bodies are equal in weight they correspond to the same Number and are to be represented by the same numeral. We then find another body equal in weight to the body 1 (and therefore also corresponding to 1) and place it on the same pan with that body; the weight of this composite body is then 2. We find another body balancing, and so equal in weight to, the composite body, place it on the pan with the body 1 and so obtain a composite body 3; and so on. So long as we can always find a body (single or composite) equal in weight to the composite body produced by "adding" the body 1 to the previous composite body, we can continue finding new bodies corresponding to the successive Numbers 1, 2, 3, 4, 5, 6, ... the weight of which is to be represented by the corresponding numerals. By this process we assign numerals in a perfectly determinate manner to an indefinitely long series of bodies which form what we shall call the standard series.

Having obtained our standard series we can also determine uniquely the weight of any body which is equal in weight to any one of the standard series, in virtue of the definition that bodies equal in weight are to be represented by the same numeral.

**The criteria of addition. The First Law.** Of course nobody who is likely to read this book requires to be told what the actual process of weighing is; if mere description had been the object there would have been no need for the long account which has just been given<sup>1</sup>. But we are inquiring why

<sup>1</sup> It may be doubted even whether the long account is accurate. We do not actually establish a standard series of quite the nature or in quite the way which has been described. We do not actually include in our standard series all the terms 1, 2, 3, 4, 5, 6, 7, 8, 9, but

such a process exists for the measurement of weight and not for the measurement of density. The answer to this question is not so familiar. I have never seen it given adequately; the omission may be due to the fact that it is too trite to be worth stating, but I have to confess that I only found it out for myself after long thought. There are many things which appear obvious when they are once stated but are not so easy to state precisely for the first time.

The answer which our recent discussion indicates is this. In respect of weighing a process of addition and a relation of equality can be found which are similar, in the sense that has been explained, to the arithmetical process of addition and the arithmetical relation of equality; but a process of addition and a relation of equality which are thus similar to the arithmetical process and relation cannot be found in respect of determinations of density. But the matter will not be completely cleared up until it is known exactly in what respect the similarity which holds in one case fails in the other.

It would seem at the outset that the difference between weight and density cannot lie in the relation of equality. For we have seen that the characteristic relation which generates order leads directly, both for weight and for density, to a relation of equality which is so far similar to the arithmetic relation that it is transitive and symmetrical, and these properties are all that are explicitly asserted for arithmetical equality. Such a relation must exist for any property which can be ordered and measured to the same extent that hardness can be measured. It may seem therefore that it is only the process of addition which is lacking. But the two are really inseparable, because the laws of addition to which the physical laws have to be similar involve the symbol of equality as well as that of addition. A process of addition which might correspond adequately to the arithmetical process with one relation of equality might not correspond adequately if another relation of equality were chosen. The possibility must be borne in mind that we could find two or more characteristic relations, each of which would represent properly the meaning of density and give rise to a transitive symmetrical relation of equality; and yet that if one of these were adopted, but not if another were adopted, a satisfactory process of addition could be found. (A proposed process of addition will be termed "satisfactory" if it is found to be similar, in the sense described, to the arithmetical relation.) The decision whether the possibility ever becomes actual must be left until we have examined further the process of addition.

Again, the characteristic relation which generates order and lies at the basis of all measurement, complete or incomplete, gives at once some indication of what processes of addition will prove satisfactory. For the process of

only 1, 2, 2, 5, ... or even 1, 2, 4, 8, ..., and we do not make the 5 by five successive steps from 1, but by making it equal to  $1 + 2 + 2$ . But this method involves the proposition, which we have not yet discussed, that  $1 + 2 + 2 = 5$ . Further we sometimes get rid of the necessity for some of these steps by placing weights on the other pan; if we do this we only need a still more restricted series; but in that case we need the proposition that, if  $x + p = q$  and  $p$  and  $q$  are known,  $x$  is known. If we only knew what we suppose that we know at present, the method given would be necessary.

addition must be such that the system which is produced by adding one body possessing the property in question to another must be greater than either of the bodies added<sup>1</sup>. It is this proposition which corresponds to the arithmetical proposition, included in the laws of addition, which states that  $a + 1$  is greater than  $a$ . This proposition will be called the First Law of addition. (Since any body<sup>2</sup> can be taken as 1 no really greater generality would be obtained by stating that  $a + b$  must not be equal to  $a$ .) Now "greater than" is an expression which has meaning quite apart from any measurement or assignment of numerals; "greater than" is the characteristic relation which underlies the very beginning of measurement; and accordingly, before we have begun to assign numerals at all, we are able to apply one test to discover whether any process of addition is satisfactory. It is true that we might be forced, in order to find a satisfactory process of addition, to change the meaning originally attributed to "equal"; but since "equal to" must always be associated in the manner discussed with "greater than" and "less than", a change in the meaning of equality will not abolish the need for a characteristic relation, "greater than", and the test can always be applied.

Moreover the law of addition,  $a + 1$  is greater than  $a$ , is not only the first law which can be tested; it is also the first law which is necessarily applied in the process of measurement. For in our description of the process of establishing a standard series of weights it was implied that the system resulting from the addition of 1 to 1 was different from 1. If this assumption had not been true, the standard series would have consisted of bodies all of which were equal, and we should not have been able to assign a numeral to any body except to that arbitrarily selected as 1; the whole proposed system of measurement would have broken down from the outset.

This, then, is the reason why the first law of addition, in the form  $1 + 1$  is greater than 1, must be fulfilled if the process of addition is to be satisfactory. And this is also the reason why it is possible to find a satisfactory process of addition for weight and not for density. We can find a process of combining two bodies of equal weight, which gives a body of a weight greater than either; but we cannot find a process of combining two bodies of equal density which gives a body of density greater than either<sup>2</sup>. However we combine two bodies of equal density we always obtain a body of the same density; however we define addition for density, we always find that  $1 + 1 = 1$ , and that the first law of addition is untrue.

<sup>1</sup> In view of the possibility of negative magnitudes, which are discussed later, it would be more accurate to say "greater or less". But at present we know nothing of sign, and the more limited expression will serve.

<sup>2</sup> By "bodies of equal density" is meant here bodies of really equal density, composed of the same substance. I say that bodies are not *really* of the same density unless they are composed of the same substance, because unless that condition is fulfilled they are not related entirely in the same way to any third body. A solution of potash and a solution of sulphuric acid may have the same density and fulfil the condition for equality in respect of the relation which characterises the property density; but if equal quantities of the two solutions are mixed with the same quantity of some third solution, say another solution of potash, the density of the resulting mixture will be different in the two cases.

**Qualities and quantities.** It will be recognised, I think, that this difference between weight and density is the true root of the matter; the criterion which is here applied to distinguish the two properties depends on their real meaning. For it is just because the proposition  $1 + 1 = 1$  is true for density and not true for weight that these two properties are important for physics and important in different ways. The distinction between the two properties corresponds to the distinction, which everyone recognises, though few analyse it, between a substance and its properties. We feel that the amount of substance in a body is something which is increased by combining two bodies, while the properties of the substance are something which are not changed at all by any combination of two similar bodies; accordingly we feel that properties for which a process of addition can be found which satisfies the first law of addition represent the amount of some kind of substance, while those for which the law is always false represent the qualities of a substance. In general we desire to be able to place a property in one or other of these two classes, either the class of amounts or quantities of substance, or the class of qualities of substance. When we discuss the matter fully in a much later chapter we shall find that no property can be regarded as a quantity of substance unless it is fully measurable and a process of addition which fulfils all the laws can be found for it. Accordingly, if we find a property such that no process of addition can be found which fulfils all those laws, so that the property is excluded from one class, we always try so to define the property so that it falls into the other class, inclusion in which implies that  $1 + 1 = 1$ . It is not mere accident that there is a large class of physical properties of which this proposition is true; besides density, there are viscosity, solubility, dielectric constant and a host of others; they are all the result of deliberate intention; they have been deliberately defined so that they are the same for any combination of similar bodies as for the component bodies. It is because they are so defined that they are important in physics and their values are tabulated in our works of reference.

We recognise then that there is a large class of properties of the greatest importance for physics which are such that no complete system of measurement, such as is available for weight, is applicable to them. Except in so far as they are "derived magnitudes", and numerals can be assigned to them which are based on the measurement of other magnitudes, they are not capable of measurement, except in the limited and incomplete way in which hardness is measured on Mohs' scale. This class of properties will be termed "qualities". The class of properties which obey the first law of addition will for the present be called "quantities", though the meaning of that term will be limited later. All fundamental magnitudes must be quantities in this sense; but the converse proposition need not be true.

**The criteria of addition. The Second Law.** So far only one of the laws of addition has been considered. We must now inquire into the other laws and ask whether, if the first law is true, the others are necessarily true;

or, if not, in what circumstances they are true and what kind of evidence of their truth is possible and sufficient.

For this purpose it will be convenient to express the laws in a manner slightly different but logically equivalent. In place of the commutative and distributive laws we shall substitute the following Second Law of addition: The magnitude of a system produced by the addition of bodies  $A, B, C, \dots$  depends only on the magnitude of those bodies and not on the order or method of their addition; it is the same so long as the magnitude of the bodies combined is the same and so long as the order and method of combination satisfies the condition laid down in the definition of the process of addition. (Note that the definition of the *same* magnitude does not require any measurement depending on addition.) A special case of this law is that if  $A_1$  is equal to  $A_2$  and  $B_1$  to  $B_1$  then the system produced by adding  $A_1$  and  $B_1$  is equal to that produced by the adding of  $A_2$  and  $B_2$ .

It will be found that the physical laws corresponding to the commutative and distributive laws as ordinarily stated are precisely the same as that corresponding to this Second Law. This is obviously true in the case of the commutative law. In the case of the distributive law, we have  $(b + c) = b + c$  by definition; hence  $a + (b + c) = a + b + c$  by second law; but  $(a + b) = a + b$ , and  $(a + b) + c = a + b + c$ . Therefore  $a + (b + c) = (a + b) + c$ .

Further, for some of our later discussions, it will be necessary to consider what form the laws of addition take when "greater than" or "less than" is substituted for "equal to". In the arithmetical propositions, we have, first, the statement that the relations greater than and less than are transitive and asymmetrical, and, second, corresponding to the commutative and distributive laws, the following:

If  $a + b \geq c$ ; then  $b + a \geq c$ .

If  $(b + c) \geq d$ ; then  $a + (b + c) \geq a + d$ .

To the first of these corresponds again the Second Law stated above. Corresponding to the second we have that the addition of equal magnitudes does not alter the order of magnitude of the bodies to which they are added. This law also must be true if the process of addition is to be satisfactory when we are considering inequalities as well as equalities.

The question is then whether these laws are necessarily true, if the first law of addition is true; it is best examined in the example of weight. We define the addition of bodies in respect of weight as the placing of them in the same pan. The second law of addition will be true if the system which will balance a given combination of bodies in a pan is the same however those bodies are placed in the pan and if, when we add to each of the previously balanced pans one of a pair of bodies of equal weight, the pans will remain balanced. Now whether these conditions are fulfilled depends entirely on what we mean by a balance. If we mean the most carefully constructed instrument made by the maker of the highest reputation, we shall probably find that they are fulfilled; but if we mean anything that a second-hand dealer in cheap apparatus would call a balance, we shall find that they are not always fulfilled. This,

of course, is obvious, but it is not an answer to the question which is being asked; for we are supposing that the process of measurement has proved satisfactory up to this point, and that supposition excludes many of the most imperfect balances. The first law of addition does not, it is true, exclude many imperfections; so long as the beam is free to swing at all the addition of some body to one pan will always make that pan sink; but the condition that the relations of greater than and equal to have the necessary properties does exclude many common imperfections. Thus, if the balance had unequal arms, the second law of addition would not be true, for interchanging the bodies on the pans (which is not excluded by the definition) would alter our judgement of equality; but it would also make the relation of greater than not always transitive and asymmetrical and the relation of equality not always transitive and symmetrical. Such an imperfection is already excluded in the process of establishing merely an order of weight. But there are imperfections which might not be discovered in establishing that order which would make the second law of addition untrue. For example, if the pans were improperly hung and their knife edges not parallel to the main knife edge, the weight of a body (as we should say now) would depend on its position in the pan. In balancing single bodies against each other we might put them always in the same place and so arrive at perfectly consistent judgements of equality; but if we are to add bodies and place two in a pan at the same time, they cannot both occupy the same position as they did when they were balanced singly; pairs of bodies which, weighed singly, were equal may not give equal sums. Or a discrepancy might appear if the nature of the bodies weighed was unsuitable, even if the balance was perfect when other bodies were used. Suppose that one body was a magnet and the other a piece of soft iron; if the piece of soft iron was placed in the same pan very near the magnet it would change the force on it due to the earth's field and would (as we should say) alter its apparent weight; if the magnet were balanced against a non-magnetic body and we placed in each pan a piece of soft iron of equal weight the balance would be disturbed.

It is such defects as these which we are considering when we speak of a "perfect balance", or say that certain conditions are necessary for accurate weighing. The experiments which are described in any adequate text-book of physics as the tests which must be made on a balance before its indications can be accepted as accurate will be found on analysis to consist almost entirely of examinations whether the second law of addition is fulfilled in all cases; and the warnings issued against the neglect of certain precautions (such as that the bodies weighed must be at the same temperature as the balance) are directed against conditions in which that law is known to be false. The only other matter which is taken into account is whether the indications of the balance are likely to be consistent if the observation is repeated. This is a subject we shall examine presently.

Our conclusion is then that the truth of the second law does not necessarily follow from that of the first; that the second law of addition is a true and

independent experimental law; and that experiment only can show if, and in what conditions, it is true<sup>1</sup>. Until we have examined carefully the property which we propose to measure we cannot be sure that measurement is possible, and the doubt may be concerned with the second law of addition as well as with the first law or with the nature of the proposed characteristic relation generating order. The possibility of measurement at every stage depends entirely upon the assumption of certain experimental laws.

But still there is a difference in this matter between the first and second laws of addition. The truth or falsity of the first law seems involved in the very nature of the property which we are proposing to measure. If the first law were not true for weight or if it were true for density, the significance of the properties of weight and density would be quite different from what it actually is. The second law in a sense seems much less important; any difficulty which is experienced in defining addition so that it is true is connected rather with experimental details rather than with fundamental principles. It will be admitted that, even if the first law is true, there are circumstances in which the second law is false; but it may yet be questioned whether it is really possible that, if the first law is true and the property concerned is so defined as to be a "quantity", it should be impossible to find any conditions in which the second law is true.

Questions which involve the hypothesis that our observations might be other than they are can never be answered definitely. But I do not think it can be judged inconceivable that we might be unable to measure a property because we could not fulfil the second law. There is no doubt whatever that we find very great difficulty in fulfilling the second law in some cases, and that in some which have occurred in the more recent developments of science the difficulties are not yet overcome. There are some properties which obey the first law and are therefore regarded as representing the quantity of some substance, but for which a really satisfactory system of measurement has not been worked out. An instance is provided by the "intensity of X-rays". We regard that property as representing the quantity of a "substance" (namely, energy) carried by the rays; but neither the method of ionisation, nor of chemical action, nor even of heat development is really satisfactory for measuring and comparing the amount carried by rays of different hardness. And if we inquire why none of the methods are completely successful we should find in each case that it was doubtful whether the second law of addition was fulfilled, that is, whether rays equal in intensity added to rays equal in intensity always produce sums equal in intensity, or whether a process of addition

<sup>1</sup> It may be thought that the conditions necessary for a perfect balance and accurate weighing could be deduced, without actual experiment, by deduction from known principles, such as the laws of statics. But investigation would show that our belief in the truth of these laws is based directly on our knowledge that measurement of weight (and other forces) is possible, and thus assumes that the second law of addition is true in certain circumstances. Indeed the experiments on which the laws of statics might be based would certainly include those which show that there is such a thing as a perfect balance.

can be defined such that the intensity of the added beams depended only on the intensity of the beams added.

In this case, however, we feel that the 'progress of knowledge may enable the difficulties to be overcome. But I think there is one case which has been so completely studied that it will be felt unlikely that any further research will remove the obstacles which still exist. This instance is "quantity of heat". The full consideration of the matter is impossible until the complete discussion of the science of heat is undertaken, but enough can be said to indicate the position, if no attempt is made to examine all the complications or to prove all the statements made.

In order to define the characteristic relation of order in this instance we may say that the quantity of heat contained in a body<sup>1</sup>  $A$  is greater than that contained in  $B$ , if  $A$  when dropped into a certain volume of water raises its temperature more than  $B$  dropped into the same volume. This relation under suitable conditions proves suitable as a characteristic relation. Quantities of heat are now defined to be added when the bodies containing them are dropped into the same liquid. The first law of addition is fulfilled, for dropping a hotter body into a colder liquid always produces a rise of temperature. But addition of equals does not always produce equals.  $A$  and  $A'$  may produce the same rise when dropped in the same volume of water, and so may  $B$  and  $B'$ , while  $A$  and  $B$  dropped together in the same volume will not produce always the same rise as  $A'$  and  $B'$ . The failure of the second law of addition is due (as we should say now that we have investigated the matter) to the fact that the bodies themselves have a finite heat capacity and do not give up all their heat to the water; the second law will not be true unless the sum of the heat capacities of  $A$  and  $B$  is the same as that of  $A'$  and  $B'$ . All this was discovered early in the history of calorimetry, and for a long time efforts were devoted to finding a method in which the second law was fulfilled. But I do not think it has ever been found; the difficulty experienced at the outset has proved insuperable. We are still not able to measure quantity of heat as a fundamental magnitude, although it is certainly and pre-eminently a quantity. We measure quantity of heat nowadays not as a fundamental magnitude but as a derived magnitude; we measure actually change of temperature or the product of current and potential difference or mechanical work, and we estimate quantity of heat from certain relations which we have discovered between these measurable magnitudes and certain assumptions concerning others. If the reader will ask himself if he can measure quantity of heat (in terms of any arbitrary unit) without measuring any of the three things just mentioned, I think he will have to confess that he cannot; and if he inquires why he cannot by any process which suggests itself, he will always find that it is because of a failure of the second law of addition which is due, according to modern methods of expression, to the fact that all bodies have finite heat

<sup>1</sup> Or, more accurately, the quantity of heat in excess of that contained in the body at the temperature of the water. It should be noted that "higher" and "lower" temperatures can be defined without any reference to measurement.

capacity. If we have not been able to find a satisfactory process of fundamental calorimetry up to the present time, it is extremely unlikely that we shall find one in the future.

**Addition and equality.** The conclusion which I want to enforce is that the second law of addition is quite as important in determining our processes of measurement as either the first law or the form of the characteristic relation, although it is not, like them, involved in the very meaning of the quantity to be measured, but appears often only in the guise of experimental difficulties. For the fact, that while it might always be impossible to fulfil the second law, even when the first law is fulfilled, actually it is in the great majority of cases possible to fulfil it, is important and suggestive. Our experience in this matter doubtless is the cause of our division of properties into quantities and qualities of substance; if we found many properties which, while obeying the first law, would not obey the second we should attribute much less importance to that distinction and to the conception of substance. It is the almost invariable connection between the obeying of the first and the second law which gives that conception such importance in physics. To this matter we shall return later.

But one further question, suggested by the examples which have just been given but unnoticed in order that the argument might not be interrupted, requires brief notice. We have spoken throughout the last section of the second law of addition being obeyed; but when difficulty was found in obeying it, the alteration in the method of measurement which was made consisted usually, not in an alteration in the definition of addition, but in an alteration of the definition of "greater than" and "equal to". A consideration of the attempts that have been made to measure the intensity of X-rays will show this, but it is also shown in the case of weight. In order that the second law should be true we must have a perfect balance satisfying certain conditions, but we can reduce the number of conditions which are necessary by changing slightly our method of weighing (i.e. judging equality) without altering our definition of addition. It is well known that a balance which is imperfect if used in the ordinary manner can be used with success if the "method of substitution" is employed<sup>1</sup>. By this method difficulties arising from inequalities of the arms are overcome; but these difficulties affect, as we saw, the nature of the characteristic relation, as well as the second law of addition, and would therefore have to be eliminated even if there were no question of addition. However it also eliminates troubles which do not affect the nature of the characteristic relation, but only the second law of addition, for instance certain flexibilities of the arms. Now the employment of the method of substitution rather than the ordinary method involves a change in the meaning of "equal to" (and also of course of "greater than"); we now say that two bodies are equal if, placed successively in the same pan, they balance the same

<sup>1</sup> The method of "double weighing" is also used to eliminate certain imperfections of the balance. But that method assumes that the standard series of weights is already in existence; it cannot be used for making a standard series.

body in the other pan. On the other hand addition has precisely the same meaning as before. We have got over a failure<sup>1</sup> of the second law, not by changing the law of addition, but by changing the meaning of equality. It is to this that reference was made on p. 281, when it was pointed out that the relation of equality and the process of addition could not be separated.

It is interesting to observe that we do usually employ this method of overcoming difficulties arising in addition, for it shows that we regard the process of addition as even more intimately involved in the meaning of the process than the relation of equality. We do not feel that in changing the meaning of equality from that characteristic of the ordinary method of weighing to that of the method of substitution we have in any way changed the meaning of "weight"; but it is difficult to see how we could change the process of addition so as to be free from our difficulties without changing that meaning.

**Is the Second Law necessary?** After the long discussion which has been devoted to the second law of addition it may appear rather startling to inquire whether that law is really of any importance and whether measurement would not be quite as satisfactory if it were not true. Nevertheless I think it can be shown that the question is not wholly unreasonable.

Our discussion started from our desire to limit much further than is possible in the case of hardness the choice of numerals which can be used to represent a given property. By the method which has been described we have reduced ourselves from an infinite choice for every property to a single choice for one property, which then fixes all the other numerals. Can we reduce this choice further? No—the full reason will appear hereafter. Well, if we cannot reduce the arbitrary choice to nothing at all, would there be any very great harm in extending it slightly and allowing ourselves (say) two arbitrary choices; for if we allow ourselves this extra liberty a great many of the conditions which have been imposed on measurement could be removed? Let us consider this matter.

The removal of the infinite choice was effected by defining the process of addition. When we have defined addition and the body which has the weight 1, we have fixed, in virtue of the definition of 2 as  $1 + 1$  and so on, which bodies are to have the weight 2, etc. It would seem then that so long as the process of addition was such that adding 1 produces a body with a weight different from 1, all that is required has been attained; for we shall then have a series of bodies all with different weights represented each by a definitely fixed numeral; the weights of all bodies which are equal in weight to any one of this series will be definitely fixed. Is not this all that we desire?<sup>2</sup>

<sup>1</sup> But not all failures. The imperfections arising from faulty hanging of the pans or of interaction between bodies placed in the same pan still remain.

<sup>2</sup> Of course there will be a large number of bodies the weights of which are not equal to those of any of the series, namely bodies which have weights which are not integral multiples of the unit. But the same is true, so far as we have gone at present, if the other laws about addition are insisted on. We shall deal with fractional weights later; for the present it may be observed that if we chose our unit weight small enough, then, whether the other laws of

Nothing has been assumed about addition, except the first law; why has the necessity for the second law been insisted on?

Suppose for example that, leaving our definitions otherwise unchanged, we allowed ourselves to use a balance with unequally flexible arms. Then the relation of equality would be transitive and symmetrical and the first law of addition would be true<sup>1</sup>; we could still calibrate weights and produce a series 1, 2, 3, 4, ... . In what respects would this series differ from that produced by a perfect balance? There are two differences which it is important to note here. In the first place, unless all balances were exactly the same in the flexibility of their arms, weights calibrated on two different balances would not agree; "weight 2" made with one balance would not be equal in weight to "weight 2" made with another. But this is not a very serious matter. After all nobody ever calibrates weights for himself; he always refers them to somebody who makes a business of calibration; and as that person has to provide himself with a standard gramme, why should he not provide himself also with a standard balance? There is a standard current balance at the Board of Trade; why not a standard weight balance? We are forced to introduce one arbitrary element into our measurement in our selection of the unit; why not two arbitrary elements? That is the question which is asked, and I think that if this were the only matter in which the new series of weights differed from the old, it would be difficult to give any answer that would be felt to be really convincing. Arguments founded merely on experimental convenience should have very little importance unless there are none others forthcoming.

But there is a second difference. If the second law were not true, we should not find, when we had calibrated the weights 2 and 5 and the weights 4 and 3, that the first two added were equal to the second two added. (The proof that, if  $2 + 5$  is to be equal to  $4 + 3$ , the second law must be true, need not be given in detail.) This appears more serious; but is it really so? Again it would be troublesome experimentally; we should not be able to weigh with the usual calibrated box containing 1, 2, 2, 5, ..., but should need a body to represent every weight, 1, 2, 3, 4, 5, .... Would there be any result which is anything worse than an experimental inconvenience? Having asked that question, we must leave its answer to a later stage.

**Is the unit the only arbitrary element in measurement?** One further question demands examination. In the system of measurement which has been described there is necessarily one arbitrary element, namely the choice of the unit<sup>2</sup>; that choice is not determined in any way by the laws addition were observed or not, all other bodies would have weights which were sensibly integral multiples of the unit. This observation is made only to show that the problem of fractions has got nothing to do with the matter discussed here.

<sup>1</sup> I am not sure that this statement is quite accurate; but it is certain that some kind of balance could be devised which would be such that everything except the second law remained true. It is such a balance that I am imagining.

<sup>2</sup> It should be observed that it has been assumed that the fulfilment of the laws of addition is independent of the choice of unit; and that, if they are fulfilled with one choice of unit, they will be fulfilled with any other. The assumption was necessarily true if a new unit is chosen

which underlie the process of measurement. Is this the only arbitrary element? It is not immediately obvious that the answer is affirmative, for the system of measurement involves the selection of one physical process rather than another for the combination of the bodies measured in the manner which is called addition. The choice of this process is not entirely arbitrary, because, as we have seen, the process must be such that the rules of numerical addition are obeyed; but is it possible that there should be two processes, both obeying these laws, and that our choice of one rather than the other is arbitrary?

The answer seems to be this. If there is only one process obeying the first law of addition, that is to say only one way of combining two systems so as to produce a third of which the magnitude is different from either, then there is nothing arbitrary in the system of measurement except the unit. But there may be as many independent processes of measurement as there are processes obeying the first law of addition.

The first part of the answer seems to follow by mere logical proof. For let us assume that there is only one way of combining two bodies so as to produce a third of different magnitude; and then, having established one satisfactory system of measurement, let us endeavour to establish another. The magnitude of a body measured according to the system already established will be represented by the numeral  $x$ ; we shall call  $x$  the old magnitude of the body. If there is some other system of measurement, according to which the magnitude is represented by the numeral  $y$ ,  $y$  will be termed the new magnitude of the body.

Now let us take two bodies of which the old and new magnitudes are respectively  $x_1, y_1$  and  $x_2, y_2$ , and try to find a process of addition which shall be characteristic of the new system of measurement and shall produce the body which has the new magnitude  $(y_1 + y_2)$ . By our fundamental assumption this body must be produced by adding, in the manner characteristic of the old system of measurement, bodies having the old magnitudes  $x_1, x_2$ . However, we need not simply add the two original bodies in this manner. For since the definition of equality of magnitudes is prior to and independent of any system of measurement, in forming the body  $(y_1 + y_2)$  I may make any number of bodies, each having the old magnitude  $x_1$ , and combine them with any number, each having the old magnitude  $x_2$ . If I combine together  $q$  bodies  $x_1$  and  $r$  bodies  $x_2$ , then the old magnitude of the combination must be  $(qx_1 + rx_2)$ , for the only way in which I can combine them so as to produce a body different in magnitude is by the method which is addition on the old system of measurement. The only latitude which I have is in the choice of  $q$  and  $r$ . If  $q$  and  $r$  are not equal I must have some rule to determine of which body  $q$  specimens are to be taken and of which  $r$ . This rule can

from the members of the standard series established according to the method of this chapter; but, as we shall see in Chapter XII, that series does not include all bodies that form the field of the characteristic relation. Of course the assumption is universally true; but its truth appears to be an experimental fact, not a logical necessity.

be founded on the difference between  $>$  and  $<$ , which again is prior to measurement. We may say for example that  $q$  of the greater and  $r$  of the lesser bodies are to be taken.

If this method is adopted, the body to which is assigned the new magnitude  $(y_1 + y_2)$  must have the old magnitude  $(qx_1 + rx_2)$ , where  $q$  and  $r$  may have any integral values. But since to every  $x$  there must correspond a  $y$  and only one  $y$ , there must be between  $x$  and  $y$  an equation of the form  $y = f(x)$ , where  $f$  may have any form so long as it is single-valued. Accordingly

$$(y_1 + y_2) = f(qx_1 + rx_2)$$

or

$$f(x_1) + f(x_2) = f(qx_1 + rx_2).$$

If  $f$  is an analytic function, having everywhere a differential coefficient (and we shall see later reason to believe that such functions alone can be significant in physics), then this relation will be satisfied for all values of  $x_1$  and  $x_2$  only if  $q = r = 1$ , and if  $f(x) \equiv Ax$ . That is to say we must have  $y = Ax$ , where  $A$  may have any value; this relation clearly states that the new magnitude can differ from the old only in the choice of unit.

If however there is some other way of combining two bodies of old magnitudes  $x_1$  and  $x_2$  so as to produce a system of magnitude different from that of either of them, then it may be possible to find a new system of measurement differing from the old otherwise than in the unit. For if we can produce from the two bodies a system of magnitude  $\phi(x_1, x_2)$ , (where  $\phi$  is not of the form  $qx_1 + rx_2$ ), and if we can find a function  $f$  such that

$$f[\phi(x_1, x_2)] = f(x_1) + f(x_2),$$

then the new magnitude  $y = f(x)$  will be satisfactory and will obey the second law of addition. Such related functions  $\phi$  and  $f$  are possible; for instance, if  $\phi = x_1x_2$ ,  $f$  will be  $\log x$ , or if

$$\phi = \frac{x_1x_2}{x_1 + x_2}, \quad f(x) = 1/x;$$

another example, which will be seen later to have some importance in connection with motion, occurs if

$$\phi = \frac{x_1 + x_2}{1 + \frac{x_1x_2}{c^2}},$$

and

$$f(x) = \log \frac{1 + \frac{x}{c}}{1 - \frac{x}{c}}.$$

It is impossible to predict *a priori* whether or no such a function  $\phi$  can be found; experiment and experiment only can decide the matter; and accordingly we must recognise the possibility that a system of measurement may be arbitrary otherwise than in the choice of unit; there may be arbitrariness in the choice of the process of addition.

On the other hand it must be noted that such alternative systems of measurement may not be regarded as measuring the same magnitude. Since

it is fundamental to all measurement that the order of the numerals assigned as the result of measurement should agree with the order in which the bodies are placed by the characteristic relation defining the property which is the magnitude, two systems of measurement will not measure the same magnitude unless the order of the numerals assigned by both is the same. This condition places an additional limitation on the forms which can be permitted for the function  $f$ , if  $y$  is to measure the same magnitude as  $x$ ;  $f$  must be such that  $df/dx$  is always positive. The condition is fulfilled if  $f = \log x$ , but not if  $f = 1/x$ ; if a second method of combining the bodies were found such that

$$\phi(x_1, x_2) = \frac{x_1 x_2}{x_1 + x_2},$$

there would be a second system of measuring a magnitude of the bodies concerned, but the magnitude  $y$  could not be regarded as the same magnitude as  $x$ .

A simple example of this last possibility is actually provided in elementary physics.

Suppose we define the electrical resistance of a body as a property which is equal when, if one body is substituted for another in a circuit containing a source of potential, the current is unchanged by the substitution. If I have two coils,  $A$  and  $B$ , I can combine them to form a body of which the resistance is not equal to that of either of them in two ways; I can place them either in parallel or in series. Investigation would show (we shall consider the matter in a later volume) that both these processes of combination obey the law of addition. Whichever I adopt, the resistance of the combined bodies is independent of the order of combination, and the resistance of the body resulting from the combination of  $C$  with a body equal in resistance to the combination of  $A$  and  $B$  is equal to the resistance of the body resulting from the combination of  $B$  with a body equal in resistance to the combination of  $A$  and  $C$ . Accordingly our principle shows that there should be two distinct systems of measurement of resistance, and that the results of one should be related to the results of the other by an equation of the form  $y = f(x)$ . And there is such a relation; if both systems adopt the same unit, the relation is simply  $y = 1/x$ . But note that the order of the  $x$ 's is not the order of the  $y$ 's; it is the inverse order. Accordingly we say that the two systems of measurement are not measuring the same magnitude; we call one magnitude the resistance and the other the conductance.

There are not many cases, similar to that which has just been considered, in which two magnitudes, not exactly the same, but as closely connected as are resistance and conductance, can both be measured fundamentally; but all such cases as exist are naturally important. Of the other class of cases, which seem *a priori* possible, where the same magnitude can be measured in two different ways, I have not so far been able to think of a single example. The absence of examples is doubtless connected with the feeling, which will be mentioned presently, that all measurements are fundamentally measure-

ments of numbers. Apparently measurement is actually unique except for the arbitrary choice of the unit, because there is only one way of combining two systems in respect of the property under consideration which obeys the first law of addition. However, a certain latitude must be allowed in interpreting the expression "one way"; several ways which differ in minor details are to be regarded as the same way, so long as they differ in no respect important for the process of addition. The criterion to be applied is this. If the two ways,  $A$  and  $B$ , of combining  $X_1$  and  $X_2$  to form a third system are such that, if  $X_1$  and  $X_2$  combined by  $A$  are equal to  $X_3$  and  $X_4$  combined by  $A$ , then  $X_1$  and  $X_2$  combined by  $B$  are equal to  $X_3$  and  $X_4$  combined by  $B$ , then  $A$  and  $B$  are to be regarded as the same way of combination, because any differences between them are immaterial for measurement.

Thus, for the property, weight, which has been used throughout as an example, there is only one way of combining two bodies so as to produce a third which differs in weight from either of them; this way is to establish a rigid connection between them so that they are not capable of relative motion or, more accurately, of relative motion in a vertical direction. But this way has really many forms; we regard as "placing in one pan with another" any method of preventing relative vertical motion of the two bodies; it does not really matter if the second body is placed actually in the pan; it will do if it is hung on a hook below. We so regard all these forms because they are equivalent for measurement, and if we analyse them, we shall find that they are distinguished from the methods that are not so regarded by the fact that they prevent relative vertical motion. All the ways that are equivalent for addition have a common property.