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Transposed Poisson and 3-Lie superalgebras

Mathematics

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TRANSPONEERITUD POISSONI JA 3-LIE SUPERALGEBRAD

Bakalaureusetöö

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Lühikokkuvõte

Käesolev töö on on pühendatud transponeeritud Poissoni superalgebrale. Selgitatakse transponeeritud Poissoni algebra mõistet ning teisi mõisteid, mis on seotud Poissoni struktuuridega. Uuritakse transponeeritud Poissoni (super)algebra seost 3-Lie (super)algebraga. Transponeeritud Poissoni superalgebra ja selle paaris derivatsiooni põhjal konstrueeritakse ternaarse Lie superalgebra. Selle konstruktsiooni korrektsust tõestatakse töö põhitulemusena viimases peatükis.

CERCS teaduseriala: P120 Arvuteooria, väljateooria, algebraline geomeetria, algebra, rühmateooria. P150 Geomeetria, algebraline topoloogia.

Märksõnad: Superalgebra, Lie algebra, Poissoni algebra, algebra derivatsioon, transponeeritud Poissoni algebra, 3-Lie algebra, transponeeritud Poissoni superalgebra, 3-Lie superalgebra.

TRANSPOSED POISSON AND 3-LIE SUPERALGEBRAS

Bachelor thesis

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Abstract

This thesis is devoted to the transposed Poisson superalgebra. The concept of transposed Poisson algebra, its motivation and the other structures, connected to Poisson structures, are explained. The connection between transposed Poisson (super)algebra and 3-Lie (super)algebra is studied. The construction of a ternary Lie superalgebra by means of transposed Poisson superalgbra and its even derivation is proposed. The consistency of the proposed structure is proven in the main theorem of the thesis.

CERCS research specialisation: P120 Number theory, field theory, algebraic geometry, algebra, group theory. P150 Geometry, algebraic topology.

Key Words: Superalgebra, Lie algebra, Poisson algebra, derivation of algebra, transposed Poisson algebra, 3-Lie algebra, transposed Poisson superalgebra, 3-Lie superalgebra.

Contents

Introduction

Poisson structures play an important role in many branches of geometry and theoretical physics. Today we see that this area of research is actively developing. More and more different generalizations of Poisson algebra are appearing. One direction of development in the theory of Poisson algebras is the connection of the concept of Poisson algebra to structures with a n-ary multiplication law. For the first time such a generalization was proposed by Nambu in his paper [10]. Later, similar generalizations of various algebraic structures to ternary and n-ary operations found wide popularity. For example, the concept of a n-Lie algebra was proposed by Filippov [5]. The elements of a n-Lie algebra satisfy the Filippov-Jacobi identity, which is a generalization of Jacobi identity to n-ary bracket.

Recently, the notion of a transposed Poisson algebra has been proposed [3]. A transposed Poisson algebra is a structure dual to the concept of a Poisson algebra. Chengming Bai with co-authors has shown in [3] that a transposed Poisson algebra is very similar to a Poisson algebra. For example, the class of transposed Poisson algebras is closed under the tensor product of such algebras. In [3] it is also indicated that there is an important connection between transposed Poisson algebras and n-Lie algebras, Novikov-Poisson algebras and some other structures. There are very useful identities in [3] which play an important role in the theory of transposed Poisson algebras. Many mathematicians are actively researching transposed Poisson algebras at the moment and there are still many open questions in this area [4].

In fundamental paper [3] the authors prove an important theorem which gives a method for constructing 3-Lie algebras by means of transposed Poisson algebras.

Theorem. Let $(L, \cdot, [\,,\,])$ be a transposed Poisson algebra and let D be a derivation of $(L, \cdot)$ and $(L, [\,,\,])$. Define a ternary operation on L as follows

$$[x, y, z] := D(x)[y, z] + D(y)[z, x] + D(z)[x, y], \quad x, y, z \in L. \tag{0.1}$$

Then $(L, [\ ,\ ,\ ])$ is a 3-Lie algebra.

The main purpose of the present thesis is to generalize this theorem to the case of transposed Poisson superalgebra and 3-Lie superalgebra. In order to do this we need to extend the ternary bracket (0.1) with the help of derivation with parity to the case of transposed Poisson superalgebra. Hence in our case there are two possibilities for the parity of a derivation D, that is, it can be either even derivation or odd derivation. In the present thesis we use even derivation because odd derivation leads to inconsistent structure in the sense of parities of ternary bracket and its arguments. For this purpose, all necessary algebraic structures that are not included in the standard bachelor's course are defined. Non-trivial examples of these structures and some features are given. The present thesis contains five chapters. The first chapter contains definitions of the basic structures that are used in this thesis. The most important structures defined in the first chapter are super vector space and superalgebra. In addition to the definitions, a simple example of this structure is given and the main method for generalizing algebraic structures to superalgebras - so-called Koszul sign rule - is explained. No less important is the introduced concept of derivation of algebra and superalgebra, since further most of the examples and the entire construction of the main result will be based precisely on derivations.

The second chapter of the thesis contains definitions and examples of Lie structures which are necessary to understand further. First the definition of Lie algebra is given. Alternative definitions of this structure are also presented, since to understand the proof of the main theorem of the thesis it is necessary to understand the degrees of freedom that exist within the framework of the choice of the type of identities. The same chapter contains the most important examples of Lie algebra. The following is the definition of a Lie superalgebra with examples. Similarly with Lie algebra, an alternative definition of the graded Jacobi identity is given. The chapter ends with the definition of ternary Lie superalgebra, to derive the bracket for which the definition of ternary Lie algebra is given as it was first defined by Filippov in his article [5]. The Filippov-Jacobi identity is not given in its standard form, since given form is much more convenient for proving the main theorem of the thesis.

The purpose of the third chapter is to introduce the recently defined transposed Poisson algebra. This chapter is divided into two paragraphs. The first paragraph of the third chapter describes Poisson algebra, since without definition of this algebra it is impossible to talk about transposed Poisson algebra. Therefore, the first paragraph provides the definition of Poisson algebra, the most important examples and properties of this structure. The second section introduces the concept of transposed Poisson algebra. The most important examples of this structure and some properties are also given, mainly in comparison with the Poisson algebra described in the first paragraph of the chapter. In addition, we give a theorem which contains a necessary and sufficient condition for a Poisson algebra to be a transposed Poisson algebra. It is worth mentioning here that, similar to Poisson algebra, transposed Poisson algebra has many generalizations and connections with other algebraic structures that are worthy of special mention, but which cannot be fully recounted within the framework of this thesis. Most of them are described in detail in the article [4]. At the end of the third chapter we recall the theorem defining the ternary Lie bracket on the transposed Poisson algebra. This theorem will be generalized to the case of superalgebra in the final chapter of the thesis.

The fourth chapter is devoted exclusively to the transposed Poisson superalgebra. The transposed Poisson superalgebra was first defined recently by V. Abramov and O. Liivapuu in the article [1]. Since the structure is very new, there are very few sources with results on transposed Poisson superalgebra[1], [11],[12]. The fourth chapter of this thesis mentions the most important results that are already known such as classification and non-trivial examples of this structure [1], [11]. The main result in this chapter is the identities that hold in the transposed Poisson superalgebra. These identities will be actively used in the proof of the main theorem of this thesis.

The final, that is, the fifth chapter of this thesis is devoted to the proof of the main theorem.

Theorem. Let $(A, \cdot, [\,,\,])$ be a transposed Poisson superalgebra and D an even derivation of a superalgebra $(A, \cdot)$ and Lie superalgebra $(A, [\,,\,])$. Define the ternary bracket

$$[x, y, z] = D(x) \cdot [y, z] + (-1)^{|x,yz|} D(y) \cdot [z, x] + (-1)^{|xy,z|} D(z) \cdot [x, y], \qquad (0.2)$$

where $x, y, z \in A$. Then $(A, [\,,\,,\,])$ is a 3-Lie superalgebra.

First of all, we will construct a ternary operation by means of even derivation of a transposed Poisson superalgebra that satisfies all the conditions for ternary bracket of 3-Lie superalgebra. The most extensive and important part of the proof is to show that the ternary operation satisfies the Filippov-Jacobi identity. The main strategy for proving the theorem is to write down, by the definition of the ternary Lie bracket, the left-hand and right-hand sides of the Filippov-Jacobi identity, then cancel all similar terms based on known identities and show that the left side is equal to the right side. To achieve this goal, many nontrivial transformations are used based on the same well-known identities. All transformations are explained in detail and some of the calculations are included in a separate lemma.

1 Superalgebra

This chapter contains definitions of the basic structures that we will use in this thesis. As the ground field, which will be denoted by $\mathbb{F}$, we will consider either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. By algebra A we mean a vector space over a field $\mathbb{F}$ equipped with a bilinear mapping $\varphi : A \times A \to A$, which is called a multiplication in A. If there are few different multiplications on a vector space A then we will use notation (A, φ), explicitly indicating which multiplication we mean. An algebra A will be referred to as an associative (commutative) algebra if for any $x, y, z \in A$ its multiplication has the property $\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$ $(\varphi(x, y) = \varphi(y, x))$. An algebra A will be referred to as a unital algebra if there exists an element $e \in A$ (identity element) such that for any $x \in A$ we have $\varphi(e, x) = \varphi(x, e) = x$. A derivation of an algebra A is a linear mapping $D : A \to A$ which satisfies the Leibniz rule, that is,

$$D(\varphi(x, y)) = \varphi(D(x), y) + \varphi(x, D(y)), \quad \forall x, y \in A. \tag{1.1}$$

The next important structure that we will use in this thesis is superalgebra. It should be noted that superalgebras began to play an important role in algebra and geometry in connection with the development of supersymmetric field theories in theoretical physics. Let us denote by $\mathbb{Z}_2 = \{0, 1\}$ the set of congruence classes of integers modulo 2. It should be noted here that we simplify the notations by identifying congruence classes modulo 2 with the numbers $0, 1$. It is well known that $\mathbb{Z}_2$ is a ring with respect to addition and multiplication of congruence classes modulo 2. Therefore, in what follows we can add and multiply congruence classes modulo 2. A super vector space is a vector space V equipped with a direct sum decomposition $V = V_0 \oplus V_1$, where V_0, V_1 are subspaces of V and $0, 1$ are congruence classes modulo 2. Those elements of a super vector space V that belong to either V_0 or V_1 will be called homogeneous. We can assign to each homogeneous element

x its parity 0 or 1 depending on which subspace it belongs to. The parity of a homogeneous element x will be denoted by $|x|$ and

$$|x| = \begin{cases} 0, & \text{if } x \in V_0, \\ 1, & \text{if } x \in V_1. \end{cases}$$

In sequel it should be taken into account that whenever we use the parity $|x|$ of an element x, we always assume that x is a homogeneous element. Obviously, if there is a set of homogeneous elements all having the same parity, then any linear combination of them will have the same parity. Elements with parity 0 are usually called even elements of a vector super space V, and elements with parity 1 are odd. We will use this terminology.

Definition 1. Let A be an algebra with multiplication denoted by $(x, y) \in A \times A \mapsto x \cdot y \in A$. Then A is said to be a *superalgebra* if

1. A is a super vector space, that is, $A = A_0 \oplus A_1$,
2. $|x \cdot y| = |x| + |y|$ for any $x, y \in A_0 \cup A_1$.

A superalgebra A is called *associative* if A is an associative algebra.

From the second condition in Definition 1 it follows that $A_i \cdot A_j \subset A_{i+j}$, where $i, j \in \mathbb{Z}_2$. Particularly if $i = j = 0$ then $A_0 \cdot A_0 \subset A_0$, i.e. the subspace of even elements is closed under multiplication in an algebra A or, by other words, A_0 is a subalgebra of an algebra A. If $i = 0, j = 1$ or $i = 1, j = 0$ then $A_0 \cdot A_1 \subset A_1,\ A_1 \cdot A_0 \subset A_1$. Now it is easy to show that in the case of an associative superalgebra the subspace of odd elements A_1 is a bimodule over the algebra of even elements A_0.

One important tool, which is widely applied for performing calculations in a superalgebra, is the *Koszul sign rule.* Generally this sign rule can be formulated as follows: if in an algebraic expression we swap two adjacent quantities of degree a and b $(a, b \in \mathbb{Z}_2)$, then this operation must be accompanied by multiplying this algebraic expression by $(-1)^{ab}$. Raising minus one to the power of an element of a ring $\mathbb{Z}_2$ is understood in the following sense: $(-1)^0 = 1$ and $(-1)^1 = -1$.

As an example, let us show how Koszul sign rule works in the case of the concept of commutativity. If $(A, \cdot)$ is an algebra then the commutativity of A means that for any $x, y \in A$ we have $x \cdot y = y \cdot x$. Applying the Koszul sign rule to this formula we get the condition of commutativity for superalgebra $x \cdot y = (-1)^{|x||y|} y \cdot x$. Hence a superalgebra A is referred to as a *commutative superalgebra* if for any two elements $x, y \in A$ it holds

$$x \cdot y = (-1)^{|x||y|} y \cdot x.$$

There is also a notion of a derivation of superalgebra. In superalgebra case it is necessary to take into account that the derivation also has parity. A derivation D is called even if it does not change the parity of homogeneous elements, that is, $|D(x)| = |x|$. Otherwise, that is, if $|D(x)| = |x| + 1$, the derivation D is called odd. A parity of a derivation D will be denoted as $|D|$. That is, $|D| = 0$ if D is an even derivation and $|D| = 1$ if D is odd. Thereby a derivation of a superalgebra A is a linear mapping $D : A \to A$ which satisfies the Leibniz rule (1.1) graded analogue

$$D(x \cdot y) = D(x) \cdot y + (-1)^{|x||D|} x \cdot D(y). \tag{1.2}$$

In what follows we will have very long expressions, where elements of an algebra will be multiplied by coefficients $(-1)^K$, where K is a very complicated combination of parities of elements. In order to simplify the way of writing K we will use the following notations for parity of elements $|xy, z| := (|x| + |y|)|z|$ and $|x, yz| := |x|(|y| + |z|)$.

To complete the chapter, we give one simple example of superalgebra.

Example 1. Consider an algebra of polynomials $A := \mathbb{F}[x]$ with decomposition $A = A_0 \oplus A_1$, where A_0 is a set of even degree polynomials and A_1 is a set of polynomials of odd degree. Then the grading of polynomial may be computed as

$|f| = \deg f \pmod 2$. Simple computation shows that compatibility condition is satisfied:

$$|f \cdot g| = \deg(f \cdot g) = \deg f + \deg g = |f| + |g| \pmod 2.$$

Thus we get a superalgebra. Differentiation of polynomials plays the role of derivation of this algebra. For example, the first derivative of a polynomial is an odd derivation, since degree of a polynomial is decreased by one. The second derivative of a polynomial is an example of even derivation.

More serious examples of superalgebras that are of interest to theoretical physicists are based on Grassmann and Clifford algebras [7],[9].

2 Lie and 3-Lie superalgebras

This chapter contains definitions of a Lie superalgebra and 3-Lie superalgebra, which will play an important role in what follows. First, we give the definition of Lie algebra with some examples. Then we move on the generalization of this structure to the super case. At the end of this chapter, in connection with the purpose of this work, we give a definition of 3-Lie superalgebra.

The definition of Lie algebra has several alternative forms. We take the most common of them.

Definition 2. Let L be a vector space over a field $\mathbb{F}$ and $[\ ,\ ] : L \times L \to L$ be a bilinear mapping. The pair $(L, [\ ,\ ])$ is called a **Lie algebra** if for any $x, y, z \in L$ the following conditions are satisfied:

$$[x, y] = -[y, x], \tag{2.1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{2.2}$$

The conditions (2.1) and (2.2) are referred to as *skew-symmetry* and *Jacobi identity*, respectively. A binary operation $[\ ,\ ]$ is called *Lie bracket*.

It is easy to see that the property (2.1) is equivalent to requirement $[x, x] = 0$ for any $x \in L$. For the computational purpose Jacobi identity is often given in alternative form. It is worth noting that in this form it is easy to see that a Lie bracket with a fixed element determines the derivation of the Lie algebra

Proposition 1. *Jacobi identity (2.2) is equivalent to the following*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \tag{2.3}$$

Proof. We begin with the Jacobi identity and put the last two terms to the right-hand side

$$[x, [y, z]] = -[z, [x, y]] - [y, [z, x]].$$

Then we use the skew-symmetry of Lie bracket and get

$$-[z,[x,y]] = [[x,y],z], \quad -[y,[z,x]] = -[y,-[x,z]] = [y,[x,z]].$$

Combining these results we get the required identity. □

The very important class of Lie algebras can be obtained by means of matrix algebras and the commutator of matrices.

Example 2. Consider some noncommutative associative algebra A (e.g. matrix algebra) with multiplication $(x,y) \in A \times A \mapsto x \cdot y \in A$ and define a Lie bracket as $[x,y] = x \cdot y - y \cdot x$. Then $(A, [\ ,\ ])$ is the Lie algebra. In this case a Lie bracket is called a *commutator*.

Remark that in case of commutative algebras the commutator of any elements is zero and the conditions of Lie algebra is satisfied. Such a structure is usually called Abelian Lie algebra.

Further let us consider the structure of Lie superalgebra.

Definition 3. Let $L = L_0 \oplus L_1$ be a super vector space and $[\,,] : L \times L \to L$ a bilinear mapping. The couple $(L, [\ ,\ ])$ is called a **Lie superalgebra** if the following conditions are satisfied for any $x, y, z \in L$:

$$|[x,y]| = |x| + |y| \tag{2.4}$$

$$[x,y] = -(-1)^{|x||y|}[y,x], \tag{2.5}$$

$$(-1)^{|x||z|}[[x,y],z] + (-1)^{|y||x|}[[y,z],x] + (-1)^{|z||y|}[[z,x],y] = 0. \tag{2.6}$$

The conditions (2.5) and (2.6) are usually referred to as, respectively, *graded skew-symmetry* and *graded Jacobi identity*. Since in this paper we consider exclusively superalgebras, the word *graded* will be omitted.

Similarly to the Lie algebra case, there is an alternative form of Jacobi identity

Proposition 2. *Jacobi identity 2.6 is equivalent to the following*

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]. \tag{2.7}$$

Proof. The proof is analogical to the one in Lie algebra case. Again we start with the original identity and move the last two terms to the right-hand side

$$(-1)^{|x||z|}[x,[y,z]] = -(-1)^{|z||y|}[z,[x,y]] - (-1)^{|y||x|}[y,[z,x]].$$

Then by using skew-symmetry of the bracket we get

$$(-1)^{|x||z|}[x,[y,z]] = (-1)^{|z||x|}[[x,y],z] + (-1)^{|y||x|+|x||z|}[y,[x,z]].$$

The required identity is derived from the previous equality by multiplying both sides of it by $(-1)^{|x||z|}$. □

There is analogical construction of commutator of associative superalgebra similar to what we had in Lie algebra case.

Example 3. Let $A = A_0 \oplus A_1$ be a superalgebra and define a mapping $[x,y] = xy - (-1)^{|x||y|}yx$. Then $(A, [\,,\,])$ is a Lie superalgebra.

More examples and remarkable results on theory of Lie superalgebras can be found in [7].

In conclusion, let us recall the definition of 3-Lie superalgebra. For this purpose we have to define 3-Lie algebra, which is a special case of n-Lie algebra, which was first defined by V.T. Filippov (see [5]).

Definition 4. A pair $(L, [\,,\,,\,])$ is said to be a **3-Lie algebra** if L is a vector space, the mapping $[\,,\,,\,] : L^3 \to L$ is skew-symmetric trilinear and satisfies so-called *Filippov-Jacobi identity*

$$[[x,y,z],u,v] = [[x,u,v],y,z] + [x,[y,u,v],z] + [x,y,[z,u,v]] \tag{2.8}$$

In case of a ternary bracket it is easier to define skew-symmetricity with two conditions ($\forall x,y,z \in L$):

$$[y,x,z] = -[x,y,z] = [x,z,y].$$

For the other permutations of arguments of a bracket the result can be computed with help of these conditions. By means of skew-symmetry one can see that Filippov-Jacobi identity (2.8) is equivalent to the following

$$[[x,y,z],u,v] = [[x,u,v],y,z] + [[y,u,v],z,x] + [[z,u,v],x,y]. \tag{2.9}$$

Thus we see that due to skew-symmetry in ternary structures we have a wide degree of freedom in choosing the form of identities. As we remember in superalgebra case for every permutation of, say, elements a,b we get an extra factor $(-1)^{|a||b|}$. For the purpose of this work we take a definition of 3-Lie superalgebra in the following form:

Definition 5. A pair $(L,[\ ,\ ,\ ])$ is said to be a **3-Lie superalgebra** if $L = L_0 \oplus L_1$ is a super vector space, the mapping $[\ ,\ ,\ ] : L^3 \to L$ is (graded) skew-symmetric, trilinear and satisfies two more conditions: relation between the parity of a bracket and the parity of its arguments (analogue of the condition (2.4))

$$|[x,y,z]| = |x| + |y| + |z| \tag{2.10}$$

and so-called *Filippov-Jacobi identity*

$$\begin{aligned} [[x,y,z],u,v] = (-1)^{|yz,uv|}[[x,u,v],y,z] + (-1)^{|x,yz|+|xz,uv|}[[y,u,v],z,x] \\ + (-1)^{|xy,zuv|}[[z,u,v],x,y]. \end{aligned} \tag{2.11}$$

By skew-symmetry of the bracket we mean the following conditions:

$$[y, x, z] = -(-1)^{|x||y|}[x, y, z], \quad [x, z, y] = -(-1)^{|y||z|}[x, y, z].$$

For the other permutations of arguments of a bracket the result can be computed with help of these conditions.

3 Transposed Poisson algebra

The purpose of this chapter is to define transposed Poisson algebra and briefly describe its structure. Since transposed Poisson algebra is very closely related to Poisson algebra, we begin this chapter by looking at the structure of Poisson algebra. In the first paragraph of this chapter we will give the definition of Poisson algebra, some of its examples, the most important properties and mention some applications. In the next paragraph we will talk in more detail about the structure of the transposed Poisson algebra, which plays a crucial role in this thesis.

3.1 Poisson algebra

The purpose of this paragraph is to give a short overview of the basic concepts associated with Poisson algebras. The origin of Poisson algebra is Hamiltonian mechanics, which is a part of classical mechanics. One of the important concepts of Hamiltonian mechanics is the Poisson bracket of two differentiable functions defined on a phase space (see Example 4 below). From algebraic point of view the set of differentiable functions defined on a phase space is a unital associative commutative algebra if we define the sum of two differentiable functions, the multiplication of functions by scalars and the product of two differentiable functions pointwise. The Poisson bracket of two functions determines the structure of a Lie algebra on the commutative algebra of functions, since the Poisson bracket is skew-symmetric and satisfies the Jacobi identity. Thus, on the commutative algebra of functions we have the additional structure of the Lie algebra. It can be shown that these two structures are compatible in the sense that the Poisson bracket is a derivation of the pointwise product of two functions. An excellent expositions of Hamiltonian mechanics can be found in [2], [6].

Let L be a vector space over real or complex numbers.

Definition 6. A vector space L is said to be a **Poisson algebra** if L is a unital associative commutative algebra, whose multiplication will be denoted by

$$(x, y) \in L \times L \mapsto x \cdot y \in L,$$

equipped with a Lie bracket $(x, y) \in L \times L \mapsto [x, y] \in L$, which satisfies the compatibility condition (also called Leibniz rule)

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z]. \tag{3.1}$$

Naturally there are trivial examples of this structure. For any commutative algebra one can define a Lie bracket as $[x, y] = 0$ and get a Poisson algebra. Analogously for any Lie algebra we can define dot multiplication as zero product and get Poisson algebra. However, such examples are not of particular interest, especially for applications. The most important example of Poisson algebra for theoretical physics, which was briefly described in the beginning of this chapter and from which the study of this structure began, is the following

Example 4. Consider the vector space $\mathbb{R}^{2n}$ ($n \in \mathbb{N}$) and write its elements as $(x_1, \ldots, x_n, p_1, \ldots, p_n)$. Let $C^\infty(\mathbb{R}^{2n})$ be a space of smooth functions on $\mathbb{R}^{2n}$. If $\cdot : C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ is a pointwise product of functions and $[\ ,\ ] : C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ is defined as

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right),$$

then $(C^\infty(\mathbb{R}^{2n}), \cdot, [\ ,\ ])$ is a Poisson algebra.

Previous example can be generalized for arbitrary commutative algebra by using notion of derivation.

Example 5. Let $(A, \cdot)$ be a commutative associative algebra and $D_1, D_2 : A \to A$ be two commuting derivations of A (that is, $D_2(D_1(a)) = D_1(D_2(a))$ for any

$a \in A$). Define a mapping

$$[\,,\,] : A \times A \to A,\ [x, y] = D_1(x)D_2(y) - D_2(x)D_1(y).$$

Then $(A, [\,,\,])$ is a Lie algebra and $(A, \cdot, [\,,\,])$ is a Poisson algebra.

Poisson algebras have appeared in an extremely wide range of areas in mathematics and physics, such as Poisson manifolds, algebraic geometry, operads, quantization theory, quantum groups, and classical and quantum mechanics [4]. The study of Poisson algebras also led to other algebraic structures, such as noncommutative Poisson algebras, generic Poisson algebras, Poisson bialgebras etc. More details about Poisson structures can be found in the book [8].

One of the most important property of Poisson algebra that is worth mentioning, which is crucial for applications in classical mechanics, is the closure under tensor products and left multiplications ([3] Proposition 2.8).

3.2 Transposed Poisson algebra

In this paragraph the definition of transposed Poisson algebra is given. The notion of a transposed Poisson algebra was first proposed in an article [3] by Chengming Bai and his colleagues. Transposed Poisson algebra is essentially the dual notion of Poisson algebra, where the roles played by the two binary operations in the Leibniz rule are switched.

Definition 7. Let L be a vector space over a field $\mathbb{F}$ and $\cdot\,, [\;,\;] : L \times L \to L$ two bilinear mappings. The triple $(L, \cdot, [\;,\;])$ is called a **transposed Poisson algebra** if $(L, \cdot)$ is a commutative associative algebra and $(L, [\;,\;])$ is a Lie algebra that satisfy compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + [y, z \cdot x]. \tag{3.2}$$

The condition (3.2) is called *transposed Leibniz rule*.

Similar to the case of Poisson algebra, if we equip any commutative algebra with zero Lie bracket or any Lie algebra with zero multiplication, we get trivially transposed Poisson algebra. To give a wide class of nontrivial examples of transposed Poisson algebra, we use the notion of derivation of algebra.

Example 6. Let $(A, \cdot)$ be a commutative associative algebra and $D : A \to A$ be a derivation of A. Define a mapping

$$[\,,\,] : A \times A \to A,\ [x, y] = xD(y) - D(x)y.$$

Then $(A, [\,,\,])$ is a Lie algebra and $(A, \cdot, [\,,\,])$ is a transposed Poisson algebra.

In fact, for any fixed commutative algebra with identity $(A, \cdot)$ there is a one-one correspondence between the set of derivations on $(A, \cdot)$ and the set of multiplications $[\,,\,]$ on A with which $(A, \cdot, [\,,\,])$ is a transposed Poisson algebra. (See Proposition 2.4 in [3] for the proof).

More specific example of such a structure was proposed by V. Abramov and O. Liivapuu in [1].

Example 7. Let M^n be a smooth n-dimensional manifold, $\mathcal{F}(M^n)$ an algebra of smooth functions on M^n and X a vector field. Then

$$[f, g]_X = fX(g) - gX(f), \quad f, g \in \mathcal{F}(M^n)$$

defines the transposed Poisson algebra of smooth functions on a manifold M^n.

The study of transposed Poisson algebra has the potential to play an important role in applications because it shares many features with Poisson algebra and other structures found in theoretical physics [3],[4]. For example, the same as Poisson algebra, it is closed under tensor products and left multiplications ([3] Theorem 2.9).

Since Poisson algebra and transposed Poisson algebra share many similarities, a natural question arises when these structures coincide. The answer to this question can be found in paper of Chengming Bai [3].

Proposition 3. *Let* $(L, \cdot)$ *be a commutative associative algebra and* $(L, [\,,\,])$ *be a Lie algebra. Then* $(L, \cdot, [\,,\,])$ *is both a Poisson algebra and transposed Poisson algebra if and only if for any* $x, y, z \in L$

$$x[y, z] = [xy, z] = 0.$$

One of the most important results for this thesis in the article [3] is the construction of the ternary Lie bracket by means of transposed Poisson algebra. Let us recall this result here in the form of a theorem.

Theorem 1. *[3] Let* $(L, \cdot, [\,,\,])$ *be a transposed Poisson algebra and let* D *be a derivation of* $(L, \cdot)$ *and* $(L, [\,,\,])$*. Define a ternary operation on* L *as follows*

$$[x, y, z] := D(x)[y, z] + D(y)[z, x] + D(z)[x, y], \quad x, y, z \in L$$

Then (L,[, ,]) is a 3-Lie algebra.

In what follows we will generalize this result to the case of transposed Poisson superalgebra and 3-Lie superalgebra.

4 Transposed Poisson superalgebra

In this chapter the definition of transposed Poisson superalgebra is given. We also present all currently known results about this structure. Transposed Poisson superalgebra was first defined by V. Abramov and O. Liivapuu in their paper [1]. This definition can be written using the notation of the present thesis as follows

Definition 8. Transposed Poisson superalgebra is a triple $(P, \cdot, [\,,\,])$, where $(P, \cdot)$ is a commutative associative superalgebra and $(P, [\,,\,])$ is a Lie superalgerba. The compatibility condition is

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{|x||z|}[x, z \cdot y]. \tag{4.1}$$

Similarly to transposed Poisson algebras we can construct a wide class of examples of transposed Poisson superalgebras by means of associative commutative superalgebra and its even derivation. It is worth noting that this construction is not consistent in the case of odd derivation because the condition of Lie superalgebra (2.4) is not satisfied.

Example 8. Let $(A, \cdot)$ be a commutative associative superalgebra and $D : A \to A$ be an even derivation of A. Define a mapping

$$[\,,\,] : A \times A \to A,\ [x, y] = xD(y) - (-1)^{|x||y|}yD(x).$$

Then $(A, \cdot, [\,,\,])$ is a transposed Poisson superalgebra.

Despite active work in the field of transposed Poisson algebras, superalgebras (and in the general case graded ones) have been studied very little. In a paper [1] V. Abramov and O. Liivapuu showed that there are many non-trivial transposed Poisson superalgebras by classifying transposed Poisson superalgebras in small dimensions. Also there is provided an example of infinite-dimensional algebra, which is closely related to geometry and promises a close connection with applications.

The most important part needed to study transposed Poisson algebras is the non-trivial identities that hold in them. Originally discovered for transposed Poisson algebras in the article [3] (Theorem 2.7), the identities were generalized for the superalgebra case by V. Abramov and O. Liivapuu in [1] (Theorem 2). For the purpose to prove the main theorem of this paper, we will recall those identities in a form of a theorem here using the notions of present thesis.

Theorem 2. *Let $(P, \cdot, [\,,\,])$ be a transposed Poisson superalgebra. Then, for any $h, x, y, z, u, v \in P$, we have the following identities:*

$$(-1)^{|x||z|}x[y,z] + (-1)^{|x||y|}y[z,x] + (-1)^{|y||z|}z[x,y] = 0 \tag{4.2}$$

$$(-1)^{|x||z|}[h\cdot[x,y],z] + (-1)^{|x||y|}[h\cdot[y,z],x] + (-1)^{|y||z|}[h\cdot[z,x],y] = 0 \tag{4.3}$$

$$(-1)^{|x||z|}[h\cdot x,[y,z]] + (-1)^{|x||y|}[h\cdot y,[z,x]] + (-1)^{|y||z|}[h\cdot z,[x,y]] = 0 \tag{4.4}$$

$$(-1)^{|x||z|}[h,x][y,z] + (-1)^{|x||y|}[h,y][z,x] + (-1)^{|y||z|}[h,z][x,y] = 0 \tag{4.5}$$

$$2u\cdot v\cdot[x,y] = (-1)^{|x||v|}[u\cdot x, v\cdot y] + (-1)^{|u,xv|}[v\cdot x, u\cdot y] = 0 \tag{4.6}$$

$$(-1)^{|u,yv|}x\cdot[u, y\cdot v] + (-1)^{|xy,v|}v\cdot[x\cdot y, u] + (-1)^{|x,yv|}y\cdot[v,x]\cdot u = 0 \tag{4.7}$$

Speaking about the results in the field of studying transposed Poisson superalgebras, it is worth mentioning the theorem proven by A. F. Ouaridi in [11], which describes simple transposed Poisson superalgebras. With the help of identities written out in the Theorem 2 the following theorem was proven.

Theorem. Suppose that $\mathbb{F}$ is algebraically closed and $char(\mathbb{F}) = 0$, then any simple finite-dimensional transposed Poisson superalgebra is trivial.

5 3-Lie superalgebra constructed by means of transposed Poisson superalgebra

In this chapter we show that given a transposed Poisson superalgebra and an even derivation of this superalgebra we can construct a 3-Lie superalgebra. We construct a ternary bracket by means of an even derivation and binary graded Lie bracket of a transposed Poisson superalgebra. This construction will be stated in the form of a theorem.

We will need the following lemma.

Lemma 1. *If* $(A,\cdot,[\,,\,])$ *is a transposed Poisson superalgebra and* D *is an even derivation of a Lie superalgebra* $(A,[\,,\,])$ *then*

$$\begin{aligned}&D(x)\cdot D([y,z])+(-1)^{|x,yz|}D(y)\cdot D([z,x])+(-1)^{|xy,z|}D(z)\cdot D([x,y])\\&\quad+x\cdot[D(y),D(z)]+(-1)^{|x,yz|}y\cdot[D(z),D(x)])+(-1)^{|xy,z|}z\cdot[D(x),D(y)]=0.\end{aligned}$$

Proof. Since $D:A\to A$ is an even derivation of a Lie superalgebra $(A,[\,,\,])$, it satisfies the Leibniz rule, for any $a,b\in A$ we have $D([a,b])=[D(a),b]+[a,D(b)]$. Since it is an even derivation we also have $|D(a)|=|a|$ for any $a\in A$. Making use of the Leibniz rule and the compatibility condition for transposed Poisson superalgebra (3.2) we get

$$\begin{aligned}D(x)\cdot D([y,z])&=D(x)\cdot[D(y),z]+D(x)\cdot[y,D(z)]\\&=\frac{1}{2}\big([D(x)\cdot D(y),z])+(-1)^{|x||y|}[D(y),D(x)\cdot z]\\&\qquad+[D(x)\cdot y,D(z)]+(-1)^{|x||y|}[y,D(x)\cdot D(z)]\big).\end{aligned}\tag{5.1}$$

Having made cyclic permutations of elements x,y,z, we get two more equations

$$\begin{aligned}D(y)\cdot D([z,x])&=\frac{1}{2}\big([D(y)\cdot D(z),x])+(-1)^{|y||z|}[D(z),D(y)\cdot x]\\&\qquad+[D(y)\cdot z,D(x)]+(-1)^{|y||z|}[z,D(y)\cdot D(x)]\big),\end{aligned}\tag{5.2}$$

and

$$D(z)\cdot D([x,y]) = \frac{1}{2}\big([D(z)\cdot D(x),y]) + (-1)^{|x||z|}[D(x),D(z)\cdot y]$$
$$+[D(z)\cdot x, D(y)] + (-1)^{|x||z|}[x, D(z)\cdot D(y)]\big). \qquad (5.3)$$

Taking the sum of the equation (5.1) with the equations (5.2), (5.3) multiplied by $(-1)^{|x,yz|}$ and $(-1)^{|xy,z|}$ respectively we get the equation whose left-hand side is

$$D(x)\cdot D([y,z]) + (-1)^{|x,yz|}D(y)\cdot D([z,x]) + (-1)^{|xy,z|}D(z)\cdot D([x,y]), \qquad (5.4)$$

and the right-hand side can be written in the form

$$-\frac{1}{2}\big([x\cdot D(y), D(z)] + (-1)^{|x||y|}[D(y), x\cdot D(z)]$$
$$+(-1)^{|x,yz|}([y\cdot D(z), D(x)] + (-1)^{|y||z|}[D(z), y\cdot D(x)]$$
$$+(-1)^{|xy,z|}([z\cdot D(x), D(x)] + (-1)^{|x||z|}[D(x), z\cdot D(y)])\big).$$

Making use of the compatibility condition for transposed Poisson superalgebra we can write the right-hand side in the form

$$-x\cdot[D(y),D(z)] - (-1)^{|x,yz|}y\cdot[D(z),D(x)]) - (-1)^{|xy,z|}z\cdot[D(x),D(y)],$$

which ends the proof of lemma. □

Theorem 3. *Let $(A,\cdot,[\,,\,])$ be a transposed Poisson superalgebra and D an even derivation of a superalgebra $(A,\cdot)$ and Lie superalgebra $(A,[\,,\,])$. Define the ternary bracket*

$$[x,y,z] := D(x)\cdot[y,z] + (-1)^{|x,yz|}D(y)\cdot[z,x] + (-1)^{|xy,z|}D(z)\cdot[x,y], \qquad (5.5)$$

where $x,y,z\in A$. Then $(A,[\,,\,,\,])$ is a 3-Lie superalgebra.

Proof. It is easy to see that ternary bracket (5.5) is trilinear. Next, we will show that the ternary bracket (5.5) has a correct structure, that is, the parity of the bracket is equal to the sum of the parities of its arguments (condition (2.10)). Taking homogeneous elements $x, y, z \in A_0 \cup A_1$ one can compute

$$|D(x) \cdot [y,z]| = |D(x)| + |[y,z]| = |x| + |y| + |z| = |D(y) \cdot [z,x]| = |D(z) \cdot [x,y]|.$$

Since all these terms have the same parity, these elements are in the same subspace of A (either in A_0 or A_1). Thus their linear combination is in the same subspace and has the same parity. That is, $|[x,y,z]| = |x| + |y| + |z|$.

Now we show that the bracket has correct symmetries, that is, how it behaves under a permutation of two arguments. For any $x, y, z \in A$ we have

$$\begin{aligned}
[y,x,z] &= D(y)[x,z] + (-1)^{|y,xz|} D(x)[z,y] + (-1)^{|yx,z|} D(z)[x,y] \\
&= -(-1)^{|y,xz|+|y||z|} D(x)[y,z] - (-1)^{|x||z|} D(y)[z,x] - (-1)^{|yx,z|+|x||y|} D(z)[y,x] \\
&= -(-1)^{|x||y|} \Big(D(x)[y,z] + (-1)^{|x,yz|} D(y)[z,x] + (-1)^{|xy,z|} D(z)[y,x] \Big) \\
&= -(-1)^{|x||y|}[x,y,z].
\end{aligned}$$

Analogously

$$[x,z,y] = -(-1)^{|y||z|}[x,y,z]$$

and combining these results

$$[z,y,x] = -(-1)^{|x||y|+|x||z|+|y||z|}[x,y,z].$$

Hence our bracket has correct symmetries.

Now we just have to prove the super Filippov-Jacobi identity (2.11)

$$\begin{aligned}
[[x,y,z],u,v] = (-1)^{|yz,uv|}[[x,u,v],y,z] + (-1)^{|x,yz|+|xz,uv|}[[y,u,v],z,x] \\
+ (-1)^{|xy,zuv|}[[z,u,v],x,y],
\end{aligned} \tag{5.6}$$

and this will complete the proof of the theorem. It should be noted that this identity is the most complex (and computationally intensive) part of the proof of the theorem.

For the beginning compute the double bracket on the left-hand side of (5.6). First, use a definition of a ternary bracket (5.5)

$$
\begin{aligned}
&[[x,y,z],u,v]=[D(x)[y,z],u,v]+(-1)^{|x,yz|}[D(y)[z,x],u,v]+(-1)^{|xy,z|}[D(z)[x,y],u,v]\\
&\;=D(D(x)[y,z])[u,v]+(-1)^{|xyz,uv|}D(u)[v,D(x)[y,z]]+(-1)^{|v,xyzu|}D(v)[D(x)[y,z],u]\\
&\qquad+(-1)^{|x,yz|}\Big(D(D(y)[z,x])[u,v]+(-1)^{|xyz,uv|}D(u)[v,D(y)[z,x]]\\
&\qquad+(-1)^{|v,xyzu|}D(v)[D(y)[z,x],u]\Big)+(-1)^{|xy,z|}\Big(D(D(z)[x,y])[u,v]\\
&\qquad+(-1)^{|xyz,uv|}D(u)[v,D(z)[x,y]]+(-1)^{|v,xyzu|}D(v)[D(z)[x,y],u]\Big)\\
&\;=D\Big(D(x)[y,z]\Big)[u,v]+(-1)^{|xyz,uv|}D(u)[v,D(x)[y,z]]+(-1)^{|v,xyzu|}D(v)[D(x)[y,z],u]\\
&\qquad+(-1)^{|x,yz|}D\Big(D(y)[z,x]\Big)[u,v]+(-1)^{|xyz,uv|+|x,yz|}D(u)[v,D(y)[z,x]]\\
&\qquad+(-1)^{|v,xyzu|+|x,yz|}D(v)[D(y)[z,x],u]+(-1)^{|xy,z|}D\Big(D(z)[x,y]\Big)[u,v]\\
&\qquad+(-1)^{|xyz,uv|+|xy,z|}D(u)[v,D(z)[x,y]]+(-1)^{|v,xyzu|+|xy,z|}D(v)[D(z)[x,y],u].
\end{aligned}
$$

Next, we apply Leibniz rule for even derivation D in the 1st, 4th and 7th terms (other terms will remain the same). We get

$$
\begin{aligned}
D\Big(D(x)[y,z]\Big)[u,v]&=D^2(x)[y,z][u,v]+\underline{D(x)D([y,z])}[u,v],\\
(-1)^{|x,yz|}D\Big(D(y)[z,x]\Big)[u,v]&=(-1)^{|x,yz|}D^2(y)[z,x][u,v]+\underline{(-1)^{|x,yz|}D(y)D([z,x])}[u,v],\\
(-1)^{|xy,z|}D\Big(D(z)[x,y]\Big)[u,v]&=(-1)^{|xy,z|}D^2(z)[x,y][u,v]+\underline{(-1)^{|xy,z|}D(z)D([x,y])}[u,v].
\end{aligned}
$$

Note that the sum of underlined terms in above equations is equal to the first row of equation in lemma 1 (multiplied by $[u,v]$). Hence using the identity of lemma 1 we can write this sum as follows

$$D(x)D([y,z])[u,v]+(-1)^{|x,yz|}D(y)D([z,x])[u,v]+(-1)^{|xy,z|}D(z)D([x,y])[u,v]=$$
$$-x[D(y),D(z)][u,v]-(-1)^{|x,yz|}y[D(z),D(x)][u,v]-(-1)^{|xy,z|}z[D(x),D(y)][u,v].$$

Using these results the double bracket at the left-hand side of Filippov-Jacobi identity can be written in the following form

$$\begin{aligned}[[x,y,z],u,v]=&\ D^2(x)[y,z][u,v]-x[D(y),D(z)][u,v]\\ &+(-1)^{|xyz,uv|}D(u)[v,D(x)[y,z]]+(-1)^{|v,xyzu|}D(v)[D(x)[y,z],u]\\ &+(-1)^{|x,yz|}D^2(y)[z,x][u,v]-(-1)^{|x,yz|}y[D(z),D(x)])[u,v]\\ &+(-1)^{|xyz,uv|+|x,yz|}D(u)[v,D(y)[z,x]]+(-1)^{|v,xyzu|+|x,yz|}D(v)[D(y)[z,x],u]\\ &+(-1)^{|xy,z|}D^2(z)[x,y][u,v]-(-1)^{|xy,z|}z[D(x),D(y)][u,v]\\ &+(-1)^{|xyz,uv|+|xy,z|}D(u)[v,D(z)[x,y]]+(-1)^{|v,xyzu|+|xy,z|}D(v)[D(z)[x,y],u].\end{aligned}$$

Let us rearrange the terms so that terms with similar structure will be grouped together and common factor will be put in front

$$\begin{aligned}&[[x,y,z],u,v]=D^2(x)[y,z][u,v]+(-1)^{|x,yz|}D^2(y)[z,x][u,v]+(-1)^{|xy,z|}D^2(z)[x,y][u,v]\\ &+(-1)^{|xyz,uv|}D(u)\big([v,D(x)[y,z]]+(-1)^{|x,yz|}[v,D(y)[z,x]]+(-1)^{|xy,z|}[v,D(z)[x,y]]\big)\\ &+(-1)^{|xyzu,v|}D(v)\big([D(x)[y,z],u]+(-1)^{|x,yz|}[D(y)[z,x],u]+(-1)^{|xy,z|}[D(z)[x,y],u]\big)\\ &-x[D(y),D(z)][u,v]-(-1)^{|x,yz|}y[D(z),D(x)][u,v]-(-1)^{|xy,z|}z[D(x),D(y)][u,v].\end{aligned} \tag{5.7}$$

In what follows we will work with the terms of this long expression and for this we will label them as follows: we assign to every term a pair (L, n), where L stands for the "left-hand side of the super Filippov-Jacobi identity" and n is the number of the term in this expression. For example, (L, 1) is a label for $D^2(x)[y,z][u,v]$.

Now consider the terms at the right-hand side of the super Filippov-Jacobi identity

(5.6). These terms can be computed with the help of the previous result (5.7). To compute $[[x,u,v],y,z]$ by means of $[[x,y,z],u,v]$ we apply the following permutation of arguments $x \to x,\ y \to u,\ z \to v,\ u \to y,\ v \to z$. The result is

$$
\begin{aligned}
[[x,u,v],y,z] = {} & D^2(x)[u,v][y,z] - x[D(u),D(v)][y,z] \\
& + (-1)^{|xuv,yz|} D(y)[z, D(x)[u,v]] + (-1)^{|z,xuvy|} D(z)[D(x)[u,v],y] \\
& + (-1)^{|x,uv|} D^2(u)[v,x][y,z] - (-1)^{|x,uv|} u[D(v),D(x)])[y,z] \\
& + (-1)^{|xuv,yz|+|x,uv|} D(y)[z, D(u)[v,x]] + (-1)^{|z,xuvy|+|x,uv|} D(z)[D(u)[v,x],y] \\
& + (-1)^{|xu,v|} D^2(v)[x,u][y,z] - (-1)^{|xu,v|} v[D(x),D(u)][y,z] \\
& + (-1)^{|xuv,yz|+|xu,v|} D(y)[z, D(v)[x,u]] + (-1)^{|z,xuvy|+|xu,v|} D(z)[D(v)[x,u],y].
\end{aligned}
\tag{5.8}
$$

Similarly to compute $[[y,u,v],z,x]$ we use a permutation $x \to y,\ y \to z,\ z \to x,$ $u \to u,\ v \to v$ and we get

$$
\begin{aligned}
[[y,u,v],z,x] = {} & D^2(y)[u,v][z,x] - y[D(u),D(v)][z,x] \\
& + (-1)^{|yuv,zx|} D(z)[x, D(y)[u,v]] + (-1)^{|x,yuvz|} D(x)[D(y)[u,v],z] \\
& + (-1)^{|y,uv|} D^2(u)[v,y][z,x] - (-1)^{|y,uv|} u[D(v),D(y)])[z,x] \\
& + (-1)^{|yuv,zx|+|y,uv|} D(z)[x, D(u)[v,y]] + (-1)^{|x,yuvz|+|y,uv|} D(x)[D(u)[v,y],z] \\
& + (-1)^{|yu,v|} D^2(v)[y,u][z,x] - (-1)^{|yu,v|} v[D(y),D(u)][z,x] \\
& + (-1)^{|yuv,zx|+|yu,v|} D(z)[x, D(v)[y,u]] + (-1)^{|x,yuvz|+|yu,v|} D(x)[D(v)[y,u],z].
\end{aligned}
\tag{5.9}
$$

To compute $[[z,u,v],x,y]$ we use the permutation $x \to y,\ y \to z,\ z \to x,\ u \to u,$

$v \to v$ and we get

$$
\begin{aligned}
[[z,u,v],x,y] = {} & D^2(z)[u,v][x,y] - z[D(u),D(v)][x,y] \\
& + (-1)^{|zuv,xy|} D(x)[y, D(z)[u,v]] + (-1)^{|y,zuvx|} D(y)[D(z)[u,v],x] \\
& + (-1)^{|z,uv|} D^2(u)[v,z][x,y] - (-1)^{|z,uv|} u[D(v),D(z)])[x,y] \\
& + (-1)^{|zuv,xy|+|z,uv|} D(x)[y,D(u)[v,z]] + (-1)^{|y,zuvx|+|z,uv|} D(y)[D(u)[v,z],x] \\
& + (-1)^{|zu,v|} D^2(v)[z,u][x,y] - (-1)^{|zu,v|} v[D(z),D(u)][x,y] \\
& + (-1)^{|zuv,xy|+|zu,v|} D(x)[y,D(v)[z,u]] + (-1)^{|y,zuvx|+|zu,v|} D(y)[D(v)[z,u],x].
\end{aligned}
\tag{5.10}
$$

In order to get the right-hand side of Filippov-Jacobi identity we have to multiply the calculated terms (5.8), (5.9), (5.10) by corresponding factors (see (5.6)). Multiplying every term by its corresponding coefficient we get the following result:

$$
\begin{aligned}
(-1)^{|uv,yz|}[[x,u,v],y,z] = {} & (-1)^{|uv,yz|} D^2(x)[u,v][y,z] - (-1)^{|uv,yz|} x[D(u),D(v)][y,z] \\
& + (-1)^{|x,yz|} D(y)[z,D(x)[u,v]] + (-1)^{|z,xy|+|uv,y|} D(z)[D(x)[u,v],y] \\
& + (-1)^{|xyz,uv|} D^2(u)[v,x][y,z] - (-1)^{|xyz,uv|} u[D(v),D(x)])[y,z] \\
& + (-1)^{|x,uvyz|} D(y)[z,D(u)[v,x]] + (-1)^{|xy,zuv|} D(z)[D(u)[v,x],y] \\
& + (-1)^{|xu,v|+|uv,yz|} D^2(v)[x,u][y,z] - (-1)^{|xu,v|+|uv,yz|} v[D(x),D(u)][y,z] \\
& + (-1)^{|x,yz|+|xu,v|} D(y)[z,D(v)[x,u]] \\
& \qquad + (-1)^{|z,xy|+|xu,v|+|uv,y|} D(z)[D(v)[x,u],y].
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
(-1)^{|x,yz|+|uv,xz|}[[y,u,v],z,x] &= (-1)^{|x,yz|+|uv,xz|}D^2(y)[u,v][z,x] \\
&- (-1)^{|x,yz|+|uv,xz|}y[D(u),D(v)][z,x] + (-1)^{|yx,z|}D(z)[x,D(y)[u,v]] \\
&+ (-1)^{|uv,z|}D(x)[D(y)[u,v],z] + (-1)^{|xzy,uv|+|x,yz|}D^2(u)[v,y][z,x] \\
&- (-1)^{|xzy,uv|+|x,yz|}u[D(v),D(y)])[z,x] + (-1)^{|xy,z|+|y,uv|}D(z)[x,D(u)[v,y]] \\
&+ (-1)^{|zy,uv|}D(x)[D(u)[v,y],z] + (-1)^{|x,yz|+|u,xzv|+|v,xyz|}D^2(v)[y,u][z,x] \\
&- (-1)^{|x,yz|+|u,xzv|+|v,xyz|}v[D(y),D(u)][z,x] + (-1)^{|xy,z|+|yu,v|}D(z)[x,D(v)[y,u]] \\
&+ (-1)^{|z,uv|+|yu,v|}D(x)[D(v)[y,u],z].
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
(-1)^{|zuv,xy|}[[z,u,v],x,y] &= (-1)^{|zuv,xy|}D^2(z)[u,v][x,y] - (-1)^{|zuv,xy|}z[D(u),D(v)][x,y] \\
&+ D(x)[y,D(z)[u,v]] + (-1)^{|x,zuvy|}D(y)[D(z)[u,v],x] \\
&+ (-1)^{|xyz,uv|+|z,xy|}D^2(u)[v,z][x,y] - (-1)^{|xyz,uv|+|z,xy|}u[D(v),D(z)])[x,y] \\
&+ (-1)^{|z,uv|}D(x)[y,D(u)[v,z]] + (-1)^{|z,xuv|+|x,yuv|}D(y)[D(u)[v,z],x] \\
&+ (-1)^{|zu,xyv|+|xy,v|}D^2(v)[z,u][x,y] - (-1)^{|zu,xyv|+|xy,v|}v[D(z),D(u)][x,y] \\
&+ (-1)^{|zu,v|}D(x)[y,D(v)[z,u]] + (-1)^{|x,yzuv|+|zu,v|}D(y)[D(v)[z,u],x].
\end{aligned}
\tag{5.13}
$$

Similarly with left-hand side case we will label the terms of these long expressions as follows: we assign to every term a pair (m,n), where m is a Roman number I, II or III (here I for (5.11), II for (5.12) and III for (5.13)), and n is the number of a term in expression itself. For example, the pair (I, 1) denotes the term $(-1)^{|uv,yz|}D^2(x)[u,v][y,z]$.

One can see that the terms (I,1) and (L, 1) are equal, because dot product is commutative. Analogously (II,1) equals to (L, 2) and (III,1) equals to (L, 3). Further these terms will not be considered as they can be cancelled.

Our next goal is to show that the rest of terms containing a square of a derivation D can be cancelled. For this purpose collect all the elements with factor $D^2(u)$ from the right-hand side. Those are (I, 5), (II, 5) and (III, 5). Their sum is

$$D^2(u)\Big((-1)^{|xyz,uv|}[v,x][y,z]+(-1)^{|xyz,uv|+|x,yz|}[v,y][z,x]+(-1)^{|xyz,uv|+|z,xy|}[v,z][x,y]\Big)$$
$$=(-1)^{|xyz,uv|}D^2(u)\Big([v,x][y,z]+(-1)^{|x,yz|}[v,y][z,x]+(-1)^{|z,xy|}[v,z][x,y]\Big).$$

This sum is equal to zero by identity (4.5).

Similarly collect all the terms with factor $D^2(v)$. These are (I,9), (II,9) and (III,9). Their sum is

$$(-1)^{|xyzu,v|}D^2(v)\Big((-1)^{|yz,u|}[x,u][y,z]+(-1)^{|x,yz|+|xz,u|}[y,u][z,x]+(-1)^{|xy,zu|}[z,u][x,y]\Big). \tag{5.14}$$

To apply the identity (4.5) here we use skew-symmetry of Lie bracket as follows

$$(-1)^{|yz,u|}[x,u]=-(-1)^{|xyz,u|}[u,x],\ (-1)^{|x,yz|+|xz,u|}[y,u]=-(-1)^{|x,yz|+|xyz,u|}[u,y]$$

and

$$(-1)^{|xy,zu|}[z,u]=-(-1)^{|xy,zu|+|u||z|}[u,z]=-(-1)^{|xyz,u|+|xy,z|}[u,z].$$

Remark that we get common factor $-(-1)^{|xyz,u|}$ in every term. So the expression (5.14) can be written in the form

$$-(-1)^{|xyzu,v|+|xyz,u|}D^2(v)\big([u,x][y,z]+(-1)^{|x,yz|}[u,y][z,x]+(-1)^{|x,yz|}[u,z][x,y]\big), \tag{5.15}$$

which equals to zero by identity (4.5).

Thus, we got rid of the terms containing the square of the derivation D, and those terms that contained $D^2(x)$, $D^2(y)$ and $D^2(z)$ were canceled due to the fact that they were in the identity on both the left and the right, and the terms containing $D^2(u)$, $D^2(v)$ in the sum gave zero due to identity (4.5).

Now collect the terms with $[D(u), D(v)]$. Those are (I, 2), (II, 2) and (III, 2). To put the common factor out of bracket we rearrange factors in the product using commutativity of dot product. Remind that derivation D is even, that is, it doesn't change a parity of an element and $|[D(u), D(v)]| = |u| + |v|$. So the term (I, 2) can be written as follows:

$$(-1)^{|uv,yz|}x[D(u), D(v)][y, z] = (-1)^{|uv,xyz|}[D(u), D(v)]x[y, z]$$

Elements (II,2) and (III, 2) can be written similarly and the common factor is $(-1)^{|uv,xyz|}[D(u), D(v)]$. So the sum of elements (I,2), (II,2) and (III, 2) is

$$-(-1)^{|uv,xyz|}[D(u), D(v)]\big(x[y, z] + (-1)^{|x,yz|}y[z, x] + (-1)^{|z,xy|}z[x, y]\big),$$

which is equal to zero by identity (4.2).

Further calculations are based on identity (4.3). This identity has three terms. We will find in expressions (5.11), (5.12), (5.13) two terms from identity (4.3) and replace them with the third.

Take a sum of elements (III,7) and (II,8).

$$\begin{aligned}(-1)^{|z,uv|}D(x)[y, D(u)[v, z]] + (-1)^{|zy,uv|}D(x)[D(u)[v, y], z] =\\ = (-1)^{|z,uv|}D(x)\big([y, D(u)[v, z]] + (-1)^{|y,uv|}[D(u)[v, y], z]\big)\end{aligned}$$

Using graded skew-symmetry of Lie bracket two times for the first term we get

$$[y, D(u)[v, z]] = (-1)^{|y,zuv|+|v||z|}[D(u)[z, v], y]$$

Thus

$$\begin{aligned}(\mathrm{III},7)+(\mathrm{II},8) &= (-1)^{|z,uv|}D(x)\big((-1)^{|y,zuv|+|v||z|}[D(u)[z,v],y]+(-1)^{|y,uv|}[D(u)[v,y],z]\big)\\ &= (-1)^{|yz,uv|+|z||v|}D(x)\big((-1)^{|y||z|}[D(u)[z,v],y]+(-1)^{|z||v|}[D(u)[v,y],z]\big).\end{aligned}$$

In this particular case the identity (4.3) takes on the form

$$(-1)^{|y||z|}[D(u)[z,v],y]+(-1)^{|z||v|}[D(u)[v,y],z]+(-1)^{|y||v|}[D(u)[y,z],v]=0.$$

Thus

$$(\mathrm{III},7)+(\mathrm{II},8) = -(-1)^{|yz,uv|+|z||v|+|y||v|}D(x)[D(u)[y,z],v] = -(-1)^{|yz,u|}D(x)[D(u)[y,z],v].$$

Analogously

$$\begin{aligned}(\mathrm{I},7)+(\mathrm{III},8) &= (-1)^{|x,uvyz|}D(y)[z,D(u)[v,x]]+(-1)^{|z,xuv|+|x,yuv|}D(y)[D(u)[v,z],x]\\ &= -(-1)^{|x,yzu|+|z||u|}D(y)[D(u)[z,x],v].\end{aligned}$$

$$\begin{aligned}(\mathrm{II},7)+(\mathrm{I},8) &= (-1)^{|xy,z|+|y,uv|}D(z)[x,D(u)[v,y]]+(-1)^{|xy,zuv|}D(z)[D(u)[v,x],y]\\ &= -(-1)^{|xy,zu|}D(z)[D(u)[x,y],v].\end{aligned}$$

The next group of terms we will analogously transform by means of identity is (I, 11), (I,12), (II,11), (II,12), (III,11) and (III,12).

$$\begin{aligned}(\mathrm{II},12)+(\mathrm{III},11) &= (-1)^{|z,uv|+|yu,v|}D(x)[D(v)[y,u],z]+(-1)^{|zu,v|}D(x)[y,D(v)[z,u]]\\ &= -(-1)^{|yzu,v|+|y||z|}D(x)[D(v)[z,y],u].\end{aligned}$$

$$\begin{aligned}(\mathrm{III},12)+(\mathrm{I},11) &= (-1)^{|x,yzuv|+|zu,v|}D(y)[D(v)[z,u],x]+(-1)^{|x,yz|+|xu,v|}D(y)[z,D(v)[x,u]]\\ &= -(-1)^{|xzu,v|+|x||y|}D(y)[D(v)[x,z],u].\end{aligned}$$

$$(\mathrm{I},12) + (\mathrm{II},11) = (-1)^{|xy,z|+|xu,v|+|y,uv|} D(z)[D(v)[x,u],y] + (-1)^{|xy,z|+|yu,v|} D(z)[x, D(v)[y,u]]$$
$$= -(-1)^{|x,yzv|+|y,zv|+|u||v|} D(z)[D(v)[y,x],u].$$

Before we proceed further let us write the terms (I,10), (II,10), (III,10) as follows (we show calculations only in the case of the first term (I,10))

$$(\mathrm{I},10) = -(-1)^{|xu,v|+|yz,uv|} v[D(x), D(u)][y,z] = (-1)^{|xyz,uv|+|u||v|} v[D(u), D(x)][y,z].$$

Now we can summarize our calculations. After all cancellations at the right-hand side of the Filippov-Jacobi identity we have the following terms

$$\begin{aligned}
&\underline{-(-1)^{|yz,u|} D(x)[D(u)[y,z],v]} - \underline{(-1)^{|x,yzu|+|z||u|} D(y)[D(u)[z,x],v]} \\
&\quad \underline{-(-1)^{|xy,zu|} D(z)[D(u)[x,y],v]} \\
&\underline{-(-1)^{|yzu,v|+|y||z|} D(x)[D(v)[z,y],u]} - \underline{(-1)^{|xzu,v|+|x||y|} D(y)[D(v)[x,z],u]} \\
&\quad \underline{-(-1)^{|x,yzv|+|y,zv|+|u||v|} D(z)[D(v)[y,x],u]} \\
&+D(x)[y, D(z)[u,v]] + (-1)^{|x,yz|} D(y)[z, D(x)[u,v]] \\
&\quad +(-1)^{|xy,z|} D(z)[x, D(y)[u,v]] \\
&+(-1)^{|z,uv|} D(x)[D(y)[u,v],z] + (-1)^{|x,yzuv|} D(y)[D(z)[u,v],x] \\
&\quad +(-1)^{|xy,z|+|y,uv|} D(z)[D(x)[u,v],y] \\
&\underline{-(-1)^{|xyz,uv|} u\big([D(v), D(x)][y,z]} + \underline{(-1)^{|x,yz|}[D(v), D(y)][z,x]} \\
&\quad + \underline{(-1)^{|xy,z|}[D(v), D(z)][x,y]}\big) \\
&+\underline{(-1)^{|xyz,uv|+|u||v|} v\big([D(u), D(x)][y,z]} + \underline{(-1)^{|x,yz|}[D(u), D(y)][z,x]} \\
&\quad + \underline{(-1)^{|xy,z|}[D(u), D(z)][x,y]}\big).
\end{aligned} \tag{5.16}$$

The left-hand side has the form

$$\underline{(-1)^{|xyz,uv|}D(u)\big([v,D(x)[y,z]]} + \underline{(-1)^{|x,yz|}[v,D(y)[z,x]]} + \underline{(-1)^{|xy,z|}[v,D(z)[x,y]]}\big)$$

$$+ \underline{(-1)^{|xyzu,v|}D(v)\big([D(x)[y,z],u]} + \underline{(-1)^{|x,yz|}[D(y)[z,x],u]} + \underline{(-1)^{|xy,z|}[D(z)[x,y],u]}\big)$$

$$- x[D(y),D(z)][u,v] - (-1)^{|x,yz|}y[D(z),D(x)][u,v] - (-1)^{|xy,z|}z[D(x),D(y)][u,v]. \tag{5.17}$$

Our further calculations will be based on identity (4.7)

$$(-1)^{|u,yv|}x\cdot[u,y\cdot v] + (-1)^{|v,xy|}v\cdot[x\cdot y,u] + (-1)^{|x,yv|}y\cdot[v,x]\cdot u = 0.$$

Substituting in this identity $x\to D(u)$, $u\to v$, $y\to[y,z]$, $v\to D(x)$, we obtain

$$(-1)^{|xyz,v|}D(u)[v,[y,z]D(x)] + (-1)^{|x,yzu|}D(x)[D(u)[y,z],v] \\ + (-1)^{|xyz,u|}[y,z][D(x),D(u)]v = 0.$$

From this it follows

$$(-1)^{|xyz,uv|}D(u)[v,D(x)[y,z]] = -(-1)^{|yz,u|}D(x)[D(u)[y,z],v] \\ + (-1)^{|xyz,uv|+|u||v|}v\,[D(u),D(x)][y,z] \tag{5.18}$$

In this equation the left-hand side is equal to the first term at the left-hand side of the Filippov-Jacobi identity (5.17) and the right-hand side appears at the right-hand side of the same identity (5.16) (underlined terms by solid black line). Hence these terms can be cancelled.

By analogy one can see that all underlined terms of (5.17) are equal to the sum of respectively underlined terms of (5.16).

Let us write remaining terms again. In left-hand side we still have a sum

$$-\underline{x[D(y),D(z)][u,v]} - (-1)^{|x,yz|}y[D(z),D(x)][u,v] - (-1)^{|xy,z|}z[D(x),D(y)][u,v] \tag{5.19}$$

and in right-hand side we have the terms

$$\begin{aligned} D(x)[y,D(z)[u,v]] + (-1)^{|x,yz|}D(y)[z,D(x)[u,v]] + \underline{(-1)^{|xy,z|}D(z)[x,D(y)[u,v]]} \\ +(-1)^{|z,uv|}D(x)[D(y)[u,v],z] + \underline{(-1)^{|x,yzuv|}D(y)[D(z)[u,v],x]} \\ +(-1)^{|xy,z|+|y,uv|}D(z)[D(x)[u,v],y] \end{aligned} \tag{5.20}$$

Consider remaining terms at the left-hand side (5.19). We will take them to the form in which factor $[u,v]$ comes at the beginning and one-element factor (x, y or z respectively) comes at the end. We will show transformations for the first term (underlined with solid line), others can be transformed similarly:

$$\begin{aligned} x[D(y),D(z)][u,v] &= (-1)^{|x,yzuv|}[D(y),D(z)][u,v]x \\ &= (-1)^{|x,yzuv|+|yz,uv|}[u,v][D(y),D(z)]x. \end{aligned}$$

By identity (4.7) one can see that

$$\begin{aligned} (-1)^{|x,yuv|}D(z)[x,[u,v]D(y)] + (-1)^{|y,zuv|}D(y)[D(z)[u,v],x] \\ + (-1)^{|z,yuv|}[u,v][D(y),D(z)]x = 0. \end{aligned}$$

From this we get

$$\begin{aligned} &-(-1)^{|x,yzuv|+|yz,uv|}[u,v][D(y),D(z)]x \\ &\quad=(-1)^{|y,uv|+|xy,z|}D(z)[x,[u,v]D(y)] + (-1)^{|x,yzuv|}D(y)[D(z)[u,v],x]. \end{aligned}$$

Now we can see that the second term $(-1)^{|x,yzuv|}D(y)[D(z)[u,v],x]$ is equal to the 5th in (5.20). With following transformations of another term

$$(-1)^{|y,uv|+|xy,z|}D(z)[x,[u,v]D(y)] = (-1)^{|xy,z|}D(z)[x,D(y)[u,v]]$$

one can see that it is equal to the 3rd term of (5.20). The other terms cancel by analogy with these (terms that cancel each other are underlined with the same line). This completes the proof of the Filippov-Jacobi identity and the entire theorem. □

Conclusion

Assuming only a basic familiarity with basic algebraic structures (field, vector space, algebra), we were able to introduce all the necessary concepts to define a completely new structure of transposed Poisson superalgebra. We have succesfully generalized the construction of 3-Lie algebra by means of transposed Poisson algebra proposed in [3] to the case of transposed Poisson superalgebra and 3-Lie superalgebra. We have extended the ternary bracket with the help of an even derivation to the case of transposed Poisson superalgebra. It is worth noting that a similar structure is not consistent in the case of odd derivation. The question of constructing examples and structures similar to those introduced in this thesis for the case of odd derivations remains open. There are also many results in the article [3] that are not generalized for the case of superalgebras. also require generalization that may not be feasible in superalgebras. The next goal in this area of research would be to try to reconcile these results with the structure of the transposed Poisson superalgebra and related superstructures.

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