

LAXMIPRIYA PATI

The effects of non-Riemannian
connection in teleparallel gravity



LAXMIPRIYA PATI

The effects of non-Riemannian
connection in teleparallel gravity



UNIVERSITY OF TARTU
Press

This study was carried out at the University of Tartu.

The dissertation was admitted on 12.02.2025 in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics, and was allowed for defense by the Council of the Institute of Physics, University of Tartu.

Supervisors: Dr. Laur Järv

Head of the Laboratory, Associate Professor in Theoretical Physics, Institute of Physics, University of Tartu, Estonia

Dr. María José Guzmán Monsalve

Associate Professor in Theoretical Physics, Institute of Physics, University of Tartu, Estonia

Opponent: Prof. Syksy Räsänen

Senior Researcher in Theoretical Physics at the University of Helsinki, Department of Physics, Division of Particle Physics and Astrophysics, and Adjunct Researcher at Helsinki Institute of Physics, Finland

Defence: 26.03.2025, University of Tartu, Estonia

The work of this dissertation was carried out at the Laboratory of Theoretical Physics Institute of Physics, University of Tartu. The author was supported by the Dora Plus program at the University of Tartu and the INSPIRE Fellowship (File No. DST/INSPIRE Fellowship/2019/IF190600) awarded by the Department of Science and Technology (DST), Government of India, New Delhi. Further support was provided by the European Regional Development Fund's "The Dark Side of the Universe" (TK133), and Estonian Research Council grants "Gauge Gravity" (PRG356), "Foundations of the Universe" (TK202U4), and "Theoretical frameworks for numerical modified gravity" (PSG910).

ISSN 1406-0647 (print)

ISBN 978-9916-27-825-3 (print)

ISSN 2806-2523 (pdf)

ISBN 978-9916-27-826-0 (pdf)

Copyright: Laxmipriya Pati, 2025

University of Tartu Press

www.tyk.ee

Contents

List of publications	8
1 Introduction	10
1.1 Motivations	10
1.2 Why Modified Theories of Gravity?	12
1.3 Aim of the Thesis	12
1.4 Statements	14
1.5 Structure of the Thesis	14
2 Geometrical Foundations	16
2.1 General Affine Connection	16
2.2 Tetrads and Noncoordinate Frames	19
2.3 Teleparallel Geometry	22
2.4 Lie Derivative and Symmetries	23
2.5 Geodesic and Autoparallel Curves	26
3 Action and Field Equations	28
3.1 Einstein–Hilbert Action and its Extension	28
3.2 Metric Teleparallel Gravity and its Extension	29
3.3 Symmetric Teleparallel Gravity and its Extension	32
3.4 GTEGR Gravity and its Extension	34
3.5 Equivalent Formulations of Modified Gravity Theories	35
3.6 Consideration of Cosmological Field Equations and Solutions	41
4 Hamiltonian Formalism in Teleparallel Theories of Gravity	43
4.1 Overview of Dirac–Bergmann Algorithm	44
4.2 3+1 Decomposition	46
4.3 The Impact of Gauge Selection on Teleparallel Equivalent to General Relativity	47
4.4 Hamiltonian Analysis for Teleparallel Equivalent to General Relativity	48
5 Cosmological Stability of the Alternative Connection in Symmetric Teleparallel Theories	52
5.1 Symmetric Teleparallel Cosmology	53
5.2 General Relativity Limit	57

5.3	Methodology	58
5.4	Stability of Scalar-Nonmetricity Cosmology	60
5.5	Stability of $f(Q)$ Cosmology	62
6	Future Directions	66
	Summary	67
	Acknowledgments	68
	Bibliography	70
	Kokkuvõte (in Estonian)	79
	Attached publications	81
7	Hamilton's Equations in the Covariant Teleparallel Equivalent of General Relativity	83
	Reprint of "Hamilton's Equations in the Covariant Teleparallel Equivalent of General Relativity"	87
	ABSTRACT	87
	I. INTRODUCTION	87
	II. TELEPARALLEL GRAVITY AND TETRAD 3+1 DECOMPOSITION	89
	III. HAMILTONIAN FOR TTEGR	92
	IV. HAMILTON'S EQUATIONS FOR TTEGR	94
	V. HAMILTON'S EQUATION FOR THE COVARIANT TTEGR	95
	VI. DISCUSSION AND COMPARISON WITH PREVIOUS WORKS	97
	VII. CONCLUSIONS	98
	ACKNOWLEDGMENTS	99
	APPENDIX A: USEFUL IDENTITIES	99
	APPENDIX B: HAMILTON'S EQUATIONS IN GENERAL RELATIVITY	99
	APPENDIX C: HAMILTON'S EQUATIONS IN COVARIANT TTEGR (PREVIOUS PARTIAL RESULTS)	101
	REFERENCES	102
8	Stability of Symmetric Teleparallel Scalar-Tensor Cosmologies with Alternative Connections	105
	Reprint of "Stability of Symmetric Teleparallel Scalar-Tensor Cosmologies with Alternative Connections"	109
	ABSTRACT	109
	I. INTRODUCTION	109
	II. SYMMETRIC TELEPARALLEL GRAVITIES	110

III. SPATIALLY HOMOGENEOUS AND ISOTROPIC FIELD CONFIGURATIONS	112
IV. SCALAR-TENSOR COSMOLOGY	114
V. LIMIT OF GENERAL RELATIVITY	116
VI. STABILITY OF STANDARD COSMOLOGICAL REGIMES	118
VII. DISCUSSION	128
ACKNOWLEDGMENTS	129
REFERENCES	129
 9 Exploring the Stability of $f(Q)$ Cosmology Near General Relativity	
Limit with Different Connections	133
Reprint of “Exploring the Stability of $f(Q)$ Cosmology Near General Relativity Limit with Different Connections”	137
ABSTRACT	137
I. INTRODUCTION	137
II. SYMMETRIC TELEPARALLEL GEOMETRY AND GRAVITY	139
III. COSMOLOGY OF SPATIALLY HOMOGENEOUS AND ISOTROPIC FIELD CONFIGURATIONS	141
IV. GENERAL RELATIVITY LIMIT	146
V. CONNECTION SET 1 IN THE GENERAL RELATIVITY LIMIT	147
VI. CONNECTION SET 2 IN THE GENERAL RELATIVITY LIMIT	149
VII. NUMERICAL EXAMPLE	156
VIII. CONCLUSIONS	157
ACKNOWLEDGMENTS	158
REFERENCES	158
 Curriculum Vitae	161
In English	161
Eesti keeles (In Estonian)	164

List of publications

This thesis is based on the following three publications:

- I Laxmipriya Pati, D. Blixt and María-José Guzmán “Hamilton’s equations in the covariant teleparallel equivalent of general relativity” *Phys. Rev. D.* **99**, 044071 (2023)

See attached publication in Chapter 7, and arXiv: 2210.07971 [INSPIRE] [ETIS]

- II Laur Järv, Laxmipriya Pati “Stability of symmetric teleparallel scalar-tensor cosmologies with alternative connections” *Phys. Rev. D.* **109**, 064069 (2024)

See attached publication in Chapter 8, and arXiv: 2309.04262 [INSPIRE] [ETIS]

- III María-José Guzmán, Laur Järv and Laxmipriya Pati “Exploring the stability of $f(Q)$ cosmology near general relativity limit with different connections” *Phys. Rev. D.* **110**, 124013 (2024)

See attached publication in Chapter 9, and arXiv: 2406.11621 [INSPIRE] [ETIS]

Author’s contribution¹

I, Laxmipriya Pati, independently computed and verified all equations in the publications I, II, and III. I actively participated in all scientific discussions from posing the research problems and developing the methodology, to analyzing the results and drawing the conclusions. I contributed substantially to the writing and editing of the publications I and II, while III was mostly written by me. I contributed significantly to the response of the peer review in I, II and III.

I presented the results of publication I at the 32nd meeting of the Indian Association for General Relativity and Gravitation (IAGRG32), Kolkata, India held on 19-21 Dec 2022, Central European Relativity Seminar (CERS13), Stockholm

¹In theoretical physics it is often common to list the authors in the alphabetical order. This was followed in the publications II and III where in fact I was the corresponding author. In publication I, the order of author names followed the Indian institutional requirements.

on 3-5 May 2023, as well as Tartu University Theoretical Physics Journal club discussion. I discussed the results of article II in CA21136 Cosmoverse@Lisbon on 30 May to 1 June 2023, a seminar during the STSM period at Lisbon on 18 Oct 2023, Geometric Foundations of Gravity at University of Tartu 19-22 June 2023, as well as Tartu University Theoretical Physics Journal club. I delivered talks on the findings of publication III in the Metric-Affine Frameworks for Gravity 2024 conference at University of Tartu, Estonia, 17-21 June 2024 and Tartu University Theoretical Physics Journal club discussion. The whole thesis was summarized in a presentation in the Theoretical physics seminar at University of Tartu on 29 Oct 2024.

Chapter 1

Introduction

1.1 Motivations

This thesis begins with a brief journey from Newtonian gravity to Einstein's general relativity (GR) and beyond while we touch upon some of the key highlights and general properties of those theories. Subsequently, we will explore the limitations of GR and examine various approaches to its modification, which are known as modified theories of gravity, one of the main topics in this research.

In the Newtonian framework, gravity is understood as a force of attraction between massive objects. The theory assumes space and time are absolute and treats mass and energy as independent quantities. The gravitational potential Φ in this scenario obeys Poisson's differential equation

$$\nabla^2\Phi = 4\pi G\rho, \tag{1.1}$$

which relates the potential to the matter density ρ , with G being Newton's gravitational constant. Newtonian gravity, however, exhibits limitations in strong gravitational fields, where relativistic effects become significant. To address these limitations and provide a more comprehensive understanding of gravity, Einstein introduced the general theory of relativity in 1915. This revolutionary theory reconceptualized gravity as a curvature of spacetime rather than a force caused by mass and energy. His identification of inertial mass and gravitational mass was a cornerstone of the Equivalence principle, which played a pivotal role in the development of gravitational theory, and the universality of free fall is a consequence of this principle. Newtonian gravity emerges as an approximation of GR under specific conditions (a static field, in the limit of weak gravitational fields, and at low velocities). Nature has four fundamental forces: strong (exists between quarks and short range) and weak (exists in leptons and very short range) operates at the subatomic level, electromagnetic (acting between charged particles and long range), and gravitational (weakest force, exist everywhere and long range).

Unlike the other three forces, gravity is distinguished by the equivalence of inertial and gravitational mass. It remains incompatible with quantum mechanics, and a full renormalizable theory of quantum gravity has not yet been found. For a comprehensive understanding, readers are encouraged to consult [1, 2].

While the dynamics of objects in flat spacetime are relatively straightforward, understanding their behavior in curved spacetime presents a significant challenge. To conceptually know the dynamics in curved spacetime, we need some knowledge of tensor calculus, differential geometry, and Riemannian geometry, partially covered (only the necessary geometrical structure in the thesis) in Chapter 2.

In curved spacetime, a metric tensor plays a fundamental role in defining the geometry of spacetime. Two assumptions are made in GR in order to obtain the unique Levi-Civita connection: the first is the metric compatibility condition, and the second is the vanishing torsion that makes the connection symmetric. Einstein explained a crucial mathematical relationship between how spacetime geometry responds to the presence of matter configuration, where gravity is measured by spacetime curvature, through Einstein's field equations

$$\mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = \frac{8\pi G}{c^4}\mathcal{T}_{\mu\nu}. \quad (1.2)$$

Here, $\mathring{G}_{\mu\nu}$ denotes the Einstein tensor, $\mathring{R}_{\mu\nu}$ the Ricci tensor constructed from Riemann tensor by contracting first and third indices, $\mathring{R}$ the Ricci scalar built from contracting a metric tensor, and $\mathcal{T}_{\mu\nu}$ source of the matter-energy momentum tensor. As the Riemann tensor involves second-order derivatives of the metric tensor, Einstein's equations (1.2) are second-order differential equations for the metric tensor.

By modifying the underlying assumptions of GR, interestingly, two alternative equivalent formulations of gravity emerge: teleparallel equivalent of general relativity (TEGR) based on torsion and symmetric teleparallel equivalent of general relativity (STEGR) based on nonmetricity, discussed in Secs. 3.2 and 3.3. It is noteworthy that metric teleparallel alternative theories were considered even alongside the development of GR, often motivated with the aim of formulating gravity as a gauge theory [3–5]. However, symmetric teleparallel theory emerged later and has a less extensive history compared to metric teleparallel theories [6, 7]. In the modern perspective, these three theories, commonly known as the ‘Geometric Trinity of Gravity’ [8], offer distinct perspectives on the nature of gravity. Recently the authors of [9] have argued that, unlike in GR, the equivalence principle is not fundamental in TEGR and STEGR, although it can be recovered through a specific gauge choice. A recent work [10] has proposed a more general theory known as a general teleparallel equivalent to GR (GTEGR), which incorporates both torsion and nonmetricity to describe the gravitational field. This theory is further explored in [11–13]. We consider these alternative formulations as starting points for modifying GR.

1.2 Why Modified Theories of Gravity?

Despite the remarkable successes of GR in explaining phenomena such as the perihelion precession of Mercury, Solar System tests, redshift distortions, gravitational lensing, prediction and detection of gravitational waves, and the existence of black holes, it faces significant challenges in addressing the quantization of gravity [14], the nature of dark matter and dark energy, inconsistencies observed in the Hubble constant (H_0 tension) [15–17] and growth index perturbation (σ_8 tension) [18], including the recent indications on variable dark energy from the Dark Energy Spectroscopic Instrument (DESI) survey [19]. To address these shortcomings, researchers have explored modified theories of gravity, as reviewed in [20–25], while preserving the core structure of GR. Some possible ways of modification include:

- modifying the underlying geometry of GR,
- incorporating extra fields into the gravitational action,
- introducing nonlinear functions to the gravitational action,
- exploring higher-dimensional gravitational theories,
- modifying the matter content of the universe,
- breaking local Lorentz invariance or diffeomorphism invariance.

This thesis aims to address the first three modifications to investigate the effects of non-Riemannian connections in teleparallel geometry. Here, it would be useful for the reader to know that the teleparallel connections, unlike the Levi–Civita one, are not fully determined once we impose vanishing curvature. Therefore, there are free functions that define the curvatureless connection, and those functions represent a gauge freedom of the theory. The thesis explores the physical meaning of gauge fixing (these free functions) and their relation to fixing dynamic variables such as the tetrad and the metric. Leaving the gauge freedom untouched is the so-called covariant approach. We follow the covariant approach to gain deeper insights into the role of teleparallel connections and if they carry extra dynamics beyond those of the fundamental dynamical variables. Furthermore, the analysis is extended to the cosmological context to explore whether the cosmological stability is influenced by the impact of the free function of the teleparallel connection.

1.3 Aim of the Thesis

Hamiltonian formalism and Dirac’s algorithm are very important tools for analyzing any physical theories and enabling the identification of physical degrees

of freedom (PDOF) through constraint classification. We applied the Hamiltonian technique to the covariant formulation of TEGR (which consists of six extra fields due to the inclusion of spin connection through the Lorentz matrix) in order to better understand the dynamical behavior of the theory, the nature of its degrees of freedom (DOF) and compared the results with the standard GR case. Our research addressed the following questions: Do the six additional fields in the spin connection carry dynamical DOF? If not, how could it be demonstrated they do not contain any new physical information? Why is gauge fixing important in teleparallel theory? Then, what are the advantages of using a gauge independent and covariant approach to construct the Hamiltonian formalism in TEGR? Are there any significant differences in covariant TEGR Hamiltonian formalism with the standard Hamiltonian formulation in GR? Does the inclusion of a boundary term in the Hamiltonian impact the PDOF and the resulting Hamilton's equations of motion? How does our methodology offer advantages over previous existing literature? A detailed discussion of these questions is found in Chapter 7 of the attached publication.

We investigated the stability of the scalar-nonmetricity cosmology within the three nontrivial branches: set 1, set 2, and set 3 corresponding to zero spatial curvature (each contained an additional free function), in the vicinity of GR limit, considering all the eras starting from the radiation dominated epoch to matter domination, to the era of dark energy. Our research aimed to address how the additional function in the connection influences typical cosmological dynamics across different evolutionary eras. Can the connection be itself a source of dark energy, or enable or disable the scalar field to behave as one? Can it allow phantom crossing for even a minimally coupled scalar field? Does the cosmological evolution spontaneously stabilize at some field values, i.e., is there an “attractor mechanism” like in the original scalar-tensor theory, or does the connection make the universe unstable? Among all the three connection branches, can we have some evidence to say which one is more physical? All these questions are addressed in the attached paper in Chapter 8.

While $f(Q)$ gravity is a subclass of scalar-nonmetricity theory, it is not immediately clear how the dynamics of the extra scalar field manifest in $f(Q)$ theory within the nontrivial connection. Furthermore, problematic features such as ghost DOF and singularities, noted in [26, 27] motivate further study in $f(Q)$ cosmology. We investigated the stability of $f(Q)$ solutions within the nontrivial connection in the close GR regime considering the radiation, dust matter, and cosmological constant dominated eras. We aimed to address the following questions in our work in the attached article in Chapter 9: Are there any general conditions that define a viable $f(Q)$ model, and are these conditions independent of the choice of connection? Even if an analytic solution exists near GR, can we say those solutions are stable? Apart from the square root model (derived from the equations of motion of set 1), why do other $f(Q)$ models also exhibit observational compatibility with

Λ CDM? Can it always be the case that a single $f(Q)$ function is able to explain all the epochs of the universe? Does the answer have any impact on the physical viability of the nontrivial connection?

1.4 Statements

This thesis presents the following significant findings:

1. The counting of the number of PDOF is independent of gauge fixing in TEGR, which also applies to other alternative and generalized modified theories.
2. In general, Hamilton's equations are affected by the inclusion of boundary terms.
3. The Hamiltonian formulation of TEGR offers a significant advantage by eliminating undetermined antisymmetric evolution equations.
4. While alternative Friedmann–Lemaître–Robertson–Walker (FLRW) connections in symmetric teleparallelism cannot be deemed outright pathological and do not necessarily make the universe unstable, they do have a very strange and possibly dangerous influence on scalar field dynamics.
5. The stability of a $f(Q)$ model cannot be assessed independently of the connection; the two aspects must be considered together.
6. The nontrivial realization of the GR regime in set 2 of $f(Q)$ probably has limited applications because different eras imply different behaviors of $f(Q)$.
7. The existence of an unstable solution signaling $f(Q)$ theory is already problematic at the background level, which further supports the existing literature results on strongly coupled fields and ghost degree of freedom.
8. The phenomenon of sudden singularities in $f(Q)$ needs further investigation, and it could turn out that the alternative connections are unable to provide reliable backgrounds for cosmology.

1.5 Structure of the Thesis

The structure of the thesis is organized as follows. Chapter 2 provides useful geometric foundations for all the papers about the remaining part of the thesis. Chapter 3 introduces the action and field equations in alternative theories and their extensions, emphasizing more on teleparallel gravity, which plays a major role in this thesis. Chapter 4 outlines the basic concepts of Hamiltonian formalism needed to understand the discussions in the attached publication in Chapter 7. We

discuss the idea of understanding the stability of a theory in cosmology using the perturbative technique around the GR limit in Chapter 5 to follow the attached publications 8 and 9. This thesis concludes with a discussion of future research directions in Chapter 6, followed by a comprehensive summary, acknowledgments, a bibliography, a summary in Estonian, and finally, my Curriculum Vitae presented in both English and Estonian.

We employ the following notation: Lorentz spacetime indices are denoted by uppercase Latin letters $A, B, \dots = 0, 1, 2, 3$, while Greek letters $\mu, \nu, \dots = 0, 1, 2, 3$ represent spacetime indices. Lowercase Latin letters $i, j, \dots = 1, 2, 3$ indicate spatial indices. Both the spacetime metric $g_{\mu\nu}$ and the Minkowski metric η_{AB} possess a positive signature $(-1, +1, +1, +1)$. To distinguish among different types of connections, we employ the following notation: $\{\tilde{\Gamma}, \tilde{\omega}, \tilde{\nabla}\}$ for the general connection, $\{\Gamma, \omega, \nabla\}$ for teleparallel connection, and $\{\overset{\circ}{\Gamma}, \overset{\circ}{\omega}, \overset{\circ}{\nabla}\}$ for the Levi-Civita connection.

Chapter 2

Geometrical Foundations

This chapter introduces the geometrical interpretation of curvature, torsion, and nonmetricity, along with presenting the structure of parallel transport and independent connections in teleparallel gravity. The notions of Lie derivatives, Killing equations, and symmetries are also given. The chapter concludes with a discussion on the trajectories of point particles in curved spacetime, emphasizing the distinction between geodesics and autoparallels in teleparallel geometries.

2.1 General Affine Connection

A manifold $\mathcal{M}$ is equipped with a line element to measure distances between points

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.1)$$

where under general coordinate transformations, the metric transforms as a tensor

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \quad (2.2)$$

The affine connection $\Gamma^\alpha_{\mu\nu}$ defines parallel transport and covariant derivatives along curves, given by

$$\begin{aligned} \nabla_\alpha T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} = & \partial_\alpha T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} + \Gamma^{\mu_1}_{\lambda\alpha} T^{\lambda \dots \mu_p}_{\nu_1 \dots \nu_q} + \dots + \Gamma^{\mu_p}_{\lambda\alpha} T^{\mu_1 \dots \mu_{p-1} \lambda}_{\nu_1 \dots \nu_q} \\ & - \Gamma^\lambda_{\nu_1\alpha} T^{\mu_1 \dots \mu_p}_{\lambda \dots \nu_q} - \dots - \Gamma^\lambda_{\nu_q\alpha} T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \lambda}, \end{aligned} \quad (2.3)$$

which generalizes the concept of partial derivatives to curved spaces. Although the connection itself does not transform as a tensor,

$$\Gamma'^\alpha_{\mu\nu}(x') = \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma^\beta_{\rho\sigma}(x) + \frac{\partial x'^\alpha}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu}, \quad (2.4)$$

the covariant derivative of any tensorial object, as shown in Eq. (2.3) does indeed transform as a tensor.

The most general affine connection can be split into

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \overset{\circ}{\Gamma}^\lambda_{\mu\nu} + K^\lambda_{\mu\nu} + L^\lambda_{\mu\nu}, \quad (2.5)$$

where $\overset{\circ}{\Gamma}^\lambda_{\mu\nu}$ denotes the Levi–Civita connection of the metric

$$\overset{\circ}{\Gamma}^\lambda_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) = \overset{\circ}{\Gamma}^\lambda_{\nu\mu}, \quad (2.6)$$

$K^\lambda_{\mu\nu}$ denotes the contortion tensor expressed in terms of torsion tensor as

$$K^\lambda_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\beta}(-T_{\mu\beta\nu} - T_{\nu\beta\mu} + T_{\beta\mu\nu}) = -K^\lambda_{\nu\mu}, \quad (2.7)$$

and the disformation tensor $L^\lambda_{\mu\nu}$ built from the nonmetricity tensor as

$$L^\lambda_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\beta}(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}) = L^\lambda_{\nu\mu}. \quad (2.8)$$

By construction, both the Levi–Civita connection and the disformation tensor are symmetric w.r.t. last two indices. The contortion tensor is constructed from the torsion tensor, which captures the antisymmetric part of the connection as

$$T^\lambda_{\mu\nu} \equiv \tilde{\Gamma}^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\nu\mu}, \quad (2.9)$$

which is indeed a tensorial quantity¹ though the connection itself is not a tensor. Geometrically, torsion quantifies the failure of a parallelogram to close when its sides are parallel and transported around a closed curve. While the nonmetricity tensor is constructed as

$$Q_{\rho\mu\nu} \equiv \tilde{\nabla}_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \tilde{\Gamma}^\beta_{\mu\rho} g_{\beta\nu} - \tilde{\Gamma}^\beta_{\nu\rho} g_{\mu\beta}, \quad (2.10)$$

which measures how the connection deviates from being metric-compatible. Geometrically, the curvature tensor measures the deviation of the direction of a vector when we parallel transported a vector along a closed loop. Mathematically, this relationship can be represented as

$$R^\sigma_{\rho\mu\nu}(\tilde{\Gamma}) \equiv \partial_\mu \tilde{\Gamma}^\sigma_{\nu\rho} - \partial_\nu \tilde{\Gamma}^\sigma_{\rho\mu} + \tilde{\Gamma}^\sigma_{\lambda\mu} \tilde{\Gamma}^\lambda_{\rho\nu} - \tilde{\Gamma}^\sigma_{\lambda\nu} \tilde{\Gamma}^\lambda_{\rho\mu}. \quad (2.11)$$

Substituting the decomposition of general affine connection (2.5) into the curvature tensor (2.11), yields the following relation between the most general curvature and the Riemannian curvature as

$$\begin{aligned} R^\alpha_{\mu\nu\rho}(\tilde{\Gamma}) = & \overset{\circ}{R}^\alpha_{\mu\nu\rho}(g) + T^\lambda_{\nu\rho} K^\alpha_{\lambda\mu} + T^\lambda_{\nu\rho} L^\alpha_{\lambda\mu} + 2\overset{\circ}{\nabla}_{[\nu} K^\alpha_{\rho]\mu} + 2\overset{\circ}{\nabla}_{[\nu} L^\alpha_{\rho]\mu} \\ & + 2K^\alpha_{[\nu|\lambda} K^\lambda_{\rho]\mu} + 2L^\alpha_{[\nu|\lambda} K^\lambda_{\rho]\mu} + 2K^\alpha_{[\nu|\lambda} L^\lambda_{\rho]\mu} + 2L^\alpha_{\nu|\lambda} L^\lambda_{\rho]\mu}, \end{aligned} \quad (2.12)$$

¹The non-homogeneous term gets canceled during the subtraction of the connections.

where the square brackets indicate antisymmetrization, and the symbol “|” within square brackets excludes the immediately adjacent index from antisymmetrization.² A contraction in the first and third indices of curvature tensor

$$R_{\mu\rho} = R^\alpha{}_{\mu\alpha\rho}, \quad (2.13)$$

yields the Ricci tensor

$$\begin{aligned} R_{\mu\rho}(\tilde{\Gamma}) = & \mathring{R}_{\mu\rho}(g) + T^\lambda{}_{\nu\rho}K^\nu{}_{\lambda\mu} + 2\mathring{\nabla}_{[\nu}K^\nu{}_{\rho]\mu} + T^\lambda{}_{\nu\rho}L^\nu{}_{\lambda\mu} + 2\mathring{\nabla}_{[\nu}L^\nu{}_{\rho]\mu} \\ & + 2K^\nu{}_{[\nu|\lambda}K^\lambda{}_{\rho]\mu} + 2L^\nu{}_{[\nu|\lambda}K^\lambda{}_{\rho]\mu} + 2K^\nu{}_{[\nu|\lambda}L^\lambda{}_{\rho]\mu} + 2L^\nu{}_{\nu|\lambda}L^\lambda{}_{\rho|\mu}, \end{aligned} \quad (2.14)$$

and upon another contraction with a metric tensor

$$R = g^{\mu\rho}R_{\mu\rho}. \quad (2.15)$$

The most general Ricci scalar is related to the Levi–Civita Ricci scalar by

$$R(\tilde{\Gamma}) = \mathring{R}(g) + T - Q + 2\mathring{\nabla}_\mu T^\mu - \mathring{\nabla}_\mu(\hat{Q}^\mu - Q^\mu), \quad (2.16)$$

where $\mathring{\nabla}$ denotes the covariant derivative w.r.t. to Levi–Civita connection.

The torsion scalar T is written in terms of a scalar combination of torsion tensor

$$T = \frac{1}{4}T^\rho{}_{\mu\nu}T^\mu{}_{\rho}{}^{\nu} + \frac{1}{2}T^\rho{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^\rho{}_{\mu\rho}T^{\sigma\mu}{}_{\sigma}, \quad (2.17)$$

and trace of torsion

$$T^\mu = T^\alpha{}_{\alpha}{}^\mu. \quad (2.18)$$

The nonmetricity scalar is defined as³

$$Q \equiv -\frac{1}{4}Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + \frac{1}{2}Q_{\lambda\mu\nu}Q^{\mu\nu\lambda} + \frac{1}{4}Q_\mu Q^\mu - \frac{1}{2}Q_\mu \hat{Q}^\mu, \quad (2.19)$$

while

$$Q_\mu = Q_{\mu\nu}{}^\nu \quad \text{and} \quad \hat{Q}_\mu = Q_{\nu\mu}{}^\nu \quad (2.20)$$

are two independent traces that arise from the symmetry properties of the nonmetricity tensor.

By imposing restrictions on the connection, we can manipulate tensorial objects such as curvature (2.11), torsion (2.9), and nonmetricity (2.10) to formulate various geometric theories. For vanishing torsion, nonmetricity, and curvature, the underlying geometry is Minkowskian. Conversely, when all three entities are

² $A_{[\alpha\beta]} = \frac{(A_{\alpha\beta} - A_{\beta\alpha})}{2}$ and $A_{[\alpha|\nu\beta]} = \frac{(A_{\alpha\nu\beta} - A_{\beta\nu\alpha})}{2}$.

³Note that some authors define Q with an overall opposite sign [28].

nonzero, the theory falls under the umbrella of metric affine geometry [22]. A particularly interesting subset, which is the focus of this thesis, is teleparallel geometry, characterized by vanishing curvature alongside nonzero torsion and nonmetricity (see, for example, the recently proposed GTEGR [10]). Furthermore, vanishing curvature and nonmetricity (or known as metric teleparallel postulates) refer to metric teleparallel geometry detailed in Sec. 3.2, while vanishing both curvature and torsion, known as symmetric teleparallel geometry (or symmetric teleparallel postulates), are covered in Sec. 3.3; this thesis is more towards the latter part. Apart from this, though, other geometric frameworks are possible, but they fall outside the scope of this thesis. For further exploration, interested readers may look at this comprehensive review [24]. The accompanying diagram in Fig. 2.1 illustrates how these different theories are interconnected.

2.2 Tetrads and Noncoordinate Frames

The alternative geometric representation of teleparallel geometry can be formulated using two independent quantities, tetrad (θ^A_μ) and spin connections ($\omega^A_{B\mu}$), instead of metric and affine connections.

Consider a manifold equipped with a non-holonomic/non-coordinate basis vector θ^A in the tangent space. Their dual covector e_A in the cotangent space are related to a coordinate basis by

$$\theta^A = \theta^A_\mu dx^\mu, \quad e_A = e^\mu_A \partial_\mu, \quad (2.21)$$

where θ^A_μ and e^μ_A are the components of the non-holonomic basis and its dual basis, respectively, expressed on a coordinate basis. Those fields are defined on each spacetime point and are related to the spacetime metric via

$$g_{\mu\nu} = \eta_{AB} \theta^A_\mu \theta^B_\nu, \quad \eta_{AB} = g_{\mu\nu} e^\mu_A e^\nu_B, \quad (2.22)$$

which implies infinite tetrads can correspond to the same spacetime manifold while the tetrad and cotetrad components satisfy the orthonormality relation

$$\theta^A_\mu e^\mu_B = \delta^A_B, \quad \theta^A_\mu e^\nu_A = \delta^\nu_\mu. \quad (2.23)$$

The spin connection defines a spacetime affine connection with connection coefficients

$$\tilde{\Gamma}^\rho_{\mu\nu} \equiv e^\rho_A (\partial_\nu \theta^A_\mu + \tilde{\omega}^A_{B\nu} \theta^B_\mu), \quad (2.24)$$

which is a unique affine connection satisfying the tetrad postulate

$$\partial_\mu \theta^A_\nu + \tilde{\omega}^A_{B\mu} \theta^B_\nu - \tilde{\Gamma}^\rho_{\nu\mu} \theta^A_\rho = 0, \quad (2.25)$$

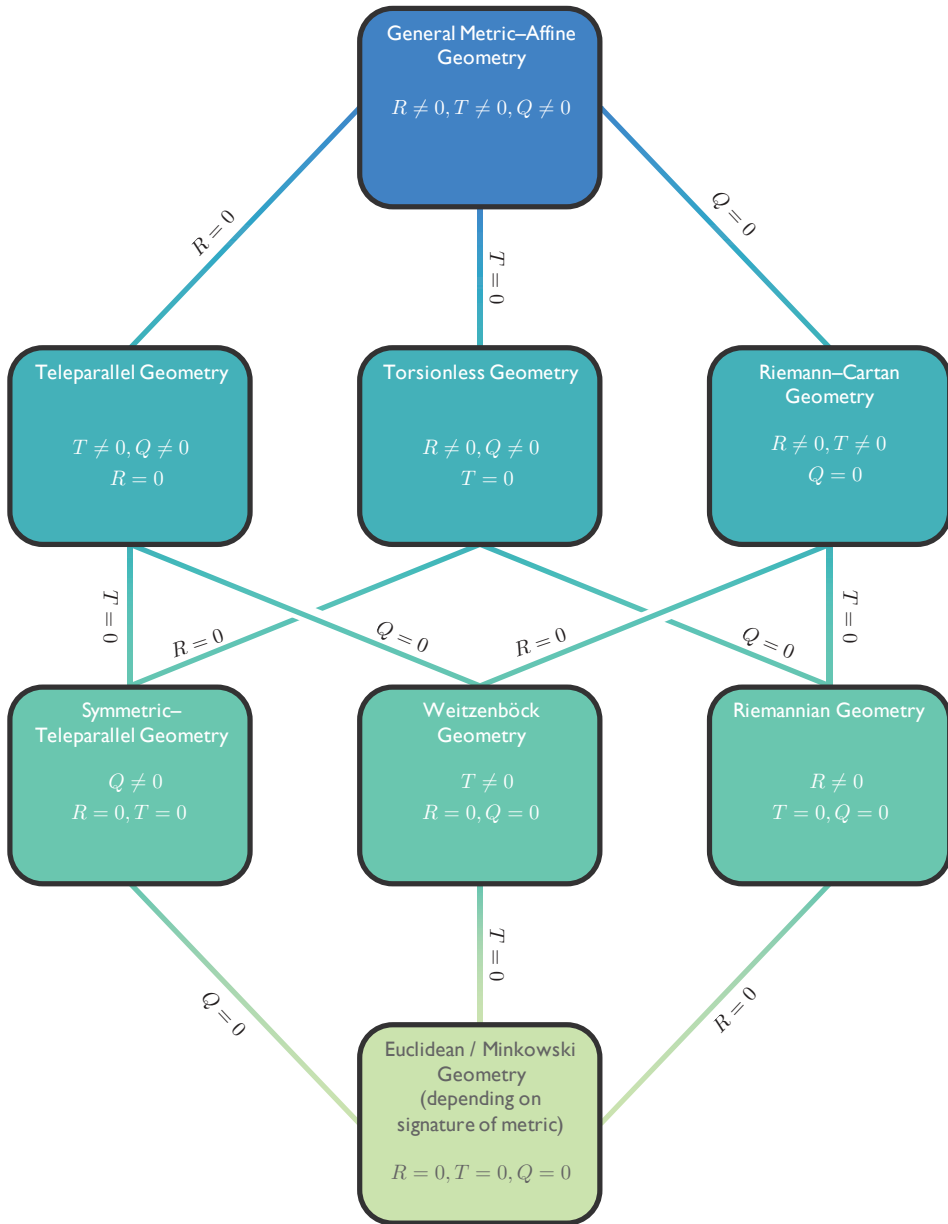


Figure 2.1: Geometric Classification Based on Restricting the Connection, taken from Ref. [25].

it is the connection that forces the distinction between curvature and teleparallel-based connections.

A remarkable feature of the above tensorial formulation is its invariance of the metric tensor (2.22) and general connection (2.24) under simultaneous local Lorentz transformations (LLT) of the tetrad and the spin connection given by

$$\theta'^A{}_\mu(x') = \Lambda^A{}_B(x) \theta^B{}_\mu(x), \quad (2.26a)$$

$$\omega'^A{}_{B\mu}(x') = \Lambda^A{}_C(x) \Lambda_B{}^D(x) \omega^C{}_{D\mu}(x) + \Lambda^A{}_C(x) \partial_\mu \Lambda_B{}^C(x), \quad (2.26b)$$

where the Lorentz matrix satisfies the fundamental relation

$$\eta_{AC} \Lambda^A{}_B \Lambda^C{}_D = \eta_{BD}, \quad (2.27)$$

and the notation $\Lambda_B{}^D$ is the inverse of $(\Lambda^{-1})^D{}_B$. It is noteworthy that an LLT on the tetrad (2.26a) alone while keeping the spin connection arbitrary, or vice versa, fails to preserve the affine connection relation (2.24) invariance.

Again, by employing a general coordinate transformation, the coordinate index of the tetrad and spin connection transforms as

$$\theta'^A{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} \theta^A{}_\nu(x), \quad \omega'^A{}_{B\mu}(x') = \frac{\partial x^\nu}{\partial x'^\mu} \omega^A{}_{B\nu}(x). \quad (2.28)$$

Both the simultaneous LLT (2.26) and coordinate transformation (2.28) leaves the metric (2.22) and general connection (2.24) invariant. Those transformations can also be viewed as gauge choices and are further explored in Sec. 3.2.

The coordinate basis tensorial quantities (2.9), (2.10) and (2.11) can also be represented on a non-coordinate basis. For instance, in a non-coordinate frame the torsion tensor is formulated as

$$T^A{}_{\mu\nu} \equiv \partial_\mu \theta^A{}_\nu - \partial_\nu \theta^A{}_\mu + \tilde{\omega}^A{}_{B\mu} \theta^B{}_\nu - \tilde{\omega}^A{}_{B\nu} \theta^B{}_\mu, \quad (2.29)$$

the nonmetricity tensor is expressed in terms of spin connection via the Minkowski metric

$$Q_{\mu AB} \equiv -\eta_{AC} \tilde{\omega}^C{}_{B\mu} - \eta_{CB} \tilde{\omega}^C{}_{A\mu}, \quad (2.30)$$

and the curvature tensor is

$$R^A{}_{B\mu\nu} \equiv \partial_\mu \tilde{\omega}^A{}_{B\nu} - \partial_\nu \tilde{\omega}^A{}_{B\mu} + \tilde{\omega}^A{}_{C\mu} \tilde{\omega}^C{}_{B\nu} - \tilde{\omega}^A{}_{C\nu} \tilde{\omega}^C{}_{B\mu}. \quad (2.31)$$

Using tetrad and cotetrad fields, tensorial objects can be seamlessly transformed between mixed tangent space and spacetime indices and vice versa. In both ways, the curvature tensor is shown as

$$R^A{}_{B\mu\nu} = \theta^A{}_\rho e^\sigma{}_B R^\rho{}_{\sigma\mu\nu}, \quad R^\mu{}_{\nu\rho\sigma} = e^\mu{}_A \theta^B{}_\nu R^A{}_{B\rho\sigma}, \quad (2.32)$$

the torsion tensor

$$T^A_{\mu\nu} = \theta^A_\rho T^\rho_{\mu\nu}, \quad T^\mu_{\nu\rho} = e^\mu_A T^A_{\nu\rho}, \quad (2.33)$$

and the nonmetricity as

$$Q_{\mu AB} = e^\nu_A e^\rho_B Q_{\mu\nu\rho}, \quad Q_{\mu\nu\rho} = \theta^A_\nu \theta^B_\rho Q_{\mu AB}. \quad (2.34)$$

2.3 Teleparallel Geometry

Teleparallel geometry, a crucial part of this thesis, is a subclass of metric affine geometry, which is characterized by 10 independent components of the metric tensor and 64 independent components of affine connection. We shall now investigate how the application of teleparallel postulates reduces the number of independent components of the connection.

We now proceed to apply the postulates of teleparallel geometry, specifically the vanishing curvature (2.11). This allows us to express the general connection in the form of [11]

$$\Gamma^\alpha_{\mu\nu} = L^\alpha_a \partial_\nu L_\mu^a, \quad (2.35)$$

where $L^\alpha_a \in GL(4, \mathcal{R})$ and the inverse is denoted as L_μ^a , carrying in total 16 independent components. The index ‘ a ’ is neither a tangent space nor a spacetime index, but under certain limiting conditions, it can be associated with either.

As previously discussed, teleparallel gravity encompasses two primary sub-branches: metric teleparallelism and symmetric teleparallelism. By incorporating the vanishing nonmetricity condition (2.10) alongside the assumption of a flat spacetime (2.11), we can express the connection within the framework of metric teleparallelism as [24]

$$\omega^A_{B\mu} = \Lambda^A_C \partial_\mu \Lambda_B^C. \quad (2.36)$$

This specific form of the connection is commonly referred to as the teleparallel spin connection and possesses a total of six independent components provided by the independent components of the matrices Λ^A_B associated with Lorentz transformations. Here, it is important to note that one can always fix a gauge to formulate the theory entirely in terms of the tetrad field. If we set $\omega^A_{B\mu} = 0$, this is known as the Weitzenböck gauge. Later, we will discuss the consequences of this gauge choice, particularly in the context of nonlinear modification of the theory, in Sec. 3.2.

Assuming a flat connection (2.11) and vanishing torsion (2.9), we can write the symmetric teleparallel connection as [29]

$$\Gamma^\alpha_{\mu\nu} = \partial_\lambda x^\alpha \partial_\mu \partial_\nu \xi^\lambda, \quad (2.37)$$

where ξ^λ are set of four independent scalar functions that can be interpreted as change of coordinates $\xi^\lambda(x^\nu)$. Like metric teleparallel, it is always possible to vanish the symmetric teleparallel connection (2.37) globally $\Gamma^\alpha_{\mu\nu} = 0$ by the specific choice of the function $\xi^\lambda = x^\lambda$, often referred to as the coincident gauge [29], ensures that the origin of the four scalar functions coincides with the spacetime coordinates and covariant derivative coincides with partial derivative.

2.4 Lie Derivative and Symmetries

A fundamental concept of how much a tensor field changes to first-order when it is dragged along by a flow generated by a vector field is measured by the Lie derivative.⁴ The Lie derivative of a tensor of rank (p, q) along the direction of the vector v reads

$$\begin{aligned}\mathcal{L}_v T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} = & v^\lambda \partial_\lambda T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} - \partial_\lambda v^{\mu_1} T^{\lambda \cdots \mu_p}_{\nu_1 \cdots \nu_q} - \cdots \\ & - \partial_\lambda v^{\mu_p} T^{\mu_1 \cdots \lambda}_{\nu_1 \cdots \nu_q} + \partial_{\nu_1} v^\lambda T^{\mu_1 \cdots \mu_p}_{\lambda \cdots \nu_q} + \cdots \\ & + \partial_{\nu_q} v^\lambda T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \lambda}.\end{aligned}\quad (2.38)$$

Using Eq. (2.38), the Lie derivatives of the fundamental geometric objects, expressed in holonomic bases, are given by:

$$\mathcal{L}_v g_{\mu\nu} = v^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu v^\lambda + g_{\mu\lambda} \partial_\nu v^\lambda, \quad (2.39a)$$

$$\begin{aligned}\mathcal{L}_v \tilde{\Gamma}^\alpha_{\mu\nu} = & v^\lambda \partial_\lambda \tilde{\Gamma}^\alpha_{\mu\nu} - \tilde{\Gamma}^\lambda_{\mu\nu} \partial_\lambda v^\alpha + \tilde{\Gamma}^\alpha_{\nu\lambda} \partial_\mu v^\lambda + \tilde{\Gamma}^\alpha_{\mu\lambda} \partial_\nu v^\lambda + \partial_\mu \partial_\nu v^\alpha \\ = & \nabla_\nu \nabla_\mu v^\alpha - v^\sigma R^\alpha_{\mu\nu\sigma} - \nabla_\nu (v^\sigma T^\alpha_{\mu\sigma}).\end{aligned}\quad (2.39b)$$

Up to this point, v represents an arbitrary vector.

Killing vectors are vector fields that are independent of spacetime coordinates, which preserve the metric along the direction of the Killing vector. Since in the teleparallel context, the connection is independent of the metric, imposing a symmetry that respects the teleparallel connection means that the Lie derivatives of the metric and affine connection (2.39) along Killing vectors v vanishes,

$$\mathcal{L}_v g_{\mu\nu} \stackrel{!}{=} 0, \quad \mathcal{L}_v \Gamma^\alpha_{\mu\nu} \stackrel{!}{=} 0, \quad (2.40)$$

which allows us to determine a maximum number of independent connection coefficients.

⁴The Lie derivative of the connection, as shown in Eq. (2.39b), is a tensor despite the connection itself not being a tensor. It can also be related to infinitesimal change of coordinates.

2.4.1 Cosmological Symmetry in Symmetric Teleparallelism

This point would be more illustrative if we consider a specific example to demonstrate which independent connections respect the symmetry (2.40) along the Killing vector. The example we have taken here is in the framework of the symmetric teleparallel case, so the connection must follow (2.40).

The spatially homogeneous and isotropic cosmological spacetimes in the spherical coordinate system

$$ds^2 = -dt^2 + a(t)^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (2.41)$$

are characterized by the Killing vectors of translations ζ_{T_i} and rotations ζ_{R_i} ,

$$\zeta_{T_x}^\mu = (0 \quad \chi \sin \theta \cos \phi \quad \frac{\chi}{r} \cos \theta \cos \phi \quad -\frac{\chi}{r} \frac{\sin \phi}{\sin \theta}), \quad (2.42a)$$

$$\zeta_{T_y}^\mu = (0 \quad \chi \sin \theta \sin \phi \quad \frac{\chi}{r} \cos \theta \sin \phi \quad \frac{\chi}{r} \frac{\cos \phi}{\sin \theta}), \quad (2.42b)$$

$$\zeta_{T_z}^\mu = (0 \quad \chi \cos \theta \quad -\frac{\chi}{r} \sin \theta \quad 0), \quad (2.42c)$$

$$\zeta_{R_x}^\mu = (0 \quad 0 \quad \sin \phi \quad \frac{\cos \phi}{\tan \theta}), \quad (2.42d)$$

$$\zeta_{R_y}^\mu = (0 \quad 0 \quad -\cos \phi \quad \frac{\sin \phi}{\tan \theta}), \quad (2.42e)$$

$$\zeta_{R_z}^\mu = (0 \quad 0 \quad 0 \quad -1), \quad (2.42f)$$

where $\chi = \sqrt{1 - kr^2}$ describes the curvature of the 3-space. In symmetric teleparallel theory, it has been found that the following three different branches of connections, when considering zero spatial curvature respect the symmetry condition (2.40) while simultaneously fulfilling the postulates of symmetric teleparallelism.

Connection set 1:

The connection set 1 is given by the expression

$$\Gamma^\rho_{\mu\nu} = \left[\begin{array}{c} \left[\begin{array}{cccc} \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \end{array} \right] \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{array} \right] \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} & \cot \theta & 0 \end{array} \right] \end{array} \right], \quad (2.43)$$

which has been previously studied in the literature, appearing as case 1 ($K_2 = K_3 = 0$) with $\gamma = K_1 = -K$ in Ref. [28], as case Γ_Q^{III} with $\gamma = C_1$ in Ref. [30], and as the trivial branch $\alpha_I = \alpha_{II} = 0$ in Ref. [27] with no restriction on the

unknown function $\gamma(t)$. Using the metric (2.41) and the affine connection (2.43), the nonmetricity scalar (2.19) is

$$Q = -6H^2. \quad (2.44)$$

Connection set 2

The following equation defines the connection set 2

$$\Gamma^\rho_{\mu\nu} = \begin{bmatrix} \left[\begin{array}{cccc} \gamma(t) + \frac{\dot{\gamma}(t)}{\gamma(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc} 0 & \gamma(t) & 0 & 0 \\ \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \end{array} \right] \\ \left[\begin{array}{cccc} 0 & 0 & \gamma(t) & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \gamma(t) & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{array} \right] \left[\begin{array}{cccc} 0 & 0 & 0 & \gamma(t) \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ \gamma(t) & \frac{1}{r} & \cot \theta & 0 \end{array} \right] \end{bmatrix}, \quad (2.45)$$

which by definition $\gamma(t) \neq 0$, corresponds to case 3 ($K_2 = 0, K_3 \neq 0$) with $\gamma = K_3 = K$ in Ref. [28] and case $\Gamma_Q^{(I)}$ with $\gamma = C_3$ in Ref. [30]. The corresponding nonmetricity scalar (2.19) using the metric (2.41) and the connection (2.45) is

$$Q = -6H^2 + 9H\gamma + 3\dot{\gamma}. \quad (2.46)$$

Connection set 3

Finally, the connection set 3 corresponds to

$$\Gamma^\rho_{\mu\nu} = \begin{bmatrix} \left[\begin{array}{cccc} -\frac{\dot{\gamma}(t)}{\gamma(t)} & 0 & 0 & 0 \\ 0 & \gamma(t) & 0 & 0 \\ 0 & 0 & r^2\gamma(t) & 0 \\ 0 & 0 & 0 & r^2\gamma(t) \sin^2 \theta \end{array} \right] \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \end{array} \right] \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{array} \right] \left[\begin{array}{cccc} 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} & \cot \theta & 0 \end{array} \right] \end{bmatrix}, \quad (2.47)$$

where by definition $\gamma(t) \neq 0$, corresponds to case 2 ($K_2 \neq 0, K_3 = 0$) with $\gamma = K_2 = -a^2 K$ in Ref. [28] and case $\Gamma_Q^{(II)}$ with $\gamma = C_2$ in Ref. [30]. Using the metric (2.41) and the affine connection (2.47), the nonmetricity scalar (2.19) obtained as

$$Q = -6H^2 + 9H\gamma + 3\dot{\gamma}. \quad (2.48)$$

Another branch exists for nonzero spatial curvature that also respects this symmetry (2.40). However, our research focuses only on the zero curvature case. Interested readers can refer to [30]. We need to point out that with a suitable transformation, one can transform all three branches of connection (2.43), (2.45), and (2.47) into the coincident gauge but at the expense of a very complicated metric, for instance, see [31].

2.5 Geodesic and Autoparallel Curves

In Euclidean geometry, the shortest path between two points is a straight line. In curved spacetime, this concept is generalized to geodesics and autoparallels. Geodesics represent the shortest paths between two points, while autoparallels are curves whose tangent vectors remain parallel along the curve. Mathematically, autoparallels are characterized by the equation:

$$x^\nu \tilde{\nabla}_\nu x^\mu = 0. \quad (2.49)$$

To mathematically characterize geodesics in any geometry, the action integral is

$$S = \int_{\lambda_2}^{\lambda_1} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} d\lambda, \quad (2.50)$$

where λ is the affine parameter. Sticking to Riemannian geometry, extremizing the action (2.50) yields the trajectory of a freely falling test particle

$$\frac{d^2 x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \iff x^\alpha \tilde{\nabla}_\alpha x^\mu = 0, \quad (2.51)$$

demonstrates that both geodesics and autoparallels coincide, implying there is no difference in shortest paths and straight lines of test particles. For a detailed derivation, see [32].

In teleparallel gravity, where the connection is no longer restricted to the Levi-Civita connection, the question arises: Do geodesics of the general connection describe the free fall motion of a test particle? Can geodesics exhibit behavior that differs from autoparallel? To investigate this, let us take the general connection (2.5) defined before. The autoparallel equation (2.49) for this connection is given by [33]:

$$\frac{d^2 x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0, \quad (2.52)$$

and the geodesics equation, on the other hand, is

$$\frac{d^2 x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = (K^\mu_{\rho\sigma} + L^\mu_{\rho\sigma}) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}. \quad (2.53)$$

By considering only the symmetric part of the general connection (2.1), Eq. (2.53) reduces to:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = L^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}. \quad (2.54)$$

Similarly, considering only the antisymmetric part of the Eq. (2.1), Eq. (2.53) becomes:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = K^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}. \quad (2.55)$$

At this point, let us recall the equation of motion for a charged particle of mass m and charge q in an electromagnetic field

$$\frac{d^2 x^\mu}{d\tau^2} + \hat{\Gamma}^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{q}{m} F^\mu_{\rho} \frac{dx^\rho}{d\tau}, \quad (2.56)$$

where the r.h.s. represents the Lorentz force. In teleparallel gravity, the distortion and contortion tensors, which appear on the r.h.s. of Eqs. (2.54) and (2.55), play a role analogous to the Lorentz force [24, 34]. Consequently, in teleparallel geometry, geodesics and autoparallels are distinct.

Chapter 3

Action and Field Equations

This chapter gives an overview of the action principle and its corresponding field equations in alternative theories of GR and their extensions. We analyzed the necessary conditions for a general model function in modified theories of gravity to define a consistent gravitational theory. Furthermore, we have identified all possible equivalent actions for some modified gravity theories that describe the same field equations. Finally, based on an analysis of previous studies, this chapter concludes with a discussion of the procedure for solving field equations in teleparallel gravity.

3.1 Einstein–Hilbert Action and its Extension

The Einstein–Hilbert action in the standard¹ formalism is given by

$$S_{\text{GR}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \mathring{R} + S_m(g), \quad (3.1)$$

with matter Lagrangian S_m , $\kappa^2 = \frac{8\pi G}{c^4}$ and $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$. Varying the action (3.1) w.r.t. the metric yields Einstein’s metric field equations of motion (EOM)

$$\mathring{G}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{1}{2} \mathring{R} g_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu}, \quad (3.2)$$

which are second-order differential equations in the metric, and $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor defined by

$$\mathcal{T}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \quad (3.3)$$

¹This theory is purely metric dependent.

The $f(\mathring{R})$ action is given by

$$S_{f(\mathring{R})} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(\mathring{R}) + S_m(g). \quad (3.4)$$

Varying the action (3.4) w.r.t. inverse metric, yields the modified Einstein field equations

$$f'(\mathring{R})\mathring{R}_{\mu\nu} - \frac{1}{2}f(\mathring{R})g_{\mu\nu} - \left(\mathring{\nabla}_\mu \mathring{\nabla}_\nu - g_{\mu\nu} \mathring{\square} \right) f'(\mathring{R}) = \kappa^2 \mathcal{T}_{\mu\nu}, \quad (3.5)$$

where ' denotes differentiation w.r.t. $\mathring{R}$ and (3.5) leads to fourth-order differential equations for the metric. This arises because the surface term of the Ricci tensor itself is second-order in metric and now becomes part of the action; hence, it is no longer solely a boundary term. This characteristic distinguishes the $f(\mathring{R})$ EOM from those arising in nonlinear teleparallel theories, which we will explore in the following sections 3.2 and 3.3. The appearance of the last two terms within the bracket on the left side introduces the extra dynamics beyond those present in GR unless $f(\mathring{R})$ is a linear function. This implies that $f(\mathring{R})$ gravity admits a richer solution space compared to GR. General relativity is recovered as a limiting case of $f(\mathring{R})$ gravity when $f(\mathring{R}) = \mathring{R}$.

Physical viability condition for $f(\mathring{R})$ theory: A viable metric $f(\mathring{R})$ theory needs to satisfy the following conditions for consistency with successful cosmological evolution, starting from radiation to matter to dark energy [35].

- To avoid ghost states, $f'(\mathring{R}) > 0$.
- To avoid the existence of a scalar degree of freedom with negative mass squared, i.e., tachyons, the model should obey the condition $f''(\mathring{R}) > 0$.
- A necessary condition for stability and the existence of a late time de Sitter solution holds $0 < \frac{\mathring{R}f''(\mathring{R})}{f'(\mathring{R})} < 1$,

for $\mathring{R} > \mathring{R}_0$, where $\mathring{R}_0$ is the present value of Ricci scalar. For a comprehensive review of $f(\mathring{R})$ theory and associated cosmological and theoretical issues, please refer to [20, 23].

3.2 Metric Teleparallel Gravity and its Extension

The TEGR provides an alternative geometric formulation of Einstein–Hilbert action

$$S_{\text{TEGR}} = -\frac{1}{2\kappa^2} \int d^4x \theta T + S_m(\theta), \quad (3.6)$$

where θ is the determinant of the tetrad and is related to the determinant of metric by

$$\theta = \det(\theta^A{}_\mu) = \sqrt{-g}, \quad (3.7)$$

and the energy-momentum tensor defined through the matter Lagrangian S_m by

$$\mathcal{T}_A{}^\lambda = \frac{1}{\theta} \frac{\delta(\theta S_m)}{\delta \theta^A{}_\lambda}. \quad (3.8)$$

Enforcing the metric teleparallel postulates on identity (2.16), the TEGR Lagrangian is related to GR

$$\mathring{R} = -T + 2\theta \partial_\mu (e T^\mu) = -T + B_T, \quad (3.9)$$

up to a boundary term B_T . Due to the above relation (3.9), the TEGR EOMs are equivalent to GR EOMs since the surface term does not contribute to the EOM (in the bulk). Moreover, the spin connection is a pure gauge degree of freedom (GDOF) or non-physical.

At this point, it is instructive to review how the theory responds under a LLT. As previously noted, a simultaneous transformation of the tetrad and spin connection (2.28) leaves the torsion tensor invariant, hence the TEGR lagrangian (3.9) remains unchanged. However, for a historical reason, it has been seen that, if we make a transformation purely on tetrad and leave the spin connection arbitrary, or purely on spin connection but leave tetrad untouched as

$$\theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu, \quad \omega'^A{}_{B\mu} \equiv 0, \quad (3.10a)$$

$$\omega'^A{}_{B\mu} = \Lambda^A{}_C \partial_\mu (\Lambda^{-1})^C{}_B, \quad \theta'^A{}_\mu \equiv \theta^A{}_\mu, \quad (3.10b)$$

this transformation keeps the lagrangian (3.9) pseudo-invariant up to a boundary term B_T . While this is not an issue in TEGR due to the isolation of problematic terms to surface integrals, it exhibits peculiar characteristics in its nonlinear modifications, as will be discussed shortly.

The torsion scalar can alternatively be written as

$$T = \frac{1}{2} T^\rho{}_{\sigma\mu} S_\rho{}^{\sigma\mu}, \quad (3.11)$$

where the so-called superpotential is defined in terms of the contortion and torsion tensor as

$$S_\rho{}^{\sigma\mu} = K_\rho{}^\sigma{}_\nu{}^\mu - \delta_\rho^\sigma T_\nu{}^{\nu\mu} + \delta_\rho^\mu T_\nu{}^{\nu\sigma}. \quad (3.12)$$

The tetrad EOM are obtained by varying the TEGR action (3.6) w.r.t. tetrad as

$$\frac{1}{\theta} \partial_\sigma (\theta S_A{}^{\mu\sigma}) - T^\sigma{}_{\nu A} S_\sigma{}^{\nu\mu} + \frac{1}{2} T e_A{}^\mu + \omega^B{}_{A\nu} S_B{}^{\nu\mu} = \kappa^2 \mathcal{T}_A{}^\mu, \quad (3.13)$$

and the spin connection EOM are identically satisfied, which can also be easily seen from the antisymmetric part of tetrad EOM (3.13), and this coincidence occurs due to Bianchi identities [24]. From (3.13), a tetradic formulation of the Einstein tensor can also be expressed as

$$\mathring{G}_{\alpha\beta} = \frac{1}{\theta} \partial_\nu (\theta S_{\beta\alpha}{}^\nu) - T^B{}_{\nu\beta} S_B{}^\nu{}_\alpha + \frac{1}{2} T g_{\alpha\beta} + \omega^B{}_{\beta\nu} S_B{}^\nu{}_\alpha. \quad (3.14)$$

The modified TEGR action is given by

$$S_{f(T)} = -\frac{1}{2\kappa^2} \int d^4x \theta f(T) + S_m(\theta). \quad (3.15)$$

Upon varying the action with respect to the tetrad and spin connection, the EOMs become

$$\begin{aligned} -\frac{1}{\theta} f_T \partial_\nu (\theta S_A{}^{\mu\nu}) - S_A{}^{\mu\nu} \partial_\nu f_T + f_T T^B{}_{\nu A} S_B{}^{\nu\mu} \\ - f_T \omega^B{}_{A\nu} S_B{}^{\nu\mu} - \frac{1}{2} f e_A{}^\mu = \kappa^2 \mathcal{T}_A{}^\mu, \end{aligned} \quad (3.16a)$$

$$S_{[AB]}{}^\nu \partial_\nu f_T = 0. \quad (3.16b)$$

Rearranging the above EOM to isolate the GR term and the modification yields a second-order differential equation for the tetrad

$$f_T \mathring{G}_{\mu\nu} + \frac{1}{2} (f - f_T T) g_{\mu\nu} + f_{TT} S_{\mu\nu\alpha} \partial^\alpha T = \kappa^2 \mathcal{T}_{\mu\nu}, \quad (3.17)$$

which is no longer equivalent to $f(\mathring{R})$ theory. The notation f_T and f_{TT} denote the first and second derivatives of the model function w.r.t. torsion scalar. The above field equations (3.16) reduce to GR when $f(T) = T$.

We shall now resume the discussion of LLTs in the nonlinear $f(T)$ theory. Unlike TEGR, the torsion tensor is now not invariant under a pure tetrad LLT since the extra term that appears inside the function is no longer a surface term. Therefore, the theory breaks local Lorentz invariance, and due to this, some consequences are seen in the literature. For instance, while solving the $f(T)$ EOM (3.16), some tetrad restricts the model function to be linear (trivially), in terminology known as “bad tetrad”, and those tetrads solve the EOM without restricting the model function trivially, in terminology are known as “good tetrad” [36]. A detailed review and examples of these issues can be found in [37]. Similar issues arise in symmetric teleparallel theory, as will be discussed in the following section.

A reasonable choice for a good tetrad-spin connection pair should obey the following two conditions:

- The tetrad must exhibit symmetry compatible with the chosen spin connection.

- The tetrad must satisfy the antisymmetric field equations (3.16) without imposing restrictions on the underlying theory trivially.

However, these peculiarities are avoided with a simultaneous tetrad and spin connection transformation (2.26) that makes the metric teleparallel theory fully covariant [38–40].

3.3 Symmetric Teleparallel Gravity and its Extension

An alternative geometric formulation of GR can be achieved using the STEGR action, given by

$$S_{STEGR} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} Q + S_m(g). \quad (3.18)$$

By imposing symmetric teleparallel postulates on (2.16), the STEGR Lagrangian relates to GR

$$\mathring{R} = Q + \mathring{\nabla}_\mu (\hat{Q}^\mu - Q^\mu) = Q + B_Q, \quad (3.19)$$

up to a boundary term B_Q . This relation implies the STEGR EOM should be equivalent to GR EOM, as the surface term does not contribute to the field equations unless working on the boundary of the manifold. In this scenario, the symmetric teleparallel connection is a pure gauge and can be chosen arbitrarily. Looking at relation (3.19), the Ricci scalar $\mathring{R}$ is invariant under diffeomorphism purely on metric (2.2), but the boundary term in r.h.s. is not, then the nonmetricity scalar also changes under such transformation. Therefore, any production of such a boundary term inside the action is integrated out. Hence, STEGR lagrangian is pseudo-invariant up to a boundary term and has some peculiarity in its nonlinear extensions, analogous to metric teleparallel theory.

Another useful tensor, the so-called superpotential (or conjugate) tensor is expressed as

$$P^\alpha{}_{\mu\nu} = -\frac{1}{4}Q^\alpha{}_{\mu\nu} + \frac{1}{2}Q_{(\mu}{}^\alpha{}_{\nu)} + \frac{1}{4}g_{\mu\nu}Q^\alpha - \frac{1}{4}(g_{\mu\nu}\hat{Q}^\alpha + \delta^\alpha{}_{(\mu}Q_{\nu)}). \quad (3.20)$$

Varying the action (3.18) w.r.t. metric the metric EOM are

$$\begin{aligned} \mathring{G}_{\mu\nu} = \frac{2}{\sqrt{-g}} \nabla_\alpha (\sqrt{-g} P^\alpha{}_{\mu\nu}) + P_{(\mu|\alpha\beta} Q_{\nu)}{}^{\alpha\beta} - 2P^{\alpha\beta}{}_{(\nu} Q_{\alpha\beta|\mu)} \\ - \frac{1}{2} Q g_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu}, \end{aligned} \quad (3.21)$$

when the matter field is not coupled with the connection, the connection field equations are

$$\nabla_\mu \nabla_\nu (\sqrt{-g} P^{\mu\nu}{}_\alpha) = 0, \quad (3.22)$$

which can either be obtained from the Levi-Civita covariant derivative of the gravitational sector of the metric equation, a consequence of Bianchi identities [41].

The $f(Q)$ action is defined as

$$S_{f(Q)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(Q) + S_m(g). \quad (3.23)$$

Varying the action (3.23) w.r.t. metric the metric equations are

$$f_Q \dot{G}_{\mu\nu} + \frac{1}{2} (Q f_Q - f(Q)) g_{\mu\nu} + 2 f_{QQ} P^\lambda_{\mu\nu} \partial_\lambda Q = \kappa^2 \mathcal{T}_{\mu\nu}, \quad (3.24)$$

and the connection equations are [29]

$$\nabla_\mu \nabla_\nu (\sqrt{-g} f_Q P^{\mu\nu}{}_\alpha) = 0. \quad (3.25)$$

GR is recovered as a special case of $f(Q)$ gravity when $f(Q) = Q$.

Unlike STEGR, a diffeomorphism purely on the metric (2.2) no longer keeps the $f(Q)$ lagrangian invariant since the extra term inside the function is no longer a surface term. Hence, this theory breaks the diffeomorphism invariance. Consequently, while solving the $f(Q)$ EOM (3.24) and (3.25), some coordinate choices restrict the model function to be linear (trivially), which are termed “bad coordinates.” Conversely, those solve the EOM without imposing restrictions trivially on the model function are called “good coordinate.” However, a simultaneous transformation of both the metric and the affine connection, as defined in equations (2.2) and (2.4), restores full covariance to the theory.

Physical viability condition for the $f(Q)$ theory: Following the established criteria for physical viability in $f(R)$ gravity, in the publication III, we investigate the conditions for a physically valid $f(Q)$ model to obtain a well-defined gravitational theory. A viable $f(Q)$ model must satisfy the following:

- To ensure the absence of antigravity, infinite gravitational constants, or singularities, the function $f(Q)$ must always be positive, i.e.

$$f_Q > 0. \quad (3.26)$$

- Ideally, $0 \leq Q f_Q - f(Q) < \infty$. However, this condition is not strictly necessary. Negative cosmological constants, while generally leading to a collapsing universe, may be acceptable in certain scenarios.

A crucial subtlety may arise when the nonmetricity scalar, undergoes a sign change during evolution or within specific configurations of a given connection. The viability of the model is dependent upon the form of the model function $f(Q)$

and whether the conditions are immune under such sign changes. For instance, models of the form $f(Q) = Q + Q^n$, where n is an even integer, can violate the conditions Eq. (3.26) when Q changes sign. Conversely, models with odd values of n remain unaffected. Therefore, it seems we can not claim that a certain model $f(Q)$ is viable by itself or that a certain connection is viable by itself, but these aspects must be investigated together.

3.4 GTEGR Gravity and its Extension

Yet another equivalent theory of GR known as GTEGR developed in recent years completes the understanding of all possible ways to formulate the equivalent theories of GR, which can be fitted another equivalence class of GR alongside the established Geometric Trinity of Gravity [10].

The dynamical fields of the theory are a metric tensor and an affine connection, which is imposed to be vanishing curvature, explicitly defined the form in (2.35). The Lagrangian is built of torsion and nonmetricity and mixed combinations of them, as given by

$$G = -Q + T + T^{\mu\nu\rho}Q_{\nu\rho\mu} - T^{\mu}_{\rho\mu}Q_{\rho\nu}{}^{\nu} + T^{\mu}_{\rho\mu}Q^{\nu}_{\nu\rho}. \quad (3.27)$$

Then the action is formulated as

$$S_{GTEGR} = -\frac{1}{2\kappa^2} \int d^4(x) \sqrt{-g} G + S_m(g), \quad (3.28)$$

and a general connection variation defined as

$$Y_{\mu}{}^{\nu\rho} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{GTEGR}}{\delta \Gamma^{\mu}_{\nu\rho}}. \quad (3.29)$$

Let us define a new tensor constructed from the combination of the contortion tensor and the disformation tensor

$$M^{\mu}_{\nu\rho} = K^{\mu}_{\nu\rho} + L^{\mu}_{\nu\rho}. \quad (3.30)$$

The Lagrangian of GTEGR is related to GR up to a boundary term expressed as

$$\overset{\circ}{R} = -G + B, \quad (3.31)$$

where,

$$B = 2 \overset{\circ}{\nabla}_{\mu} M^{[\nu\mu]}_{\nu} = \overset{\circ}{\nabla}_{\mu} \left(\tilde{Q}^{\mu} - Q^{\mu} - 2T^{\mu} \right). \quad (3.32)$$

The metric field equations turn out to be identical to those of GR (3.2). Since the connection contributes to the action only as a total derivative term, which does not alter the equation of motions within the bulk of spacetime, the connection

can be treated as a GDOF. Given the absence of coupling between matter and the connection, the connection equations yield

$$\nabla_\tau Y_\mu^{\nu\tau} - M^\omega_{\tau\omega} Y_\mu^{\nu\tau} = 0, \quad (3.33)$$

where

$$Y_\mu^{\nu\rho} = \frac{1}{2\kappa^2} (M^{\nu\sigma} \delta_\mu^\rho + M^\sigma_{\mu\sigma} g^{\nu\rho} - M^{\nu\rho}_\mu - M^\rho_\mu{}^\nu). \quad (3.34)$$

The modified general teleparallel action takes the form

$$S = -\frac{1}{2\kappa^2} \int d^4(x) \sqrt{-g} f(G) + S_m(g). \quad (3.35)$$

Upon varying the action with respect to the metric and flat connection, the respective resulting field equations are [11, 42, 43]

$$\begin{aligned} f'(G) \mathring{G}_{\mu\nu} - M^\rho_{(\mu\nu)} \mathring{\nabla}_\rho f'(G) + \mathring{\nabla}_{(\mu} f'(G) M^\sigma_{\nu)\sigma} + M^\sigma_{[\rho\sigma]} g_{\mu\nu} \mathring{\nabla}_\rho f'(G) + \\ \frac{1}{2} (f(G) - f'(G)G) g_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu}, \end{aligned} \quad (3.36a)$$

$$M^{\nu\rho}_\rho \nabla_\mu f'(G) + M^\sigma_{\mu\sigma} g^{\nu\rho} \nabla_\rho f'(G) - M^{\nu\rho}_\mu \nabla_\rho f'(G) - M^\rho_\mu{}^\nu \nabla_\rho f'(G) = 0. \quad (3.36b)$$

The $f(G)$ theories differ from GTEGR primarily due to the nature of the connection term within the action (3.35). In GTEGR, this term is a total derivative, rendering the connection non-dynamical. Conversely, in $f(G)$ theories with $f''(G) \neq 0$, the connection contribution to the action ceases to be a total derivative, thereby introducing the connection as a dynamical field and necessitating additional non-trivial connection field equations. From this modified general teleparallel theory, $f(T)$ and $f(Q)$ gravity emerge as limiting cases when either metric compatibility or vanishing torsion is imposed. Furthermore, a linear $f(T)$ model within a metric-compatible framework or a linear $f(Q)$ model within a torsionless framework reduces to GR.

An alternative action formulation involves using Lagrange multipliers to enforce the geometric postulates through coupling with Lagrange multipliers while leaving the connection free [30]. However, both approaches yield identical equations of motion.

3.5 Equivalent Formulations of Modified Gravity Theories

This section explores equivalent action principles of $f(\mathring{R})$, $f(T)$, and $f(Q)$ gravitational theories, which can also be extended to establish equivalence at the level of field equations.

3.5.1 Scalar-Curvature Representation of $f(\mathring{R})$ Gravity

First, we turn our attention to a dynamically equivalent formulation of the $f(\mathring{R})$ action (3.4), where, under suitable field redefinitions of both gravity and matter, the resulting action is identical to the original $f(\mathring{R})$ theory.

This involves introducing a pair of auxiliary fields ζ and Φ and writing the dynamically equivalent $f(\mathring{R})$ action

$$S_{EQf(\mathring{R})} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\zeta(\mathring{R} - \Phi) + f(\Phi) \right) + S_m(g), \quad (3.37)$$

where “EQ $f(\mathring{R})$ ” denotes the equivalent formulation of $f(\mathring{R})$ gravity. Varying w.r.t. the auxiliary field ζ yields the constraint

$$\mathring{R} = \Phi, \quad (3.38)$$

effectively restricting the action (3.37) to be equivalent to that given by (3.4), unless $f''(\mathring{R}) = 0$ in which case the theory reduces to Einstein–Hilbert.

Upon varying (3.37) w.r.t. Φ , yields the algebraic equation $f'(\Phi) = \zeta$. Introducing a new field $f'(\Phi) = \mathcal{A}(\Phi)$, the action (3.37) can be recast as

$$S_{EQf(\mathring{R})} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\mathcal{A}(\Phi) \mathring{R} - \mathcal{V}(\Phi) \right) + S_m(g), \quad (3.39)$$

where

$$\mathcal{V}(\Phi) = \Phi \mathcal{A}(\Phi) - f(\Phi) \quad (3.40)$$

is the potential of the scalar field. By utilizing (3.38) and redefining $\mathcal{A}(\Phi) = f'(\mathring{R})$ the action (3.39) can be seen to be equivalent to the original $f(\mathring{R})$ action, where $\mathcal{A}(\Phi)$ and $\mathcal{V}(\Phi)$, respectively, represent gravitational coupling and potential functions. If we choose $\mathcal{A}(\Phi) = \Phi$, the action (3.39) is equivalent to Brans–Dicke theory (BD) [44] with a vanishing BD parameter, which means that the action lacks a kinetic term.

A generalized formulation of scalar-tensor action can be expressed as [45]

$$S_{JF} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\mathcal{A}(\Phi) \mathring{R} - \mathcal{B}(\Phi) g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - 2\mathcal{V}(\Phi) \right) + S_m(g), \quad (3.41)$$

where “JF” denotes the Jordan frame scalar-curvature representation. Here, the scalar field introduces an additional DOF beyond the two tensor DOF that appeared in GR.

Varying the action (3.41) w.r.t. the metric $g^{\mu\nu}$ and scalar field, the metric and scalar field EOM are, respectively [46],

$$\begin{aligned} \mathcal{A}(\Phi)\overset{\circ}{G}_{\mu\nu} + g_{\mu\nu}\overset{\circ}{\square}\mathcal{A}(\Phi) - \overset{\circ}{\nabla}_{\mu}\overset{\circ}{\nabla}_{\nu}\mathcal{A}(\Phi) + \frac{1}{2}g_{\mu\nu}\left(\overset{\circ}{\nabla}_{\alpha}\Phi\overset{\circ}{\nabla}^{\alpha}\Phi\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi)\right) \\ - \overset{\circ}{\nabla}_{\mu}\Phi\overset{\circ}{\nabla}_{\nu}\Phi\mathcal{B}(\Phi) = \kappa^2 T_{\mu\nu}, \end{aligned} \quad (3.42a)$$

$$2\mathcal{B}(\Phi)\overset{\circ}{\square}\Phi + \mathcal{B}'(\Phi)\overset{\circ}{\nabla}_{\alpha}\Phi\overset{\circ}{\nabla}^{\alpha}\Phi + \mathcal{A}'(\Phi)\overset{\circ}{R} - 2\mathcal{V}'(\Phi) = 0. \quad (3.42b)$$

Further, a conformal transformation can be used to break non-minimal coupling in curvature and the scalar field in the generalized scalar-tensor action. Under a conformal transformation of the metric

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \hat{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}, \quad (3.43)$$

where Ω^2 is the conformal factor, and an overhat notation indicates quantities in what we will refer to as the Einstein frame. After evaluating all necessary quantities by using (3.43), an action is obtained in the Einstein frame as [20]

$$\begin{aligned} S_{EF} = \int d^4x \sqrt{-\hat{g}} \left[\frac{\overset{\circ}{R}}{2\kappa^2} - \frac{1}{2}\hat{g}^{\mu\nu} (\partial_{\mu}\Psi) (\partial_{\nu}\Psi) - U(\Psi) \right] \\ + S_m \left(e^{-\sqrt{\frac{2}{3}}\kappa\Psi} \hat{g}_{\mu\nu} \right), \end{aligned} \quad (3.44)$$

with

$$2U(\Psi) = \mathcal{V}(\Phi)\mathcal{A}^{-2}(\Phi), \quad \mathcal{A}(\Phi) = \Omega^2(x). \quad (3.45)$$

For a detailed review of conformal transformation, see Ref. [20, 47]. We stress once more that actions (3.39) are equivalent to (3.4). However, the generalized scalar-curvature JF action (3.41) and Einstein frame action (3.44) have more functional freedom compared to $f(\overset{\circ}{R})$ theory. In fact, $f(\overset{\circ}{R})$ gravity can be viewed as a limiting case of these more general formulations obtained through a particular substitution within the arbitrary model function.

3.5.2 Scalar-Torsion Representation of $f(T)$ Gravity

Analogous to the scalar-curvature theory, $f(T)$ gravity action can be recast in an equivalent form by introducing two auxiliary fields, ζ and Φ , as

$$S_{EQf(T)} = \frac{1}{2\kappa^2} \int d^4x \theta [\zeta(\Phi - T) + f(\Phi)]. \quad (3.46)$$

Varying the above action w.r.t. ζ enforces the constraint $\Phi = T$ which means the action (3.46) is equivalent to (3.15), unless $f_{TT} = 0$, in which case the action reduces to TEGR.

Varying the action w.r.t. Φ yield, $\zeta = -f'(\Phi)$ and redefining $\mathcal{A}(\Phi) = f'(\Phi)$, the action (3.46) is rewritten as

$$S_{EQf(T)} = \frac{1}{2\kappa^2} \int d^4x \theta [\mathcal{A}(\Phi)T - \mathcal{V}(\Phi)], \quad (3.47)$$

where

$$\mathcal{V}(\Phi) = \Phi f'(\Phi) - f(\Phi). \quad (3.48)$$

This reformulated action (3.47) is equivalent to the standard $f(T)$ (3.15) action. Notably, it lacks a kinetic term for the scalar field Φ , rendering it equivalent to a (teleparallel) BD theory with a vanishing BD parameter and $\mathcal{A}(\Phi) = \Phi$.

Like scalar-curvature action, a more general $f(T)$ equivalent scalar-torsion action can be formed by introducing arbitrary scalar functions as

$$S_{JF} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\mathcal{A}(\Phi)T - \mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - 2\mathcal{V}(\Phi) \right) + S_m(g). \quad (3.49)$$

Varying the action (3.49) w.r.t. the tetrad, the symmetric part of tetrad field equations yield [48, 49]

$$\begin{aligned} \mathcal{A}(\Phi)\overset{\circ}{G}_{\mu\nu} + \mathcal{A}'(\Phi)\partial_\rho S_{(\mu\nu)}{}^\rho + \frac{1}{2}g_{\mu\nu}(\mathcal{B}(\Phi)g^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi + 2\mathcal{V}(\Phi)) \\ - \mathcal{B}(\Phi)\partial_\mu\Phi\partial_\nu\Phi = \kappa^2 T_{\mu\nu}, \end{aligned} \quad (3.50)$$

the antisymmetric part of the tetrad field equations

$$\mathcal{A}'(\Phi)T^\rho{}_{[\mu\nu}\partial_{\rho]}\Phi = 0, \quad (3.51)$$

which is identical to the connection equation as expected from Bianchi identities and the scalar field equation is

$$2\mathcal{B}(\Phi)\overset{\circ}{\square}\Phi + \mathcal{B}'(\Phi)g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \mathcal{A}'(\Phi)T - 2\mathcal{V}'(\Phi) = 0. \quad (3.52)$$

Limiting cases

- When the scalar field is globally constant, Eq. (3.50) reduces to Einstein's equations in GR with the value of the potential playing the role of the cosmological constant, while (3.51) and (3.52) immediately vanish.
- By setting the nonminimal gravitational coupling function $\mathcal{A}(\Phi)$ to unity and eliminating both the kinetic and potential terms of the scalar field $\mathcal{B}(\Phi) = \mathcal{V}(\Phi) = 0$ reduces the theory to the TEGR (3.6).

- The $f(T)$ gravity is recovered by choosing

$$\mathcal{A}(\Phi) = f'(T), \quad \mathcal{B}(\Phi) = 0, \quad 2\mathcal{V}(\Phi) = Qf'(T) - f(T), \quad (3.53)$$

as previously established at the action level.

Under the same conformal transformation (3.43), the tetrad and its inverse must transform as

$$\hat{\theta}^A{}_\mu = \Omega(\phi)\theta^A{}_\mu, \quad \hat{e}_A{}^\mu = \Omega^{-1}(\phi)e_A{}^\mu. \quad (3.54)$$

After applying this conformal transformation to all the tensorial objects and scalars, the conformally transformed $f(T)$ action without matter content in the Einstein frame assumes the following form

$$\mathcal{S}_{EF} = \frac{1}{2\kappa^2} \int d^4x \hat{\theta} \left[-\hat{T} + \frac{2}{\mathcal{A}} \hat{T}^\mu \partial_\mu \mathcal{A} + \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - U(\psi) \right]. \quad (3.55)$$

It is crucial to note that the presence of the coupling term involving the scalar field and torsion vector in the second term of the action prevents a straightforward transition to the Einstein frame via the conformal transformation, at least not in the way that it occurs in $f(\hat{R})$ theory. A detailed discussion of conformal transformations in $f(T)$ gravity can be found in Refs. [50–52] and references therein.

3.5.3 Scalar-Nonmetricity Representation of $f(Q)$ Gravity

Following the approach outlined in Subsec. (3.5.1) and (3.5.2) the $f(Q)$ gravity action can be recast in an equivalent form by introducing a pair of auxiliary fields

$$S_{EQf(Q)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [\zeta(Q - \Phi) + f(\Phi)] + S_m(g). \quad (3.56)$$

Varying w.r.t. ζ leads to the constraint $Q = \Phi$, enforcing the action (3.56) is equivalent to (3.23), unless $f_{QQ} = 0$. Further varying the action (3.56) w.r.t. Φ gives $\zeta = f'(\Phi)$ and by redefining $f'(\zeta) = \mathcal{A}(\Phi)$, the action can be rewritten as

$$S_{EQf(Q)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (\mathcal{A}(\Phi)Q - (\Phi f'(\Phi) - f(\Phi))) + S_m(g), \quad (3.57)$$

with

$$V(\Phi) = \Phi f'(\Phi) - f(\Phi). \quad (3.58)$$

Now the most general scalar-nonmetricity action can be expressed as

$$S_{JF} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\mathcal{A}(\Phi)Q - \mathcal{B}(\Phi)g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - 2\mathcal{V}(\Phi) \right) + S_m(g). \quad (3.59)$$

Varying the action (3.59) w.r.t. metric, the metric field equations yield [53, 54]

$$\mathcal{A}(\Phi)\mathring{G}_{\mu\nu} + 2\mathcal{A}'(\Phi)P^\lambda{}_{\mu\nu}\partial_\lambda\Phi + \frac{1}{2}g_{\mu\nu}\left(\mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + 2\mathcal{V}(\Phi)\right) - \mathcal{B}(\Phi)\partial_\mu\Phi\partial_\nu\Phi = \kappa^2\mathcal{T}_{\mu\nu}, \quad (3.60)$$

the symmetric teleparallel connection equations

$$\left(\frac{1}{2}Q_\beta + \nabla_\beta\right)\left[\partial_\alpha\mathcal{A}\left(\frac{1}{2}Q_\mu g^{\alpha\beta} - \frac{1}{2}\delta_\mu^\alpha Q^\beta - Q_\mu{}^{\alpha\beta} + \delta_\mu^\alpha Q_\gamma{}^{\gamma\beta}\right)\right] = 0, \quad (3.61)$$

and the scalar field equation

$$2\mathcal{B}\mathring{\square}\Phi + \mathcal{B}'(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + \mathcal{A}'(\Phi)Q - 2\mathcal{V}'(\Phi) = 0. \quad (3.62)$$

Limiting cases

- When the scalar field is globally constant, Eqs. (3.60) reduces to Einstein's equations in GR with the value of the potential playing the role of the cosmological constant, while (3.61) and (3.62) immediately vanish.
- STEGR is recovered if

$$\mathcal{A}(\Phi) = 1, \quad \mathcal{B}(\Phi) = \mathcal{V}(\Phi) = 0. \quad (3.63)$$

- The $f(Q)$ gravity is recovered by setting

$$\mathcal{A}(\Phi) = f'(Q), \quad \mathcal{B}(\Phi) = 0, \quad 2\mathcal{V}(\Phi) = Qf'(Q) - f(Q). \quad (3.64)$$

By using the conformal transformation, in Einstein frame $f(Q)$ theory yields [53, 55, 56]

$$S_{EF} = \frac{1}{2} \int \sqrt{-\hat{g}} \left[\hat{Q} + \sqrt{\frac{2}{3}} \left(\hat{Q}_\sigma{}^{\sigma\mu} - \hat{Q}^{\mu\sigma}{}_\sigma \right) \partial_\mu\psi + \hat{g}^{\mu\nu} \partial_\mu\psi \partial_\nu\psi - 2U(\psi) \right] + S_m \left(e^{-\sqrt{\frac{2}{3}}\kappa\Psi} \hat{g}_{\mu\nu} \right), \quad (3.65)$$

where

$$\psi = \sqrt{\frac{3}{2}} \ln \Omega^2(x). \quad (3.66)$$

Similar to $f(T)$ gravity, the presence of a coupling term between gravity and the scalar field in the second term, the Einstein frame in $f(Q)$ theory is unable to cleanly separate the extra DOF from gravity.

3.6 Consideration of Cosmological Field Equations and Solutions

Having established the geometrical foundations, formulated the action, and derived the most general field equations, we now turn our attention to the cosmological solutions of the modified gravity theories that are discussed in the previous Sec. 3. The motivation for finding those solutions is crucial for understanding the late time accelerated expansion of the universe and the nature of dark energy. Cosmological solutions in any gravity theory are typically derived under the assumption of the cosmological principle (i.e., homogeneity and isotropy on large scales) and a suitable matter content consistent with this symmetry.

While this thesis focuses on teleparallel gravity, interested readers may refer to the extensive literature on cosmological solutions in metric $f(\mathring{R})$ [57–60], Palatini $f(R)$ [61, 62], and scalar-curvature theories [63] and references therein for a broader perspective. Unlike metric $f(\mathring{R})$ gravity, modified teleparallel theories incorporate the flat spin connection (flat affine connection) alongside the tetrad (metric) in field equations, where both of them are independent fields. The nonlinear $f(T)$ and $f(Q)$ have different cosmological solutions from each other unless working in specific gauge choices, such as the Weitzenböck gauge and coincident gauge, respectively, and their cosmological solutions have been extensively studied in [22, 24, 25, 64–67] and the references therein. A priori without fixing a gauge, different strategies are used in [30, 68] where it has been imposed that the tetrad (metric) shares the same symmetry with the spin connection (affine connection), with the aim of finding the most general spin connection (affine connection), which satisfies the teleparallel field equations.

To better understand the symmetric teleparallel case, it is instructive to briefly review the corresponding results in metric teleparallelism. In metric teleparallelism, three distinct branches of the connection have been identified that respect the imposed symmetry: one branch corresponds to zero spatial curvature, while the remaining two correspond to non-zero spatial curvature scenarios. Notably, all connection coefficients in metric teleparallelism are entirely determined by the tetrad field, implying that the connection does not introduce any additional dynamical DOF [30, 68].

We now focus on the symmetric teleparallel case. As demonstrated in Subsec. 2.4.1, three distinct connection branches fulfill the symmetry requirement. These branches are associated with spatial curvature, and all incorporate an additional function that consequently appears in the cosmological field equations. The cosmological equations derived from the first connection branch (2.43) are found to be identical to those obtained from the coincident gauge $f(Q)$ theory, and are often referred to as the trivial branch [28, 30], while the other two branches are known as the nontrivial connection branch. In our publication II and III,

we investigate analytical cosmological solutions within scalar-nonmetricity and $f(Q)$ gravity by employing a perturbative approach near GR cosmology, focusing specifically on the nontrivial connection branches, as detailed in Chapter 5.

Chapter 4

Hamiltonian Formalism in Teleparallel Theories of Gravity

Hamiltonian analysis is an important tool in understanding the viability of any physical theory, identifying all constraints, and predicting the correct number of DOF.¹ Furthermore, the analysis provides valuable insights into energy, entropy, and canonical quantum gravity from a Hamiltonian perspective, which is essential for understanding a physical system. For instance, interested readers may take a look Refs. [69–74], where the authors employed the Hamiltonian formalism in different modified theories of gravity to gain deeper insights into the theory. The application of this theory could also be extended to numerical relativity.

A linear perturbative analysis around a fixed background reveals both $f(T)$ and $f(Q)$ theories are strongly coupled,² as demonstrated in [27, 56], in the sense that the counting of perturbative modes does not match with the Hamiltonian analysis result [75, 76]. In this situation, one might look into higher-order perturbation theory or a careful Hamiltonian analysis to gain insights into the correct dynamics of the theory. Unfortunately, there is no clear result on the number of correct PDOF of $f(T)$ [72, 73, 77] and $f(Q)$ [25, 78, 79] theory, except the information of carrying some extra dynamics as compared to GR. This arising peculiarity might be due to the symmetry violations in those theories.

Let us examine some counting PDOF and GDOF that are confirmed from Hamiltonian analysis. GR, Coincident General Relativity (CGR) [80], andTEGR [81] are found to exhibit two PDOF and four GDOF. While $f(R)$ gravity introduces an additional PDOF [82], it preserves diffeomorphism symmetry, making it equivalent to a scalar-tensor theory. Recent work by [83] has clarified the possible

¹Determine the number of independent dynamical DOF contained in the canonical fields.

²When the interactions between particles within a system become so significant that they cannot be accurately described using linear perturbation theory, it becomes necessary to use higher order perturbation methods.

PDOF in $f(Q)$ theory and shown no GDOF exists, a result deviating the $f(Q)$ theory from the standard GR and $f(R)$. However, this remains a subject of debate, and the findings require further validation.

This thesis presents the derivation of Hamilton's equations for both covariant (including spin connection) and Weitzenböck gauge ofTEGR, and we argue that counting of PDOF is independent of gauge choice. To achieve this, we first formulated a covariant Hamiltonian, hence a basic understanding of the Dirac–Bergmann algorithm to count DOF and a 3+1 split of the Lagrangian to analyze the time evolution of canonical variables and momenta is needed, which is explained in the subsequent Subsec. 4.1 and 4.2.

4.1 Overview of Dirac–Bergmann Algorithm

This section gives a brief introduction to the Dirac–Bergmann algorithm, which provides a method for going from Lagrangian formalism to Hamiltonian formalism. Our main motivations for studying this algorithm are to classify between the constraints that help in counting the PDOF in modified theories of gravity and identifying the potential pathologies. Before going to review the Dirac–Bergmann algorithm, one important assumption often considered is that the Lagrangian only depends on first-order derivatives of the phase space variables. For Lagrangians of higher-order derivatives, it is necessary to generalize Dirac and Bergmann's recipe and consider the Ostrogradsky procedure for systems with higher derivatives, which has been much less studied in the literature and out of the scope of this thesis. The procedure can be summarized with the following steps:

1. The equations of motion for a physical system are derived by extremizing the action, defined by the Lagrangian $L(q^i, \dot{q}^i)$ as

$$S = \int_{t_1}^{t_2} L(q^i, \dot{q}^i) dt, \quad i = 1, \dots, N \quad (4.1)$$

where $q^i = q^i(t)$ defines the state of the system and $\dot{q}^i$ is the velocity.

2. Then, the Euler–Lagrange equations can be obtained by performing a variation of the dynamical variables that help in identifying the Hessian, which plays a crucial role in Hamiltonian formalism.
3. The canonical momenta are determined by taking the derivative of the Lagrangian with respect to the time derivatives of the fundamental fields

$$p_i(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}. \quad (4.2)$$

4. Then, from the definitions of canonical momenta, we can identify primary constraints as

$$\phi_\rho(q, p) = 0, \quad (4.3)$$

where the index ρ labels the primary constraints.

5. The Poisson bracket of two functions, F, G , in phase space, crucial for deriving Hamilton's equations, is defined as

$$\{F, G\} = \frac{\partial F}{\partial q^k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q^k}. \quad (4.4)$$

6. To derive the primary Hamiltonian (H_p), we begin by constructing the canonical Hamiltonian (H_c), defined as

$$H_c = \dot{q}^k p_k - L(q, \dot{q}), \quad (4.5)$$

where the first term is always written in the form of the product of velocities and the respective canonical momenta (for instance, the number of terms = number of primary constraints). Then, the primary Hamiltonian is expressed as

$$H_p = H_c + u^\rho \phi_\rho, \quad (4.6)$$

where u^ρ denote Lagrange multipliers.

7. Once the primary Hamiltonian is obtained, one can compute the Hamilton's equation of motion

$$\dot{q}_i = \frac{\delta H_p}{\delta q^i}, \quad \dot{p}_i = -\frac{\delta H_p}{\delta q^i}. \quad (4.7)$$

8. Ensuring the time evolution of the constraints in equation (4.6) is equivalent to enforcing that the Poisson bracket between the constraints and the primary Hamiltonian vanishes when evaluated on the constraint surface

$$\{H_p, \phi_\rho\} \stackrel{!}{\approx} 0. \quad (4.8)$$

This process can yield either identities or restrictions on the Lagrange multipliers u^ρ . If nontrivial results emerge from the Poisson bracket calculation, a secondary constraint is identified. These new constraints are then incorporated into the Hamiltonian, and the process is repeated. This iterative procedure continues until no new constraints are generated.

9. Though the classifications of primary, secondary, and tertiary constraints do not give any physical insights, in fact, the identifications of first-class and second-class are useful for counting the number of DOF. A phase space function $\mathcal{F}(q, p)$ is classified as a first-class if its Poisson brackets with the remaining constraints weakly vanish, i.e.,

$$\{\mathcal{F}(q, p), \phi_\rho\} \approx 0, \quad (4.9)$$

if not satisfied (4.9) then it is second class.

10. First-class constraints are related to gauge symmetries and manifest the redundancy in the description of the system, but a gauge transformation is not necessarily always linked to the existence of a corresponding first-class constraint. However, the second-class constraints represent physical restrictions and do not correspond to gauge symmetries.
11. After arriving at this stage, the number of physical DOF can be readily calculated using the following formula:

$$\begin{aligned} \text{Number of physical DOF} = & \frac{1}{2} (\text{No. of canonical variables} - \\ & \text{No. of second class constraint} - \\ & 2 \times \text{No. of first-class constraint}) \end{aligned} \quad (4.10)$$

A remark here is that the aforementioned procedure has been described for mechanical systems, but we have used its (not so straightforward) extension to field theory. However, in our article 7, the algorithm is used until the 7th step, as our primary objective was to derive Hamilton's equations within the framework of covariant TEGR.

4.2 3+1 Decomposition

An Arnowitt–Deser–Misner (ADM) split is usually adopted for performing a 3+1 decomposition, which is useful for separating the time evolution of the gravitational field; this split parametrizes the spacetime into three spatial hypersurfaces with constant time slices and a normal vector to the hypersurfaces. The ADM variable consists of a three-dimensional intrinsic metric³ γ_{ij} on the hypersurfaces of constant time slices and the normal vector ξ^μ , whose components are written in terms of new variable lapse α and shift β^i as

$$\xi^0 = -\frac{1}{\alpha}, \quad \xi^i = \frac{\beta^i}{\alpha}, \quad (4.11)$$

³Some literature also calls it induced metric.

along with these variables the covariant ADM metric

$$g_{00} = -\alpha^2 + \beta^i \beta^j \gamma_{ij}, \quad g_{0i} = \beta_i, \quad g_{ij} = \gamma_{ij}, \quad (4.12)$$

and the inverse metric is expressed as

$$\alpha^2 g^{00} = -\frac{1}{\alpha^2}, \quad g^{0i} = \frac{\beta^i}{\alpha^2}, \quad g^{ij} = \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2}. \quad (4.13)$$

The teleparallel fundamental variable, such as tetrad, is split into the temporal and spatial parts as

$$\theta^A_0 = \alpha \xi^A + \beta^i \theta^A_i, \quad (4.14)$$

in the same manner, the cotetrad is defined as

$$e_A^0 = -\frac{1}{\alpha} \xi_A, \quad e_A^i = \theta_A^i + \xi_A \frac{\beta^i}{\alpha}, \quad (4.15)$$

satisfying the relation

$$\theta^A_i \theta^B_j \eta_{AB} = \gamma_{ij}, \quad (4.16)$$

along with the following orthogonal

$$\eta_{AB} \xi^A \xi^B = \xi_A \xi^A = -1, \quad (4.17)$$

and orthonormal relation

$$\eta_{AB} \xi^B \theta^A_i = \xi_A \theta^A_i = 0. \quad (4.18)$$

In order to recover the ADM metric from the tetrad (4.14), the normal vector needs to satisfy the condition

$$\xi^A = -\frac{1}{6} \epsilon^A_{BCD} \theta^B_i \theta^C_j \theta^D_k \epsilon^{ijk}. \quad (4.19)$$

4.3 The Impact of Gauge Selection on Teleparallel Equivalent to General Relativity

The minimal way to work with the Weitzenböck gauge and including spin connection (covariant formalism) in TEGR has been discussed in Sec. 3.2. As we pointed out before, in the Weitzenböck gauge, the TEGR lagrangian is pseudo-invariant up to a boundary term; hence the theory is not covariant under LLTs, although it is easier to work with, and many Hamiltonian analyses have been done with this gauge choice [73, 84]. The advantage of not imposing gauge fixing is that the theory is fully covariant and the opportunity to study the impact of the physical role of the extra fields (Lorentz matrices) in the Hamiltonian analysis.

In Chapter 7, we argued the gauge choice is independent of the counting DOF obtained from the Dirac–Bergmann algorithm. In Hamiltonian analysis, though there are different fields being treated as fundamental, in our approach, both tetrad and Lorentz matrices are treated as fundamental. From the beginning, the tetrad has 16 independent components, along with six extra from the spin connection. Furthermore, Chapter 7 includes a brief discussion of the final counting of physical DOF.

4.4 Hamiltonian Analysis for Teleparallel Equivalent to General Relativity

Employing the 3+1 decomposition defined in Sec. 4.2, the TEGR Lagrangian can be expressed as

$$\begin{aligned} \mathcal{L}_{\text{TEGR}} = & \frac{\sqrt{\gamma}}{2\alpha} M^i{}_A{}^j{}_B T^A{}_{0i} T^B{}_{0j} - \frac{\sqrt{\gamma}}{\alpha} T^A{}_{0i} T^B{}_{kl} \cdot \left[M^i{}_A{}^l{}_B \beta^k \right. \\ & \left. - \frac{\alpha}{\kappa} \gamma^{il} \left(\frac{1}{2} \xi_B \theta_A{}^k - \xi_A \theta_B{}^k \right) \right] + L_S, \end{aligned} \quad (4.20)$$

while the tensor $M^i{}_A{}^j{}_B$ that multiplies the quadratic velocity terms is given by

$$M^i{}_A{}^j{}_B = \frac{1}{\kappa} \left(\frac{1}{2} \gamma^{ij} \eta_{AB} + \frac{1}{2} \xi_A \xi_B \gamma^{ij} + \frac{1}{2} \theta_A{}^j \theta_B{}^i - \theta_A{}^i \theta_B{}^j \right), \quad (4.21)$$

and the spatial term of the Lagrangian, collectively denoted L_S , is explicitly expressed as

$$L_S = \frac{\sqrt{\gamma}}{\alpha} T^A{}_{ij} T^B{}_{kl} \beta^i \left[\frac{1}{2} M^j{}_A{}^l{}_B \beta^k - \frac{\alpha}{\kappa} \gamma^{jl} \left(\frac{1}{2} \xi_B \theta_A{}^k - \xi_A \theta_B{}^k \right) \right] + \frac{\alpha \sqrt{\gamma}}{2\kappa} {}^3\mathbb{T}, \quad (4.22)$$

where the notation ${}^3\mathbb{T}$ explicitly represents

$$\begin{aligned} {}^3\mathbb{T} = & H_{AB} {}^{ijkl} T^A{}_{ij} T^B{}_{kl} \\ = & \left(-\frac{1}{4} \eta_{AB} \gamma^{k[i} \gamma^{j]l} + \frac{1}{2} \theta_B{}^{[i} \gamma^{j][k} \theta_A{}^{l]} - \theta_A{}^{[i} \gamma^{j][k} \theta_B{}^{l]} \right) T^A{}_{ij} T^B{}_{kl}. \end{aligned} \quad (4.23)$$

We found that the conjugate momenta associated with the lapse and shift form part of primary constraints

$${}^\alpha \pi = \frac{\partial L}{\partial \partial_0 \alpha} \approx 0, \quad (4.24)$$

$${}^\beta \pi_i = \frac{\partial L}{\partial \partial_0 \beta^i} \approx 0, \quad (4.25)$$

and a new momenta appeared associated with the Lorentz matrix of the spin connection denoted by

$$P^B{}_A = \frac{\partial L}{\partial_0 \Lambda^A{}_B} = \pi_C^i \eta_{AD} (\Lambda^{-1})^B{}_E \eta^{C[E} \theta^{D]}{}_i, \quad (4.26)$$

and these primary constraints read

$$C^{AB} = P^{[A}{}_D \eta^{B]C} \Lambda_C{}^D + \pi_C^i \eta^{C[B} \theta^{A]}{}_i \approx 0. \quad (4.27)$$

The conjugate momenta with respect to the time evolution of the tetrad, derived from the Lagrangian, are given by

$$\begin{aligned} \pi_A{}^i &= \frac{\partial L}{\partial_0 \theta^A{}_\mu} \\ &= \frac{\sqrt{\gamma}}{\alpha} \left[M^i{}_A{}^l{}_B (T^B{}_{0l} - T^B{}_{ml} \beta^m) + \frac{\alpha}{\kappa} T^B{}_{ml} \gamma^{il} \left(\frac{1}{2} \xi_B \theta_A{}^m - \xi_A \theta_B{}^m \right) \right]. \end{aligned} \quad (4.28)$$

4.4.1 Irreducible Decomposition

To construct the Hamiltonian presented in the following Subsec. 4.4.2, it is necessary to express all velocities in terms of their conjugate momenta. This is achieved by employing an irreducible decomposition technique, where velocities are decomposed into vectorial, antisymmetric, symmetric, and trace-free components under the rotation group. This approach facilitates the calculation of the Moore–Penrose pseudoinverse, which is essential for this process. The velocities of the tetrad are decomposed as

$$\dot{\theta}^A{}_i = \mathcal{V} \dot{\theta}_i \xi^A + \mathcal{A} \dot{\theta}_{ji} \gamma^{kj} \theta^A{}_k + \mathcal{S} \dot{\theta}_{ji} \gamma^{kj} \theta^A{}_k + \mathcal{T} \dot{\theta} \theta^A{}_i, \quad (4.29)$$

and the conjugate momenta is,

$$\pi_A{}^i = \mathcal{V} \pi^i \xi_A + \mathcal{A} \pi^{ji} \theta^B{}_j \eta_{AB} + \mathcal{S} \pi^{ji} \theta^B{}_j \eta_{AB} + \mathcal{T} \pi \theta^B{}_j \eta_{AB} \gamma^{ij}. \quad (4.30)$$

With this irreducible decomposition at hand, the TEGR primary constraints are obtained from (4.28) as

$$\mathcal{V} C^i = \frac{\mathcal{V} \pi^i \kappa}{\sqrt{\gamma}} + T^B{}_{jk} \gamma^{ik} \gamma^{jl} \theta^A{}_l \eta_{AB} \approx 0, \quad (4.31)$$

and

$$\mathcal{A} C^{ij} = \frac{\mathcal{A} \pi^{ij} \kappa}{\sqrt{\gamma}} - \frac{1}{2} \gamma^{ik} \gamma^{jl} T^B{}_{kl} \xi_B \approx 0. \quad (4.32)$$

Following the decomposition, the Moore–Penrose inverse can be expressed as

$$(M^{-1})^A{}_i{}^C{}_k = \frac{\kappa}{2} (\gamma_{ik} \gamma^{mn} \theta^A{}_m \theta^C{}_n + \theta^A{}_k \theta^C{}_i - \theta^A{}_i \theta^C{}_k). \quad (4.33)$$

4.4.2 Hamiltonian for TEGR

After some computations, we obtain the primary Hamiltonian for covariant TEGR:

$$\begin{aligned}
 H_{\text{TEGR}} = & \alpha \left[\frac{\kappa}{2\sqrt{\gamma}} \mathcal{S} \pi_{ij} \mathcal{S} \pi^{ij} - \frac{3\kappa}{4\sqrt{\gamma}} \mathcal{T} \pi \mathcal{T} \pi - \frac{\sqrt{\gamma}}{2\kappa} {}^3\mathbb{T} - \xi^A \partial_i \pi_A^i + \pi_A^i \omega^A{}_{B i} \xi^B \right] \\
 & + \beta^j \left[-\theta^A{}_j \partial_i \pi_A^i + \pi_A^i \omega^A{}_{C i} \theta^C{}_j - \pi_A^i T^A{}_{ij} \right] - \lambda^A C_A \\
 & - \lambda_{AB} \left(P^{[A}{}_D \eta^{B]C} \Lambda_C{}^D + \pi_C^i \eta^{C[B} \theta^{A]}{}_i \right) \\
 & - \nu \lambda_i \left(\frac{\nu \pi^i \kappa}{\sqrt{\gamma}} + T^B{}_{jk} \gamma^{ik} \gamma^{jl} \theta^A{}_l \eta_{AB} \right) \\
 & - {}^A \lambda_{ij} \left(\frac{{}^A \pi^{ij} \kappa}{\sqrt{\gamma}} - \frac{1}{2} \gamma^{ik} \gamma^{jl} T^B{}_{kl} \xi_B \right).
 \end{aligned} \tag{4.34}$$

In the Weitzenböck gauge the expression for the Hamiltonian reduces to

$$\begin{aligned}
 H_{\text{TEGR}} = & \alpha \left[\frac{\kappa}{2\sqrt{\gamma}} \mathcal{S} \pi_{ij} \mathcal{S} \pi^{ij} - \frac{3\kappa}{4\sqrt{\gamma}} \mathcal{T} \pi \mathcal{T} \pi - \frac{\sqrt{\gamma}}{2\kappa} {}^3\mathbb{T} - \xi^A \partial_i \pi_A^i \right] - \lambda^A C_A \\
 & + \beta^j \left[-\theta^A{}_j \partial_i \pi_A^i - \pi_A^i T^A{}_{ij} \right] - \nu \lambda_i \left(\frac{\nu \pi^i \kappa}{\sqrt{\gamma}} + T^B{}_{jk} \gamma^{ik} \gamma^{jl} \theta^A{}_l \eta_{AB} \right) \\
 & - {}^A \lambda_{ij} \left(\frac{{}^A \pi^{ij} \kappa}{\sqrt{\gamma}} - \frac{1}{2} \gamma^{ik} \gamma^{jl} T^B{}_{kl} \xi_B \right).
 \end{aligned} \tag{4.35}$$

Starting from the Hamiltonian (4.34) and (4.35), we derived Hamilton's equations in both the covariant formalism and the Weitzenböck gauge in TEGR. This is the primary objective of our article, as we aimed to investigate the role of the additional dynamical field associated with the spin connection in Hamilton's equations. Given the extensive length of these equations, we refer interested readers to our attached article in Chapter 7 for detailed calculations.

4.4.3 Outcomes

In the attached publication in Chapter 7, we discussed and commented on some existing research done in the Hamiltonian formulation of TEGR. Compared to the work done in the literature, we argue the advantages of our formalism being gauge-independent and covariant. Because the problem that occurs in TEGR is that there are no antisymmetric field equations while there is antisymmetric evolution [85]. Thus, it is unclear what one should do with the antisymmetric evolution equations. So, this kind of difficulty is avoided in our approach. This is the first

time that it has been presented a novel derivation of Hamilton equations for the covariant version of TEGR, with an arbitrary spin connection depending on Lorentz matrices as well as the Hamilton equations in Weitzenböck gauge.

We observed two main differences between the (covariant) TEGR Hamilton equations: the first difference is related to which fields are treated as canonical. Again, tetrads have six additional independent components compared to the metric, and the spin connection (or rather the Lorentz matrices) introduces six more independent components [86]. Hence, Hamilton's equations exist for all of the canonical fields and their conjugate momenta. The second main difference is that Hamilton's equations for TEGR are more numerous than those in the standard formulation of GR. Despite these discrepancies, both (covariant) TEGR and GR exhibit the same number of DOF: four DOF in the phase space, equivalent to two physical DOF in the configuration space. This equivalence demonstrates that the gauge choice does not affect the counting of physical DOF provided a careful implementation of all the computations.

Chapter 5

Cosmological Stability of the Alternative Connection in Symmetric Teleparallel Theories

The second part of this research is motivated by the recent growing interest in symmetric teleparallel theory, driven by both theoretical [29, 67] and observational [87–89] considerations, particularly in response to the challenges of ghosts [27] and strong coupling [56, 78, 90] in $f(Q)$ gravity. Unlike TEGR and STEGR cases, the extra bits of connection in modified symmetric teleparallel (3.23) and scalar-nonmetricity (3.59) actions now have their own equation, as discussed in Sec. 3.3 and Subsec. 3.5.3. This raises the issue of how to analytically solve the combined cosmological equations to find solutions. We addressed this in the attached publications in Chapter 8 and 9.

There have been few studies within the nontrivial connection in the symmetric teleparallel theory: some of them have explored specific cosmological solutions and analyzed their properties [91–93], phase space features via fixed point analysis [94, 95], and constraints from observational data [96, 97]. Some research even suggests that these alternative connections fit the observational Hubble data better than the Λ CDM and other models [89]. However, these studies typically consider specific model functions. In our publications II and III, the analytical cosmological solutions are derived, and the cosmological stability has been analyzed near the vicinity of the GR cosmology. We keep the model functions arbitrary to encompass a wide range of models. Furthermore, in III, we argued why so many model solutions perform better in fitting with observational data. Recently, Ref. [98] derived the exact analytical Λ CDM cosmological solutions by reconstructing the model function and studied the stability of the solution in the nontrivial connection sets, which is a subcase of our analysis and confirmed our findings.

In this work, we investigate the stability of symmetric teleparallel cosmology

in an FLRW background with a nontrivial connection. The stability is analyzed near the vicinity of the GR limit (a limit under which the modified cosmological equations approach GR, discussed in later Sec. 5.2) across all the evolutionary epochs of the universe, starting from radiation to matter dominated eras to dark energy dominated eras, by using a perturbative technique. Our aim is to check whether a solution of these theories smoothly approaches the GR limit as an attractor while all perturbations decay over time. For that, first, we aim to find the analytical cosmological solution by employing a linear perturbation around the GR limit. Due to the strong coupling in $f(Q)$, we anticipate encountering some kind of instability in our cosmological solutions. Given the success of GR in describing phenomena such as the Solar System [99], astrophysical and cosmological tests [100], the prediction and detection of gravitational waves, Big Bang nucleosynthesis, etc., this nearby regime is of particular importance for analyzing the stability of any theory.

5.1 Symmetric Teleparallel Cosmology

In this section, we aim to present and analyze the cosmological equations of the most general scalar-nonmetricity action (3.59) by using metric field equations (3.60), connection field equations (3.61) and the scalar field equation (3.62). We then subsequently discuss the independent dynamics of the physical system and some potential instability by analyzing the scalar-nonmetricity cosmological equations and the $f(Q)$ cosmological equations (a limiting case of the generalized scalar-nonmetricity theory, as pointed out in Sec. 3.5.3).

Let us assume a perfect fluid matter energy-momentum tensor consistent with the assumed symmetries (2.40) given by

$$\mathcal{T}_{\mu\nu} = \text{diag}(\rho(t), \quad a^2(t)p(t), \quad r^2 a^2(t)p(t), \quad r^2 a^2(t)p(t) \sin^2 \theta), \quad (5.1)$$

with the barotropic equation of state

$$p = w\rho, \quad (5.2)$$

where w is the equation of state parameter, and the scalar field depends on the time

$$\Phi = \Phi(t). \quad (5.3)$$

Upon substituting the metric (2.41), connection set 1 (2.43), perfect fluid matter (5.1), scalar field (5.3) into the field equations (3.60), (3.61) and (3.62) yields two nontrivial metric equations, scalar field equation, and matter continuity equation as

follows

$$6H^2\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho, \quad (5.4a)$$

$$-4H\dot{\Phi}\mathcal{A}'(\Phi) - (6H^2 + 4\dot{H})\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2 w\rho, \quad (5.4b)$$

$$-6H^2\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0, \quad (5.4c)$$

$$\dot{\rho} + 3H(1 + w)\rho = 0. \quad (5.4d)$$

Compared to scalar-curvature cosmology [101], the main difference lies in the absence of matter sources from the scalar field equation (5.4c), provided that, for minimally coupled theories, the equations fully coincide. Moreover, the above Eqs. (5.4) also coincide with the scalar-torsion cosmological equations [40]. In this branch, the connection equations (3.61) are trivially satisfied, leaving $\gamma(t)$ completely arbitrary by the field equations. Upon expressing the above cosmological equations into a system of first-order differential equations, we observed that the phase space is characterized by three independent variables (Φ, ρ, H) .

Substituting the FLRW metric (2.41), connection set 2 (2.45), matter (5.1) and scalar field (5.3) into the field equations (3.60), (3.61) and (3.62) yields two nontrivial metric equations (3.60), one nontrivial connection equation (3.61), the scalar field equation (3.62) and the matter continuity equation. These equations are given by:

$$3\dot{\Phi}\gamma\mathcal{A}'(\Phi) + 6H^2\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho, \quad (5.5a)$$

$$(3\dot{\Phi}\gamma - 4H\dot{\Phi})\mathcal{A}'(\Phi) - (6H^2 + 4\dot{H})\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2 w\rho, \quad (5.5b)$$

$$3\gamma(\ddot{\Phi}\mathcal{A}'(\Phi) + 3H\dot{\Phi}\mathcal{A}'(\Phi) + \dot{\Phi}^2\mathcal{A}''(\Phi)) = 0, \quad (5.5c)$$

$$(-6H^2 + 9H\gamma + 3\dot{\gamma})\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0, \quad (5.5d)$$

$$\dot{\rho} + 3H(1 + w)\rho = 0. \quad (5.5e)$$

Notice, the connection function $\gamma(t)$ appears in the equations only in association with derivatives of $\mathcal{A}(\Phi)$, thus affecting only nonminimally coupled theories. Comparing the γ terms in Eqs. (5.5a) and (5.5b) with the ρ terms reveals an effective barotropic index of +1 for γ , rather than the expected -1 or 0 . Consequently, it can not contribute to effective dark matter or dark energy in the cosmological equations. Furthermore, we observe that though the connection equation (5.5c) does not provide dynamics for the independent connection function γ , it does

influence the scalar field dynamics to

$$\ddot{\Phi} = -3H\dot{\Phi} - \frac{\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)}\dot{\Phi}^2. \quad (5.6)$$

This equation effectively acts as a constraint and helps to reduce the system of equations to four independent variables: $\{\Phi, \rho, H, \gamma\}$. To analyze scalar field dynamics, we draw an analogy to a mechanical equation in physics where the l.h.s. of (5.6) is like acceleration, while the r.h.s. is like the total force acting on the system. The first term of the force acts like friction, opposing the motion, while the second term can act as extra friction or anti-friction, depending on the alignment of signs. Now, let us extract the stability information by simply analyzing the simplified connection equation (5.6). For lower speed $\dot{\Phi}$, all solutions are convergent to some arbitrary value of Φ . However, sufficiently high speeds lead to the dominance of the second term, potentially causing the acceleration and, thus, the speed itself to approach infinity, hence inducing possible instability. Unstable solutions could also arise near the extrema of the gravitational coupling function at some value of Φ_s .

Finally, upon substituting the FLRW metric (2.41), connection set 3 (2.47), matter (5.1) and scalar field (5.3) to the field equations (3.60), (3.61) and (3.62) yields the two nontrivial metric equations (3.60), one nontrivial connection equation (3.61), the scalar field equation (3.62) and the matter continuity equation as

$$6H^2\mathcal{A}(\Phi) - 3\bar{\gamma}\dot{\Phi}\mathcal{A}'(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho, \quad (5.7a)$$

$$(\bar{\gamma}\dot{\Phi} - 4H\dot{\Phi})\mathcal{A}'(\Phi) - (6H^2 + 4\dot{H})\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2 w\rho, \quad (5.7b)$$

$$-6\dot{\bar{\gamma}}\dot{\Phi}\mathcal{A}'(\Phi) - 3\bar{\gamma}\left(\ddot{\Phi}\mathcal{A}'(\Phi) + 5H\dot{\Phi}\mathcal{A}'(\Phi) + \dot{\Phi}^2\mathcal{A}''(\Phi)\right) = 0, \quad (5.7c)$$

$$(-6H^2 + 9H\bar{\gamma} + 3\dot{\bar{\gamma}})\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0, \quad (5.7d)$$

$$\dot{\rho} + 3H(1 + w)\rho = 0. \quad (5.7e)$$

The cosmological equations for connection set 2 (5.5) share a similar structure with those for connection set 3 (5.7), allowing the previous analysis to be readily extended. In contrast to the set 2 case, the $\bar{\gamma}$ term in (5.7a) carries an opposite sign. Nevertheless, $\bar{\gamma}$ still cannot contribute to effective dark matter or dark energy, as the associated effective barotropic index in Equations (5.7a)–(5.7b) is $-\frac{1}{3}$, analogous to spatial curvature. Like connection set 2, the above system (5.7) exhibits a four-dimensional phase space with four independent dynamical variables $\{\Phi, \rho, H, \bar{\gamma}\}$.

Furthermore, the connection equation can be expressed as

$$\ddot{\Phi} = - \left(\frac{2\dot{\gamma}}{\bar{\gamma}} + 5H \right) \dot{\Phi} - \frac{\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)} \dot{\Phi}^2, \quad (5.8)$$

containing an additional friction term; hence, all the remarks about the possible instabilities considerations are applicable here as well.

While analyzing the stability of scalar-nonmetricity cosmology around the GR limit, without loss of generality, we introduce a reparameterized scalar field $\phi(\Phi)$ that preserves all physical measurable quantities. For simplicity, we choose a parametrization that yields the kinetic energy, i.e., $\mathcal{B}(\phi) \equiv 1$. Furthermore, we express the nonminimal coupling to gravity as $\mathcal{A}(\phi) = 1 + f(\phi)$, separating out a constant term, and redefine the potential as $V(\phi)$ in terms of the reparameterized field.

5.1.1 $f(Q)$ cosmology

As demonstrated in Sec. 3.5.3, by employing the identification (3.64) and utilizing a parameterization

$$f(Q) = Q + F(Q), \quad (5.9)$$

without loss of generality, it can be easily verified that each of the three sets of scalar-nonmetricity cosmological equations (5.4), (5.5) and (5.7) reduce to the corresponding branch of the $f(Q)$ cosmological equations, which is the starting point of the publication III. The primary motivation for investigating the independent dynamics of $f(Q)$ cosmological equations lies in elucidating how the additional scalar degree of freedom inherent in scalar-nonmetricity theories can be realized in $f(Q)$ cosmology.

In connection set 1, $f(Q)$ and $f(T)$ cosmologies with identical symmetry share the same background equations, although the evolution of perturbations could differ. We verified that the cosmological dynamics in connection set 1 can be expressed in terms of a single independent dynamical variable (either ρ or H) in the phase space, like in $f(T)$. Depending on the specific form of the function $F(Q)$, the system of differential equations may encounter a singularity where $\dot{H}$ diverges [102].

Unlike connection set 1, the nonmetricity scalar (2.46) in set 2 can vanish or remain constant for any Hubble function $H(t)$ depending on $\gamma(t)$. This allows the cosmological equations to reduce to GR, or to GR with a cosmological constant while the connection equation is identically satisfied. Notice that the nonmetricity scalar Q in set 2 contains additional dynamics of γ that prevent the Friedmann equation from being a constraint equation. Notably, while keeping arbitrary model functions, the system is characterized by four independent dynamical variables: $\{\gamma, \dot{\gamma}, H, \rho\}$, defining a four-dimensional phase space. Analysis of the equations,

particularly the Hubble function's derivative, reveals potential singularities when γ approaches zero or $1 + F_Q$ vanishes. The latter agrees with the general physical viability condition discussed in Sec. 3.3. The connection equation can be equivalently written as

$$\ddot{Q} = -3H\dot{Q} - \frac{F_{QQQ}}{F_{QQ}}\dot{Q}^2, \quad (5.10)$$

demonstrating a structural similarity to the scalar-nonmetricity connection equation (5.6); indeed, a similar instability conclusion can be drawn, as follows from the previous explanation.

Analogous to the nonmetricity scalar (2.46) in the connection set 2, the nonmetricity scalar (2.48) in set 3 exhibits a similar structure. In cosmology, when (2.48) vanishes or is constant, the cosmological equations recover GR or GR with a cosmological constant, with the connection equation being identically satisfied. The system, represented by the four independent variables with $\{\bar{\gamma}, \dot{H}, \rho, \bar{\gamma}\}$ spans a four-dimensional phase space (to visualize, please refer to Chapter 9). Just looking at the system of equations reveals potential singularities when $\bar{\gamma}$ approaches zero or $1 + F_Q$ vanishes, likely leading to a divergence in the Hubble parameter and thus a singularity in the system. The connection equation is expressible as

$$\ddot{Q} = -\left(\frac{2\dot{\bar{\gamma}}}{\bar{\gamma}} + 5H\right)\dot{Q} - \frac{F_{QQQ}}{F_{QQ}}\dot{Q}^2, \quad (5.11)$$

exhibits a structure similar to that of Eq. (5.8), suggesting analogous instability behavior.

5.2 General Relativity Limit

The Friedmann equations for a spatially flat universe containing a single barotropic fluid matter component in GR are

$$3H_*^2 = 8\pi G_N \rho_* + \Lambda, \quad (5.12a)$$

$$2\dot{H}_* + 3H_*^2 = -8\pi G_N w \rho_* + \Lambda, \quad (5.12b)$$

$$\dot{\rho}_* + 3H_*(1 + w)\rho_* = 0, \quad (5.12c)$$

where G_N represents the Newtonian gravitational constant and Λ denotes the cosmological constant.

For non-relativistic matter (dust) with an equation of state parameter $w = 0$, solving Eqs. (5.12), yields the Hubble function and matter energy density

$$H_*(t) = \frac{2}{3(t - t_s)}, \quad \rho_*(t) = \frac{4}{3\kappa^2(t - t_s)^2}. \quad (5.13)$$

For relativistic matter (radiation), with an equation of state parameter $w = \frac{1}{3}$ the Hubble function and energy density evolve as

$$H_*(t) = \frac{1}{2(t - t_s)}, \quad \rho_*(t) = \frac{3}{4\kappa^2(t - t_s)^2}, \quad (5.14)$$

where t_s is an integration constant, representing the moment of the initial singularity, and will be set to zero for simplicity.

For a cosmological constant (Λ) dominating the universe $w = 0$ with the absence of matter and radiation, the Hubble parameter and energy density are given by

$$H_*(t) = \pm \sqrt{\frac{\kappa^2 \Lambda}{3}}, \quad \rho_*(t) = 0. \quad (5.15)$$

Alternatively, considering dark energy as a fluid with constant energy density with an equation of state $w = -1$, the Hubble parameter and energy density are expressed as

$$H_*(t) = \pm \sqrt{\frac{\kappa^2 \Lambda_{DE}}{3}}, \quad \rho_* = \Lambda_{DE}. \quad (5.16)$$

5.3 Methodology

This section provides a brief overview of the common methodology employed in the publications II and III.

1. The initial step involves identifying the conditions under which the modified cosmological field equations reduce to the standard GR (5.12), referred to as the GR limit.
2. To see whether a small perturbation to the limit of GR is dynamically stable, first, we perturbed all independent variables involved in the cosmological equations around the GR cosmology. For instance, in scalar-nonmetricity cosmology, this involves perturbing the scalar field, Hubble function, matter-energy density, and the connection function as follows:

$$\begin{aligned} \phi(t) &= \phi_* + x(t), & H(t) &= H_*(t) + h(t), \\ \gamma(t) &= \gamma_*(t) + g(t), & \rho(t) &= \rho_*(t) + r(t), \end{aligned} \quad (5.17)$$

where ϕ_* is constant, and $x(t)$, $h(t)$, $g(t)$, $r(t)$ represent small perturbations assumed to be of the same order of magnitude. A Taylor series is often

employed to expand the generic functions (here it is gravitational coupling and potential function) around the constant scalar field

$$f(\phi(t)) = f_* + f'_*x(t) + \frac{f''_*(x(t))^2}{2}, \quad (5.18a)$$

$$V(\phi(t)) = V_* + V'_*x(t) + \frac{V''_*(x(t))^2}{2}. \quad (5.18b)$$

Analogously, in $f(Q)$ cosmology along with the above (5.17), the independent variable is the nonmetricity scalar instead of being the scalar field, and it perturbed as

$$Q(t) = Q_*(t) + q(t), \quad (5.19)$$

as $q(t)$ is a small perturbation, and the arbitrary model function is expanded as

$$F(Q(t)) = F^*(t) + F_Q^*(t)q(t) + \frac{1}{2}F_{QQ}^*(t)q(t)^2. \quad (5.20)$$

3. To derive an approximate field equation and analyze the behavior of the perturbed functions in each evolutionary epoch, the expansions Eqs. (5.17), (5.18), Eqs. (5.19) and (5.20) are substituted into the respective cosmological equations for scalar-nonmetricity and $f(Q)$ gravity.
4. Then, we proceeded to solve the approximate field equations analytically, order by order, encompassing all cosmological phases of the universe as outlined in (5.13), (5.14), (5.15), (5.16).
5. After finding all the perturbations, to investigate an era to be stable, all perturbed functions should exhibit a decrease with time. Furthermore, the presence of constant perturbations necessitates further investigation as they may potentially induce a system collapse towards a singularity.
6. While analyzing the connection equation, we drew some conclusions regarding possible instability. Hence, we expect to see some unstable solutions at the perturbative level.

For a detailed description of how the aforementioned steps are applied to analyze the stability of symmetric teleparallel theory on specific cosmological backgrounds, please refer to the publications attached in Chapters 8 and 9. In the following, we discuss the conditions under which the general function leads to successful stability eras around GR within the framework of scalar-nonmetricity theory.

5.4 Stability of Scalar-Nonmetricity Cosmology

The fundamental geometric structure underlying the symmetric teleparallel gravity studied in the second publication II is provided in Sec. 2 with the action (3.59), the nontrivial connection is presented in Subsec. (2.4) and the corresponding cosmological field equations are given in Sec. 5.1. To encompass a broad range of models, we keep nonminimal coupling and potential function arbitrary.

5.4.1 Conditions on the Generic Scalar Functions and Analyzing the Stability of Each Era

This subsection summarizes the key findings, which consist of the restrictions on the model functions to ensure the stability of the standard cosmological sequence: radiation domination, followed by dust matter domination and scalar potential domination.

Connection set 1:

To recover standard GR, a necessary condition found from the background scalar field equation is a simultaneous occurrence of critical points for the derivatives of the gravitational coupling, f'_* , and the potential, V'_* , at the same value of ϕ_* . Our results demonstrate that for the models at the simultaneous critical point, all perturbations near the GR limit are stable, i.e. solutions eventually approach GR. This confirms the successful recovery of a stable standard cosmic evolution of the universe.

Connection set 2:

In this branch, the background scalar field cosmological equation for matter, radiation, and potential dominated era demands three options. While finding better stability in the solutions, sometimes we consider that the second derivative of the potential also vanishes. The stability results for all the cosmic epochs for long run time are summarized in the given Table 5.1 and finite time singularities are discussed in the Chapter 8. The blue color indicates that, although the background connection grows over time, its coupling with the evolving scalar field mitigates its influence in the cosmological field Eqs. (5.5), preventing unphysical behavior. The red color is used to denote unstable or marginally stable perturbations, while the magenta color is used to represent undetermined perturbations where we cannot conclude whether the respective dominating era is stable.

Near the simultaneous critical point, defined by

$$f'_* = 0, \quad V'_* = 0, \quad (5.21)$$

Cases	Matter domination	Radiation domination	Potential domination
$f'_* \neq 0$, $V'_* \neq 0$	$\gamma_*(t) \sim t$, $x(t) \sim t^{-1}$, $h(t) \sim t^0$, $g(t) \sim t^2$, $r(t) \sim t^{-1}$	$\gamma_*(t) \sim t$, $x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{\frac{1}{2}}$, $g(t) \sim t^{\frac{5}{2}}$, $r(t) \sim t^{-\frac{1}{2}}$	$\gamma_*(t) \sim t^0$, $x(t) \sim e^{-3H_*t}$, $h(t) \sim e^{-3H_*t}$, $g(t) \sim te^{-3H_*t}$, $r(t) \sim e^{-3H_*t}$
$f'_* \neq 0$, $V'_* = 0$	$\gamma_*(t) \sim t^{-1}$, $x(t) \sim t^{-1}$, $h(t) \sim t^{-2}$, $g(t) \sim t^0$, $r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^{-1}$, $x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{-2} \ln t$, $g(t) \sim t^{\frac{1}{2}}$, $r(t) \sim t^{-3} \ln t$	$\gamma_*(t) \sim t^0$, $x(t) \sim e^{-3H_*t}$, $h(t) \sim e^{-3H_*t}$, $g(t) \sim te^{-3H_*t}$, $r(t) \sim e^{-3H_*t}$
$f'_* \neq 0$, $V'_* = V''_* = 0$	$\gamma_*(t) \sim t^{-1}$, $x(t) \sim t^{-1}$, $h(t) \sim t^{-2}$, $g(t) \sim t^{-2}$, $r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^{-1}$, $x(t) \sim t^{-\frac{1}{2}}$, $h(t) \sim t^{-2} \ln t$, $g(t) \sim t^{-\frac{3}{2}}$, $r(t) \sim t^{-3} \ln t$	$\gamma_*(t) \sim t^0$, $x(t) \sim e^{-3H_*t}$, $h(t) \sim e^{-3H_*t}$, $g(t) \sim te^{-3H_*t}$, $r(t) \sim e^{-3H_*t}$
$f'_* = 0$, $V'_* = 0$	$\gamma_*(t) \sim t$, $x(t) \sim t^0$, $h(t) \sim t^{-2}$, $g(t) \sim \text{undetermined}$, $r(t) \sim t^{-3}$	$h(t) \sim t^{-2}$, $r(t) \sim t^{-3}$, $x(t) \sim t^0$, $\gamma_*(t) \sim t$, $g(t) \sim \text{undetermined}$	$h(t) \sim e^{-3H_*t}$, $r(t) \sim e^{-3H_*t}$, $x(t) \sim t^0$, $g(t) \sim \text{undetermined}$, $\gamma_*(t) \sim t^0$

Table 5.1: Stability results in connection set 2

a broad range of initial conditions leads to singular solutions at finite time, as demonstrated by the analysis of scalar field perturbation in Sec. B of Chapter 8. Although some solutions converge to constant values, they deviate from the GR limit and violate the initial linearized perturbation assumptions (5.17), thus also being classified as unstable. Furthermore, in this limit, the background connection becomes undetermined. While a single solution does approach the GR limit, the associated connection perturbation remains undetermined, precluding a definitive conclusion about the stability of the standard cosmic history within connection set 2. A detailed phase space portrait analysis of these results is presented in Fig. 1 of paper II.

Connection set 3:

Given the analogous structure of the scalar field equation in both connection sets 2 and 3, the background scalar field equation again demands the same three distinct options. The results of the stability analysis in all the cosmological epochs for long run time are summarized in Table 5.2 given below, and finite time singularities are discussed in the Chapter 8. Combining the stability results of all three eras, it appears that a cosmic history from radiation to dust to dark energy domination cannot be realized in a stable manner in the vicinity of some fixed scalar field value ϕ_* . Meanwhile, the dust matter and potential domination eras are stable around $f'_* \neq 0$, $V'_* = V''_* = 0$.

5.4.2 Outcomes

In summary, the additional connection function in sets 2 and 3 increases the independent cosmological phase space dimensions by one but cannot take the role of dark matter or dark energy. Instead, it manifests as a stiff fluid (set 2) or spatial

Cases	Matter domination	Radiation domination	Potential domination
$f'_* \neq 0, V'_* \neq 0$	$\gamma_*(t) \sim t, x(t) \sim t^{-\frac{13}{3}},$ $h(t) \sim t^{-2}, g(t) \sim t^0, r(t) \sim t^{-3}$	$\gamma_*(t) \sim t, x(t) \sim t^{-\frac{7}{2}},$ $h(t) \sim t^{-2}, g(t) \sim t^0, r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^0, x(t) \sim e^{-5H_*t},$ $h(t) \sim e^{-3H_*t}, g(t) \sim te^{-3H_*t},$ $r(t) \sim e^{-3H_*t}$
$f'_* \neq 0, V'_* = 0$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{-\frac{1}{3}},$ $h(t) \sim t^{-\frac{4}{3}}, g(t) \sim t^{\frac{2}{3}}, r(t) \sim t^{-\frac{7}{3}}$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{\frac{1}{2}},$ $h(t) \sim t^{-2}, g(t) \sim t^{\frac{3}{2}}, r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^0, x(t) \sim e^{-5H_*t},$ $h(t) \sim e^{-3H_*t}, g(t) \sim te^{-3H_*t},$ $r(t) \sim e^{-3H_*t}$
$f'_* \neq 0, V'_* = V''_* = 0$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{-\frac{1}{3}},$ $h(t) \sim t^{-\frac{4}{3}}, g(t) \sim t^{-\frac{4}{3}}, r(t) \sim t^{-\frac{7}{3}}$	$\gamma_*(t) \sim t^{-1}, x(t) \sim t^{\frac{1}{2}},$ $h(t) \sim t^{-2}, g(t) \sim t^{-\frac{1}{2}}, r(t) \sim t^{-3}$	$\gamma_*(t) \sim t^0, x(t) \sim e^{-5H_*t},$ $h(t) \sim e^{-3H_*t}, g(t) \sim te^{-3H_*t},$ $r(t) \sim e^{-3H_*t}$
$f'_* = 0, V'_* = 0$	$\gamma_*(t) \sim t^{-\frac{5}{3}}, h(t) \sim t^{-2},$ $r(t) \sim t^{-3},$ $x(t) \sim \text{See Chapter 8 text C.1}$	$h(t) \sim t^{-2}, r(t) \sim t^{-3},$ $\gamma_*(t) \sim t^{-\frac{5}{4}},$ $x(t) \sim \text{See Chapter 8 text C.2}$	$\gamma_*(t) \sim e^{-\frac{5H_*t}{2}}, h(t) \sim e^{-3H_*t},$ $r(t) \sim e^{-3H_*t},$ $x(t) \sim \text{See Chapter 8 text C.3}$

Table 5.2: Stability results in connection set 3.

curvature (set 3). The connection equation significantly affects the scalar field dynamics, leading to conditions on the model function and the scalar field evolution where the scalar field becomes singular. As expected, even small perturbations to the scalar field trigger these singularities. In sets 2 and 3, near a simultaneous critical point of the coupling and potential functions, a class of singular solutions emerges due to the divergence of the scalar field perturbation at finite time, consequently driving other physical quantities (energy density, connection function, and scalar field) to become singular as well. Comparing all three branches of connection, set 1 maintains a standard, stable cosmic history at the simultaneous critical point, whereas sets 2 and 3 exhibit a class of singular solutions there. Notably, in sets 2 and 3, the background equations fail to uniquely determine the background connection function, and the scalar field derivatives become singular in a simultaneous critical point, preventing smooth convergence to the GR limit. This suggests a fundamental issue with scalar-nonmetricity cosmology at the background level. Hence, we claim that sets 2 and 3 appear less physically viable than set 1.

5.5 Stability of $f(Q)$ Cosmology

In the attached publication (Chapter 9), we employ the geometrical formalism outlined in Chapter 2, the $f(Q)$ action (3.23) and the corresponding field equations for the metric (3.24) and connection (3.25). Utilizing the metric (2.41) and the affine connections defined in Eqs. (2.43), (2.45), and (2.47) into the metric field equations (3.24) and connection equation (3.25) yields three distinct sets of cosmological equations, as detailed in Chapter 9. The appearance of the additional free function $\gamma(t)$ in the $f(Q)$ cosmological equations motivates an investigation of its independent dynamics in the physical space. Furthermore, recent studies [27]

have reported the presence of a ghost degree of freedom in the ultraviolet regime within the second connection branch. This finding provides further motivation for our analysis of connection set 2, specifically to examine whether any potential instability arises from the dynamics of $\gamma(t)$.

5.5.1 GR Limit and Stability in Connection Set 1

We found three distinct scenarios where the modified cosmological Friedmann equations in symmetric teleparallel gravity reduce to the standard GR equations (5.12). First, the trivial case:

$$F^*(t) = 0. \quad (5.22)$$

The second also occurs trivially when the connection evolves in a manner whereby the nonmetricity scalar

$$Q_* = \text{constant}, \quad \text{or} \quad 0, \quad (5.23)$$

and the third, the nontrivial one

$$F^*(t) = \alpha \sqrt{-Q_*(t)}. \quad (5.24)$$

A perturbative analysis around the nontrivial GR limit (5.24) shows stable radiation, matter, and geometric dark energy eras, transitioning to a marginally stable dark energy era, as summarized the results in Table 5.3. However, the second option is quite restrictive under the FLRW symmetry assumption for the connection, as these two conditions together select only specific Hubble functions, namely those corresponding to Minkowski spacetime. Our study of generic $f(Q)$ models shows that a large class of solutions possess a GR attractor. Hence, we conclude that every model must have a value Q_* at which the model function must be equal to $Q + \sqrt{Q}$, whereby the cosmological equations reduce to the form coinciding with GR. This explains why many models can fit well with observations [87, 89, 103–105] even though these models are different from the exact GR solution for other values of Q .

5.5.2 GR Limit and Stability in Connection Set 2

Connection set 2 also has three ways to recover GR. First, the modified $f(Q)$ cosmological equations trivially reduce to GR when

$$F^*(t) = 0. \quad (5.25)$$

Second, in contrast to set 1, the condition in

$$Q_* = \text{constant}, \quad \text{or} \quad 0, \quad (5.26)$$

Value			
$F^*(t) = \alpha \sqrt{-Q_*(t)}$			
Different phase			
Matter	Radiation	Dark energy	Geometric dark energy
$h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$h(t) \sim t^{-2},$ $r(t) \sim t^{-3}$	$h(t) \sim t^0,$ $r(t) \sim t^0$	$h(t) \sim e^{-3H_*t},$ $r(t) \sim e^{-3H_*t}$

Table 5.3: Stability results of $f(Q)$ cosmology in connection set 1.

Matter	Radiation	Dark energy	Geometric dark energy
$F^*(t) = \tilde{c}_2 Q_*(t)$ $+ \frac{3\tilde{c}_2^2 Q_*(t)^2}{16\tilde{c}_2}$	$F^*(t) = Q_*(t)\tilde{c}_2$ $-\frac{3}{2}\tilde{c}_1 Q_*(t)\sqrt{-\frac{\tilde{c}_1 Q_*(t)}{\tilde{c}_2}}$	$F^*(t) = \hat{c}_3 (Q_*(t) - 2\kappa^2 \Lambda_{DE})$, $F^*(t) = \hat{c}_4 e^{-\frac{Q_*(t)}{2\Lambda_{DE}\kappa^2}}$	$F^*(t) = 6\kappa^2 \Lambda_{GDE}$ $-4Q_*(t)$
$r(t) \sim t^{-2},$ $h(t) \sim t^{-2},$ $g(t) \sim t^0$	$r(t) \sim t^{-2},$ $h(t) \sim t^{-2},$ $g(t) \sim t^{-\frac{1}{2}}$	$r(t) \sim t^0, \quad h(t) \sim t^0,$ $g(t) \sim \text{undetermined.}$ $h(t) \sim t^0, \quad g(t) \sim t^0,$ $r(t) \sim \text{undetermined}$	$r(t) \sim e^{-3H_*t},$ $h(t) \sim e^{-3H_*t}$ $g(t) \sim \text{undetermined}$

Table 5.4: Stability results of $f(Q)$ cosmology in connection set 2.

only allows de Sitter solution while respecting cosmological symmetry. The third option provides a nontrivial realization of the modified Friedmann equations. However, unlike connection set 1, solving the modification terms is difficult due to the presence of $\gamma_*(t)$ and its time derivatives up to $\ddot{\gamma}_*(t)$ in the cosmological equations. Therefore, we began by solving the background connection equation, which can be simplified to

$$\ln \left(F_{QQ}^*(t) \dot{Q}_*(t) \right) = -3 \int H_*(t) dt. \quad (5.27)$$

As the above expression shows, the dependence on the different background Hubble expansion (5.13), (5.14), (5.15) and (5.16) leads to a different condition for each cosmological epoch.

The stability results in the trivial regime (5.25) coincide with those of set 1. In the nontrivial regime of the dust dominant era, we found a quadratic function containing the exact GR-type solution. Applying small perturbations around this value, we demonstrate that the perturbations exhibit asymptotic decay over time, signifying the stability of Hubble expansion and matter energy density. However, the marginal stability of the background connection raises concerns about the long term stability of a matter dominated era around the GR limit. To investigate

the potential impact of this marginal stability of the connection, we conducted a numerical example; see Sec. VII of Chapter 9. Similarly, the stability results for other cosmological epochs are summarized in Table 5.4. Unlike connection set 1, a single value of the nonmetricity function cannot capture the entire evolutionary history of the universe, from radiation to dark energy to geometric dark energy domination.

5.5.3 Outcomes

Let us summarize the key findings in the attached publication in Chapter 9: the extra connection function brings two additional directions to the phase space in sets 2 and 3, making the phase space four-dimensional. By analyzing the conditions that a physically viable model should obey, our result confirmed that we can not say that a certain model $f(Q)$ is viable or a certain connection is viable, but these two aspects must be considered together. The nontrivial realization of the GR regime in set 2 might have less applications because different eras imply different behavior of $f(Q)$, and the stability is only guaranteed in the radiation domination epoch. Then, an inspection of the field equations in the form of a dynamical system and an analysis of the connection equation showed the conditions for possible singularity. The numerical evolution of the equations, starting near a matter solution in GR, explicitly demonstrated this, showing the connection function hitting a singularity at a finite time and consequently driving the entire system to singularity. These singular solutions highlight that cosmologies using the alternative FLRW connection sets 2 (and, by analogy, set 3) must be handled with utmost care.

Chapter 6

Future Directions

As discussed in Chapter 4, there is compelling evidence that $f(Q)$ and $f(T)$ theories of gravity are strongly coupled. The Hamiltonian analysis developed during this PhD will be instrumental in addressing this ongoing controversy. Beyond determining the correct number of PDOF, we will identify and characterize perturbative modes. This includes classifying them into vector, tensor, and scalar modes, as well as identifying and eliminating any unphysical (unhealthy) modes. This analysis will be performed using perturbation theory. Future work will also explore the potential of Hamiltonian analysis in the context of numerical relativity.

With the exception of a few studies [13], the Hamiltonian analysis of the recent proposed GTEGR theories [11, 12] remains unexplored. This provides a promising avenue for future research. In our forthcoming article, we aim to derive Hamilton's equations of STEGR in coincident gauge without omitting the boundary term in order to gain deeper insights into the role of boundary term. This analysis aims to enhance the impact of boundary terms in the Hamiltonian, Hamilton's equations, and the resulting constraint structure. We will then compare our findings with those presented in [80, 106]. Furthermore, based on these findings, we intend to investigate the potential impact of boundary terms on the breaking of diffeomorphism symmetry in modified theories. We anticipate that this research will significantly contribute to determining the correct number of PDOF in the $f(Q)$ theory.

When a theory incorporates additional dynamics beyond GR, it is crucial to understand the behavior of these extra dynamics in the vicinity of the GR limit. To assess the stability of such modified theories, we can extend our research by employing cosmological perturbation techniques around GR background, provided the theory admits a well-defined GR limit. It is possible to generalize this approach to study the dynamics of the theory in the presence of hyperfluids (allowing coupling of connection to matter) and how it impacts the stability of the system.

Summary

The effects of non-Riemannian connection in teleparallel gravity

In general, this thesis aims to explore modifications to general relativity, mainly emphasizing the modification of the underlying geometric sector motivated by the limitations of general relativity, such as dark energy, dark matter, fundamental cosmological challenges, and the Hubble tension. We mostly concentrate on the teleparallel theory, where the flatness condition does not completely fix the teleparallel connection, making it a gauge degree of freedom. While one can choose a gauge where the connection vanishes (e.g., the Weitzenböck gauge in metric teleparallelism or the coincident gauge in symmetric teleparallelism), this thesis instead explores the impact of a nonvanishing connection to investigate its influence on the theory and cosmology.

The main focus of the thesis work can be split into two parts. To understand the physical viability and dynamics of any theory, a well-respected procedure to utilize is the Hamiltonian approach. We apply this analysis to covariant teleparallel equivalent to general relativity. Our primary motivation for avoiding gauge fixing is to preserve covariance and gain deeper insights into how the additional field (related to the spin connection) influences Hamilton's equations. We concluded that while the extra field alters the constraint structure and Hamilton's equations, the result does not impact the total number of physical degrees of freedom.

The second part of this work focuses on the stability of symmetric teleparallel theories with nontrivial connections across all cosmic eras near the general relativity limit. The objective is to investigate the influence of the extra connection function on cosmological dynamics and stability. We found the connection function does extend the phase space but does not take the role of dark matter and dark energy, and it drastically changes the dynamics of the scalar field in scalar-nonmetricity cosmology. A cosmological perturbative analysis confirms that the scalar field experiences singularity, which ultimately makes other quantities singular as well. A numerical example in $f(Q)$ cosmology demonstrates that the connection function hits a singularity at finite time. Furthermore, in the nontrivial branch, one single model function can not able to reproduce all the cosmic epochs of the universe. Therefore, we conclude that nontrivial connections might be less physically viable for cosmology compared to the trivial branch, although further research is needed to rule it out completely.

Acknowledgments

I extend sincere gratitude to my supervisor, Dr. Laur Järv and Dr. María José Guzmán Monsalve, for supervising, consistently reviewing my results, providing constructive feedback, and giving valuable time for scientific discussion, which ensured the research progressed in the right direction. Their patience in answering my questions, dedication to helping me broaden my understanding of physics beyond the research topic, their effort in developing my critical thinking and improving writing skills, and consistent encouragement in all situations have been a great influence during my PhD. I am especially grateful to Dr. María José Guzmán Monsalve for the financial support provided at the final stage of my PhD. I am grateful to my collaborator, Dr. Daniel Blixt, for his assistance with the xAct computations done in publication I, insightful discussions, and for making my visit to Tartu productive and enjoyable as well as thanks for the thesis template.

I would like to thank Dr. Jose Pedro Mimoso, Dr. Tiago Barreiro, Dr. Nelson Nunes, and Dr. Bruno J. Barros for the insightful discussions I had during my short-term visit to the University of Lisbon. I am grateful to Dr. Daniel Blixt and Dr. Salvatore Capozziello for assistance and scientific discussion during my short-term visit to Scuola Superiore Meridionale, Largo S. Marcellino, Napoli, Italy.

I would like to acknowledge the understanding physics book examiners Dr. Kalev Tarkpea, Dr. Laur Järv, and Dr. María José Guzmán Monsalve for their insightful feedback and the decision. Their evaluation significantly contributed to my deeper understanding of the subject matter. I am grateful to Dr. Manuel Hohmann for assisting with the Mathematica notebook and to all members of the Laboratory of Theoretical Physics for the friendly and supportive environment.

I would like to express my sincere appreciation to my former supervisor, Dr. Bivudutta Mishra, for his guidance in research. I acknowledge all my collaborators with whom I worked in BITS, Hyderabad, and Dr. S. K. Tripathy from IGIT Sarang. A special thanks to Santosh V. Lohakare for his technical expertise and patience, for supporting my decision, and for guiding me in the right direction. I am grateful to Sayantun Ghosh for generously sharing his time and helping to clarify physics concepts. Additionally, I want to thank all my friends at BITS, Hyderabad, for their collective support and encouragement throughout this journey. Although not specifically named, each played an important role in my journey and success.

I am grateful to my laboratory friends, Debora, Ilaria, Luxi, Maria Naeem, Priidik, Sofia, and Vasiliki, as well as visiting fellows Adrián and Mercè, and Mousumi from Tartu observatory for their support and insightful formal and informal discussions. I also extend thanks to postdoc fellows Alejandro, Damianos, Jorge, and Pepe for their assistance.

Growing up in a small village where higher education for women was uncommon, I am eternally grateful to my family for their unwavering support, and their encouragement of my decision to study abroad has become my greatest support and strength. Their immense belief, along with their support in every situation, has never let my inner strength down even a little. Therefore, my deep appreciation goes to each member of my family, who championed me from afar for their emotional support. I dedicate this thesis to my grandfather, who sadly passed away on 24th January 2025, as I was finishing the thesis. He was so looking forward to seeing me as a doctoral graduate.

I heartily thank all of you for never letting me feel alone, standing with me in all the difficult situations I faced, and being treated like a family member.

Bibliography

- [1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. John Wiley and Sons, New York, 1972.
- [2] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, 2003. <http://www.amazon.com/Spacetime-Geometry-Introduction-General-Relativity/dp/0805387323>.
- [3] L. O’Raifeartaigh and N. Straumann, “Early history of gauge theories and Kaluza-Klein theories,” [arXiv:hep-ph/9810524](https://arxiv.org/abs/hep-ph/9810524).
- [4] K. Hayashi and T. Shirafuji, “New general relativity,” *Physical Review D* **19** (1979) no. 12, 3524.
- [5] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction*. Springer, 2013.
- [6] J. M. Nester and H.-J. Yo, “Symmetric teleparallel general relativity,” *Chin. J. Phys.* **37** (1999) 113, [arXiv:gr-qc/9809049](https://arxiv.org/abs/gr-qc/9809049).
- [7] M. Adak and O. Sert, “A Solution to symmetric teleparallel gravity,” *Turk. J. Phys.* **29** (2005) 1–7, [arXiv:gr-qc/0412007](https://arxiv.org/abs/gr-qc/0412007).
- [8] J. Beltrán Jiménez, L. Heisenberg, and T. S. Koivisto, “The Geometrical Trinity of Gravity,” *Universe* **5** (2019) no. 7, 173, [arXiv:1903.06830](https://arxiv.org/abs/1903.06830) [hep-th].
- [9] C. Mancini, G. M. Tino, and S. Capozziello, “Equivalent Gravities and Equivalence Principle: Foundations and experimental implications,” [arXiv:2501.06487](https://arxiv.org/abs/2501.06487) [gr-qc].
- [10] J. Beltrán Jiménez, L. Heisenberg, D. Iosifidis, A. Jiménez-Cano, and T. S. Koivisto, “General teleparallel quadratic gravity,” *Phys. Lett. B* **805** (2020) 135422, [arXiv:1909.09045](https://arxiv.org/abs/1909.09045) [gr-qc].

- [11] D. A. Gomes, J. Beltrán Jiménez, and T. S. Koivisto, “General parallel cosmology,” *JCAP* **12** (2023) 010, arXiv:2309.08554 [gr-qc].
- [12] M. Adak, T. Dereli, T. S. Koivisto, and C. Pala, “General teleparallel metrical geometries,” *Int. J. Geom. Meth. Mod. Phys.* **20** (2023) no. Supp01, 2350215, arXiv:2303.17812 [gr-qc].
- [13] F. Bajardi and D. Blixt, “Primary constraints in general teleparallel quadratic gravity,” *Phys. Rev. D* **109** (2024) no. 8, 084078, arXiv:2401.11591 [gr-qc].
- [14] K. S. Stelle, “Renormalization of Higher Derivative Quantum Gravity,” *Phys. Rev. D* **16** (1977) 953–969.
- [15] L. Perivolaropoulos and F. Skara, “Challenges for Λ CDM: An update,” *New Astron. Rev.* **95** (2022) 101659, arXiv:2105.05208 [astro-ph.CO].
- [16] E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D. F. Mota, A. G. Riess, and J. Silk, “In the realm of the Hubble tension—a review of solutions,” *Class. Quant. Grav.* **38** (2021) no. 15, 153001, arXiv:2103.01183 [astro-ph.CO].
- [17] E. Abdalla, G. F. Abellán, A. Aboubrahim, A. Agnello, *et al.*, “Cosmology intertwined: A review of the particle physics, astrophysics, and cosmology associated with the cosmological tensions and anomalies,” *Journal of High Energy Astrophysics* **34** (2022) 49–211, arXiv:2203.06142 [astro-ph.CO].
- [18] N.-M. Nguyen, D. Huterer, and Y. Wen, “Evidence for Suppression of Structure Growth in the Concordance Cosmological Model,” *Phys. Rev. Lett.* **131** (2023) no. 11, 111001, arXiv:2302.01331 [astro-ph.CO].
- [19] **DESI** Collaboration, A. G. Adame *et al.*, “DESI 2024 VI: Cosmological Constraints from the Measurements of Baryon Acoustic Oscillations,” arXiv:2404.03002 [astro-ph.CO].
- [20] A. De Felice and S. Tsujikawa, “ $f(R)$ theories,” *Living Rev. Rel.* **13** (2010) 3, arXiv:1002.4928 [gr-qc].
- [21] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, “Modified Gravity and Cosmology,” *Phys. Rept.* **513** (2012) 1–189, arXiv:1106.2476 [astro-ph.CO].
- [22] **CANTATA** Collaboration, Y. Akrami *et al.*, *Modified Gravity and Cosmology. An Update by the CANTATA Network*. Springer, 2021. arXiv:2105.12582 [gr-qc].

- [23] T. P. Sotiriou and V. Faraoni, “ $f(R)$ Theories Of Gravity,” *Rev. Mod. Phys.* **82** (2010) 451–497, arXiv:0805.1726 [gr-qc].
- [24] S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud, and E. Di Valentino, “Teleparallel gravity: from theory to cosmology,” *Rept. Prog. Phys.* **86** (2023) no. 2, 026901, arXiv:2106.13793 [gr-qc].
- [25] L. Heisenberg, “Review on $f(Q)$ gravity,” *Phys. Rept.* **1066** (2024) 1–78, arXiv:2309.15958 [gr-qc].
- [26] L. Jarv and L. Pati, “Stability of symmetric teleparallel scalar-tensor cosmologies with alternative connections,” *Phys. Rev. D* **109** (2024) no. 6, 064069, arXiv:2309.04262 [gr-qc].
- [27] D. A. Gomes, J. Beltrán Jiménez, A. J. Cano, and T. S. Koivisto, “Pathological Character of Modifications to Coincident General Relativity: Cosmological Strong Coupling and Ghosts in $f(Q)$ Theories,” *Phys. Rev. Lett.* **132** (2024) no. 14, 141401, arXiv:2311.04201 [gr-qc].
- [28] M. Hohmann, “General covariant symmetric teleparallel cosmology,” *Phys. Rev. D* **104** (2021) no. 12, 124077, arXiv:2109.01525 [gr-qc].
- [29] J. B. Jiménez, L. Heisenberg, and T. Koivisto, “Coincident General Relativity,” *Phys. Rev. D* **98** (2018) no. 4, 044048, arXiv:1710.03116 [gr-qc].
- [30] F. D’Ambrosio, L. Heisenberg, and S. Kuhn, “Revisiting cosmologies in teleparallelism,” *Class. Quant. Grav.* **39** (2022) no. 2, 025013, arXiv:2109.04209 [gr-qc].
- [31] E. Jensko, “Spatial curvature in coincident gauge $f(Q)$ cosmology,” arXiv:2407.17568 [gr-qc].
- [32] V. Faraoni and S. Capozziello, *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics*. Springer, Dordrecht, 2011.
- [33] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984.
- [34] V. C. de Andrade and J. G. Pereira, “Gravitational Lorentz force and the description of the gravitational interaction,” *Phys. Rev. D* **56** (1997) 4689–4695, arXiv:gr-qc/9703059.
- [35] A. A. Starobinsky, “Disappearing cosmological constant in $f(R)$ gravity,” *JETP Lett.* **86** (2007) 157–163, arXiv:0706.2041 [astro-ph].

- [36] N. Tamanini and C. G. Boehmer, “Good and bad tetrads in $f(T)$ gravity,” *Phys. Rev. D* **86** (2012) 044009, arXiv:1204.4593 [gr-qc].
- [37] M. Krssak, R. J. Van Den Hoogen, J. G. Pereira, C. G. Boehmer, and A. A. Coley, “Teleparallel Theories of Gravity: Illuminating a Fully Invariant Approach,” arXiv:1810.12932 [gr-qc].
- [38] M. Krššák and E. N. Saridakis, “The covariant formulation of $f(T)$ gravity,” *Class. Quant. Grav.* **33** (2016) no. 11, 115009, arXiv:1510.08432 [gr-qc].
- [39] A. Golovnev, T. Koivisto, and M. Sandstad, “On the covariance of teleparallel gravity theories,” *Class. Quant. Grav.* **34** (2017) no. 14, 145013, arXiv:1701.06271 [gr-qc].
- [40] M. Hohmann, L. Järv, and U. Ualikhanova, “Covariant formulation of scalar-torsion gravity,” *Phys. Rev. D* **97** (2018) no. 10, 104011, arXiv:1801.05786 [gr-qc].
- [41] T. Koivisto, “Covariant conservation of energy momentum in modified gravities,” *Class. Quant. Grav.* **23** (2006) 4289–4296, arXiv:gr-qc/0505128.
- [42] L. Heisenberg, M. Hohmann, and S. Kuhn, “Homogeneous and isotropic cosmology in general teleparallel gravity,” *Eur. Phys. J. C* **83** (2023) no. 4, 315, arXiv:2212.14324 [gr-qc].
- [43] M. Hohmann, “Teleparallel Gravity,” *Lect. Notes Phys.* **1017** (2023) 145–198, arXiv:2207.06438 [gr-qc].
- [44] C. Brans and R. H. Dicke, “Mach’s principle and a relativistic theory of gravitation,” *Phys. Rev.* **124** (1961) 925–935.
- [45] Y. Fujii and K. Maeda, *The scalar-tensor theory of gravitation*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 7, 2007.
- [46] G. Esposito-Farese and D. Polarski, “Scalar tensor gravity in an accelerating universe,” *Phys. Rev. D* **63** (2001) 063504, arXiv:gr-qc/0009034.
- [47] D. Wands, “Extended gravity theories and the Einstein-Hilbert action,” *Class. Quant. Grav.* **11** (1994) 269–280, arXiv:gr-qc/9307034.
- [48] M. Hohmann and C. Pfeifer, “Scalar-torsion theories of gravity II: $L(T, X, Y, \phi)$ theory,” *Phys. Rev. D* **98** (2018) no. 6, 064003, arXiv:1801.06536 [gr-qc].

- [49] M. Hohmann, “Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants,” *Phys. Rev. D* **98** (2018) no. 6, 064004, arXiv:1801.06531 [gr-qc].
- [50] R.-J. Yang, “Conformal transformation in $f(T)$ theories,” *EPL* **93** (2011) no. 6, 60001, arXiv:1010.1376 [gr-qc].
- [51] S. Bahamonde and M. Wright, “Teleparallel quintessence with a nonminimal coupling to a boundary term,” *Phys. Rev. D* **92** (2015) no. 8, 084034, arXiv:1508.06580 [gr-qc]. [Erratum: *Phys.Rev.D* 93, 109901 (2016)].
- [52] M. Wright, “Conformal transformations in modified teleparallel theories of gravity revisited,” *Phys. Rev. D* **93** (2016) no. 10, 103002, arXiv:1602.05764 [gr-qc].
- [53] L. Järv, M. Rünkla, M. Saal, and O. Vilson, “Nonmetricity formulation of general relativity and its scalar-tensor extension,” *Phys. Rev. D* **97** (2018) no. 12, 124025, arXiv:1802.00492 [gr-qc].
- [54] S. Bahamonde, J. Gigante Valcarcel, L. Järv, and J. Lember, “Black hole solutions in scalar-tensor symmetric teleparallel gravity,” *JCAP* **08** (2022) 082, arXiv:2206.02725 [gr-qc].
- [55] V. Gakis, M. Krššák, J. Levi Said, and E. N. Saridakis, “Conformal gravity and transformations in the symmetric teleparallel framework,” *Phys. Rev. D* **101** (2020) no. 6, 064024, arXiv:1908.05741 [gr-qc].
- [56] D. Zhao, “Conformal transformation of $f(Q)$ gravity and its cosmological perturbations,” arXiv:2404.16299 [gr-qc].
- [57] S. Capozziello, A. Stabile, and A. Troisi, “Spherical symmetry in $f(R)$ -gravity,” *Class. Quant. Grav.* **25** (2008) 085004, arXiv:0709.0891 [gr-qc].
- [58] M. C. B. Abdalla, S. Nojiri, and S. D. Odintsov, “Consistent modified gravity: Dark energy, acceleration and the absence of cosmic doomsday,” *Class. Quant. Grav.* **22** (2005) L35, arXiv:hep-th/0409177.
- [59] S. Capozziello and A. De Felice, “ $f(R)$ cosmology by Noether’s symmetry,” *JCAP* **08** (2008) 016, arXiv:0804.2163 [gr-qc].
- [60] H. Mohseni Sadjadi, “Multicomponent solution in a modified theory of gravity in the early universe,” *Phys. Rev. D* **77** (2008) 103501.

- [61] E. Barausse, T. P. Sotiriou, and J. C. Miller, “A No-go theorem for polytropic spheres in Palatini $f(R)$ gravity,” *Class. Quant. Grav.* **25** (2008) 062001, arXiv:gr-qc/0703132.
- [62] D. E. Barraco and V. H. Hamity, “Spherically symmetric solutions in $f(R)$ theories of gravity obtained using the first order formalism,” *Phys. Rev. D* **62** (2000) 044027.
- [63] V. Faraoni, *Cosmology in scalar tensor gravity*. 2004.
- [64] R. Myrzakulov, “Accelerating universe from $F(T)$ gravity,” *Eur. Phys. J. C* **71** (2011) 1752, arXiv:1006.1120 [gr-qc].
- [65] P. Wu and H. W. Yu, “The dynamical behavior of $f(T)$ theory,” *Phys. Lett. B* **692** (2010) 176–179, arXiv:1007.2348 [astro-ph.CO].
- [66] S.-H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, “Cosmological perturbations in $f(T)$ gravity,” *Phys. Rev. D* **83** (2011) 023508, arXiv:1008.1250 [astro-ph.CO].
- [67] J. Beltrán Jiménez, L. Heisenberg, T. S. Koivisto, and S. Pekar, “Cosmology in $f(Q)$ geometry,” *Phys. Rev. D* **101** (2020) no. 10, 103507, arXiv:1906.10027 [gr-qc].
- [68] M. Hohmann, “Metric-affine Geometries With Spherical Symmetry,” *Symmetry* **12** (2020) no. 3, 453, arXiv:1912.12906 [math-ph].
- [69] R. L. Arnowitt, S. Deser, and C. W. Misner, “Dynamical Structure and Definition of Energy in General Relativity,” *Phys. Rev.* **116** (1959) 1322–1330.
- [70] S. W. Hawking and G. T. Horowitz, “The Gravitational Hamiltonian, action, entropy and surface terms,” *Class. Quant. Grav.* **13** (1996) 1487–1498, arXiv:gr-qc/9501014.
- [71] T. Multamaki, J. Vainio, and I. Vilja, “Hamiltonian perturbation theory in $f(R)$ gravity,” *Phys. Rev. D* **81** (2010) 064025, arXiv:0910.5659 [gr-qc].
- [72] R. Ferraro and M. J. Guzmán, “Hamiltonian formalism for $f(T)$ gravity,” *Phys. Rev. D* **97** (2018) no. 10, 104028, arXiv:1802.02130 [gr-qc].
- [73] M. Blagojević and J. M. Nester, “Local symmetries and physical degrees of freedom in $f(T)$ gravity: a Dirac Hamiltonian constraint analysis,” *Phys. Rev. D* **102** (2020) no. 6, 064025, arXiv:2006.15303 [gr-qc].

- [74] D. Blixt, M.-J. Guzmán, M. Hohmann, and C. Pfeifer, “Review of the Hamiltonian analysis in teleparallel gravity,” *Int. J. Geom. Meth. Mod. Phys.* **18** (2021) no. supp01, 2130005, arXiv:2012.09180 [gr-qc].
- [75] A. Golovnev and M.-J. Guzman, “Nontrivial Minkowski backgrounds in $f(T)$ gravity,” *Phys. Rev. D* **103** (2021) no. 4, 044009, arXiv:2012.00696 [gr-qc].
- [76] A. Golovnev and M.-J. Guzmán, “Foundational issues in $f(T)$ gravity theory,” *Int. J. Geom. Meth. Mod. Phys.* **18** (2021) no. supp01, 2140007, arXiv:2012.14408 [gr-qc].
- [77] M. Li, R.-X. Miao, and Y.-G. Miao, “Degrees of freedom of $f(T)$ gravity,” *JHEP* **07** (2011) 108, arXiv:1105.5934 [hep-th].
- [78] L. Heisenberg, M. Hohmann, and S. Kuhn, “Cosmological teleparallel perturbations,” *JCAP* **03** (2024) 063, arXiv:2311.05495 [gr-qc].
- [79] F. D’Ambrosio, L. Heisenberg, and S. Zentarra, “Hamiltonian Analysis of $f(Q)$ Gravity and the Failure of the Dirac–Bergmann Algorithm for Teleparallel Theories of Gravity,” *Fortsch. Phys.* **71** (2023) no. 12, 2300185, arXiv:2308.02250 [gr-qc].
- [80] F. D’Ambrosio, M. Garg, L. Heisenberg, and S. Zentarra, “ADM formulation and Hamiltonian analysis of Coincident General Relativity,” arXiv:2007.03261 [gr-qc].
- [81] R. Ferraro and M. J. Guzmán, “Hamiltonian formulation of teleparallel gravity,” *Phys. Rev. D* **94** (2016) no. 10, 104045, arXiv:1609.06766 [gr-qc].
- [82] D. Liang, Y. Gong, S. Hou, and Y. Liu, “Polarizations of gravitational waves in $f(R)$ gravity,” *Phys. Rev. D* **95** (2017) no. 10, 104034, arXiv:1701.05998 [gr-qc].
- [83] K. Tomonari and S. Bahamonde, “Dirac–Bergmann analysis and degrees of freedom of coincident $f(Q)$ -gravity,” *Eur. Phys. J. C* **84** (2024) no. 4, 349, arXiv:2308.06469 [gr-qc]. [Erratum: Eur.Phys.J.C 84, 508 (2024)].
- [84] M. J. Guzmán and R. Ferraro, “Degrees of freedom and Hamiltonian formalism for $f(T)$ gravity,” *Int. J. Mod. Phys. A* **35** (2020) no. 02n03, 2040022, arXiv:1910.03100 [gr-qc].
- [85] S. Capozziello, A. Finch, J. L. Said, and A. Magro, “The 3+1 formalism in teleparallel and symmetric teleparallel gravity,” *Eur. Phys. J. C* **81** (2021) no. 12, 1141, arXiv:2108.03075 [gr-qc].

- [86] A. Golovnev and M.-J. Guzman, “Lorentz symmetries and primary constraints in covariant teleparallel gravity,” *Phys. Rev. D* **104** (2021) no. 12, 124074, arXiv:2110.11273 [gr-qc].
- [87] J. Shi, “Cosmological constraints in covariant $f(Q)$ gravity with different connections,” *Eur. Phys. J. C* **83** (2023) no. 10, 951, arXiv:2307.08103 [gr-qc].
- [88] G. Subramaniam, A. De, T.-H. Loo, and Y. K. Goh, “Energy condition bounds on $f(Q)$ model parameters in a curved FLRW Universe,” *Phys. Dark Univ.* **41** (2023) 101243, arXiv:2304.05031 [gr-qc].
- [89] Y. Yang, X. Ren, B. Wang, Y.-F. Cai, and E. N. Saridakis, “Data reconstruction of the dynamical connection function in $f(Q)$ cosmology,” arXiv:2404.12140 [astro-ph.CO].
- [90] J. Beltrán Jiménez and T. S. Koivisto, “Accidental gauge symmetries of Minkowski spacetime in Teleparallel theories,” *Universe* **7** (2021) no. 5, 143, arXiv:2104.05566 [gr-qc].
- [91] N. Dimakis, A. Paliathanasis, M. Roumeliotis, and T. Christodoulakis, “FLRW solutions in $f(Q)$ theory: The effect of using different connections,” *Phys. Rev. D* **106** (2022) no. 4, 043509, arXiv:2205.04680 [gr-qc].
- [92] N. Dimakis, M. Roumeliotis, A. Paliathanasis, P. S. Apostolopoulos, and T. Christodoulakis, “Self-similar cosmological solutions in symmetric teleparallel theory: Friedmann-Lemaître-Robertson-Walker spacetimes,” *Phys. Rev. D* **106** (2022) no. 12, 123516, arXiv:2210.10295 [gr-qc].
- [93] A. Paliathanasis, “Stability Properties of Self-Similar Solutions in Symmetric Teleparallel $f(Q)$ -Cosmology,” *Symmetry* **15** (2023) no. 2, 529.
- [94] H. Shabani, A. De, and T.-H. Loo, “Phase-space analysis of a novel cosmological model in $f(Q)$ theory,” *Eur. Phys. J. C* **83** (2023) no. 6, 535, arXiv:2304.02949 [gr-qc].
- [95] A. Paliathanasis, “Dynamical analysis of $f(Q)$ -cosmology,” *Phys. Dark Univ.* **41** (2023) 101255, arXiv:2304.04219 [gr-qc].
- [96] Q. Wang, X. Ren, Y.-F. Cai, W. Luo, and E. N. Saridakis, “Observational Test of $f(Q)$ Gravity with Weak Gravitational Lensing,” *Astrophys. J.* **974** (2024) no. 1, 7, arXiv:2406.00242 [astro-ph.CO].
- [97] I. R. Vasquez and A. Oliveros, “Analysis of the cosmological evolution parameters, energy conditions, and linear matter perturbations of an exponential-type model in $f(Q)$ gravity,” arXiv:2501.12585 [gr-qc].

- [98] S. Chakraborty, J. Dutta, D. Gregoris, K. Karwan, and W. Khyllep, “Reproducing Λ CDM-like Solutions in $f(Q)$ Gravity: A Comprehensive Study Across All Connection Branches,” [arXiv:2501.15159](#) [gr-qc].
- [99] C. M. Will, “The Confrontation between General Relativity and Experiment,” *Living Rev. Rel.* **17** (2014) 4, [arXiv:1403.7377](#) [gr-qc].
- [100] J.-P. Uzan, “Varying Constants, Gravitation and Cosmology,” *Living Rev. Rel.* **14** (2011) 2, [arXiv:1009.5514](#) [astro-ph.CO].
- [101] L. J  rv, P. Kuusk, M. Saal, and O. Vilson, “Transformation properties and general relativity regime in scalar–tensor theories,” *Class. Quant. Grav.* **32** (2015) 235013, [arXiv:1504.02686](#) [gr-qc].
- [102] M. Hohmann, L. Jarv, and U. Ualikhanova, “Dynamical systems approach and generic properties of $f(T)$ cosmology,” *Phys. Rev. D* **96** (2017) no. 4, 043508, [arXiv:1706.02376](#) [gr-qc].
- [103] F. K. Anagnostopoulos, S. Basilakos, and E. N. Saridakis, “First evidence that non-metricity $f(Q)$ gravity could challenge Λ CDM,” *Phys. Lett. B* **822** (2021) 136634, [arXiv:2104.15123](#) [gr-qc].
- [104] S. Capozziello and R. D’Agostino, “Model-independent reconstruction of $f(Q)$ non-metric gravity,” *Phys. Lett. B* **832** (2022) 137229, [arXiv:2204.01015](#) [gr-qc].
- [105] J. Ferreira, T. Barreiro, J. P. Mimoso, and N. J. Nunes, “Testing Λ -free $f(Q)$ cosmology,” *Phys. Rev. D* **108** (2023) no. 6, 063521, [arXiv:2306.10176](#) [astro-ph.CO].
- [106] M.-J. Guzman, “The Hamiltonian constraint in the symmetric teleparallel equivalent of general relativity,” [arXiv:2311.01424](#) [gr-qc].

Kokkuvõte (in Estonian)

Mitte-Riemanni seostuse efektid teleparalleelses gravitatsioonis

Käesoleva väitekirja eesmärgiks on uurida üldrelatiivsusteooria modifikatsioone rõhudes peamiselt aluseks oleva geomeetrilise sektori muutmisele, mida motiveerib üldrelatiivsusteooria piiratud võime seletada nähtuseid nagu tumeenergia, tumeaine, fundamentaalsed kosmoloogilised väljakutsed ja Hubble'i pinge Universumi paisumise andmestikus. Me keskendume enamasti teleparalleelsetele teooriatele, kus tasasuse tingimus ei fikseeri seostust täielikult vaid jätab selle kalibratsiooniliseks vabadusastmeks. Kuigi on võimalik valida kalibratsioon, kus seostus on null (nt Weitzenböcki kalibratsioon meetrilises teleparallelismis või kokkulangev kalibratsioon sümmeetrilises teleparallelismis), uurib doktoritöö mitte-nullise seostuse tähendust teooriale ülesehitusel ning kosmoloogilistes rakendustes.

Töö põhifookuse võib jagada kaheks osaks. Esiteks, hindamaks mistahes teooria füüsikalist paikapidavust ja dünaamikat pakub hästitutud protseduuri Hamiltoni formalism. Töös on vastavat analüüsi rakendatud kovariantsele üldrelatiivsusteooria teleparalleelsele ekvivalendile. Peamine motivatsioon kalibratsiooni fikseerimise vältimiseks on kovariatsuse säilitamine ilmutatud kujul, et saada sügavam ülevaad sellest, kuidas lisaväli (spinn-seostus) mõjutab Hamiltoni võrrandeid. Lõpuks nägime, et kuigi lisaväli muudab tingimuste avaldiste struktuuri ja Hamiltoni võrrandit, ei mõjuta see füüsikaliste vabadusastmete koguarvu.

Töö teine osa keskendub sümmeetriliste teleparalleelsete kosmoloogiliste mudelite üldrelatiivsusteooria piiri stabiilsuse uurimisele, hõlmates mittetriviaalset seostust kõigi kosmiliste ajastute kirjeldamise juures. Eesmärgiks oli jälgida täiendava seostusest lähtuva funktsiooni mõju universumi dünaamikale. Leidsime, et vastav funktsioon laiendab faasiruumi, kuid ei saa endale võtta tumeaine ega tumeenergia rolli. Lisaks muudab see drastiliselt skalaarvälja käitumist skalaar-mitte-meetrilisuse kosmoloogias. Kosmiliste häirituste analüüs näitab, et skalaarväli jõuab singulaarsusse, mis viib singulaarseks ka teised suurused. Seda ilmestab $f(Q)$ kosmoloogia numbriline näide, kus seostuses sisaldunud funktsioon läheb singulaarseks lõpliku aja jooksul. Täiendavalt tuli mittetriviaalse seostuse korral

välja, et üks mudelfunktsioon ei suuda reprodutseerida kõiki kosmilisi ajastuid. Seetõttu järelendasime, et mittetriviaalsed seostused võivad olla olla kosmoloogia jaoks füüsikalises mõttes vähem sobivad võrreldes triviaalse seostusega, kuigi nende täielikuks välistamiseks on vaja täiendavaid uuringuid.

Attached publications

- I Hamilton's equations in the covariant teleparallel equivalent of general relativity 83
- II Stability of symmetric teleparallel scalar-tensor cosmologies with alternative connections 105
- III Exploring the stability of cosmology near general relativity limit with different connections 133

Curriculum Vitae

In English

Personal information:

Name	Laxmipriya Pati
Date of birth	13.10.1995
Address	Raatuse 22-353, 51009 Tartu, Estonia
Telephone	+37256755076, +918882115220
E-mail	lpriyapati1995@gmail.com

Education:

2023 – 2025	Doctoral studies (physics 80342), University of Tartu
2022 – 2022	A visiting doctoral student at the University of Tartu
2020 – 2022	Doctoral studies (cosmology), BITS-Pilani, Hyderabad campus, India
2017 – 2019	Master's studies (Mathematics), Berhampur University, Ganjam, Odisha, India
2013 – 2016	Bachelor's studies (Mathematics), J.K.B.K. Govt. College, Cuttack, Odisha, India
2011 – 2013	Intermediate studies (Science), SRI A.I. of H.S.R. Matrubhaban College, Cuttack, Odisha, India
2002 – 2011	Elementary school, Rasol, Dhenkanal, Odisha, India

Awards and stipends:

E-COST-GRANT-CA21136:	Short-term scientific mission (STSM) grant to visit Dr. Daniel Blixt at the Scuola Superiore Meridionale University of Naples from September 23 to 30, 2024.
-----------------------	--

E-COST-GRANT-CA21136:	Short-term scientific mission (STSM) grant to visit Dr. Jose Pedro Mimoso at the Institute of Astrophysics and Space Science at the University of Lisbon from October 16 to 27, 2023.
-----------------------	--

E-COST-CA18108:	E-COST-MEETING-CA18108 to participate in the Final Annual Conference at the Faculty of Physics University of Rijeka, Croatia, from July 11 to 14, 2023.
E-COST-CA18108	E-COST-TRAINING-SCHOOL-CA18108 to participate in Theoretical and Experimental Approaches to Quantum Gravity Phenomenology at the Faculty of Physics, Belgrade, Serbia, from September 3 to 10, 2022.
Dora-Plus-Grant:	Awarded a Dora Plus Scholarship for a research visit to the University of Tartu.
DST-INSPIRE:	Awarded an INSPIRE Fellowship (File No. DST/INSPIRE Fellowship/2019/IF190600) for doctoral research from 2020 to 2022.

Publications:

1. María-José Guzmán, Laur Järv and Laxmipriya Pati “Exploring the stability of $f(Q)$ cosmology near general relativity limit with different connections” *Phys. Rev. D.* **110**, 124013 (2024) , arXiv: 2406.11621 [INSPIRE] [ETIS]
2. Laur Järv, Laxmipriya Pati “Stability of symmetric teleparallel scalar-tensor cosmologies with alternative connections” *Phys. Rev. D.* **109**, 064069 (2024), arXiv:2309.04262 [INSPIRE] [ETIS]
3. Laxmipriya Pati, D. Blixt and María-José Guzmán “Hamilton’s equations in the covariant teleparallel equivalent of general relativity” *Phys. Rev. D.* **99**, 044071 (2023), arXiv:2210.07971 [INSPIRE] [ETIS]
4. Laxmipriya Pati, S. A. Narawade, S. K. Tripathy and B. Mishra “Evolutionary behaviour of cosmological parameters with dynamical system analysis in $f(Q, T)$ gravity ” *Eur. Phys. J. C.* **83**, 445 (2023), arXiv:2206.11928 [INSPIRE]
5. S. A. Narawade, Laxmipriya Pati, B. Mishra and S. K. Tripathy “Dynamical system analysis for accelerating models in non-metricity $f(Q)$ gravity ” *Phys. Dark Univ.* **36**, 101020 (2022), arXiv:2203.14121 [INSPIRE]

6. Laxmipriya Pati, S. A. Kadam, S. K. Tripathy and B. Mishra “Rip cosmological models in extended symmetric teleparallel gravity ” *Phys. Dark Univ.* **35**, 100925 (2022), arXiv:2112.00271 [INSPIRE]
7. A. S. Agrawal, Laxmipriya Pati, S. K. Tripathy and B. Mishra “Matter bounce scenario and the dynamical aspects in $f(Q, T)$ gravity ” *Phys. Dark Univ.* **33**, 100863 (2021), arXiv:2108.02575 [INSPIRE]
8. Laxmipriya Pati, B. Mishra and S. K. Tripathy “Model parameters in the context of late time cosmic acceleration in $f(Q, T)$ gravity ” *Phys. Dark Univ.* **96**, 105003 (2021), arXiv:2107.12790 [INSPIRE]

Curriculum Vitae

eesti keeles

Isiklik informatsioon:

Nimi	Laxmipriya Pati
Sünniaeg	13.10.1995
Aadress	Raatuse 22-353, 51009 Tartu, Eesti
Telefon	+37256755076, +918882115220
E-mail	lpriyapati1995@gmail.com

Hariduskäik:

2023 – 2025	Doktorantuur (füüsika 80342), Tartu Ülikool
2022 – 2022	Tartu Ülikooli külalisdoktorant
2020 – 2022	Doktorantuur (kosmoloogia), BITS-Pilani ülikool, Hyderabad, India
2017 – 2019	Magistratuur (mathemaatika), Berhampuri ülikool, Ganjam, Odisha, India
2013 – 2016	Bakalaureuse õpingud (mathemaatika), J.K.B.K. Govt. College, Cuttack, Odisha, India
2011 – 2013	Vaheastme õpingud (loodusteadused), SRI A.I. of H.S.R. Matrubhaban College, Cuttack, Odisha, India
2002 – 2011	Põhikool, Rasol, Dhenkanal, Odisha, India

Uurimistoetused ja stipendiumid:

E-COST-GRANT-CA21136:	Lühiajalise teadusvisiidi (STSM) grant, et külastada Dr. Daniel Blixiti Napolis Scuola Superiore Meridionale ülikoolis 23.–30. septembril 2024.
E-COST-GRANT-CA21136:	Lühiajalise teadusvisiidi (STSM) grant, et külastada Dr. Jose Pedro Mimosot Lissaboni ülikooli astrofüüsika ja kosmoseteaduse instituudis 16.–27. septembril 2023.

E-COST-CA18108:	E-COST-MEETING-CA18108 toetus, et osaleda programmi viimasel aastakonverentsil Horvaatias Rijeka ülikooli füüsika teaduskonnas 11.-14. juulil 2023.
E-COST-CA18108	E-COST-TRAINING-SCHOOL-CA18108 toetus, et osaleda kvantgravitatsiooni teoreetiliste ja eksperimentaalsete aspektide suvekoolis Serbias Belgradi ülikooli füüsika teaduskonnas 3.–10. septembril 2022.
Dora-Plus-Grant:	Külalisdoktorandi stipendium õpinguteks Tartu ülikoolis.
DST-INSPIRE:	INSPIRE Fellowship (File No. DST/INSPIRE Fellowship/2019/IF190600) doktoriõpinguteks aastatel 2020 kuni 2022.

Publikatsioonid:

1. María-José Guzmán, Laur Järv and Laxmipriya Pati “Exploring the stability of $f(Q)$ cosmology near general relativity limit with different connections” *Phys. Rev. D.* **110**, 124013 (2024) , arXiv: 2406.11621 [INSPIRE] [ETIS]
2. Laur Järv, Laxmipriya Pati “Stability of symmetric teleparallel scalar-tensor cosmologies with alternative connections” *Phys. Rev. D.* **109**, 064069 (2024), arXiv:2309.04262 [INSPIRE] [ETIS]
3. Laxmipriya Pati, D. Blixt and María-José Guzmán “Hamilton’s equations in the covariant teleparallel equivalent of general relativity” *Phys. Rev. D.* **99**, 044071 (2023), arXiv:2210.07971 [INSPIRE] [ETIS]
4. Laxmipriya Pati, S. A. Narawade, S. K. Tripathy and B. Mishra “Evolutionary behaviour of cosmological parameters with dynamical system analysis in $f(Q, T)$ gravity ” *Eur. Phys. J. C.* **83**, 445 (2023), arXiv:2206.11928 [INSPIRE]
5. S. A. Narawade, Laxmipriya Pati, B. Mishra and S. K. Tripathy “Dynamical system analysis for accelerating models in non-metricity $f(Q)$ gravity ” *Phys. Dark Univ.* **36**, 101020 (2022), arXiv:2203.14121 [INSPIRE]
6. Laxmipriya Pati, S. A. Kadam, S. K. Tripathy and B. Mishra “Rip cosmological models in extended symmetric teleparallel gravity ” *Phys. Dark Univ.* **35**, 100925 (2022), arXiv:2112.00271 [INSPIRE]

7. A. S. Agrawal, Laxmipriya Pati, S. K. Tripathy and B. Mishra “Matter bounce scenario and the dynamical aspects in $f(Q, T)$ gravity ” *Phys. Dark Univ.* **33**, 100863 (2021), arXiv:2108.02575 [INSPIRE]
8. Laxmipriya Pati, B. Mishra and S. K. Tripathy “Model parameters in the context of late time cosmic acceleration in $f(Q, T)$ gravity ” *Phys. Dark Univ.* **96**, 105003 (2021), arXiv:2107.12790 [INSPIRE]

DISSERTATIONES PHYSICAE UNIVERSITATIS TARTUENSIS

1. **Andrus Ausmees.** XUV-induced electron emission and electron-phonon interaction in alkali halides. Tartu, 1991.
2. **Heiki Sõnajalg.** Shaping and recalling of light pulses by optical elements based on spectral hole burning. Tartu, 1991.
3. **Sergei Savihhin.** Ultrafast dynamics of F-centers and bound excitons from picosecond spectroscopy data. Tartu, 1991.
4. **Ergo Nõmmiste.** Leelishalogeniidide röntgenelektronemissioon kiiritamisel footonitega energiaga 70–140 eV. Tartu, 1991.
5. **Margus Rätsep.** Spectral gratings and their relaxation in some low-temperature impurity-doped glasses and crystals. Tartu, 1991.
6. **Tõnu Pullerits.** Primary energy transfer in photosynthesis. Model calculations. Tartu, 1991.
7. **Olev Saks.** Attoampri diapasoonis soovitud mõõtmise füüsikalised alused. Tartu, 1991.
8. **Andres Virro.** AlGaAsSb/GaSb heterostructure injection lasers. Tartu, 1991.
9. **Hans Korge.** Investigation of negative point discharge in pure nitrogen at atmospheric pressure. Tartu, 1992.
10. **Jüri Maksimov.** Nonlinear generation of laser VUV radiation for high-resolution spectroscopy. Tartu, 1992.
11. **Mark Aizengendler.** Photostimulated transformation of aggregate defects and spectral hole burning in a neutron-irradiated sapphire. Tartu, 1992.
12. **Hele Siimon.** Atomic layer molecular beam epitaxy of A^2B^6 compounds described on the basis of kinetic equations model. Tartu, 1992.
13. **Tõnu Reinot.** The kinetics of polariton luminescence, energy transfer and relaxation in anthracene. Tartu, 1992.
14. **Toomas Rõõm.** Paramagnetic H^{2-} and F^+ centers in CaO crystals: spectra, relaxation and recombination luminescence. Tallinn, 1993.
15. **Erko Jalviste.** Laser spectroscopy of some jet-cooled organic molecules. Tartu, 1993.
16. **Alvo Aabloo.** Studies of crystalline celluloses using potential energy calculations. Tartu, 1994.
17. **Peeter Paris.** Initiation of corona pulses. Tartu, 1994.
18. **Павел Рубин.** Локальные дефектные состояния в CuO_2 плоскостях высокотемпературных сверхпроводников. Тарту, 1994.
19. **Olavi Ollikainen.** Applications of persistent spectral hole burning in ultrafast optical neural networks, time-resolved spectroscopy and holographic interferometry. Tartu, 1996.
20. **Ülo Mets.** Methodological aspects of fluorescence correlation spectroscopy. Tartu, 1996.
21. **Mikhail Danilkin.** Interaction of intrinsic and impurity defects in CaS:Eu luminophors. Tartu, 1997.

22. **Ирина Кудрявцева.** Создание и стабилизация дефектов в кристаллах KBr, KCl, RbCl при облучении ВУФ-радиацией. Тарту, 1997.
23. **Andres Osvet.** Photochromic properties of radiation-induced defects in diamond. Tartu, 1998.
24. **Jüri Örd.** Classical and quantum aspects of geodesic multiplication. Tartu, 1998.
25. **Priit Sarv.** High resolution solid-state NMR studies of zeolites. Tartu, 1998.
26. **Сергей Долгов.** Электронные возбуждения и дефектообразование в некоторых оксидах металлов. Тарту, 1998.
27. **Kaupo Kukli.** Atomic layer deposition of artificially structured dielectric materials. Tartu, 1999.
28. **Ivo Heinmaa.** Nuclear resonance studies of local structure in $\text{RBa}_2\text{Cu}_3\text{O}_{6+x}$ compounds. Tartu, 1999.
29. **Aleksander Shelkan.** Hole states in CuO_2 planes of high temperature superconducting materials. Tartu, 1999.
30. **Dmitri Nevedrov.** Nonlinear effects in quantum lattices. Tartu, 1999.
31. **Rein Ruus.** Collapse of 3d (4f) orbitals in 2p (3d) excited configurations and its effect on the x-ray and electron spectra. Tartu, 1999.
32. **Valter Zazubovich.** Local relaxation in incommensurate and glassy solids studied by Spectral Hole Burning. Tartu, 1999.
33. **Indrek Reimand.** Picosecond dynamics of optical excitations in GaAs and other excitonic systems. Tartu, 2000.
34. **Vladimir Babin.** Spectroscopy of exciton states in some halide macro- and nanocrystals. Tartu, 2001.
35. **Toomas Plank.** Positive corona at combined DC and AC voltage. Tartu, 2001.
36. **Kristjan Leiger.** Pressure-induced effects in inhomogeneous spectra of doped solids. Tartu, 2002.
37. **Helle Kaasik.** Nonperturbative theory of multiphonon vibrational relaxation and nonradiative transitions. Tartu, 2002.
38. **Tõnu Laas.** Propagation of waves in curved spacetimes. Tartu, 2002.
39. **Rünno Lõhmus.** Application of novel hybrid methods in SPM studies of nanostructural materials. Tartu, 2002.
40. **Kaido Reivelt.** Optical implementation of propagation-invariant pulsed free-space wave fields. Tartu, 2003.
41. **Heiki Kasemägi.** The effect of nanoparticle additives on lithium-ion mobility in a polymer electrolyte. Tartu, 2003.
42. **Villu Repän.** Low current mode of negative corona. Tartu, 2004.
43. **Алексей Котлов.** Оксианионные диэлектрические кристаллы: зонная структура и электронные возбуждения. Тарту, 2004.
44. **Jaak Talts.** Continuous non-invasive blood pressure measurement: comparative and methodological studies of the differential servo-oscillometric method. Tartu, 2004.
45. **Margus Saal.** Studies of pre-big bang and braneworld cosmology. Tartu, 2004.

46. **Eduard Gerškevičs.** Dose to bone marrow and leukaemia risk in external beam radiotherapy of prostate cancer. Tartu, 2005.
47. **Sergey Shchemelyov.** Sum-frequency generation and multiphoton ionization in xenon under excitation by conical laser beams. Tartu, 2006.
48. **Valter Kiisk.** Optical investigation of metal-oxide thin films. Tartu, 2006.
49. **Jaan Aarik.** Atomic layer deposition of titanium, zirconium and hafnium dioxides: growth mechanisms and properties of thin films. Tartu, 2007.
50. **Astrid Rekker.** Colored-noise-controlled anomalous transport and phase transitions in complex systems. Tartu, 2007.
51. **Andres Punning.** Electromechanical characterization of ionic polymer-metal composite sensing actuators. Tartu, 2007.
52. **Indrek Jõgi.** Conduction mechanisms in thin atomic layer deposited films containing TiO_2 . Tartu, 2007.
53. **Aleksei Krasnikov.** Luminescence and defects creation processes in lead tungstate crystals. Tartu, 2007.
54. **Küllike Rägo.** Superconducting properties of MgB_2 in a scenario with intra- and interband pairing channels. Tartu, 2008.
55. **Els Heinsalu.** Normal and anomalously slow diffusion under external fields. Tartu, 2008.
56. **Kuno Kooser.** Soft x-ray induced radiative and nonradiative core-hole decay processes in thin films and solids. Tartu, 2008.
57. **Vadim Boltrushko.** Theory of vibronic transitions with strong nonlinear vibronic interaction in solids. Tartu, 2008.
58. **Andi Hektor.** Neutrino Physics beyond the Standard Model. Tartu, 2008.
59. **Raavo Josepson.** Photoinduced field-assisted electron emission into gases. Tartu, 2008.
60. **Martti Pärs.** Study of spontaneous and photoinduced processes in molecular solids using high-resolution optical spectroscopy. Tartu, 2008.
61. **Kristjan Kannike.** Implications of neutrino masses. Tartu, 2008.
62. **Vigen Issahhanjan.** Hole and interstitial centres in radiation-resistant MgO single crystals. Tartu, 2008.
63. **Veera Krasnenko.** Computational modeling of fluorescent proteins. Tartu, 2008.
64. **Mait Müntel.** Detection of doubly charged higgs boson in the CMS detector. Tartu, 2008.
65. **Kalle Kepler.** Optimisation of patient doses and image quality in diagnostic radiology. Tartu, 2009.
66. **Jüri Raud.** Study of negative glow and positive column regions of capillary HF discharge. Tartu, 2009.
67. **Sven Lange.** Spectroscopic and phase-stabilisation properties of pure and rare-earth ions activated ZrO_2 and HfO_2 . Tartu, 2010.
68. **Aarne Kasikov.** Optical characterization of inhomogeneous thin films. Tartu, 2010.
69. **Heli Valtna-Lukner.** Superluminally propagating localized optical pulses. Tartu, 2010.

70. **Artjom Vargunin.** Stochastic and deterministic features of ordering in the systems with a phase transition. Tartu, 2010.
71. **Hannes Liivat.** Probing new physics in e^+e^- annihilations into heavy particles via spin orientation effects. Tartu, 2010.
72. **Tanel Mullari.** On the second order relativistic deviation equation and its applications. Tartu, 2010.
73. **Aleksandr Lissovski.** Pulsed high-pressure discharge in argon: spectroscopic diagnostics, modeling and development. Tartu, 2010.
74. **Aile Tamm.** Atomic layer deposition of high-permittivity insulators from cyclopentadienyl-based precursors. Tartu, 2010.
75. **Janek Uin.** Electrical separation for generating standard aerosols in a wide particle size range. Tartu, 2011.
76. **Svetlana Ganina.** Hajusandmetega ülesanded kui üks võimalus füüsikaõppe efektiivsuse tõstmiseks. Tartu, 2011
77. **Joel Kuusk.** Measurement of top-of-canopy spectral reflectance of forests for developing vegetation radiative transfer models. Tartu, 2011.
78. **Raul Rammula.** Atomic layer deposition of HfO_2 – nucleation, growth and structure development of thin films. Tartu, 2011.
79. **Сергей Наконечный.** Исследование электронно-дырочных и интерстициал-вакансионных процессов в монокристаллах MgO и LiF методами термоактивационной спектроскопии. Тарту, 2011.
80. **Niina Voropajeva.** Elementary excitations near the boundary of a strongly correlated crystal. Tartu, 2011.
81. **Martin Timusk.** Development and characterization of hybrid electro-optical materials. Tartu, 2012, 106 p.
82. **Merle Lust.** Assessment of dose components to Estonian population. Tartu, 2012, 84 p.
83. **Karl Kruusamäe.** Deformation-dependent electrode impedance of ionic electromechanically active polymers. Tartu, 2012, 128 p.
84. **Liis Rebane.** Measurement of the $W \rightarrow \tau\nu$ cross section and a search for a doubly charged Higgs boson decaying to τ -leptons with the CMS detector. Tartu, 2012, 156 p.
85. **Jevgeni Šablonin.** Processes of structural defect creation in pure and doped MgO and NaCl single crystals under condition of low or super high density of electronic excitations. Tartu, 2013, 145 p.
86. **Riho Vendt.** Combined method for establishment and dissemination of the international temperature scale. Tartu, 2013, 108 p.
87. **Peeter Piksarv.** Spatiotemporal characterization of diffractive and non-diffractive light pulses. Tartu, 2013, 156 p.
88. **Anna Šugai.** Creation of structural defects under superhigh-dense irradiation of wide-gap metal oxides. Tartu, 2013, 108 p.
89. **Ivar Kuusik.** Soft X-ray spectroscopy of insulators. Tartu, 2013, 113 p.
90. **Viktor Vabson.** Measurement uncertainty in Estonian Standard Laboratory for Mass. Tartu, 2013, 134 p.

91. **Kaupo Voormansik.** X-band synthetic aperture radar applications for environmental monitoring. Tartu, 2014, 117 p.
92. **Deivid Pugal.** hp-FEM model of IPMC deformation. Tartu, 2014, 143 p.
93. **Siim Pikker.** Modification in the emission and spectral shape of photo-stable fluorophores by nanometallic structures. Tartu, 2014, 98 p.
94. **Mihkel Pajusalu.** Localized Photosynthetic Excitons. Tartu, 2014, 183 p.
95. **Taavi Vaikjärv.** Consideration of non-adiabaticity of the Pseudo-Jahn-Teller effect: contribution of phonons. Tartu, 2014, 129 p.
96. **Martin Vilbaste.** Uncertainty sources and analysis methods in realizing SI units of air humidity in Estonia. Tartu, 2014, 111 p.
97. **Mihkel Rähn.** Experimental nanophotonics: single-photon sources- and nanofiber-related studies. Tartu, 2015, 107 p.
98. **Raul Laasner.** Excited state dynamics under high excitation densities in tungstates. Tartu, 2015, 125 p.
99. **Andris Slavinskis.** EST Cube-1 attitude determination. Tartu, 2015, 104 p.
100. **Karlis Zalite.** Radar Remote Sensing for Monitoring Forest Floods and Agricultural Grasslands. Tartu, 2016, 124 p.
101. **Kaarel Piip.** Development of LIBS for *in-situ* study of ITER relevant materials. Tartu, 2016, 93 p.
102. **Kadri Isakar.** ^{210}Pb in Estonian air: long term study of activity concentrations and origin of radioactive lead. Tartu, 2016, 107 p.
103. **Artur Tamm.** High entropy alloys: study of structural properties and irradiation response. Tartu, 2016, 115 p.
104. **Rasmus Talviste.** Atmospheric-pressure He plasma jet: effect of dielectric tube diameter. Tartu, 2016, 107 p.
105. **Andres Tiko.** Measurement of single top quark properties with the CMS detector. Tartu, 2016, 161 p.
106. **Aire Olesk.** Hemiboreal Forest Mapping with Interferometric Synthetic Aperture Radar. Tartu, 2016, 121 p.
107. **Fred Valk.** Nitrogen emission spectrum as a measure of electric field strength in low-temperature gas discharges. Tartu, 2016, 149 p.
108. **Manoop Chenchiliyan.** Nano-structural Constraints for the Picosecond Excitation Energy Migration and Trapping in Photosynthetic Membranes of Bacteria. Tartu, 2016, 115p.
109. **Lauri Kaldamäe.** Fermion mass and spin polarisation effects in top quark pair production and the decay of the higgs boson. Tartu, 2017, 104 p.
110. **Marek Oja.** Investigation of nano-size α - and transition alumina by means of VUV and cathodoluminescence spectroscopy. Tartu, 2017, 89 p.
111. **Viktoriia Levushkina.** Energy transfer processes in the solid solutions of complex oxides. Tartu, 2017, 101 p.
112. **Mikk Antsov.** Tribomechanical properties of individual 1D nanostructures: experimental measurements supported by finite element method simulations. Tartu, 2017, 101 p.
113. **Hardi Veermäe.** Dark matter with long range vector-mediated interactions. Tartu, 2017, 137 p.

114. **Aris Auzans.** Development of computational model for nuclear energy systems analysis: natural resources optimisation and radiological impact minimization. Tartu, 2018, 138 p.
115. **Aleksandr Gurev.** Coherent fluctuating nephelometry application in laboratory practice. Tartu, 2018, 150 p.
116. **Ardi Loot.** Enhanced spontaneous parametric downconversion in plasmonic and dielectric structures. Tartu, 2018, 164 p.
117. **Andreas Valdmann.** Generation and characterization of accelerating light pulses. Tartu, 2019, 85 p.
118. **Mikk Vahtrus.** Structure-dependent mechanical properties of individual one-dimensional metal-oxide nanostructures. Tartu, 2019, 110 p.
119. **Ott Vilson.** Transformation properties and invariants in scalar-tensor theories of gravity. Tartu, 2019, 183 p.
120. **Indrek Sünter.** Design and characterisation of subsystems and software for ESTCube-1 nanosatellite. Tartu, 2019, 195 p.
121. **Marko Eltermann.** Analysis of samarium doped TiO₂ optical and multi-response oxygen sensing capabilities. Tartu, 2019, 113 p.
122. **Kalev Erme.** The effect of catalysts in plasma oxidation of nitrogen oxides. Tartu, 2019, 114 p.
123. **Sergey Koshkarev.** A phenomenological feasibility study of the possible impact of the intrinsic heavy quark (charm) mechanism on the production of doubly heavy mesons and baryons. Tartu, 2020, 134 p.
124. **Kristi Uudeberg.** Optical Water Type Guided Approach to Estimate Water Quality in Inland and Coastal Waters. Tartu, 2020, 222 p.
125. **Daniel Blixt.** Hamiltonian analysis of covariant teleparallel theories of gravity. Tartu, 2021, 142 p.
126. **Ulbossyn Ualikhanova.** Gravity theories based on torsion: theoretical and observational constraints. Tartu, 2021, 154 p.
127. **Iaroslav Iakubivskyi.** Nanospacecraft for Technology Demonstration and Science Missions. Tartu, 2021, 177 p.
128. **Heido Trofimov.** Polluted clouds at air pollution hot spots help to better understand anthropogenic impacts on Earth's climate. Tartu, 2022, 96 p.
129. **Ott Rebane.** *In situ* non-contact sensing of microbiological contamination by fluorescence spectroscopy. Tartu, 2022, 157 p.
130. **Juhan Saaring.** Ultrafast Relaxation Processes in Ternary Hexafluorides Studied under Synchrotron Radiation Excitation. Tartu, 2022, 106 p.
131. **Ahmet Ilker Topuz.** Quantitative and qualitative investigations for muon scattering tomography via GEANT4 simulations: A computational study. Tartu, 2023, 163 p.
132. **Nico Benincasa.** Phase transitions and gravitational waves in models of dark matter. Tartu, 2023, 206 p.
133. **Kaja Pae.** Electron-phonon interactions in local degenerate electronic states in solids. Tartu, 2024, 201 p.
134. **Kristjan Müürsepp.** Phenomenological implications of Standard Model extensions. Tartu, 2024, 136 p.
135. **Ye Wang.** Investigating the properties of metal surfaces under high electric fields based on ab initio calculations. Tartu, 2024, 107 p.