University of Tartu
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Arithmetic Asian options pricing using general lattice method

Actuarial and Financial Engineering

Master's Thesis (30 ECTS)

Supervisors: Toomas Raus

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Tartu 2025

ARITHMETIC ASIAN OPTIONS PRICING USING GENERAL LATTICE METHOD

Master thesis

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Abstract

The aim of this thesis is to study the pricing of path-dependent discrete arithmetic Asian options. In the case of arithmetic Asian options, the option price cannot be found analytically and various numerical methods must be used to find the price.

The theoretical part of this thesis focuses on the general lattice method proposed in 2020 by Gambaro, Kyriakou, and Fusai for pricing European and American style Asian options with fixed and floating strike prices. The novelty of this approach is that by a change of numeraire, lattice becomes one dimensional, while previous lattice methods were two dimensional. In addition, this thesis also examines the pricing of Asian options using Monte Carlo simulations and Hull-White method.

Numerical examples compare the accuracy and computation speed of option prices obtained by general lattice method, the Hull-White method and the Monte Carlo method.

CERCS research specialisation: P160 Statistics, operations research, programming, financial and actuarial mathematics.

Key Words: Arithmetic Asian option, Monte Carlo, Hull-White method, general lattice method

ARITMEETILISE AASIA OPTSIOONI HINNA LEIDMINE ÜLDISE VÕREMEETODIGA

Magistritöö

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Lühikokkuvõte

Selle lõputöö eesmärk on uurida teest sõltuvate diskreetsete aritmeetiliste Aasia optsioonide hinnastamist. Aritmeetilise Aasia optsiooni korral ei õnnestu optsiooni hinda leida analüütiliselt ning hinna leidmiseks tuleb kasutada erinevaid numbrilisi meetodeid.

Töö teoreetilises osas on põhirõhk 2020.aastal kirjanduses väljapakutud üldise võremeetodi tutvustamisel nii fikseeritud kui ujuva täitmishinnaga Euroopa ja Ameerika tüüpi Aasia optsioonide hinnastamiseks. Meetod on huvipakkuv selle poolest, et võimaldab *numeraire* vahetusega muuta võremeetodi ühemõõtmeliseks, samal ajal kui varasemad võremeetodid Aasia optsiooni hinnastamiseks olid kahemõõtmelised. Lisaks sellele meetodile vaadeldakse töös Aasia optsiooni hinna leidmist ka Monte-Carlo simulatsioonidega ning Hull-White meetodil.

Numbriliste eksperimentide käigus võrreldakse nii üldise võremeetodiga, Hull-White meetodiga ja Monte-Carlo meetodiga leitud optsiooni hindade täpsust ning arvutuste kiirust.

CERCS teaduseriala: P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika.

Märksõnad: Aritmeetilise keskmisega Aasia optsioon, Monte Carlo, Hull-White meetod, üldine võremeetod

Contents

Introduction

Financial markets offer multiple financial instruments, among which options play an important role. Options provide the right but not the obligation to buy or sell an asset at predetermined strike price and on predetermined exercise date (or before). Options are classified into vanilla and exotic options, depending on the structure of payoff and/or exercise features. Payoff of a vanilla option depends on the price of underlying asset at the day of realization. Although vanilla options are widely known and traded, there has been growing interest in exotic options.

Asian options are path-dependent exotic options. Their payoff depends on the average price of the underlying asset over some period. This averaging mechanism reduces volatility of an underlying asset and that is why Asian options are cheaper than vanilla options with the same strike price and maturity. Asian options are widely used in the market of commodities, such as gas, oil, metal or grain, and for hedging in energy markets (McHich, 2020); usage of average price of an asset, reduces the effect from short-term price fluctuations. This makes Asian options less sensitive to market shocks compare to vanilla options.

There are different ways of calculating the average. In practice, monitoring of asset's price is almost always discrete. While the geometric-average European-style Asian option has closed-form solution, pricing of arithmetic Asian option requires numerical methods.

The objective of this thesis is to study the Asian options' theory and their pricing methods, in particular, a new general lattice method presented in the paper "General lattice methods for arithmetic Asian optins" by Gambaro, Kyriakou, and Fusai, 2020. Furthermore, we compare efficiency of the new method to standard lattice Hull-White method and Monte Carlo simulations via numerical examples

Thesis consists of an introduction, 4 chapters, a conclusion, references and an appendix. In the first chapter, theoretical background of options and option pricing is given. The second chapter introduces Asian options and several Asian option pricing methods: exact formula for discrete geometric average Asian option, Turnbull-Wakeman approximation and Hull-White method. In the third chapter, the new general lattice method is explained in detail for European-style fixed strike call option and given for put option. Also, in the third chapter we provide formulas in case of floating strike option, American-style fixed strike option and forward start option. Chapter four consists of numerical examples for Asian options pricing with Monte Carlo, Hull-White and general lattice method, described in chapter 3, as well as comparison of obtained results. Additional results of numerical examples are presented in Appendix 1.

1 Theoretical foundations of options and option pricing

Options are financial derivatives that grant the holder the right (but not the obligation) to buy or sell an underlying asset. The underlying asset can be a financial instrument, physical goods or other measurable quantities, for example, stocks, oil, gold, electricity or even weather indexes. Pricing models aim to determine the fair price of an option. Reliable pricing is crucial for the issuer of an option, since the issuer takes on the responsibility to fulfill the agreement if the holder decides to exercise the option. The theoretical background presented in this chapter is primarily based on the book "Options, Futures, and Other Derivatives" (Hull and Basu, 2016).

1.1 Options

There are two types of options: **call** and **put**. A call option gives the holder the right to buy the underlying asset, while a put option gives the right to sell the asset. Both the purchase and sale of the underlying asset must be exercised by a specified date and at a predetermined price. The date and the price are specified in the contract. The date is referred to as the **exercise date** or **maturity**, while the price is known as the **exercise price** or **strike price**.

First, we introduce the necessary notations:

- T - time to maturity,
- S_t - the underlying asset's price at time t $(0 \leq t \leq T)$,
- S_0 - the initial asset price,
- σ - volatility, measured by the standard deviation of asset return,

- r - continuous, annual risk-free interest rate,
- K - strike price,
- V_C and V_P - prices of call and put options.

Call and put options can be considered as **European options** and **American options**. Subsequently, a description of each will be presented.

A **European option** is a type of option that can only be exercised at maturity. This characteristic makes European options easier to analyze mathematically, as their value solely depends on the underlying asset's price at maturity.

Definition 1. *The payoff of European call options is defined by*

$$P = P(S_T) = \max(S_T - K, 0) \tag{1}$$

and in case of put option

$$P = P(S_T) = \max(K - S_T, 0) \tag{2}$$

where S_T is the asset price at maturity.

An **American option** can be exercised at any time up to and including the expiration date. This flexibility makes American options generally more valuable and more complex to price.

Definition 2. *The payoff of American call options is defined by*

$$P = P(S_t) = \max(S_t - K, 0) \tag{3}$$

and in case of put option

$$P = P(S_t) = \max(K - S_t, 0), \tag{4}$$

where S_t is the asset price and $0 \leq t \leq T$.

Although European options are typically more straightforward to analyze, the majority of exchange-traded options are American in nature. Nevertheless, many theoretical properties of American options are derived from the characteristics of European options.

1.2 Black-Scholes equation

If the market is weakly efficient, then the continuous return of asset $\ln S(t)$ can be described by the following differential equation:

$$d \ln S(t) = \rho(t)dt + \sigma(t)dW(t),$$

where $\rho(t)$ is expected rate of return, $\sigma(t)$ is the volatility of asset price $S(t)$ and $dW(t)$ is a Wiener process.

Applying Ito's lemma we obtain:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \tag{5}$$

where $\mu(t) = \rho(t) + 0.5\sigma^2(t)$.

Market model is arbitrage-free if the risk-neutral probability exists, therefore we introduce the risk-neutral probability measure Q. Then differential equation (5) takes the following form:

$$d \ln S(t) = \left(r(t) - \frac{1}{2}\sigma^2(t)\right) dt + \sigma dW_Q(t),$$

where $dW_Q(t)$ is a Wiener process under the risk-neutral measure Q.

The Black-Scholes partial differential equation provides a foundational framework for pricing options. To derive the Black-Scholes partial differential equation, the following assumptions are made:

- The price of the underlying asset follows the lognormal random walk (5).
- The risk-free interest rate $r(t)$ and volatility of an asset $\sigma(t)$ are known at time $t = 0$.
- It is assumed that the market is arbitrage-free.
- The underlying asset does not pay dividends during the life of the option.
- There are no taxes or transaction costs, and trading is continuous.
- Short selling is unrestricted and the assets are divisible.

Suppose that option price $V = V(S,t)$ depends on S and t. Under the assumptions listed above, it is possible to show that option price V satisfies the Black-Scholes partial differential equation:

$$\frac{\partial V(S(t),t)}{\partial t} + \frac{1}{2}\sigma^2(t)S^2(t)\frac{\partial^2 V(S(t),t)}{\partial S(t)^2} + r(t)S(t)\frac{\partial V(S(t),t)}{\partial S(t)} - r(t)V(S(t),t) = 0 \tag{6}$$

Equation (6) has infinitely many solutions, in order to find a unique, we must specify an additional condition at maturity, so as we know that option price at $t = T$ is equal to payoff: $V(S(T),T) = P(S(T))$.

In case of European call and put options, equation (6) can be solved analytically. If volatility and interest rate are constants, the prices of European call and put options are given using the Black-Scholes formulas:

$$V_C = S_0\Phi(d_1) - K\exp(-rT)\Phi(d_2),$$

$$V_P = K\exp(-rT)\Phi(-d_2) - S_0\Phi(-d_1),$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \; d_2 = d_1 - \sigma\sqrt{T},$$

and $\Phi()$ denotes the cumulative distribution function of the standard normal distribution.

In addition to vanilla options, there also exist **exotic options** that have more complex features, for example, payoff that depends on asset's value at several times during the life of the contract or, on maximal or minimal asset's price over the option's lifetime. One notable category of exotic options is Asian options, which will be discussed in the next chapter.

In some cases, like for European options, the price can be found analytically, however, it is not the case for every type of options. Therefore, numerical methods are required. Common approaches include lattice methods (such as binomial and trinomial), finite difference methods, and Monte Carlo simulations.

1.3 Binomial model

The binomial model, introduced by Cox, Ross and Rubinstein (Cox, Ross, and Rubinstein, 1979), is a discrete-time framework for option pricing, offering an effective approach for valuing both European and American options.

The binomial model assumes that the price of the underlying asset follows a multiplicative binomial process over discrete time intervals. For the model, the lifetime of an option $[0,\ T]$ is be divided into N equal time steps of size Δt. At each time step, the asset price can move either up by a factor u with probability p, or down by a factor d with probability $1-p$. Thus, if the current asset price is S, its value at the next time step becomes either Su or Sd as shown in Figure 1.

Within the framework of the model, a risk-neutral world is assumed, in which investors are considered to be indifferent to risk. One of the fundamental principles of option pricing is that a model is arbitrage-free and complete if there exists a unique risk-neutral probability measure with respect to the chosen numeraire

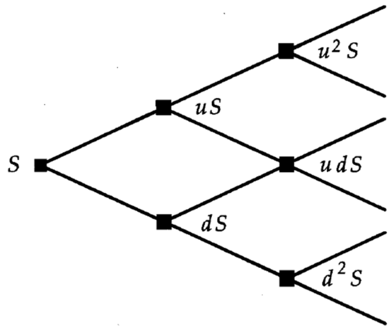

Figure 1: Tree of asset prices for binomial model (Wilmott, Dewynne, and Howison, 1993).

(Pascucci, 2011). In case of the binomial model, bank account process $B(t)$, which models a short term risk-free investment ($B(0) = 1$), can serve as the numeraire, meaning that all asset prices are evaluated relative to $B(t)$. Another possible choice of numeraire is price $S(t)$.

Under the risk-neutral measure, the probability p is the risk-neutral probability, chosen such that the expected discounted price of the asset equals its current price. Taking into account risk-neutral measure, the conditional expected value of asset price at the step $t + \Delta t$ must be risk-free return multiplied by asset price at time t:

$$\mathbb{E}(S_{t+\Delta t}|S_t) = \exp(r\Delta t)S_t.$$

Using binomial model, the same result is found by:

$$\mathbb{E}(S_{t+\Delta t}|S_t) = puS_t + (1-p)dS_t.$$

From the two equalities presented above, we obtain the following equation:

$$pu + (1-p)d = \exp(r\Delta t), \tag{7}$$

which leads to:

$$p = \frac{\exp(r\Delta t) - d}{u - d}. \tag{8}$$

To ensure that $0 < p < 1$, parameters u and d must satisfy following inequality: $0 < d < R < u$, where risk free return $R = \exp(r\Delta t)$.

If asset price behavior is given by the expression (5), then the conditional variance of asset price S_t at time step $t + \Delta t$ is given by:

$$Var(S_{t+\Delta t}|S_t) = \sigma^2 S_t^2 \Delta t + o(\Delta t),$$

where $o(\Delta t)$ indicates that $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.

In the binomial model, the corresponding conditional variance is:

$$Var(S_{t+\Delta t}|S_t) = S_t^2(pu^2 + (1-p)d^2 - (pu + (1-p)d)^2).$$

From the two equations of variance follows that

$$p(1-p)(u-d)^2 = \sigma^2 \Delta t + o(\Delta t). \tag{9}$$

The parameters u and d must satisfy (9), however, there is no unique solution for u and d, multiple valid specifications exist for the up and down factors. For example, by setting additional condition $u = \frac{1}{d}$, we obtain Cox-Ross-Rubinstein (CRR) model (Cox, Ross, and Rubinstein, 1979) as follows:

$$\begin{cases} u = \exp(\sigma\sqrt{\Delta t}) \\ d = \exp(-\sigma\sqrt{\Delta t}). \end{cases} \tag{10}$$

Additionally, we present the setup for the Jarrow–Rudd model (Jarrow and Rudd, 1982):

$$\begin{cases} u = \exp((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}) \\ d = \exp((r - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}). \end{cases}$$

If the price process $S(t)$ is chosen as numeraire, then the risk-neutral probabilities can be found from the equation:

$$\frac{B(t)}{S(t)} = \bar{p}_1 \frac{RB(t)}{uS(t)} + (1 - \bar{p}_1) \frac{RB(t)}{dS(t)}$$

or

$$1 = \bar{p}_1 \frac{R}{u} + (1 - \bar{p}_1) \frac{R}{d}. \tag{11}$$

Then solving equation we obtain

$$\bar{p}_1 = \frac{u(R-d)}{R(u-d)} \text{ and } \bar{p}_2 = 1 - \bar{p}_1.$$

Next, we will discuss the application of the binomial model to the pricing of European and American options. As it was discussed previously, the price of an asset can go up by u or down by d, therefore the asset's price $S_{i,j}$ at each node (i, j) i.e., at time step i and state j, has the form:

$$S_{i,j} = S(0)u^j d^{i-j}, \quad i = 0, 1, ..., N \text{ and } 0 \le j \le i.$$

For European options the first step is computing the option's price $V_{N,j}$ at the terminal nodes of the tree:

$$V_{N,j} = P(S_{N,j}), \quad 0 \le j \le N.$$

Then, computing backward to the present, at each node (i, j) price of the option $V_{i,j}$ is given by:

$$V_{i,j} = R^{-1}(p_1 V_{i+1,j+1} + p_2 V_{i+1,j}), \quad i = N-1,\ N-2, ..., 0 \text{ and } 0 \le j \le i. \tag{12}$$

By backward computations, we obtain current price $V_{0,0}$ of the option at time $t = 0$.

For the pricing of American options, the procedure is to work backward through the binomial tree from maturity to the initial node where $i = 0$ and $j = 0$, evaluating at each step whether early exercise is optimal. The price of the option at maturity is the same as for European option:

$$V_{N,j} = P(S_{N,j}), \quad 0 \leq j \leq N.$$

For earlier nodes it is the maximum of the payoff from early exercise and the value given by formula of European option (see (12)):

$$V_{i,j} = \max(P(S_{i,j}),\ R^{-1}(p_1 V_{i+1,j+1} + p_2 V_{i+1,j})), \quad i = N-1,\ N-2, ..., 0 \text{ and } 0 \leq j \leq i.$$

Then current price of the option at time $t = 0$ is found by $V_{0,0}$.

1.4 Options pricing with Monte Carlo simulations

Another common approach in options pricing is the use of Monte Carlo simulations. The theoretical framework in this section is primarily based on the book: "Monte Carlo Methods in Financial Engineering" (Glasserman, 2004).

To estimate option prices using Monte Carlo method, we use the stochastic differential equation, which describes evolution of asset price under risk-neutral probability measure (see (5)). The equation has a closed-form solution, which allows us to express the price of asset at maturity:

$$S_T = S_0 \exp[(r - 0.5\sigma^2)T + \sigma W_Q(T)]. \tag{13}$$

The Brownian motion $W_Q(T)$ is normally distributed with a mean of 0 and a standard deviation of $\sqrt{T}$. To express $W_Q(T)$ using the standard normal distribution, we can apply a variable change, as follows:

$$W_Q(T) = Z(T) * \sqrt{T}, \tag{14}$$

where $Z(T)$ is a standard normal random variable with mean of 0 and standard deviation of 1. Then the formula (13) can be rewritten as:

$$S_T = S_0 \exp[(r - 0.5\sigma^2)T + \sigma\sqrt{T} * Z(T)]. \tag{15}$$

This formula will be used further for Monte Carlo simulations.

The core idea of the Monte Carlo method is random sampling from a given distribution to generate the required estimations. For pricing the options, we generate independent standard normal variable. Formula (15) is then used to calculate a stock price. Finally, the discounted payoff $\exp(-rT)P(S_T)$ is calculated. This process is repeated n times, where n represents the number of Monte Carlo simulations. The average of the n discounted payoffs is the result of the simulations i.e. price $\bar{V}$ of the option at time $t = 0$.

The standard error of option price $\bar{V}$ at time $t = 0$ is estimated with the following formula:

$$\sqrt{\frac{1}{(n-1)n}\sum_{i=1}^{n}((\exp(-rT)P(S_T))_i - \bar{V})^2}.$$

As the number of Monte Carlo simulations increases, the accuracy of the model improves. However, variance can also be reduced using variance reduction techniques. Below, we describe the antithetic variates and control variates methods.

Antithetic variates

The antithetic variates method works by generating pairs of correlated random variables whose means are opposite. Specifically, if we generate a random variable Z, the antithetic pair is given by $-Z$. The idea is to use both the original and the antithetic variables to obtain an average. This approach results in variance reduction and lowers the number of required samples.

To price options using the antithetic variates method, a standard normal random variable Z is generated and its opposite is denoted by $-Z$. Then, the asset price paths are calculated using both Z and $-Z$. After that, discounted payoffs for both the original and antithetic paths are obtained. The payoffs are denoted as $P(S_T)$ and $\tilde{P}(S_T)$, respectively. As a last step, average of the discounted payoffs is calculated to obtain the antithetic estimate:

$$\frac{1}{2n}\sum_{i=1}^{n} \exp(-rT)\left(P(S_T) + \tilde{P}(S_T)\right).$$

The antithetic variates method helps to reduce variance, which in turn accelerates convergence and decreases the number of simulations needed.

Control variates

To describe the control variates method, let Y_i denote the discounted payoff in the ith Monte Carlo simulation. We suppose that Y_i are independent and identically distributed (i.i.d.). After n replications of the simulation, we obtain the values $Y_1, ..., Y_n$. Our objective is to estimate $\mathbb{E}[Y]$, for which we use the unbiased estimator $\hat{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. It can be shown that as $n \to \infty$, $\hat{Y}$ converges to $\mathbb{E}[Y]$ with probability 1. Now, suppose that for each of the n independent simulation runs, we generate an additional output X_i. Assume that the pairs (X_i, Y_i), $i = 1, ..., n$, are i.i.d and that $\mathbb{E}[X]$ is known. Then, for any fixed b, we can construct the following unbiased control variate estimator:

$$\hat{Y}(b) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - b(X_i - \mathbb{E}[X])). \tag{16}$$

The optimal coefficient b that minimizes the variance is given by:

$$b = \frac{Cov(X,Y)}{Var(X)}. \tag{17}$$

Therefore, the greater the correlation between the Monte Carlo estimate and the control variate, the greater the variance reduction.

2 Asian options

Asian options are a type of exotic options in which the payout is determined by the underlying asset's average price over a given time period. This averaging mechanism makes them less volatile compared to standard European and American options. Additionally, attempts to influence the asset price right before expiration will typically have little to no impact on the option's value if it is based on an average (Haug, 2007). In this chapter, the main characteristics of Asian options are discussed. The theory and mathematical concepts related to Asian options presented in this chapter are primarily based on the book "Mathematical Models of Financial Derivatives" (Kwok, 2008).

2.1 Classifications of Asian options

To begin the classification of Asian options, we first examine how their payoffs are defined. Let us denote A_T as the average price of the underlying asset over the relevant period. Asian options that can be exercised only at maturity are called **European-style** Asian options. However, if an Asian option can be exercised at any time up to and including maturity, then it is called **American-style** Asian option.

Definition 3. *The payoff P of European-style Asian call option at time T with a fixed strike price is defined by*

$$P = P(A_T) = \max(A_T - K, 0) \tag{18}$$

and in case of put option

$$P = P(A_T) = \max(K - A_T, 0). \tag{19}$$

Definition 4. *The payoff P of European-style Asian call option at time T with a* *floating strike price are defined by*

$$P = P(A_T) = \max(\lambda S_T - A_T, 0) \tag{20}$$

and in case of put option

$$P = P(A_T) = \max(A_T - \lambda S_T, 0), \tag{21}$$

where S_T is the price of the underlying asset at maturity and $\lambda \in \mathbb{R}$ is the number of units of stock received or bought, usually $\lambda = 1$ (Henderson and Wojakowski, 2002).

To define the average price A_T used in Asian options, several factors must be considered, including the selection of data points and the method used to compute the average. Arithmetic and geometric averages are the two most basic and widely used methods to calculate the average (Han and Hong, 2022). They can be separated into two categories: discrete and continuous. This thesis specifically examines the discrete averaging method.

In the case of discrete averaging, it is possible to include or exclude the initial price S_0 from the calculation of the average price. When the initial price is included in the average (i.e., $\alpha = 0$), the option is referred to as a **Standard Asian option**. Conversely, when the averaging starts at some date in the future, it is known as a **Forward-starting Asian option** ($\alpha > 0$ and $\alpha \in \mathbb{N}$).

We introduce an additional notation: let N denote the number of equal-length periods partitioning the time period $[0,\ T]$ and discrete time points $t_i = i\Delta t$, $i = 0, 1, ..., N$.

Definition 5. *The discrete arithmetic average of Asian options over the period $[0,\ T]$ is defined by*

$$A_T = \frac{1}{N - \alpha + 1} \sum_{i=\alpha}^{N} S_{t_i}. \tag{22}$$

Definition 6. *The discrete geometric average of Asian options over the period* $[0,\ T]$ *is defined by*

$$G_T = \left(\prod_{i=\alpha}^{N} S_{t_i} \right)^{1/(N-\alpha+1)}. \tag{23}$$

2.2 Asian options pricing

Asian option pricing is first considered in the context of geometric average European-style Asian option, for which a closed-form solution exists. The geometric average follows a lognormal distribution, allowing its expected value and variance to be calculated explicitly (Kemna and Vorst, 1990).

The price of the **discrete geometric European-style average Asian option** with fixed strike price K is given by (Han and Hong, 2022):

$$V_C = S_0 \exp(\mu T)\Phi(d_1) - K\Phi(d_2) \tag{24}$$

$$V_P = K\Phi(-d_2) - S_0 \exp(\mu T)\Phi(-d_1),$$

where

$$\sigma_{adj}^2 = \frac{(2N+1)(N+1)^\alpha}{6N^\alpha(N+1-\alpha)}\sigma^2,\ \mu = \frac{N+1}{2(N+1-\alpha)}\left(r - \frac{\sigma^2}{2}\right) + \frac{\sigma_{adj}^2}{2},$$

$$d_1 = \frac{\ln(S_0/K) + (\mu + \frac{\sigma_{adj}^2}{2}T)}{\sigma_{adj}\sqrt{T}},\ d_2 = d_1 - \sigma_{adj}\sqrt{T},$$

and $\alpha = 0$ for standard Asian options, $\alpha = 1$ for forward-starting Asian options.

While geometric Asian options have a closed-form solution, arithmetic Asian options lack an exact formula. Therefore, arithmetic average Asian options can be priced using several numerical methods.

Next, we will examine commonly used approximate formula for pricing arithmetic

European-style Asian options, as presented in "The Complete Guide to Option Pricing Formulas" (Haug, 2007).

The **Turnbull and Wakeman Approximation** adjusts the mean and variance in such a way that they align with the exact moments of the arithmetic average, providing a more accurate estimate for the option price. The adjusted mean and variance are then used in the Black-Scholes model formula:

$$V_C(0) \approx \exp(-rT)(\mathbb{E}[A_T]\Phi(d_1) - K\Phi(d_2)),$$

$$V_P(0) \approx \exp(-rT)(K\Phi(-d_2) - \mathbb{E}[A_T]\Phi(-d_1)),$$

where

$$d_1 = \frac{\ln\left(\frac{\mathbb{E}[A_T]}{K}\right) + \frac{1}{2}\sigma_A^2 T}{\sigma_A\sqrt{T}}, \; d_2 = d_1 - \sigma_A\sqrt{T},$$

$$\sigma_A = \sqrt{\frac{\ln(\mathbb{E}[A_T^2]) - 2\ln(\mathbb{E}[A_T])}{T}}$$

and T is the time to maturity. The first and second moments of the arithmetic average are given by:

$$\mathbb{E}[A_T] = \frac{S_0(1 - \exp(rhm))}{m(1 - \exp(rh))},$$

$$\mathbb{E}[A_T^2] = \frac{S_0}{m^2}\left(\frac{1 - \exp((2r + \sigma^2)hm)}{1 - \exp((2r + \sigma^2)h)} + \right.$$
$$\left. + \frac{2}{1 - \exp((r + \sigma^2)h)}\left(\frac{1 - \exp(rhm)}{1 - \exp(rh)} - \frac{1 - \exp((2r + \sigma^2)hm)}{1 - \exp((2r + \sigma^2)h)}\right)\right),$$

where $h = \frac{T}{m-1}$ and m is number of time points equally spaced between $[0, T]$. The formula is given in case when averaging starts from 0, i.e. parameter $\alpha = 0$.

Asian options pricing with Hull and White method

The method proposed by Hull and White (Hull and White, 1993) is lattice-based approach for the pricing of path-dependent options. In this section, it will be presented for fixed strike price K but it is possible to transform the model for the case when the strike price is floating.

The usual binomial tree (see section 1.3) is modified by the state vector that for Asian options tracks an average price over time. For options based on arithmetic averages, the number of possible values for the average grows exponentially with the number of time steps. It is impractical to track every average, therefore, a grid with the range of possible averages is constructed. To estimate option values, an interpolation is used when working backwards through the tree.

Using the CRR binomial tree model (see section 1.3), the values of average price A and asset price S are given by

$$S_j^n = S_0 \exp(j\sigma\sqrt{\Delta t}),\ n = 0,\ ...,\ N;\ j = -n,\ -n+2,\ ...,\ n-2,\ n,$$

$$A_k^n = S_0 \exp(kh),\ n = 0,\ ...,\ N;\ k = k_{min}(n),\ ...,\ k_{max}(n), \tag{25}$$

the formulas for $k_{min}(n)$ and $k_{max}(n)$ will be given later. Scaling factor h is selected such that the number of average nodes at $t = T$ stays the same across different maturities and volatilities (Forsyth, Vetzal, and Zvan, 2002):

$$h = \alpha\sqrt{\frac{0.25}{T}\sigma^2\Delta t}, \tag{26}$$

where parameter α determines the granularity of the grid in the average direction.

Following the approach of Chalasani (Chalasani et al., 1999) the number of nodal averages at time step n can be estimated (for large enough n). For a lattice after

n steps, the maximum possible average value is:

$$A^n_{max} = \frac{\sum_{k=0}^{n} \exp(k\sigma\sqrt{\Delta t})}{n+1} = \frac{1-\exp(\sigma\sqrt{\Delta t}(n+1))}{(n+1)(1-\exp(\sigma\sqrt{\Delta t}))}.$$

The minimum possible value after n steps is:

$$A^n_{min} = \frac{\sum_{k=0}^{n} \exp(-k\sigma\sqrt{\Delta t})}{n+1} = \frac{1-\exp(-\sigma\sqrt{\Delta t}(n+1))}{(n+1)(1-\exp(-\sigma\sqrt{\Delta t}))}.$$

Using A^N_{max} and A^N_{min} we can define the initial values of k index:

$$k_{min}(N) = \left[\frac{1}{h}\frac{\ln(A^N_{min})}{\ln(S_0)}\right] - 1,$$

$$k_{max}(N) = \left[\frac{1}{h}\frac{\ln(A^N_{max})}{\ln(S_0)}\right] + 1.$$

Then it is possible to calculate the values of average price A^n_k (see (25)), $n = 0, .., N$.

In binomial approximation price changes from S^n_j up to S^{n+1}_{j+1} or down to S^{n+1}_{j-1} with risk neutral probability p or $1-p$ accordingly. From the transitions, the average is updated:

$$A^{n+1}_{k^+(j,k)} = A^n_k + \frac{S^{n+1}_{j+1} - A^n_k}{n+2},$$

$$A^{n+1}_{k^-(j,k)} = A^n_k + \frac{S^{n+1}_{j-1} - A^n_k}{n+2}.$$

The Figure 2 illustrates that asset price S has set of average values A^n_k and approximate prices of the option V^n_k at each tree node.

The averages $A^{n+1}_{k^+(j,k)}$ and $A^{n+1}_{k^-(j,k)}$ might not lie on the lattice grid. Therefore, to find the values, it is possible to use interpolation.

We define the indices for the lattice average:

$$k^{\pm}_{floor}(j,k) = \left\lfloor \frac{1}{h}\frac{\ln(A^{n+1}_{k^{\pm}(j,k)})}{\ln(S_0)} \right\rfloor$$

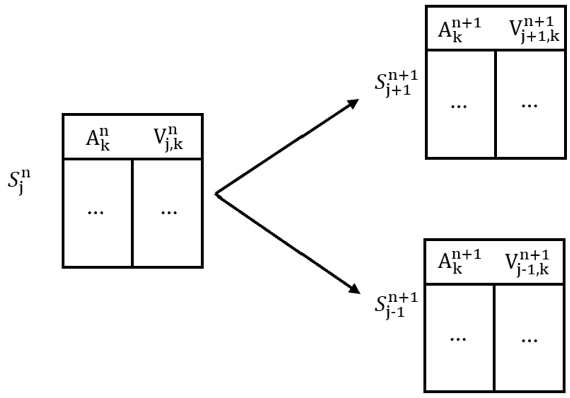

Figure 2: Tree of asset prices with a set of discrete averages and approximate option values (Forsyth, Vetzal, and Zvan, 2002).

$$k^{\pm}_{ceil}(j,k) = k^{\pm}_{floor}(j,k) + 1.$$

By the backward recursion from terminal condition $V^N_{j,k} = P(A^N_k)$, we can find the value of the option. The estimated value $V^n_{j,k}$ is calculated by linear interpolation:

$$V^n_{j,k} = \exp(-r\Delta t)\Bigg[p\left(\alpha^{n+1}_{k^+_{floor}(j,k)} V^{n+1}_{j+1,k^+_{floor}(j,k)} + \left(1-\alpha^{n+1}_{k^+_{floor}(j,k)}\right) V^{n+1}_{j+1,k^+_{ceil}(j,k)}\right) + \\ +(1-p)\left(\alpha^{n+1}_{k^-_{floor}(j,k)} V^{n+1}_{j-1,k^-_{floor}(j,k)} + \left(1-\alpha^{n+1}_{k^-_{floor}(j,k)}\right) V^{n+1}_{j-1,k^-_{ceil}(j,k)}\right)\Bigg]$$

$$n = N-1,\ldots,0; \quad j = -n,\ -n+2,\ \ldots,\ n-2,\ n; \quad k = k_{min}(n),\ldots,k_{max}(n),$$

where p is risk neutral probability and

$$\alpha^{n+1}_{k^{\pm}_{floor}(j,k)} = \frac{A^{n+1}_{k^{\pm}(j,k)} - A^{n+1}_{k^{\pm}_{floor}(j,k)}}{A^{n+1}_{k^{\pm}_{ceil}(j,k)} - A^{n+1}_{k^{\pm}_{floor}(j,k)}}.$$

By backward recursion we obtain the price of an European-style Asian option at time $t = 0$, i.e. $V^0_{0,0}$. To find the price of American-style Asian option, it is necessary to take the maximum of the continuation value $V_{j,k}^n$ and immediate exercise value:

$$\max(P(A_k^n), V_{j,k}^n),$$

$$n = N-1, \ldots, 0; \quad j = -n,\ -n+2,\ ...,\ n-2,\ n; \quad k = k_{min}(n), \ldots, k_{max}(n).$$

3 General lattice methods for arithmetic Asian options

The general lattice method is a new discrete-time approach for Asian option pricing, developed by Gambaro, Kyriakou, and Fusai, 2020. It offers a flexible tree method for European and American option style with fixed or floating strike. In this thesis we consider the method in case of constant volatility but it can be also extended to constant elasticity of volatility (CEV) models and models with stochastic volatility.

In case of standard lattice methods we consider risk-neutral probabilities with the bank account process $B(t)$ as numeraire and find option price recursively backward using original prices. In contrast, for the new method we take asset price $S(t)$ as numeraire and find option price recursively backward using relative prices. In the case of Asian options, this approach for simplest European-style options and American-style options with floating strike price reduces the problem to one-dimensional. However, in standard approaches - such as the Hull-White method, the problem remains two-dimensional, as a grid over average prices must be considered for each underlying asset price.

We consider an N-period discrete-time model, where the time interval $[0,\ T]$ is divided into N equal steps. The length of each time step is denoted by $\Delta t := T/N$. Under the risk-neutral probability measure Q, the price of the underlying asset at an arbitrary time step $n\Delta t \leq T$ is given by:

$$S_n = S_0 \exp\left(\sum_{j=0}^{n} \xi_j\right), \tag{27}$$

where $\xi_0 := 0$ and $\{\xi_j\}_{j=1}^{N}$ is a sequence of discrete random variables with a probability distribution. To proceed, we recall the key structure of the CRR binomial model (see (10)) and define ξ_j as follows:

$$\xi_j = \begin{cases} x_1 = \exp(\sigma\sqrt{\Delta t}), & p_1 = \frac{R-x_2}{x_1-x_2} \\ x_2 = \exp(-\sigma\sqrt{\Delta t}), & p_2 = 1 - p_1, \end{cases} \tag{28}$$

where p_1 and p_2 are risk-neutral probabilities if numeraire is bank account process $B(t)$.

3.1 European-style fixed strike call option

First, we consider a European-style Asian option with a fixed strike price. Let us define the payoff P_N of a fixed-strike Asian call option at maturity $T = N\Delta t$:

$$P_N := \left(\frac{\sum_{n=0}^{N} S_n}{N+1} - K \right)^+ = \left(\frac{\sum_{n=0}^{N} S_n \Delta t}{T + \Delta t} - K \right)^+,$$

where $(x)^+$ denotes the positive part of x: $(x)^+ = \max(x, 0)$.

The process Z is defined as

$$Z_j = \frac{\sum_{n=0}^{j} S_n \Delta t - K(T + \Delta t)}{S_j}, \ 0 \le j \le N. \tag{29}$$

So, then

$$Z_0 = \Delta t - \frac{K(T + \Delta t)}{S_0}$$

and

$$Z_N = \frac{\sum_{n=0}^{N} S_n \Delta t - K(T + \Delta t)}{S_N}.$$

It is easy to check that

$$P_N = \frac{S_N Z_N^+}{T + \Delta t},$$

where Z_N^+ is $1/(T + \Delta t)$ payoff in relative prices if numeraire is price process $S(t)$.

Using equalities (27) and (29) follows that:

$$Z_j = \frac{\sum_{n=0}^{j-1} S_n \Delta t + S_j \Delta t - K(T+\Delta t)}{S_{j-1}\exp(\xi_j)} = \frac{Z_{j-1}}{\exp(\xi_j)} + \Delta t,\ 0 < j \leq N. \tag{30}$$

To find the Z_N, let us write Z_{j+1}, Z_{j+2} and Z_{j+3}:

$$Z_{j+1} = \frac{Z_j}{\exp(\xi_{j+1})} + \Delta t,$$

$$Z_{j+2} = \frac{Z_{j+1}}{\exp(\xi_{j+2})} + \Delta t = \frac{1}{\exp(\xi_{j+2})}\left(\frac{Z_j}{\exp(\xi_{j+1})} + \Delta t\right) + \Delta t,$$

$$Z_{j+3} = \frac{Z_{j+2}}{\exp(\xi_{j+3})} + \Delta t = \frac{1}{\exp(\xi_{j+3})}\left(\frac{1}{\exp(\xi_{j+2})}\left(\frac{Z_j}{\exp(\xi_{j+1})} + \Delta t\right) + \Delta t\right) + \Delta t.$$

Using recursive substitution, for every $0 \leq j \leq N-1$, the following equality can be obtained:

$$Z_N = Z_j \prod_{k=j+1}^{N} \exp(-\xi_k) + \Delta t \sum_{i=j+1}^{N-1} \prod_{k=i+1}^{N} \exp(-\xi_k) + \Delta t. \tag{31}$$

So the behavior of Z_j, $j = 1, ..., N$, depends on Z_0 and behavior of random variable ξ_k.

The price at time T of the option is defined as follows:

$$\mathbb{E}(P_N) = \frac{\mathbb{E}(S_N Z_N^+)}{T+\Delta t} = \frac{S_0 \exp(rT)}{T+\Delta t}\bar{\mathbb{E}}(Z_N^+), \tag{32}$$

where $\bar{\mathbb{E}}$ is the expected value under the new risk-neutral measure $\bar{Q}$ if the numeraire is given by the price process $S(t)$. For binomial model (see (11)) we get following risk neutral probabilities:

$$\begin{cases} \bar{p}_1 = p_1 \frac{x_1}{R} \\ \bar{p}_2 = 1 - p_1. \end{cases} \tag{33}$$

Based on results (30) and (31), the price (32) can be determined through backward recursion from maturity.

First, we compute the truncated ranges for z at each time step j. We set $z\ell_N = 0$ and $zu_0 = 0$ and get the lower and upper cut-off points by reverse recursion ($z\ell$) or forward propagation (zu) from equality (30):

$$z\ell_{j-1} = (z\ell_j - \Delta t)x_1, \quad 0 < j \leq N, \tag{34}$$

$$zu_j = \frac{zu_{j-1}}{x_2} + \Delta t, \quad 0 < j \leq N. \tag{35}$$

where x_1 and x_2 are given by formula (28). In case when $Z_0 \geq 0$, the inverse relation (30) yields the $Z_j > 0$ and the option is exercised surely in this case. If $Z_0 < z\ell_0$ then also $Z_N < 0$ and immediate exercise of the option would not result in gain. Now we can find the option relative prices by recursively working backwards. For $j = N-1,\ N-2,\ ...,\ 0$ follows:

$$c(z, N) = z^+, \tag{36}$$

$$c(z, j) = \sum_{i=1}^{2} \bar{p}_i \, c\left(\frac{z}{x_i} + \Delta t, j+1\right), \qquad z\ell_j \leq z \leq zu_j, \tag{37}$$

where $\{\bar{p}_1,\ \bar{p}_2\}$ is the probability distribution of ξ_j under the $\bar{Q}$ and $c\left(\frac{z}{x_i} + \Delta t, j+1\right)$ is relative price of the option if numeraire is price process $S(t)$.

The price V_C of the call option at time $t = 0$ is then given by (32)

$$V_C = \frac{S_0}{T + \Delta t} c\left(Z_0, 0\right) = \frac{S_0}{T + \Delta t} c\left(\Delta t - \frac{K}{S_0}(T + \Delta t), 0\right).$$

Returning to the recursive framework (see (37)), we construct equally spaced grids for the z at each time step j:

$$z_{m,j} := z\ell_j + m\Delta z,\ \ m = 0, 1, ..., n_z. \tag{38}$$

In the paper (Gambaro, Kyriakou, and Fusai, 2020) the information about choice of the grid size n_z is not provided. In numerical examples we take following grid size n_j that depends on j:

$$n_j = \left\lfloor \frac{zu_j - z\ell_j}{\Delta z} \right\rfloor + 1$$

and choice of Δz will be discussed later.

Since the tree model is non-recombining, to find values $c\left(\frac{z_{m,j}}{x_i} + \Delta t, j+1\right)$, $i = 1, 2$, we use linear interpolation. Assume that

$$\frac{z_{m,j}}{x_i} + \Delta t \in (z\ell_{j+1} + m^i_{j+1}\Delta z,\ z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z),\ i = 1, 2, \tag{39}$$

where

$$m^i_{j+1} := \left\lfloor \frac{z_{m,j}/x_i + \Delta t - z\ell_{j+1}}{\Delta z} \right\rfloor.$$

Then for a twice differentiable function $c(z, j+1)$ in z, bounded second derivative and for each j,

$$\begin{aligned} c(\frac{z_{m,j}}{x_i} + \Delta t,\ j+1) = \alpha^i_{j+1} c(z\ell_{j+1} + m^i_{j+1}\Delta z, j+1) + \\ + (1 - \alpha^i_{j+1}) c(z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z, j+1) + \gamma^i_{j+1} \end{aligned} \tag{40}$$

where

$$\alpha^i_{j+1} = \frac{z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z - z_{m,j}/x_i - \Delta t}{\Delta z}$$

and γ^i_{j+1} denotes an interpolation error.

In the paper (Gambaro, Kyriakou, and Fusai, 2020) there is no discussion about if it is possible to find indexes m^i_{j+1} (see (39)) so that $m^i_{j+1} \geq 0$. Let us check if

$$\frac{z_{m,j}}{x_i} + \Delta t \geq z\ell_{j+1},\ i = 1, 2$$

holds. From the equation of equally spaced grids (38) follows:

$$\frac{z\ell_j + m\Delta z}{x_i} + \Delta t \geq z\ell_{j+1}$$

and, using formula of lower cut-off point, $z\ell_j$ can be rewritten as $(z\ell_{j+1} - \Delta t)x_1$:

$$\frac{(z\ell_{j+1} - \Delta t)x_1 + m\Delta z}{x_i} + \Delta t \geq z\ell_{j+1}.$$

If $i = 1$, then

$$z\ell_{j+1} - \Delta t + \frac{m\Delta z}{x_1} + \Delta t \geq z\ell_{j+1}$$

holds. If $i = 2$, then

$$(z\ell_{j+1} - \Delta t)x_1 + m\Delta z + x_2\Delta t \geq x_2 z\ell_{j+1}$$

$$m\Delta z \geq (x_2 - x_1)z\ell_{j+1} + (x_1 - x_2)\Delta t.$$

It mean that inequality holds only if

$$m \geq \frac{(x_1 - x_2)(\Delta t - z\ell_{j+1})}{\Delta z}.$$

If value of m does not satisfy the condition above, then we take $c(z\ell_{j+1} + m^i_{j+1}\Delta z, j+1) = 0$ in formula (40). Now we consider the case when $m^i_{j+1} \leq n_j$ and assume that $z_{m,j} \leq zu_j$. We check if

$$\frac{z_{m,j}}{x_i} + \Delta t \leq zu_{j+1}.$$

From the equation of equally spaced grids (38) follows:

$$z_{m,j} = z\ell_j + m\Delta z = zu_j - (n_j - m)\Delta z.$$

Using formula for upper cut-off points (35) we get:

$$\frac{z_{m,j}}{x_i} + \Delta t \leq \frac{zu_j}{x_2} + \Delta t$$

Then it follows that

$$\frac{zu_j - (n_j - m)\Delta z}{x_i} + \Delta t \leq \frac{zu_j}{x_2} + \Delta t.$$

If we take $i = 2$, then

$$\frac{zu_j}{x_2} - \frac{(n_j - m)\Delta z}{x_2} \leq \frac{zu_j}{x_2} \text{ holds.}$$

If $i = 1$ as $zu_j \geq 0$ and $x_1 \geq 0$ then

$$\frac{zu_j}{x_1} - \frac{(n_j - m)\Delta z}{x_1} \leq \frac{zu_j}{x_2} \text{ holds.}$$

So, only the last point $\frac{z_{n_j,j}}{x_i} + \Delta t$ may lie outside of constructed grid. In such case, authors of the paper (Gambaro, Kyriakou, and Fusai, 2020) propose following solution:

$$c(z_{m,j}, j) = z_{m,j}\bar{\mu} + \Delta t,$$

where $\bar{\mu}$ is:

$$\bar{\mu} = \bar{p}_1 \frac{1}{x_1} + \bar{p}_2 \frac{1}{x_2}.$$

Returning to the interpolation (see (40)), the error γ^i_{j+1} is given by:

$$\gamma^i_{j+1} := \frac{1}{2} \frac{\partial^2 c(z^*_{m^i_{j+1}}, j+1)}{\partial z^2} \left(\frac{z_{m,j}}{x_i} + \Delta t - z\ell_{j+1} - m^i_{j+1}\Delta z \right) * \\ * \left(z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z - \frac{z_{m,j}}{x_i} - \Delta t \right)$$

for $z^*_{m^i_{j+1}} \in (z\ell_{j+1} + m^i_{j+1}\Delta z,\ z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z)$. The accumulated interpolation error $\varepsilon_{m,j}$ at node m and time step j:

$$\varepsilon_{m,j} := \sum_{i=1}^{2} \bar{p}_i[\alpha^i_{j+1}\varepsilon_{m^i_{j+1},j+1} + (1-\alpha^i_{j+1})\varepsilon_{m^i_{j+1},j+1} + \gamma^i_{j+1}].$$

Let M be defined as:

$$M = \max_j \max_m \left|\frac{\partial^2 c(z^*_m, j+1)}{\partial z^2}\right|.$$

The norm of the interpolation error is:

$$\|\varepsilon_j\| := \max_m |\varepsilon_{m,j}|, \tag{41}$$

and

$$|\gamma^x_{j+1}| \leq M(\Delta z)^2. \tag{42}$$

From (41) and (42) following inequality for accumulated interpolation error holds:

$$\|\varepsilon_j\| \leq \|\varepsilon_{j+1}\| + M(\Delta z)^2,$$

and, at time zero

$$\|\varepsilon_0\| \leq \|\varepsilon_N\| + NM(\Delta z)^2 = \frac{TM(\Delta z)^2}{\Delta t} \tag{43}$$

since the terminal payoff is evaluated exactly. It can be concluded that if order of $(\Delta z)^2$ is larger than Δt, i.e., $\Delta z < \sqrt{\Delta t}$, then the norm of the accumulated interpolation error ε converges to zero as $\Delta t \to 0$. To ensure accurate interpolation it is better to choose Δz small enough.

As the number of periods $N \to \infty$, the option price obtained from the lattice method converges to the theoretical option price with continuous average given by:

$$A_T := \frac{1}{T}\int_0^T S(t)dt.$$

3.2 European-style fixed strike put option

In the description of general lattice method (Gambaro, Kyriakou, and Fusai, 2020) the focus was on the formulation for call options. We will now show the corresponding expressions for European-style Asian fixed strike put options. Let us define the payoff of a fixed-strike Asian put option at maturity:

$$P_N := \left(K - \frac{\sum_{n=0}^{N} S_n}{N+1}\right)^+ = \left(K - \frac{\sum_{n=0}^{N} S_n \Delta t}{T+\Delta t}\right)^+.$$

The process Z changes to:

$$Z_j = \frac{K(T+\Delta t) - \sum_{n=0}^{j} S_n \Delta t}{S_j},\ 0 < j \leq N,$$

and we can express process Z in such form:

$$Z_j = \frac{K(T+\Delta t) - \sum_{n=0}^{j-1} S_n \Delta t - S_j \Delta t}{S_{j-1}\exp(\xi_j)} = \frac{Z_{j-1}}{\exp(\xi_j)} - \Delta t,\ 0 < j \leq N. \tag{44}$$

Firstly, we define the upper bound zu_j. Initial value is found by following equation:

$$zu_0 = Z_0 + \Delta t = \frac{K}{S_0}(T+\Delta t).$$

Then recursively we can find the upper bound for every j using formula:

$$zu_j = \frac{zu_{j-1}}{x_2} - \Delta t,\ j = 1,\ 2,\ ...,\ N.$$

For lower bound $z\ell$ we set initial value $z\ell_0 = 0$ and find other values by formula:

$$z\ell_j = \frac{z\ell_{j-1}}{x_1} - \Delta t,\ j = 1,\ 2,\ ...,\ N.$$

The expression for $c(z, j)$ can be modified analogously to the corresponding changes in process Z_j for $j = N-1,\ N-2,\ ...,\ 0$ follows:

$$c(z, N) = z^+, \tag{45}$$

$$c(z, j) = \sum_{i=1}^{2} \bar{p}_i\ c\left(\frac{z}{x_i} - \Delta t, j+1\right) \qquad z\ell_j \leq z \leq zu_j. \tag{46}$$

The price of the put option at time $t = 0$ is then given by

$$V_P = \frac{S_0}{T + \Delta t} c\left(\frac{K}{S_0}(T + \Delta t) - \Delta t, 0\right).$$

The same as for call option, linear interpolation is used to find values $c\left(\frac{z_{m,j}}{x_i} - \Delta t, j+1\right)$, $i = 1, 2$. Again, we assume that

$$\frac{z_{m,j}}{x_i} - \Delta t \in (z\ell_{j+1} + m^i_{j+1}\Delta z,\ z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z),\ i = 1, 2, \tag{47}$$

where

$$m^i_{j+1} := \left\lfloor \frac{z_{m,j}/x_i - \Delta t - z\ell_{j+1}}{\Delta z} \right\rfloor.$$

Then the formula for interpolation changes to:

$$\begin{aligned} c(\frac{z_{m,j}}{x_i} - \Delta t,\ j+1) = \alpha^i_{j+1} c(z\ell_{j+1} + m^i_{j+1}\Delta z, j+1) + \\ + (1 - \alpha^i_{j+1}) c(z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z, j+1) + \gamma^i_{j+1} \end{aligned} \tag{48}$$

where

$$\alpha^i_{j+1} = \frac{z\ell_{j+1} + (m^i_{j+1} + 1)\Delta z - z_{m,j}/x_i + \Delta t}{\Delta z}.$$

Analogously to the case of call option, it can be checked whether it is possible to find indexes m^i_{j+1} so that $0 \leq m^i_{j+1}$. If we examine

$$\frac{z_{m,j}}{x_i} - \Delta t \geq z\ell_{j+1},$$

then it holds for $i = 1$. In case when $i = 2$, holds only if

$$m \geq \frac{z\ell_j x_2}{x_1 \Delta z} - \frac{z\ell_j}{\Delta z}.$$

If inequality does not hold, then we take the value of function $c()$ (48) for the corresponding index equal to 0. If we consider the case when $m^i_{j+1} \leq n_j$ and assume that $z_{m,j} \leq zu_j$ then

$$\frac{z_{m,j}}{x_i} - \Delta t \leq zu_{j+1}$$

holds for $i = 1, 2$,

3.3 Floating strike option

The process Z in case of an Asian option with floating strike price is defined as:

$$Z_j = \frac{\frac{1}{j+1}\sum_{n=0}^{j} S_n}{S_j} = \frac{j}{j+1}\frac{Z_{j-1}}{\exp(\xi_j)} + \frac{1}{j+1}, \qquad 0 < j \leq N. \tag{49}$$

To obtain the lower and upper cut-off points, we take $z\ell_0 = 1 - \Delta t$ and $zu_0 = 1 + \Delta t$. Then

$$z\ell_j = \frac{(j-1)z\ell_{j-1}}{jx_1} + \frac{1}{j}, \quad 0 < j \leq N,$$

$$zu_j = \frac{j}{j+1}\frac{zu_{j-1}}{x_2} + \frac{1}{j+1}, \quad 0 < j \leq N.$$

For Asian put option with European type of exercise the process $c(z, j)$ is:

$$c(z, N) = (z - \lambda)^+ \,, \tag{50}$$

$$c(z, j) = \sum_{i=1}^{2} \bar{p}_i \; c\left(\frac{j+1}{j+2}\frac{z}{x_i} + \frac{1}{j+2}, j+1\right) \qquad z\ell_j \le z \le zu_j \tag{51}$$

where $j = N-1,\ N-2,\ ..., 0$.

Since for Asian option of American style, the early exercise payoff is compared with the continuation value of the option, then process $c(z, j)$ for put option has following form:

$$c(z, N) = (z - \lambda)^+ \,, \tag{52}$$

$$c(z, j) = \max\left[\sum_{i=1}^{2} \bar{p}_i \; c\left(\frac{j+1}{j+2}\frac{z}{x_i} + \frac{1}{j+2}, j+1\right),\ z - \lambda\right] \quad z\ell_j \le z \le zu_j \tag{53}$$

where $j = N-1,\ N-2,\ ..., 0$.

The pricing of American-style Asian put option with floating strike becomes a one-dimensional problem due to a change of measure and nature of the payoff, therefore there is no need to track the asset price S.

For both European and American style put options the price is given by:

$$V_C = S_0 c(1, 0).$$

For the call option, expressions described above change from $z - \lambda$ to $\lambda - z$ in formulas (52) and (53).

3.4 American-style fixed strike option

An Asian call option of American style can be exercised at any time up to and including the maturity. Therefore, the process Z is re-defined in the same way as for floating option (see (49)), using weights $\frac{1}{j+1}$ for arbitrary j:

$$Z_j = \frac{\frac{1}{j+1}\sum_{n=0}^{j} S_n}{S_j} = \frac{j}{j+1}\frac{Z_{j-1}}{\exp(\xi_j)} + \frac{1}{j+1}, \qquad 0 < j \le N.$$

Next, we change recursion process $c(S, z, j)$ in case of call option so that it takes into account early-exercise:

$$c(S, z, N) = \left(z - \frac{K}{S}\right)^{+}, \tag{54}$$

$$c(S, z, j) = \max\left[\sum_{i=1}^{2} \bar{p}_i \; c\left(S_{j+1}, \frac{j+1}{j+2}\frac{z}{x_i} + \frac{1}{j+2}, j+1\right),\; z - \frac{K}{S}\right] \quad z\ell_j \le z \le zu_j \tag{55}$$

where $j = N-1,\; N-2,\; ..., 0$.

The price of the call option is then given by:

$$V_C = S_0 \; c(S_0, 1, 0).$$

However, the pricing problem for American options with fixed strike is two-dimensional, therefore increases computational complexity.

3.5 Forward start option

For forward-starting Asian call option, only part of asset's prices is used when calculating the average. We denote monitored time interval $[\alpha\Delta t,\; N\Delta t]$, $0 < \alpha < N$.

It is possible to adapt the described pricing approach in case when averaging does not start from $t = 0$. Authors (Gambaro, Kyriakou, and Fusai, 2020) modify the process Z (see (29)) as follows:

$$Z_j := \frac{\frac{1}{N-\alpha+1}\sum_{n=\alpha}^{j} S_n - K}{S_j} = \frac{Z_{j-1}}{\exp(\xi_j)} + \frac{1}{N-\alpha+1}, \qquad \alpha < j \le N.$$

Then we can show that

$$Z_\alpha = \frac{\frac{1}{N-\alpha+1}\sum_{n=\alpha}^{\alpha} S_n - K}{S_\alpha} = \frac{1}{N-\alpha+1} - \frac{K}{S_\alpha}.$$

By tower property of expected value, the price of the call option is:

$$\mathbb{E}(S_N Z_N^+) = S_0 \bar{\mathbb{E}}\left[c\left(\frac{1}{N-\alpha+1} - \frac{K}{S_\alpha}, \alpha\right)\right].$$

Using the following formula:

$$c\left(\frac{1}{N-\alpha+1} - \frac{K}{s}\right) = \sum_{i=1}^{2} \bar{p}_i \; c\left(\frac{z}{x_i} + \Delta t, j+1\right), \; z\ell_j \le z \le z u_j,$$

we find prices for the call option at time $t = \alpha$. Price at time $t = 0$ can be found recurrently, using standard binomial method.

4 Numerical examples

In this section, numerical examples will be presented, to show the performance of the Asian options pricing methods described in the previous sections. To perform the calculations, the following software was used: R version 4.3.1 using RStudio on a Windows 11 64-bit system with an AMD Ryzen 7 6800HS Creator Edition processor (3.20 GH, 8 cores) and 16 GB of RAM.

4.1 Monte Carlo simulations

For pricing the discrete arithmetic Asian options, we generate N independent standard normal variables $Z_i(t),\ i = 1, ..., N$, where N denotes the number of periods equally spaced throughout the time period $[0, T]$ with a size of $\Delta t = \frac{T}{N}$. Then each variable $Z_i(t)$ is used to calculate the price of an asset by formula:

$$S_i = S_{i-1} \exp[(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t} * Z_i(t)], \quad 0 < i \leq N. \tag{56}$$

The average A_T is then determined by applying the formula for the standard discrete arithmetic average (see (22)):

$$A_T = \frac{1}{N+1} \sum_{i=0}^{N} S_i. \tag{57}$$

After that, the discounted payoff $\exp(-rT)P(A_T)$ is calculated. The described process is repeated n times, where n denotes number of Monte Carlo simulations. The average of obtained discounted payoffs is the option price at time $t = 0$. The application of antithetic and control variates is explained in chapter 1.4. As shown by Kemna and Vorst (Kemna and Vorst, 1990), in case of arithmetic Asian option, a suitable choice for control variate is the geometric Asian option (see (23)). It serves as a close proxy for the value of an arithmetic Asian option and has a closed-form solution (see (24)).

Table 1: Monte Carlo method for arithmetic average Asian call options with $S_0 = 100$, $r = 0.05$, $\sigma = 0.05$, $T = 1$ and 50,000 simulations.

		K			**Time (s)**
Model	**N**	**95**	**100**	**105**	
Arithmetic Asian options	20	7.16858 (12.557)	2.71911 (10.630)	0.33225 (4.102)	2.455
	30	7.16294 (12.521)	2.70689 (10.640)	0.33462 (4.084)	3.130
	40	7.19959 (12.581)	2.71913 (10.718)	0.33146 (4.0847)	3.833
Arithmetic Asian options Antithetic variates	20	7.17848 (0.455)	2.70644 (2.665)	0.33081 (2.693)	4.706
	30	7.17808 (0.443)	2.71059 (2.685)	0.33543 (2.697)	5.889
	40	7.17855 (0.473)	2.71333 (2.694)	0.33155 (2.691)	7.076
Arithmetic Asian options Control variates	20	7.17819 (0.105)	2.70828 (0.092)	0.32856 (0.083)	2.695
	30	7.17825 (0.105)	2.71076 (0.091)	0.33129 (0.081)	3.511
	40	7.17799 (0.104)	2.71214 (0.092)	0.33283 (0.082)	4.346

The Tables 1 and 2 present the results of the 50,000 and 100,000 Monte Carlo simulations for a standard arithmetic European-style Asian call option with different strike prices K and number of fixing dates N. The standard errors ($\times 10^{-3}$) of the estimates are shown in parentheses. The corresponding computation times (in seconds) are presented alongside for each model and scenario.

It can be concluded that pricing accuracy improves with increased number of simulations. However, this leads to an increase in computation time. To enhance accuracy and efficiency, variance reduction techniques were implemented. The control variates method, where geometric Asian option is used as control variate, demonstrated better results in terms of both computational time and accuracy compare to the results obtained using antithetic variates.

Table 2: Monte Carlo method for arithmetic average Asian call options with $S_0 = 100$, $r = 0.05$, $\sigma = 0.05$, $T = 1$ and 100,000 simulations.

Model	N	K 95	K 100	K 105	Time (s)
Arithmetic Asian options	20	7.18175 (8.852)	2.69260 (7.486)	0.33101 (2.882)	4.341
	30	7.18568 (8.869)	2.70696 (7.548)	0.33269 (2.874)	5.976
	40	7.16755 (8.888)	2.70307 (7.565)	0.32972 (2.864)	7.528
Arithmetic Asian options Antithetic variates	20	7.17830 (0.322)	2.70759 (1.881)	0.32611 (1.874)	8.882
	30	7.17781 (0.319)	2.71439 (1.917)	0.33050 (1.894)	10.767
	40	7.17834 (0.326)	2.71691 (1.914)	0.33178 (1.902)	13.588
Arithmetic Asian options Control variates	20	7.17846 (0.075)	2.70808 (0.065)	0.32855 (0.058)	5.437
	30	7.17817 (0.073)	2.71073 (0.065)	0.33140 (0.058)	6.938
	40	7.17809 (0.073)	2.71203 (0.065)	0.33281 (0.058)	8.725

4.2 General lattice method

We start the numerical examples of general lattice method with analysis of price behavior depending on the grid granularity. The parameter τ is introduced to ensure that Δz is small enough for accurate interpolation, where $\Delta z = \frac{\sqrt{\Delta t}}{\tau}$. We consider several possibilities of Δz value, because in the source (Gambaro, Kyriakou, and Fusai, 2020) there was no discussion about the choice of Δz.

In the Tables 3, 4 and 9a (tables marked with 'a' are presented in appendix) are presented the prices of European-style Asian call options with fixed strike price. The prices are computed by general lattice and Monte Carlo (control variate) methods. In addition to prices, the corresponding computational times are presented. In parentheses, the maximal size of the grid n_j is given (maximal, because size of grid can differ for each state) for general lattice method and standard error for Monte

Carlo.

Table 3: General lattice and Monte Carlo method with $S_0 = 100$, $K = 95$, $r = 0.05$, $\sigma = 0.05$, $T = 1$.

	τ				
N	**100**	**300**	**500**	**1000**	**MCCV**
30	7.17807 0.152 sec (633)	7.17783 0.179 sec (1899)	7.17783 0.221 sec (3165)	7.17782 0.304 sec (6330)	7.17824 9.926 sec SE 7.37 x10^{-5}
50	7.17807 0.173 sec (852)	7.17783 0.252 sec (2554)	7.17781 0.319 sec (4257)	7.17780 0.514 sec (8513)	7.17804 18.620 sec SE 7.36 x10^{-5}
100	7.17806 0.255 sec (1301)	7.17780 0.491 sec (3903)	7.17778 0.908 sec (6504)	7.17777 1.221 sec (13007)	7.17795 26.843 sec SE 7.26 x10^{-5}
200	7.17805 0.495 sec (2060)	7.17778 1.158 sec (6180)	7.17776 2.100 sec (10300)	7.17775 3.543 sec (20599)	7.17770 37.824 sec SE 7.25 x10^{-5}
500	7.17805 1.904 sec (4123)	7.17778 5.086 sec (12367)	7.17775 9.705 sec (20612)	7.17774 24.089 sec (41223)	7.17779 1.532 min SE 7.20 x10^{-5}

Table 4: General lattice and Monte Carlo method with $S_0 = 100$, $K = 90$, $r = 0.09$, $\sigma = 0.5$, $T = 1$.

	τ				
N	**100**	**300**	**500**	**1000**	**MCCV**
30	18.14342 0.188 sec (3028)	18.14294 0.271 sec (9082)	18.14294 0.376 sec (15136)	18.14292 0.611 sec (30272)	18.11589 13.156 sec SE 0.00523
50	18.16543 0.282 sec (6902)	18.16470 0.567 sec (20704)	18.16465 0.886 sec (34506)	18.16462 1.651 sec (69011)	18.14297 20.481 sec SE 0.00514
100	18.17830 1.165 sec (30226)	18.17755 3.198 sec (90678)	18.17748 5.366 sec (151130)	18.17745 11.535 sec (302259)	18.16951 35.258 sec SE 0.00507
200	18.18414 12.110 sec (239465)	18.18332 38.359 sec (718394)	18.18325 1.081 min (1197324)	18.18323 2.632 min (2394647)	18.17580 40.867 sec SE 0.00511

From the results in Table 3, we can conclude that with lower volatility, general lattice method computes the results faster than Monte Carlo method even for large N and τ. However, from the Table 4 we can see that when σ is high, then with increasing N computational time of general lattice method slows down due to large grid size. Monte Carlo results are obtained faster than for lattice method and even with $\tau = 1000$ the price value of general lattice method differs from Monte Carlo result in the second decimal place. From Tables 3 and 4 we can see that price stays relatively stable for $\tau > 300$.

Next, we will compare general lattice method with Hull-White method and Monte Carlo; approximate formula will serve as a benchmark. We choose grid step $\Delta z = \sqrt{0.00001\Delta t}$, this corresponds to $\tau \approx 316$ (and then accumulated interpolation error is less or equal $10^{-5}M$ (see (43))). In the Tables 5, 6 and 10a are presented the prices of European-style Asian call options with fixed strike price computed by general lattice, Hull-White and Monte Carlo methods. For Hull-White method,

Table 5: Comparison of general lattice, Hull-White and Monte Carlo methods with $S_0 = 100$, $K = 95$, $r = 0.05$, $\sigma = 0.05$, $T = 1$, $\Delta z = \sqrt{0.00001\Delta t}$.

N	**General lattice**	**Hull-White $\alpha = 40$**	**Hull-White $\alpha = 20$**	**MCCV**	**Approx. formula**
30	7.17784 0.389 sec (2002)	7.17797 1.179 sec (167)	7.17789 4.876 sec (331)	7.17824 9.926 sec (7.37 x10^{-5})	7.17842
50	7.17783 0.433 sec (2692)	7.17787 13.098 sec (356)	7.17782 1.176 min (709)	7.17804 18.620 sec (7.36 x10^{-5})	7.17824
100	7.17779 0.702 sec (4114)	7.17779 9.239 min (1002)	7.17777 37.426 min (2002)	7.17795 26.843 sec (7.26 x10^{-5})	7.17811
120	7.17780 1.160 sec (4624)	7.17778 25.251 min (1316)	7.17777 1.906 hours (2631)	7.17786 32.731 sec (7.24 x10^{-5})	7.17809

the code from bachelor thesis of Rain Kask (Kask, 2013) was used (transformed from Python to R). The parameter α, which determines the granularity of the grid in Hull-White method, is equal to 40 and 20 for the results presented in Table 5. Together with option price, the time of computations is presented and grid size in parenthesis for general lattice and Hull-White methods; for Monte Carlo, standard error is shown in parentheses.

Table 6: Comparison of general lattice, Hull-White and Monte Carlo methods with $S_0 = 100$, $K = 90$, $r = 0.09$, $\sigma = 0.5$, $T = 1$, $\Delta z = \sqrt{0.00001\Delta t}$.

N	**General lattice**	**Hull-White $\alpha = 40$**	**Hull-White $\alpha = 3$**	**MCCV**	**Approx. formula**
30	18.14302 0.489 sec (9573)	19.02499 0.22732 sec (19)	18.17001 2.277 sec (221)	18.11589 13.156 sec (0.00523)	18.37953
50	18.16468 1.055 sec (21823)	18.54151 0.48092 sec (37)	18.17609 33.221 sec (474)	18.14297 20.481 sec (0.00514)	18.40241
100	18.17754 5.182 sec (95583)	18.37487 7.032 sec (102)	18.18228 16.490 min (1335)	18.16142 26.519 sec (0.00509)	18.41965
120	18.17949 8.924 sec (154104)	18.35949 22.602 sec (133)	18.18342 42.193 min (1754)	18.16590 34.124 sec (0.00516)	18.42254

Comparison between general lattice, Hull-White and Monte Carlo methods (see Tables 5, 6) again shows that results depend on volatility, specifically, when σ is low, then general lattice method gives the result much faster than both Hull-White and Monte Carlo methods (see Table 5). However, if σ is high, then grid size for general lattice method increases, which requires additional computational sources for obtaining the result. For Hull-White method, when σ is large, the grid size is small, so the computed option prices differ from the Monte Carlo and general lattice results. The grid size of Hull-White method is affected by stepping parameter h (see 26), which depends on σ^2. Therefore, when volatility is low, higher value of α can be chosen, as seen from Table 5 - option prices are close to general lattice and Monte Carlo prices and results are computed faster. However, in case when volatility is high (see Table 6), parameter $\alpha = 3$ was chosen to obtain results that are close to general lattice and Monte Carlo values.

Additionally, simulations for put options were performed for different parameters (see Appendix 1. Table 11a). The results follow the same logic as for call option.

Now, we consider discrete arithmetic European-style Asian put option with floating strike price.

Table 7: Floating put option. General lattice, Hull-White ($\alpha = 40$) and Monte Carlo method with $S_0 = 100$, $\lambda = 1$, $r = 0.05$, $\sigma = 0.05$, $T = 1$, $\Delta z = \frac{\sqrt{\Delta t}}{\tau}$.

	τ					
N	**100**	**300**	**500**	**1000**	**H-W**	**MC**
30	0.29941 0.187 sec (152)	0.29564 0.315 sec (456)	0.29531 0.432 sec (760)	0.29511 0.802 sec (1521)	0.29683 1.422 sec (167)	0.29608 7.636 sec (0.00169)
50	0.30915 0.288 sec (253)	0.29887 0.609 sec (759)	0.29861 0.961 sec (1266)	0.29848 1.859 sec (2532)	0.29958 13.871 sec (356)	0.29844 12.342 sec (0.00168)
100	0.37646 1.738 sec (510)	0.30147 2.076 sec (1532)	0.30114 3.207 sec (2554)	0.30104 6.296 sec (5108)	0.30157 9.039 min (1002)	0.29955 16.377 sec (0.00169)
120	0.37021 1.057 sec (615)	0.30208 2.901 sec (1846)	0.30157 4.735 sec (3077)	0.30147 9.472 sec (6154)	0.30193 29.218 min (1316)	0.30238 30.406 sec (0.00170)

Table 8: Floating put option. General lattice, Hull-White ($\alpha = 3$) and Monte Carlo method with $S_0 = 100$, $\lambda = 1$, $r = 0.09$, $\sigma = 0.5$, $T = 1$, $\Delta z = \frac{\sqrt{\Delta t}}{\tau}$.

	τ					
N	**100**	**300**	**500**	**1000**	**H-W**	**MC**
30	9.03390 0.891 sec (2765)	9.03155 2.390 sec (8297)	9.03113 4.094 sec (13828)	9.03082 7.816 sec (27657)	9.04688 2.331 sec (221)	8.99058 8.634 sec (0.01586)
50	9.04973 2.786 sec (6591)	9.04801 8.949 sec (19774)	9.04779 15.675 sec (32957)	9.04764 28.152 sec (65914)	9.05718 31.446 sec (474)	9.01206 10.235 sec (0.01590)
100	9.10879 20.605 sec (29749)	9.06093 58.296 sec (89248)	9.06083 2.013 min (148747)	9.06077 2.961 min (297494)	9.06556 14.790 min (1335)	9.03135 15.780 sec (0.01606)
120	9.15930 38.570 sec (48154)	9.06313 1.927 min (144462)	9.06305 3.051 min (240770)	9.06300 5.445 min (481541)	9.06700 43.204 min (1754)	9.04702 24.042 sec (0.01611)

For this type of option, results for Monte Carlo method are obtained using antithetic variance reduction technique, since the geometric Asian option price is not suitable control in case of floating strike price. Similar to previous tables, together with option price, the computation time and grid size will be presented (in parenthesis); standard error (in parenthesis) for Monte Carlo.

From the Tables 7 we can see that for lower volatility, general lattice method computes option value faster than Hull-White or Monte Carlo method, even when value of τ is high. In opposite, from table 8 we see how computational time of general lattice method increases with increase of number of periods N and τ, however, the computation time for Hull-White stays higher.

Finally, a some results for prices of Americal-style Asian option can be seen in Table 13a in appendix. Compare to European-style Asian options, prices of American-style options are higher.

Conclusions

In this thesis, the problem of discrete arithmetic Asian options pricing was studied. Due to Asian options' payoffs dependence on the average of underlying asset's price, they are one of the most popular path-dependent options traded in commodity or exchange rate market. For discrete arithmetic Asian options there is no closed-form solution, but exist several numerical pricing approaches.

Main focus of the thesis was on the new discrete-time model approach or general lattice method for arithmetic Asian options, developed by Gambaro, Kyriakou, and Fusai(Gambaro, Kyriakou, and Fusai, 2020). The change of numeraire allows lattice to become one dimensional, therefore the price of the options can be computed faster. In the thesis we describe the process of European-style fixed strike Asian call and put options' pricing under general lattice method. Additionally, we give the formulas adapted to American-style fixed strike option, floating strike option and forward start option.

In numerical examples we compare the results obtained by the new general lattice method with results of Monte Carlo simulations, Hull-White method and approximate formula. Additionally, for general lattice method we show the convergence of option price depending on the grid size and number of periods.

The new general lattice method shows precise performance and in case of low volatility, lower computational times than other methods. However, in case when volatility is high and if the step size for grid setting is chosen very small, then it leads to increase in computational time. Although it is still more efficient, in terms of speed, than Hull-White method. Therefore, it can be concluded, that the performance of general lattice method depends on the volatility level and the optimal choice of step size for the grid needs further investigation.

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Appendix 1. Results of simulations

Table 9a: Fixed strike, call option. General lattice and Monte Carlo method with $S_0 = 100$, $K = 105$, $r = 0.05$, $\sigma = 0.05$, $T = 1$.

	τ				
N	**100**	**300**	**500**	**1000**	**MCCV**
30	0.31241 0.147 sec (633)	0.30701 0.192 sec (1899)	0.30687 0.214 sec (3165)	0.30667 0.330 sec (6330)	0.33126 10.259 sec (5.74 x10^{-5})
50	0.32665 0.166 sec (852)	0.32064 0.256 sec (2554)	0.32008 0.322 sec (4257)	0.31991 0.511 sec (8513)	0.33371 14.429 sec (5.75 x10^{-5})
100	0.33566 0.256 sec (1301)	0.32958 0.459 sec (3903)	0.32903 0.686 sec (6504)	0.32881 1.704 sec (13007)	0.33534 31.779 sec (5.64 x10^{-5})
200	0.34029 0.478 sec (2060)	0.33386 1.226 sec (6180)	0.33332 2.027 sec (10300)	0.33310 4.113 sec (20599)	0.33639 54.245 sec (5.62 x10^{-5})
500	0.34305 1.902 sec (4123)	0.33643 5.221 sec (12367)	0.33588 9.790 sec (20612)	0.33565 24.826 sec (41223)	0.33686 1.584 min (5.63 x10^{-5})

Table 10a: Fixed strike, call option. Comparison of general lattice, Hull-White and Monte Carlo methods with $S_0 = 100$, $K = 105$, $r = 0.05$, $\sigma = 0.05$, $T = 1$.

N	**General lattice**	**Hull-White**	**MCCV**
30	0.30719 0.308 sec (2002)	0.31148 1.447 sec (167)	0.33126 10.259 sec (5.74 x10^{-5})
50	0.32061 0.498 sec (2692)	0.32171 14.706 sec (356)	0.33371 14.429 sec (5.75 x10^{-5})
100	0.32949 0.810 sec (4114)	0.32947 8.991 min (1002)	0.33534 31.779 sec (5.64 x10^{-5})
120	0.33092 1.149 sec (4624)	0.33076 25.585 min (1316)	0.33575 33.651 sec (5.65 x10^{-5})

Table 11a: Comparison of European-style Asian put option pricing methods with different parameters. $T = 1$, $N = 100$, $\Delta z = \sqrt{0.00001\Delta t}$.

S_0	K	r	σ	α	**General lattice**	**Hull-White$_\alpha$**	**Approx. f-la**	**MC CV**
100	95	0.05	0.05	40	0.00324 1.323 sec	0.00324 19.427 min	0.00355	0.00330 44.722 sec (0.53 x10^{-5})
100	100	0.05	0.05	40	0.29617 1.419 sec	0.29603 19.696 min	0.29813	0.29612 47.932 sec (1.68 x10^{-5})
100	105	0.05	0.05	40	2.66734 1.494 sec	2.66721 19.806 min	2.67131	2.67320 47.343 sec (5.71 x10^{-5})
100	95	0.09	0.5	4	6.62302 18.255 sec	6.63179 9.748 min	6.83633	6.60905 24.160 sec (0.00221)
100	105	0.09	0.5	4	11.24851 21.181 sec	11.25757 9.701 min	11.37606	11.23215 27.434 sec (0.00243)
50	40	0.1	0.3	7	0.15911 2.098 sec	0.15996 8.196 min	0.18666	0.159543 38.035 sec (0.00024)
50	50	0.1	0.3	7	2.18431 2.761 sec	2.18700 12.202 min	2.21085	2.18111 38.429 sec (0.00040)
50	60	0.1	0.3	7	7.89432 3.326 sec	7.89645 10.314 min	7.87521	7.89603 38.142 sec (0.00064)

Table 12a: Comparison of European-style Asian call option pricing methods with different parameters and floating strike price. $T = 1$, $N = 100$, $\Delta z = \sqrt{0.00001\Delta t}$.

S_0	r	σ	α	**General lattice**	**Hull-White$_\alpha$**	**MC**	**MC Std. Error x10^{-2}**
100	0.05	0.05	40	2.75984 2.964 sec	2.76022 8.481 min	2.75925 27.068 sec	0.10373
100	0.09	0.5	3	13.42797 1.020 min	13.43290 29.156 min	13.40283 20.597 sec	4.80097
50	0.1	0.3	3	4.68145 24.684 sec	4.68201 59.432 min	4.68086 28.044 sec	1.16004
50	0.1	0.15	3	3.14880 8.462 sec	3.14886 3.810 hours	3.15632 26.797 sec	0.44295

Table 13a: Comparison of European and American style Asian put option with floating strike price $S_0 = 100$, $r = 0.05$, $\sigma = 0.05$, $T = 1$, $\Delta z = \frac{\sqrt{\Delta t}}{300}$.

N	European-style	American-style
30	0.29564 0.298 sec (456)	0.82530 0.300 sec (456)
50	0.29887 0.567 sec (759)	0.86123 0.574 sec (759)
100	0.30147 2.076 sec (1532)	0.89682 2.089 sec (1532)
120	0.30208 2.901 sec (1864)	0.90366 2.976 sec (1846)

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