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MARIA NAEEM

First order electroweak radiative corrections to the decay of the polarised *W* boson

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Contents

List of publications

This thesis is based on the following three publications:

I G. Priidik, S. Groote and M. Naeem "Gauge Dependence of the Gauge Boson Projector" *Particles* **3**, 543–561 (2020)

See attached publication in Article I, and arXiv: 2001.04106 [INSPIRE]

II S. Groote, L. Kaldamäe and M. Naeem "Identical Particle and Lepton Mass Effects in the Decay $H \to Z^*(\to \tau^+\tau^-) + Z^*(\to \tau^+\tau^-)$ " *Braz. J. Phys.* **53**, 107 (2023)

See attached publication in Article II, and arXiv: 2206.05901 [INSPIRE]

III M. Naeem and S. Groote "First order electroweak radiative corrections to the decay of the polarized W boson" *Phys. Rev. D.* **112**, 073009 (2025)

See attached publication in Article III , and arXiv: 2407.13810 [INSPIRE]

Author's contribution:

I, Maria Naeem, independently computed and verified all equations in the publications I, II, and III. For our main III, I calculated all the integrals, perform the renormalisation of the theory and wrote the publication. For II I calculated the matrix elements. For I my contribution is to check the unitary gauge independence by calculating the self energy diagrams.

I also presented an invited talk at the University of Wah, Pakistan, supported by a DORA short-term scholarship on 20th September 2022. I also delivered a talk at the Abdus Salam International Centre for Theoretical Physics (ICTP) Summer School on Particle Physics, (June 19-30, 2023), Trieste, Italy. I delivered an invited talk at the National Centre for Nuclear Research (NCBJ), Warsaw, Poland at the invitation of Andrzej Kupść on 3rd Oct 2024.

The whole thesis was summarized in a presentation in the Theoretical physics seminar at University of Tartu on 3rd Feb 2026.

1 Introduction

The electroweak (EW) theory unifies two fundamental forces of nature: the electromagnetic and weak interactions. In 1967, Glashow, Weinberg, and Salam formulated a comprehensive theory describing the interactions and masses of fermions and bosons, laying the foundation of the Standard Model (SM). This theory describes how elementary particles interact through the exchange of gauge bosons: the photon γ for electromagnetism, and the W and Z bosons for weak interactions. These points highlight the development of the EW theory as a major milestone in modern physics explaining both particles interaction and the origin of mass. Experimental studies of the electroweak interaction to date show full consistency with the Glashow–Salam–Weinberg (GSW) model, which forms the foundation of what is now known as the SM. Despite its remarkable experimental success, the SM does not explain everything. The gaps have led physicists to search for physics beyond the Standard Model (BSM) in order to explain the unsolved problems. Improving the precision of the SM is therefore very important. It helps ensure that the effects we observe are not taken to be new physics beyond the SM. The mass effects have not been considered in much detail in previous publications. This motivates us to include the masses of the decay particles in our analysis. In this framework, the study of the W and Z boson is of particular importance. In this thesis, I will address these issues.

The EW theory is a gauge theory. Physics must be independent of the choice of gauge. Different gauges can be used to describe the theory, such as the Feynman gauge, the Landau gauge which includes only transverse components and the unitary gauge. In contrast to the photon, the W and Z bosons are massive gauge bosons. Therefore, unitary gauge is more appropriate in our case. The unitary gauge has an additional benefit, it removes unphysical degrees of freedom means Goldstone bosons, which are the massless scalar fields that appear due to spontaneous symmetry breaking. We have dealt with this issue in our first publication.

Spontaneous symmetry breaking, also known under the name of the Higgs mechanism, is a mandatory element of EW theory. In the Higgs mechanism, a scalar field with a non-zero vacuum expectation value spontaneously breaks electroweak symmetry. Before spontaneous symmetry breaking, the bosons mediating the electroweak force, the W and Z bosons, were massless. The Higgs mechanism generates their masses while preserving the gauge invariance and leaving the photon massless. Spontaneous symmetry breaking also gives masses to the fermions (via Yukawa couplings). We addressed the decay of the Higgs boson via the so-called "golden channel" $H \to ZZ$ in detail in our second publication.

1.1 Statements

This thesis presents the following significant findings:

1. The pragmatic approach can be used to find the completeness relation for the vector boson.

2. Mass effects are essential, as they influence the phase space but also give rise to the new diagrams.

3. The results we obtained are independent of the particular gauge we use (e.g. unitary vs. Feynman gauge).

4. The polarisation degree of freedom gives additional information for the decay process (e.g new observables).

5. Even though the weak coupling is of the order of the fine structure constant $\alpha = 1/137.036$, the electroweak correction turn out to be not that small but of the order of a few percent.

6. Despite of our hopes, we were not able to explain the enlarged decay rates for the massive final states via our methods.

1.2 Structure of the Thesis

The structure of the thesis is organized as follows. Chapter 2 provides an introduction and an overview of the third publication. Chapter 3 introduces the kinematics of the three-particle process, which plays a major role in this thesis. In Chapter 4 we calculated the integrals, and we discuss about the convergent and divergent integral technique. While most of the calculations in Chapter 3 and Chapter 4 can be performed for a more general final state, in the following we return to our particular process $W \to c + \bar{b}$. In Chapter 5 we calculated the loop contributions. In Chapter 6 we did the renormatisation by counter terms and the cancellation of UV parts. In Chapter 7 we calculated the helicity bilinears and did the cancellation of IR singularities. In Chapter 8 we investigated about the collinear singularities in order to compare our results with the massless case. In Chapter 9 we interpret our results by using different observables and different schemes. Chapter 10 and 11 provide the remaining parts of the second and third publications. This thesis concludes by a comprehensive summary, acknowledgments, biblography, a summary in Estonian and also a Curriculum Vitae both in English and Estonian.

2 First order EW radiative corrections to the decay of the polarized W boson

2.1 Introduction

The main SM channel for the single W boson is the decay process $t \to W^+ + X_b$ is analysed for instance in Refs. [1–5]. The primary focus is to study the properties of W boson. In this work we analyse the decay of the W boson into to real fermion. Accordingly, we take the charm and bottom quark as the heaviest fermions as $W \to c\bar{b}$. As we are searching for mass effects in the decay of an on-shell W boson, this is the most sensitive process. However, the result can easily be translated into other decay channels. The W boson has been subject to intense investigation recently. For instance in the measured decay width of the W decay of $\Gamma_W = 2.137 \pm 0.032\mathrm{GeV}$ [6, 7], contrasted with the SM prediction of $\Gamma_W = 2.0892 \pm 0.0008\mathrm{GeV}$ [8, 9] that is 1.5σ smaller than the measured result. The experimentally measured larger decay rate might indicate that the difference is due to the mass effects and the radiative corrections to the leading order (LO) Born-term result not taken properly into account. Polarization describes the alignment or orientation of a particle's spin or charge. Deviations in polarization behavior could suggest new interactions or particles in SM. Radiative corrections account for the effects of higher-order processes in particle interactions, like emission of additional particles (e.g the photons) they might modify the process. In general, the mass effects of the decay particles have not been extensively studied analytically, so our aim is to investigate them using an analytical approach.

2.2 Overview

The central object of interest is the angular dependence of the decay rate for the polarised W boson, as it can be inferred by looking at publications of the supervisor together with Jürgen Körner and Priit Tuvike [10–12] where I have got the understanding from Eq. (1) in Ref. [10] is that the main quantity we are looking at, contains the spin-density matrix ρ that describes the polarisation of the W boson, the decay functions $H_{mm'}$, also known as the helicity bilinears and calculated for the decay channel of the W boson, and the polar angle dependence on θ. The azimuthal angle dependence is found in the References [28, 30] cited in Ref. [10]. Therefore, the central quantity to look at for now is the decay rate function [10]

We want to calculate the angular distribution of our decay,

$$
\begin{aligned}
W(\theta) &= \sum_{m=0,\pm 1} \rho_{mm} H^{mm}(\theta) = \sum_{m,m'=0,\pm 1} \rho_{mm}\, d^1_{mm'}(\theta)\, d^1_{mm'}(\theta)\; H^{m'm'} \\
&= \frac{3}{8}(1+\cos^2\theta)\Big((\rho_{++}+\rho_{--})\,(H^{++}+H^{--}) + 2\rho_{00}H^{00}\Big)
\end{aligned}
$$

$$+\frac{3}{4}\cos\theta\left((\rho_{++}-\rho_{--})\left(H^{++}-H^{--}\right)\right)$$
$$+\frac{3}{4}\sin^2\theta\left((\rho_{++}+\rho_{--})H^{00}+\rho_{00}(H^{++}+H^{--}-H^{00})\right), \quad (1)$$

using the normalized spin density matrix elements $\hat{\rho}_{mm} = \rho_{mm}/\sum_{m'}\rho_{m'm'}$ with $\hat{\rho}_{++}+\hat{\rho}_{00}+\hat{\rho}_{--}=1$ and normalized decay functions given by $\hat{H}_{mm} = H_{mm}/\sum_{m'}H_{m'm'}$ such that $\hat{H}_{++}+\hat{H}_{00}+\hat{H}_{--}=1$, where the θ angle here is the polar angle between the direction of flight of the W boson, transformed to the rest frame of the W boson, and the direction of the quark in the final state. While the spin density matrix elements are inputs for our calculation from the production of the W boson in top quark decay (cf., e.g., Eq. (27) in Ref. [10]).

2.3 The decay rate

In this work we are dealing with the main observable which is also measurable in the experiment, the decay rate of the W boson which decays into two particles (in our case: two quarks). In order to do this, we will first focus on the conventions needed to carry out the calculation. Calculations are usually performed in momentum space. q, p_1 and p_2 are the momenta of the W boson, the quark Q and the antiquark $\bar{q}$. In order for electric charge to be conserved, the W boson decays into an up-type quark Q with electric charge $+2/3$ (in units of the elementary charge $e = 1.6022\cdot 10^{-19}C$) and a down-type antiquark $\bar{q}$ with electric charge $+1/3$ (the corresponding down-type quark q has electric charge $-1/3$). To calculate the total decay rate, we have to integrate over all momenta p_1, p_2 which are not observed. However, sometimes it happens that a kinematic parameter (for instance, a decay angle) is observed in the experiment. In this case, we keep this quantity unintegrated and obtain the differential decay rate with respect to it.

The integration measure is three-dimensional and space-like. In four space-time dimensions, we use the following identity for this measure.

$$\int\frac{d^3p}{2E} = \int d^4p\delta(p^2-m^2)\theta(p_0), \quad (2)$$

where $\theta(x)$ is the step function (equal to 1 for $x > 0$, else 0), and $\delta(x)$ is Dirac's delta distribution.

2.4 Born term contribution

For the calculation of the phase space, we need to explicitly describe the kinematics of the decay in the rest frame of the W boson. In order to calculate the decay rate, we have to integrate the absolute square of the matrix element over the phase space. For the calculation of the phase space, we first need to make the kinematics of the decay explicit in the rest frame of the W boson. In the two-body decay of the W

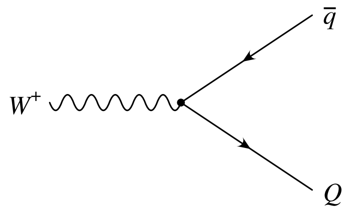


Figure 1: Born-term contribution

boson into up-type quark $Q(p_1)$ and down-type antiquark $\bar{q}(p_2)$, the two quarks are produced back to back. Here we can choose two frames. In the *quark frame* we take the positive z axis to be the direction of flight of the charm quark. One obtains

$$p_1 = (E_1; 0, 0, |\vec{p}\,|), \qquad p_2 = (E_2; 0, 0, -|\vec{p}\,|), \tag{3}$$

where $p_1 + p_2 = q = (\sqrt{q^2}; 0, 0, 0)$ and

$$E_1 = \frac{1}{2}(1+\mu_1-\mu_2)\sqrt{q^2}, \quad E_2 = \frac{1}{2}(1-\mu_1+\mu_2)\sqrt{q^2}, |\vec{p}\,| = \frac{1}{2}\sqrt{\lambda(1,\mu_1,\mu_2)}\sqrt{q^2}, \tag{4}$$

with the Källén function given by $\lambda(a,b,c) := a^2+b^2+c^2-2ab-2ac-2bc$. Taking into account that the axis of flight of the W boson produced in e.g. the dominant process $t \to W^+ + b$, this defines another axis. Considering the polarisation of the W boson as important parameter for our analysis, the polar angle θ between the quark frame and the axis of flight of the W is of importance here. The rest frame of the W boson with the direction of flight pointing in positive z direction is called the *W frame* in the following. The two-particle phase space depends on the polar angle and is given by

$$dPS_2 = \frac{1}{16\pi q^2}\sqrt{\lambda(q^2, m_1^2, m_2^2)}\; d(\cos\theta) = \frac{1}{16\pi}\sqrt{\lambda(1,\mu_1,\mu_2)}\; d(\cos\theta). \tag{5}$$

According to Fermi's golden rule, for the calculation of the decay rate we have to combine the absolute square of the matrix element and the phase space factor to obtain

$$d\Gamma = \frac{1}{2m_W}|\mathcal{M}|^2 dPS. \tag{6}$$

The absolute square of the matrix element $\mathcal{M}$ is the main part of the calculation. The Born term contribution for the decay process is shown in Fig. 1. In order to build up the matrix element $\mathcal{M}$ corresponding to the decay, we write the corresponding diagram where the fermion lines constitute chains, containing lines (propagators) and points (vertices). According to the rule, fermion lines are supplemented with arrows to indicate the fermion current, i.e., fermions move in arrow direction and antifermions against this arrow direction. Only a fermion–antifermion pair can be produced from the decaying W boson. For the calculation of $\mathcal{M}$, the fermion lines are traced in opposite direction (starting with an antispinor and ending with a spinor), while the boson lines are added as factors. $u(p_1)$ is the spinor of the quark Q and $v(p_2)$ is the spinor of the antiquark $\bar{q}$. Between them the W boson vertex is located, which is given by the factor

$$i\frac{e}{\sqrt{2}s_W}V_{cb}\gamma^\mu\frac{\mathbb{1}-\gamma_5}{2}, \tag{7}$$

where e is the unit charge, $s_W = \sin\theta_W$ is the sine of the Weinberg angle, and V_{cb} is the Kobayashi–Maskawa mixing matrix element, while γ^μ and γ_5 are the known Dirac matrices.

The index μ corresponds to the wave function $\varepsilon_\mu(q)$ of the W boson. We obtain the zeroth order, i.e., the Born term level matrix element as

$$\mathcal{M}_0 = \bar{u}(p_1,s_1)\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^\mu\frac{\mathbb{1}-\gamma_5}{2}\right)v(p_2,s_2)\varepsilon_\mu(q,\lambda). \tag{8}$$

In order to calculate the absolute square of $\mathcal{M}$ we also need $\mathcal{M}_0^*$. Taking into account the algebra of the Dirac matrices and using $\bar{u}(p,s) = u^\dagger(p,s)\gamma^0$, we obtain

$$\mathcal{M}_0^* = \bar{v}(p_2,s_2)\left(\frac{-ieV_{cb}^*}{\sqrt{2}s_W}\gamma^\mu\frac{\mathbb{1}-\gamma_5}{2}\right)u(p_1,s_1)\varepsilon_\mu^*(q,\lambda). \tag{9}$$

Finally, we have to sum over non-observed spins. For this we use

$$\sum_s u(p,s)\bar{u}(p,s) = (\not{p}+m), \qquad \sum_s v(p,s)\bar{v}(p,s) = (\not{p}-m), \tag{10}$$

where we used the slash notation $\not{p} := p_\mu\gamma^\mu$. The result is

$$\begin{aligned}|\mathcal{M}_0|^2 &= \sum_{s_1,s_2}\mathcal{M}_0^*\mathcal{M}_0 \\ &= \frac{e^2}{8s_W^2}|V_{cb}|^2\mathrm{tr}\Big((\not{p}_2-m_2)\gamma^\mu(\mathbb{1}-\gamma_5) \\ &\quad\times(\not{p}_1+m_1)\gamma^\nu(\mathbb{1}-\gamma_5)\Big)\varepsilon_\mu^*(q,\lambda)\varepsilon_\nu(q,\lambda).\end{aligned} \tag{11}$$

By using the trace rules we got $H^{\mu\nu}(Born)$ the Born term hadron tensor expressed as

$$H^{\mu\nu}(Born) \;=\; 8(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu}(p_2p_1) - p_{2\kappa}p_{1\lambda}i\epsilon^{\kappa\lambda\mu\nu}). \tag{12}$$

The squared absolute value of matrix element of the Born term is given by

$$\overline{|\mathcal{M}_0|}^2 = \frac{e^2|V_{cb}|^2q^2}{3s_W^2}H^{\mu\nu}(Born)\varepsilon_\mu^*(q,\lambda)\varepsilon_\nu(q,\lambda), \tag{13}$$

The final state spins are summed over and the mean value is taken over the polarisation states of the W boson as incoming particle, the latter resulting in a factor 1/3. For an unpolarised W boson we summed over the polarisations λ, and choosing the unitary gauge $\xi_W = \infty$, in Eq. (20) of Ref. [13] we obtained

$$\sum_\lambda \varepsilon^*_\mu(q,\lambda)\varepsilon_\nu(q,\lambda) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2}. \tag{14}$$

On mass shell we can use $q^2 = m_W^2$. Using $q = p_1 + p_2$, by rearranging accordingly and squaring one obtains

$$p_1p_2 = \frac{q^2}{2}(1-\mu_1-\mu_2), \quad p_1q = \frac{q^2}{2}(1+\mu_1-\mu_2), \quad p_2q = \frac{q^2}{2}(1-\mu_1+\mu_2). \tag{15}$$

where $m_1^2 = p_1^2 = \mu_1 q^2$ and $m_2^2 = p_2^2 = \mu_2 q^2$ introduces two dimensionless mass parameters μ_1 and μ_2. Therefore,

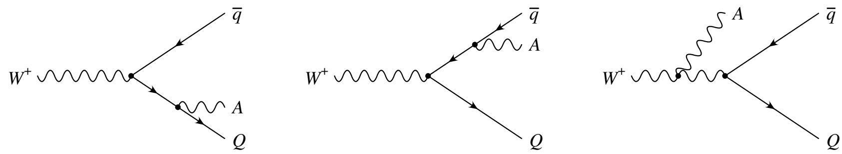
$$\begin{aligned}
&|\mathcal{M}_0|^2 \\
&= \frac{e^2|V_{cb}|^2}{3s_W^2}\left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - p_1p_2 g^{\mu\nu} + ip_{1\kappa}p_{2\lambda}\epsilon^{\kappa\lambda\mu\nu}\right)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \\
&= \frac{e^2|V_{cb}|^2}{3s_W^2}\left((D-2)p_1p_2 + \frac{1}{q^2}\left(2(p_1q)(p_2q) - (p_1p_2)q^2\right)\right) \\
&= \frac{e^2|V_{cb}|^2q^2}{6s_W^2}\left((D-3)(1-\mu_1-\mu_2) + (1+\mu_1-\mu_2)(1-\mu_1+\mu_2)\right) \\
&\overset{D=4}{=} \frac{e^2|V_{cb}|^2q^2}{6s_W^2}\left(2-\mu_1-\mu_2-(\mu_1-\mu_2)^2\right).
\end{aligned} \tag{16}$$

After integrating over the phase space dPS, the unpolarised Born term decay rate with an on-shell W boson ($q^2 = m_W^2$) is given by

$$\Gamma_0 = \frac{e^2|V_{cb}|^2 m_W}{96\pi s_W^2}\sqrt{\lambda(1,\mu_1,\mu_2)}\left(2-\mu_1-\mu_2-(\mu_1-\mu_2)^2\right). \tag{17}$$

3 Tree contributions

In this work we deal with the Next-to-Leading Order (NLO) electroweak corrections. For this we take into account the tree corrections with the real photon as shown in Fig. 2. In order to calculate the decay rate, we calculate $|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$. The complex conjugated matrix element is the corresponding inverse process. Therefore, the absolute value of the squared matrix element can be represented by a Feynman diagram combined of the original diagram (lower part) and the inverse diagram (upper part). Each of the quarks now can radiate a photon, and if the photon is



(1) (2) (3)

Figure 2: First order electroweak tree corrections

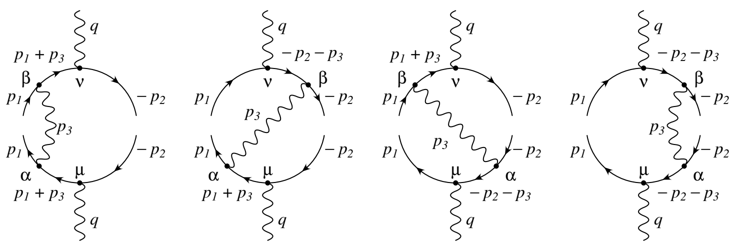
sufficiently soft (i.e., the energy is quite low), this photon will not be observed in the experiment. Therefore, in order to describe the process $W \to q\bar{q}$ to first order, we call the corresponding matrix elements $\mathcal{M}_1$ and $\mathcal{M}_2$ and $\mathcal{M}_3$. In order to calculate radiative contributions to the first order, we sum up the matrix elements to obtain

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 \tag{18}$$

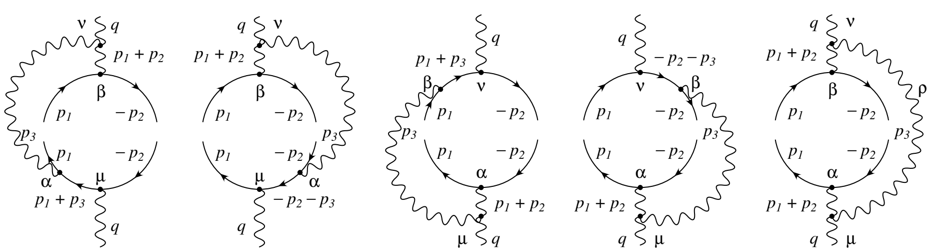
In calculating the absolute square of this matrix element we obtain

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + \mathcal{M}_1^*\mathcal{M}_2 + \mathcal{M}_2^*\mathcal{M}_1 + |\mathcal{M}_2|^2 + \dots \tag{19}$$

Now the different contributions like $|\mathcal{M}_1|^2$ can be represented by the Cutkosky diagram. For the nine contributions we can plot the following Cutkosky diagrams: (the photon line is not cut for matter of convenience only).



(11) (12) (21) (22)



(13) (23) (31) (32) (33)

New elements in these diagrams are lines between two points which are called propagators because the particle state propagates between these points (vertices).

As the photon line is cut (even though this is not shown), the propagator for this line is replaced by the completeness relation

$$\sum_{\lambda_3=0}^{3} \varepsilon_\mu^*(p_3,\lambda_3)\varepsilon_\nu(p_3,\lambda_3) = -g_{\mu\nu}. \tag{20}$$

However, there are also Dirac propagators. These are given by the factor

$$S_F(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \tag{21}$$

where p is the momentum and m is the mass of the fermion. Note that the particle on a propagator line need not be on the mass shell, therefore in general $p^2 \neq m^2$. The addition $i\epsilon$ is an infinitesimal contribution which helps to decide where the poles of the propagator, i.e., the zeros of the numerator are located in the complex plane. This choice is due to Richard P. Feynman himself. For a first approach we can simply skip this imaginary part. Finally, the vertex between fermion with electric charge Q_f (in units of the elementary charge $e = 1.6022 \times 10^{-19}$C) and the photon is given by

$$-ieQ_f\gamma^\alpha. \tag{22}$$

3.1 Kinematics for three body decay

In order to calculate the decay rate we have to integrate the absolute square of matrix element over the phase space. In order to perform the phase space integration it would be appropriate to have the denominator factors in the easiest shape. For this we define the dimensionless quantities y_1 and y_2 by

$$\begin{aligned} y_1q^2 &:= (p_1+p_3)^2 - m_1^2 = p_1^2 + 2p_1p_3 + p_3^2 - m_1^2 = 2p_1p_3 + p_3^2, \\ y_2q^2 &:= (p_2+p_3)^2 - m_2^2 = p_2^2 + 2p_2p_3 + p_3^2 - m_2^2 = 2p_2p_3 + p_3^2 \end{aligned} \tag{23}$$

(note that the outer quark lines are on mass shell, so that $p_i^2 = m_i^2$). Accordingly, we write

$$p_1^2 = m_1^2 = \mu_1 q^2, \qquad p_2^2 = m_2^2 = \mu_2 q^2, \tag{24}$$

As the integration over the phase space will result in infrared (IR) divergences, we use the photon mass regularisation as

$$p_3^2 = m_A^2 = \Lambda q^2. \tag{25}$$

The missing scalar product p_1p_2 can be obtained by the expansion

$$q^2 = (p_1+p_2+p_3)^2 = p_1^2 + 2p_1p_2 + 2p_1p_3 + p_2^2 + 2p_2p_3 + p_3^2. \tag{26}$$

To conclude, we have

$$\begin{aligned}
p_1p_1 &= \mu_1 q^2, \\
p_1p_2 &= \frac{1}{2}\left(1-(\mu_1+y_1)-(\mu_2+y_2)+\Lambda\right)q^2, \\
p_1p_3 &= \frac{1}{2}(y_1-\Lambda)q^2, \\
p_2p_2 &= \mu_2 q^2, \\
p_2p_3 &= \frac{1}{2}(y_2-\Lambda)q^2, \\
p_3p_3 &= \Lambda q^2.
\end{aligned} \tag{27}$$

The general ansatz for the kinematics in the quark frame is given by $p_1 = (E_1; 0, 0, |\vec{p}_1|)$,

$$p_2 = (E_2; |\vec{p}_2|\sin\theta_{12}, 0, |\vec{p}_2|\cos\theta_{12}), \qquad p_3 = (E_3; |\vec{p}_3|\sin\theta_{13}, 0, |\vec{p}_3|\cos\theta_{13}) \tag{28}$$

with

$$\begin{aligned}
E_1 &= \frac{1}{2}\left(1+\mu_1-(\mu_2+y_2)\right)\sqrt{q^2} & |\vec{p}_1| &= \frac{1}{2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\sqrt{q^2} \\
E_2 &= \frac{1}{2}\left(1-(\mu_1+y_1)+\mu_2\right)\sqrt{q^2} & |\vec{p}_2| &= \frac{1}{2}\sqrt{\lambda(1,\mu_1+y_1,\mu_2)}\sqrt{q^2} \\
E_3 &= \frac{1}{2}(y_1+y_2)\sqrt{q^2} & |\vec{p}_3| &= \frac{1}{2}\sqrt{(y_1+y_2)^2-4\Lambda}\sqrt{q^2}
\end{aligned} \tag{29}$$

and the cosines and sines of the relative angles given by

$$\begin{aligned}
\cos\theta_{12} &= \frac{y_1y_2+(1-\mu_1+\mu_2)y_1+(1+\mu_1-\mu_2)y_2-\lambda-2\Lambda}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\sqrt{\lambda(1,\mu_1+y_1,\mu_2)}}, \\
\sin\theta_{12} &= \frac{2\sqrt{N(y_1,y_2)}}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\sqrt{\lambda(1,\mu_1+y_1,\mu_2)}}, \\
\cos\theta_{13} &= \frac{-y_1(1-\mu_1+\mu_2+y_2)+(1+\mu_1-\mu_2-y_2)y_2+2\Lambda}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\sqrt{(y_1+y_2)^2-4\Lambda}}, \\
\sin\theta_{13} &= \frac{-2\sqrt{N(y_1,y_2)}}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\sqrt{(y_1+y_2)^2-4\Lambda}},
\end{aligned} \tag{30}$$

with

$$\begin{aligned}
N(y_1,y_2) = {}& (1-\mu_1-\mu_2)y_1y_2 - y_1^2(\mu_2+y_2) - (\mu_1+y_1)y_2^2 \\
&+ \Lambda\left((1-\mu_1+\mu_2)y_1+(1+\mu_1-\mu_2)y_2+y_1y_2-\lambda\right) - \Lambda^2.
\end{aligned} \tag{31}$$

The three-body phase space is given by

$$dPS_3 = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3-q)\prod_{i=1}^{3}\frac{d^4p_i}{(2\pi)^4}(2\pi)\delta(p_i^2-m_i^2)\theta(p_i^0). \quad (32)$$

which in the quark frame simplifies to

$$dPS_3 = dPS_2 \times \frac{q^2}{(4\pi)^2\sqrt{\lambda(1,\mu_1,\mu_2)}}dy_1dy_2, \quad (33)$$

3.2 Three-particle process

In order to calculate the absolute square of the matrix element of three particles, we obtain nine diagrams which we name (from top to bottom) "11", "12", "21" and "22",... corresponding to the starting and final points of the photon.

$$\begin{aligned} &iH_{11}^{\mu\nu} \\ &= \operatorname{tr}\Bigg(\frac{ie\gamma^\nu}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}\frac{i(\not p_1+\not p_3+m_1)}{(p_1+p_3)^2-m_1^2+i\varepsilon}(-ie\gamma^\beta Q_1)(\not p_1+m_1) \\ &\quad\times(-ie\gamma^\alpha Q_1)\frac{i(\not p_1+\not p_3+m_1)}{(p_1+p_3)^2-m_1^2+i\varepsilon}\frac{ie\gamma^\mu}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}(\not p_2-m_2)\Bigg)g_{\alpha\beta}, \end{aligned}$$

$$\begin{aligned} &iH_{12}^{\mu\nu} \\ &= \operatorname{tr}\Bigg(\frac{ie\gamma^\nu}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}(\not p_1+m_1)(-ie\gamma^\alpha Q_1)\frac{i(\not p_1+\not p_3+m_1)}{(p_1+p_3)^2-m_1^2+i\varepsilon} \\ &\quad\times\frac{ie\gamma^\mu}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}(\not p_2-m_2)(-ie\gamma^\beta Q_2)\frac{i(-\not p_2-\not p_3+m_2)}{(-p_2-p_3)^2-m_2^2+i\varepsilon}\Bigg)g_{\alpha\beta}, \end{aligned}$$

$$\begin{aligned} &iH_{21}^{\mu\nu} \\ &= \operatorname{tr}\Bigg(\frac{ie\gamma^\nu}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}\frac{i(\not p_1+\not p_3+m_1)}{(p_1+p_3)^2-m_1^2+i\varepsilon}(-ie\gamma^\beta Q_1)(\not p_1+m_1) \\ &\quad\times\frac{ie\gamma^\mu}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}\frac{i(-\not p_2-\not p_3+m_2)}{(-p_2-p_3)^2-m_2^2+i\varepsilon}(-ie\gamma^\alpha Q_2)(\not p_2-m_2)\Bigg)g_{\alpha\beta}, \end{aligned}$$

$$\begin{aligned} &iH_{22}^{\mu\nu} \\ &= \operatorname{tr}\Bigg(\frac{ie\gamma^\nu}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}(\not p_1+m_1)\frac{ie\gamma^\mu}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}\frac{i(-\not p_2-\not p_3+m_2)}{(-p_2-p_3)^2-m_2^2+i\varepsilon} \end{aligned}$$

$$\times (-ie\gamma^\alpha Q_2)(\not{p}_2 - m_2)(-ie\gamma^\beta Q_2)\frac{i(-\not{p}_2 - \not{p}_3 + m_2)}{(-p_2-p_3)^2 - m_2^2 + i\varepsilon}\Bigg) g_{\alpha\beta},$$

$$\begin{aligned}
&iH_{13}^{\mu\nu} \\
&= \mathrm{tr}\Bigg(\frac{ie\gamma^\beta}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}(\not{p}_1 + m_1)(-ie\gamma^\alpha Q_1)\frac{i(\not{p}_1 + \not{p}_3 + m_1)}{(p_1+p_3)^2 - m_1^2 + i\varepsilon} \\
&\quad \times \frac{ie\gamma^\mu}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}(\not{p}_2 - m_2)\Bigg)\frac{-i}{(p_1+p_2)^2 - m_W^2 + i\varepsilon} \\
&\quad \times ieQ_W\Big(g_{\alpha\beta}(p_3 - p_1 - p_2)^\nu + g_\beta{}^\nu(2p_1 + 2p_2 + p_3)_\alpha \\
&\quad + g^\nu{}_\alpha(-p_1 - p_2 - 2p_3)_\beta\Big),
\end{aligned}$$

$$\begin{aligned}
&iH_{23}^{\mu\nu} \\
&= \mathrm{tr}\Bigg(\frac{ie\gamma^\beta}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}(\not{p}_1 + m_1)\frac{ie\gamma^\mu}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}\frac{i(-\not{p}_2 - \not{p}_3 + m_2)}{(-p_2-p_3)^2 - m_2^2 + i\varepsilon} \\
&\quad \times (-ie\gamma^\alpha Q_2)(\not{p}_2 - m_2)\Bigg)\frac{-i}{(p_1+p_2)^2 - m_W^2 + i\varepsilon} \\
&\quad \times ieQ_W\Big(g_{\alpha\beta}(p_3 - p_1 - p_2)^\nu + g_\beta{}^\nu(2p_1 + 2p_2 + p_3)_\alpha \\
&\quad + g^\nu{}_\alpha(-p_1 - p_2 - 2p_3)_\beta\Big),
\end{aligned}$$

$$\begin{aligned}
&iH_{31}^{\mu\nu} \\
&= \mathrm{tr}\Bigg(\frac{ie\gamma^\nu}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}\frac{i(\not{p}_1 + \not{p}_3 + m_1)}{(p_1+p_3)^2 - m_1^2 + i\varepsilon}(-ie\gamma^\beta Q_1)(\not{p}_1 + m_1) \\
&\quad \times \frac{ie\gamma^\alpha}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}(\not{p}_2 - m_2)\Bigg)\frac{-i}{(p_1+p_2)^2 - m_W^2 + i\varepsilon} \\
&\quad \times ieQ_W\Big(g_\beta{}^\mu(-p_1 - p_2 - 2p_3)_\alpha + g^\mu{}_\alpha(2p_1 + 2p_2 + p_3)_\beta \\
&\quad + g_{\alpha\beta}(-p_1 - p_2 + p_3)^\mu\Big),
\end{aligned}$$

$$\begin{aligned}
&iH_{32}^{\mu\nu} \\
&= \mathrm{tr}\Bigg(\frac{ie\gamma^\nu}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}(\not{p}_1 + m_1)\frac{ie\gamma^\alpha}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}(\not{p}_2 - m_2)
\end{aligned}$$

$$\times\,(-ie\gamma^\beta Q_2)\frac{i(-\not p_2-\not p_3+m_2)}{(-p_2-p_3)^2-m_2^2+i\varepsilon}\Bigg)\frac{-i}{(p_1+p_2)^2-m_W^2+i\varepsilon}$$
$$\times\,ieQ_W\Big(g_\beta{}^\mu(-p_1-p_2-2p_3)_\alpha+g^\mu{}_\alpha(2p_1+2p_2+p_3)_\beta$$
$$+\,g_{\alpha\beta}(-p_1-p_2+p_3)^\mu\Big),$$

$$\begin{aligned} iH_{33}^{\mu\nu} & \\ = \;& \mathrm{tr}\left(\frac{ie\gamma^\beta}{\sqrt{2}s_W}V_{ji}^\dagger\frac{1-\gamma_5}{2}(\not p_1+m_1)\frac{ie\gamma^\alpha}{\sqrt{2}s_W}V_{ij}\frac{1-\gamma_5}{2}(\not p_2-m_2)\right)\\ & \times\frac{-i}{(p_1+p_2)^2-m_W^2+i\varepsilon}\times\frac{-i}{(p_1+p_2)^2-m_W^2+i\varepsilon}\\ & \times\,ieQ_W\Big(g^\rho{}_\beta(p_3-p_1-p_2)^\nu+g_\beta{}^\nu(2p_1+2p_2+p_3)^\rho\\ & +\,g_{\nu\rho}(-p_1-p_2-2p_3)_\beta\Big)\times ieQ_W\Big(g_\rho{}^\mu(-2p_3-p_1-p_2)_\alpha\\ & +\,g^\mu{}_\alpha(2p_1+2p_2+p_3)_\rho+g_{\alpha\rho}(-p_1-p_2+p_3)^\mu\Big) \end{aligned} \tag{34}$$

We have used here

$$\begin{aligned}\gamma^0\left(\frac{i}{\not p-m}\right)^\dagger\gamma^0 &= \gamma^0\frac{(i(\not p+m))^\dagger}{p^2-m^2}\gamma^0=-i\frac{\gamma^0\not p^\dagger\gamma^0+m}{p^2-m^2}\\ &= -i\frac{\not p+m}{p^2-m^2}=\frac{-i}{\not p-m},\end{aligned} \tag{35}$$

but also $\gamma^0(-i\gamma^\alpha)^\dagger\gamma^0=i\gamma^0\gamma^\alpha\gamma^0=i\gamma^\alpha$. Therefore, the two signs cancel. For the phase space limits, the condition is

$$-1\le\cos\theta_2=\frac{E_1E_2-\mu_{12}q^2}{|\vec p_1||\vec p_2|}\le+1 \tag{36}$$

or equivalently $E_1E_2-\mu_{12}q^2=\pm|\vec p_1||\vec p_2|$ describes the phase space limits. Solving for y_1 we obtain

$$\begin{aligned} & y_{1\pm}(y_2)\\ = \;& \frac{1}{2(\mu_2+y_2)}\Big(y_2\,(1-\mu_1-(\mu_2+y_2))+\Lambda\,(1-\mu_1+(\mu_2+y_2))\\ & \pm\sqrt{(y_2-\Lambda)^2-4\Lambda\mu_2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\Big)\\ =: \;& \frac{A(y_2)\pm B(y_2)}{C(y_2)}. \end{aligned} \tag{37}$$

The condition that the square roots are real specifies the integration range. We obtain

$$y_2 \geq y_{20} = \Lambda + 2\sqrt{\Lambda\mu_2}, \qquad y_2 \leq y_{2-} = (1-\sqrt{\mu_1})^2 - \mu_2 \leq 1. \tag{38}$$

The IR singularity resides in the phase space at $y_1 = y_2 = 0$ and is regularised by the parameter Λ.

4 Calculation of Integrals

Now we will discuss the details about the calculation of the phase space integrals. The integrals necessary for the tree corrections are of the form

$$I_{WW}(n_1, n_2; \Lambda) = \int_{y_{20}}^{y_{2-}} dy_2 \int_{y_{1-}}^{y_{1+}} dy_1 \frac{y_1^{n_1} y_2^{n_2}}{(y_1+y_2)^2}. \tag{39}$$

and are obtained from the product of $\mathcal{M}_3\mathcal{M}_3$, while the integrals are regular for $n_1 + n_2 > 0$ and can be calculated taking $\Lambda = 0$, the integrals at the border line $n_1 + n_2 = 0$ are IR singular. From the products of $\mathcal{M}_1\mathcal{M}_3$ and $\mathcal{M}_2\mathcal{M}_3$ the integrals we obtained are in the form of

$$I_W(n_1, n_2; \Lambda) = \int_{y_{20}}^{y_{2-}} dy_2 \int_{y_{1-}}^{y_{1+}} dy_1 \frac{y_1^{n_1} y_2^{n_2}}{(y_1+y_2)}. \tag{40}$$

And from the products of $\mathcal{M}_1\mathcal{M}_2$ we obtain integrals. As an example in most cases, the previous integrals reduce to the ones shown below, which we calculate in the following section. While all integrals to be calculated can be transformed to integrals $I_{WW}(n_1, n_2; \Lambda)$, at least the divergent integrals can be reduced to $I(n_1, n_2; \Lambda)$. Therefore, as examples we consider in the following the integrals $I(n_1, n_2; \Lambda)$ only, skipping the additional argument Λ for the convergent integrals.

4.1 Performing the integration – the example $I(-1, 0)$

As an example we calculate the integral

$$I(-1,0) = \int_{y_{20}}^{y_{2-}} dy_2 \int_{y_{1-}}^{y_{1+}} \frac{dy_1}{y_1} = \int_{y_{20}}^{y_{2-}} dy_2 \ln\left(\frac{y_{1+}(y_2)}{y_{1-}(y_2)}\right). \tag{41}$$

This integral turns out to be finite. Therefore, in this case we can put $\Lambda = 0$. The limits simplify to $y_{20} \to 0$ and $y_{2-} \to (1-\sqrt{\mu_1})^2 - \mu_2$, and for the argument of the logarithm one obtains

$$\frac{y_{1+}(y_2)}{y_{1-}(y_2)} = \frac{1 - \mu_1 - \mu_2 - y_2 + \sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}}{1 - \mu_1 - \mu_2 - y_2 - \sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}}. \tag{42}$$

The expression above is difficult to integrate over y_2. In this case we will simplify the expression especially the square root. Written explicitly, one obtains

$$\begin{aligned}
&\lambda(1,\mu_1,\mu_2+y_2)\\
&= (1-\mu_1)^2+(\mu_2+y_2)^2-2\mu_1-2(\mu_2+y_2)-2\mu_1(\mu_2+y_2)\\
&= (1+\mu_1-\mu_2-y_2)^2-(1+\mu_1)^2+(1-\mu_1)^2\\
&= (1+\mu_1-\mu_2-y_2)^2-4\mu_1\\
&= \left((1+\sqrt{\mu_1})^2-\mu_2-y_2\right)\left((1-\sqrt{\mu_1})^2-\mu_2-y_2\right)\\
&= (y_{2+}-y_2)(y_{2-}-y_2).
\end{aligned} \tag{43}$$

y_{2-} is introduced before already as upper limit, while
$y_{2+}=(1+\sqrt{\mu_1})^2-\mu_2$. So in the next step we do replacements like
$y_{2-}-y_2=y_2'$. We obtain

$$\begin{aligned}
I(-1,0) &= \int_0^{y_{2-}} \ln\left(\frac{y_{1+}(y_2)}{y_{1-}(y_2)}\right)dy_2\\
&= \int_0^{y_{2-}} \ln\left(\frac{1-\mu_1-\mu_2-y_2+\sqrt{(y_{2+}-y_2)(y_{2-}-y_2)}}{1-\mu_1-\mu_2-y_2-\sqrt{(y_{2+}-y_2)(y_{2-}-y_2)}}\right)dy_2\\
&= \int_0^{y_{2-}} \ln\left(\frac{2\sqrt{\mu_1}(1-\sqrt{\mu_1})+y_2'+\sqrt{(4\sqrt{\mu_1}+y_2')y_2'}}{2\sqrt{\mu_1}(1-\sqrt{\mu_1})+y_2'-\sqrt{(4\sqrt{\mu_1}+y_2')y_2'}}\right)dy_2'.
\end{aligned} \tag{44}$$

Next we replace
$y_2'=4\sqrt{\mu_1}\sinh^2\tau$ to obtain (τ_- defined by $y_{2-}=:4\sqrt{\mu_1}\sinh^2\tau_-$)

$$\begin{aligned}
&I(-1,0)\\
&= 8\sqrt{\mu_1}\int_0^{\tau_-}\ln\left(\frac{1-\sqrt{\mu_1}+2\sinh^2\tau+2\sinh\tau\cosh\tau}{1-\sqrt{\mu_1}+2\sinh^2\tau-2\sinh\tau\cosh\tau}\right)\sinh\tau\cosh\tau\,d\tau\\
&= 2\sqrt{\mu_1}\int_0^{\tau_-}\ln\left(\frac{1-\sqrt{\mu_1}+e^{2\tau}-1}{1-\sqrt{\mu_1}-1+e^{-2\tau}}\right)(e^{2\tau}-e^{-2\tau})d\tau\\
&= 2\sqrt{\mu_1}\int_0^{\tau_-}\ln\left(\frac{e^{2\tau}-\sqrt{\mu_1}}{e^{-2\tau}-\sqrt{\mu_1}}\right)(e^{2\tau}-e^{-2\tau})d\tau,
\end{aligned} \tag{45}$$

where we have used $\sinh\tau=\frac{1}{2}(e^\tau-e^{-\tau})$, $\cosh\tau=\frac{1}{2}(e^\tau+e^{-\tau})$.
The last substitution is given by $z=e^{-2\tau}$ which leads to ($z_-:=e^{-2\tau_-}$)

$$\begin{aligned}
I(-1,0) &= \sqrt{\mu_1}\int_{z_-}^1\ln\left(\frac{1-\sqrt{\mu_1}z}{(z-\sqrt{\mu_1})z}\right)\left(\frac{1}{z^2}-1\right)dz\\
&= \sqrt{\mu_1}\times\Bigg[-\left(\frac{1}{z}-\sqrt{\mu_1}\right)\ln(1-\sqrt{\mu_1}z)-\sqrt{\mu_1}\ln z
\end{aligned}$$

$$
\begin{aligned}
& -\left(z-\frac{1}{\sqrt{\mu_1}}\right)\ln(1-\sqrt{\mu_1}z)+z+\left(\frac{1}{z}-\frac{1}{\sqrt{\mu_1}}\right)\ln(z-\sqrt{\mu_1}) \\
& +\frac{1}{\sqrt{\mu_1}}\ln z+(z-\sqrt{\mu_1})\ln(z-\sqrt{\mu_1})-z+\frac{1}{z}\ln z+\frac{1}{z} \\
& \left.+z\ln z-z\right]_{z_-}^{1} = \qquad \text{(each line for one of the logarithms)} \\
= \; & \sqrt{\mu_1}\Bigg[\left(\sqrt{\mu_1}+\frac{1}{\sqrt{\mu_1}}-\frac{1}{z}-z\right)\ln\left(\frac{1-\sqrt{\mu_1}z}{(z-\sqrt{\mu_1})z}\right) \\
& \left.+\frac{2}{\sqrt{\mu_1}}\ln z+\frac{1}{z}-z\right]_{z_-}^{1} \\
= \; & -\sqrt{\mu_1}\Bigg[\left(\sqrt{\mu_1}+\frac{1}{\sqrt{\mu_1}}-z_+-z_-\right)\ln\left(\frac{z_+-\sqrt{\mu_1}}{z_--\sqrt{\mu_1}}\right) \\
& -\frac{2}{\sqrt{\mu_1}}\ln(z_-)+z_+-z_-\Bigg] \\
= \; & -\mu_2\ln\left(\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right)+\ln\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{1+\mu_1-\mu_2-\sqrt{\lambda}}\right)-\sqrt{\lambda},
\end{aligned}
\tag{46}
$$

where we have used $z_+z_-=1$, $z_\pm=(1+\mu_1-\mu_2\pm\sqrt{\lambda})/(2\sqrt{\mu_1})$.

4.2 Calculating divergent integrals – the example $I(-2,0)$

For $n_1+n_2=-2$ the integrals are divergent and this integral turn out to be IR-divergent. In principal all integrals can be calculated by using net substitutions.

$$
\begin{aligned}
I(-2,0) &= \int_0^{y_{2-}} dy_2 \int_{y_{1-}(y_2)}^{y_{1+}(y_2)} \frac{dy_1}{y_1^2} \\
&= \int_0^{y_{2-}} dy_2 \left(\frac{1}{y_{1-}(y_2)}-\frac{1}{y_{1+}(y_2)}\right)
\end{aligned}
\tag{47}
$$

After performing the integration over y_1, the integration over y_2 cannot be calculated analytically for a general parameter Λ in a closed form.

$$
\begin{aligned}
I(-2,0) &= \int_0^{y_{2-}} dy_2 \left(\frac{C(y_2)}{A(y_2)-B(y_2)}-\frac{C(y_2)}{A(y_2)+B(y_2)}\right) \\
&= \int_0^{y_{2-}} \frac{2B(y_2)C(y_2)dy_2}{A(y_2)^2-B(y_2)^2}
\end{aligned}
\tag{48}
$$

This region is the infrared region close to $y_2 = 0$. The integral $I(-2,0)$ with exact regularisation parameter Λ cannot be calculated analytically. However, instead of

$$f(y_2,\Lambda) = \frac{2B(y_2)C(y_2)}{A(y_2)^2 - B(y_2)^2} \approx \frac{\sqrt{(y_2-\Lambda)^2 - 4\Lambda\mu_2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}}{\mu_1 y_2^2} \tag{49}$$

(the corrections of $O(\Lambda)$ in the denominator could be neglected), one can find an integrand with the same singular behaviour close to the lower boundary $y_{20} = \Lambda + 2\sqrt{\Lambda\mu_2}$, namely

$$f_0(y_2,\Lambda) = \frac{\sqrt{(y_2-\Lambda)^2 - 4\Lambda\mu_2}\sqrt{\lambda(1,\mu_1,\mu_2)}}{\mu_1 y_2^2}. \tag{50}$$

In subtracting this simplified integrand and additing it again, one obtains

$$\begin{aligned} I(-2,0,\Lambda) &= \int_{y_{20}}^{y_{2-}} f(y_2,\Lambda)dy_2 \\ &= \int_{y_{20}}^{y_{2-}} \left(f(y_2,\Lambda) - f_0(y_2,\Lambda)\right)dy_2 + \int_{y_{20}}^{y_{2-}} f_0(y_2,\Lambda)dy_2. \end{aligned} \tag{51}$$

Up to this point this equality is exact. First, we calculate the last integral which is the divergent part $I_D(-2,0,\Lambda)$. Second, in the difference integral the singularity will drop out. Therefore, this integral is calculable even for $\Lambda = 0$. This is the convergent part $I_C(-2,0)$. The sum of both is equal to $I(-2,0,\Lambda)$ for small values of Λ.

For the calculation of divergent part we can use the universal substitution to the new variable ζ, but first we define

$$y_2 = \Lambda + 2\sqrt{\Lambda\mu_2}\cosh\eta, \qquad dy_2 = 2\sqrt{\Lambda\mu_2}\sinh\eta\, d\eta, \tag{52}$$

where ζ is defined as $\zeta = e^{-2\eta}$ then we get the substitution in form of ζ as

$$y_2 = \Lambda + \sqrt{\Lambda\mu_2}\frac{1+\zeta}{\sqrt{\zeta}}, \qquad dy_2 = -\sqrt{\Lambda\mu_2}\frac{1-\zeta}{2\zeta\sqrt{\zeta}}d\zeta \tag{53}$$

leading to

$$\begin{aligned} I_D(-2,0,\Lambda) &= \frac{\sqrt{\lambda}}{\mu_1}\int_{\Lambda+2\sqrt{\Lambda\mu_2}}^{y_{2-}} \frac{\sqrt{(y_2-\Lambda)^2 - 4\Lambda\mu_2}}{y_2^2}dy_2 \\ &= \int_{\zeta_-}^{1} \frac{(1-\zeta)^2}{2\zeta(1+\zeta)^2}d\zeta = \int_{\zeta_-}^{1}\left(\frac{1}{2\zeta} - \frac{1}{(1+\zeta)^2}\right)d\zeta \end{aligned} \tag{54}$$

After simplification the limit $\Lambda \to 0$ is performed, leading to an expression in terms of ζ which can be partitioned fractionally (with or without a common logarithm). We obtain

$$I_s(-2,0,\zeta) = \frac{\sqrt{\lambda}}{2\mu_1\zeta} - \frac{1-\mu_1-\mu_2}{2\mu_1}\left(\frac{1}{\zeta+\zeta_0} - \frac{\zeta_0}{\zeta_0\zeta+1}\right) \tag{55}$$

After integration we obtain

$$I_D(-2,0) = \frac{\sqrt{\lambda}}{2\mu_1} I_\zeta^0 - \frac{1-\mu_1-\mu_2}{2\mu_1} I_\zeta^{-1}, \tag{56}$$

where

$$I_\zeta^0 \;=\; \int_{\zeta_-}^1 \frac{d\zeta}{\zeta} = \ln \zeta_-, \qquad I_\zeta^{-1} \;=\; -\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \tag{57}$$

with $\zeta_- := \Lambda\mu_2/\left((1-\sqrt{\mu_1})^2-\mu_2\right)^2$, a quantity set to zero in all uncritical cases. Now for the calculation of counterterm, we replace back ζ to y_2 as

$$\zeta = \frac{y_2^2 - 2\Lambda\mu_2 - y_2\sqrt{y_2^2-4\Lambda\mu_2}}{2\Lambda\mu_2}, \qquad d\zeta = \frac{(y_2-\sqrt{y_2^2-4\Lambda\mu_2})^2 dy_2}{2\Lambda\mu_2\sqrt{y_2^2-4\Lambda\mu_2}}, \tag{58}$$

and calculate the limit $\Lambda \to 0$. In doing so, we obtain

$$I_C(-2,0) = \int_{y_{20}}^{y_{2-}} \frac{\sqrt{\lambda} dy_2}{\mu_1 y_2}, \tag{59}$$

It is interesting that both I_D and I_C vanish for $n_1 + n_2 > -2$. Therefore, the procedure can be performed without distinguishing between divergent and convergent integrals. The counterterm is now subtracted from the original integrand and transformed to z. The convergent part is given by

$$\begin{aligned} I_C(-2,0) &= \int_0^{y_{2-}} \left(f(y_2,0) - f_0(y_2,0)\right) dy_2 \\ &= \int_0^{y_{2-}} \frac{dy_2}{\mu_1 y_2}\left(\sqrt{\lambda(1,\mu_1,\mu_2+y_2)} - \sqrt{\lambda(1,\mu_1,\mu_2)}\right) \\ &= \int_{z_-}^1 \left(\frac{-2\sqrt{\lambda}}{\mu_1(z-z_+)} - \frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{\mu_1 z} - \frac{1}{\sqrt{\mu_1}}\left(1+\frac{1}{z^2}\right)\right) dz \end{aligned} \tag{60}$$

In integrated form one obtains accordingly

$$I_C(-2,0) \;=\; -\frac{1}{\mu_1} I_z^{1+} - \frac{1+\mu_1-\mu_2}{\mu_1} I_z^0 - \frac{2\sqrt{\lambda}}{\mu_1} I_z^+, \tag{61}$$

where

$$\begin{aligned} I_z^{1+} \;&=\; \sqrt{\mu_1}\int_{z_-}^1 \left(\frac{1}{z^2}+1\right) dz \;=\; \sqrt{\mu_1}(z_+ - z_-) \;=\; \sqrt{\lambda}, \\ I_z^0 \;&=\; \int_{z_-}^1 \frac{dz}{z} \;=\; -\ln(z_-) \;=\; \frac{1}{2}\ln\left(\frac{z_+}{z_-}\right) \end{aligned}$$

$$\begin{aligned} I_z^+ &= \int_{z_-}^{1} \frac{dz}{z-z_+} \\ &= -\frac{1}{2}\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}}\right) - \frac{1}{4}\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right). \end{aligned} \tag{62}$$

The dilogarithmic terms contained in some expressions are given by

$$\begin{aligned} t_\zeta^\ell &= \int_{\zeta_-}^{1} \left(\frac{1}{\zeta} - \frac{2}{1+\zeta}\right) L_\zeta(\zeta) d\zeta, \\ t_{\zeta W}^\ell &= \int_{\zeta_-}^{1} \left(\frac{1}{\zeta} - \frac{2}{1+\zeta}\right) L_{\zeta W}(\zeta) d\zeta, \end{aligned} \tag{63}$$

with

$$L_\zeta(\zeta) = \ln\left(\frac{\zeta+\zeta_0}{1+\zeta_0\zeta}\right) \qquad L_{\zeta W}(\zeta) = \ln\left(\frac{\zeta+\zeta_{0W}}{1+\zeta_{0W}\zeta}\right) \tag{64}$$

$$\zeta_0 = \frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}, \qquad \zeta_{0W} = \frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}} \tag{65}$$

and

$$\begin{aligned} t_z^{\ell-} &= \int_{z_-}^{1} \left[\left(\frac{1}{z-z_-} - \frac{1}{z-z_+}\right) L_z(z) - \frac{L_z(z_-)}{z-z_+}\right] dz, \\ t_z^{\ell+} &= \int_{z_-}^{1} \left[\left(\frac{1}{z-z_-} + \frac{1}{z-z_+} - \frac{1}{z}\right) L_z(z) - \frac{L_z(z_-)}{z-z_+}\right] dz, \\ t_{zW}^{\ell-} &= \int_{z_-}^{1} \left[\left(\frac{1}{z-z_-} - \frac{1}{z-z_+}\right) L_{zW}(z) - \frac{L_{zW}(z_-)}{z-z_+}\right] dz, \\ t_{zW}^{\ell+} &= \int_{z_-}^{1} \left[\left(\frac{1}{z-z_-} + \frac{1}{z-z_+} - \frac{1}{z}\right) L_{zW}(z) - \frac{L_{zW}(z_-)}{z-z_+}\right] dz, \\ t_z^{\pm\ell} &= \int_{z_-}^{1} \frac{L_z(z)}{1\pm z} dz, \qquad t_z^\ell = \int_{z_-}^{1} \frac{L_z(z)}{z} dz, \\ t_{zW}^{\pm\ell} &= \int_{z_-}^{1} \frac{L_{zW}(z)}{1\pm z} dz, \qquad t_{zW}^\ell = \int_{z_-}^{1} \frac{L_{zW}(z)}{z} dz, \end{aligned} \tag{66}$$

with

$$L_z(z) = \ln\left(\frac{1-z\sqrt{\mu_1}}{(z-\sqrt{\mu_1})z}\right), \qquad L_{zW}(z) = \ln\left(\frac{z-\sqrt{\mu_1}}{z(1-z\sqrt{\mu_1})}\right) \tag{67}$$

and

$$\begin{aligned} \zeta_- &= \frac{\Lambda\mu_2}{((1-\sqrt{\mu_1})^2-\mu_2)^2}, \\ z_\pm &= \frac{1}{2\sqrt{\mu_1}}(1+\mu_1-\mu_2\pm\sqrt{\lambda}), \\ \sqrt{\lambda} &= \sqrt{\lambda(1,\mu_1,\mu_2)}. \end{aligned} \tag{68}$$

The logarithmic terms ℓ_ζ, ℓ_0, ℓ_+, ℓ_1 and ℓ_{1W} are listed below. A large portion of the ζ- and z-integrals can be expressed in terms of rational functions and logarithms, containing

$$\begin{aligned}
\ell_\zeta &= \ln\left(\frac{\lambda^2}{\Lambda\mu_1\mu_2}\right), & \ell_0 &= \ln\left(\frac{1-\sqrt{\mu_1}}{\sqrt{\mu_2}}\right), \\
\ell_+ &= \ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}}\right), & \ell_1 &= \ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right), \\
\ell_{1W} &= \ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right).
\end{aligned} \tag{69}$$

Exceptional are integrals containing a logarithm in the integrand together with ζ or z to the power of -1. These integrals contain dilogarithms and are kept as closed form terms t_ζ and t_z, the analytic expressions for these found in the following. The remaining z-terms read

$$\begin{aligned}
t_z^{\ell-} \\
= {} & \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) - \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) \\
& +\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \ell_1\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{(1+\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}\right),
\end{aligned} \tag{70}$$

$$\begin{aligned}
t_z^{\ell+} \\
= {} & \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) + \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) \\
& +\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) - \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right)\\
&+2\mathrm{Li}_2(\sqrt{\mu_1})-\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)\\
&-\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right)-\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right)\\
&-\ell_1\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}}\right)-\ln^2\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{(1+\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}\right), \qquad (71)
\end{aligned}
$$

$$
\begin{aligned}
&t_z^{-\ell}\\
&= \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right)\\
&\quad+\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}\right), \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
&t_z^{+\ell}\\
&= -\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2+\sqrt{\lambda}}{2(1+\sqrt{\mu_1})}\right)+\mathrm{Li}_2\left(-\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}(1+\sqrt{\mu_1})}\right)\\
&\quad+\mathrm{Li}_2\left(-\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(-\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right)\\
&\quad-\mathrm{Li}_2(-1)+\frac{1}{2}\ell_1\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}(1+\sqrt{\mu_1})}\right)-\frac{1}{2}\ell_{1W}\ln\left(\frac{1+\sqrt{\mu_1}}{\sqrt{\mu_1}}\right)\\
&\quad+\frac{1}{4}\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)\\
&\quad+\ln(\sqrt{\mu_1})\ln\left(\frac{1-\sqrt{\mu_1}}{\sqrt{\mu_2}}\right), \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
&t_z^{\ell}\\
&= \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right)-2\mathrm{Li}_2(\sqrt{\mu_1})\\
&\quad+\frac{1}{4}\ln^2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right), \qquad (74)
\end{aligned}
$$

$$
\begin{aligned}
& t_{zW}^{\ell -} \\
= \; & \mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2-\sqrt{\lambda}}\right) \\
& +\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\
& +\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)-\ell_{1W}\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{(1+\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}\right), \qquad (75)
\end{aligned}
$$

$$
\begin{aligned}
& t_{zW}^{\ell +} \\
= \; & \mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2-\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& +\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\
& +\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\
& -\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\ell_{1W}\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}}\right) \\
& +\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right) \\
& -2\mathrm{Li}_2(\sqrt{\mu_1}), \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
& t_{zW}^{-\ell} \\
= \; & \mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right) \\
& -\mathrm{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}(1-\sqrt{\mu_1})}\right), \qquad (77)
\end{aligned}
$$

$$
\begin{aligned}
& t_{zW}^{+\ell} \\
= \; & \mathrm{Li}_2\left(-\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2+\sqrt{\lambda}}{2(1+\sqrt{\mu_1})}\right) \\
& -\mathrm{Li}_2\left(-\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}(1+\sqrt{\mu_1})}\right)+\mathrm{Li}_2\left(-\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right) \\
& +\ln(1+\sqrt{\mu_1})\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}(1-\sqrt{\mu_1})}\right)-\mathrm{Li}_2(-1) \\
& -\ln\left(1+\frac{1}{\sqrt{\mu_1}}\right)\ln\left(\frac{1-\mu_1+\mu_2+\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right) \\
& -\ell_{1W}\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right),
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
& t_{zW}^{\ell} \\
= \; & -\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right)-\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right)+2\mathrm{Li}_2(\sqrt{\mu_1})
\end{aligned} \tag{79}
$$

4.3 Summary of the procedure

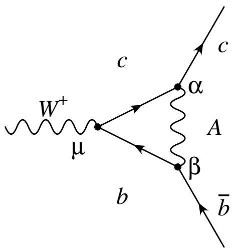
As for now, the general procedure consists of five steps:

1. Integration over y_1 with regularised limits
2. Transformation $y_2 \to \zeta$ in order to obtain the divergent parts
3. Transformation $\zeta \to y_2$ back to obtain the counterparts
4. Transformation $y_2 \to z$ for the difference to obtain the convergent parts
5. Summation of divergent and convergent parts

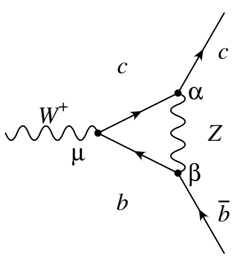
In this way we can calculate the more integrals in detail.

5 Loop contributions

The tree contributions are not the only first order corrections. In addition we have to take into account a large number of loop corrections. The detail calculation of the loop corrections to the vertex are below,

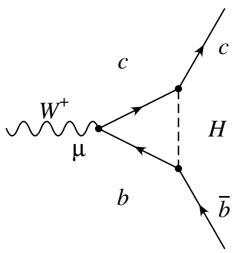


$$
\begin{aligned}
& \bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^{\mu}_{a1}v(p_2) \\
= & \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\left(-ieQ_c\gamma^{\alpha}\right) \\
& \times \frac{i(\not{k}+\not{p}_1+m_c)}{(k+p_1)^2-m_c^2}\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^{\mu}\Lambda_-\right)\frac{i(\not{k}-\not{p}_2+m_b)}{(k-p_2)^2-m_b^2}\left(-ieQ_b\gamma^{\beta}\right) \\
& \times v(p_2)\frac{-ig_{\alpha\beta}}{k^2-m_A^2} \\
= & \frac{ieV_{cb}}{\sqrt{2}s_W}\times -ie^2Q_cQ_b \\
& \times \int \frac{d^Dk}{(2\pi)^D}\frac{1}{((k+p_1)^2-m_c^2)\,((k-p_2)^2-m_b^2)\,(k^2-m_A^2)} \\
& \times \bar{u}(p_1)\gamma^{\alpha}(\not{k}+\not{p}_1+m_c)\gamma^{\mu}\Lambda_-(\not{k}-\not{p}_2+m_b)\gamma_{\alpha}v(p_2), \qquad (80)
\end{aligned}
$$

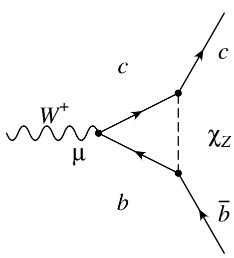


$$
\begin{aligned}
& \bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^{\mu}_{a2}v(p_2) \\
= & \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)ie\gamma^{\alpha}\left(g_c^-\Lambda_-+g_c^+\Lambda_+\right) \\
& \times \frac{i(\not{k}+\not{p}+m_c)}{(k+p_1)^2-m_c^2}\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^{\mu}\Lambda_-\right)\frac{i(\not{k}-\not{p}_2+m_b)}{(k-p_2)^2-m_b^2}ie\gamma^{\beta} \\
& \times \left(g_b^-\Lambda_-+g_b^+\Lambda_+\right)v(p_2)\frac{-ig_{\alpha\beta}}{k^2-m_Z^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{ieV_{cb}}{\sqrt{2}s_W} \times -ie^2 \int \frac{d^Dk}{(2\pi)^D} \frac{1}{((k+p_1)^2 - m_c^2)\,((k-p_2)^2 - m_b^2)\,(k^2 - m_Z^2)} \\
&\quad \times \bar{u}(p_1)\gamma^\alpha \left(g_c^- \Lambda_- + g_c^+ \Lambda_+\right) (\not{k} + \not{p}_1 + m_c)\gamma^\mu \Lambda_- \\
&\qquad \times (\not{k} - \not{p}_2 + m_b)\gamma_\alpha \left(g_b^- \Lambda_- + g_b^+ \Lambda_+\right) v(p_2), \qquad (81)
\end{aligned}
$$

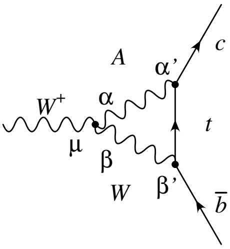


$$
\begin{aligned}
&\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{a3}v(p_2) \\
&= \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\left(\frac{-iem_c}{2s_W m_W}\right)\frac{i(\not{k}+\not{p}_1+m_c)}{(k+p_1)^2-m_c^2}\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^\mu\Lambda_-\right) \\
&\quad \times \frac{i(\not{k}-\not{p}_2+m_b)}{(k-p_2)^2-m_b^2}\left(\frac{-iem_b}{2s_W m_W}\right)v(p_2)\frac{i}{k^2-m_H^2} \\
&= \frac{ie}{\sqrt{2}s_W} \times \frac{ie^2 m_c m_b}{4s_W^2 m_W^2} \\
&\quad \times \int \frac{d^Dk}{(2\pi)^D} \frac{1}{((k+p_1)^2 - m_c^2)\,((k-p_2)^2 - m_b^2)\,(k^2 - m_H^2)} \\
&\quad \times \bar{u}(p_1)(\not{k}+\not{p}_1+m_c)\gamma^\mu\Lambda_-(\not{k}-\not{p}_2+m_b)v(p_2), \qquad (82)
\end{aligned}
$$

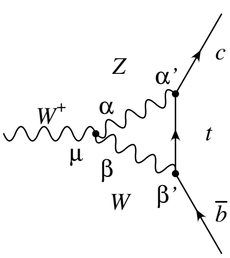


$$
\begin{aligned}
&\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{a4}v(p_2) \\
&= \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\left(\frac{em_c}{2s_W m_W}\gamma_5\right)\frac{i(\not{k}+\not{p}_1+m_c)}{(k+p_1)^2-m_c^2}\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^\mu\Lambda_-\right) \\
&\quad \times \frac{i(\not{k}-\not{p}_2+m_b)}{(k-p_2)^2-m_b^2}\left(-\frac{em_b}{2s_W m_W}\gamma_5\right)v(p_2)\frac{i}{k^2-m_Z^2} \\
&= \frac{ieV_{cb}}{\sqrt{2}s_W} \times \frac{ie^2 m_c m_b}{4s_W^2 m_W^2}
\end{aligned}
$$

$$\times \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left((k+p_1)^2 - m_c^2\right)\left((k-p_2)^2 - m_b^2\right)\left(k^2 - m_Z^2\right)}$$
$$\times\, \bar{u}(p_1)\gamma_5(\not{k} + \not{p}_1 + m_c)\gamma^\mu \Lambda_-(\not{k} - \not{p}_2 + m_b)\gamma_5 v(p_2), \qquad (83)$$



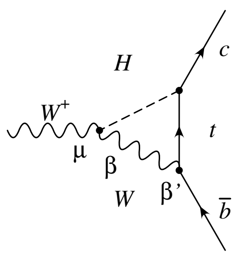
$$\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{b1}v(p_2)$$
$$= \int \frac{d^D k}{(2\pi)^D}\bar{u}(p_1)\left(-ieQ_c\gamma^{\alpha'}\right)\frac{i(-\not{k}+m_c)}{k^2-m_c^2}\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^{\beta'}\Lambda_-\right)v(p_2)$$
$$\times \frac{-ig_{\alpha\alpha'}}{(k+p_1)^2-m_A^2}\frac{-ig_{\beta\beta'}}{(k-p_2)^2-m_W^2} \times ie\Big(g^{\alpha\mu}(-k-p_1-p_1-p_2)^\beta$$
$$+ g^{\mu\beta}(p_1+p_2-k+p_2)^\alpha + g^{\beta\alpha}(k-p_2+k+p_1)^\mu\Big)$$
$$= \frac{ieV_{cb}}{\sqrt{2}s_W} \times ie^2 Q_c$$
$$\int \frac{d^D k}{(2\pi)^D}\frac{\bar{u}(p_1)\gamma_\alpha(-\not{k}+m_c)\gamma_\beta\Lambda_- v(p_2)}{\left((k+p_1)^2-m_A^2\right)\left((k-p_2)^2-m_W^2\right)\left(k^2-m_c^2\right)}$$
$$\times \Big(g^{\alpha\mu}(k+2p_1+p_2)^\beta + g^{\mu\beta}(k-p_1-2p_2)^\alpha$$
$$- g^{\beta\alpha}(2k+p_1-p_2)^\mu\Big), \qquad (84)$$



$$\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{b2}v(p_2)$$
$$= \int \frac{d^D k}{(2\pi)^D}\bar{u}(p_1)ie\gamma^{\alpha'}\left(g_c^-\Lambda_- + g_c^+\Lambda_+\right)\frac{i(-\not{k}+m_c)}{k^2-m_c^2}\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^{\beta'}\Lambda_-\right)$$
$$\times\, v(p_2)\frac{-ig_{\alpha\alpha'}}{(k+p_1)^2-m_Z^2}\frac{-ig_{\beta\beta'}}{(k-p_2)^2-m_W^2} \times \frac{-iec_W}{s_W}$$
$$\times \Big(g^{\alpha\mu}(-k-p_1-p_1-p_2)^\beta + g^{\mu\beta}(p_1+p_2-k+p_2)^\alpha$$

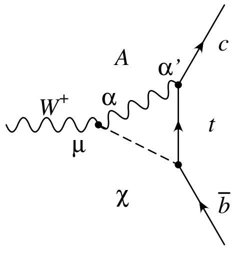
$$+ g^{\beta\alpha}(k - p_2 + k + p_1)^\mu\Big)$$

$$= \frac{ieV_{cb}}{\sqrt{2}s_W} \times \frac{ie^2 c_W}{s_W}$$
$$\times \int \frac{d^Dk}{(2\pi)^D} \frac{\bar{u}(p_1)\gamma_\alpha \left(g_c^- \Lambda_- + g_c^+ \Lambda_+\right)(-\not{k} + m_c)\gamma_\beta \Lambda_- v(p_2)}{\left((k+p_1)^2 - m_Z^2\right)\left((k-p_2)^2 - m_W^2\right)\left(k^2 - m_c^2\right)}$$
$$\times \Big(g^{\alpha\mu}(k + 2p_1 + p_2)^\beta + g^{\mu\beta}(k - p_1 - 2p_2)^\alpha$$
$$- g^{\beta\alpha}(2k + p_1 - p_2)^\mu\Big), \quad (85)$$



$$\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{b3}v(p_2)$$

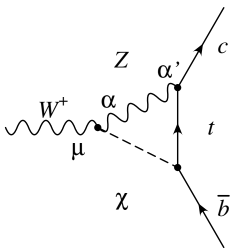
$$= \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\left(\frac{-iem_c}{2s_W m_W}\right)\frac{i(-\not{k} + m_c)}{k^2 - m_c^2}\left(\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^{\beta'}\Lambda_-\right)$$
$$\times v(p_2)\frac{i}{(k+p_1)^2 - m_H^2}\frac{-ig_{\beta\beta'}}{(k-p_2)^2 - m_W^2}\left(\frac{ieg^{\mu\beta}m_W}{s_W}\right)$$
$$= \frac{ieV_{cb}}{\sqrt{2}s_W} \times \frac{ie^2 m_c}{2s_W^2}$$
$$\times \int \frac{d^Dk}{(2\pi)^D}\frac{1}{\left((k+p_1)^2 - m_H^2\right)\left((k-p_2)^2 - m_W^2\right)\left(k^2 - m_c^2\right)}$$
$$\times \bar{u}(p_1)(-\not{k} + m_c)\gamma^\mu \Lambda_- v(p_2), \quad (86)$$



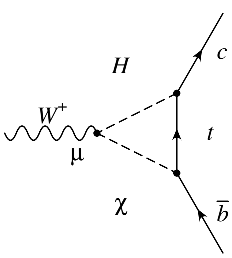
$$\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{b4}v(p_2)$$

$$= \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\left(-ieQ_c\gamma^{\alpha'}\right)\frac{i(-\not{k} + m_c)}{k^2 - m_c^2}\frac{ieV_{cb}}{\sqrt{2}s_W m_W}\left(m_c\Lambda_- - m_b\Lambda_+\right)$$
$$\times v(p_2)\frac{-ig_{\alpha\alpha'}}{(k+p_1)^2 - m_A^2}\frac{i}{(k-p_2)^2 - m_W^2}\left(-ieg^{\mu\alpha}m_W\right)$$

$$
\begin{aligned}
= \quad & \frac{ieV_{cb}}{\sqrt{2}s_W} \times -ie^2 Q_c \\
& \times \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left((k+p_1)^2 - m_A^2\right)\left((k-p_2)^2 - m_W^2\right)\left(k^2 - m_c^2\right)} \\
& \times \bar{u}(p_1)\gamma^\mu(-\not{k} + m_c)\left(m_c\Lambda_- - m_b\Lambda_+\right)v(p_2),
\end{aligned} \tag{87}
$$

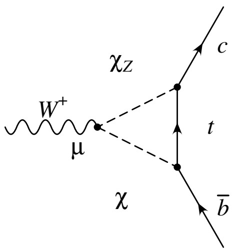


$$
\begin{aligned}
& \bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{b5}v(p_2) \\
= \quad & \int \frac{d^D k}{(2\pi)^D}\bar{u}(p_1) ie\gamma^{\alpha'}\left(g_c^-\Lambda_- + g_c^+\Lambda_+\right)\frac{i(-\not{k}+m_c)}{k^2 - m_c^2} \\
& \times \frac{ieV_{cb}}{\sqrt{2}s_W m_W}\left(m_c\Lambda_- - m_b\Lambda_+\right)v(p_2) \times \\
& \times \frac{-ig_{\alpha\alpha'}}{(k+p_1)^2 - m_Z^2}\frac{i}{(k-p_2)^2 - m_W^2}\left(\frac{-ies_W g^{\mu\alpha} m_W}{c_W}\right) \\
= \quad & \frac{ieV_{cb}}{\sqrt{2}s_W} \times \frac{ie^2 s_W}{c_W} \\
& \times \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left((k+p_1)^2 - m_Z^2\right)\left((k-p_2)^2 - m_W^2\right)\left(k^2 - m_c^2\right)} \\
& \times \bar{u}(p_1)\gamma^\mu\left(g_c^-\Lambda_- + g_c^+\Lambda_+\right)(-\not{k} + m_c)\left(m_c\Lambda_- - m_b\Lambda_+\right)v(p_2),
\end{aligned} \tag{88}
$$



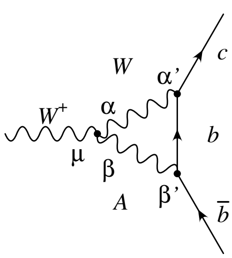
$$
\begin{aligned}
& \bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{b6}v(p_2) \\
= \quad & \int \frac{d^D k}{(2\pi)^D}\bar{u}(p_1)\left(\frac{-iem_c}{2s_W m_W}\right)\frac{i(-\not{k}+m_c)}{k^2 - m_c^2}\frac{ieV_{cb}}{\sqrt{2}s_W m_W} \\
& \times \left(m_c\Lambda_- - m_b\Lambda_+\right)v(p_2)\frac{i}{(k+p_1)^2 - m_H^2}\frac{i}{(k-p_2)^2 - m_W^2} \\
& \times \left(\frac{-ie}{2s_W}(k - p_2 + k + p_1)^\mu\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{ieV_{cb}}{\sqrt{2}s_W} \times \frac{ie^2 m_c}{4s_W^2 m_W^2} \\
&\times \int \frac{d^D k}{(2\pi)^D} \frac{\bar{u}(p_1)(-\not{k} + m_c)\left(m_c \Lambda_- - m_b \Lambda_+\right) v(p_2)}{\left((k+p_1)^2 - m_H^2\right)\left((k-p_2)^2 - m_W^2\right)\left(k^2 - m_c^2\right)} \\
&\times (2k + p_1 - p_2)^\mu, \qquad (89)
\end{aligned}
$$



$$
\bar{u}(p_1) \frac{ieV_{cb}}{\sqrt{2}s_W} \Gamma^\mu_{b7} v(p_2)
$$

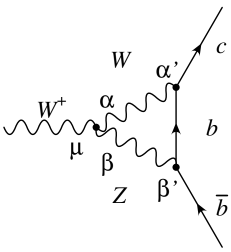
$$
\begin{aligned}
&= \int \frac{d^D k}{(2\pi)^D} \bar{u}(p_1) \left(\frac{-e m_c \gamma_5}{2 s_W m_W}\right) \frac{i(-\not{k} + m_c)}{k^2 - m_c^2} \frac{ieV_{cb}}{\sqrt{2}s_W m_W} \\
&\times \left(m_c \Lambda_- - m_b \Lambda_+\right) v(p_2) \frac{i}{(k+p_1)^2 - m_Z^2} \frac{i}{(k-p_2)^2 - m_W^2} \\
&\times \left(\frac{e}{2s_W}(k - p_2 + k + p_1)^\mu\right) \\
&= \frac{ieV_{cb}}{\sqrt{2}s_W} \times \frac{ie^2 m_c}{4s_W^2 m_W^2} \\
&\times \int \frac{d^D k}{(2\pi)^D} \frac{\bar{u}(p_1)\gamma_5(-\not{k} + m_c)\left(m_c \Lambda_- - m_b \Lambda_+\right) v(p_2)}{\left((k+p_1)^2 - m_Z^2\right)\left((k-p_2)^2 - m_W^2\right)\left(k^2 - m_c^2\right)} \\
&\times (2k + p_1 - p_2)^\mu, \qquad (90)
\end{aligned}
$$



$$
\bar{u}(p_1) \frac{ieV_{cb}}{\sqrt{2}s_W} \Gamma^\mu_{c1} v(p_2)
$$

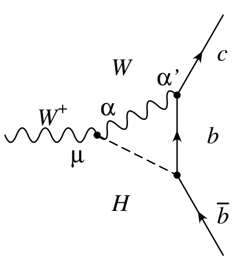
$$
\begin{aligned}
&= \int \frac{d^D k}{(2\pi)^D} \bar{u}(p_1) \frac{ieV_{cb}}{\sqrt{2}s_W} \gamma^{\alpha'} \Lambda_- \frac{i(-\not{k} + m_b)}{k^2 - m_b^2} \left(-ieQ_b \gamma^{\beta'}\right) v(p_2) \\
&\times \frac{-ig_{\alpha\alpha'}}{(k+p_1)^2 - m_W^2} \frac{-ig_{\beta\beta'}}{(k-p_2)^2 - m_A^2} ie\Big(g^{\beta\mu}(k - p_2 - p_1 - p_2)^\alpha \\
&+ g^{\mu\alpha}(p_1 + p_2 + k + p_1)^\beta + g^{\alpha\beta}(-k - p_1 - k + p_2)^\mu\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{ieV_{cb}}{\sqrt{2}s_W} \times -ie^2 Q_b \\
&\times \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left((k+p_1)^2 - m_W^2\right)\left((k-p_2)^2 - m_A^2\right)\left(k^2 - m_b^2\right)} \\
&\times \bar{u}(p_1)\gamma_\alpha \Lambda_- (-\not{k} + m_b)\gamma_\beta v(p_2) \Big(g^{\beta\mu}(k - p_1 - 2p_2)^\alpha \\
&+ g^{\mu\alpha}(k + 2p_1 + p_2)^\beta - g^{\alpha\beta}(2k + p_1 - p_2)^\mu \Big),
\end{aligned} \tag{91}
$$



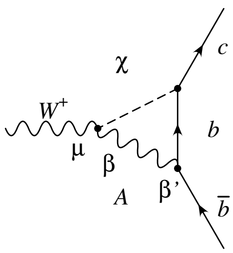
$$
\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{c2} v(p_2)
$$

$$
\begin{aligned}
&= \int \frac{d^D k}{(2\pi)^D} \bar{u}(p_1) \frac{ieV_{cb}}{\sqrt{2}s_W} \gamma^{\alpha'} \Lambda_- \frac{i(-\not{k} + m_b)}{k^2 - m_b^2} ie\gamma^{\beta'} \left(g_b^- \Lambda_- + g_b^+ \Lambda_+ \right) \\
&\times v(p_2) \frac{-ig_{\alpha\alpha'}}{(k+p_1)^2 - m_W^2} \frac{-ig_{\beta\beta'}}{(k-p_2)^2 - m_Z^2} \\
&\times \frac{-iec_W}{s_W} \Big(g^{\beta\mu}(k - p_2 - p_1 - p_2)^\alpha + g^{\mu\alpha}(p_1 + p_2 + k + p_1)^\beta \\
&+ g^{\alpha\beta}(-k - p_1 - k + p_2)^\mu \Big) \\
&= \frac{ieV_{cb}}{\sqrt{2}s_W} \times \frac{-ie^2 c_W}{s_W} \\
&\times \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left((k+p_1)^2 - m_W^2\right)\left((k-p_2)^2 - m_Z^2\right)\left(k^2 - m_b^2\right)} \\
&\times \bar{u}(p_1)\gamma_\alpha \Lambda_- (-\not{k} + m_b)\gamma_\beta \left(g_b^- \Lambda_- + g_b^+ \Lambda_+ \right) v(p_2) \\
&\times \left(g^{\beta\mu}(k - p_1 - 2p_2)^\alpha + g^{\mu\alpha}(k + 2p_1 + p_2)^\beta - g^{\alpha\beta}(2k + p_1 - p_2)^\mu \right),
\end{aligned} \tag{92}
$$



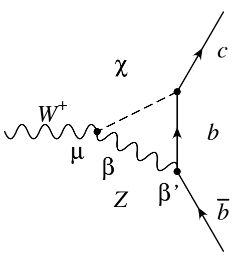
$$
\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{c3} v(p_2)
$$

$$
\begin{aligned}
&= \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\gamma^{\alpha'}\Lambda_-\frac{i(-\not{k}+m_b)}{k^2-m_b^2}\frac{-iem_b}{2s_Wm_W}v(p_2) \\
&\quad \times \frac{-ig_{\alpha\alpha'}}{(k+p_1)^2-m_W^2}\frac{i}{(k-p_2)^2-m_H^2}\left(\frac{ieg^{\alpha\mu}m_W}{s_W}\right) \\
&= \frac{ieV_{cb}}{\sqrt{2}s_W}\times\frac{ie^2m_b}{2s_W^2} \\
&\quad \times \int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left((k+p_1)^2-m_W^2\right)\left((k-p_2)^2-m_H^2\right)\left(k^2-m_b^2\right)} \\
&\quad \times \bar{u}(p_1)\gamma^\mu\Lambda_-(-\not{k}+m_b)v(p_2), \qquad (93)
\end{aligned}
$$



$$
\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{c4}v(p_2)
$$

$$
\begin{aligned}
&= \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_Wm_W}\left(m_c\Lambda_- - m_b\Lambda_+\right)\frac{i(-\not{k}+m_b)}{k^2-m_b^2} \\
&\quad \times\left(-ieQ_b\gamma^{\beta'}\right)v(p_2)\frac{i}{(k+p_1)^2-m_W^2}\frac{-ig_{\beta\beta'}}{(k-p_2)^2-m_A^2}\left(-ieg^{\mu\beta}m_W\right) \\
&= \frac{ieV_{cb}}{\sqrt{2}s_W}\times -ie^2Q_b \\
&\quad \times \int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left((k+p_1)^2-m_W^2\right)\left((k-p_2)^2-m_A^2\right)\left(k^2-m_b^2\right)} \\
&\quad \times \bar{u}(p_1)\left(m_c\Lambda_- - m_b\Lambda_+\right)(-\not{k}+m_b)\gamma^\mu v(p_2), \qquad (94)
\end{aligned}
$$



$$
\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{c5}v(p_2)
$$

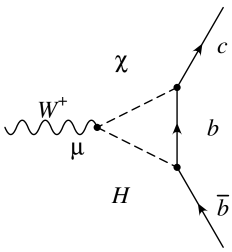
$$
\begin{aligned}
&= \int \frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_Wm_W}\left(m_c\Lambda_- - m_b\Lambda_+\right)\frac{i(-\not{k}+m_b)}{k^2-m_b^2} \\
&\quad \times ie\gamma^{\beta'}\left(g_c^-\Lambda_- + g_b^+\Lambda_+\right)v(p_2)\frac{i}{(k+p_1)^2-m_W^2}
\end{aligned}
$$

$$\times \frac{-ig_{\beta\beta'}}{(k-p_2)^2-m_Z^2}\left(\frac{-ies_W g^{\mu\beta}m_W}{c_W}\right)$$

$$= \frac{ieV_{cb}}{\sqrt{2}s_W}\times\frac{ie^2 s_W}{c_W}$$

$$\times\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left((k+p_1)^2-m_W^2\right)\left((k-p_2)^2-m_Z^2\right)\left(k^2-m_b^2\right)}$$

$$\times\,\bar{u}(p_1)\left(m_c\Lambda_- - m_b\Lambda_+\right)(-\not{k}+m_b)\gamma^\mu\left(g_b^-\Lambda_- + g_b^+\Lambda_+\right)v(p_2), \tag{95}$$



$$\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{c6}v(p_2)$$

$$= \int\frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W m_W}\left(m_c\Lambda_- - m_b\Lambda_+\right)\frac{i(-\not{k}+m_b)}{k^2-m_b^2}$$

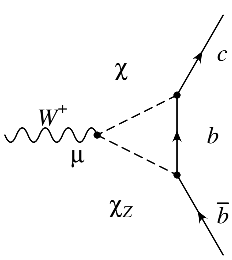
$$\times\left(\frac{-iem_b}{2s_W m_W}\right)v(p_2)\frac{i}{(k+p_1)^2-m_W^2}\frac{i}{(k-p_2)^2-m_H^2}$$

$$\times\left(\frac{-ie}{2s_W}(-k-p_1-k+p_2)^\mu\right)$$

$$= \frac{ieV_{cb}}{\sqrt{2}s_W}\times\frac{-ie^2 m_b}{4s_W^2 m_W^2}$$

$$\times\int\frac{d^Dk}{(2\pi)^D}\frac{\bar{u}(p_1)\left(m_c\Lambda_- - m_b\Lambda_+\right)(-\not{k}+m_b)v(p_2)}{\left((k+p_1)^2-m_W^2\right)\left((k-p_2)^2-m_H^2\right)\left(k^2-m_b^2\right)}$$

$$\times\,(2k+p_1-p_2)^\mu, \tag{96}$$



$$\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W}\Gamma^\mu_{c7}v(p_2)$$

$$= \int\frac{d^Dk}{(2\pi)^D}\bar{u}(p_1)\frac{ieV_{cb}}{\sqrt{2}s_W m_W}\left(m_c\Lambda_- - m_b\Lambda_+\right)\frac{i(-\not{k}+m_b)}{k^2-m_b^2}$$

$$
\begin{aligned}
& \times \left(\frac{e m_b \gamma_5}{2 s_W m_W}\right) v(p_2) \frac{i}{(k+p_1)^2 - m_W^2} \frac{i}{(k-p_2)^2 - m_Z^2} \\
& \times \left(\frac{e}{2 s_W}(-k - p_1 - k + p_2)^\mu\right) \\
= \; & \frac{i e V_{cb}}{\sqrt{2} s_W} \times \frac{i e^2 m_b}{4 s_W^2 m_W^2} \\
& \times \int \frac{d^D k}{(2\pi)^D} \frac{\bar{u}(p_1)\left(m_c \Lambda_- - m_b \Lambda_+\right)(-\not{k} + m_b)\gamma_5 v(p_2)}{\left((k+p_1)^2 - m_W^2\right)\left((k-p_2)^2 - m_Z^2\right)\left(k^2 - m_b^2\right)} \\
& \times (2k + p_1 - p_2)^\mu . \qquad (97)
\end{aligned}
$$

From these diagrams we actually have this scalar, vector and tensor integrals and in order to calculate this we reduce to the one, two and three point functions.

5.1 Three-point integrals

The principal idea of the method developed by Giampiero Passarino and Moshe Veltman is that vector and tensor integrals obtained from the vertex corrections can be expressed only by the outer vectors and tensors at hand. The three-point vector integral reads as

$$
\begin{aligned}
& C^\mu(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) \\
= \; & \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{\left((k+p_1)^2 - m_1^2 + i\epsilon\right)\left((k-p_2)^2 - m_2^2 + i\epsilon\right)\left(k^2 - m_3^2 + i\epsilon\right)}
\end{aligned} \qquad (98)
$$

can only be expressed in terms of p_1^μ and p_2^μ, i.e.,

$$
C^\mu = C_1 p_1^\mu + C_2 p_2^\mu \qquad (99)
$$

In contracting this equation with $p_{1\mu}$ and $p_{2\mu}$, we obtain the system of equations

$$
C^1 := C^\mu p_{1\mu} = C_1 p_1^2 + C_2 p_1 p_2, \qquad C^2 := C^\mu p_{2\mu} = C_1 p_1 p_2 + C_2 p_2^2. \qquad (100)
$$

This system of equation can be solved for C_1 and C_2. However, before we do so, we calculate the contractions called C^1 and C^2 on the left to obtain (skipping $i\epsilon$)

$$
\begin{aligned}
& 2C^1(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) \\
= \; & \int \frac{d^D k}{(2\pi)^D} \frac{2 k p_1}{\left((k+p_1)^2 - m_1^2\right)\left((k-p_2)^2 - m_2^2\right)\left(k^2 - m_3^2\right)} \\
= \; & \int \frac{d^D k}{(2\pi)^D} \frac{\left((k+p_1)^2 - m_1^2\right) - \left(k^2 - m_3^2\right) - p_1^2 + m_1^2 - m_3^2}{\left((k+p_1)^2 - m_1^2\right)\left((k-p_2)^2 - m_2^2\right)\left(k^2 - m_3^2\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\left((k-p_2)^2-m_2^2\right)\left(k^2-m_3^2\right)} \\
&\quad - \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\left((k+p_1)^2-m_1^2\right)\left((k-p_2)^2-m_2^2\right)} \\
&\quad - \int \frac{d^Dk}{(2\pi)^D} \frac{p_1^2-m_1^2+m_3^2}{\left((k+p_1)^2-m_1^2\right)\left((k-p_2)^2-m_2^2\right)\left(k^2-m_3^2\right)} \\
&= B(p_2^2;m_2,m_3) - B(p_3^2;m_1,m_2) \\
&\quad -(p_1^2-m_1^2+m_3^2)C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3), \qquad (101)
\end{aligned}
$$

$$
\begin{aligned}
&2C^2(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) \\
&= \int \frac{d^Dk}{(2\pi)^D} \frac{2kp_2}{\left((k+p_1)^2-m_1^2\right)\left((k-p_2)^2-m_2^2\right)\left(k^2-m_3^2\right)} \\
&= \int \frac{d^Dk}{(2\pi)^D} \frac{(k^2-m_3^2)-\left((k-p_2)^2-m_2^2\right)+p_2^2-m_2^2+m_3^2}{\left((k+p_1)^2-m_1^2\right)\left((k-p_2)^2-m_2^2\right)\left(k^2-m_3^2\right)} \\
&= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\left((k+p_1)^2-m_1^2\right)\left((k-p_2)^2-m_2^2\right)} \\
&\quad - \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\left((k+p_1)^2-m_1^2\right)\left(k^2-m_3^2\right)} \\
&\quad + \int \frac{d^Dk}{(2\pi)^D} \frac{p_2^2-m_2^2+m_3^2}{\left((k+p_1)^2-m_1^2\right)\left((k-p_2)^2-m_2^2\right)\left(k^2-m_3^2\right)} \\
&= B(p_3^2;m_1,m_2) - B(p_1^2;m_1,m_3) \\
&\quad + (p_2^2-m_2^2+m_3^2)C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) \qquad (102)
\end{aligned}
$$

The system of equations (100) is solved by ($\lambda = \lambda(p_1^2,p_2^2,p_3^2) = 4\left((p_1p_2)^2 - p_1^2p_2^2\right)$)

$$
\begin{aligned}
\lambda C_1 &= 4p_1p_2C^2 - 4p_2^2C^1 \\
&= (p_3^2-p_1^2-p_2^2)\Big(B(p_3^2;m_1,m_2) - B(p_1^2;m_1,m_3) \\
&\quad + (p_2^2-m_2^2+m_3^2)C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3)\Big) \\
&\quad - 2p_2^2\Big(B(p_2^2;m_2,m_3) - B(p_3^2;m_1,m_2) \\
&\quad - (p_1^2-m_1^2+m_3^2)C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3)\Big), \qquad (103)
\end{aligned}
$$

$$
\begin{aligned}
\lambda C_2 &= 4p_1p_2C^1 - 4p_1^2C^2 \\
&= (p_3^2 - p_1^2 - p_2^2)\Big(B(p_2^2;m_2,m_3) - B(p_3^2;m_1,m_2) \\
&\quad - (p_1^2 - m_1^2 + m_3^2)C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3)\Big) \\
&\quad - 2p_1^2\Big(B(p_3^2;m_1,m_2) - B(p_1^2;m_1,m_3) \\
&\quad + (p_2^2 - m_2^2 + m_3^2)C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3\Big).
\end{aligned}
\tag{104}
$$

Scalar one-, two- and three-point functions are indicated by the capital letters A, B and C, respectively. However, there is a much more instructive notation which is more acquainted for the Passarino–Veltman method, namely a notation in terms of powers of denominator factors. In this notation (based on the scalar three-point function as "mother" of all what follows) one has

$$
\begin{aligned}
C(1,1,1) &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{((k+p_1)^2-m_1^2)\,((k-p_2)^2-m_2^2)\,(k^2-m_3^2)} \\
&= C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3), \\
C(1,1,0) &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{((k+p_1)^2-m_1^2)\,((k-p_2)^2-m_2^2)} \\
&= B(p_3^2;m_1,m_2), \\
C(1,0,1) &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{((k+p_1)^2-m_1^2)\,(k^2-m_3^2)} \\
&= B(p_1^2;m_1,m_3), \\
C(0,1,1) &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{((k-p_2)^2-m_2^2)\,(k^2-m_3^2)} \\
&= B(p_2^2;m_2,m_3), \\
C(0,0,1) &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_3^2} = A(m_3),
\end{aligned}
$$

$$
C(-1,1,1) = \int \frac{d^Dk}{(2\pi)^D}\frac{(k+p_1)^2-m_1^2}{((k-p_2)^2-m_2^2)\,(k^2-m_3^2)} = \ldots \tag{105}
$$

Now we deal with the three-point tensor integral

$$
\begin{aligned}
&C^{\mu\nu}(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) \\
&= \int \frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{((k+p_1)^2-m_1^2+i\epsilon)\,((k-p_2)^2-m_2^2+i\epsilon)\,(k^2-m_3^2+i\epsilon)}
\end{aligned}
\tag{106}
$$

which according to Passarino and Veltman can be expressed as

$$C^{\mu\nu} = C_{11} p_1^\mu p_1^\nu + C_{12} p_1^\mu p_2^\nu + C_{21} p_2^\mu p_1^\nu + C_{22} p_2^\mu p_2^\nu + C_g g^{\mu\nu}. \tag{107}$$

The system of equations

$$\begin{aligned}
C^{11} &:= p_{1\mu} p_{1\nu} C^{\mu\nu} \\
&= C_{11} (p_1^2)^2 + C_{12} p_1^2 (p_1 p_2) + C_{21} (p_1 p_2) p_1^2 + C_{22} (p_1 p_2)^2 + C_g p_1^2 \\
C^{12} &:= p_{1\mu} p_{2\nu} C^{\mu\nu} \\
&= C_{11} p_1^2 (p_1 p_2) + C_{12} p_1^2 p_2^2 + C_{21} (p_1 p_2)^2 + C_{22} (p_1 p_2) p_2^2 + C_g (p_1 p_2) \\
C^{21} &:= p_{2\mu} p_{1\nu} C^{\mu\nu} \\
&= C_{11} (p_1 p_2) p_1^2 + C_{12} (p_1 p_2)^2 + C_{21} p_2^2 p_1^2 + C_{22} p_2^2 (p_1 p_2) + C_g (p_1 p_2) \\
C^{22} &:= p_{2\mu} p_{2\nu} C^{\mu\nu} \\
&= C_{11} (p_1 p_2)^2 + C_{12} (p_1 p_2) p_2^2 + C_{21} p_2^2 (p_1 p_2) + C_{22} (p_2)^2 + C_g p_2^2 \\
C^g &:= g_{\mu\nu} C^{\mu\nu} \\
&= C_{11} p_1^2 + C_{12} (p_1 p_2) + C_{21} (p_1 p_2) + C_{22} p_2^2 + C_g D
\end{aligned} \tag{108}$$

obtained after contraction can be solved for C_{11}, C_{12}, C_{21}, C_{22} and C_g by using

$$C_{00} := \frac{C^{11} p_2^2 - C^{12} p_1 p_2 - C^{21} p_1 p_2 + C^{22} p_1^2}{(p_1 p_2)^2 - p_1^2 p_2^2} \tag{109}$$

to obtain

$$\begin{aligned}
C_{11} &= \frac{1}{(p_1 p_2)^2 - p_1^2 p_2^2} \left(C^{22} + (C^g + (D-1) C_{00}) \frac{p_2^2}{D-2} \right), \\
C_{12} &= \frac{-1}{(p_1 p_2)^2 - p_1^2 p_2^2} \left(C^{12} + (C^g + (D-1) C_{00}) \frac{p_1 p_2}{D-2} \right), \\
C_{21} &= \frac{-1}{(p_1 p_2)^2 - p_1^2 p_2^2} \left(C^{21} + (C^g + (D-1) C_{00}) \frac{p_1 p_2}{D-2} \right), \\
C_{22} &= \frac{1}{(p_1 p_2)^2 - p_1^2 p_2^2} \left(C^{11} + (C^g + (D-1) C_{00}) \frac{p_1^2}{D-2} \right), \\
C_g &= \frac{C^g + C_{00}}{D-2}.
\end{aligned} \tag{110}$$

we obtain

$$
\begin{aligned}
&4C^{11} \\
&\quad = C(-1,1,1) - 2C(0,1,0) - 2(p_1^2 - m_1^2 + m_3^2)C(0,1,1) + C(1,1,-1) \\
&\qquad + 2(p_1^2 - m_1^2 + m_3^2)C(1,1,0) + (p_1^2 - m_1^2 + m_3^2)^2 C(1,1,1) \\
&\quad = -2A(m_2) - 2(p_1^2 - m_1^2 + m_3^2)\Big(B(p_2^2; m_2, m_3) - B(p_3^2; m_1, m_2)\Big) \\
&\qquad + (p_1^2 - m_1^2 + m_3^2)^2 C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) \\
&\qquad + C(-1,1,1) + C(1,1,-1), \\
&4C^{12} = 4C^{21} \\
&\quad = C(0,1,0) - C(0,0,1) + (p_2^2 - m_2^2 + m_3^2)C(0,1,1) - C(1,1,-1) \\
&\qquad + C(1,0,0) - (p_2^2 - m_2^2 + m_3^2)C(1,1,0) - (p_1^2 - m_1^2 + m_3^2)C(1,1,0) \\
&\qquad + (p_1^2 - m_1^2 + m_3^2)C(1,0,1) - (p_1^2 - m_1^2 + m_3^2)(p_2^2 - m_2^2 + m_3^2) \\
&\qquad \times C(1,1,1), \\
&\quad = A(m_1) + A(m_2) - A(m_3) \\
&\qquad + (p_2^2 - m_2^2 + m_3^2)\Big(B(p_2^2; m_2, m_3) - B(p_3^2; m_1, m_2)\Big) \\
&\qquad + (p_1^2 - m_1^2 + m_3^2)\Big(B(p_1^2; m_1, m_3) - B(p_3^2, m_1, m_2)\Big) \\
&\qquad - (p_1^2 - m_1^2 + m_3^2)(p_2^2 - m_2^2 + m_3^2)C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) \\
&\qquad - C(1,1,-1), \\
&4C^{22} \\
&= C(1,1,-1) - 2C(1,0,0) + 2(p_2^2 - m_2^2 + m_3^2)C(1,1,0) + C(1,-1,1) \\
&\quad - 2(p_2^2 - m_2^2 + m_3^2)C(1,0,1) + (p_2^2 - m_2^2 + m_3^2)^2 C(1,1,1) \\
&= -2A(m_1) - 2(p_2^2 - m_2^2 + m_3^2)\Big(B(p_1^2; m_1, m_3) - B(p_3^2; m_1, m_2)\Big) \\
&\quad + (p_2^2 - m_2^2 + m_3^2)^2 C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) + C(1,-1,1) + C(1,1,-1), \\
&C^g \\
&= C(1,1,0) + m_3^2 C(1,1,1) \\
&= B(p_3^2; m_1, m_2) + m_3^2 C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3).
\end{aligned}
\tag{111}
$$

Solving the systems of equations and inserting the contractions, we obtained the results as follows

$$
\begin{aligned}
&\lambda C_1 \\
&= -(p_3^2 - p_1^2 - p_2^2)B(p_1^2; m_1, m_3) - 2p_2^2 B(p_2^2; m_2, m_3) \\
&\quad - (p_1^2 - p_2^2 - p_3^2)B(p_3^2; m_1, m_2) + \Big((p_2^2 - m_2^2 + m_3^2)(p_3^2 - p_1^2 - p_2^2)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(p_1^2 - m_1^2 + m_3^2)p_2^2\Big)C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3),\\
&\lambda C_2\\
&= 2p_1^2 B(p_1^2; m_1, m_3) + (p_3^2 - p_1^2 - p_2^2)B(p_2^2; m_2, m_3)\\
&\quad + (p_2^2 - p_3^2 - p_1^2)B(p_3^2; m_1, m_2) - \Big((p_1^2 - m_1^2 + m_3^2)(p_3^2 - p_1^2 - p_2^2)\\
&\quad + 2(p_2^2 - m_2^2 + m_3^2)p_1^2\Big)C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3),\\
&\lambda^2 C_{11}\\
&= \lambda\frac{\lambda - (p_2^2 - p_3^2 - p_1^2)p_2^2}{2p_3^2 p_1^2}A(m_1) - \lambda\frac{p_1^2 - p_2^2 - p_3^2}{2p_3^2}A(m_2) - \lambda\frac{p_3^2 - p_1^2 - p_2^2}{2p_1^2}\\
&\quad \times A(m_3) + \frac{B(p_1^2; m_1, m_3)}{2(D-2)p_1^2}\Big[(D-2)\lambda\Big((p_1^2 - m_1^2 + m_3^2)(p_3^2 - p_1^2 - p_2^2)\\
&\quad - 2(p_2^2 - m_2^2 + m_3^2)p_1^2\Big) + 4(D-1)p_1^2 p_2^2\Big(2m_2^2 p_1^2 + m_1^2(p_3^2 - p_1^2 - p_2^2)\\
&\quad + m_3^2(p_2^2 - p_3^2 - p_1^2) + (p_1^2 - p_2^2 - p_3^2)p_1^2\Big)\Big] - \frac{2(D-1)p_2^2 B(p_2^2; m_2, m_3)}{D-2}\\
&\quad \times \Big[(p_2^2 - m_2^2 + m_3^2)(p_3^2 - p_1^2 - p_2^2) + 2(p_1^2 - m_1^2 + m_3^2)p_2^2\Big] + \frac{B(p_3^2; m_1, m_2)}{2(D-2)p_3^2}\\
&\quad \times \Big[(D-2)\lambda\,\Big((p_3^2 - m_1^2 + m_3^2)(p_1^2 - p_2^2 - p_3^2) - 2(p_2^2 - m_3^2 + m_2^2)p_3^2\Big)\\
&\quad + 4(D-1)p_2^2 p_3^2\Big(2m_3^2 p_3^2 + m_1^2(p_1^2 - p_2^2 - p_3^2) + m_2^2(p_2^2 - p_3^2 - p_1^2)\\
&\quad + (p_3^2 - p_1^2 - p_2^2)p_3^2\Big)\Big] + \frac{C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3)}{D-2}\Big[(D-2)\lambda(p_2^2, m_2^2, m_3^2)\lambda\\
&\quad + 4(D-1)p_2^2\Big(p_1^2 p_2^2 p_3^2 + m_2^4 p_1^2 + m_1^4 p_2^2 + m_3^4 p_3^2 + (m_2^2 p_1^2 + m_1^2 m_3^2)\\
&\quad \times (p_1^2 - p_2^2 - p_3^2) + (m_1^2 p_2^2 + m_2^2 m_3^2)(p_2^2 - p_3^2 - p_1^2)\\
&\quad + (m_3^2 p_3^2 + m_1^2 m_2^2)(p_3^2 - p_1^2 - p_2^2)\Big)\Big],
\end{aligned}
\tag{112}
$$

$$
\begin{aligned}
&\lambda^2 C_{12}\\
&= \frac{\lambda(p_1^2 - p_2^2 - p_3^2)}{2p_3^2}A(m_1) + \frac{\lambda(p_2^2 - p_3^2 - p_1^2)}{2p_3^2}A(m_2) + \lambda A(m_3)\\
&\quad + \frac{B(p_1^2; m_1, m_3)}{D-2}\Big[\lambda(p_1^2 - m_1^2 + m_3^2) - 2(D-1)p_1^2\\
&\quad \times \Big(2m_1^2 p_2^2 + m_2^2(p_3^2 - p_1^2 - p_2^2) + m_3^2(p_1^2 - p_2^2 - p_3^2) + (p_2^2 - p_3^2 - p_1^2)p_2^2\Big)\Big]\\
&\quad + \frac{B(p_2^2; m_2, m_3)}{D-2}\Big[\lambda(p_2^2 - m_2^2 + m_3^2) - 2(D-1)p_2^2\\
&\quad \times \Big(2m_2^2 p_1^2 + m_1^2(p_3^2 - p_1^2 - p_2^2) + m_3^2(p_2^2 - p_3^2 - p_1^2) + (p_1^2 - p_2^2 - p_3^2)p_1^2\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{B(p_3^2; m_1, m_2)}{2(D-2)p_3^2}\Big[2\lambda(m_1^2 + m_2^2 - p_3^2)p_3^2 \\
&+ (D-2)\lambda\Big(2m_3^2 p_3^2 - m_1^2(p_1^2 - p_2^2 - p_3^2) - m_2^2(p_2^2 - p_3^2 - p_1^2) \\
&+ (p_3^2 - p_1^2 - p_2^2)p_3^2\Big) - 4(D-1)p_3^2\Big(2p_1^2 p_2^2 p_3^2 + m_2^2 p_1^2(p_1^2 - p_2^2 - p_3^2) \\
&+ m_1^2 p_2^2(p_2^2 - p_3^2 - p_1^2) + m_3^2 p_3^2(p_3^2 - p_1^2 - p_2^2)\Big)\Big] \\
&+ \frac{C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3)}{D-2}\Big[(D-2)\lambda\left(p_1^2 p_2^2 - m_3^2(p_1^2 + p_2^2 - p_3^2) + m_3^4\right) \\
&+ D\lambda(m_2^2 p_1^2 + m_1^2 p_2^2 - m_3^2 p_3^2 - m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) \\
&- 2(D-1)\Big((p_1^2 p_2^2 p_3^2 + m_2^4 p_1^2 + m_1^4 p_2^2 + m_3^4 p_3^2)(p_3^2 - p_1^2 - p_2^2) \\
&+ 4(m_3^2 p_3^2 + m_1^2 m_2^2)p_1^2 p_2^2 + 2(m_1^2 p_2^2 + m_2^2 m_3^2)p_1^2(p_1^2 - p_2^2 - p_3^2) \\
&+ 2(m_2^2 p_1^2 + m_1^2 m_3^2)p_2^2(p_2^2 - p_3^2 - p_1^2)\Big)\Big], \qquad (113)
\end{aligned}
$$

$$
\begin{aligned}
&\lambda^2 C_{22} \\
&= -\lambda\frac{p_2^2 - p_3^2 - p_1^2}{2p_3^2}A(m_1) + \lambda\frac{\lambda - (p_1^2 - p_2^2 - p_3^2)p_1^2}{2p_2^2 p_3^2}A(m_2) \\
&- \lambda\frac{p_3^2 - p_1^2 - p_2^2}{2p_2^2}A(m_3) - \frac{2(D-1)p_1^2 B(p_1^2; m_1, m_3)}{D-2} \\
&\times \left[(p_1^2 - m_1^2 + m_3^2)(p_3^2 - p_1^2 - p_2^2) + 2(p_2^2 - m_2^2 + m_3^2)p_1^2\right] \\
&+ \frac{B(p_2^2; m_2, m_3)}{2(D-2)p_2^2}\Big[(D-2)\lambda\Big((p_2^2 - m_2^2 + m_3^2)(p_3^2 - p_1^2 - p_2^2) \\
&- 2(p_1^2 - m_1^2 + m_3^2)p_2^2\Big) + 4(D-1)p_1^2 p_2^2\Big(2m_1^2 p_2^2 + m_2^2(p_3^2 - p_1^2 - p_2^2) \\
&+ m_3^2(p_1^2 - p_2^2 - p_3^2) + (p_2^2 - p_3^2 - p_1^2)p_2^2\Big)\Big] + \frac{B(p_3^2; m_1, m_2)}{2(D-2)p_3^2} \\
&\times \Big[(D-2)\lambda\left((p_3^2 + m_1^2 - m_2^2)(p_2^2 - p_3^2 - p_1^2) - 2(p_1^2 + m_1^2 - m_3^2)p_3^2\right) \\
&+ 4(D-1)p_1^2 p_3^2\Big(2m_3^2 p_3^2 + m_1^2(p_1^2 - p_2^2 - p_3^2) + m_2^2(p_2^2 - p_3^2 - p_1^2) \\
&+ (p_3^2 - p_1^2 - p_2^2)p_3^2\Big)\Big] + \frac{C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3)}{D-2} \\
&\times \Big[(D-2)\lambda(p_1^2, m_1^2, m_3^2)\lambda + 4(D-1)p_1^2 \\
&\times \Big(p_1^2 p_2^2 p_3^2 + m_2^4 p_1^2 + m_1^4 p_2^2 + m_3^4 p_3^2 + (m_2^2 p_1^2 + m_1^2 m_3^2)(p_1^2 - p_2^2 - p_3^2) \\
&+ (m_1^2 p_2^2 + m_2^2 m_3^2)(p_2^2 - p_3^2 - p_1^2) + (m_3^2 p_3^2 + m_1^2 m_2^2)(p_3^2 - p_1^2 - p_2^2)\Big)\Big], \qquad (114)
\end{aligned}
$$

$$
\begin{aligned}
&\lambda C_g \\
&= -\frac{B(p_1^2;m_1,m_3)}{2(D-2)}\Big[(p_1^2-m_1^2+m_3^2)(p_3^2-p_1^2-p_2^2)+2(p_2^2-m_2^2+m_3^2)p_1^2\Big] \\
&\quad - \frac{B(p_2^2;m_2,m_3)}{2(D-2)}\Big[(p_2^2-m_2^2+m_3^2)(p_3^2-p_1^2-p_2^2)+2(p_1^2-m_1^2+m_3^2)p_2^2\Big] \\
&\quad - \frac{B(p_3^2;m_1,m_2)}{2(D-2)}\Big[(m_1^2+m_2^2-p_3^2)(p_3^2-p_1^2-p_2^2) \\
&\quad + 2(m_2^2p_1^2+m_1^2p_2^2-m_3^2p_3^2)\Big] \\
&\quad + \frac{C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3)}{D-2}\Big[p_1^2p_2^2p_3^2+m_2^4p_1^2+m_1^4p_2^2+m_3^4p_3^2 \\
&\quad + (m_3^2p_3^2+m_1^2m_2^2)(p_3^2-p_1^2-p_2^2)+(m_2^2p_1^2+m_1^2m_3^2)(p_1^2-p_2^2-p_3^2) \\
&\quad + (m_1^2p_2^2+m_2^2m_3^2)(p_2^2-p_3^2-p_1^2)\Big]. \qquad (115)
\end{aligned}
$$

We can calculate the integral in a more general fashion, proposed by Gerard ’t Hooft and Moshe Veltman [14]. The scalar integral reads

$$
\begin{aligned}
&C(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) \\
&= \frac{i}{(4\pi)^2}C_f(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) = \frac{i}{(4\pi)^2}\sum_{i=1}^{3}C_i(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3)
\end{aligned} \qquad (116)
$$

with

$$
\begin{aligned}
C_i(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) = {} & \frac{1}{\sqrt{\lambda'}}\int_0^1 \frac{dy_i}{y_i-y_{i0}} \\
& \times\Big[\ln\big((y_i-y_{i+})(y_i-y_{i-})-i\epsilon\big) \\
& - \ln\big((y_{i0}-y_{i+})(y_{i0}-y_{i-})-i\epsilon\big)\Big],
\end{aligned} \qquad (117)
$$

where in choosing the sign of the (renormalised) ε it is assumed that all outer momenta are on mass shell ($p_1^2, p_2^2, p_3^2 > 0$). Using

$$
\int \frac{dy_i}{y_i-y_{i0}}\left[\ln(y_i-y_{i\pm})-\ln(y_{i0}-y_{i\pm})\right] = -\mathrm{Li}_2\left(\frac{y_i-y_{i0}}{y_{i\pm}-y_{i0}}\right), \qquad (118)
$$

where

$$
\begin{aligned}
y_{10} &= \frac{(p_2^2+m_2^2-m_3^2)\left(\sqrt{\lambda'}+p_1^2-p_2^2-p_3^2\right)+2p_2^2(p_3^2-m_1^2+m_2^2)}{2p_2^2\sqrt{\lambda'}}, \\
y_{20} &= \frac{(p_3^2+m_1^2-m_2^2)\left(\sqrt{\lambda'}-p_1^2+p_2^2-p_3^2\right)+2p_3^2(p_1^2-m_3^2+m_1^2)}{2p_3^2\sqrt{\lambda'}},
\end{aligned}
$$

$$y_{30} = \frac{(p_1^2+m_3^2-m_1^2)\left(\sqrt{\lambda'}-p_1^2-p_2^2+p_3^2\right)+2p_1^2(p_2^2-m_2^2+m_3^2)}{2p_1^2\sqrt{\lambda'}}, \tag{119}$$

The arguments of the logarithms are treated as a whole and leading to

$$\begin{aligned} y_{1\pm} &= \frac{p_2^2+m_2^2-m_3^2 \pm \sqrt{\lambda(p_2^2,m_2^2,m_3^2)}}{2p_2^2}, \\ y_{2\pm} &= \frac{p_3^2+m_1^2-m_2^2 \pm \sqrt{\lambda(p_3^2,m_1^2,m_2^2)}}{2p_3^2}, \\ y_{3\pm} &= \frac{p_1^2+m_3^2-m_1^2 \pm \sqrt{\lambda(p_1^2,m_3^2,m_1^2)}}{2p_1^2} \end{aligned} \tag{120}$$

we obtain

$$\begin{aligned} &C_f(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) \\ &= \frac{1}{\sqrt{\lambda'}}\sum_{i=1}^{3}\int_0^1 \frac{dy_i}{y_i-y_{i0}}\left[\ln\left((y_i-y_{i+})(y_i-y_{i-})\right) – \ln\left((y_{i0}-y_{i+})(y_{i0}-y_{i-})\right)\right] \\ &= \frac{1}{\sqrt{\lambda'}}\sum_{i=1}^{3}\sum_{\pm}\int_0^1 \frac{dy_i}{y_i-y_{i0}}\left[\ln(y_i-y_{i\pm})-\ln(y_{i0}-y_{i\pm})\right] \\ &= \frac{-1}{\sqrt{\lambda'}}\sum_{i=1}^{3}\sum_{\pm}\left[\mathrm{Li}_2\left(\frac{y_i-y_{i0}}{y_{i\pm}-y_{i0}}\right)\right]_{y_i=0}^{1} \\ &= \frac{-1}{\sqrt{\lambda'}}\sum_{i=1}^{3}\sum_{\pm}\left[\mathrm{Li}_2\left(\frac{1-y_{i0}}{y_{i\pm}-y_{i0}}\right)-\mathrm{Li}_2\left(\frac{-y_{i0}}{y_{i\pm}-y_{i0}}\right)\right]. \end{aligned} \tag{121}$$

With more care for the imaginary parts we obtain

$$\begin{aligned} &C_f(p_1^2,p_2^2,p_3^2;m_1,m_2,m_3) \\ &= \frac{-1}{\sqrt{\lambda'}}\sum_{i=1}^{3}\sum_{\pm}\left[\mathrm{Li}_2\left(\frac{1-y_{i0}}{y_{i\pm}\pm i\epsilon-y_{i0}}\right)-\mathrm{Li}_2\left(\frac{-y_{i0}}{y_{i\pm}\pm i\epsilon-y_{i0}}\right)\right] \end{aligned} \tag{122}$$

which is a special case of the compact result given by Ansgar Denner [9].

5.2 Two-point integrals

The two-point contributions to the W self energy correction (which can be seen in later section) can be expressed in terms of tensor, vector and scalar two-point integrals.

$$\begin{aligned}
&B^{\mu\nu}(q^2;m_1,m_2)\\
&=\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}=B_g g^{\mu\nu}+B_{11}q^\mu q^\nu,\\
&B^\mu(q^2;m_1,m_2)\\
&=\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}=B_1 q^\mu
\end{aligned}\tag{123}$$

by solving the systems of equations

$$\begin{aligned}
B^g&=B_gD+B_{11}q^2\\
&=\int\frac{d^Dk}{(2\pi)^D}\frac{k^2}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}\\
&=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left((q+k)^2-m_1^2\right)}+m_2^2\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}\\
&=B(1,0)+m_2^2B(1,1),
\end{aligned}\tag{124}$$

$$\begin{aligned}
B^q&=B_gq^2+B_{11}q^4\\
&=\int\frac{d^Dk}{(2\pi)^D}\frac{(qk)^2}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}\\
&=\frac{1}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{\left(\left((q+k)^2-m_1^2\right)-\left(k^2-m_2^2\right)-\left(q^2-m_1^2+m_2^2\right)\right)^2}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}\\
&=\frac{1}{4}\Big(B(-1,1)-2B(0,0)-2(q^2-m_1^2+m_2^2)B(0,1)\\
&\quad+B(1,-1)+2(q^2-m_1^2+m_2^2)B(1,0)+(q^2-m_1^2+m_2^2)^2B(1,1)\Big)
\end{aligned}\tag{125}$$

and

$$\begin{aligned}
B^1&=B_1q^2\\
&=\int\frac{d^Dk}{(2\pi)^D}\frac{qk}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}\\
&=\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{\left((q+k)^2-m_1^2\right)-\left(k^2-m_2^2\right)-\left(q^2-m_1^2+m_2^2\right)}{\left((q+k)^2-m_1^2\right)\left(k^2-m_2^2\right)}\\
&=\frac{1}{2}\Big(B(0,1)-B(1,0)-(q^2-m_1^2+m_2^2)B(1,1)\Big).
\end{aligned}\tag{126}$$

One obtains

$$
\begin{aligned}
B_{11} &= \frac{1}{4(D-1)q^4}\Big(\Big(D(3q^2-m_1^2+m_2^2)-4q^2\Big)A(m_1)-D(q^2-m_1^2+m_2^2) \\
&\quad\times A(m_2)+\Big(D(q^2-m_1^2+m_2^2)^2-4m_2^2q^2\Big)B(q^2;m_1,m_2)\Big), \\
B_g &= \frac{1}{4(D-1)q^2}\Big((q^2+m_1^2-m_2^2)A(m_1)+(q^2-m_1^2+m_2^2)A(m_2) \\
&\quad-\lambda(q^2,m_1^2,m_2^2)B(q^2;m_1,m_2)\Big), \\
B_1 &= \frac{1}{2q^2}\Big(A(m_2)-A(m_1)-(q^2-m_1^2+m_2^2)B(q^2;m_1,m_2)\Big). \qquad (127)
\end{aligned}
$$

In order to obtain this result, we have used

$$
\begin{aligned}
B(0,0) &= \int\frac{d^Dk}{(2\pi)^D}=0, \\
B(0,1) &= \int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_2^2}=A(m_2) \\
B(1,0) &= \int\frac{d^Dk}{(2\pi)^D}\frac{1}{(q+k)^2-m_1^2}=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_1^2}=A(m_1), \\
B(-1,1) &= \int\frac{d^Dk}{(2\pi)^D}\frac{(q+k)^2-m_1^2}{k^2-m_2^2} \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{2qk+k^2-m_2^2+q^2-m_1^2+m_2^2}{k^2-m_2^2} \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{2qk}{k^2-m_2^2}+\int\frac{d^Dk}{(2\pi)^D} \\
&\quad+(q^2-m_1^2+m_2^2)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_2^2} \\
&= (q^2-m_1^2+m_2^2)A(m_2), \\
B(1,-1) &= \int\frac{d^Dk}{(2\pi)^D}\frac{k^2-m_2^2}{(q+k)^2-m_1^2} \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{(k-q)^2-m_2^2}{k^2-m_1^2} \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{-2kq+k^2-m_1^2+q^2+m_1^2-m_2^2}{k^2-m_1^2} \\
&= \int\frac{d^Dk}{(2\pi)^D}\frac{-2qk}{k^2-m_1^2}+\int\frac{d^Dk}{(2\pi)^D} \\
&\quad+(q^2+m_1^2-m_2^2)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_1^2}
\end{aligned}
$$

$$
\begin{aligned}
&= (q^2+m_1^2-m_2^2)A(m_1),\\
B(1,1) &= \int\frac{d^Dk}{(2\pi)^D}\frac{1}{((q+k)^2-m_1^2)\,(k^2-m_2^2)}\\
&= B(q^2;m_1,m_2). \qquad (128)
\end{aligned}
$$

Now we have to expand the Passarino–Veltman method to tensors of third rank because the integrals contain up to three momentum factors. From

$$
\begin{aligned}
&\int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu k^\rho}{((q+k)^2-m_1^2)(k^2-m_2^2)}\\
&= B_{111}q^\mu q^\nu q^\rho + B_{1g1}q^\mu g^{\nu\rho} + B_{1g2}q^\nu g^{\mu\rho} + B_{1g3}q^\rho g^{\mu\nu} \qquad (129)
\end{aligned}
$$

one obtains the system of equations

$$
\begin{aligned}
B^{111} &= B_{111}q^6 + B_{1g1}q^4 + B_{1g2}q^4 + B_{1g3}q^4,\\
B^{1g} &= B_{111}q^4 + B_{1g1}q^2D + B_{1g2}q^2 + B_{1g3}q^2,\\
B^{1g} &= B_{111}q^4 + B_{1g1}q^2 + B_{1g2}q^2D + B_{1g3}q^2,\\
B^{1g} &= B_{111}q^4 + B_{1g1}q^2 + B_{1g2}q^2 + B_{1g3}q^2D \qquad (130)
\end{aligned}
$$

which is solved by

$$
B_{1g} := B_{1g1} = B_{1g2} = B_{1g3} = \frac{B^{1g}q^2 - B^{111}}{(D-1)q^4},\ B_{111} = \frac{(D+2)B^{111} - 3B^{1g}q^2}{(D-1)q^6} \qquad (131)
$$

We obtain

$$
\begin{aligned}
B^{111} &= \int\frac{d^Dk}{(2\pi)^D}\frac{(qk)^3}{((q+k)^2-m_1^2)(k^2-m_2^2)}\\
&= \frac{1}{8}\Big(B(-2,1) - 3B(-1,0) - 3(q^2-m_1^2+m_2^2)B(-1,1) + 3B(0,-1)\\
&\quad + 6(q^2-m_1^2+m_2^2)B(0,0) + 3(q^2-m_1^2+m_2^2)^2B(0,1) - B(1,-2)\\
&\quad - 3(q^2-m_1^2+m_2^2)B(1,-1) - 3(q^2-m_1^2+m_2^2)^2B(1,0)\\
&\quad - (q^2-m_1^2+m_2^2)^3B(1,1)\Big),\\
B^{1g} &= \int\frac{d^Dk}{(2\pi)^D}\frac{(qk)k^2}{((q+k)^2-m_1^2)(k^2-m_2^2)}\\
&= \frac{1}{2}\Big(B(0,0) - B(1,-1) - (q^2-m_1^2+m_2^2)B(1,0)\\
&\quad + m_2^2\Big(B(0,1) - B(1,0) - (q^2-m_1^2+m_2^2)B(1,1)\Big)\Big). \qquad (132)
\end{aligned}
$$

One has to add the new integrals

$$
\begin{aligned}
B(-2,1) &= \int \frac{d^Dk}{(2\pi)^D} \frac{((q+k)^2 - m_1^2)^2}{k^2 - m_2^2} \\
&= \int \frac{d^Dk}{(2\pi)^D} \frac{(q^2 - m_1^2 + m_2^2 + 2qk + k^2 - m_2^2)^2}{k^2 - m_2^2} \\
&= (q^2 - m_1^2 + m_2^2)^2 \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 - m_2^2} \\
&\quad + 4(q^2 - m_1^2 + m_2^2) \int \frac{d^Dk}{(2\pi)^D} \frac{qk}{k^2 - m_1^2} \\
&\quad + 2(q^2 - m_1^2 + m_2^2) \int \frac{d^Dk}{(2\pi)^D} + 4 \int \frac{d^Dk}{(2\pi)^D} \frac{(qk)^2}{k^2 - m_2^2} \\
&\quad + 4 \int \frac{d^Dk}{(2\pi)^D} qk + \int \frac{d^Dk}{(2\pi)^D} (k^2 - m_2^2) \\
&= (q^2 - m_1^2 + m_2^2)^2 \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 - m_2^2} + 4 q_\mu q_\nu \int \frac{d^Dk}{(2\pi)^D} \\
&\quad \times \frac{k^\mu k^\nu}{k^2 - m_2^2} \\
&= (q^2 - m_1^2 + m_2^2)^2 A(m_2) + \frac{4q^2 m_2^2}{D} A(m_2) \\
B(-1,0) &= \int \frac{d^Dk}{(2\pi)^D} \left((q+k)^2 - m_1^2\right) = 0, \\
B(0,-1) &= \int \frac{d^Dk}{(2\pi)^D} (k^2 - m_2^2) = 0, \\
B(1,-2) &= \int \frac{d^Dk}{(2\pi)^D} \frac{(k^2 - m_2^2)^2}{(q+k)^2 - m_1^2} \\
&= \int \frac{d^Dk}{(2\pi)^D} \frac{((k-q)^2 - m_2^2)^2}{k^2 - m_1^2} \\
&= \int \frac{d^Dk}{(2\pi)^D} \frac{(k^2 - m_1^2 - 2kq + q^2 + m_1^2 - m_2^2)^2}{k^2 - m_1^2} \\
&= \frac{4q^2 m_1^2}{D} A(m_1) + (q^2 + m_1^2 - m_2^2)^2 A(m_1).
\end{aligned}
\tag{133}
$$

The scalar two-point function $B(p^2; m_1, m_2)$ is given by

$$\begin{aligned}
B(p^2; m_1, m_2) &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{\Big((k+p)^2 - m_1^2 + i\epsilon\Big)(k^2 - m_2^2 + i\epsilon)} \\
&= \frac{i\mu^{-2\varepsilon}}{(4\pi)^2} \Bigg[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + 2 - \frac{1}{2}\left(\ln\left(\frac{m_1^2}{\mu^2}\right) + \ln\left(\frac{m_2^2}{\mu^2}\right)\right) \\
&\quad - \frac{m_1^2 - m_2^2}{2p^2} \ln\left(\frac{m_1^2}{m_2^2}\right) + \frac{\sqrt{\lambda(p^2, m_1^2, m_2^2)}}{2p^2} \\
&\quad \times \ln\left(\frac{p^2 - m_1^2 - m_2^2 - \sqrt{\lambda(p^2, m_1^2, m_2^2)} - i\epsilon}{p^2 - m_1^2 - m_2^2 + \sqrt{\lambda(p^2, m_1^2, m_2^2)} + i\epsilon} \right) + O(\varepsilon) \Bigg] \\
&=: \frac{i\mu^{-2\varepsilon}}{(4\pi)^2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + B_f(p^2; m_1, m_2) \right].
\end{aligned} \tag{134}$$

5.3 One-point integral

The tensor integral is given by

$$A^{\mu\nu}(m^2) = \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{k^2 - m^2} = A_g g^{\mu\nu}. \tag{135}$$

by solving

$$A_g D = \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{k^2 - m^2} = \int \frac{d^D k}{(2\pi)^D} + m^2 \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m^2} = m^2 A(m), \tag{136}$$

one obtains $A_g = m^2 A(m)/D$.

The scalar one-point function is given by

$$\begin{aligned}
A(m) &= \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m^2} \;=\; -\int \frac{d^D k}{(2\pi)^D} \frac{1}{-k^2 + m^2} \\
&= -\frac{i\Gamma(1 - D/2)}{(4\pi)^{D/2}} (m^2)^{D/2-1} \\
&= -\frac{i\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}(\varepsilon - 1)\varepsilon} (m^2)^{1-\varepsilon} \\
&= \frac{im^2 \mu^{-2\varepsilon}}{(4\pi)^2} \left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{m^2}{\mu^2}\right) + 1 \right).
\end{aligned} \tag{137}$$

5.4 Results for the form factors

The results we obtained from the loop corrections are both ultraviolet (UV) and IR singular. First we will deal with UV divergences. We calculated the vertex correction, the result for the bare vertex correction V_b^μ can in principle be split up into six contributions according to

$$\begin{aligned} V_b^\mu &= V_-^0 \bar{u}(p_1)\gamma^\mu \Lambda_- v(p_2) + V_+^0 \bar{u}(p_1)\gamma^\mu \Lambda_+ v(p_2) \\ &\quad + V_-^1 \bar{u}(p_1) p_1^\mu \Lambda_- v(p_2) + V_+^1 \bar{u}(p_1) p_1^\mu \Lambda_+ v(p_2) \\ &\quad + V_-^2 \bar{u}(p_1) p_2^\mu \Lambda_- v(p_2) + V_+^2 \bar{u}(p_1) p_2^\mu \Lambda_+ v(p_2) \end{aligned} \tag{138}$$

with $\Lambda_\pm = (1 \pm \gamma_5)/2$. However, considering only on-shell W boson decays, the renormalised loop corrections are divided up into four form factors V_-, V_+, V_1 and V_2 only, defined by

$$\begin{aligned} \frac{\alpha}{4\pi} V_- &= \mathrm{Re}(V_-^0 + \delta Z_{WW} + \delta Z_{cc}^L + \delta Z_{bb}^L + \delta Z_{\mathrm{CKM}} + \delta Z_e - \frac{\delta s_W}{s_W}), \\ \frac{\alpha}{4\pi} \mu_1 \mu_2 V_+ q^2 &= m_1 m_2 \,\mathrm{Re}\, V_+^0, \\ \frac{\alpha}{4\pi} \mu_1 V_1 &= m_1 \,\mathrm{Re}(V_-^1 - V_-^2), \\ \frac{\alpha}{4\pi} \mu_2 V_2 &= m_2 \,\mathrm{Re}(V_+^1 - V_+^2), \end{aligned} \tag{139}$$

6 Renormalization by counter terms

The UV singularities which we want to subtract are absorbed into the counter terms and the counter terms are related to all the constants which has coefficients: the charge, the electroweak mixing angle and the outer lines, which contains the self energy diagrams. All these should be renormalised. δZ_{CKM} is the counter term for the CKM matrix. For the latter, we have to apply the renormalisation procedure. In order to perform the renormalisation of UV divergences, we need the self energy corrections for all outer particles. Therefore, we start with the self energy corrections of the W boson.

6.1 Self energy corrections for the W boson

The necessary diagrams for the self energy corrections of the W boson are divided up into two groups, namely the ones containing a two-point function and the ones containing a one-point function. The Feynman rules will again be taken from Appendix A.2 in the handbook of Böhm, Denner and Joos [15]. In Fig. 3 the momentum and index configurations are shown. In naming the self energy

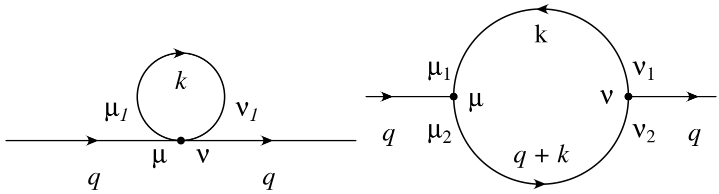
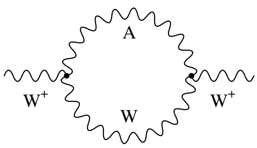
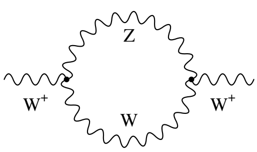


Figure 3: Conventions for self energy diagrams with one and two-point functions

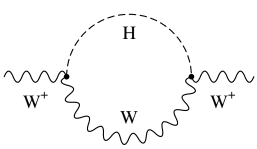
corrections by Σ, one obtains



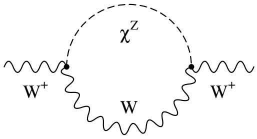
$$
\begin{aligned}
&-i\Sigma^{\mu\nu}_{a1}(q) \\
= &\int \frac{d^Dk}{(2\pi)^D} ie\left(g^{\nu_1\nu_2}(-2k-q)^{\nu} + g^{\nu_2\nu}(2q+k)^{\nu_1} + g^{\nu\nu_1}(-q+k)^{\nu_2}\right) \\
&\times \frac{-ig_{\mu_2\nu_2}}{(q+k)^2 - m_W^2} \\
&\times ie\left(g^{\mu_1\mu}(k-q)^{\mu_2} + g^{\mu\mu_2}(2q+k)^{\mu_1} + g^{\mu_2\mu_1}(-q-2k)^{\mu}\right)\frac{-ig_{\mu_1\nu_1}}{k^2 - m_A^2},
\end{aligned}
$$



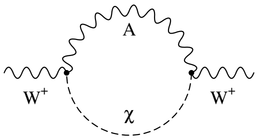
$$
\begin{aligned}
&-i\Sigma^{\mu\nu}_{a2}(q) \\
= &\int \frac{d^Dk}{(2\pi)^D} \frac{-iec_W}{s_W}\left(g^{\nu_1\nu_2}(-2k-q)^{\nu} + g^{\nu_2\nu}(2q+k)^{\nu_1} + g^{\nu\nu_1}(-q+k)^{\nu_2}\right) \\
&\times \frac{-ig_{\mu_2\nu_2}}{(q+k)^2 - m_W^2}\frac{-iec_W}{s_W} \\
&\times \left(g^{\mu_1\mu}(k-q)^{\mu_2} + g^{\mu\mu_2}(2q+k)^{\mu_1} + g^{\mu_2\mu_1}(-q-2k)^{\mu}\right)\frac{-ig_{\mu_1\nu_1}}{k^2 - m_Z^2},
\end{aligned}
$$



$$
\begin{aligned}
&-i\Sigma^{\mu\nu}_{b1}(q) \\
= &\int \frac{d^Dk}{(2\pi)^D}\left(\frac{iem_W g^{\nu\nu_2}}{s_W}\right)\frac{-ig_{\mu_2\nu_2}}{(q+k)^2 - m_W^2}\left(\frac{iem_W g^{\mu\mu_2}}{s_W}\right)\frac{i}{k^2 - m_H^2},
\end{aligned}
$$

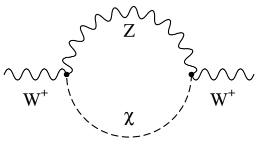


$$- i\Sigma^{\mu\nu}_{b2}(q) \; = \; 0$$



$$- i\Sigma^{\mu\nu}_{c1}(q)$$

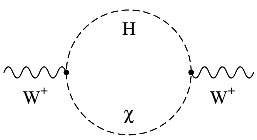
$$= \int \frac{d^Dk}{(2\pi)^D}(-iem_W g^{\nu\nu_1})\frac{i}{(q+k)^2 - m_W^2}(-iem_W g^{\mu\mu_1})\frac{-ig_{\mu_1\nu_1}}{k^2 - m_A^2},$$



$$- i\Sigma^{\mu\nu}_{c2}(q)$$

$$= \int \frac{d^Dk}{(2\pi)^D}\left(-\frac{iem_W s_W g^{\nu\nu_1}}{c_W}\right)\frac{i}{(q+k)^2 - m_W^2}\left(-\frac{iem_W s_W g^{\mu\mu_1}}{c_W}\right)$$

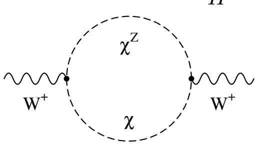
$$\times \frac{-ig_{\mu_1\nu_1}}{k^2 - m_Z^2},$$



$$- i\Sigma^{\mu\nu}_{d1}(q)$$

$$= \int \frac{d^Dk}{(2\pi)^D}\left(\frac{ie}{2s_W}(q+2k)^\nu\right)\frac{i}{(q+k)^2 - m_W^2}\left(-\frac{ie}{2s_W}(-q-2k)^\mu\right)$$

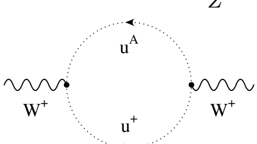
$$\times \frac{i}{k^2 - m_H^2},$$



$$- i\Sigma^{\mu\nu}_{d2}(q)$$

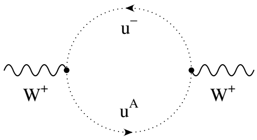
$$= \int \frac{d^Dk}{(2\pi)^D}\left(\frac{e}{2s_W}(q+2k)^\nu\right)\frac{i}{(q+k)^2 - m_W^2}\left(\frac{e}{2s_W}(-q-2k)^\mu\right)$$

$$\times \frac{i}{k^2 - m_Z^2},$$



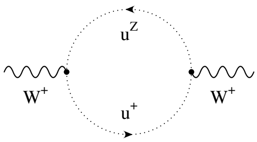
$$- i\Sigma^{\mu\nu}_{e1}(q)$$

$$= \quad -\int \frac{d^D k}{(2\pi)^D} \left(-iek^\nu\right) \frac{i}{(q+k)^2 - m_W^2} \left(-ie(q+k)^\mu\right) \frac{i}{k^2 - m_A^2},$$



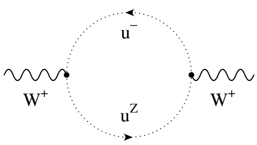
$$-\, i\Sigma_{e2}^{\mu\nu}(q)$$

$$= \quad -\int \frac{d^D k}{(2\pi)^D} \left(iek^\nu\right) \frac{i}{(q+k)^2 - m_A^2} \left(ie(q+k)^\mu\right) \frac{i}{k^2 - m_W^2},$$



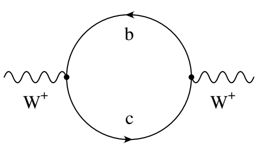
$$-\, i\Sigma_{e3}^{\mu\nu}(q)$$

$$= \quad -\int \frac{d^D k}{(2\pi)^D} \left(\frac{iec_W}{s_W} k^\nu\right) \frac{i}{(q+k)^2 - m_W^2} \left(\frac{iec_W}{s_W}(q+k)^\mu\right) \frac{i}{k^2 - m_Z^2},$$



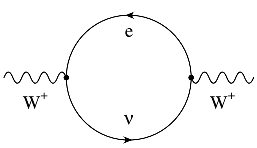
$$-\, i\Sigma_{e4}^{\mu\nu}(q)$$

$$= \quad -\int \frac{d^D k}{(2\pi)^D} \left(-\frac{iec_W}{s_W} k^\nu\right) \frac{i}{(q+k)^2 - m_Z^2} \left(-\frac{iec_W}{s_W}(q+k)^\mu\right) \frac{i}{k^2 - m_W^2},$$



$$-\, i\Sigma_{f1}^{\mu\nu}(q)$$

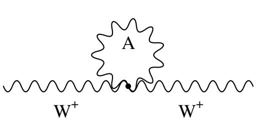
$$= \quad -\int \frac{d^D k}{(2\pi)^D} \mathrm{tr}\left(\frac{ie\gamma^\nu V_{cb}^*}{\sqrt{2} s_W} \Lambda_- \frac{i}{\not{q} + \not{k} - m_c} \frac{ie\gamma^\mu V_{cb}}{\sqrt{2} s_W} \Lambda_- \frac{i}{\not{k} - m_b}\right),$$



$$-\, i\Sigma_{f2}^{\mu\nu}(q)$$

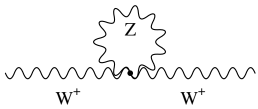
$$= \quad -\int \frac{d^D k}{(2\pi)^D} \mathrm{tr}\left(\frac{ie\gamma^\nu}{\sqrt{2} s_W} \Lambda_- \frac{i}{\not{q} + \not{k} - m_\nu} \frac{ie\gamma^\mu}{\sqrt{2} s_W} \Lambda_- \frac{i}{\not{k} - m_e}\right)$$

The one-point contributions are as follows and lead to one-point integrals,

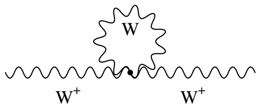


$$-\, i\Sigma_{g1}^{\mu\nu}(q)$$

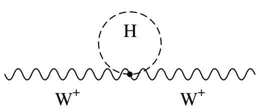
$$= \quad \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} (-ie^2) \left(2g^{\mu\nu} g^{\nu_1\mu_1} - g^{\nu\mu_1} g^{\mu\nu_1} - g^{\mu_1\mu} g^{\nu\nu_1}\right) \frac{-ig_{\mu_1\nu_1}}{k^2 - m_A^2},$$



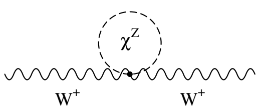
$$- i\Sigma^{\mu\nu}_{g2}(q)$$
$$= \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\left(\frac{-ie^2c_W^2}{s_W^2}\right)\left(2g^{\mu\nu}g^{\nu_1\mu_1} - g^{\nu\mu_1}g^{\mu\nu_1} - g^{\mu_1\mu}g^{\nu\nu_1}\right)\frac{-ig_{\mu_1\nu_1}}{k^2-m_Z^2},$$



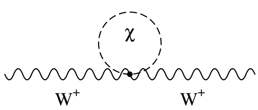
$$- i\Sigma^{\mu\nu}_{g3}(q)$$
$$= \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\left(\frac{-ie^2}{s_W^2}\right)\left(2g^{\mu\nu}g^{\nu_1\mu_1} - g^{\nu\mu_1}g^{\mu\nu_1} - g^{\mu_1\mu}g^{\nu\nu_1}\right)\frac{-ig_{\mu_1\nu_1}}{k^2-m_W^2},$$



$$- i\Sigma^{\mu\nu}_{h1}(q) = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{ie^2g^{\mu\nu}}{2s_W^2}\frac{i}{k^2-m_H^2},$$



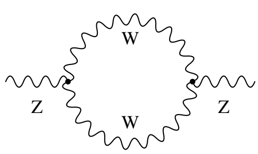
$$- i\Sigma^{\mu\nu}_{h2}(q) = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{ie^2g^{\mu\nu}}{2s_W^2}\frac{i}{k^2-m_Z^2},$$



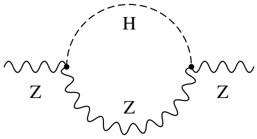
$$- i\Sigma^{\mu\nu}_{h3}(q) = \int \frac{d^Dk}{(2\pi)^D}\frac{ie^2g^{\mu\nu}}{2s_W^2}\frac{i}{k^2-m_W^2} \quad (140)$$

6.2 Self energy correction for the photon and the Z boson

The numbering of the diagrams for the Z boson self energy diagrams is slightly different from the ones for the W boson, as the Z boson is neutral. We obtain

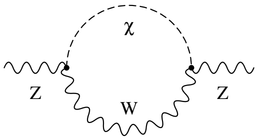


$$- i\Sigma^{\mu\nu}_{a1}(q)$$
$$= \int \frac{d^Dk}{(2\pi)^D}\frac{-iec_W}{s_W}\Big(g^{\nu\nu_2}(-2q-k)^{\nu_1} + g^{\nu_2\nu_1}(q+2k)^{\nu} + g^{\nu_1\nu}(-k+q)^{\nu_2}\Big)$$
$$\times \frac{-ig_{\mu_2\nu_2}}{(q+k)^2-m_W^2} \times -\frac{iec_W}{s_W}$$
$$\times \left(g^{\mu\mu_1}(q-k)^{\mu_2} + g^{\mu_1\mu_2}(2k+q)^{\mu} + g^{\mu_2\mu}(-2q-k)^{\mu_1}\right)\frac{-ig_{\mu_1\nu_1}}{k^2-m_W^2},$$



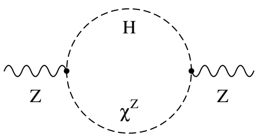
$$-i\Sigma_{b1}^{\mu\nu}(q)$$

$$= \int \frac{d^Dk}{(2\pi)^D} \frac{iem_W}{c_W^2 s_W} g^{\nu\nu_2} \frac{i}{(q+k)^2 - m_H^2} \frac{iem_W}{c_W^2 s_W} g^{\mu\mu_2} \frac{-ig_{\mu_2\nu_2}}{k^2 - m_Z^2},$$



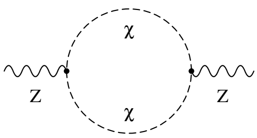
$$-i\Sigma_{b2}^{\mu\nu}(q)$$

$$= 2\int \frac{d^Dk}{(2\pi)^D} \frac{-ies_W m_W}{c_W} g^{\nu\nu_2} \frac{i}{(q+k)^2 - m_W^2} \frac{-ies_W m_W}{c_W} g^{\mu\mu_2} \frac{-ig_{\mu_2\nu_2}}{k^2 - m_W^2},$$



$$-i\Sigma_{d1}^{\mu\nu}(q)$$

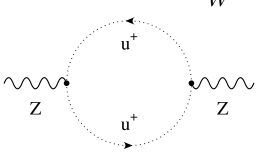
$$= \int \frac{d^Dk}{(2\pi)^D} \frac{e}{2c_W s_W} (q+2k)^\nu \frac{i}{(q+k)^2 - m_Z^2} \frac{e}{2c_W s_W} (-q-2k)^\mu \frac{i}{k^2 - m_H^2},$$



$$-i\Sigma_{d2}^{\mu\nu}(q)$$

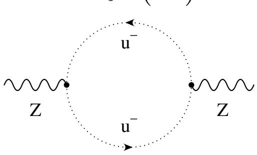
$$= \int \frac{d^Dk}{(2\pi)^D} \frac{ie(c_W^2 - s_W^2)}{2c_W s_W} (q+2k)^\nu \frac{i}{(q+k)^2 - m_W^2} \frac{ie(c_W^2 - s_W^2)}{2c_W s_W} (q+2k)^\mu$$

$$\times \frac{i}{k^2 - m_W^2},$$



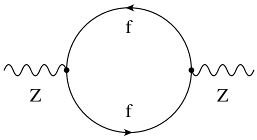
$$-i\Sigma_{e1}^{\mu\nu}(q)$$

$$= -\int \frac{d^Dk}{(2\pi)^D} \frac{-iec_W}{s_W} k^\nu \frac{i}{(q+k)^2 - m_W^2} \frac{-iec_W}{s_W} (q+k)^\mu \frac{i}{k^2 - m_W^2},$$



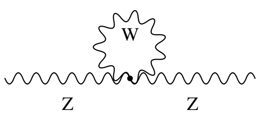
$$-i\Sigma_{e2}^{\mu\nu}(q)$$

$$= -\int \frac{d^Dk}{(2\pi)^D} \frac{iec_W}{s_W} k^\nu \frac{i}{(q+k)^2 - m_W^2} \frac{iec_W}{s_W} (q+k)^\mu \frac{i}{k^2 - m_W^2},$$

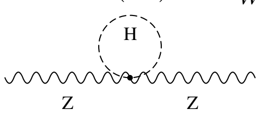


$$-\,i\Sigma^{\mu\nu}_{f1}(q)$$

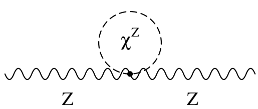
$$= \; -\int \frac{d^Dk}{(2\pi)^D}\mathrm{tr}\Big(ie\gamma^{\nu}(g_f^{-}\Lambda_- + g_f^{+}\Lambda_+)\frac{i(\not{q}+\not{k}+m_f)}{(q+k)^2-m_f^2}$$
$$\times\, ie\gamma^{\mu}(g_f^{-}\Lambda_- + g_f^{+}\Lambda_+)\frac{i(\not{k}+m_f)}{k^2-m_f^2}\Big),$$



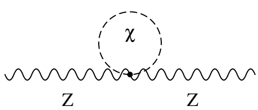
$$-\,i\Sigma^{\mu\nu}_{g1}(q)$$
$$= \int \frac{d^Dk}{(2\pi)^D}\frac{-ie^2c_W^2}{s_W^2}\left(2g^{\mu_1\nu_1}g^{\mu\nu} - g^{\mu_1\nu}g^{\nu_1\mu} - g^{\mu_1\mu}g^{\nu_1\nu}\right)\frac{-ig_{\mu_1\nu_1}}{k^2-m_W^2},$$



$$-\,i\Sigma^{\mu\nu}_{h1}(q) = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{ie^2}{c_W^2s_W^2}g^{\mu\nu}\frac{i}{k^2-m_H^2},$$



$$-\,i\Sigma^{\mu\nu}_{h2}(q) = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{ie^2}{c_W^2s_W^2}g^{\mu\nu}\frac{i}{k^2-m_Z^2},$$



$$-\,i\Sigma^{\mu\nu}_{h3}(q) = \int \frac{d^Dk}{(2\pi)^D}\frac{ie^2(c_W^2-s_W^2)^2}{2c_W^2s_W^2}g^{\mu\nu}\frac{i}{k^2-m_W^2}. \tag{141}$$

In order to obtain the self energy corrections for the photon or the AZ mixture, the nine diagrams contributing to the self energy correction of the photon are shown in Fig. 4. By adding all these contributions we will obtain

$$\Sigma^{\mu\nu}_{AA}(q) = \left(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2}\right)\Sigma^T_{AA}(q^2) + \frac{q^{\mu}q^{\nu}}{q^2}\Sigma^L_{AA}(q^2). \tag{142}$$

In the same way we also calculate the mixture between photon and Z boson where the right photon A is replaced by a Z boson.

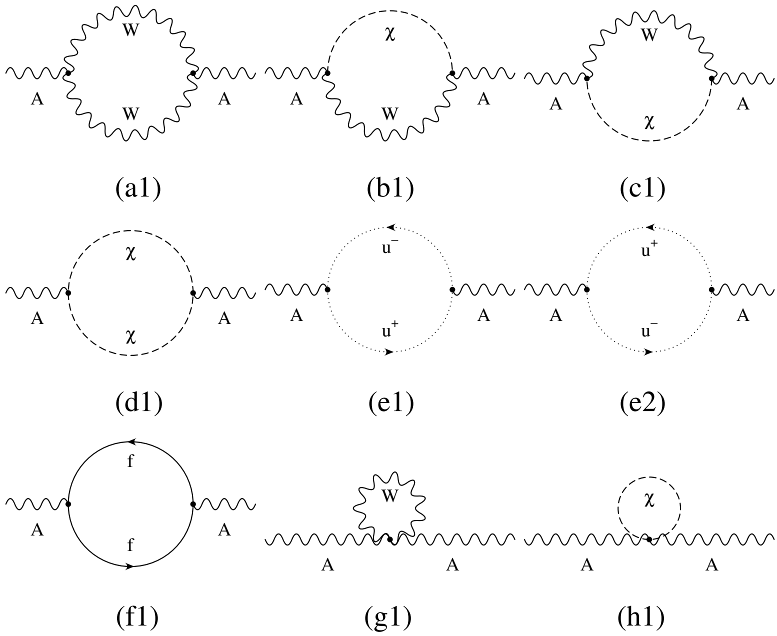
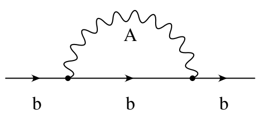


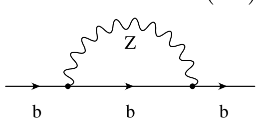
Figure 4: photon self energy diagrams

6.3 Self energy corrections for the bottom quark

The self energy corrections for the bottom quark are given on a more elementary level as

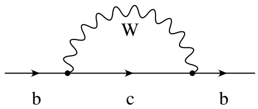


$$i\Sigma^b_{a1}(\not{q})$$
$$= \int \frac{d^D k}{(2\pi)^D} (-ieQ_b\gamma^\nu) \frac{i(\not{q}+\not{k}+m_b)}{(q+k)^2 - m_b^2} (-ieQ_b\gamma^\mu) \frac{-ig_{\mu\nu}}{k^2 - m_A^2}$$
$$= -e^2 Q_b^2 \int \frac{d^D k}{(2\pi)^D} \frac{\gamma^\nu(\not{q}+\not{k}+m_b)\gamma^\mu}{((q+k)^2 - m_b^2)(k^2 - m_A^2)} g_{\mu\nu},$$



$$i\Sigma^b_{a2}(\not{q})$$
$$= \int \frac{d^D k}{(2\pi)^D} ie\gamma^\nu(g_b^- \Lambda_- + g_b^+ \Lambda_+) \frac{i(\not{q}+\not{k}+m_b)}{(q+k)^2 - m_b^2} ie\gamma^\mu(g_b^- \Lambda_- + g_b^+ \Lambda_+)$$
$$\times \frac{-ig_{\mu\nu}}{k^2 - m_Z^2}$$

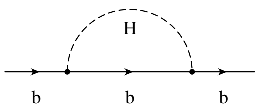
$$= -e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma^\nu (g_b^- \Lambda_- + g_b^+ \Lambda_+)(\not{q} + \not{k} + m_b)\gamma^\mu (g_b^- \Lambda_- + g_b^+ \Lambda_+)}{((q+k)^2 - m_b^2)(k^2 - m_Z^2)} g_{\mu\nu},$$



$$i\Sigma_{a3}^{ij}(\not{q})$$

$$= \int \frac{d^Dk}{(2\pi)^D} \sum_k \frac{ieV_{ik}}{\sqrt{2}s_W} \gamma^\nu \Lambda_- \frac{i(\not{q} + \not{k} + m_k)}{(q+k)^2 - m_k^2} \frac{ieV_{kj}}{\sqrt{2}s_W} \gamma^\mu \Lambda_- \frac{-ig_{\mu\nu}}{k^2 - m_W^2}$$

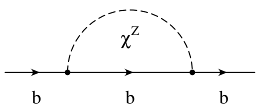
$$= -\sum_k \frac{e^2 V_{ik} V_{kj}}{2s_W^2} \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma^\nu \Lambda_- (\not{q} + \not{k} + m_k)\gamma^\mu \Lambda_-}{((q+k)^2 - m_k^2)(k^2 - m_W^2)} g_{\mu\nu},$$



$$i\Sigma_{b1}^{b}(\not{q})$$

$$= \int \frac{d^Dk}{(2\pi)^D} \frac{-iem_b}{2s_W m_W} \frac{i(\not{q} + \not{k} + m_b)}{(q+k)^2 - m_b^2} \frac{-iem_b}{2s_W m_W} \frac{i}{k^2 - m_H^2}$$

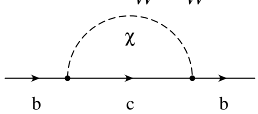
$$= \frac{e^2 m_b^2}{4s_W^2 m_W^2} \int \frac{d^Dk}{(2\pi)^D} \frac{\not{q} + \not{k} + m_b}{((q+k)^2 - m_b^2)(k^2 - m_H^2)},$$



$$i\Sigma_{b2}^{b}(\not{q})$$

$$= \int \frac{d^Dk}{(2\pi)^D} \frac{em_b}{2s_W m_W} \gamma_5 \frac{i(\not{q} + \not{k} + m_b)}{(q+k)^2 - m_b^2} \frac{em_b}{2s_W m_W} \gamma_5 \frac{i}{k^2 - m_Z^2}$$

$$= -\frac{e^2 m_b^2}{4s_W^2 m_W^2} \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma_5 (\not{q} + \not{k} + m_b)\gamma_5}{((q+k)^2 - m_b^2)(k^2 - m_Z^2)},$$



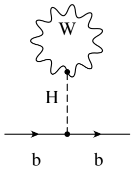
$$i\Sigma_{b3}^{ij}(\not{q})$$

$$= \int \frac{d^Dk}{(2\pi)^D} \sum_k \frac{ieV_{ik}}{\sqrt{2}s_W m_W} (-m_i \Lambda_- + m_k \Lambda_+) \times$$

$$\times \frac{i(\not{q} + \not{k} + m_k)}{(q+k)^2 - m_k^2} \frac{ieV_{kj}}{\sqrt{2}s_W m_W} (m_k \Lambda_- - m_j \Lambda_+) \frac{i}{k^2 - m_W^2}$$

$$= \sum_k \frac{e^2 V_{ik} V_{kj}}{2s_W^2 m_W^2}$$

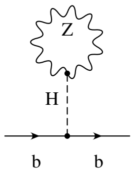
$$\times \int \frac{d^Dk}{(2\pi)^D} \frac{(-m_i \Lambda_- + m_k \Lambda_+)(\not{q} + \not{k} + m_k)(m_k \Lambda_- - m_j \Lambda_+)}{((q+k)^2 - m_k^2)(k^2 - m_W^2)},$$



$$i\Sigma^{b}_{t1}(\not{q})$$

$$= \frac{-em_b}{2s_W m_W m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{iem_W g^{\mu\nu}}{s_W}\frac{-ig_{\mu\nu}}{k^2-m_W^2}$$

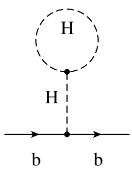
$$= \frac{-e^2m_b}{2s_W^2 m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{g^{\mu\nu}}{k^2-m_W^2}g_{\mu\nu},$$



$$i\Sigma^{b}_{t2}(\not{q})$$

$$= \frac{-em_b}{2s_W m_W m_H^2}\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{iem_W g^{\mu\nu}}{c_W^2 s_W}\frac{-ig_{\mu\nu}}{k^2-m_Z^2}$$

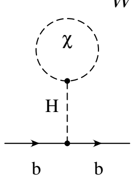
$$= \frac{-e^2m_b}{4c_W^2 s_W^2 m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{g^{\mu\nu}}{k^2-m_Z^2}g_{\mu\nu},$$



$$i\Sigma^{b}_{t3}(\not{q})$$

$$= \frac{-em_b}{2s_W m_W m_H^2}\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{-3iem_H^2}{2s_W m_W}\frac{i}{k^2-m_H^2}$$

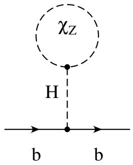
$$= \frac{-3e^2m_b}{8s_W^2 m_W^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_H^2},$$



$$i\Sigma^{b}_{t4}(\not{q})$$

$$= \frac{-em_b}{2s_W m_W m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{-iem_H^2}{2s_W m_W}\frac{i}{k^2-m_W^2}$$

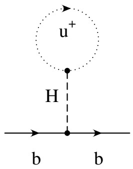
$$= \frac{-e^2m_b}{4s_W^2 m_W^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_W^2},$$



$$i\Sigma^b_{t5}(\not{q})$$

$$= \frac{-em_b}{2s_W m_W m_H^2}\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\frac{-iem_H^2}{2s_W m_W}\frac{i}{k^2-m_Z^2}$$

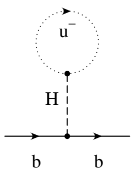
$$= \frac{-e^2m_b}{8s_W^2 m_W^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_Z^2},$$



$$i\Sigma^b_{t6}(\not{q})$$

$$= -\frac{-em_b}{2s_W m_W m_H^2}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{-iem_W}{2s_W}\right)\frac{i}{k^2-m_W^2}$$

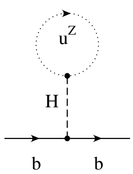
$$= \frac{e^2m_b}{2s_W^2 m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_W^2},$$



$$i\Sigma^b_{t7}(\not{q})$$

$$= -\frac{-em_b}{2s_W m_W m_H^2}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{-iem_W}{2s_W}\right)\frac{i}{k^2-m_W^2}$$

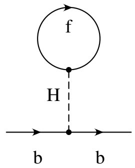
$$= \frac{e^2m_b}{2s_W^2 m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_W^2},$$



$$i\Sigma^b_{t8}(\not{q})$$

$$= -\frac{-em_b}{2s_W m_W m_H^2}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{-iem_W}{2c_W^2 s_W}\right)\frac{i}{k^2-m_Z^2}$$

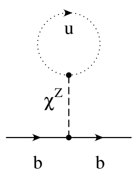
$$= \frac{e^2m_b}{4c_W^2 s_W^2 m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_Z^2},$$



$$i\Sigma^b_{t9}(\not{q})$$

$$= -\frac{-iem_b}{2s_W m_W}\frac{i}{-m_H^2}\int\frac{d^Dk}{(2\pi)^D}\mathrm{tr}\left(\frac{-iem_f}{2s_W m_W}\frac{i(\not{k}+m_f)}{k^2-m_f^2}\right)$$

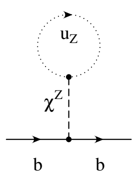
$$= \frac{e^2 m_b m_f}{4s_W^2 m_W^2 m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{\mathrm{tr}(\not{k}+m_f)}{k^2-m_f^2},$$



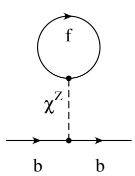
$$i\Sigma^b_{t10}(\not{q})$$

$$= \frac{-iem_b}{2s_W m_W m_Z^2}\gamma_5$$

$$\times\int\frac{d^Dk}{(2\pi)^D}\left(-\left(\frac{em_W}{2s_W}\right)\frac{i}{k^2-m_W^2}-\left(\frac{-em_W}{2s_W}\right)\frac{i}{k^2-m_W^2}\right) = 0,$$



$$i\Sigma^b_{t11}(\not{q}) = 0,$$



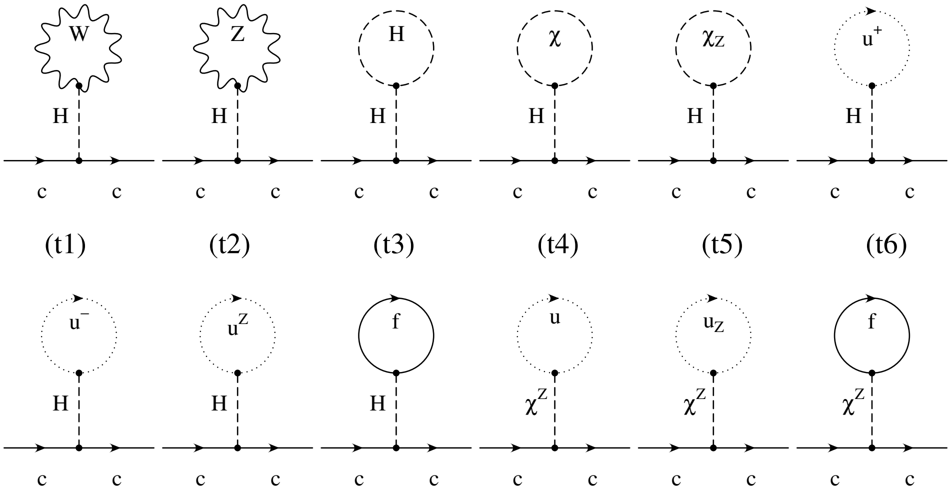
$$i\Sigma^b_{t12}(\not{q})$$

$$= \frac{em_b}{2s_W m_W}\gamma_5\frac{i}{-m_Z^2}\int\frac{d^Dk}{(2\pi)^D}\sum_f -\mathrm{tr}\left(\frac{-em_f I_f}{s_W m_W}\gamma_5\frac{i(\not{k}+m_f)}{k^2-m_f^2}\right) = 0. \tag{143}$$

6.4 Self energy corrections for the charm quark

The charm quark self energy tadpole diagrams are shown in Fig. 5, where the baseline has to be exchanged. The baseline vertex together with the (momentumless) Higgs boson propagator gives

$$\frac{iem_W}{c_W^2 s_W}g^{\mu\nu}\frac{i}{-m_H^2} = \frac{em_W g^{\mu\nu}}{c_W^2 s_W m_H^2}, \tag{144}$$



(t1) (t2) (t3) (t4) (t5) (t6)

(t7) (t8) (t9) (t10) (t11) (t12)

Figure 5: charm quark self energy tadpole diagrams

and the first nine contributions read

$$
\begin{aligned}
-i\Sigma^{\mu\nu}_{t1}(q) =& \frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \int \frac{d^Dk}{(2\pi)^D} \frac{iem_W}{s_W} g^{\mu_1\nu_1} \frac{-ig_{\mu_1\nu_1}}{k^2-m_W^2} \\
=& \frac{De^2 m_W^2}{c_W^2 s_W^2 m_H^2} A(m_W), \\
-i\Sigma^{\mu\nu}_{t2}(q) =& \frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{iem_W}{c_W^2 s_W} g^{\mu_1\nu_1} \frac{-ig_{\mu_1\nu_1}}{k^2-m_Z^2} \\
=& \frac{De^2 m_W^2}{c_W^4 s_W^2 m_H^2} A(m_Z), \\
-i\Sigma^{\mu\nu}_{t3}(q) =& \frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{-3iem_H^2}{2s_W m_W} \frac{i}{k^2-m_H^2} \\
=& \frac{3e^2 m_W^2}{4c_W^2 s_W^2} A(m_H), \\
-i\Sigma^{\mu\nu}_{t4}(q) =& \frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \int \frac{d^Dk}{(2\pi)^D} \frac{-iem_H^2}{2s_W m_W} \frac{i}{k^2-m_W^2} \\
=& \frac{e^2}{2c_W^2 s_W^2} A(m_W), \\
-i\Sigma^{\mu\nu}_{t5}(q) =& \frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{-iem_H^2}{2s_W m_W} \frac{i}{k^2-m_Z^2}
\end{aligned}
$$

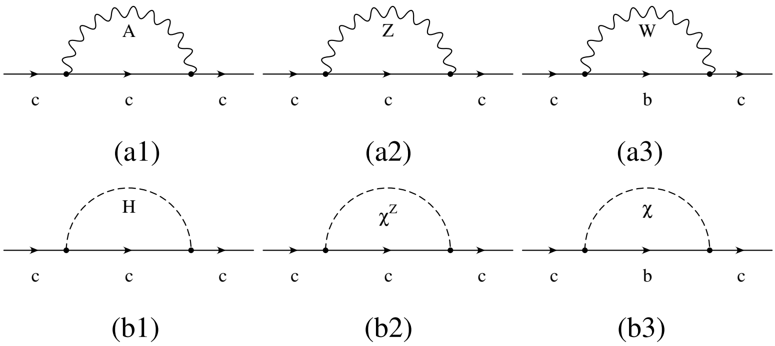


Figure 6: charm quark self energy diagrams

$$
\begin{aligned}
&= \frac{e^2}{4c_W^2 s_W^2} A(m_Z), \\
-i\Sigma_{t6}^{\mu\nu}(q) &= -\frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \int \frac{d^D k}{(2\pi)^D} \frac{-iem_W}{2s_W} \frac{i}{k^2 - m_W^2} \\
&= -\frac{e^2 m_W^2}{2c_W^2 s_W^2 m_H^2} A(m_W) = -i\Sigma_{t7}^{\mu\nu}(q),
\end{aligned}
$$

$$
\begin{aligned}
-i\Sigma_{t8}^{\mu\nu}(q) &= -\frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \int \frac{d^D k}{(2\pi)^D} \frac{-iem_W}{2c_W^2 s_W} \frac{i}{k^2 - m_Z^2} \\
&= -\frac{e^2 m_W^2}{2c_W^4 s_W^2 m_H^2} A(m_Z),
\end{aligned}
$$

$$
\begin{aligned}
-i\Sigma_{t9}^{\mu\nu}(q) &= -\frac{em_W}{c_W^2 s_W m_H^2} g^{\mu\nu} \sum_f \int \frac{d^D k}{(2\pi)^D} \frac{-iem_f}{2s_W m_W} \mathrm{tr}\left(\frac{i(\not{k} + m_f)}{k^2 - m_f^2} \right) \\
&= -\frac{2e^2 m_f^2}{c_W^2 s_W^2 m_H^2} A(m_f).
\end{aligned}
\qquad (145)
$$

Self energy contributions for the charm quark are shown in Fig. 6. As examples we will calculate the charm quark diagrams (a2) and (b2),

$$
\begin{aligned}
i\Sigma_{a2}^c(\not{q}) &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} ie\gamma^\nu \left(g_c^- \Lambda_- + g_c^+ \Lambda_+ \right) \frac{i(\not{q} + \not{k} + m_c)}{(q+k)^2 - m_c^2} \\
&\quad \times ie\gamma^\mu \left(g_c^- \Lambda_- + g_c^+ \Lambda_+ \right) \frac{-ig_{\mu\nu}}{k^2 - m_Z^2}, \\
i\Sigma_{b2}^c(\not{q}) &= \int \frac{d^D k}{(2\pi)^D} \frac{em_c}{2s_W m_W} \gamma_5 \frac{i(\not{q} + \not{k} + m_c)}{(q+k)^2 - m_c^2} \frac{em_c}{2s_W m_W} \gamma_5 \frac{i}{k^2 - m_Z^2}.
\end{aligned}
\qquad (146)
$$

6.5 The comparison of Feynman and unitary gauge

Now by using the unitary gauge one obtains

$$
\begin{aligned}
i\Sigma_{a2}^{c\prime}(\not{q}) &= \int \frac{d^Dk}{(2\pi)^D} \int \frac{d^Dk}{(2\pi)^D} ie\gamma^\nu \left(g_c^-\Lambda_- + g_c^+\Lambda_+\right) \frac{i(\not{q}+\not{k}+m_c)}{(q+k)^2-m_c^2} \\
&\times ie\gamma^\mu \left(g_c^-\Lambda_- + g_c^+\Lambda_+\right) \frac{-i}{k^2-m_Z^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_Z^2}\right). \qquad (147)
\end{aligned}
$$

by using $q^2 = m_c^2$, the results we obtain are

$$
\begin{aligned}
i\Sigma_{a2}^{c}(\not{q}) &= \frac{-e^2}{4m_c}\Big[4Dg_c^-g_c^+m_c^2B(m_c^2;m_c,m_Z) + (D-2)\left((g_c^-)^2+(g_c^+)^2\right) \\
&\times\Big(A(m_c) - A(m_Z) - (2m_c^2-m_Z^2)B(m_c^2;m_c,m_Z)\Big)\Big], \\
i\Sigma_{b2}^{c}(\not{q}) &= \frac{-e^2m_c}{8m_W^2s_W^2}\Big[2m_c^2B(m_c^2;m_c,m_Z) + \Big(A(m_c) - A(m_Z) \\
&-(2m_c^2-m_Z^2)B(m_c^2;m_c,m_Z)\Big)\Big], \\
i\Sigma_{a2}^{c\prime}(\not{q}) &= \frac{-e^2}{4m_cm_Z^2}\Big[2(g_c^- - g_c^+)^2m_c^2A(m_c) + 2m_c^2m_Z^2 \\
&\times\Big((g_c^-)^2 + (g_c^+)^2 + 2(D-1)g_c^-g_c^+\Big)B(m_c^2;m_c,m_Z) \\
&+(D-2)m_Z^2\left((g_c^-)^2+(g_c^+)^2\right)\Big(A(m_c) - A(m_Z) \\
&-(2m_c^2-m_Z^2)B(m_c^2;m_c,m_Z)\Big)\Big]. \qquad (148)
\end{aligned}
$$

Using explicitly $g_c^- = (\frac{1}{2} - s_W^2Q_c)m_Z/(s_Wm_W)$ and $g_c^+ = -s_W^2Q_cm_Z/(s_Wm_W)$ one obtains

$$
i\Sigma_{a2}^{c\prime}(\not{q}) - i\Sigma_{a2}^{c}(\not{q}) - i\Sigma_{b2}^{c}(\not{q}) = \frac{-e^2m_c}{8m_W^2s_W^2}A(m_Z). \qquad (149)
$$

We got the nonvanishing contribution (149), as it contains solely a one-point integral. The tadpole contributions (t2) with Z boson and (t5) with Z Goldstone boson are given by

$$
\begin{aligned}
i\Sigma_{t2}^{c}(\not{q}) &= \frac{1}{2}\left(\frac{-iem_c}{2s_Wm_W}\right)\frac{i}{-m_H^2}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{iem_Wg^{\mu\nu}}{c_W^2s_W}\right)\frac{-ig_{\mu\nu}}{k^2-m_Z^2} \\
&= \frac{-De^2m_c}{4c_W^2s_W^2m_H^2}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m_Z^2}, \qquad (150) \\
i\Sigma_{t5}^{c}(\not{q}) &= \frac{1}{2}\left(\frac{-iem_c}{2s_Wm_W}\right)\frac{i}{-m_H^2}\int\frac{d^Dk}{(2\pi)^D}\left(\frac{-iem_H^2}{2s_Wm_W}\right)\frac{i}{k^2-m_Z^2}
\end{aligned}
$$

$$= \frac{-e^2 m_c}{8s_W^2 m_W^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m_Z^2}. \tag{151}$$

The factor $1/2$ is a combinatorical factor due to the fact that the Z boson is its own antiparticle. As the Goldstone boson is absent for unitary gauge (i.e., does not propagate), the contribution in (151) is the one which compensates the difference on the side of the Feynman gauge. Therefore, we have to deal with the (150) contribution which will be different for unitary gauge. Via the Higgs boson a ghost field can be coupled to the fermion. For the tadpole loop with a u_Z one obtains

$$i\Sigma_{t8}^c(\not{q}) = -\left(\frac{-iem_c}{2s_W m_W}\right) \frac{i}{-m_H^2} \int \frac{d^D k}{(2\pi)^D} \left(\frac{-iem_W \xi_Z}{2c_W^2 s_W}\right) \frac{i}{k^2 - \xi_Z m_Z^2}, \tag{152}$$

where the minus sign comes from the closed ghost loop. For unitary gauge ($\xi_Z \to \infty$) the contribution remains finite. However, the dependence on the inner momentum k disappears and, therefore, there is no ghost contribution either. On the other hand, for Feynman gauge ($\xi_Z = 1$) one obtains

$$i\Sigma_{t8}^c(\not{q}) = \frac{e^2 m_c}{4c_W^2 s_W^2 m_H^2} A(m_Z). \tag{153}$$

This ghost contribution again is the same as the additional contribution from unitary gauge in

$$i\Sigma_{t2}^{c\prime}(\not{q})$$
$$= \frac{-e^2 m_c}{4c_W^2 s_W^2 m_H^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m_Z^2} \left(D - \frac{k^2}{m_Z^2}\right) = \frac{-(D-1)e^2 m_c}{4c_W^2 s_W^2 m_H^2} A(m_Z). \tag{154}$$

From this we finally conclude that the sum of "baseline" self energy contributions of vector and Goldstone bosons and the tadpole contributions of vector and Goldstone bosons and ghosts turns out to be the same for Feynman and unitary gauge.

It is necessary for the renormalisation to obtain the momentum and mass parts separately. For the charm quark, one obtains

$$i\Sigma_{a1}^c(\not{q})$$
$$= \frac{e^2 Q_c^2}{2q^2} \Big[(D-2)\Big(A(m_A) - A(m_c) + (q^2 + m_c^2 - m_A^2) B(q^2; m_c, m_A)\Big) \not{q}$$
$$- 2Dq^2 m_c B(q^2; m_c, m_A)\Big],$$

$$
\begin{aligned}
& i\Sigma^c_{a2}(\not{q}) \\
&= \frac{e^2}{2q^2}\Big[(D-2)(g_c^-)^2\Big(A(m_Z)-A(m_c)+(q^2+m_c^2-m_Z^2)B(q^2;m_c,m_Z)\Big) \\
&\quad \times \not{q}\Lambda_- + (D-2)(g_c^+)^2\Big(A(m_Z)-A(m_c) \\
&\quad + (q^2+m_c^2-m_Z^2)B(q^2;m_c,m_Z)\Big)\not{q}\Lambda_+ \\
&\quad - 2Dq^2 g_c^- g_c^+ m_c B(q^2;m_c,m_Z)\Big],
\end{aligned}
$$

$$
\begin{aligned}
& i\Sigma^c_{a3}(\not{q}) \\
&= \frac{(D-2)e^2|V_{cb}|^2}{4q^2 s_W^2}\Big(A(m_W)-A(m_b)+(q^2+m_b^2-m_W^2)B(q^2;m_b,m_W)\Big) \\
&\quad \times \not{q}\Lambda_-,
\end{aligned}
$$

$$
\begin{aligned}
& i\Sigma^c_{b1}(\not{q}) \\
&= \frac{e^2 m_c^2}{8q^2 s_W^2 m_W^2}\Big[\Big(A(m_H)-A(m_c)+(q^2+m_c^2-m_H^2)B(q^2;m_c,m_H)\Big)\not{q} \\
&\quad + 2q^2 m_c B(q^2;m_c,m_H)\Big],
\end{aligned}
$$

$$
\begin{aligned}
& i\Sigma^c_{b2}(\not{q}) \\
&= \frac{e^2 m_c^2}{8q^2 s_W^2 m_W^2}\Big[\Big(A(m_Z)-A(m_c)+(q^2+m_c^2-m_Z^2)B(q^2;m_c,m_Z)\Big)\not{q} \\
&\quad - 2q^2 m_c B(q^2;m_c,m_Z)\Big],
\end{aligned}
$$

$$
\begin{aligned}
& i\Sigma^c_{b3}(\not{q}) \\
&= \frac{e^2|V_{cb}|^2}{4q^2 s_W^2 m_W^2}\Big[m_b^2\Big(A(m_W)-A(m_b)+(q^2+m_b^2-m_W^2) \\
&\quad \times B(q^2;m_b,m_W)\Big)\not{q}\Lambda_- \\
&\quad + m_c^2\Big(A(m_W)-A(m_b)+(q^2+m_b^2-m_W^2)B(q^2;m_b,m_W)\Big)\not{q}\Lambda_+ \\
&\quad - 2q^2 m_b^2 m_c B(q^2;m_b,m_W)\Big], \qquad (155)
\end{aligned}
$$

For the flavour changing currents, i.e., for those self energy corrections containing a W vector or Goldstone boson correction. We recalculate (a3) and (b3) (with j the initial, i the final, and k the intermediate fermion) to obtain (cf. Fig. 7)

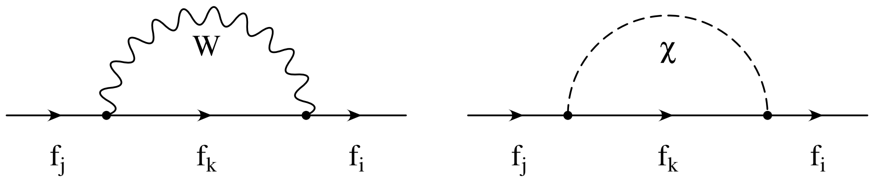


Figure 7: Fermion self energy diagrams for changing flavour

$$
\begin{aligned}
& i\Sigma_{a3}^{ij}(\not{q}) \\
&= \sum_k \frac{(D-2)e^2 V_{ik} V_{kj}^\dagger}{4q^2 s_W^2}\Big(A(m_W) - A(m_k) + (q^2 + m_k^2 - m_W^2) \\
&\quad \times B(q^2; m_k, m_W)\Big)\not{q}\Lambda_-,
\end{aligned}
$$

$$
\begin{aligned}
& i\Sigma_{b3}^{ij}(\not{q}) \\
&= \sum_k \frac{e^2 V_{ik} V_{kj}^\dagger}{4q^2 s_W^2 m_W^2}\Big[m_k^2\Big(A(m_W) - A(m_k) + (q^2 + m_k^2 - m_W^2) \\
&\quad \times B(q^2; m_k, m_W)\Big)\not{q}\Lambda_- \\
&\quad + m_i m_j \Big(A(m_W) - A(m_k) + (q^2 + m_k^2 - m_W^2)B(q^2; m_k, m_W)\Big) \\
&\quad \times \not{q}\Lambda_+ - m_i m_k^2 B(q^2; m_k, m_W)\Lambda_- - m_j m_k^2 B(q^2; m_k, m_W)\Lambda_+\Big].
\end{aligned} \tag{156}
$$

Summing up all contributions, we end up with

$$
\Sigma_{ij}(\not{q}) = \not{q}\Lambda_- \Sigma_{ij}^L(q^2) + \not{q}\Lambda_+ \Sigma_{ij}^R(q^2) + \Lambda_- \Sigma_{ij}^l(q^2) + \Lambda_+ \Sigma_{ij}^r(q^2). \tag{157}
$$

6.6 Calculation of the counter terms

The NLO result of the decay rate is contributed by a matrix element $\mathcal{M}(q,\lambda) = \mathcal{M}^\mu \varepsilon_\mu(q,\lambda)$, where $\varepsilon(q,\lambda)$ is the polarisation vector of the W boson, and

$$
\begin{aligned}
\mathcal{M}^\mu &= \frac{ie}{\sqrt{2}s_W}\Bigg\{\gamma^\mu \Lambda_- \Bigg[V_{cb}\left(1 + \delta Z_e - \frac{\delta s_W}{s_W} + \delta Z_{WW}\right) \\
&\qquad + \delta V_{cb} + \sum_k (\delta Z_{ck}^{L*} V_{kb} + V_{ck}\delta Z_{kb}^L)\Bigg] + V_{cb} V_b^\mu\Bigg\},
\end{aligned} \tag{158}
$$

where

$$\delta V_{cb} = \frac{1}{2}\sum_k \Big((\delta Z^L_{ck} - \delta Z^{L*}_{ck})V_{kb} - V_{ck}(\delta Z^L_{kb} - \delta Z^{L*}_{kb})\Big)\,, \tag{159}$$

The renormalisation factor also contains the IR singularities. As the CKM matrix element V_{cb} dominating the decay $W^+ \to c\bar{b}$ is not close to unit, we have to take into account the mixing of quark states and, related to this, the renormalisation of the mixing matrix [9, 16–23]. Using

$$\delta V_{cb} = \frac{1}{2}\sum_k \Big((\delta Z^L_{ck} - \delta Z^{L*}_{ck})V_{kb} - V_{ck}(\delta Z^L_{kb} - \delta Z^{L*}_{kb})\Big)\,, \tag{160}$$

one obtains

$$\begin{aligned}\mathcal{M}^\mu = \frac{ie}{\sqrt{2}s_W}\Bigg\{\gamma^\mu\Lambda_-\Bigg[V_{cb}\left(1+\delta Z_e - \frac{\delta s_W}{s_W} + \delta Z_{WW} + \delta Z^L_{cc} + \delta Z^L_{bb}\right)\\ + \operatorname{Re}\delta Z^L_{cu}V_{ub} + \operatorname{Re}\delta Z^L_{ct}V_{tb} + V_{cd}\operatorname{Re}\delta Z^L_{db} + V_{cs}\operatorname{Re}\delta Z^L_{sb}\Bigg] + V_{cb}V^\mu_b\Bigg\}.\end{aligned} \tag{161}$$

Defining $\delta Z_{\rm CKM} := \Big(\operatorname{Re}\delta Z^L_{cu}V_{ub}+\operatorname{Re}\delta Z^L_{ct}V_{tb}+V_{cd}\operatorname{Re}\delta Z^L_{db}+V_{cs}\operatorname{Re}\delta Z^L_{sb}\Big)/V_{cb}$, one ends up with V_- as given in Eq. (139). In the $\alpha(0)$ scheme (which we will see in later section), the UV singular part within V^μ_b is given by $V^\mu_s = \gamma^\mu\Lambda_- V^0_s$ with

$$V^0_s = \frac{e^2}{4m^2_W s^2_W\varepsilon}\left[m^2_c + m^2_b + 12m^2_W - m^2_Z\Big(1-2(Q_c - Q_b + 2Q_cQ_b)s^2_W\Big)\right] \tag{162}$$

with space-time dimension $D = 4-2\varepsilon$, and $Q_c = 2/3$ and $Q_b = -1/3$ for the electric charges of the quarks, while $\delta Z^s_e + \delta Z^s_{WW} - \delta s^s_W/s_W = -2e^2/(s^2_W\varepsilon)$ and [15]

$$\begin{aligned}\delta Z^{Ls}_{bb} &= -\frac{e^2}{8m^2_W s^2_W\varepsilon}\left[m^2_b + \sum_k |V_{kb}|^2(m^2_k + 2m^2_W) + (1+4Q_cQ_bs^2_W)m^2_Z\right],\\ \delta Z^{Ls}_{cc} &= -\frac{e^2}{8m^2_W s^2_W\varepsilon}\left[m^2_c + \sum_k |V_{ck}|^2(m^2_k + 2m^2_W) + (1+4Q_cQ_bs^2_W)m^2_Z\right],\\ \delta Z^{Ls}_{ij} &= \frac{e^2}{4m^2_W s^2_W\varepsilon}\sum_k V_{ik}V_{kj}\frac{m^2_im^2_j - 2m^2_im^2_k - m^2_jm^2_k + 2m^2_jm^2_W}{m^2_i - m^2_j}.\end{aligned} \tag{163}$$

Note that according to Eq. (4.5.27) in Ref. [15], one has $\delta Z^{Ls*}_{ij} = \delta Z^{Ls}_{ij}|_{m^2_i\leftrightarrow m^2_j}$. Therefore,

$$\operatorname{Re}\delta Z^{Ls}_{ij} = \frac{-e^2}{8m^2_W s^2_W\varepsilon}\sum_k V_{ik}V_{kj}(m^2_k + 2m^2_W), \qquad V_{kj} = V^*_{jk}. \tag{164}$$

Without the mixed contributions (and without the general factor V_{cb}), one first obtains

$$V_s^0 + \delta Z_e^s - \frac{\delta s_W^s}{s_W} + \delta Z_{WW}^s + \delta Z_{cc}^{Ls} + \delta Z_{bb}^{Ls}$$
$$= \frac{e^2}{8m_W^2 s_W^2 \varepsilon}\Big[m_c^2 + m_b^2 + 4m_W^2 - \sum_k |V_{ck}|^2(m_k^2 + 2m_W^2)$$
$$- \sum_k |V_{kb}|^2(m_k^2 + 2m_W^2)\Big]. \tag{165}$$

On the other hand, the mixed part gives

$$\mathrm{Re}\,\delta Z_{cu}^{Ls} V_{ub} + \mathrm{Re}\,\delta Z_{ct}^{Ls} V_{tb} + V_{cd}\,\mathrm{Re}\,\delta Z_{db}^{Ls} + V_{cs}\,\mathrm{Re}\,\delta Z_{sb}^{Ls}$$
$$= \frac{-e^2}{8m_W^2 s_W^2 \varepsilon}\sum_k (V_{ck}V_{ku}V_{ub} + V_{ck}V_{kt}V_{tb} + V_{cd}V_{dk}V_{kb} + V_{cs}V_{sk}V_{kb})$$
$$(m_k^2 + 2m_W^2)$$
$$= \frac{-e^2 V_{cb}}{8m_W^2 s_W^2 \varepsilon}\Big[m_b^2 + 2m_W^2 - \sum_k |V_{ck}|^2(m_k^2 + 2m_W^2) + m_c^2 + 2m_W^2$$
$$- \sum_k |V_{kb}|^2(m_k^2 + 2m_W^2)\Big]. \tag{166}$$

This cancels exactly against the previous contribution, gaining an UV finite form factor.
The renormalisation of the main form factor is done by using the bare self energy correction for the fermion f,

$$\Sigma_{ff}^b(\not{q}) = \not{q}\Lambda_- \Sigma_{ff}^{bL}(q^2) + \not{q}\Lambda_+ \Sigma_{ff}^{bR}(q^2) + \Lambda_- \Sigma_{ff}^{bl}(q^2) + \Lambda_+ \Sigma_{ff}^{br}(q^2), \tag{167}$$

the fermion wave function counter terms are given by

$$2\delta Z_{ff}^L = -\Sigma_{ff}^{bL}(m_f^2) - 2m_f\left(m_f \Sigma_{ff}^{bL\prime}(m_f^2) + \Sigma_{ff}^{br\prime}(m_f^2)\right). \tag{168}$$

For the counter terms of the boson wave functions is given as

$$2\delta Z_{WW} = -\frac{\partial \Sigma_{WW}^{Tb}(k^2)}{\partial k^2}\Big|_{k^2=m_W^2}, \quad 2\delta Z_{AA} = -\frac{\partial \Sigma_{AA}^{Tb}(k^2)}{\partial k^2}\Big|_{k^2=m_A^2},$$
$$\delta Z_{ZA} = \frac{\Sigma_{AZ}^{Tb}(m_A^2)}{m_Z^2}, \tag{169}$$

from this we can calculate the counter term for the charge

$$\delta Z_e = -\delta Z_{AA} - \delta Z_{ZA}\frac{s_W}{c_W}, \tag{170}$$

And the counter term for the weak mixing angle is given by

$$\frac{\delta s_W}{s_W} = -\frac{c_W^2}{2s_W^2}(\delta Z_W^2 - \delta Z_Z^2). \tag{171}$$

where the mass renormalisation factors are

$$\delta Z_W^2 = \frac{\Sigma_{WW}^{Tb}(m_W^2)}{m_W^2}, \qquad \delta Z_Z^2 = \frac{\Sigma_{ZZ}^{Tb}(m_Z^2)}{m_Z^2}, \tag{172}$$

and

$$\begin{aligned} Z_{ij}^L & \\ &= \frac{1}{m_i^2 - m_j^2}\Big(m_j^2\Sigma_{ij}^{bL}(m_j^2) + m_i m_j \Sigma_{ij}^{bR}(m_j^2) + m_i \Sigma_{ij}^{bl}(m_j^2) + m_j \Sigma_{ij}^{br}(m_j^2)\Big), \end{aligned} \tag{173}$$

6.7 Cancellation of the UV divergent parts

Before we calculate the full renormalisation factors in the mass shell scheme, we first sketch the divergent parts. The only UV-divergent vertex contribution is found in the main part V_-^{0s} and reads

$$V_-^{0s} = \frac{e^2}{4(4\pi)^2 m_W^2 s_W^2 \varepsilon}\Big(m_c^2 + m_b^2 + 11m_W^2 + (2Q_c - 1)(2Q_b + 1)m_Z^2 s_W^2\Big) \tag{174}$$

where in the end we can put $2Q_c - 1 = Q_c + Q_b = 2Q_b + 1$. For this we obtain term by term ($\lambda' = \lambda(m_W^2, m_c^2, m_b^2)$). For this we use our results for the two-point functions to obtain the following coefficients for $1/\varepsilon$

$$\begin{aligned} \delta Z_{Ws}^2 &= \frac{-e^2}{6m_W^2 s_W^2}\left(31m_W^2 - 6m_Z^2 + \sum_f (3m_f^2 - m_W^2)\right) \\ &\quad - \frac{e^2}{4m_H^2 m_W^2 s_W^2}\Big(3m_H^4 + (2m_W^2 + m_Z^2)m_H^2 + 12m_W^4 + 6m_Z^4 - 8\sum_f m_f^4\Big), \\ \delta Z_{Zs}^2 &= \frac{-e^2}{6m_Z^2 m_W^2 s_W^2}\Big(42m_W^4 - 10m_W^2 m_Z^2 - 7m_Z^4 \\ &\quad + 4\sum_f \Big(3(g_f^- - g_f^+)^2 m_f^2 - \Big((g_f^-)^2 + (g_f^+)^2\Big) m_Z^2\Big) m_W^2 s_W^2\Big) \\ &\quad - \frac{e^2}{4m_H^2 m_W^2 s_W^2}\Big(3m_H^4 + (2m_W^2 + m_Z^2)m_H^2 + 12m_W^4 + 6m_Z^4 - 8\sum_f m_f^4\Big). \end{aligned} \tag{175}$$

It is obvious that the tadpole contributions for the W and Z self energy are exactly the same. Therefore, they will cancel in the difference necessary for the calculation of $\delta s_W/s_W$. One obtains ($(g_f^- - g_f^+)^2 = 1/4/c_W^2/s_W^2$, $4\left((g_f^-)^2 + (g_f^+)^2\right) = (1 - 8I_f^3 Q_f s_W^2 + 8Q_f^2 s_W^4)/c_W^2/s_W^2$)

$$\begin{aligned}
\frac{\delta s_{Ws}}{s_W} &= \frac{-e^2}{12 m_Z^4 s_W^4}\Big(42 m_W^4 - 41 m_W^2 m_Z^2 - m_Z^4 \\
&\quad + \sum_f \Big(3m_f^2 - 4\left((g_f^-)^2 + (g_f^+)^2\right) m_W^2 s_W^2\Big) m_Z^2 \\
&\quad - \sum_f \Big(3m_f^2 - m_W^2\Big) m_Z^2\Big) \\
&= \frac{e^2}{12 m_Z^4 s_W^4}\Big((42 m_W^2 + m_Z^2) m_Z^2 s_W^2 \\
&\quad + \sum_f \Big(4\left((g_f^-)^2 + (g_f^+)^2\right) s_W^2 - 1\Big) m_W^2 m_Z^2\Big) \\
&= \frac{e^2}{12 m_Z^2 s_W^2}\Big(42 m_W^2 + m_Z^2 + \sum_f \Big(1 - 8I_f^3 Q_f + 8Q_f^2 s_W^2\Big) m_Z^2\Big). \qquad (176)
\end{aligned}$$

For the summation over the fermions one has ($Q_c - Q_b = Q_\nu - Q_e = 1$)

$$\begin{aligned}
&\sum_f \Big(1 - 8I_f^3 Q_f + 8Q_f^2 s_W^2\Big) \\
&= 3\Big(1 + 1 - 4 + 8s_W^2 + N_c\Big(1 - 4Q_c + 8Q_c^2 s_W^2 + 1 + 4Q_b + 8Q_b^2 s_W^2\Big)\Big) \\
&= -6\Big(1 - 4s_W^2 + N_c\Big(1 - 4(Q_c^2 + Q_b^2)s_W^2\Big)\Big). \qquad (177)
\end{aligned}$$

Finally, for the W boson wave function renormalisation we obtain the singular part

$$\begin{aligned}
\delta Z_{WWs} &= -\frac{1}{2}\frac{\partial \Sigma_{WW}^{Tb}(k^2)}{\partial k^2}\Big|_{k^2=m_W^2} = \frac{e^2}{12 s_W^2}\left(19 - 6(1 + N_c)\right), \\
\delta Z_{WWs} - \frac{\delta s_{Ws}}{s_W} &= \frac{-e^2}{2 m_Z^2 s_W^2}\Big(3m_W^2 + m_Z^2 + 4N_c(Q_c^2 + Q_b^2) m_Z^2 s_W^2\Big). \qquad (178)
\end{aligned}$$

Actually, it is worth to even unite this result with the charge renormalisation factor

$$\begin{aligned}
\delta Z_{es} &= \frac{2e^2}{3}\sum_f Q_f^2 - \frac{7e^2}{2} = \frac{2e^2}{3}\Big(3N_c(Q_c^2 + Q_b^2) + 3\Big) - \frac{7e^2}{2} \\
&= \frac{e^2}{2}\Big(4N_c(Q_c^2 + Q_b^2) - 3\Big) \qquad (179)
\end{aligned}$$

to obtain

$$\delta Z_{WWs} - \frac{\delta s_{Ws}}{s_W} + \delta Z_{es} = \frac{-e^2}{2m_Z^2 s_W^2}\left(3m_W^2 + m_Z^2 + 3m_Z^2 s_W^2\right) = -\frac{2e^2}{s_W^2}. \quad (180)$$

For the vertex correction we have

$$\begin{aligned}\Gamma_s^\mu &= \frac{e^2\gamma^\mu\Lambda_-}{4m_W^2 s_W^2}\left(m_t^2 + m_b^2 + 12m_W^2 - m_Z^2\left(1 - 2(Q_t - Q_b + 2Q_tQ_b)s_W^2\right)\right)\\ &= \Gamma_s\gamma^\mu\Lambda_-.\end{aligned} \quad (181)$$

Adding the subtractions, we finally find that

$$\Gamma_s + \delta Z_{WWs} - \frac{\delta s_{Ws}}{s_W} + \delta Z_{es} + \delta Z_{ccs}^L + \delta Z_{bbs}^L = -\frac{e^2 m_Z^2}{2m_W^2}(1 + Q_b - Q_t)^2. \quad (182)$$

This expression vanishes because of $Q_t - Q_b = Q_W = 1$. Therefore, the UV singularities cancel. The UV singularity cancellation takes place only in the part proportional to $\gamma^\mu\Lambda_-$, i.e., the part proportional to the Born term result. All other vertex corrections should be UV finite. That can be seen in the section below.

6.8 UV finite parts

The UV finite parts of the UV counter terms that latter will be used for the IR subtracted vertex correction V_-^* read as

$$\delta Z_e^f = -\frac{1}{3}\left\{1 - \frac{21}{2}\left(1 - A_f(m_W)\right) + 2\sum_f Q_f^2\left(1 - A_f(m_f)\right)\right\},$$

$$\begin{aligned}&\frac{\delta s_W^f}{s_W}\\ &= \frac{-1}{72m_W^2 m_Z^6 s_W^4}\Bigg\{4m_W^2(m_W^2 - m_Z^2)(36m_W^4 + 24m_W^2m_Z^2 + m_Z^4)\\ &\quad + 3(4m_W^2 - m_Z^2)(12m_W^4 + 20m_W^2m_Z^2 + m_Z^4)\\ &\quad \times\Big(m_W^2 B_f(m_Z^2; m_W, m_W) - m_W^2 A_f(m_W) - m_Z^2 B_f(m_W^2; m_W, m_Z)\\ &\quad + m_Z^2 A_f(m_Z)\Big) + 3m_Z^4\Big((12m_W^4 - 4m_W^2m_H^2 + m_H^4)B_f(m_W^2; m_W, m_H)\\ &\quad - m_W^2 m_H^2(A_f(m_H) - A_f(m_W))\Big) - 3m_W^2m_Z^2\Big((12m_Z^4 - 4m_Z^2m_H^2 + m_H^4)\\ &\quad \times B_f(m_Z^2; m_Z, m_H) - m_Z^2m_H^2(A_f(m_H) - A_f(m_Z))\Big)\\ &\quad + 144m_W^4m_Z^2(m_W^2 - m_Z^2)B_f(m_W^2, m_W, m_A) + 6m_W^4m_Z^2(42m_W^2 - m_Z^2)\end{aligned}$$

$$
\begin{aligned}
&\times (A_f(m_W) - A_f(m_Z)) + 6m_W^2 m_Z^2 (m_W^2 - m_Z^2)(18m_W^2 - 5m_Z^2) A_f(m_Z) \\
&+ 3(m_W^2 - m_Z^2) m_Z^2 m_H^4 A_f(m_H) - m_Z^4 \sum_i \Big[4m_W^2 (m_W^2 - 3m_{\nu_i}^2 - 3m_{\ell_i}^2) \\
&+ 6 \left((m_{\nu_i}^2 - m_{\ell_i}^2)^2 + (m_{\nu_i}^2 + m_{\ell_i}^2 - 2m_W^2) m_W^2 \right) B_f(m_W^2; m_{\nu_i}, m_{\ell_i}) \\
&- 6(m_{\nu_i}^2 - m_{\ell_i}^2) \left(m_{\nu_i}^2 A_f(m_{\nu_i}) - m_{\ell_i}^2 A_f(m_{\ell_i}) \right) + 12 m_W^2 \Big(m_{\nu_i}^2 A_f(m_{\nu_i}) \\
&+ m_{\ell_i}^2 A_f(m_{\ell_i}) \Big) \Big] - m_Z^4 \sum_{i,j} |V_{ij}|^2 \Big[4m_W^2 (m_W^2 - 3m_i^2 - 3m_j^2) \\
&+ 6 \left((m_i^2 - m_j^2)^2 + (m_i^2 + m_j^2 - 2m_W^2) m_W^2 \right) B_f(m_W^2; m_i, m_j) \\
&- 6(m_i^2 - m_j^2) \left(m_i^2 A_f(m_i) - m_j^2 A_f(m_j) \right) + 12 m_W^2 \Big(m_i^2 A_f(m_i) \\
&+ m_j^2 A_f(m_j) \Big) \Big] - 8 m_W^4 (m_W^2 - m_Z^2) \\
&\times \sum_f \Big[(g_f^{-2} + g_f^{+2}) \left(m_Z^2 - 6m_f^2 + 6m_f^2 A_f(m_f) \right) \\
&+ 3 \left((g_f^{-2} + g_f^{+2} - 6 g_f^- g_f^+) m_f^2 - (g_f^{-2} + g_f^{+2}) m_Z^2 \right) B_f(m_Z^2; m_f, m_f) \Big] \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
&\delta Z_{WW}^f \\
&= \frac{1}{72 m_W^4 m_Z^2 s_W^2} \Bigg\{ 4 m_W^4 m_Z^2 \\
&\quad + 3(48 m_W^6 - 16 m_W^4 m_Z^2 + 6 m_W^2 m_Z^4 + m_Z^6) \\
&\quad \times B_f(m_W^2; m_W, m_Z) - 3 m_Z^2 (2m_W^2 - m_H^2) m_H^2 B_f(m_W^2; m_W, m_H) \\
&\quad - 144 m_W^4 (m_W^2 - m_Z^2) B_f(m_W^2; m_W, m_A) + 3 m_W^2 (4 m_W^2 - m_Z^2) \\
&\quad \times (12 m_W^4 + 20 m_W^2 m_Z^2 + m_Z^4) B'_f(m_W^2; m_W, m_Z) \\
&\quad - 3 m_W^2 m_Z^2 (12 m_W^4 - 4 m_W^2 m_H^2 + m_H^4) B'_f(m_W^2; m_W, m_H) \\
&\quad + 144 m_W^6 m_Z^2 s_W^2 B'_f(m_W^2; m_W, m_A) \\
&\quad - 3 m_W^2 m_Z^2 (2 m_W^2 - m_Z^2 - m_H^2) A_f(m_W) \\
&\quad + 3 m_Z^2 (m_W^2 - m_Z^2)(8 m_W^2 + m_Z^2) A_f(m_Z) \\
&\quad + 3 m_Z^2 (m_W^2 - m_H^2) m_H^2 A_f(m_H) + \\
&\quad - m_Z^2 \sum_i \Big[3 \Big(2 \left((m_{\nu_i}^2 - m_{\ell_i}^2)^2 + 2 m_W^4 \right) B_f(m_W^2; m_{\nu_i}, m_{\ell_i}) \\
&\quad - 2 m_W^2 \left((m_{\nu_i}^2 - m_{\ell_i}^2)^2 + (m_{\nu_i}^2 + m_{\ell_i}^2 - 2 m_W^2) m_W^2 \right) B'_f(m_W^2; m_{\nu_i}, m_{\ell_i}) \\
&\quad - 2 (m_{\nu_i}^2 - m_{\ell_i}^2)(m_{\nu_i}^2 A_f(m_{\nu_i}) - m_{\ell_i}^2 A_f(m_{\ell_i})) \Big) - 4 m_W^4 \Big]
\end{aligned}
$$

$$
\begin{aligned}
& - m_Z^2 \sum_{i,j} |V_{ij}|^2 \Big[3 \Big(2 \left((m_i^2 - m_j^2)^2 + 2m_W^4 \right) B_f(m_W^2; m_i, m_j) \\
& - 2m_W^2 \left((m_i^2 - m_j^2)^2 + (m_i^2 + m_j^2 - 2m_W^2) m_W^2 \right) B_f'(m_W^2; m_i, m_j) \\
& - 2(m_i^2 - m_j^2)(m_i^2 A_f(m_i) - m_j^2 A_f(m_j)) \Big) - 4m_W^4 \Big] \Big\},
\end{aligned}
$$

$$
\begin{aligned}
& \delta Z_{cc}^{Lf} \\
& = \frac{-1}{16 m_c^2 m_W^2 s_W^2} \Bigg\{ 2m_c^4 + 2m_c^2 \left(4(Q_c^2 + g_c^{-2}) m_W^2 s_W^2 + m_c^2 \right) (A_f(m_c) - 1) \\
& + m_Z^2 (8 g_c^{-2} m_W^2 s_W^2 + m_c^2) \left(B_f(m_c^2; m_c, m_Z) - A_f(m_Z) \right) \\
& + m_c^2 m_H^2 \left(B_f(m_c^2; m_c, m_H) - A_f(m_H) \right) + 2m_c^2 \Big(8 g_c^{-2} (2m_c^2 - m_Z^2) m_W^2 \\
& \times s_W^2 - 32 g_c^- g_c^+ m_c^2 m_W^2 s_W^2 - m_c^2 m_Z^2 \Big) B_f'(m_c^2; m_c, m_Z) \\
& + 2m_c^4 (4m_c^2 - m_H^2) B_f'(m_c^2; m_c, m_H) - 32 m_c^4 m_W^2 s_W^2 Q_c^2 B_f'(m_c^2; m_c, m_A) \\
& + \sum_k |V_{ck}|^2 \Big[- 4m_c^2 m_W^2 + 2(m_k^2 + 2m_W^2) \\
& \times \Big((m_c^2 - m_k^2 + m_W^2) B_f(m_c^2; m_k, m_W) + m_k^2 A_f(m_k) - m_W^2 A_f(m_W) \Big) \\
& - 4m_c^2 (m_c^2 m_k^2 - m_k^4 - 2m_c^2 m_W^2 - m_k^2 m_W^2 + 2m_W^4) B_f'(m_c^2; m_k, m_W) \Big] \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
& \delta Z_{bb}^{Lf} \\
& = \frac{-1}{16 m_b^2 m_W^2 s_W^2} \Bigg\{ 2m_b^4 + 2m_b^2 \left(4(Q_b^2 + g_b^{-2}) m_W^2 s_W^2 + m_b^2 \right) (A_f(m_b) - 1) \\
& + m_Z^2 (8 g_b^{-2} m_W^2 s_W^2 + m_b^2) \left(B_f(m_b^2; m_b, m_Z) - A_f(m_Z) \right) \\
& + m_b^2 m_H^2 \left(B_f(m_b^2; m_b, m_H) - A_f(m_H) \right) + 2m_b^2 \Big(8 g_b^{-2} (2m_b^2 - m_Z^2) m_W^2 \\
& \times s_W^2 - 32 g_b^- g_b^+ m_b^2 m_W^2 s_W^2 - m_b^2 m_Z^2 \Big) B_f'(m_b^2; m_b, m_Z) \\
& + 2m_b^4 (4m_b^2 - m_H^2) B_f'(m_b^2; m_b, m_H) - 32 m_b^4 m_W^2 s_W^2 Q_b^2 B_f'(m_b^2; m_b, m_A) \\
& + \sum_k |V_{kb}|^2 \Big[- 4m_b^2 m_W^2 + 2(m_k^2 + 2m_W^2) \\
& \times \Big((m_b^2 - m_k^2 + m_W^2) B_f(m_b^2; m_k, m_W) + m_k^2 A_f(m_k) - m_W^2 A_f(m_W) \Big) \\
& - 4m_b^2 (m_b^2 m_k^2 - m_k^4 - 2m_b^2 m_W^2 - m_k^2 m_W^2 + 2m_W^4) B_f'(m_b^2; m_k, m_W) \Big] \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
&\delta Z_{ij}^{Lf} \\
&= \frac{-1}{8m_W^2 s_W^2} \sum_k V_{ik}V_{kj} \Bigg\{ m_k^2 A_f(m_k) - m_W^2 A_f(m_W) - 2m_W^2 \\
&\quad + \frac{B_f(m_i^2; m_k, m_W)}{m_i^2 - m_j^2} \Big((m_i^2 - m_k^2)(m_j^2 - m_k^2) \\
&\quad + (2m_i^2 - m_j^2 + m_k^2 - 2m_W^2) m_W^2 \Big) \\
&\quad + \frac{B_f(m_j^2; m_k, m_W)}{m_j^2 - m_i^2} \Big((m_j^2 - m_k^2)(m_i^2 - m_k^2) \\
&\quad + (2m_j^2 - m_i^2 + m_k^2 - 2m_W^2) m_W^2 \Big) \Bigg\}. \qquad (183)
\end{aligned}
$$

Here, δZ_e and $\delta s_W/s_W$ are the counter terms for the electric charge and the sine of the Weinberg angle. δZ_{WW}, δZ_{cc}^L and δZ_{bb}^L are counter terms for the renormalisation of the wave functions of W boson, left handed charm and bottom quarks, respectively. For the CKM matrix we need flavour changing wave function counter terms δZ_{ij}^L. We have checked all these results against the results given in Ref. [9] and found agreement.

In order to save the symmetry, the expressions presented here contain also vanishing contributions from e.g. the neutrinos. However, in order to handle two-point functions containing the neutrino masses appropriately, a couple of limiting cases has to be calculated. Calculating $B_f(m_W^2; m_\nu, m_\ell)$ and the derivative $B_f'(m_W^2; m_\nu, m_\ell)$, we start with the expansion of the square root of Källén function for this particular mass configuration,

$$
\sqrt{\lambda(m_W^2, m_\nu^2, m_\ell^2)} = m_W^2 - m_\ell^2 - \frac{p^2 + m_\ell^2}{p^2 - m_\ell^2} m_\nu^2 + O(m_\nu^4). \qquad (184)
$$

The finite part of the two-point function $B(p^2; m_1, m_2) = i\bar{\mu}^{-2\varepsilon}/(4\pi)^2(1/\varepsilon + B_f(p^2; m_1, m_2))$ (for simplicity, $\bar{\mu}$ is taken to be the $\overline{\rm MS}$ scale) is written as (cf. (B.1) in Ref. [24])

$$
\begin{aligned}
&B_f(p^2; m_1, m_2) \\
&= \quad 2 - \frac{1}{2}\left(\ln\left(\frac{m_1^2}{\bar{\mu}^2}\right) + \ln\left(\frac{m_2^2}{\bar{\mu}^2}\right)\right) - \frac{m_1^2 - m_2^2}{2p^2} \ln\left(\frac{m_1^2}{m_2^2}\right) + \frac{1}{p^2} \times
\end{aligned}
$$

$$
\begin{cases}
-\sqrt{(m_1+m_2)^2-p^2}\sqrt{(m_1-m_2)^2-p^2} \\
\qquad \times \ln\left(\frac{\sqrt{(m_1+m_2)^2-p^2}-\sqrt{(m_1-m_2)^2-p^2}}{\sqrt{(m_1+m_2)^2-p^2}+\sqrt{(m_1-m_2)^2-p^2}}\right) \\
-2\sqrt{(m_1+m_2)^2-p^2}\sqrt{p^2-(m_1-m_2)^2} \\
\qquad \times \arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right) \\
\sqrt{p^2-(m_1+m_2)^2}\sqrt{p^2-(m_1-m_2)^2} \\
\qquad \times \left(\ln\left(\frac{\sqrt{p^2-(m_1+m_2)^2}-\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{p^2-(m_1+m_2)^2}+\sqrt{p^2-(m_1-m_2)^2}}\right)+i\pi\right)
\end{cases} \tag{185}
$$

in the cases $p^2 < (m_1 - m_2)^2$, $(m_1 - m_2)^2 < p^2 < (m_1 + m_2)^2$, and $p^2 > (m_1+m_2)^2$, respectively. The derivative of this finite part with respect to p^2 results in

$$
\begin{aligned}
& B'_f(p^2; m_1, m_2) \\
&= \frac{m_1^2 - m_2^2}{2p^4}\ln\left(\frac{m_1^2}{m_2^2}\right) - \frac{1}{p^2} + \frac{1}{p^4} \times \\
& \begin{cases}
\frac{(m_1^2-m_2^2)^2-(m_1^2+m_2^2)p^2}{\sqrt{(m_1+m_2)^2-p^2}\sqrt{(m_1-m_2)^2-p^2}} \\
\qquad \times \ln\left(\frac{\sqrt{(m_1+m_2)^2-p^2}-\sqrt{(m_1-m_2)^2-p^2}}{\sqrt{(m_1+m_2)^2-p^2}+\sqrt{(m_1-m_2)^2-p^2}}\right) \\
-2\frac{(m_1^2-m_2^2)^2-(m_1^2+m_2^2)p^2}{\sqrt{(m_1+m_2)^2-p^2}\sqrt{p^2-(m_1-m_2)^2}} \\
\qquad \times \arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right) \\
-\frac{(m_1^2-m_2^2)^2-(m_1^2+m_2^2)p^2}{\sqrt{p^2-(m_1+m_2)^2}\sqrt{p^2-(m_1-m_2)^2}} \\
\qquad \times \left(\ln\left(\frac{\sqrt{p^2-(m_1+m_2)^2}-\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{p^2-(m_1+m_2)^2}+\sqrt{p^2-(m_1-m_2)^2}}\right)+i\pi\right)
\end{cases}
\end{aligned} \tag{186}
$$

in the same cases. The power series expansions are given by

$$
\begin{aligned}
B_f(m_W^2; m_\nu, m_\ell) &= 2 - \frac{m_\ell^2}{m_W^2}\ln\left(\frac{m_\ell^2}{\bar{\mu}^2}\right) - \frac{m_W^2 - m_\ell^2}{m_W^2}\left(\ln\left(\frac{m_W^2 - m_\ell^2}{\bar{\mu}^2}\right) - i\pi\right) \\
&\quad + O(m_\nu^2), \\
B'_f(m_W^2; m_\nu, m_\ell) &= -\frac{1}{m_W^2} + \frac{m_\ell^2}{m_W^4}\ln\left(\frac{m_\ell^2}{\bar{\mu}^2}\right) - \frac{m_\ell^2}{m_W^4}\left(\ln\left(\frac{m_W^2 - m_\ell^2}{\bar{\mu}^2}\right) - i\pi\right) \\
&\quad + O(m_\nu^2).
\end{aligned} \tag{187}
$$

This is sufficient for the parts coming from the W boson self energy. However, for the parts from the Z boson self energy in δs_W we have to expand $B_f(m_Z^2; m_\nu, m_\nu)$ and $B_f'(m_Z^2; m_\nu, m_\nu)$ in m_ν. With

$$\sqrt{\lambda(m_Z^2, m_\nu^2, m_\nu^2)} = \sqrt{m_Z^2(m_Z^2 - 4m_\nu^2)} = m_Z^2 - 2m_\nu^2 - \frac{2m_\nu^4}{m_Z^2} + O(m_\nu^6) \tag{188}$$

one obtains

$$\begin{aligned} & B_f(m_Z^2; m_\nu, m_\nu) \\ = \quad & 2 - \ln\left(\frac{m_\nu^2}{\bar{\mu}^2}\right) + \sqrt{1 - \frac{4m_\nu^2}{m_Z^2}}\left(\ln\left(\frac{1-\sqrt{1-4m_\nu^2/m_Z^2}}{1+\sqrt{1-4m_\nu^2/m_Z^2}}\right) + i\pi\right), \end{aligned} \tag{189}$$

$$\begin{aligned} & B_f'(m_Z^2; m_\nu, m_\nu) \\ = \quad & -\frac{1}{m_Z^2} + \frac{2m_\nu^2}{m_Z^4\sqrt{1-4m_\nu^2/m_Z^2}}\left(\ln\left(\frac{1-\sqrt{1-4m_\nu^2/m_Z^2}}{1+\sqrt{1-4m_\nu^2/m_Z^2}}\right) + i\pi\right). \end{aligned} \tag{190}$$

For the G_μ scheme dealt with later, we have to calculate $B_f(0; m_1, m_2)$, As $p^2 = 0$, we have to use the first case. Without loss of generality, we assume that $m_1^2 > m_2^2$ (the opposite inequality will lead to the same result). We have

$$\begin{aligned} \sqrt{\lambda(p^2, m_1^2, m_2^2)} &= \sqrt{(m_1^2 - m_2^2)^2 - 2p^2(m_1^2 + m_2^2) + p^4} \\ &= m_1^2 - m_2^2 - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2}p^2 + O(p^4). \end{aligned} \tag{191}$$

In this case $p^2 < (m_1 - m_2)$. Therefore

$$\begin{aligned} & B_f(p^2; m_1, m_2) \\ = \quad & 2 - \frac{1}{2}\left(\ln\left(\frac{m_1^2}{\bar{\mu}^2}\right) + \ln\left(\frac{m_2^2}{\bar{\mu}^2}\right)\right) - \frac{m_1^2 - m_2^2}{2p^2}\ln\left(\frac{m_1^2}{m_2^2}\right) + \\ & -\frac{1}{2p^2}\left(m_1^2 - m_2^2 - p^2\frac{m_1^2 + m_2^2}{m_1^2 - m_2^2}\right)\left(\ln\left(\frac{m_2^2}{m_1^2}\right) + \frac{2p^2}{m_1^2 - m_2^2}\right) \\ = \quad & 1 - \frac{1}{m_1^2 - m_2^2}\left(m_1^2\ln\left(\frac{m_1^2}{\bar{\mu}^2}\right) - m_2^2\ln\left(\frac{m_2^2}{\bar{\mu}^2}\right)\right) + O(p^2) \end{aligned} \tag{192}$$

and

$$B_f(0;m_1,m_2)=\frac{m_1^2A_f(m_1)-m_2^2A_f(m_2)}{m_1^2-m_2^2}, \tag{193}$$

were the finite part $A_f(m)$ of the one-point function is defined via

$$A(m)=\frac{im^2\bar{\mu}^{-2\varepsilon}}{(4\pi)^2}\left(\frac{1}{\varepsilon}+A_f(m)\right),\qquad A_f(m)=1-\ln\left(\frac{m^2}{\bar{\mu}^2}\right). \tag{194}$$

The same approximation we need for equal masses. Starting with

$$\begin{aligned}\sqrt{\lambda(p^2,m^2,m^2)}&=\sqrt{(4m^2-p^2)p^2}=\sqrt{4m^2p^2}\sqrt{1-\frac{p^2}{4m^2}}\\&=p^2\sqrt{\frac{4m^2}{p^2}}\left(1-\frac{p^2}{8m^2}\right)+O(p^2),\end{aligned} \tag{195}$$

with a short Taylor series expansion one obtains

$$\begin{aligned}B_f(p^2;m,m)&=2-\ln\left(\frac{m^2}{\bar{\mu}^2}\right)-2\sqrt{\frac{4m^2}{p^2}}\arctan\left(\sqrt{\frac{p^2}{4m^2}}\right)+O(p^2)\\&=-\ln\left(\frac{m^2}{\bar{\mu}^2}\right)+O(p^2)\end{aligned} \tag{196}$$

and, therefore, $B_f(0;m,m)=A_f(m)-1$. With a longer expansion one has

$$\begin{aligned}B'_f(m_A^2;m,m)&=-\frac{1}{p^2}+\frac{4m^2}{p^2\sqrt{(4m^2-p^2)p^2}}\arctan\left(\sqrt{\frac{p^2}{4m^2-p^2}}\right)\\&=-\frac{1}{p^2}+\frac{1}{p^2}\sqrt{\frac{4m^2}{p^2}}\left(1+\frac{p^2}{8m^2}\right)\sqrt{\frac{p^2}{4m^2}}\left(1+\frac{p^2}{24m^2}\right)+O(p^2)\\&=\frac{1}{6m^2}+O(p^2)\end{aligned} \tag{197}$$

and $B'_f(0;m,m)=1/(6m^2)$.

7 Helicity bilinears

The result we want to obtain in the end is a (differential) decay rate for the polarised W boson. On the pathway to this result we use all the elements we have calculated so far. As seen in Eq. (13), already on Born term level the absolute square of the matrix element depends on the polarisation of the W boson. If we intend not to measure the polarisation of the W boson, we can sum over all polarisations and use the completeness relation (20) to calculate the result. However, the angular decay distribution with respect to the initial movement axis of the W boson gives clues about the polarisation of the W. Therefore, our next step is to calculate the different so-called helicity bilinears. We use the helicity basis in the rest frame of the decaying W boson with the z axis as the initial movement direction of the W boson, given by

$$\varepsilon(q,\pm)=\frac{1}{\sqrt{2}}(0;\mp 1,-i,0),\quad \varepsilon(q,0)=(0;0,0,1),\quad \varepsilon(q,t)=(1;0,0,0). \tag{198}$$

At Born term level, up to a general factor $e^2q^2|V_{cb}|^2/s_W^2$ the helicity bilinears $H^{\lambda_1\lambda_2}$, are calculated by contracting the Eq. (13) with the product $\varepsilon^*_\mu(q,\lambda_1)\varepsilon_\nu(q,\lambda_2)$ as

$$q^2H^{\lambda_1\lambda_2}=H^{\mu\nu}\varepsilon_\mu(q,\lambda_1)\varepsilon^*_\nu(q,\lambda_2). \tag{199}$$

In order to calculate the scalar products, we have to make explicit the kinematics of the decay. In the so-called W *frame* the (common) axis of the two-body decay into up-type quark $Q(p_1)$ and down-type antiquark $\bar{q}(p_2)$ is turned against the initial movement direction of the W boson by the polar angle θ. One obtains

$$p_1=(E_1;|\vec{p}|\sin\theta,0,|\vec{p}|\cos\theta),\quad p_2=(E_2;-|\vec{p}|\sin\theta,0,-|\vec{p}|\cos\theta), \tag{200}$$

This results in the W frame read

$$\begin{aligned}
H^{tt}(\theta) &= \frac{1}{2}\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right),\\
H^{t0}(\theta) &= H^{0t}(\theta) \;=\; \frac{1}{2}(\mu_1-\mu_2)\sqrt{\lambda}\cos\theta,\\
H^{t\pm}(\theta) &= H^{\pm t}(\theta) \;=\; \mp\frac{1}{2\sqrt{2}}(\mu_1-\mu_2)\sqrt{\lambda}\sin\theta,\\
H^{00}(\theta) &= \frac{1}{2}\left(1-\mu_1-\mu_2-\lambda\cos^2\theta\right),\\
H^{0\pm}(\theta) &= H^{\pm 0}(\theta) \;=\; -\frac{1}{2\sqrt{2}}\sqrt{\lambda}(1\mp\sqrt{\lambda}\cos\theta)\sin\theta,\\
H^{\pm\pm}(\theta) &= \frac{1}{4}(1+\mu_1-\mu_2\mp\sqrt{\lambda}\cos\theta)(1-\mu_1+\mu_2\mp\sqrt{\lambda}\cos\theta),\\
H^{\pm\mp}(\theta) &= \frac{1}{4}\lambda\sin^2\theta.
\end{aligned} \tag{201}$$

7.1 Helicity bilinears from the tree corrections

At tree level this helicity bilinears can be obtained by replacing $|\mathcal{M}_0|^2$ by $|\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3|^2$ and extracting the hadron tensor, where the matrix elements $\mathcal{M}_i$ ($i=1,2,3$) correspond to the Feynman diagrams in Figure 2. We have calculated the helicity bilinears both for Feynman and unitary gauge and the results for the helicity bilinears $H^{00}(tree)$, $H^{\pm\pm}(tree)$ and $H^{\pm\mp}(tree)$ turn out to be exactly the same. The NLO tree contributions to the helicity bilinears up to a general factor $e^2|V_{cb}|^2/s_W^2 \times \alpha q^2/(4\pi\sqrt{\lambda})$ read

$$
\begin{aligned}
H^{00}(tree) \;&=\; \tfrac{1}{2}\Big[Q_c^2+Q_b^2+Q_W^2\Big]\sqrt{\lambda}(-\mu_1+\mu_1^2-\mu_2-2\mu_1\mu_2+\mu_2^2)\ell_\zeta \\
&+\tfrac{1}{2}\Big[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\Big](-\mu_1+\mu_1^2-\mu_2-2\mu_1\mu_2+\mu_2^2) \\
&\times(t_\zeta^\ell-2t_z^{\ell+}) \\
&-\tfrac{1}{2}\Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big](-\mu_1+\mu_1^2-\mu_2-2\mu_1\mu_2+\mu_2^2) \\
&\times(t_{\zeta W}^\ell+2t_{zW}^{\ell+}) \\
&-2\mu_1\Big[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)\Big](t_z^\ell+t_z^{-\ell}-t_z^{+\ell}) \\
&-\sqrt{\mu_1}\Big[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
&-2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)\Big](t_z^{-\ell}+t_z^{+\ell})+2\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
&-(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2\Big](t_{zW}^\ell+t_{zW}^{-\ell}-t_{zW}^{+\ell}) \\
&-\tfrac{1}{\sqrt{\mu_1}}(1+\mu_1-\mu_2) \\
&\quad\times\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2\Big](t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
&+\tfrac{1}{4}\Big[(4-25\mu_1-12\mu_1^2-9\mu_2+26\mu_1\mu_2-2\mu_1^2\mu_2+6\mu_2^2-3\mu_1\mu_2^2-\mu_2^3)Q_c^2 \\
&-(4+6\mu_1+6\mu_1^2-\mu_1^3-6\mu_2-2\mu_1\mu_2+9\mu_1^2\mu_2-4\mu_1\mu_2^2+2\mu_2^3)Q_b^2 \\
&-2(4+9\mu_1+3\mu_1^2+\mu_1^3-7\mu_2-7\mu_1\mu_2-2\mu_1^2\mu_2+2\mu_2^2-2\mu_1\mu_2^2+\mu_2^3) \\
&\times Q_W^2\Big]\ell_1-\tfrac{1}{4}\Big[(4-23\mu_1-16\mu_1^2+2\mu_1^3-7\mu_2+12\mu_1\mu_2-6\mu_1^2\mu_2+4\mu_2^2 \\
&+6\mu_1\mu_2^2-2\mu_2^3)Q_c^2-(4+5\mu_1+8\mu_1^2-2\mu_1^3-3\mu_2-4\mu_1\mu_2 \\
&+6\mu_1^2\mu_2-4\mu_2^2-6\mu_1\mu_2^2+2\mu_2^3)Q_b^2 \\
&-2(4+8\mu_1+5\mu_1^2-12\mu_2-12\mu_1\mu_2+7\mu_2^2)Q_W^2\Big]\ell_{1W} \\
&+\tfrac{1}{8}\Big[(16-67\mu_1-15\mu_1^2-3\mu_2+38\mu_1\mu_2-11\mu_2^2)Q_c^2 \\
&-(8+11\mu_1+11\mu_1^2-21\mu_2-38\mu_1\mu_2+15\mu_2^2)Q_b^2 \\
&-2(24+\mu_1+9\mu_1^2-31\mu_2-14\mu_1\mu_2+9\mu_2^2)Q_W^2\Big]\sqrt{\lambda}, \qquad (202)
\end{aligned}
$$

$$
\begin{aligned}
H^{++}(tree) \;=\; & -\tfrac{1}{2}\Big[Q_c^2+Q_b^2+Q_W^2\Big]\sqrt{\lambda}\left((1-\mu_1-\mu_2-\sqrt{\lambda})\ell_\zeta+2\sqrt{\lambda}\ell_+\right) \\
& -\tfrac{1}{2}\Big[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\Big]\Big((1-\mu_1-\mu_2)(t_\zeta^\ell-2t_z^{\ell+}) \\
& -\sqrt{\lambda}(t_\zeta^\ell-2t_z^{\ell-})\Big) \\
& +\tfrac{1}{2}\Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big]\Big((1-\mu_1-\mu_2)(t_{\zeta W}^\ell+2t_{zW}^{\ell+}) \\
& -\sqrt{\lambda}(t_{\zeta W}^\ell+2t_{zW}^{\ell-})\Big) \\
& +\Big[(1-2\mu_1-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_c^2-\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)\Big]t_z^\ell \\
& +\mu_1\Big[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)\Big](t_z^\ell+t_z^{-\ell}-t_z^{+\ell}) \\
& +\tfrac{1}{2}\sqrt{\mu_1}\Big[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)\Big](t_z^{-\ell}+t_z^{+\ell})-\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
& -(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2\Big](2t_{zW}^\ell+t_{zW}^{-\ell}-t_{zW}^{+\ell}) \\
& +\tfrac{1}{2\sqrt{\mu_1}}(1+\mu_1-\mu_2)\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
& -2(1+2\mu_1-\mu_2)Q_W^2\Big]\quad(t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& +\tfrac{1}{2}\Big[(1-\mu_1)(5-3\mu_1+4\mu_2)Q_c^2-(9-10\mu_1+\mu_1^2+6\mu_2-2\mu_1\mu_2)Q_b^2 \\
& -2(1-\mu_1)(5-\mu_1+\mu_2)Q_W^2\Big]\ell_0 \\
& -\tfrac{1}{4}\Big[\left(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2+(1+5\mu_1-\mu_2)\sqrt{\lambda}\right)Q_c^2 \\
& -\mu_1(4+7\mu_1-4\mu_2)Q_b^2-2(1-\mu_2)(1+5\mu_1-\mu_2)Q_W^2\Big]\ell_1 \\
& +\tfrac{1}{4}\Big[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2+8\sqrt{\lambda})Q_c^2 \\
& -(1+2\mu_1+8\mu_1^2+6\mu_2-8\mu_1\mu_2+8\sqrt{\lambda})Q_b^2 \\
& -2(2+3\mu_1+\mu_1^2-5\mu_2-6\mu_1\mu_2+5\mu_2^2+4\sqrt{\lambda})Q_W^2\Big]\ell_{1W} \\
& -\tfrac{1}{4}\Big[(17+7\mu_1-8\mu_2)Q_c^2-(13-3\mu_1+2\mu_2)Q_b^2-6(3+\mu_1-\mu_2)Q_W^2\Big] \\
& \times\lambda_-+\tfrac{1}{24}\Big[9(1+5\mu_1+\mu_2)Q_c^2+3(15+7\mu_1-29\mu_2)Q_b^2 \\
& +4(29-10\mu_1-\mu_1^2-34\mu_2+2\mu_1\mu_2-\mu_2^2)Q_W^2\Big]\sqrt{\lambda} \\
& +\tfrac{1}{4}\Big[Q_c^2-9Q_b^2+12Q_W^2\Big]\lambda, \qquad (203)
\end{aligned}
$$

$$
\begin{aligned}
H^{+-}(tree) \;=\; H^{-+}(tree)& \\
= \quad & -\mu_1\Big[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)\Big](t_z^\ell+t_z^{-\ell}-t_z^{+\ell}) \\
& -\tfrac{1}{2}\sqrt{\mu_1}\Big[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)\Big](t_z^{-\ell}+t_z^{+\ell})+\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
& -(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2\Big](t_{zW}^\ell+t_{zW}^{-\ell}-t_{zW}^{+\ell}) \\
& -\tfrac{1}{2\sqrt{\mu_1}}(1+\mu_1-\mu_2)\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2\Big] \\
& \times(t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& +\tfrac{1}{2}\Big[(1-6\mu_1-3\mu_1^2-2\mu_2+6\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -(1+\mu_1+2\mu_1^2-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_b^2-2(1+\mu_1-\mu_2)^2Q_W^2\Big]\ell_1 \\
& -\tfrac{1}{2}\Big[(1-6\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -(1+\mu_1+2\mu_1^2-\mu_2-2\mu_1\mu_2)Q_b^2 \\
& -2(1+\mu_1-2\mu_2)(1+\mu_1-\mu_2)Q_W^2\Big]\ell_{1W}+\tfrac{1}{2}\Big[2(1-5\mu_1-\mu_2)Q_c^2 \\
& -(1+3\mu_1-\mu_2)Q_b^2-2(3+\mu_1-3\mu_2)Q_W^2\Big]\sqrt{\lambda}, \qquad (204)
\end{aligned}
$$

$$
\begin{aligned}
H^{--}(tree)& \\
= \quad & -\tfrac{1}{2}\Big[Q_c^2+Q_b^2+Q_W^2\Big]\sqrt{\lambda}\,\Big((1-\mu_1-\mu_2+\sqrt{\lambda})\ell_\zeta-2\sqrt{\lambda}\ell_+\Big) \\
& -\tfrac{1}{2}\Big[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\Big]\Big((1-\mu_1-\mu_2)(t_\zeta^\ell-2t_z^{\ell+}) \\
& +\sqrt{\lambda}(t_\zeta^\ell-2t_z^{\ell-})\Big) \\
& +\tfrac{1}{2}\Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big]\Big((1-\mu_1-\mu_2)(t_{\zeta W}^\ell+2t_{zW}^{\ell+}) \\
& +\sqrt{\lambda}(t_{\zeta W}^\ell+2t_{zW}^{\ell-})\Big) \\
& -\Big[(1-2\mu_1-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_c^2-\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)\Big]t_z^\ell \\
& +\mu_1\Big[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)\Big](t_z^\ell+t_z^{-\ell}-t_z^{+\ell}) \\
& +\tfrac{1}{2}\sqrt{\mu_1}\Big[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)\Big](t_z^{-\ell}+t_z^{+\ell}) \\
& -\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\times (t_{zW}^{-\ell} - t_{zW}^{+\ell}) \\
&+ \tfrac{1}{2\sqrt{\mu_1}}(1+\mu_1-\mu_2)\Big[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) - 2(1+2\mu_1-\mu_2)Q_W^2\Big] \\
&\times (t_{zW}^{-\ell} + t_{zW}^{+\ell}) \\
&- \tfrac{1}{2}\Big[(1-\mu_1)(5-3\mu_1+4\mu_2)Q_c^2 - (9-10\mu_1+\mu_1^2+6\mu_2-2\mu_1\mu_2)Q_b^2 \\
&- 2(1-\mu_1)(5-\mu_1+\mu_2)Q_W^2\Big]\ell_0 \\
&- \tfrac{1}{4}\Big[\Big(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2-(1+5\mu_1-\mu_2)\sqrt{\lambda}\Big)\,Q_c^2 \\
&- \mu_1(4+7\mu_1-4\mu_2)Q_b^2 - 2(1-\mu_2)(1+5\mu_1-\mu_2)Q_W^2\Big]\ell_1 + \\
&+ \tfrac{1}{4}\Big[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2-8\sqrt{\lambda})Q_c^2 \\
&- (1+2\mu_1+8\mu_1^2+6\mu_2-8\mu_1\mu_2-8\sqrt{\lambda})Q_b^2 \\
&- 2(2+3\mu_1+\mu_1^2-5\mu_2-6\mu_1\mu_2+5\mu_2^2-4\sqrt{\lambda})Q_W^2\Big]\ell_{1W} \\
&+ \tfrac{1}{4}\Big[(17+7\mu_1-8\mu_2)Q_c^2 - (13-3\mu_1+2\mu_2)Q_b^2 - 6(3+\mu_1-\mu_2)Q_W^2\Big] \\
&\times \lambda_- + \tfrac{1}{24}\Big[9(1+5\mu_1+\mu_2)Q_c^2 + 3(15+7\mu_1-29\mu_2)Q_b^2 \\
&+ 4(29-10\mu_1-\mu_1^2-34\mu_2+2\mu_1\mu_2-\mu_2^2)Q_W^2\Big]\sqrt{\lambda} \\
&- \tfrac{1}{4}\Big[Q_c^2 - 9Q_b^2 + 12Q_W^2\Big]\lambda
\end{aligned} \tag{205}
$$

with $\lambda_\pm := (1 \pm \sqrt{\mu_1})^2 - \mu_2$ and $\lambda_-\lambda_+ = \lambda := \lambda(1,\mu_1,\mu_2)$.

7.2 Helicity bilinears from the loop corrections

Again up to a general factor $e^2 q^2 |V_{cb}|^2 / s_W^2 \times \alpha/(4\pi\sqrt{\lambda})$,
the NLO loop contributions read

$$
\begin{aligned}
H^{00}(loop) &= -\frac{1}{2}(-\mu_1+\mu_1^2-\mu_2-2\mu_1\mu_2+\mu_2^2)\sqrt{\lambda}V_- + \\
&\quad + \mu_1\mu_2\sqrt{\lambda}V_+ - \frac{1}{4}\lambda\sqrt{\lambda}(\mu_1 V_1 + \mu_2 V_2), \\
H^{++}(loop) &= \frac{1}{2}(1-\mu_1-\mu_2-\sqrt{\lambda})\sqrt{\lambda}V_- + \mu_1\mu_2\sqrt{\lambda}V_+, \\
H^{+-}(loop) &= H^{-+}(loop) \;=\; 0, \\
H^{--}(loop) &= \frac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\sqrt{\lambda}V_- + \mu_1\mu_2\sqrt{\lambda}V_+.
\end{aligned} \tag{206}
$$

7.3 IR singularities from tree contributions

Both tree and loop contributions contains IR singularities. According to the Lee–Nauenberg theorem the singularities contained in tree contributions have to cancel against singularities in the loop contributions to the helicity bilinears. For the example H^{--} that we look at in this chapter, the NLO tree contributions to the helicity bilinears read

$$\begin{aligned} & H^{--}(tree) \\ &= -\Big[Q_c^2+Q_b^2+Q_W^2\Big]\sqrt{\lambda}\left((1-\mu_1-\mu_2+\sqrt{\lambda})\ell_\zeta-2\sqrt{\lambda}\ell_+\right) \\ &\quad -\Big[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\Big]\Big((1-\mu_1-\mu_2)(t_\zeta^\ell-2t_z^{\ell+}) \\ &\quad +\sqrt{\lambda}(t_\zeta^\ell-2t_z^{\ell-})\Big)+\Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big] \\ &\quad \times\Big((1-\mu_1-\mu_2)(t_{\zeta W}^\ell+2t_{zW}^{\ell+})+\sqrt{\lambda}(t_{\zeta W}^\ell+2t_{zW}^{\ell-})\Big)+\ldots \end{aligned} \tag{207}$$

where $\lambda=\lambda(1,\mu_1,\mu_2)$. The IR singular parts are contained in ℓ_ζ, t_ζ^ℓ and $t_{\zeta W}^\ell$. In order to deal with the IR singularities in a consistent way, we need a way to regularise them. ℓ_ζ itself can be used as counter term, for the other two expressions in the first order tree contributions we defined as

$$t_\zeta^\ell=t_\zeta^{\ell *}+\ell_1\ell_\zeta,\qquad t_{\zeta W}^\ell=t_{\zeta W}^{\ell *}+\ell_{1W}\ell_\zeta \tag{208}$$

with

$$\begin{aligned} t_\zeta^{\ell *} &= I_\zeta^{\ell 0*}-\ell_1\ell_\zeta \;=\; I_\zeta^{\ell 0}-2I_\zeta^{\ell 1}-\ell_1\ell_\zeta \\ &= \mathrm{Li}_2\left(-\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(-\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) \\ &\quad -2\mathrm{Li}_2\left(\frac{\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+2\mathrm{Li}_2\left(\frac{-\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) \\ &\quad +2\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-2\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) \\ &\quad +\ell_1\ln\left(\frac{4\mu_1\left((1-\sqrt{\mu_1})^2-\mu_2\right)^2}{\lambda^2}\right), \end{aligned}$$

$$\begin{aligned} t_{\zeta W}^{\ell *} &= I_{\zeta W}^{\ell 0*}-\ell_{1W}\ell_\zeta \;=\; I_{\zeta W}^{\ell 0}-2I_{\zeta W}^{\ell 1}-\ell_{1W}\ell_\zeta \\ &= \mathrm{Li}_2\left(-\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(-\frac{1-\mu_1+\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\ &\quad -2\mathrm{Li}_2\left(\frac{\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+2\mathrm{Li}_2\left(\frac{-\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \end{aligned}$$

$$+ 2\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) - 2\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right)$$
$$+ \ell_{1W}\ln\left(\frac{4\mu_1\left((1-\sqrt{\mu_1})^2-\mu_2\right)^2}{\lambda^2}\right). \quad (209)$$

The IR singularities from the tree contributions helicity bilinears proportional to ℓ_ζ read

$$\begin{aligned}
&H_{\mathrm{IR}}^{--}(tree) \\
&= -(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[Q_c^2+Q_b^2+Q_W^2\Big]\sqrt{\lambda}\ell_\zeta \\
&\quad -(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\Big]\ell_1\ell_\zeta \\
&\quad +(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big]\ell_{1W}\ell_\zeta \qquad (210)
\end{aligned}$$

7.4 IR singularities from loop contributions

The helicity bilinears from the loop contributions read

$$H^{--}(loop) = (1-\mu_1-\mu_2+\sqrt{\lambda})\sqrt{\lambda}V_- + 2\mu_1\mu_2\sqrt{\lambda}V_+. \quad (211)$$

The vertex form factor is defined as

$$\begin{aligned}
V_- &= 2(m_c^2-m_b^2+m_W^2)Q_cC_f(m_c^2,m_b^2,m_W^2;m_A,m_W,m_c) \\
&\quad + 2(m_c^2-m_b^2-m_W^2)Q_bC_f(m_c^2,m_b^2,m_W^2;m_W,m_A,m_b) \\
&\quad + 2(m_c^2+m_b^2-m_W^2)Q_cQ_bC_f(m_c^2,m_b^2,m_W^2;m_c,m_b,m_A) \\
&\quad + \delta_{\mathrm{CKM}}^f + \delta Z_e^f - \frac{\delta s_W^f}{s_W} + \delta Z_{WW}^f + \delta Z_{cc}^{Lf} + \delta Z_{bb}^{Lf} \\
&\quad + \frac{2}{\lambda'}B_f(m_c^2;m_c,m_A)Q_c(Q_b+1)\Big(m_c^2(m_b^2-m_c^2+m_W^2)+\lambda'\Big) \\
&\quad + \frac{2}{\lambda'}B_f(m_b^2;m_b,m_A)Q_b(Q_c-1)\Big(m_b^2(m_c^2-m_b^2+m_W^2)+\lambda'\Big) \\
&\quad + \ldots \qquad (212)
\end{aligned}$$

where $\lambda' = \lambda(m_W^2, m_c^2, m_b^2)$.
In contrast to the UV singularities, IR singularities do not only reside in the main contribution V_-^0 but also in other form factors. For the loop contributions the source of IR singularities are the scalar three-point functions containing a photon in the loop for this we used Denner's compact formula (122) in order to split off the IR singular contribution.The results read as

$$
\begin{aligned}
&C_f(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) \\
&= \frac{-1}{\sqrt{\lambda'}}\Bigg[\ln\left(\frac{m_A^2}{m_W m_c}\right)\ln\left(\frac{m_W}{m_c}\right)+\ln\left(\frac{m_A^2 m_b^2}{\lambda'}\right) \\
&\quad\times\ln\left(\frac{m_W^2-m_c^2-m_b^2-\sqrt{\lambda'}}{m_W^2-m_c^2+m_b^2-\sqrt{\lambda'}}\right) \\
&\quad-\ln^2\left(\frac{2\sqrt{\lambda'}}{m_W^2-m_c^2-m_b^2-\sqrt{\lambda'}}\right)-\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda'}}{m_W^2-m_c^2-m_b^2-\sqrt{\lambda'}}\right) \\
&\quad+\ln^2\left(\frac{2\sqrt{\lambda'}}{m_W^2-m_c^2+m_b^2-\sqrt{\lambda'}}\right)+\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda'}}{m_W^2-m_c^2+m_b^2-\sqrt{\lambda'}}\right)\Bigg], \\
&C_f(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) \\
&= \frac{-1}{\sqrt{\lambda'}}\Bigg[\ln\left(\frac{m_A^2}{m_b m_W}\right)\ln\left(\frac{m_b}{m_W}\right)+\ln\left(\frac{m_A^2 m_c^2}{\lambda'}\right) \\
&\quad\times\ln\left(\frac{m_W^2+m_c^2-m_b^2+\sqrt{\lambda'}}{m_W^2-m_c^2-m_b^2+\sqrt{\lambda'}}\right) \\
&\quad-\ln^2\left(\frac{2\sqrt{\lambda'}}{m_W^2+m_c^2-m_b^2+\sqrt{\lambda'}}\right)-\mathrm{Li}_2\left(\frac{2\sqrt{\lambda'}}{m_W^2+m_c^2-m_b^2+\sqrt{\lambda'}}\right) \\
&\quad+\ln^2\left(\frac{2\sqrt{\lambda'}}{m_W^2-m_c^2-m_b^2+\sqrt{\lambda'}}\right)+\mathrm{Li}_2\left(\frac{2\sqrt{\lambda'}}{m_W^2-m_c^2-m_b^2+\sqrt{\lambda'}}\right)\Bigg], \\
&C_f(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) \\
&= \frac{-1}{\sqrt{\lambda'}}\Bigg[\ln\left(\frac{m_A^2}{m_c m_b}\right)\ln\left(\frac{m_c}{m_b}\right)+\ln\left(\frac{m_A^2 m_W^2}{\lambda'}\right) \\
&\quad\times\left(\ln\left(\frac{m_W^2-m_c^2+m_b^2+\sqrt{\lambda'}}{m_W^2+m_c^2-m_b^2-\sqrt{\lambda'}}\right)-i\pi\right)+\pi^2 \\
&\quad-\ln^2\left(\frac{2\sqrt{\lambda'}}{m_W^2-m_c^2+m_b^2+\sqrt{\lambda'}}\right)-\mathrm{Li}_2\left(\frac{2\sqrt{\lambda'}}{m_W^2-m_c^2+m_b^2+\sqrt{\lambda'}}\right) \\
&\quad+\ln^2\left(\frac{2\sqrt{\lambda'}}{m_W^2+m_c^2-m_b^2-\sqrt{\lambda'}}\right)+\mathrm{Li}_2\left(\frac{-2\sqrt{\lambda'}}{m_W^2+m_c^2-m_b^2-\sqrt{\lambda'}}\right)\Bigg].
\end{aligned}
\tag{213}
$$

There are different ways to transform the results further. By using the property

$$\begin{aligned}
&\ell_1 - \ell_{1W} \\
&= \ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) - \ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) \\
&= \ln\left(\frac{(1-\mu_1-\mu_2-\sqrt{\lambda})(1-\mu_1+\mu_2+\sqrt{\lambda})}{(1-\mu_1-\mu_2+\sqrt{\lambda})(1-\mu_1+\mu_2-\sqrt{\lambda})}\right) \\
&= \ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \qquad (214)
\end{aligned}$$

($\mu_1 = m_c^2/m_W^2$, $\mu_2 = m_b^2/m_W^2$) and the "symmetrisations"

$$\begin{aligned}
\ln\left(\frac{1-\mu_1-\mu_2\pm\sqrt{\lambda}}{2\sqrt{\mu_1\mu_2}}\right) &= \mp\frac{1}{2}\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) = \mp\frac{1}{2}\ell_1, \\
\ln\left(\frac{1-\mu_1+\mu_2\pm\sqrt{\lambda}}{2\sqrt{\mu_2}}\right) &= \mp\frac{1}{2}\ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) = \mp\frac{1}{2}\ell_{1W}, \\
\ln\left(\frac{1+\mu_1-\mu_2\pm\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) &= \mp\frac{1}{2}\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) = \mp\frac{1}{2}(\ell_1-\ell_{1W}), \\
& \qquad (215)
\end{aligned}$$

for the IR divergent parts of these three-point functions one obtains

$$\begin{aligned}
&\ln\left(\frac{m_A^2}{m_W m_c}\right)\ln\left(\frac{m_W}{m_c}\right) + \ln\left(\frac{m_A^2 m_b^2}{\lambda'}\right)\ln\left(\frac{m_W^2-m_c^2-m_b^2-\sqrt{\lambda'}}{m_W^2-m_c^2+m_b^2-\sqrt{\lambda'}}\right) \\
&= \ln\left(\frac{m_A^2}{m_W m_c}\right)\ln\left(\frac{m_W}{m_c}\right) + \ln\left(\frac{m_A^2 m_b^2}{\lambda'}\right)\left(\frac{1}{2}\ell_1 - \frac{1}{2}\ell_{1W} + \ln\left(\frac{m_c}{m_W}\right)\right) \\
&= \ln\left(\frac{\lambda'}{m_b^2 m_W m_c}\right)\ln\left(\frac{m_W}{m_c}\right) + \frac{1}{2}\ln\left(\frac{m_A^2 m_b^2}{\lambda'}\right)(\ell_1 - \ell_{1W}), \\
&\ln\left(\frac{m_A^2}{m_b m_W}\right)\ln\left(\frac{m_b}{m_W}\right) + \ln\left(\frac{m_A^2 m_c^2}{\lambda'}\right)\ln\left(\frac{m_W^2+m_c^2-m_b^2+\sqrt{\lambda'}}{m_W^2-m_c^2-m_b^2+\sqrt{\lambda'}}\right) \\
&= \ln\left(\frac{m_A^2}{m_b m_W}\right)\ln\left(\frac{m_b}{m_W}\right) + \ln\left(\frac{m_A^2 m_c^2}{\lambda'}\right) \\
&\quad \times\left(-\frac{1}{2}(\ell_1 - \ell_{1W}) + \frac{1}{2}\ell_1 - \ln\left(\frac{m_b}{m_W}\right)\right) \\
&= \ln\left(\frac{\lambda'}{m_c^2 m_b m_W}\right)\ln\left(\frac{m_b}{m_W}\right) + \frac{1}{2}\ln\left(\frac{m_A^2 m_c^2}{\lambda'}\right)\ell_{1W}, \\
&\ln\left(\frac{m_A^2}{m_c m_b}\right)\ln\left(\frac{m_c}{m_b}\right) + \ln\left(\frac{m_A^2 m_W^2}{\lambda'}\right)\left(\ln\left(\frac{m_W^2-m_c^2+m_b^2+\sqrt{\lambda'}}{m_W^2+m_c^2-m_b^2-\sqrt{\lambda'}}\right) - i\pi\right)
\end{aligned}$$

$$
\begin{aligned}
&= \ln\left(\frac{m_A^2}{m_c m_b}\right)\ln\left(\frac{m_c}{m_b}\right) + \ln\left(\frac{m_A^2 m_W^2}{\lambda'}\right) \\
&\quad\times\left(-\frac{1}{2}\ell_{1W} - \frac{1}{2}(\ell_1 - \ell_{1W}) + \ln\left(\frac{m_b}{m_c}\right) - i\pi\right) \\
&= \ln\left(\frac{\lambda'}{m_W^2 m_c m_b}\right)\ln\left(\frac{m_c}{m_b}\right) - \frac{1}{2}\ln\left(\frac{m_A^2 m_W^2}{\lambda'}\right)(\ell_1 + 2\pi i),
\end{aligned}
\tag{216}
$$

where the IR-divergent second part of these expressions can be expressed as combinations of a IR-finite term and $\ell_{1W}\ell_\zeta$, $\ell_1\ell_\zeta$. Using dilog relations one can also "symmetrise" the dilogs like

$$
\begin{aligned}
&\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \ln^2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
&\quad= \frac{\pi^2}{6} - \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \ln\left(\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{2\sqrt{\lambda}}\right) \\
&\qquad\times\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\lambda}}\right) \\
&\quad= \mathrm{Li}_1(1) - \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \frac{1}{4}\left(\ln^2\left(\frac{\mu_1\mu_2}{\lambda}\right) - \ell_1^2\right)
\end{aligned}
\tag{217}
$$

and

$$
\begin{aligned}
&-\mathrm{Li}_2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) - \ln^2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
&= -\frac{\pi^2}{6} + \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) - \ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\lambda}}\right) \\
&\quad\times\ln\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2\sqrt{\lambda}}\right) \\
&= -\mathrm{Li}_1(1) + \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) - \frac{1}{4}\left(\ln^2\left(\frac{\mu_1}{\lambda}\right) - (\ell_{1W}-\ell_1)^2\right).
\end{aligned}
\tag{218}
$$

Using

$$
\Lambda = \frac{m_A^2}{m_W^2}, \quad \mu_1 = \frac{m_c^2}{m_W^2}, \quad \mu_2 = \frac{m_b^2}{m_W^2}, \quad \frac{\lambda^2}{\Lambda\mu_1\mu_2} = \frac{\lambda'^2}{m_A^2 m_c^2 m_b^2 m_W^2}
\tag{219}
$$

one obtains ($\sqrt{\lambda'} = \sqrt{\lambda(m_c^2, m_b^2, m_W^2)}$ is replaced by the dimensionless $\sqrt{\lambda} = \sqrt{\lambda(1, \mu_1\,\mu_2)}$)

$$
C_f^{\mathrm{IR}}(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) = \frac{\ell_1 - \ell_{1W}}{2m_W^2\sqrt{\lambda}}\ell_\zeta,
$$

$$
\begin{aligned}
C_f^{\rm IR}(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) &= \frac{\ell_{1W}}{2m_W^2\sqrt{\lambda}}\ell_\zeta, \\
C_f^{\rm IR}(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) &= -\frac{\ell_1 + 2\pi i}{2m_W^2\sqrt{\lambda}}\ell_\zeta.
\end{aligned} \tag{220}
$$

For the IR divergent three-point functions contained only in the first (main) factor we obtain

$$
\begin{aligned}
&C_f(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) \\
&\qquad = C_f^*(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) + \frac{\ell_1 - \ell_{1W}}{2m_W^2\sqrt{\lambda}}\ell_\zeta, \\
&C_f(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) \\
&\qquad = C_f^*(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) + \frac{\ell_{1W}}{2m_W^2\sqrt{\lambda}}\ell_\zeta, \\
&C_f(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) \\
&\qquad = C_f^*(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) - \frac{\ell_1 + 2\pi i}{2m_W^2\sqrt{\lambda}}\ell_\zeta,
\end{aligned} \tag{221}
$$

where

$$
\begin{aligned}
&C_f^*(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) \\
&= -\frac{1}{m_W^2\sqrt{\lambda}}\Bigg[\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) - \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
&\quad - \frac{1}{4}\ell_1^2 + \frac{1}{4}\ell_{1W}^2 - \frac{1}{2}\ln\left(\frac{\mu_1}{\lambda}\right)(\ell_1 - \ell_{1W})\Bigg], \\
&C_f^*(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) \\
&= -\frac{1}{m_W^2\sqrt{\lambda}}\Bigg[\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) - \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
&\quad - \frac{1}{4}\ell_1^2 + \frac{1}{4}(\ell_1 - \ell_{1W})^2 - \frac{1}{2}\ln\left(\frac{\mu_2}{\lambda}\right)\ell_{1W}\Bigg], \\
&C_f^*(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) \\
&= -\frac{1}{m_W^2\sqrt{\lambda}}\Bigg[\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
&\quad + \frac{2\pi^2}{3} + \frac{1}{4}(\ell_1 - \ell_{1W})^2 + \frac{1}{4}\ell_{1W}^2 + \frac{1}{2}\ln\left(\frac{\mu_1\mu_2}{\lambda}\right)(\ell_1 + 2\pi i)\Bigg].
\end{aligned} \tag{222}
$$

Inserted into the form factor V_- one has

$$
\begin{aligned}
&\sqrt{\lambda}V_- \\
&= \sqrt{\lambda}V_-^* + (Q_c^2 + Q_b^2 + Q_W^2)\sqrt{\lambda}\ell_\zeta + (1+\mu_1-\mu_2)Q_cQ_W(\ell_1 - \ell_{1W})\ell_\zeta \\
&\quad - (1-\mu_1+\mu_2)Q_bQ_W\ell_{1W}\ell_\zeta + (1-\mu_1-\mu_2)Q_cQ_b(\ell_1 + 2\pi i)\ell_\zeta, \quad (223)
\end{aligned}
$$

where the starred form factor V_-^* contains the starred three-point functions while the other form factors remain unchanged.

However, it becomes clear by now that the pure form factors can cancel only the double logarithmic IR singularities like $\ell_{1W}\ell_\zeta$ and $\ell_1\ell_\zeta$ and not the single logarithmic ℓ_ζ. This cancellation will be achieved from the renormalisation of the main form factor V_-^0. To repeat, the renormalised main form factor reads

$$
V = V_-^0 + \delta Z_e - \frac{\delta s_W}{s_W} + \delta Z_{WW} + \delta Z_{cc}^L + \delta Z_{bb}^L. \quad (224)
$$

To begin with, the conter terms from the two fermions are given by

$$
\begin{aligned}
\delta Z_{ff}^L \;=\; & \frac{-e^2}{16 m_f^2 m_W^2 s_W^2}\Big[2\left(m_f^2 + 2(D-2)\left(Q_f^2 + (g_f^-)^2\right)m_W^2 s_W^2\right)A(m_f) \\
& -\left(m_f^2 + 4(D-2)(g_f^-)^2 m_W^2 s_W^2\right)\left(A(m_Z) - m_Z^2 B(m_f^2; m_f, m_Z)\right) \\
& -2m_f^2\Big(m_f^2 m_Z^2 - 4(D-2)(g_f^-)^2(2m_f^2 - m_Z^2)m_W^2 s_W^2 \\
& +8D g_f^- g_f^+ m_f^2 m_W^2 s_W^2\Big)B'(m_f^2; m_f, m_Z) \\
& -m_f^2\left(A(m_H) - m_H^2 B(m_f^2; m_f, m_H)\right) + 2m_f^4(4m_f^2 - m_H^2) \\
& \times B'(m_f^2; m_f, m_H) \\
& +2\sum_k |V_{fk}|^2\left(m_k^2 + (D-2)m_W^2\right)\Big(A(m_k) - A(m_W) \\
& +(m_f^2 - m_k^2 + m_W^2)B(m_f^2; m_k, m_W)\Big) \\
& -4m_f^2\sum_k |V_{fk}|^2\Big((m_f^2 - m_k^2 + m_W^2)m_k^2 \\
& -(D-2)(m_f^2 + m_k^2 - m_W^2)m_W^2\Big)B'(m_f^2; m_k, m_W) \\
& -32 m_f^4 m_W^2 s_W^2 Q_f^2 B'(m_f^2; m_f, m_A)\Big]. \quad (225)
\end{aligned}
$$

For the counter term of the W boson wave function we have

$$\begin{aligned}
\delta Z_{WW} &= \frac{e^2}{8(D-1)m_W^4(m_Z^2-m_W^2)}\Big[2m_Z^2\sum_\ell(m_\ell^2-m_\nu^2)\left(A(m_\ell)-A(m_\nu)\right) \\
&-2m_Z^2\sum_\ell\Big((m_\ell^2-m_\nu^2)^2+(D-2)m_W^4\Big)B(m_W^2;m_\nu,m_\ell) \\
&+2m_W^2m_Z^2\sum_\ell\Big((m_\ell^2-m_\nu^2)^2+(D-3)(m_\ell^2+m_\nu^2)m_W^2-(D-2)m_W^4\Big) \\
&\times B'(m_W^2;m_\nu,m_\ell) \\
&+2m_Z^2\sum_{i,j}|V_{ij}|^2(m_i^2-m_j^2)\left(A(m_i)-A(m_j)\right) \\
&-2m_Z^2\sum_{i,j}|V_{ij}|^2\Big((m_i^2-m_j^2)^2+(D-2)m_W^4\Big)B(m_W^2;m_i,m_j) \\
&+2m_W^2m_Z^2\sum_{i,j}|V_{ij}|^2\Big((m_i^2-m_j^2)^2 \\
&+(D-3)(m_i^2+m_j^2)m_W^2-(D-2)m_W^4\Big)B'(m_W^2;m_i,m_j) \\
&+(m_Z^2-m_W^2)\Big(m_Z^2A(m_W)-(m_Z^2+4(D-2)m_W^2)A(m_Z)\Big) \\
&+\Big(m_Z^6+2(2D-5)m_W^2m_Z^4-8(D-2)m_W^4m_Z^2+16(D-1)m_W^6\Big) \\
&\times B(m_W^2;m_W,m_Z) \\
&-m_W^2(m_Z^2-4m_W^2)\Big(m_Z^4+4(2D-3)m_W^2m_Z^2+4(D-1)m_W^4\Big) \\
&\times B'(m_W^2;m_W,m_Z) \\
&-m_Z^2(m_H^2-m_W^2)\left(A(m_H)-A(m_W)\right)+m_Z^2m_H^2(m_H^2-2m_W^2) \\
&\times B(m_W^2;m_W,m_H) \\
&-m_W^2m_Z^2\Big(m_H^4-4m_H^2m_W^2+4(D-1)m_W^4\Big)B'(m_W^2;m_W,m_H) \\
&+16(D-1)m_W^4(m_Z^2-m_W^2) \\
&\times\Big(B(m_W^2;m_W,m_A)+m_W^2B'(m_W^2;m_W,m_A)\Big)\Big] \qquad (226)
\end{aligned}$$

($\ell=e,\mu,\tau$, $\nu=\nu_\ell$, $i=u,c,t$, $j=d,s,b$). While $B(m_W^2;m_W,m_A)$ is IR finite, the very last term is IR singular,

$$B'_f(m_W^2;m_W,m_A)=\frac{1}{m_W^2}\left(\ln\left(\frac{m_W}{m_A}\right)-1\right). \qquad (227)$$

For the next contribution $\delta Z_e=-\delta Z_{AA}-\delta Z_{ZA}s_W/c_W$ we need δZ_{AA} and δZ_{ZA}. We obtain

$$
\begin{aligned}
&\delta Z_{AA} \\
&= \frac{-e^2}{2(D-1)}\Big[2\sum_f Q_f^2\left((D-2)B(m_A^2;m_f,m_f)+4m_f^2B'(m_A^2;m_f,m_f)\right) \\
&\quad -(D-1)\left(3B(m_A^2;m_W,m_W)+4m_W^2B'(m_A^2;m_W,m_W)\right)\Big], \qquad (228)
\end{aligned}
$$

$$
\begin{aligned}
\delta Z_{ZA} &= \frac{e^2}{2(D-1)m_Z^4c_Ws_W}\Big[4(D-2)m_Z^2\sum_f(g_f^-+g_f^+)c_Ws_WQ_fA(m_f) \\
&\quad -8m_Z^2\sum_f m_f^2(g_f^-+g_f^+)c_Ws_WQ_fB(m_A^2;m_f,m_f) \\
&\quad +2(D-2)\left(m_Z^2-2(D-1)m_W^2\right)A(m_W) \\
&\quad +4m_W^2\left(2(D-1)m_W^2+(D-2)m_Z^2\right)B(m_A^2;m_W,m_W)\Big]. \qquad (229)
\end{aligned}
$$

Note that $(g_f^-+g_f^+)c_Ws_W = I_f^3-2s_W^2Q_f$. Using in a later step

$$
\begin{aligned}
A(m) &= \frac{im^2\bar{\mu}^{-2\varepsilon}}{(4\pi)^2}\left(\frac{1}{\varepsilon}-\ln\left(\frac{m^2}{\bar{\mu}^2}\right)+1\right), \\
B(m_A^2;m,m) &= \frac{i\bar{\mu}^{-2\varepsilon}}{(4\pi)^2}\left(\frac{1}{\varepsilon}-\ln\left(\frac{m^2}{\bar{\mu}^2}\right)\right), \\
B'(m_A^2;m,m) &= \frac{i\bar{\mu}^{-2\varepsilon}}{(4\pi)^2}\left(\frac{1}{6m^2}\right) \qquad (230)
\end{aligned}
$$

where $\bar{\mu}$ is the $\overline{\mathrm{MS}}$ scale, one obtains

$$
\begin{aligned}
\delta Z_e &= \frac{e^2}{2(D-1)m_W^2m_Z^2}\Big[-4(D-2)m_Z^2\sum_f(I_f^3-2s_W^2Q_f)Q_fA(m_f) \\
&\quad +8m_Z^2\sum_f m_f^2(I_f^3-2s_W^2Q_f)Q_fB(m_A^2;m_f,m_f) \\
&\quad +2m_W^2m_Z^2\sum_f Q_f^2\left((D-2)B(m_A^2;m_f,m_f)+4m_f^2B'(m_A^2;m_f,m_f)\right) \\
&\quad -2(D-2)(m_Z^2-2(D-1)m_W^2)A(m_W) \qquad (231) \\
&\quad -4m_W^2\left((D-2)m_Z^2+2(D-1)m_W^2\right)B(m_A^2;m_W,m_W) \\
&\quad -(D-1)m_W^2m_Z^2\left(3B(m_A^2;m_W,m_W)+4m_W^2B'(m_A^2;m_W,m_W)\right)\Big] \\
&= \frac{-e^2\bar{\mu}^{-2\varepsilon}}{6(2\pi)^2}\left[\frac{1}{\varepsilon}\left(21-4\sum_f Q_f^2\right)+4\sum_f Q_f^2\ln\left(\frac{m_f^2}{\bar{\mu}^2}\right)-21\ln\left(\frac{m_W^2}{\bar{\mu}^2}\right)+2\right].
\end{aligned}
$$

In order to determine $\frac{\delta s_W}{s_W} = -\frac{c_W^2}{2s_W^2}(\delta Z_W^2 - \delta Z_Z^2)$ we have to calculate

$$
\begin{aligned}
\delta Z_W^2 \; = \; & \frac{e^2}{4(D-1)m_W^4(m_Z^2-m_W^2)} \\
& \times\Big[2m_Z^2\sum_{i,j}|V_{ij}|^2\Big((m_i^2-m_j^2)(A(m_i)-A(m_j)) \\
& -(D-2)m_W^2(A(m_i)+A(m_j))\Big) \\
& -2m_Z^2\sum_{i,j}|V_{ij}|^2\Big((m_i^2-m_j^2)^2-m_W^4 \\
& +(D-3)(m_i^2+m_j^2-m_W^2)m_W^2\Big)B(m_W^2;m_i,m_j) \\
& +2m_Z^2\sum_{\ell}\Big((m_\nu^2-m_\ell^2)(A(m_\nu)-A(m_\ell)) \\
& -(D-2)m_W^2(A(m_\nu)+A(m_\ell))\Big) \\
& -2m_Z^2\sum_{\ell}\Big((m_\nu^2-m_\ell^2)^2-m_W^4+(D-3)(m_\nu^2+m_\ell^2-m_W^2)m_W^2\Big) \\
& \times B(m_W^2;m_\nu,m_\ell) \\
& +m_Z^2\left(m_Z^2+(D-1)(4D-9)m_W^2\right)A(m_W) \\
& -m_Z^2\left(m_H^2-(D-1)m_W^2\right)(A(m_H)-A(m_W)) \\
& -\left(m_Z^2-(D-1)m_W^2\right)\left(m_Z^2+4(D-2)m_W^2\right)A(m_Z) \\
& -16(D-1)m_W^4(m_Z^2-m_W^2)B(m_W^2;m_W,m_A) \\
& +m_Z^2\left(m_H^4-4m_H^2m_W^2+4(D-1)m_W^4\right)B(m_W^2;m_W,m_H) \\
& +(m_Z^2-4m_W^2)\left(m_Z^4+4(2D-3)m_W^2m_Z^2+4(D-1)m_W^4\right) \\
& \times B(m_W^2;m_W,m_Z)\Big], \qquad (232)
\end{aligned}
$$

$$
\begin{aligned}
\delta Z_Z^2 \; = \; & \frac{e^2}{4(D-1)m_W^2m_Z^2(m_Z^2-m_W^2)} \\
& \times\Big[-8(D-2)m_W^2(m_Z^2-m_W^2)\sum_f\left((g_f^-)^2+(g_f^+)^2\right)A(m_f) \\
& -4m_W^2(m_Z^2-m_W^2)\sum_f\Big(2(D-1)(g_f^--g_f^+)^2m_f^2 \\
& -((g_f^-)^2+(g_f^+)^2)(4m_f^2+(D-2)m_Z^2)\Big)B(m_Z^2;m_f,m_f) \\
& +2(D-2)\left(m_Z^4-4m_W^2m_Z^2+4(D-1)m_W^4\right)A(m_W)
\end{aligned}
$$

$$
\begin{aligned}
&-m_Z^2\left(m_H^2-(D-1)m_Z^2\right)A(m_H)+m_Z^2\left(m_H^2+(D-3)m_Z^2\right)A(m_Z)\\
&+(m_Z^2-4m_W^2)\left(4(D-1)m_W^4+4(2D-3)m_W^2m_Z^2+m_Z^4\right)\\
&\times B(m_Z^2;m_W,m_W)\\
&+m_Z^2\left(m_H^4-4m_H^2m_Z^2+4(D-1)m_Z^4\right)B(m_Z^2;m_Z,m_H)\Big]. \qquad (233)
\end{aligned}
$$

We find that the IR singularities only reside in the counter term δZ_{bb}^L, δZ_{cc}^L and δZ_{WW}, i.e., in the leg contributions

$$
\begin{aligned}
\delta Z_{\mathrm{leg}} &:= \delta Z_{cc}^L+\delta Z_{bb}^L+\delta Z_{WW}\\
&= \frac{\alpha Q_c^2}{4\pi}\ln\left(\frac{m_c^2}{m_A^2}\right)+\frac{\alpha Q_b^2}{4\pi}\ln\left(\frac{m_b^2}{m_A^2}\right)+\frac{\alpha Q_W^2}{4\pi}\ln\left(\frac{m_W^2}{m_A^2}\right)+\dots
\end{aligned}
\qquad (234)
$$

Combining Eq. (234) with the unrenormalised main vertex factor V_-^0 and taking into account only the IR parts, one obtains

$$
\begin{aligned}
\mathrm{Re}\,V^{\mathrm{IR}} &= \mathrm{Re}(V_-^{0\mathrm{IR}}+\delta Z_{\mathrm{leg}}^{\mathrm{IR}})\\
&= \frac{\alpha}{2\pi}(m_c^2-m_b^2+m_W^2)Q_c\,\mathrm{Re}\,C_f^{\mathrm{IR}}(m_c^2,m_b^2,m_W^2;m_A,m_W,m_c)\\
&\quad+\frac{\alpha}{2\pi}(m_c^2-m_b^2-m_W^2)Q_b\,\mathrm{Re}\,C_f^{\mathrm{IR}}(m_c^2,m_b^2,m_W^2;m_W,m_A,m_b)\\
&\quad+\frac{\alpha}{2\pi}(m_c^2+m_b^2-m_W^2)Q_cQ_b\,\mathrm{Re}\,C_f^{\mathrm{IR}}(m_c^2,m_b^2,m_W^2;m_c,m_b,m_A)\\
&\quad+\frac{\alpha}{4\pi}(Q_c^2+Q_b^2+Q_W^2)\ell_\zeta\\
&= \frac{\alpha}{2\pi}m_W^2(1+\mu_1-\mu_2)Q_c\,\mathrm{Re}\,C_f^{\mathrm{IR}}(m_c^2,m_b^2,m_W^2;m_A,m_W,m_c)\\
&\quad-\frac{\alpha}{2\pi}m_W^2(1-\mu_1+\mu_2)Q_b\,\mathrm{Re}\,C_f^{\mathrm{IR}}(m_c^2,m_b^2,m_W^2;m_W,m_A,m_b)\\
&\quad-\frac{\alpha}{2\pi}m_W^2(1-\mu_1-\mu_2)Q_cQ_b\,\mathrm{Re}\,C_f^{\mathrm{IR}}(m_c^2,m_b^2,m_W^2;m_c,m_b,m_A)\\
&\quad+\frac{\alpha}{4\pi}(Q_c^2+Q_b^2+Q_W^2)\ell_\zeta\\
&= \frac{\alpha}{4\pi\sqrt{\lambda}}\Big[(1+\mu_1-\mu_2)Q_cQ_W(\ell_1-\ell_{1W})-(1-\mu_1+\mu_2)Q_bQ_W\ell_{1W}\\
&\quad+(1-\mu_1-\mu_2)Q_cQ_b\ell_1+\sqrt{\lambda}(Q_c^2+Q_b^2+Q_W^2)\Big]\ell_\zeta\\
&= \frac{\alpha}{4\pi\sqrt{\lambda}}\Big[\left((1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\right)\ell_1-\\
&\quad\left(Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\right)\ell_{1W}\ +\sqrt{\lambda}(Q_c^2+Q_b^2+Q_W^2)\Big]\ell_\zeta, \qquad (235)
\end{aligned}
$$

where we have made use of the different squared versions of $Q_c-Q_b=Q_W$,

$$
Q_cQ_b=\tfrac{1}{2}(Q_c^2+Q_b^2-Q_W^2)\qquad Q_c^2=Q_cQ_b+Q_cQ_W
$$

$$Q_cQ_W = \tfrac{1}{2}(Q_c^2 - Q_b^2 + Q_W^2) \qquad Q_b^2 = Q_cQ_b - Q_bQ_W$$
$$Q_bQ_W = \tfrac{1}{2}(Q_c^2 - Q_b^2 - Q_W^2) \qquad Q_W^2 = Q_cQ_W - Q_bQ_W \tag{236}$$

From Eq. (206) we read as

$$H^{--}(loop) \;=\; \frac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\sqrt{\lambda}V_- + \mu_1\mu_2\sqrt{\lambda}V_+. \tag{237}$$

The IR singular part of the loop contribution reads $(1-\mu_1-\mu_2-\sqrt{\lambda})\ell_\zeta$ times the expression

$$\begin{aligned}
&(1+\mu_1-\mu_2)Q_c(\ell_1-\ell_{1W}) + (1-\mu_1-\mu_2)Q_cQ_b\ell_1 - (1-\mu_1+\mu_2)Q_b\ell_{1W}\\
&\quad+\sqrt{\lambda}(Q_c^2+Q_b^2+Q_W^2)\\
&= \Big[(1-\mu_2)Q_c^2 - \mu_1(Q_b^2-Q_W^2)\Big]\ell_1 - \Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big]\ell_{1W}\\
&\quad+\sqrt{\lambda}\Big[Q_c^2+Q_b^2+Q_W^2\Big],
\end{aligned} \tag{238}$$

where the general factor $\alpha q^2/(4\pi\sqrt{\lambda})$ has been skipped. Adding up the first order tree and loop corrections for H^{--} helicity bilinears, we get

$$\begin{aligned}
H^{--}_{\rm IR}(\alpha) &= H^{--}_{\rm IR}(tree) + H^{--}_{\rm IR}(loop)\\
&= -\tfrac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[Q_c^2+Q_b^2+Q_W^2\Big]\sqrt{\lambda}\ell_\zeta +\\
&- \tfrac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\Big]\ell_1\ell_\zeta +\\
&+ \tfrac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big]\ell_{1W}\ell_\zeta +\\
&+ \tfrac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)\Big]\ell_1\ell_\zeta +\\
&- \tfrac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2\Big]\ell_{1W}\ell_\zeta +\\
&+ \tfrac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\Big[Q_c^2+Q_b^2+Q_W^2\Big]\sqrt{\lambda}\ell_\zeta \;=\; 0.
\end{aligned} \tag{239}$$

According to the Lee–Nauenberg theorem the cancellation of an IR singularity of the H^{--} works out. The same can be done for the H^{++} and H^{00}. The helicity bilinear $H^{+-} = H^{-+}$ does not contain any IR singularity, neither from the tree contribution nor from the loop contribution.

7.5 Form factors for the vertex correction

Analytical expressions for the form factors are given as V_-^*, V_+, V_1 and V_2,

$$V_-^* \;=\; \frac{m_c^2+m_b^2+2m_W^2-2(2Q_c-1)(2Q_b+1)m_Z^2s_W^2}{4m_W^2s_W^2}$$

$$
\begin{aligned}
&+\delta^f_{\mathrm{CKM}} + \delta Z^f_e - \frac{\delta s^f_W}{s_W} + \delta Z^f_{WW} + \delta Z^{Lf}_{cc} + \delta Z^{Lf}_{bb} \\
&+2(m_c^2 - m_b^2 + m_W^2) Q_c C_f^*(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) \\
&+2(m_c^2 - m_b^2 - m_W^2) Q_b C_f^*(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) \\
&+2(m_c^2 + m_b^2 - m_W^2) Q_c Q_b C_f^*(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) \\
&-\frac{C_f(m_c^2, m_b^2, m_W^2; m_Z, m_W, m_c)}{4\lambda' m_W^2 s_W^2} \\
&\times\Big(m_c^2 \left(m_b^2(3m_Z^2 - 4m_W^2)m_Z^2 + (4m_W^2 + m_Z^2)\lambda'\right) \\
&+2\Big(m_b^2(m_c^2 - m_b^2 + 3m_W^2)m_Z^4 + 2(m_c^2 - m_b^2 + m_W^2)m_W^2\lambda' \\
&+2\left(m_b^2(m_c^2 - m_b^2 + m_W^2) + 2\lambda'\right) m_W^2 m_Z^2\Big)(2Q_c s_W^2 - 1)\Big) \\
&-\frac{C_f(m_c^2, m_b^2, m_W^2; m_W, m_Z, m_b)}{4\lambda' m_W^2 s_W^2} \\
&\times\Big(m_b^2 \left(m_c^2(3m_Z^2 - 4m_W^2)m_Z^2 + (4m_W^2 + m_Z^2)\lambda'\right) \\
&-2\Big(m_c^2(m_b^2 - m_c^2 + 3m_W^2)m_Z^4 + 2(m_b^2 - m_c^2 + m_W^2)m_W^2\lambda' \\
&+2\left(m_c^2(m_b^2 - m_c^2 + m_W^2) + 2\lambda'\right) m_W^2 m_Z^2\Big)(2Q_b s_W^2 + 1)\Big) \\
&+\frac{m_Z^2 C_f(m_c^2, m_b^2, m_W^2; m_c, m_b, m_Z)}{2\lambda' m_W^2 s_W^2} \\
&\times(m_c^2 + m_b^2 - m_W^2 - m_Z^2)\Big(m_W^2 m_Z^2 + \lambda'\Big)(2Q_c s_W^2 - 1)(2Q_b s_W^2 + 1) \\
&+\frac{m_c^2 C_f(m_c^2, m_b^2, m_W^2; m_H, m_W, m_c)}{4\lambda' m_W^2 s_W^2}(m_H^2 - 4m_W^2)\Big(m_b^2 m_H^2 + \lambda'\Big) \\
&+\frac{m_b^2 C_f(m_c^2, m_b^2, m_W^2; m_W, m_H, m_b)}{4\lambda' m_W^2 s_W^2}(m_H^2 - 4m_W^2)\Big(m_c^2 m_H^2 + \lambda'\Big) \\
&-\frac{C_f(m_c^2, m_b^2, m_W^2; m_c, m_b, m_H)}{\lambda' m_W^2 s_W^2} m_c^2 m_b^2 \Big(m_W^2 m_H^2 + \lambda'\Big) + \\
&\frac{2}{\lambda'} B_f(m_c^2; m_c, m_A) Q_c(Q_b + 1)\Big(m_c^2(m_b^2 - m_c^2 + m_W^2) + \lambda'\Big) \\
&+\frac{2}{\lambda'} B_f(m_b^2; m_b, m_A) Q_b(Q_c - 1)\Big(m_b^2(m_c^2 - m_b^2 + m_W^2) + \lambda'\Big) \\
&+\frac{2m_W^2}{\lambda'}(m_c^2 + m_b^2 - m_W^2)(Q_c - Q_b) B_f(m_W^2; m_W, m_A) \\
&-\frac{B_f(m_c^2; m_c, m_Z)}{8\lambda' m_W^2 s_W^2}\Big(m_c^2(m_c^2 + m_b^2 - m_W^2)(4m_W^2 - 3m_Z^2) + \\
&+4\Big(2m_c^2(m_b^2 - m_c^2 + m_W^2)m_W^2 + 2m_W^2\lambda'
\end{aligned}
$$

$$
\begin{aligned}
&-(m_c^2-m_W^2)(m_c^2-m_b^2+m_W^2)m_Z^2\Big)(2Q_cs_W^2-1)\\
&-2\left(2m_c^2(m_b^2-m_c^2+m_W^2)+(m_c^2-m_b^2+m_W^2)m_Z^2+2\lambda'\right)\\
&\times m_Z^2(2Q_cs_W^2-1)(2Q_bs_W^2+1)\Big)\\
&-\frac{B_f(m_b^2;m_b,m_Z)}{8\lambda' m_W^2s_W^2}\Big(m_b^2(m_c^2+m_b^2-m_W^2)(4m_W^2-3m_Z^2)\\
&-4\Big(2m_W^2m_b^2(m_c^2-m_b^2+m_W^2)+2m_W^2\lambda'\\
&-(m_b^2-m_W^2)(m_b^2-m_c^2+m_W^2)m_Z^2\Big)(2Q_bs_W^2+1)\\
&-2\left(2m_b^2(m_c^2-m_b^2+m_W^2)+(m_b^2-m_c^2+m_W^2)m_Z^2+2\lambda'\right)\\
&\times m_Z^2(2Q_cs_W^2-1)(2Q_bs_W^2+1)\Big)\\
&+\frac{B_f(m_W^2;m_W,m_Z)}{8\lambda' m_W^2s_W^2}\Big(\left((m_c^2+m_b^2-m_W^2)m_W^2+\lambda'\right)(4m_W^2-3m_Z^2)\\
&-4\left(2(m_c^2+m_b^2-m_W^2)m_W^4+2(m_c^2-m_W^2)m_W^2m_Z^2+m_Z^2\lambda'\right)\\
&\times(2Q_cs_W^2-1)+4\Big(2(m_c^2+m_b^2-m_W^2)m_W^4\\
&+2(m_b^2-m_W^2)m_W^2m_Z^2+m_Z^2\lambda'\Big)(2Q_bs_W^2+1)\Big)\\
&-\frac{m_c^2B_f(m_c^2;m_c,m_H)}{8\lambda' m_W^2s_W^2}\\
&\times\Big((m_c^2+m_b^2-m_W^2)m_H^2+4m_c^2(m_b^2-m_c^2+m_W^2)+4\lambda'\Big)\\
&-\frac{m_b^2B_f(m_b^2;m_b,m_H)}{8\lambda' m_W^2s_W^2}\\
&\times\Big((m_c^2+m_b^2-m_W^2)m_H^2+4m_b^2(m_c^2-m_b^2+m_W^2)+4\lambda'\Big)\\
&+\frac{B_f(m_W^2;m_W,m_H)}{8\lambda' m_W^2s_W^2}(m_H^2-4m_W^2)\Big((m_c^2+m_b^2-m_W^2)m_W^2+\lambda'\Big)\\
&+\frac{B_f(m_c^2;m_b,m_W)}{4\lambda' m_W^2s_W^2}\Big(m_c^2m_b^2(m_H^2-3m_Z^2)+(m_b^2+6m_W^2)\lambda'\\
&+2m_c^2(3m_b^2-3m_c^2+5m_W^2)m_W^2\\
&+2m_c^2(m_b^2-m_c^2+3m_W^2)m_Z^2(2Q_b+1)s_W^2\Big)\\
&+\frac{B_f(m_b^2;m_W,m_c)}{4\lambda' m_W^2s_W^2}\Big(m_c^2m_b^2(m_H^2-3m_Z^2)+(m_c^2+6m_W^2)\lambda'\\
&+2m_b^2(3m_c^2-3m_b^2+5m_W^2)m_W^2
\end{aligned}
$$

$$
\begin{aligned}
& -2m_b^2(m_c^2 - m_b^2 + 3m_W^2)m_Z^2(2Q_c - 1)s_W^2\Big) \\
& +\frac{B_f(m_W^2; m_c, m_b)}{4\lambda' m_W^2 s_W^2}\Big(\Big(\lambda' - 4m_c^2 m_b^2 - 2(m_c^2 + m_b^2 - 2m_W^2)m_W^2\Big)m_W^2 \\
& +4m_W^4 m_Z^2(1 + Q_b - Q_c)s_W^2 - \Big(\lambda' - 2(m_c^2 + m_b^2 - m_Z^2)m_W^2\Big) \\
& \times m_Z^2(2Q_c - 1)(2Q_b + 1)s_W^2\Big),
\end{aligned}
\tag{240}
$$

$$
\begin{aligned}
V_+ \;=\; & -\frac{C_f(m_c^2, m_b^2, m_W^2; m_Z, m_W, m_c)}{4\lambda' s_W^2} \\
& \times\Big((4m_W^2 + m_Z^2)\lambda' - 2(m_c^2 + m_b^2 - m_W^2)m_W^2 m_Z^2 \\
& +(2m_c^2 + m_b^2 - 2m_W^2)m_Z^4 \\
& -2(m_b^2 - m_c^2 + m_W^2)(2m_W^2 + m_Z^2)m_Z^2(2Q_c s_W^2 - 1)\Big) \\
& -\frac{C_f(m_c^2, m_b^2, m_W^2; m_W, m_Z, m_b)}{4\lambda' s_W^2} \\
& \times\Big((4m_W^2 + m_Z^2)\lambda' - 2(m_c^2 + m_b^2 - m_W^2)m_W^2 m_Z^2 \\
& +(2m_b^2 + m_c^2 - 2m_W^2)m_Z^4 \\
& +2(m_c^2 - m_b^2 + m_W^2)(2m_W^2 + m_Z^2)m_Z^2(2Q_b s_W^2 + 1)\Big) \\
& +\frac{m_Z^2 C_f(m_c^2, m_b^2, m_W^2; m_c, m_b, m_Z)}{4\lambda' s_W^2}\Big(m_W^2 m_Z^2 + 2\lambda' + 2\lambda'(2Q_c s_W^2 - 1) \\
& -2\lambda'(2Q_b s_W^2 + 1) + 4m_W^2 m_Z^2(2Q_c s_W^2 - 1)(2Q_b s_W^2 + 1)\Big) \\
& -\frac{m_H^2 C_f(m_c^2, m_b^2, m_W^2; m_H, m_W, m_c)}{4\lambda' s_W^2} \\
& \times\Big(m_b^2 m_H^2 + \lambda' + 2(m_c^2 + m_b^2 - m_W^2)m_W^2\Big) \\
& -\frac{m_H^2 C_f(m_c^2, m_b^2, m_W^2; m_W, m_H, m_b)}{4\lambda' s_W^2} \\
& \times\Big(m_c^2 m_H^2 + \lambda' + 2(m_c^2 + m_b^2 - m_W^2)m_W^2\Big) \\
& -\frac{m_H^2 C_f(m_c^2, m_b^2, m_W^2; m_c, m_b, m_H)}{4\lambda' s_W^2} \\
& \times\Big(m_W^2 m_H^2 + 2\lambda' + 2(m_c^2 + m_b^2 - m_W^2)m_W^2\Big) \\
& -\frac{2m_W^2}{\lambda'}(m_c^2 - m_b^2 + m_W^2)Q_c(Q_b + 1)B_f(m_c^2; m_c, m_A) \\
& -\frac{2m_W^2}{\lambda'}(m_b^2 - m_c^2 + m_W^2)Q_b(Q_c - 1)B_f(m_b^2; m_b, m_A)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4m_W^4}{\lambda'}(Q_c-Q_b)B_f(m_W^2;m_W,m_A)\\
&-\frac{B_f(m_c^2;m_c,m_Z)}{4\lambda' s_W^2}\Big(m_c^2(4m_W^2-3m_Z^2)-2(m_c^2-m_b^2+m_W^2)\\
&\times(2m_W^2+m_Z^2)(2Q_cs_W^2-1)\\
&+2(m_c^2-m_b^2+m_W^2)m_Z^2(2Q_cs_W^2-1)(2Q_bs_W^2+1)\Big)\\
&-\frac{B_f(m_b^2;m_b,m_Z)}{4\lambda' s_W^2}\Big(m_b^2(4m_W^2-3m_Z^2)\\
&+2(m_b^2-m_c^2+m_W^2)(2m_W^2+m_Z^2)(2Q_bs_W^2+1)\\
&+2(m_b^2-m_c^2+m_W^2)m_Z^2(2Q_cs_W^2-1)(2Q_bs_W^2+1)\Big)\\
&+\frac{m_W^2B_f(m_W^2;m_W,m_Z)}{4\lambda' s_W^2}\Big(4m_W^2-5m_Z^2\\
&-4(2m_W^2+m_Z^2)(2Q_cs_W^2-1)+4(2m_W^2+m_Z^2)(2Q_bs_W^2+1)\Big)\\
&+\frac{m_c^2B_f(m_c^2;m_c,m_H)}{4\lambda' s_W^2}\Big(m_H^2+2(m_c^2-m_b^2+m_W^2)\Big)\\
&+\frac{m_b^2B_f(m_b^2;m_b,m_H)}{4\lambda' s_W^2}\Big(m_H^2+2(m_b^2-m_c^2+m_W^2)\Big)\\
&+\frac{m_W^2B_f(m_W^2;m_W,m_H)}{4\lambda' s_W^2}\Big(m_H^2-4m_W^2\Big)\\
&-\frac{B_f(m_c^2;m_b,m_W)}{4\lambda' s_W^2}\\
&\times\Big(m_c^2(m_H^2+3m_Z^2)+4(m_c^2-m_b^2+m_W^2)(Q_bm_Z^2s_W^2+m_W^2)\Big)\\
&-\frac{B_f(m_b^2;m_W,m_c)}{4\lambda' s_W^2}\\
&\times\Big(m_b^2(m_H^2+3m_Z^2)-4(m_b^2-m_c^2+m_W^2)(Q_cm_Z^2s_W^2-m_W^2)\Big)\\
&-\frac{B_f(m_W^2;m_c,m_b)}{4\lambda' s_W^2}\\
&\times\Big(m_W^2(m_H^2-m_Z^2)+2\lambda'+2(m_c^2+m_b^2+m_W^2)m_W^2\\
&-4m_Z^2m_W^2s_W^2(2Q_c-1)(2Q_b+1)\Big),
\end{aligned}
\tag{241}
$$

$$
\begin{aligned}
V_1 \;=\;& \frac{C_f(m_c^2,m_b^2,m_W^2;m_Z,m_W,m_c)}{2\lambda'^2s_W^2}\\
&\times\Big\{\lambda'(\lambda'+2m_b^2m_Z^2)(4m_W^2+m_Z^2)
\end{aligned}
$$

$$+(7\lambda' + 3m_b^2 m_Z^2)(m_W^2 - m_c^2 - m_b^2)m_W^2 m_Z^2$$

$$+2\Big[\lambda'\left(\lambda' - 2m_b^2(2m_W^2 - m_Z^2)\right)$$

$$-2(\lambda' + 3m_b^2 m_Z^2)(m_W^2 - m_c^2 + m_b^2)m_W^2\Big] m_Z^2(2Q_c s_W^2 - 1)\Big\}$$

$$+\frac{m_Z^2 C_f(m_c^2, m_b^2, m_W^2; m_W, m_Z, m_b)}{2\lambda'^2 s_W^2}$$

$$\times\Big\{m_b^2\left(\lambda'(2m_W^2 + m_Z^2) + 6m_c^2 m_W^2 m_Z^2\right) + 2\Big[\lambda'^2 - 4\lambda'(m_c^2 - m_W^2)m_W^2$$

$$+2\left(\lambda'(m_c^2 + m_W^2) + 3m_c^2(m_W^2 - m_c^2 + m_b^2)m_W^2\right) m_Z^2\Big](2Q_b s_W^2 + 1)\Big\}$$

$$+\frac{m_Z^2 C_f(m_c^2, m_b^2, m_W^2, m_c, m_b, m_Z)}{2\lambda'^2 s_W^2}$$

$$\times\Big\{m_b^2(m_W^2 - m_b^2 + m_c^2)(\lambda' + 3m_W^2 m_Z^2)$$

$$+2\lambda'(\lambda' + 2m_W^2 m_Z^2)(2Q_b s_W^2 + 1)$$

$$-2(m_W^2 - m_c^2 + m_b^2)(2\lambda' + 3m_W^2 m_Z^2)m_Z^2(2Q_c s_W^2 - 1)(2Q_b s_W^2 + 1)\Big\}$$

$$+\frac{m_H^2 C_f(m_c^2, m_b^2, m_W^2, m_H, m_W, m_c)}{2\lambda'^2 s_W^2}\times$$

$$\times\Big\{\lambda'\left(\lambda' + 4m_c^2(m_W^2 - m_c^2 + m_b^2) - 3(m_W^2 - m_c^2 - m_b^2)m_W^2\right)$$

$$-m_b^2\left(\lambda' - 3(m_c^2 - m_b^2)(m_W^2 - m_c^2 - m_b^2)\right) m_H^2\Big\}$$

$$-\frac{m_b^2 m_H^2 C_f(m_c^2, m_b^2, m_W^2; m_W, m_H, m_b)}{}$$

$$\times 2\lambda'^2 s_W^2\Big\{2\lambda'(m_W^2 - 2m_b^2 + 2m_c^2) - \left(\lambda' - 6m_c^2(m_c^2 - m_b^2)\right) m_H^2\Big\}$$

$$-\frac{3m_b^2 m_H^2 C_f(m_c^2, m_b^2, m_W^2; m_c, m_b, m_H)}{2\lambda'^2 s_W^2}(m_W^2 - m_b^2 + m_c^2)$$

$$\times\left(\lambda' + m_W^2 m_H^2\right) + \frac{4m_W^2}{\lambda'}Q_c(Q_b + 1)(m_W^2 - m_c^2 - m_b^2)$$

$$\times B_f(m_c^2; m_c, m_A) + \frac{8m_W^2}{\lambda'}(Q_c - 1)Q_b m_b^2 B_f(m_b^2; m_b, m_A)$$

$$-\frac{4m_W^2}{\lambda'}(Q_c - Q_b)(m_W^2 - m_c^2 + m_b^2)B_f(m_W^2; m_W, m_A)$$

$$+\frac{B_f(m_c^2; m_c, m_Z)}{4\lambda'^2 m_c^2 s_W^2}\Big\{\lambda' m_c^2\Big[m_c^2(8m_W^2 + m_Z^2) + 8(m_W^2 - m_b^2 + m_c^2)m_W^2$$

$$+3(m_W^2 - m_c^2 - m_b^2)m_Z^2\Big] - 4\lambda' m_c^2(m_W^2 - m_b^2 + m_c^2)m_Z^2(2Q_b s_W^2 + 1)$$

$$-4\Big[\lambda'(2m_c^2 - m_Z^2)(m_W^2 - m_c^2 - m_b^2)m_W^2$$

$$+\left(\lambda' m_c^2(m_c^2+m_b^2)+6m_c^2m_b^2(m_W^2-m_b^2+m_c^2)m_W^2\right)m_Z^2\Big]$$

$$\times(2Q_cs_W^2-1)+2\Big[2\lambda' m_c^2(m_W^2-m_c^2-m_b^2)$$

$$-\left(\lambda'(m_W^2-m_b^2+2m_c^2)-6m_c^2(m_W^2-m_c^2-m_b^2)m_W^2\right)m_Z^2\Big]$$

$$\times m_Z^2(2Q_cs_W^2-1)(2Q_bs_W^2+1)\Big\}$$

$$-\frac{B_f(m_b^2;m_b,m_Z)}{4\lambda'^2s_W^2}\Big\{\lambda' m_b^2m_Z^2$$

$$-4\Big[4\lambda' m_b^2m_W^2-\left(\lambda'(2m_b^2-m_W^2)-6m_b^2(m_W^2-m_b^2+m_c^2)m_W^2\right)m_Z^2\Big]$$

$$\times(2Q_bs_W^2+1)$$

$$-2\Big[4\lambda' m_b^2+(\lambda'+12m_b^2m_W^2)m_Z^2\Big]m_Z^2(2Q_cs_W^2-1)(2Q_bs_W^2+1)\Big\}$$

$$-\frac{B_f(m_W^2;m_W,m_Z)}{4\lambda'^2s_W^2}\Big\{\lambda' m_W^2(16m_W^2+m_Z^2)$$

$$+4\lambda'(m_W^2-m_b^2+m_c^2)m_W^2$$

$$+3\left(\lambda'(m_W^2-m_c^2+m_b^2)+4m_b^2(m_W^2-m_b^2+m_c^2)m_W^2\right)m_Z^2$$

$$-4\Big[2\lambda'(m_W^2-m_c^2+m_b^2)m_W^2+\left(\lambda'(m_c^2-m_b^2)+12m_b^2m_W^4\right)m_Z^2\Big]$$

$$\times(2Q_cs_W^2-1)+4\Big[2\lambda'(m_W^2-m_c^2+m_b^2)m_W^2$$

$$+\left(\lambda'(m_c^2-m_b^2)+6(m_W^2-m_c^2-m_b^2)m_W^4\right)m_Z^2\Big](2Q_bs_W^2+1)\Big\}$$

$$+\frac{B_f(m_c^2;m_c,m_H)}{4\lambda'^2s_W^2}\Big\{\lambda' m_c^2m_H^2+$$

$$-\lambda'(4m_c^2+m_H^2)(m_W^2-m_c^2-m_b^2)+12m_c^2m_b^2(m_W^2-m_b^2+m_c^2)m_H^2\Big\}$$

$$-\frac{m_b^2B_f(m_b^2;m_b,m_H)}{4\lambda'^2s_W^2}\Big\{8\lambda' m_b^2-3\left(\lambda'+4m_b^2(m_W^2-m_b^2+m_c^2)\right)m_H^2\Big\}$$

$$+\frac{B_f(m_W^2;m_W,m_H)}{4\lambda' s_W^2}\Big\{4(m_W^2-m_c^2+m_b^2)m_W^2-(m_c^2-m_b^2)m_H^2\Big\}$$

$$-\frac{B_f(m_c^2;m_b,m_W)}{2\lambda'^2m_c^2s_W^2}\Big\{\lambda'^2m_W^2+(\lambda'^2-12m_c^4m_W^4)(m_W^2-m_c^2+m_b^2)$$

$$+\lambda' m_c^2\left(2m_b^2m_W^2-(5m_W^2-m_c^2)(m_W^2-m_b^2+m_c^2)\right)$$

$$-m_c^2m_b^2(\lambda'+6m_c^2m_W^2)m_Z^2-m_c^2m_b^2\left(\lambda'-6m_c^2(m_c^2-m_b^2)\right)m_H^2$$

$$-4m_c^2\Big[\lambda'(m_c^2+m_W^2)+3m_c^2(m_W^2-m_c^2+m_b^2)m_W^2\Big]m_Z^2(2Q_b+1)s_W^2\Big\}$$

$$
\begin{aligned}
& -\frac{B_f(m_b^2; m_W, m_c)}{2\lambda'^2 s_W^2}\Big\{\lambda'^2 - \lambda'(3m_W^2 + m_b^2)(m_W^2 - m_c^2 + m_b^2) \\
& -12m_b^2(m_W^2 - m_c^2 + m_b^2)m_W^4 - m_b^2\left(2\lambda' + 3(m_W^2 - m_c^2 - m_b^2)m_W^2\right) \\
& \times m_Z^2 + m_b^2\left(\lambda' - 3(m_c^2 - m_b^2)(m_W^2 - m_c^2 - m_b^2)\right)m_H^2 \\
& -4m_b^2\Big[\lambda' - 3(m_W^2 - m_c^2 + m_b^2)m_W^2\Big]m_Z^2(2Q_c - 1)s_W^2\Big\} \\
& +\frac{B_f(m_W^2; m_c, m_b)}{2\lambda'^2 s_W^2} \\
& \times\Big\{\lambda'(\lambda' + 4m_W^4) + \left(\lambda'(m_c^2 + m_b^2) + 6m_W^6\right)(m_W^2 - m_c^2 + m_b^2) \\
& +3m_b^2(m_W^2 - m_b^2 + m_c^2)m_W^2(m_Z^2 - m_H^2) \\
& -6(m_W^2 - m_c^2 + m_b^2)m_W^4 m_Z^2(2Q_c - 1)s_W^2 \\
& +2\left(2\lambda' + 3(m_W^2 - m_c^2 + m_b^2)m_W^2\right)m_W^2 m_Z^2(2Q_b + 1)s_W^2 \\
& -(m_W^2 - m_c^2 + m_b^2)(\lambda' + 6m_W^2 m_Z^2 s_W^2)m_Z^2(2Q_c - 1)(2Q_b + 1)s_W^2\Big\} \\
& +\frac{\ln(m_c/\bar{\mu}) - 1}{\lambda' s_W^2}\Big\{m_c^4 - m_c^2 m_b^2 + 4m_c^2 m_W^2 - m_b^2 m_W^2 + m_W^4 \\
& +(m_W^2 - m_c^2 - m_b^2)m_Z^2(2Q_c - 1)(2Q_b + 1)s_W^2\Big\} \\
& -\frac{m_b^2(\ln(m_b/\bar{\mu}) - 1)}{\lambda' m_c^2 s_W^2} \\
& \times\Big\{m_b^2(m_W^2 - m_c^2 + m_b^2) - 2m_W^4 - 2m_c^2 m_Z^2(2Q_c - 1)(2Q_b + 1)s_W^2\Big\} \\
& -\frac{m_W^2(\ln(m_W/\bar{\mu}) - 1)}{\lambda' m_c^2 s_W^2} \\
& \times\Big\{2m_c^4 - m_c^2 m_b^2 + 2m_c^2 m_W^2 - m_b^4 - m_b^2 m_W^2 + 2m_W^4\Big\} \\
& -\frac{m_Z^2(2\ln(m_Z/\bar{\mu}) - 1)}{4\lambda' m_c^2 s_W^2} \\
& \times\Big\{m_c^2(m_W^2 - m_c^2 + m_b^2) - 4(m_W^2 - m_c^2 - m_b^2)m_W^2(2Q_c s_W^2 - 1) \\
& +8m_c^2 m_W^2(2Q_b s_W^2 + 1) \\
& +2(m_W^2 - m_b^2 + m_c^2)m_Z^2(2Q_c s_W^2 - 1)(2Q_b s_W^2 + 1)\Big\} \\
& -\frac{m_H^2(2\ln(m_H/\bar{\mu}) - 1)}{4\lambda' s_W^2}(m_W^2 - m_c^2 + m_b^2) - \frac{m_b^2 + 2m_W^2}{2m_c^2 s_W^2},
\end{aligned} \tag{242}
$$

$$
V_2 = \frac{m_Z^2 C_f(m_c^2, m_b^2, m_W^2, m_Z, m_W, m_c)}{2\lambda'^2 s_W^2}
$$

$$
\begin{aligned}
&\times\Big\{m_c^2\left(\lambda'(2m_W^2+m_Z^2)+6m_b^2m_W^2m_Z^2\right)-2\Big[\lambda'^2+4\lambda'(m_W^2-m_b^2)m_W^2\\
&+2\left(\lambda'(m_W^2+m_b^2)+3m_b^2(m_W^2-m_b^2+m_c^2)m_W^2\right)m_Z^2\Big](2Q_cs_W^2-1)\Big\}\\
&+\frac{C_f(m_c^2,m_b^2,m_W^2,m_W,m_Z,m_b)}{2\lambda'^2s_W^2}\Big\{\lambda'^2(4m_W^2+m_Z^2)+8\lambda'm_c^2m_W^2m_Z^2\\
&+7\lambda'(m_W^2-m_c^2-m_b^2)m_W^2m_Z^2+m_c^2\left(2\lambda'+3(m_W^2-m_c^2-m_b^2)m_W^2\right)\\
&\times m_Z^4-2\Big[\lambda'^2-2\lambda'm_c^2(4m_W^2-m_Z^2)-2\lambda'(m_W^2-m_c^2-m_b^2)m_W^2\\
&-6m_c^2(m_W^2-m_b^2+m_c^2)m_W^2m_Z^2\Big]m_Z^2(2Q_bs_W^2+1)\Big\}\\
&+\frac{m_Z^2C_f(m_c^2,m_b^2,m_W^2;m_c,m_b,m_Z)}{2\lambda'^2s_W^2}\\
&\times\Big\{m_c^2(m_W^2-m_c^2+m_b^2)(\lambda'+3m_W^2m_Z^2)\\
&-2\lambda'(\lambda'+2m_W^2m_Z^2)(2Q_cs_W^2-1)\\
&-2(m_W^2-m_b^2+m_c^2)(2\lambda'+3m_W^2m_Z^2)m_Z^2(2Q_bs_W^2+1)(2Q_cs_W^2-1)\Big\}\\
&-\frac{m_c^2m_H^2C_f(m_c^2,m_b^2,m_W^2;m_H,m_W,m_c)}{2\lambda'^2s_W^2}\\
&\times\Big\{2\lambda'(m_W^2-2m_c^2+2m_b^2)-\left(\lambda'+6m_b^2(m_c^2-m_b^2)\right)m_H^2\Big\}\\
&+\frac{m_H^2C_f(m_c^2,m_b^2,m_W^2;m_W,m_H,m_b)}{2\lambda'^2s_W^2}\\
&\times\Big\{\lambda'\Big[\lambda'+4m_b^2(m_W^2-m_b^2+m_c^2)-3(m_W^2-m_c^2-m_b^2)m_W^2\Big]\\
&-m_c^2\left(\lambda'+3(m_c^2-m_b^2)(m_W^2-m_c^2-m_b^2)\right)m_H^2\Big\}\\
&-\frac{3m_c^2m_H^2C_f(m_c^2,m_b^2,m_W^2;m_c,m_b,m_H)}{2\lambda'^2s_W^2}\\
&\times(m_W^2-m_c^2+m_b^2)(\lambda'+m_W^2m_H^2)\\
&+\frac{8m_W^2}{\lambda'}Q_c(Q_b+1)m_c^2B_f(m_c^2;m_c,m_A)\\
&+\frac{4m_W^2}{\lambda'}(Q_c-1)Q_b(m_W^2-m_c^2-m_b^2)B_f(m_b^2;m_b,m_A)\\
&-\frac{4m_W^2}{\lambda'}(Q_c-Q_b)(m_W^2-m_b^2+m_c^2)B_f(m_W^2;m_W,m_A)+\\
&-\frac{B_f(m_c^2;m_c,m_Z)}{4\lambda'^2s_W^2}\Big\{\lambda'm_c^2m_Z^2+4\Big[4\lambda'm_c^2m_W^2\\
&+\left(\lambda'(m_W^2-2m_c^2)+6m_c^2(m_W^2-m_c^2+m_b^2)m_W^2\right)m_Z^2\Big](2Q_cs_W^2-1)
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(4\lambda' m_c^2 + (\lambda' + 12 m_c^2 m_W^2) m_Z^2\right) m_Z^2 (2Q_b s_W^2 + 1)(2Q_c s_W^2 - 1)\Big\}\\
&+\frac{B_f(m_b^2; m_b, m_Z)}{4\lambda'^2 m_b^2 s_W^2}\Big\{\lambda' m_b^2\Big[m_b^2(8m_W^2 + m_Z^2) + 8(m_W^2 - m_c^2 + m_b^2) m_W^2\\
&+3(m_W^2 - m_c^2 - m_b^2) m_Z^2\Big] - 4\Big[\lambda'(m_Z^2 - 2m_b^2)(m_W^2 - m_c^2 - m_b^2) m_W^2\\
&-\left(\lambda' m_b^2 (m_c^2 + m_b^2) + 6 m_c^2 m_b^2 (m_W^2 - m_c^2 + m_b^2) m_W^2\right) m_Z^2\Big]\\
&\times(2Q_b s_W^2 + 1) + 4\lambda' m_b^2 (m_W^2 - m_c^2 + m_b^2) m_Z^2 (2Q_c s_W^2 - 1)\\
&+2\Big[2m_b^2(m_W^2 - m_c^2 - m_b^2)(\lambda' + 3m_W^2 m_Z^2)\\
&-\lambda'(m_W^2 - m_c^2 + 2m_b^2) m_Z^2\Big] m_Z^2 (2Q_b s_W^2 + 1)(2Q_c s_W^2 - 1)\Big\}\\
&-\frac{B_f(m_W^2; m_W, m_Z)}{4\lambda'^2 s_W^2}\\
&\times\Big\{\lambda' m_W^2 (16 m_W^2 + m_Z^2) + 4\lambda'(m_W^2 - m_c^2 + m_b^2) m_W^2\\
&+3\left(\lambda'(m_W^2 - m_b^2 + m_c^2) + 4m_c^2 (m_W^2 - m_c^2 + m_b^2) m_W^2\right) m_Z^2\\
&+4\Big[2\lambda'(m_W^2 - m_b^2 + m_c^2) m_W^2 - \left(\lambda'(m_c^2 - m_b^2) - 12 m_c^2 m_W^4\right) m_Z^2\Big]\\
&\times(2Q_b s_W^2 + 1) - 4\Big[2\lambda'(m_W^2 - m_b^2 + m_c^2) m_W^2
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\lambda'(m_c^2 - m_b^2) - 6(m_W^2 - m_c^2 - m_b^2) m_W^4\right) m_Z^2\Big](2Q_c s_W^2 - 1)\Big\}\\
&-\frac{m_c^2 B_f(m_c^2; m_c, m_H)}{4\lambda'^2 s_W^2}\\
&\times\Big\{\lambda'(8m_c^2 - 3m_H^2) - 12 m_c^2 (m_W^2 - m_c^2 + m_b^2) m_H^2\Big\}\\
&-\frac{B_f(m_b^2; m_b, m_H)}{4\lambda'^2 s_W^2}\Big\{\lambda'(4m_b^2 + m_H^2)(m_W^2 - m_c^2 - m_b^2)\\
&-m_b^2\left(\lambda' + 12 m_c^2 (m_W^2 - m_c^2 + m_b^2)\right) m_H^2\Big\}\\
&+\frac{B_f(m_W^2; m_W, m_H)}{4\lambda' s_W^2}\Big\{4(m_W^2 - m_b^2 + m_c^2) m_W^2 + (m_c^2 - m_b^2) m_H^2\Big\}\\
&-\frac{B_f(m_c^2; m_b, m_W)}{2\lambda'^2 s_W^2}\Big\{\lambda'^2 - \lambda'(3m_W^2 + m_c^2)(m_W^2 - m_b^2 + m_c^2)\\
&-12 m_c^2 (m_W^2 - m_b^2 + m_c^2) m_W^4\\
&-m_c^2\Big[2\lambda' + 3(m_W^2 - m_c^2 - m_b^2) m_W^2\Big] m_Z^2\\
&+m_c^2\left(\lambda' + 3(m_c^2 - m_b^2)(m_W^2 - m_c^2 - m_b^2)\right) m_H^2
\end{aligned}
$$

$$
\begin{aligned}
&+4m_c^2\left(\lambda' - 3(m_W^2 - m_b^2 + m_c^2)m_W^2\right)m_Z^2(2Q_b+1)s_W^2\Big\} + \\
&-\frac{B_f(m_b^2; m_W, m_c)}{2\lambda'^2 m_b^2 s_W^2}\Big\{\lambda'^2 m_W^2 + (\lambda'^2 - 12m_b^4 m_W^4)(m_W^2 - m_b^2 + m_c^2) \\
&+\lambda' m_b^2\left(2m_c^2 m_W^2 - (5m_W^2 - m_b^2)(m_W^2 - m_c^2 + m_b^2)\right) \\
&-m_c^2 m_b^2(\lambda' + 6m_b^2 m_W^2)m_Z^2 - m_c^2 m_b^2\left(\lambda' + 6m_b^2(m_c^2 - m_b^2)\right)m_H^2 + \\
&+4m_b^2\Big[\lambda'(m_W^2 + m_b^2) + 3m_b^2(m_W^2 - m_b^2 + m_c^2)m_W^2\Big]m_Z^2(2Q_c-1)s_W^2\Big\} \\
&+\frac{B_f(m_W^2; m_c, m_b)}{2\lambda'^2 s_W^2} \\
&\times\Big\{\lambda'(\lambda' + 4m_W^4) + \left(\lambda'(m_c^2 + m_b^2) + 6m_W^6\right)(m_W^2 - m_b^2 + m_c^2) \\
&+3m_c^2(m_W^2 - m_c^2 + m_b^2)m_W^2(m_Z^2 - m_H^2) \\
&+6(m_W^2 - m_b^2 + m_c^2)m_W^4 m_Z^2(2Q_b+1)s_W^2 \\
&-2\left(2\lambda' + 3(m_W^2 - m_b^2 + m_c^2)m_W^2\right)m_W^2 m_Z^2(2Q_c-1)s_W^2 \\
&-(m_W^2 - m_b^2 + m_c^2)(\lambda' + 6m_W^2 m_Z^2 s_W^2)m_Z^2(2Q_b+1)(2Q_c-1)s_W^2\Big\} \\
&+\frac{m_c^2(\ln(m_c/\bar{\mu}) - 1)}{\lambda' m_b^2 s_W^2}\Big\{2m_W^4 - m_c^2(m_W^2 - m_b^2 + m_c^2) \\
&+2m_b^2 m_Z^2(2Q_b+1)(2Q_c-1)s_W^2\Big\} \\
&+\frac{\ln(m_b/\bar{\mu}) - 1}{\lambda' s_W^2}\Big\{2m_b^2 m_W^2 + (m_W^2 + m_b^2)(m_W^2 - m_c^2 + m_b^2) \\
&+(m_W^2 - m_c^2 - m_b^2)m_Z^2(2Q_b+1)(2Q_c-1)s_W^2\Big\} \\
&+\frac{m_W^2(\ln(m_W/\bar{\mu}) - 1)}{\lambda' m_b^2 s_W^2} \\
&\times\Big\{\lambda' - 3m_b^2(m_W^2 - m_c^2 + m_b^2) - 3(m_W^2 - m_c^2 - m_b^2)m_W^2\Big\} \\
&-\frac{m_Z^2(2\ln(m_Z/\bar{\mu}) - 1)}{4\lambda' m_b^2 s_W^2}\Big\{m_b^2(m_W^2 - m_b^2 + m_c^2) + \\
&+4(m_W^2 - m_c^2 - m_b^2)m_W^2(2Q_b s_W^2 + 1) - 8m_b^2 m_W^2(2Q_c s_W^2 - 1) \\
&+2(m_W^2 - m_c^2 + m_b^2)m_Z^2(2Q_b s_W^2 + 1)(2Q_c s_W^2 - 1)\Big\} \\
&-\frac{m_H^2(2\ln(m_H/\bar{\mu}) - 1)}{4\lambda' s_W^2}(m_W^2 - m_b^2 + m_c^2) - \frac{2m_W^2 + m_c^2}{2m_b^2 s_W^2},
\end{aligned}
\tag{243}
$$

where $\lambda' = \lambda(m_W^2, m_c^2, m_b^2)$.

7.6 Analytic results for the $O(\alpha)$ helicity bilinears

Adding up first order tree and loop corrections for all five helicity bilinears, the IR singularities cancel and we obtain

$$
\begin{aligned}
& H^{00}(\alpha) \\
= \; & -\tfrac{1}{2}(-\mu_1+\mu_1^2-\mu_2-2\mu_1\mu_2+\mu_2^2)\sqrt{\lambda}V_-^* + \mu_1\mu_2\sqrt{\lambda}V_+ \\
& -\tfrac{1}{4}\sqrt{\lambda^3}(\mu_1 V_1+\mu_2 V_2) + \tfrac{1}{2}(-\mu_1+\mu_1^2-\mu_2-2\mu_1\mu_2+\mu_2^2)Q_c \\
& \times\Big[(1-\mu_1-\mu_2)Q_b+(1+\mu_1-\mu_2)Q_W\Big](t_\zeta^{\ell *}-2t_z^{\ell +}) \\
& -\tfrac{1}{2}(-\mu_1+\mu_1^2-\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W \\
& \times\Big[(1+\mu_1-\mu_2)Q_c+(1-\mu_1+\mu_2)Q_b\Big](t_{\zeta W}^{\ell *}+2t_{zW}^{\ell +}) \\
& -2\mu_1 Q_c\Big[(1+7\mu_1-\mu_2)Q_b+(1+3\mu_1-\mu_2)Q_W\Big](t_z^{-\ell}-t_z^{+\ell}+t_z^{\ell}) \\
& -\sqrt{\mu_1}Q_c\Big[(1-12\mu_1-5\mu_1^2-2\mu_2+4\mu_1\mu_2+\mu_2^2)Q_b \\
& +(1-8\mu_1-\mu_1^2-2\mu_2+\mu_2^2)Q_W\Big](t_z^{-\ell}+t_z^{+\ell}) \\
& -2Q_W\Big[(2+\mu_1+\mu_1^2-3\mu_2-2\mu_1\mu_2+\mu_2^2)Q_c \\
& -(4+3\mu_1+\mu_1^2-5\mu_2-2\mu_1\mu_2+\mu_2^2)Q_b\Big](t_{zW}^{-\ell}-t_{zW}^{+\ell}+t_{zW}^{\ell}) \\
& +\frac{1+\mu_1-\mu_2}{\sqrt{\mu_1}}Q_W\Big[(1+3\mu_1-\mu_2)Q_c-(3+5\mu_1-3\mu_2)Q_b\Big] \\
& \times(t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& +\tfrac{1}{4}\Big[\Big((4-\mu_2)\lambda-\mu_1(17+16\mu_1-32\mu_2+\mu_1\mu_2+5\mu_2^2)\Big)Q_c^2 \\
& -\Big((4-\mu_1+2\mu_2)\lambda+\mu_1(15+8\mu_2+5\mu_1\mu_2+\mu_2^2)\Big)Q_b^2 \\
& -2\Big((4+\mu_1+\mu_2)\lambda+\mu_1(16+\mu_1+5\mu_2-\mu_1\mu_2-\mu_2^2)\Big)Q_W^2\Big]\ell_1 \\
& -\tfrac{1}{4}\Big[\Big(2(2+\mu_1-\mu_2)\lambda-17\mu_1-16\mu_1^2+3\mu_2+20\mu_1\mu_2-4\mu_2^2\Big)Q_c^2 \\
& -\Big(2(2-\mu_1+\mu_2)\lambda+15\mu_1+3\mu_2+4\mu_1\mu_2-4\mu_2^2\Big)Q_b^2 \\
& -2(4+8\mu_1+5\mu_1^2-12\mu_2-12\mu_1\mu_2+7\mu_2^2)Q_W^2\Big]\ell_{1W} \\
& +\tfrac{1}{8}\Big[(16-67\mu_1-15\mu_1^2-3\mu_2+38\mu_1\mu_2-11\mu_2^2)Q_c^2 \\
& -(8+11\mu_1+11\mu_1^2-21\mu_2-38\mu_1\mu_2+15\mu_2^2)Q_b^2 \\
& -2(24+\mu_1+9\mu_1^2-31\mu_2-14\mu_1\mu_2+9\mu_2^2)Q_W^2\Big]\sqrt{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
&H^{++}(\alpha) \\
&= \quad \tfrac{1}{2}(1-\mu_1-\mu_2-\sqrt{\lambda})\sqrt{\lambda}V_-^* + \mu_1\mu_2\sqrt{\lambda}V_+ - (Q_c^2+Q_b^2+Q_W^2)\lambda\ell_+ \\
&\quad -\tfrac{1}{2}Q_c\Big[(1-\mu_1-\mu_2)Q_b + (1+\mu_1-\mu_2)Q_W\Big] \\
&\quad \times\Big((1-\mu_1-\mu_2)(t_\zeta^{\ell *}-2t_z^{\ell +}) - \sqrt{\lambda}(t_\zeta^{\ell *}-2t_z^{\ell -})\Big) \\
&\quad +\tfrac{1}{2}Q_W\Big[(1+\mu_1-\mu_2)Q_c + (1-\mu_1+\mu_2)Q_b\Big] \\
&\quad \times\Big((1-\mu_1-\mu_2)(t_{\zeta W}^{\ell *}+2t_{zW}^{\ell +}) - \sqrt{\lambda}(t_{\zeta W}^{\ell *}+2t_{zW}^{\ell -})\Big) \\
&\quad +Q_c\Big[(1-3\mu_1-\mu_1^2-2\mu_2+\mu_2^2)Q_b \\
&\quad +(1-\mu_1+\mu_1^2-2\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W\Big]t_z^\ell \\
&\quad +\mu_1 Q_c\Big[(1+7\mu_1-\mu_2)Q_b + (1+3\mu_1-\mu_2)Q_W\Big](t_z^{-\ell}-t_z^{+\ell}+t_z^\ell) \\
&\quad +\tfrac{1}{2}\sqrt{\mu_1}Q_c\Big[(1-12\mu_1-5\mu_1^2-2\mu_2+4\mu_1\mu_2+\mu_2^2)Q_b \\
&\quad +(1-8\mu_1-\mu_1^2-2\mu_2+\mu_2^2)Q_W\Big](t_z^{-\ell}+t_z^{+\ell}) \\
&\quad +Q_W\Big[(2+\mu_1+\mu_1^2-3\mu_2-2\mu_1\mu_2+\mu_2^2)Q_c \\
&\quad -(4+3\mu_1+\mu_1^2-5\mu_2-2\mu_1\mu_2+\mu_2^2)Q_b\Big](t_{zW}^{-\ell}-t_{zW}^{+\ell}+2t_{zW}^\ell) \\
&\quad -\frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}}Q_W\Big[(1+3\mu_1-\mu_2)Q_c-(3+5\mu_1-3\mu_2)Q_b\Big](t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
&\quad +\tfrac{1}{2}\Big[(1-\mu_1)(5-3\mu_1+4\mu_2)Q_c^2-(9-10\mu_1+\mu_1^2+6\mu_2-2\mu_1\mu_2)Q_b^2 \\
&\quad -2(1-\mu_1)(5-\mu_1+\mu_2)Q_W^2\Big]\ell_0 + \\
&\quad -\tfrac{1}{4}\Big[\Big(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2+(1+5\mu_1-\mu_2)\sqrt{\lambda}\Big)Q_c^2 \\
&\quad -\mu_1(4+7\mu_1-4\mu_2)Q_b^2-2(1+5\mu_1-\mu_2)(1-\mu_2)Q_W^2\Big]\ell_1 \\
&\quad +\tfrac{1}{4}\Big[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2+8\sqrt{\lambda})Q_c^2 \\
&\quad -(1+2\mu_1+8\mu_1^2+6\mu_2-8\mu_1\mu_2+8\sqrt{\lambda})Q_b^2 + \\
&\quad -2(2+3\mu_1+\mu_1^2-5\mu_2-6\mu_1\mu_2+5\mu_2^2+4\sqrt{\lambda})Q_W^2\Big]\ell_{1W} \\
&\quad -\tfrac{1}{4}\Big[(17+7\mu_1-8\mu_2)Q_c^2-(13-3\mu_1+2\mu_2)Q_b^2-6(3+\mu_1-\mu_2)Q_W^2\Big] \\
&\quad \times\lambda_- + \tfrac{1}{24}\Big[3(3+15\mu_1+3\mu_2+\sqrt{\lambda})Q_c^2+3(15+7\mu_1-29\mu_2-9\sqrt{\lambda}) \\
&\quad \times Q_b^2+4(29-10\mu_1-\mu_1^2-34\mu_2+2\mu_1\mu_2-\mu_2^2-9\sqrt{\lambda})Q_W^2\Big]\sqrt{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
H^{+-}(\alpha) \; &= \; H^{-+}(\alpha) \\
= \quad & -\mu_1 Q_c \Big[(1+7\mu_1-\mu_2)Q_b + (1+3\mu_1-\mu_2)Q_W\Big](t_z^{-\ell} - t_z^{+\ell} + t_z^{\ell}) \\
& -\tfrac{1}{2}\sqrt{\mu_1}Q_c\Big[(1-12\mu_1-5\mu_1^2-2\mu_2+4\mu_1\mu_2+\mu_2^2)Q_b \\
& +(1-8\mu_1-\mu_1^2-2\mu_2+\mu_2^2)Q_W\Big](t_z^{-\ell}+t_z^{+\ell}) \\
& -Q_W\Big[(2+\mu_1+\mu_1^2-3\mu_2-2\mu_1\mu_2+\mu_2^2)Q_c \\
& -(4+3\mu_1+\mu_1^2-5\mu_2-2\mu_1\mu_2+\mu_2^2)Q_b\Big](t_{zW}^{-\ell}-t_{zW}^{+\ell}+t_{zW}^{\ell}) \\
& +\frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}}Q_W\Big[(1+3\mu_1-\mu_2)Q_c-(3+5\mu_1-3\mu_2)Q_b\Big] \\
& \times(t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& +\tfrac{1}{2}\Big[(1-6\mu_1-3\mu_1^2-2\mu_2+6\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -(1+\mu_1+2\mu_1^2-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_b^2-2(1+\mu_1-\mu_2)^2Q_W^2\Big]\ell_1 \\
& -\tfrac{1}{2}\Big[(1-6\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -(1+\mu_1+2\mu_1^2-\mu_2-2\mu_1\mu_2)Q_b^2 \\
& -2(1+\mu_1-2\mu_2)(1+\mu_1-\mu_2)Q_W^2\Big]\ell_{1W}+\tfrac{1}{2}\Big[2(1-5\mu_1-\mu_2)Q_c^2 \\
& -(1+3\mu_1-\mu_2)Q_b^2-2(3+\mu_1-3\mu_2)Q_W^2\Big]\sqrt{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
& H^{--}(\alpha) \\
= \quad & \tfrac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\sqrt{\lambda}V_-^* + \mu_1\mu_2\sqrt{\lambda}V_+ + (Q_c^2+Q_b^2+Q_W^2)\lambda\ell_+ \\
& -\tfrac{1}{2}Q_c\Big[(1-\mu_1-\mu_2)Q_b+(1+\mu_1-\mu_2)Q_W\Big] \\
& \times\Big((1-\mu_1-\mu_2)(t_\zeta^{\ell *}-2t_z^{\ell +})+\sqrt{\lambda}(t_\zeta^{\ell *}-2t_z^{\ell -})\Big) \\
& +\tfrac{1}{2}Q_W\Big[(1+\mu_1-\mu_2)Q_c+(1-\mu_1+\mu_2)Q_b\Big] \\
& \times\Big((1-\mu_1-\mu_2)(t_{\zeta W}^{\ell *}+2t_{zW}^{\ell +})+\sqrt{\lambda}(t_{\zeta W}^{\ell *}+2t_{zW}^{\ell -})\Big) \\
& -Q_c\Big[(1-3\mu_1-\mu_1^2-2\mu_2+\mu_2^2)Q_b \\
& +(1-\mu_1+\mu_1^2-2\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W\Big]t_z^{\ell} \\
& +\mu_1 Q_c\Big[(1+7\mu_1-\mu_2)Q_b+(1+3\mu_1-\mu_2)Q_W\Big](t_z^{-\ell}-t_z^{+\ell}+t_z^{\ell}) \\
& +\tfrac{1}{2}\sqrt{\mu_1}Q_c\Big[(1-12\mu_1-5\mu_1^2-2\mu_2+4\mu_1\mu_2+\mu_2^2)Q_b
\end{aligned}
$$

$$+(1-8\mu_1-\mu_1^2-2\mu_2+\mu_2^2)Q_W\Big](t_z^{-\ell}+t_z^{+\ell})$$
$$+Q_W\Big[(2+\mu_1+\mu_1^2-3\mu_2-2\mu_1\mu_2+\mu_2^2)Q_c$$
$$-(4+3\mu_1+\mu_1^2-5\mu_2-2\mu_1\mu_2+\mu_2^2)Q_b\Big](t_{zW}^{-\ell}-t_{zW}^{+\ell})$$
$$-\frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}}Q_W\Big[(1+3\mu_1-\mu_2)Q_c-(3+5\mu_1-3\mu_2)Q_b\Big]$$
$$\times(t_{zW}^{-\ell}+t_{zW}^{+\ell})$$
$$-\tfrac{1}{2}\Big[(1-\mu_1)(5-3\mu_1+4\mu_2)Q_c^2-(9-10\mu_1+\mu_1^2+6\mu_2-2\mu_1\mu_2)Q_b^2$$
$$-2(1-\mu_1)(5-\mu_1+\mu_2)Q_W^2\Big]\ell_0$$
$$-\tfrac{1}{4}\Big[\left(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2-(1+5\mu_1-\mu_2)\sqrt{\lambda}\right)Q_c^2$$
$$-\mu_1(4+7\mu_1-4\mu_2)Q_b^2-2(1+5\mu_1-\mu_2)(1-\mu_2)Q_W^2\Big]\ell_1$$
$$+\tfrac{1}{4}\Big[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2-8\sqrt{\lambda})Q_c^2$$
$$-(1+2\mu_1+8\mu_1^2+6\mu_2-8\mu_1\mu_2-8\sqrt{\lambda})Q_b^2$$
$$-2(2+3\mu_1+\mu_1^2-5\mu_2-6\mu_1\mu_2+5\mu_2^2-4\sqrt{\lambda})Q_W^2\Big]\ell_{1W}$$
$$+\tfrac{1}{4}\Big[(17+7\mu_1-8\mu_2)Q_c^2-(13-3\mu_1+2\mu_2)Q_b^2-6(3+\mu_1-\mu_2)Q_W^2\Big]$$
$$\times\lambda_-+\tfrac{1}{24}\Big[3(3+15\mu_1+3\mu_2-\sqrt{\lambda})Q_c^2+3(15+7\mu_1-29\mu_2+9\sqrt{\lambda})$$
$$\times Q_b^2+4(29-10\mu_1-\mu_1^2-34\mu_2+2\mu_1\mu_2-\mu_2^2+9\sqrt{\lambda})Q_W^2\Big]\sqrt{\lambda}. \quad (244)$$

8 Collinear singularities

8.1 Investigation of collinear singularities

In order to calculate the helicity amplitudes in the massless limit $\mu_1, \mu_2 \to 0$, we found that the tree contribution to H^{--} results in collinear divergences results as,

$$\begin{aligned} &H^{--*}(tree)\\ &= \frac{1}{2}\left(7-\frac{10\pi^2}{6}-\ln\mu_1-2\ln\mu_2-3\ln^2\mu_1-2\ln\mu_1\ln\mu_2\right)Q_c^2\\ &\quad+\frac{1}{2}\left(\frac{3}{2}-\pi^2-2\ln\mu_1-\ln\mu_2-2\ln\mu_1\ln\mu_2-3\ln^2\mu_2\right)Q_b^2\\ &\quad+\frac{1}{2}\left(\frac{23}{3}-2\ln\mu_1-2\ln\mu_2\right)Q_W^2+O(\sqrt{\mu_i}). \end{aligned} \quad (245)$$

H^{--} is the helicity component that contains $2V_-$ and is contributed also by the Born term. Therefore, we presume that corresponding collinear logarithms are found also in the form factor. In order to investigate this possibility, I have expanded the subtracted results found in Eqs. (222) in μ_1 and obtained

$$\begin{aligned}
&C_f^*(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) \\
&= \frac{1}{4(1-\mu_2)}\Big[\Big(\ln\mu_1 - 2\ln(1-\mu_2)\Big)\Big(3\ln\mu_1 + 2\ln\mu_2 - 6\ln(1-\mu_2)\Big) \\
&\quad -4\mathrm{Li}_2(\mu_2)\Big] + O(\mu_1),
\end{aligned}$$

$$\begin{aligned}
&C_f^*(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) \\
&= \frac{\ln\mu_2}{4(1-\mu_2)}\left(2\ln\mu_1 + 3\ln\mu_2 - 8\ln(1-\mu_2)\right) + O(\mu_1),
\end{aligned}$$

$$\begin{aligned}
&C_f^*(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) \\
&= \frac{-1}{12(1-\mu_2)}\Big[9\left(\ln\mu_1 - 2\ln(1-\mu_2)\right)^2 + 12\left(\ln\mu_1 - 2\ln(1-\mu_2)\right)\ln\mu_2 \\
&\quad +9\ln^2\mu_2 + 12\pi i\left(\ln\mu_1 + \ln\mu_2 - 2\ln(1-\mu_2)\right) + 8\pi^2 + 12\mathrm{Li}_2(\mu_2)\Big] \\
&\quad +O(\mu_1). \qquad (246)
\end{aligned}$$

Finally, expanding the combination found in V_- into μ_2, one has

$$\begin{aligned}
&2(1-\mu_2)Q_cQ_WC_f^*(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) \\
&-2(1+\mu_2)Q_bQ_WC_f^*(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) \\
&-2(1-\mu_2)Q_cQ_bC_f^*(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) \\
&= \frac{4\pi^2}{3}Q_cQ_b + 2\pi iQ_cQ_b\ln(\mu_1\mu_2) + \frac{3}{2}Q_c^2\ln^2\mu_1 \\
&\quad +\frac{3}{2}Q_b^2\ln^2\mu_2 + (Q_c^2+Q_b^2)\ln\mu_1\ln\mu_2 + O(\mu_i). \qquad (247)
\end{aligned}$$

In order to deal with the single logarithms, we turn back to the vertex corrections the V_-^*, V_+, V_1 and V_2 contain these contributions. In the mass zero limit the contributions are as follows:

$$\begin{aligned}
V_-^* &= 2Q_c^2B_f(m_c^2; m_c, m_A) + 2Q_b^2B_f(m_b^2; m_b, m_A) \\
&\quad - 2Q_W^2B_f(m_W^2; m_W, m_A) + \ldots, \\
V_+ &= -2Q_c^2B_f(m_c^2, m_c, m_A) - 2Q_b^2B_f(m_b^2; m_b, m_A) \\
&\quad + 4Q_W^2B_f(m_W^2; m_W, m_A) + \ldots, \\
V_1 &= 4Q_c^2B_f(m_c^2; m_c, m_A) - 4Q_W^2B_f(m_W^2; m_W, m_A), \\
V_2 &= 4Q_b^2B_f(m_b^2; m_b, m_A) - 4Q_W^2B_f(m_W^2; m_W, m_A). \qquad (248)
\end{aligned}$$

As these collinear singularities should be the only ones for the IR-finite form factors V_+, V_1 and V_2. In order to study the dependence on μ_1 and μ_2. One obtains $V_1 \to -4Q_c^2 \ln \mu_1$, $V_2 \to -4Q_b^2 \ln \mu_2$, $V_+ \to 2Q_c^2 \ln \mu_1 + 2Q_b^2 \ln \mu_2$ and

$$\begin{aligned} V_- \to Q_c^2 &\left(\frac{3}{2} \ln^2 \mu_1 + \ln \mu_1 \ln \mu_2 - 2 \ln \mu_1 \right) \\ +Q_b &\left(\ln \mu_1 \ln \mu_2 + \frac{3}{2} \ln^2 \mu_2 - 2 \ln \mu_2 \right). \end{aligned} \tag{249}$$

Now finding the collinear singularities in the finite parts of the renormalisation factors. Following the principle that the collinear singularities are closely related to the IR singularities, we looked for the (IR-subtracted) contributions $2m^2Q^2B_f'^*(m^2; m, m_A)$ from δZ_{ff}^L, $f = c, b$ and δZ_{WW} given by

$$\begin{aligned} 2m^2Q^2B_f'^*(m^2; m, m_A) &= \left(\ln \left(\frac{m^2 m_b^2 m_c^2 m_W^2}{\lambda(m_c^2, m_b^2, m_W^2)^2} \right) - 2 \right) Q^2 \\ &= \big(\ln \mu_i + \ln \mu_1 + \ln \mu_2 - 4 \ln(1 - \mu_2) \\ &\quad - 2 + O(\mu_1) \big) Q^2, i = 1, 2, W \end{aligned} \tag{250}$$

will contribute. Therefore, one is left with

$$\begin{aligned} \delta Z_{ii}^L / e^2 &= -\frac{1}{2}(Q_i^2 + g_i^{-2}) \left(A_f(m_i) - 1 \right) \\ &\quad - \frac{1}{2} g_i^{-2} \frac{m_Z^2}{m_i^2} \left(B_f(m_i^2; m_i, m_Z) - A_f(m_Z) \right) \\ &\quad + 2m_i^2 Q_i^2 B_f'^*(m_i^2; m_i, m_A) + \dots \\ &= \frac{1}{2}(Q_i^2 + g_i^{-2}) \ln \left(\frac{m_i^2}{\mu^2} \right) - \frac{1}{2} g_i^{-2} \ln \left(\frac{m_i^2}{\mu^2} \right) + Q_i^2 \Bigg(\ln \left(\frac{m_i^2}{\mu^2} \right) \\ &\quad + \ln \left(\frac{m_b^2}{\mu^2} \right) + \ln \left(\frac{m_c^2}{\mu^2} \right) \Bigg) + \dots \\ &= \frac{1}{2} Q_i^2 \left(3 \ln \left(\frac{m_i^2}{\mu^2} \right) + 2 \ln \left(\frac{m_b^2}{\mu^2} \right) + 2 \ln \left(\frac{m_c^2}{\mu^2} \right) \right) + \dots \end{aligned} \tag{251}$$

where the ellipsis stands for non-collinear parts.

8.2 Subtraction of collinearities

For the subtraction of collinear singularities, we proceed in the same way as we have done this for IR singularities, namely that each of the parts the helicity bilinears is composed of, is independently collinear finite. This means that we have to subtract the collinear singularities listed in Table 1. The remainder will then automatically be finite. The procedure is as follows. For the vertex contribution we have calculated the collinear subtracted three-point function as,

$$
\begin{aligned}
&C_f^*(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) \\
&= C_f^c(m_c^2, m_b^2, m_W^2; m_A, m_W, m_c) + \frac{1}{\sqrt{\lambda}}\left(\frac{3}{4}\ln^2\mu_1 + \frac{1}{2}\ln\mu_1\ln\mu_2\right) \\
&C_f^*(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) \\
&= C_f^c(m_c^2, m_b^2, m_W^2; m_W, m_A, m_b) + \frac{1}{\sqrt{\lambda}}\left(\frac{1}{2}\ln\mu_1\ln\mu_2 + \frac{3}{4}\ln^2\mu_2\right) \\
&C_f^*(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) \\
&= C_f^c(m_c^2, m_b^2, m_W^2; m_c, m_b, m_A) - \frac{1}{\sqrt{\lambda}}\left(\frac{3}{4}\ln^2\mu_1 + \ln\mu_1\ln\mu_2 + \frac{3}{4}\ln^2\mu_2\right)
\end{aligned} \tag{252}
$$

For the single logarithms in the vertex,

$$
\begin{aligned}
2V_-^* \;=\; & \ldots + \frac{4}{\lambda}\left(\lambda + (1-\mu_1+\mu_2)\mu_1\right) Q_c^2 B_f(m_c^2; m_c, m_A) \\
& + \frac{4}{\lambda}\left(\lambda + (1+\mu_1-\mu_2)\mu_2\right) Q_b^2 B_f(m_b^2; m_b, m_A) \\
& - \frac{4}{\lambda}(1-\mu_1-\mu_2) Q_W^2 B_f(m_W^2; m_W, m_A) + \ldots
\end{aligned} \tag{253}
$$

a replacement of $B_f(m_i^2; m_i, m_A) = 2 - \ln(m_i^2/\mu^2)$ by[1]

$$
B_f^c(m_i^2; m_i, m_A) = B_f(m_i^2; m_i, m_A) + \ln\mu_i = 2 - \ln(m_W^2/\mu^2) \tag{254}
$$

will account for these collinear singularities for $i = c, b$, being trivial for $i = W$. Going into more details, we have analysed the collinear singular behaviour of all the dilogarithmic and logarithmic terms separately to obtain

$$
\begin{aligned}
t_\zeta^{\ell *} &= t_\zeta^{\ell c} + \frac{3}{2}\ln^2\mu_1 + 2\ln\mu_1\ln\mu_2 + \frac{1}{2}\ln^2\mu_2 \\
t_{\zeta W}^{\ell *} &= t_{\zeta W}^{\ell c} + \ln\mu_1\ln\mu_2 + \frac{1}{2}\ln^2\mu_2
\end{aligned} \tag{255}
$$

and

$$
\begin{aligned}
-\frac{1}{2}\ln^2\mu_2 &\approx t_z^{\ell +} \qquad t_{zW}^{\ell +} \approx \frac{1}{2}\ln^2\mu_2 \\
\frac{1}{4}\ln^2\mu_1 - \frac{1}{2}\ln^2\mu_2 &\approx t_z^{\ell -} \qquad t_{zW}^{\ell -} \approx \frac{1}{2}\ln^2\mu_2 \\
\frac{1}{4}\ln^2\mu_1 &\approx t_z^{\ell} \qquad t_{zW}^{\ell} = 0
\end{aligned} \tag{256}
$$

[1] I use the upper label "c" for the collinearly subtracted terms that are IR finite.

$$-\ln\Lambda-\ln\mu_1-\ln\mu_2\approx\ell_\zeta\qquad \ell_0\approx-\frac12\ln\mu_2$$
$$\ln\mu_1+\ln\mu_2\approx\ell_1\qquad \ell_+\approx-\frac12\ln\mu_1$$
$$\ln\mu_2\approx\ell_{1W}\qquad \ell_-\approx-\frac12\ln\mu_1 \tag{257}$$

The same has been done for the remaining integrals. Before subtracting these collinear singularities, we have converted the arguments of the dilogarithms to such that remain finite for the collinear limit, in order to obtain

$$t_z^{\ell c}=\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right)$$
$$-2\mathrm{Li}_2(\sqrt{\mu_1})+\frac14(\ell_1^c-\ell_{1W}^c)^2+\ln\sqrt{\mu_1}(\ell_1^c-\ell_{1W}^c) \tag{258}$$

with $\ell_1^c=-2\ln((1-\mu_1-\mu_2+\sqrt{\lambda})/2)$ and $\ell_{1W}^c=-2\ln((1-\mu_1+\mu_2+\sqrt{\lambda})/2)$, while one has $t_{zW}^{\ell c}=t_{zW}^{\ell}=2\mathrm{Li}_2(\sqrt{\mu_1})-\mathrm{Li}_2((1+\mu_1-\mu_2+\sqrt{\lambda})/2)-\mathrm{Li}_2((1+\mu_1-\mu_2-\sqrt{\lambda})/2)$ without counter terms. On the other hand, a lot of the other dilogarithmic terms have to be rewritten in order to move the collinear singularity from the dilogarithms to the products of logarithms. Following this, we can subtract the singular products of logarithms listed in Eq. (256) from these terms.

$$t_\zeta^{\ell c}=2\mathrm{Li}_2\left(-\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-2\mathrm{Li}_2\left(-\frac{1-\mu_1-\mu_2}{\sqrt{\lambda}}\right)$$
$$-2\mathrm{Li}_2\left(\frac{\sqrt{\lambda}}{1-\mu_1-\mu_2}\right)-4\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\frac{\pi^2}{2}$$
$$-\frac12(\ell_1^c)^2+2\ln\left(\frac{\sqrt{\mu_1}(1-\mu_1-\mu_2)}{\lambda((1+\sqrt{\mu_1})^2-\mu_2)}\right)\ell_1^c-\ln^2\left(\frac{1-\mu_1-\mu_2}{\sqrt{\lambda}}\right)$$
$$-2\ln(\mu_1\mu_2)\ln\left(\frac{\lambda((1+\sqrt{\mu_1})^2-\mu_2)}{1-\mu_1-\mu_2}\right)\ \to\ \frac{\pi^2}{3},$$
$$t_{\zeta W}^{\ell c}=2\mathrm{Li}_2\left(-\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)-2\mathrm{Li}_2\left(\frac{\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)$$
$$-2\mathrm{Li}_2\left(-\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{\sqrt{\lambda}}\right)-4\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)$$
$$+\frac{\pi^2}{2}-\frac34(\ell_{1W}^c)^2-\ln\left(\frac{\sqrt{\lambda^3}((1+\sqrt{\mu_1})^2-\mu_2)^2}{2\mu_1}\right)\ell_{1W}^c$$
$$-\ln^2\left(\frac{\sqrt{\lambda}}{2}\right)-2\ln\mu_2\ln\left(\sqrt{\lambda}((1+\sqrt{\mu_1})^2-\mu_2)\right)\ \to\ \frac{\pi^2}{3},$$

$$
\begin{aligned}
t_z^{\ell-c} &= \mathrm{Li}_2\left(\frac{2\sqrt{\mu_1}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)\\
&- \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)\\
&- \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}\right)\\
&+ \mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right)\\
&+ \mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)-\frac{\pi^2}{2}-\frac{1}{4}(\ell_1^c)^2+\frac{1}{4}\ell_1^c\ell_{1W}^c\\
&- \frac{1}{4}(\ell_{1W}^c)^2-\frac{1}{2}\ln\left(\frac{\mu_2}{\sqrt{\mu_1}(1-\sqrt{\mu_1})}\right)(\ell_1^c+\ell_{1W}^c)\\
&- \frac{1}{2}\ln\left(\sqrt{\mu_1}((1+\sqrt{\mu_1})^2-\mu_2)^2\right)\ell_1^c\\
&- \ln(\mu_1\mu_2)\ln((1+\sqrt{\mu_1})^2-\mu_2)+\ln\sqrt{\mu_1}\ln(1-\sqrt{\mu_1})\\
&+ \ln^2(1-\sqrt{\mu_1}) \;\to\; -\frac{\pi^2}{3},\\
t_z^{\ell+c} &= 2\mathrm{Li}_2(\sqrt{\mu_1})\\
&- \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(\frac{2\sqrt{\mu_1}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)\\
&+ \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)\\
&+ \mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right)\\
&- \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{2}\right)\\
&+ \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)\\
&+ \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)-\frac{2\pi^2}{3}+\frac{1}{2}(\ell_1^c)^2-\frac{1}{2}\ell_1^c\ell_{1W}^c\\
&+ \frac{1}{2}(\ell_{1W}^c)^2-\frac{1}{2}\ln\left(\sqrt{\mu_1}(1-\sqrt{\mu_1})\right)(\ell_1^c-\ell_{1W}^c)\\
&+ \frac{1}{2}\ln\left(\sqrt{\mu_1^3}((1-\sqrt{\mu_1})^2-\mu_2)^2\right)\ell_1^c
\end{aligned}
$$

$$
\begin{aligned}
&+\ln(\mu_1\mu_2)\ln((1-\sqrt{\mu_1})^2-\mu_2)+\ln\sqrt{\mu_1}\ln(1-\sqrt{\mu_1}) \;\to\; -\frac{\pi^2}{2},\\
t^{\ell-c}_{zW} =& -\mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}\right)\\
&+\mathrm{Li}_2\left(\frac{2\sqrt{\mu_1}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)\\
&-\mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right)\\
&+\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)\\
&-\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\frac{\pi^2}{6}+\frac{1}{4}\ell_1^c\ell_{1W}^c\\
&+\frac{1}{2}\ln\left(\frac{\mu_2}{1-\sqrt{\mu_1}}\right)\ell_1^c+\frac{1}{2}\ln\left(\frac{\mu_2((1+\sqrt{\mu_1})^2-\mu_2)^2}{\sqrt{\mu_1}(1-\sqrt{\mu_1})}\right)\ell_{1W}^c\\
&-\ln\sqrt{\mu_1}\ln(1-\sqrt{\mu_1})+\ln\mu_2\ln((1+\sqrt{\mu_1})^2-\mu_2)\\
&-\ln^2(1-\sqrt{\mu_1})^2 \;\to\; 0,
\end{aligned}
$$

$$
\begin{aligned}
t^{\ell+c}_{zW} =& -2\mathrm{Li}_2(\sqrt{\mu_1})+\frac{\pi^2}{6}\\
&+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{2\sqrt{\mu_1}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)\\
&-\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)\\
&-\mathrm{Li}_2\left(\frac{2(1-\sqrt{\mu_1})}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right)\\
&-\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)\\
&+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right)\\
&+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right)-\frac{3}{4}\ell_1^c\ell_{1W}^c+\frac{1}{4}(\ell_{1W}^c)^2\\
&-\frac{1}{2}\ln\left(\frac{\mu_2}{1-\sqrt{\mu_1}}\right)(\ell_1^c-\ell_{1W}^c)-\frac{1}{2}\ln\left(\sqrt{\mu_1^3}(1-\sqrt{\mu_1})^2-\mu_2)^2\right)\ell_{1W}^c\\
&-\ln\sqrt{\mu_1}\ln(1-\sqrt{\mu_1})-\ln\mu_2\ln((1-\sqrt{\mu_1})^2-\mu_2) \;\to\; \frac{\pi^2}{6}. \qquad (259)
\end{aligned}
$$

In $(t_{zW}^{-\ell}+t_{zW}^{+\ell})/\sqrt{\mu_1}$, the collinear singularities are suppressed by $\sqrt{\mu_1}$ and will only contribute if it is divided by $\sqrt{\mu_1}$ which is the case for this particular combination. One has

$$
\begin{aligned}
t_{zW}^{-\ell c} = &-\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right)\\
&+\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}\right)+\frac{1}{8}\ell_1^c\ell_{1W}^c-\frac{1}{8}(\ell_{1W}^c)^2\\
&+\frac{1}{4}\ln\left(\frac{\mu_2}{(1-\sqrt{\mu_1})^3}\right)(\ell_1^c-\ell_{1W}^c)+\frac{1}{4}\ln\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2}{(1-\sqrt{\mu_1})^2}\right)\ell_{1W}^c\\
&-\ln\sqrt{\mu_1}(\ln(1-\sqrt{\mu_1})+\sqrt{\mu_1})\\
&+\frac{1}{2}\ln\left(\frac{\mu_2}{1-\sqrt{\mu_1}}\right)\ln\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2}{(1-\sqrt{\mu_1})^2}\right),\\
t_{zW}^{+\ell c} = &\,\mathrm{Li}_2\left(-\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2(1+\sqrt{\mu_1})}\right)\\
&-\mathrm{Li}_2\left(-\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2(1+\sqrt{\mu_1})\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(-\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right)-\mathrm{Li}_2(-1)\\
&+\mathrm{Li}_2\left(\frac{2\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right)-\frac{1}{8}\ell_1^c\ell_{1W}^c+\frac{1}{8}(\ell_{1W}^c)^2\\
&-\frac{1}{4}\ln\left(\frac{\mu_2}{(1+\sqrt{\mu_1})^3}\right)(\ell_1^c-\ell_{1W}^c)-\frac{1}{4}\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{(1+\sqrt{\mu_1})^2}\right)\ell_{1W}^c\\
&+\ln\sqrt{\mu_1}(\ln(1+\sqrt{\mu_1})-\sqrt{\mu_1})\\
&-\frac{1}{2}\ln\left(\frac{\mu_2}{1+\sqrt{\mu_1}}\right)\ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{(1+\sqrt{\mu_1})^2}\right)\\
&+\ln^2(1+\sqrt{\mu_1})-\ln(1+\sqrt{\mu_1})\ln(1-\sqrt{\mu_1})+\ln 2\ln\left(\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right),\\
&t_{zW}^{-\ell c\prime}:=\frac{t_{zW}^{-\ell c}}{\sqrt{\mu_1}}\;\to\;-2,\qquad t_{zW}^{+\ell c\prime}:=\frac{t_{zW}^{+\ell c}}{\sqrt{\mu_1}}\;\to\;0. \qquad (260)
\end{aligned}
$$

For the vertex contribution, the collinear subtracted three-point functions related to photon as

$$
\begin{aligned}
&m_W^2C_f^c(m_c^2,m_b^2,m_W^2;m_A,m_W,m_c)\\
&= \quad m_W^2C_f^*(m_c^2,m_b^2,m_W^2;m_A,m_W,m_c)+\frac{1}{\sqrt{\lambda}}\left(-\frac{3}{4}\ln^2\mu_1-\frac{1}{2}\ln\mu_1\ln\mu_2\right)\\
&= \quad -\frac{1}{\sqrt{\lambda}}\Bigg(\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{4}(\ell_1^c)^2+\frac{1}{4}(\ell_{1W}^c)^2-\frac{1}{2}(\ell_1^c-\ell_{1W}^c)\ln\left(\frac{\mu_1\mu_2}{\lambda}\right) \\
& -\ln\sqrt{\mu_1}(\ell_1^c-\ln\lambda)\Bigg) \;\to\; 0, \\
& m_W^2 C_f^c(m_c^2,m_b^2,m_W^2;m_W,m_A,m_b) \\
= \;& m_W^2 C_f^*(m_c^2,m_b^2,m_W^2;m_W,m_A,m_b)+\frac{1}{\sqrt{\lambda}}\left(-\frac{1}{2}\ln\mu_1\ln\mu_2-\frac{3}{4}\ln^2\mu_2\right) \\
= \;& -\frac{1}{\sqrt{\lambda}}\Bigg(\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)-\mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& -\frac{1}{2}\ell_1^c\ell_{1W}^c+\frac{1}{4}(\ell_{1W}^c)^2-\frac{1}{2}\ell_{1W}^c\ln\left(\frac{\mu_1\mu_2}{\lambda}\right)-\ln\sqrt{\mu_2}(\ell_1^c-\ln\lambda)\Bigg) \;\to\; 0, \\
& m_W^2 C_f^c(m_c^2,m_b^2,m_W^2;m_c,m_b,m_A) \\
= \;& m_W^2 C_f^*(m_c^2,m_b^2,m_W^2;m_c,m_b,m_A) \\
& +\frac{1}{\sqrt{\lambda}}\left(\frac{3}{4}\ln^2\mu_1+\ln\mu_1\ln\mu_2+\frac{3}{4}\ln^2\mu_2\right) \\
= \;& -\frac{1}{\sqrt{\lambda}}\Bigg(\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& +\frac{2\pi^2}{3}+\frac{1}{4}(\ell_1^c)^2-\frac{1}{2}\ell_1^c\ell_{1W}^c+\frac{1}{2}(\ell_{1W}^c)^2+\frac{1}{2}\ell_1^c\ln\left(\frac{\mu_1\mu_2}{\lambda}\right) \\
& +\ln\sqrt{\mu_1}(\ell_1^c-\ell_{1W}^c-\ln\lambda)+\ln\sqrt{\mu_2}(\ell_{1W}^c-\ln\lambda) \\
& +i\pi\ln\left(\frac{\mu_1\mu_2}{\lambda}\right)\Bigg) \;\to\; -\frac{\pi^2}{3}+i\pi\ln(\mu_1\mu_2). \qquad (261)
\end{aligned}
$$

Note that the imaginary part of the latter is still collinear singular. However, we do not need this imaginary part for the helicity bilinears. The results for the collinear, IR and UV subtracted counter terms read

$$
\begin{aligned}
\delta Z_{cc}^{Lc} &= \frac{(1-2Q_c s_W^2)^2 e^2 m_Z^2}{8m_W^2 s_W^2}\left(\ln\left(\frac{m_Z^2}{m_W^2}\right)+\frac{1}{2}\right)+\frac{1}{8s_W^2}\sum_k |V_{ck}|^2, \\
\delta Z_{bb}^{Lc} &= \frac{(1+2Q_b s_W^2)^2 e^2 m_Z^2}{8m_W^2 s_W^2}\left(\ln\left(\frac{m_Z^2}{m_W^2}\right)+\frac{1}{2}\right)+\frac{1}{8x_W^2}\sum_k |V_{kb}|^2 \\
& -\frac{e^2 m_t^2 |V_{tb}|^2}{16(m_t^2-m_W^2)^2 m_W^2 s_W^2} \\
& \times\left(3(m_t^4-m_W^4)-2m_t^2(m_t^2+2m_W^2)\ln\left(\frac{m_t^2}{m_W^2}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\delta Z^{c}_{WW} &= \frac{e^2}{72 m_W^4 m_Z^2 s_W^2}\Bigg\{4m_W^4 m_Z^2 + 144 m_W^4 m_Z^2 s_W^2 \\
&\quad +3(48m_W^6 - 16m_W^4 m_Z^2 + 6m_W^2 m_Z^4 + m_Z^6)B_f(m_W^2; m_W, m_Z) \\
&\quad -3m_Z^2(2m_W^2 - m_H^2)m_H^2 B_f(m_W^2; m_W, m_H) \\
&\quad +3m_W^2(4m_W^2 - m_Z^2)(12m_W^4 + 20m_W^2 m_Z^2 + m_Z^4) \\
&\quad \times B'_f(m_W^2; m_W, m_Z) \\
&\quad -3m_W^2 m_Z^2(12m_W^4 - 4m_W^2 m_H^2 + m_H^4)B'_f(m_W^2; m_W, m_H) \\
&\quad -3m_W^2 m_Z^2(2m_W^2 - m_Z^2 - m_H^2)A_f(m_W) \\
&\quad +3m_Z^2(m_W^2 - m_Z^2)(8m_W^2 + m_Z^2)A_f(m_Z) \\
&\quad +3m_Z^2(m_W^2 - m_H^2)m_H^2 A_f(m_H) \\
&\quad -\frac{2m_Z^2}{m_W^2}\Bigg[2\sum_{i,j}|V_{ij}|^2(2+3\pi i)m_W^6 \\
&\quad +3\sum_k |V_{tk}|^2\Bigg(m_t^2(2m_t^2 + m_W^2)m_W^2 - 2\pi i m_W^6 \\
&\quad +2(m_t^6 - m_W^6)\ln\left(\frac{m_t^2 - m_W^2}{m_W^2}\right) - 2m_t^6 \ln\left(\frac{m_t^2}{m_W^2}\right)\Bigg)\Bigg]\Bigg\},
\end{aligned}
$$

$$
\delta Z^{\alpha(m_W^2)}_{ec} = -\frac{e^2}{3} + 6Q_t^2 \ln\left(\frac{m_t^2}{m_W^2}\right), \qquad \delta Z^{G_\mu}_{ec} = 0,
$$

$$
\begin{aligned}
\frac{\delta s^{\alpha(m_W^2)}_{Wc}}{s_W} &= \frac{-e^2}{72 m_W^2 m_Z^6 s_W^4} \\
&\quad \times\Bigg\{4m_W^2(m_W^2 - m_Z^2)(36m_W^4 + 96m_W^2 m_Z^2 + m_Z^4) \\
&\quad +3(4m_W^2 - m_Z^2)(12m_W^4 + 20m_W^2 m_Z^2 + m_Z^4) \\
&\quad \times\Big(m_W^2 B_f(m_Z^2; m_W, m_W) - m_W^2 A_f(m_W) \\
&\quad -m_Z^2 B_f(m_W^2; m_W, m_Z) + m_Z^2 A_f(m_Z)\Big) \\
&\quad +3m_Z^4\Big((12m_W^4 - 4m_W^2 m_H^2 + m_H^4)B_f(m_W^2; m_W, m_H) \\
&\quad -m_W^2 m_H^2(A_f(m_H) - A_f(m_W))\Big) \\
&\quad -3m_W^2 m_Z^2\Big((12m_Z^4 - 4m_Z^2 m_H^2 + m_H^4)B_f(m_Z^2; m_Z, m_H)
\end{aligned}
$$

$$
\begin{aligned}
& -m_Z^2 m_H^2 (A_f(m_H) - A_f(m_Z))\Big) \\
& +6m_W^4 m_Z^2 (42m_W^2 - m_Z^2)(A_f(m_W) - A_f(m_Z)) \\
& +6m_W^2 m_Z^2 (m_W^2 - m_Z^2)(18m_W^2 - 5m_Z^2) A_f(m_Z) \\
& +3(m_W^2 - m_Z^2) m_Z^2 m_H^4 A_f(m_H) \\
& +12 m_W^4 m_Z^4 (5 + 3\pi i)\left(1 + \sum_{i,j} |V_{ij}|^2\right) \\
& -\frac{18 m_Z^4}{m_W^2} \sum_k |V_{tk}|^2 \Bigg(m_t^2 (m_t^2 + 2m_W^2) m_W^2 + 2\pi i m_W^6 \\
& +(m_t^6 - 3m_t^2 m_W^4 + 2m_W^6) \ln\left(\frac{m_t^2 - m_W^2}{m_W^2}\right) - m_t^6 \ln\left(\frac{m_t^2}{m_W^2}\right)\Bigg) \\
& -8 m_W^4 m_Z^4 s_W^2 \sum_f (g_f^{-2} + g_f^{+2}) \left[5 - 3\left(\ln\left(\frac{m_Z^2}{m_W^2}\right) - i\pi\right)\right] \\
& -24 m_W^4 m_Z^2 s_W^2 (g_t^{-2} + g_t^{+2}) \Bigg(-2m_t^2 + (3m_t^2 - m_Z^2) \ln\left(\frac{m_t^2}{m_W^2}\right) \\
& +m_Z^2 \left(\ln\left(\frac{m_Z^2}{m_W^2}\right) - i\pi\right) \\
& +2(m_t^2 - m_Z^2)\sqrt{\frac{4m_t^2}{m_Z^2} - 1}\left(\frac{\pi}{2} - \arctan\left(\sqrt{\frac{4m_t^2}{m_Z^2} - 1}\right)\right)\Bigg) \\
& -144 m_W^4 m_Z^2 s_W^2 g_t^- g_t^+ m_t^2 \Bigg[2 - 2\sqrt{\frac{4m_t^2}{m_Z^2} - 1} \\
& \times\left(\frac{\pi}{2} - \arctan\left(\sqrt{\frac{4m_t^2}{m_Z^2} - 1}\right)\right) - \ln\left(\frac{m_t^2}{m_W^2}\right)\Bigg]\Bigg\}, \\
\frac{\delta s_{Wc}^{G_\mu}}{s_W} = {} & \frac{-e^2}{72 m_W^4 m_Z^2 s_W^4}\Bigg\{ + 4m_W^4 s_W^2 (36 m_W^2 - 91 m_Z^2 - \sum_f m_Z^2) \\
& +3m_W^4 (103 m_W^2 - 22 m_Z^2) A_f(m_W) \\
& -3m_W^4 (64 m_W^2 + 17 m_Z^2) A_f(m_Z) \\
& -\frac{3 s_W^2}{m_W^2 - m_Z^2}(4m_W^2 - m_Z^2)(12 m_W^4 + 20 m_W^2 m_Z^2 + m_Z^4) \\
& \times\Big((m_W^2 - m_Z^2) B_f(m_W^2; m_W, m_Z) - m_W^2 A_f(m_W) \\
& +m_Z^2 A_f(m_Z)\Big) - \frac{3 m_W^4 m_Z^2 s_W^2}{m_W^2 - m_H^2}\Big(m_H^2 A_f(m_H) - m_W^2 A_f(m_W)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3m_Z^2 s_W^2}{m_W^2-m_H^2}(12m_W^4-4m_W^2m_H^2+m_H^4)\\
&\times\Big((m_W^2-m_H^2)B_f(m_W^2;m_W,m_H)-m_W^2A_f(m_W)\\
&+m_H^2A_f(m_H)\Big)+36m_W^4m_Z^2s_W^2(2+i\pi)(1+\sum_{ij}|V_{ij}|^2)\\
&-\frac{18m_Z^2s_W^2}{m_W^2}\sum_k|V_{tk}|^2\Bigg(m_t^2(m_t^2+m_W^2)m_W^2\\
&+(m_t^6-3m_t^2m_W^4+2m_W^6)\ln\left(\frac{m_t^2-m_W^2}{m_W^2}\right)\\
&-m_t^2(m_t^4-3m_W^4)\ln\left(\frac{m_t^2}{m_W^2}\right)+2\pi i m_W^6\Bigg)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\delta Z_{cu}^{Lc} &= \delta Z_{ct}^{Lc} = 0,\\
\delta Z_{db}^{Lc} &= \frac{e^2V_{td}^*V_{tb}}{8s_W^2}\left[\frac{m_t^4(m_t^2+2m_W^2)}{(m_t^2-m_W^2)^2m_W^2}\ln\left(\frac{m_t^2}{m_W^2}\right)-\frac{3m_t^2(m_t^2+m_W^2)}{2(m_t^2-m_W^2)m_W^2}\right],\\
\delta Z_{sb}^{Lc} &= \frac{e^2V_{ts}^*V_{tb}}{8s_W^2}\left[\frac{m_t^4(m_t^2+2m_W^2)}{(m_t^2-m_W^2)^2m_W^2}\ln\left(\frac{m_t^2}{m_W^2}\right)-\frac{3m_t^2(m_t^2+m_W^2)}{2(m_t^2-m_W^2)m_W^2}\right],
\end{aligned}
$$

$$
\begin{aligned}
V_-^c &= \frac{m_W^2+2m_Z^2}{m_W^2s_W^2}(1-2Q_cs_W^2)\\
&\times\left(\mathrm{Li}_2\left(\frac{2}{1-\sqrt{1-4m_W^2/m_Z^2}}\right)+\mathrm{Li}_2\left(\frac{2}{1+\sqrt{1-4m_W^2/m_Z^2}}\right)\right.\\
&\left.-\mathrm{Li}_2\left(\frac{2(1-m_W^2/m_Z^2)}{1-\sqrt{1-4m_W^2/m_Z^2}}\right)-\mathrm{Li}_2\left(\frac{2(1-m_W^2/m_Z^2)}{1+\sqrt{1-4m_W^2/m_Z^2}}\right)\right)\\
&+\frac{m_W^2+2m_Z^2}{m_W^2s_W^2}(1+2Q_bs_W^2)\\
&\times\left(\mathrm{Li}_2\left(\frac{2m_W^2/m_Z^2}{1-\sqrt{1-4m_W^2/m_Z^2}}\right)\right.\\
&\left.+\mathrm{Li}_2\left(\frac{2m_W^2/m_Z^2}{1+\sqrt{1-4m_W^2/m_Z^2}}\right)\right)\\
&-\frac{(m_W^2+m_Z^2)^2}{2m_W^6s_W^2}m_Z^2(1-2Q_cs_W^2)(1+2Q_bs_W^2)\times
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\mathrm{Li}_2\left(-\frac{m_W^2}{m_Z^2}\right)+\ln\left(1+\frac{m_W^2}{m_Z^2}\right)\ln\left(\frac{m_W^2}{m_Z^2}\right)\right)\\
&+2\frac{2m_W^2+m_Z^2}{m_W^4 s_W^2}(1-(Q_c-Q_b)s_W^2)\\
&\times\sqrt{4m_W^2m_Z^2-m_Z^4}\arctan\left(\sqrt{\frac{2m_W-m_Z}{2m_W+m_Z}}\right)\\
&-\frac{2m_W^2+m_Z^2}{2m_W^4}\Big(2m_Z^2-(Q_c-Q_b)(3m_Z^2-2m_W^2)\\
&-4Q_cQ_b(m_Z^2-m_W^2)\Big)\ln\left(\frac{m_W^2}{m_Z^2}\right)\\
&+\frac{i\pi m_Z^2}{2m_W^6 s_W^2}(1-2Q_cs_W^2)(1+2Q_bs_W^2)\\
&\times(m_W^2+m_Z^2)^2\ln\left(1+\frac{m_W^2}{m_Z^2}\right)\\
&+\frac{i\pi m_Z^4}{2m_W^4 s_W^2}(1-2Q_cs_W^2)(1+2Q_bs_W^2)\\
&+\frac{3\pi i m_Z^2}{4m_W^2 s_W^2}(1-2Q_cs_W^2+2Q_bs_W^2-4Q_cQ_bs_W^2)+Q_cQ_b\frac{4\pi^2}{3}\\
&+\frac{2m_W^2+m_Z^2}{2m_W^4}m_Z^2(1-2Q_cs_W^2+2Q_bs_W^2-4Q_cQ_bs_W^2)\\
&+\frac{4m_W^2-m_Z^2}{2m_W^2s_W^2},
\end{aligned}
$$

$$
\mu_1\mu_2V_+^c \;=\; \mu_1V_1^c \;=\; \mu_2V_2^c \;=\; 0. \tag{262}
$$

By this way the collinear singularities are cancelled against each other from tree, loop and renormalisation factors.

9 Schemes and results

In the EW theory with its multiple parameters depending on each other, it is essential to decide which of the parameters are independent means which parameters are taken as physical parameters. The correction depend on the subtraction scheme. Here we used three different schemes as $\alpha(0)$, $\alpha(m_W^2)$ and G_μ scheme. In the α scheme(s) there is the fine structure constant as measure for the coupling. In the so-called α scheme, the coupling and the masses of the particles are used. In the $\alpha(m_W^2)$ scheme, all light fermion loops occurring in the counter terms δZ_e and $\delta s_W/s_W$ are resummed into the coupling constant, as explained in Refs. [9, 15, 23].

Table 1: Contributions to the collinear singularities from tree, vertex and renormalisation

Q_c^2	tree	$-\ln\mu_1 - 2\ln\mu_2 - 3\ln^2\mu_1 - 2\ln\mu_1\ln\mu_2$
	loop	$3\ln^2\mu_1 + 2\ln\mu_1\ln\mu_2 - 4\ln\mu_1$
	renomalisation	$5\ln\mu_1 + 2\ln\mu_2$
Q_b^2	tree	$-2\ln\mu_1 - \ln\mu_2 - 2\ln\mu_1\ln\mu_2 - 3\ln^2\mu_2$
	loop	$2\ln\mu_1\ln\mu_2 + 3\ln^2\mu_2 - 4\ln\mu_2$
	renomalisation	$2\ln\mu_1 + 5\ln\mu_2$
Q_W^2	tree	$-2\ln\mu_1 - 2\ln\mu_2$
	loop	0
	renomalisation	$2\ln\mu_1 + 2\ln\mu_2$

However, depending on energy of the process, more and more fermion loops can be resummed to the coupling, so that $\alpha(0)$ is changed to e.g. $\alpha(m_W^2)$. This is shown in Sec. 8.2.1 of Ref. [9]. The Fermi constant is preferable as physical parameter, as it is measured quite precisely, $G_F = 1.1663787(6)\times 10^{-5}\,\mathrm{GeV}^{-2}$ by using the μ decay (thus, G_μ) this can be seen in the quite detailed review article in Ref. [23]. For the processes involving the electroweak vector bosons, the G_F (or G_μ) scheme based on the Fermi constant (in muon decay) and discussed in Ref. [25] is preferable. Therefore, for our work we will work with this scheme (which was already mentioned in Ref. [9]). From Eqs. (426) and (423) of Ref. [23] we deduce that

$$\begin{aligned}\delta Z_e\Big|_{G_\mu} &= \frac{1}{2}\Pi^T_{AA}(m_A^2) - \frac{s_W}{c_W}\frac{\Sigma^T_{AZ}(m_A^2)}{m_Z^2} - \frac{1}{2}\Delta r\Big|_{G_\mu}\\ &= \frac{1}{2}\Bigg(\frac{c_W^2}{s_W^2}\left(\frac{\Sigma^T_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Sigma^T_{WW}(m_W^2)}{m_W^2}\right) - \frac{\Sigma^T_{WW}(0) - \Sigma^T_{WW}(m_W^2)}{m_W^2}\\ &\quad -\frac{2\Sigma^T_{AZ}(m_A^2)}{s_W c_W m_Z^2} - \frac{\alpha(0)}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}(A_f(m_Z) - A_f(m_W))\right)\Bigg)\end{aligned} \tag{263}$$

In order to calculate this quantity, we obviously have to calculate $\Sigma_T^{WW}(p^2)$ at $p^2 = m_A^2$. (cf. also Eq. (4.6.11) in Ref. [15]). For the self energy of the W boson at $p^2 = 0$ one obtains

$$\begin{aligned}&\Sigma^T_{WW}(0)\\ &= \frac{\alpha}{4\pi}\Bigg[\frac{-1}{2m_W^2 s_W^2\varepsilon}\Big(4m_W^2 - 2m_Z^2 + \sum_i(m_{\nu_i}^2 + m_{\ell_i}^2)\\ &\quad +N_c\sum_{i,j}|V_{ij}|^2(m_i^2 + m_j^2)\Big)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{6m_W^2 s_W^2}\Big(18m_W^2 + m_Z^2 + m_H^2 - 2\sum_i (m_{\nu_i}^2 + m_{\ell_i}^2) \\
&-2N_c \sum_{i,j} |V_{ij}|^2 (m_i^2 + m_j^2)\Big) \\
&-\frac{(14m_W^2 + m_Z^2)m_W^2 - (16m_W^2 - m_Z^2)m_H^2}{3(m_H^2 - m_W^2)m_Z^2 s_W^4} A_f(m_W) \\
&-\frac{(4m_W^2 - m_Z^2)(2m_W^2 + 3m_Z^2)}{3m_W^2 m_Z^2 s_W^4} A_f(m_Z) + \frac{2m_H^2 A_f(m_H)}{3(m_H^2 - m_W^2)s_W^2} \\
&-\frac{1}{3m_W^2 s_W^2} \sum_i \Big(m_{\nu_i}^2 A_f(m_{\nu_i}) + m_{\ell_i}^2 A_f(m_{\ell_i}) \\
&+\frac{m_{\nu_i}^2 + m_{\ell_i}^2}{2(m_{\nu_i}^2 - m_{\ell_i}^2)} \Big(m_{\nu_i}^2 A_f(m_{\nu_i}) - m_{\ell_i}^2 A_f(m_{\ell_i})\Big)\Big) \\
&-\frac{N_c}{3m_W^2 s_W^2} \sum_{ij} |V_{ij}|^2 \Big(m_i^2 A_f(m_i) + m_j^2 A_f(m_j) \\
&+\frac{m_i^2 + m_j^2}{2(m_i^2 - m_j^2)} \Big(m_i^2 A_f(m_i) - m_j^2 A_f(m_j)\Big)\Big)\Bigg] \\
&+\frac{\alpha}{4\pi}\Bigg[\frac{-1}{4m_H^2 m_W^2 s_W^2 \varepsilon} \\
&\times \Bigg(6(2m_W^4 + m_Z^4) + (2m_W^2 + m_Z^2)m_H^2 + 3m_H^4 - 8\sum_f m_f^4\Bigg) \\
&+\frac{1}{4m_H^2 m_W^2 s_W^2}\Bigg(4(2m_W^4 + m_Z^4) - 2m_W^2(m_H^2 + 6m_W^2)A_f(m_W) \\
&-m_Z^2(m_H^2 + 6m_Z^2)A_f(m_Z) - 3m_H^4 A_f(m_H) + 8\sum_f m_f^4 A_f(m_f)\Bigg)\Bigg].
\end{aligned}
\tag{264}
$$

It can be shown that

$$
\begin{aligned}
\Delta r\Big|_{G_\mu} \;=\;& \Pi_{AA}^T(m_A^2) + \frac{2c_W}{s_W}\frac{\Sigma_{AZ}^T(m_A^2)}{m_Z^2} + \frac{\Sigma_{WW}^T(0) - \Sigma_{WW}^T(m_W^2)}{m_W^2} \\
&-\frac{c_W^2}{s_W^2}\Big(\frac{\Sigma_{ZZ}^T(m_Z^2)}{m_Z^2} - \frac{\Sigma_{WW}^T(m_W^2)}{m_W^2}\Big) + \frac{\alpha}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}\ln c_W^2\right)
\end{aligned}
\tag{265}
$$

is UV-finite. This has been done by using $\sum_f (1 - 4I_f^3 Q_f) = 0$, which in detail results in $Q_c - Q_b = Q_W$ for all quark and lepton generations. In replacing the

$\alpha(0)$-scheme result $\delta Z_e = -\delta Z_{AA} - \delta Z_{ZA} s_W/c_W$ by Eq. (263) means that the contribution $-\delta Z_{AA}$ producing the large mass logarithms no longer appears. We can use Eq. (263) to see that in this new scheme δZ_e together with the counter term for the sine of the Weinberg angle simplifies to

$$\begin{aligned}
\left[-\frac{\delta s_W}{s_W} + \delta Z_e\right]_{G_\mu} &= \frac{c_W^2}{2s_W^2}\left(\frac{\Sigma^T_{WW}(m_W^2)}{m_W^2} - \frac{\Sigma^T_{ZZ}(m_Z^2)}{m_Z^2}\right) + \\
&+\frac{1}{2}\left[\frac{c_W^2}{s_W^2}\left(\frac{\Sigma^T_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Sigma^T_{WW}(m_W^2)}{m_W^2}\right) - \frac{\Sigma^T_{WW}(0) - \Sigma^T_{WW}(m_W^2)}{m_W^2}\right. \\
&\left.-\frac{2\Sigma^T_{AZ}(m_A^2)}{s_W c_W m_Z^2} - \frac{\alpha(0)}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}(A_f(m_Z) - A_f(m_W)\right)\right] \\
&= -\frac{1}{2}\left[\frac{2\Sigma^T_{AZ}(m_A^2)}{s_W c_W m_Z^2} + \frac{\Sigma^T_{WW}(0) - \Sigma^T_{WW}(m_W^2)}{m_W^2}\right. \\
&\left.+\frac{\alpha(0)}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}(A_f(m_Z) - A_f(m_W)\right)\right], \qquad (266)
\end{aligned}$$

which means that in this scheme we obtain

$$\begin{aligned}
\left.\frac{\delta s_W}{s_W}\right|_{G_\mu} &= \frac{1}{2}\left[\frac{\Sigma^T_{WW}(0) - \Sigma^T_{WW}(m_W^2)}{m_W^2}\right. \\
&\left.+\frac{\alpha(0)}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}(A_f(m_Z) - A_f(m_W)\right)\right], \\
\delta Z_e|_{G_\mu} &= -\frac{\Sigma^T_{AZ}(m_A^2)}{s_W c_W m_Z^2} \qquad (267)
\end{aligned}$$

with UV singular contributions

$$\left.\frac{\delta s_W^s}{s_W}\right|_{G_\mu} = \frac{e^2}{12 s_W^2 \varepsilon}\left(19 - \sum_f 1\right), \qquad \left.\delta Z_e^s\right|_{G_\mu} = -\frac{2e^2}{s_W^2 \varepsilon} \qquad (268)$$

and UV finite contributions

$$\begin{aligned}
\left.\frac{\delta s_W^f}{s_W}\right|_{G_\mu} = e^2 &\left[\frac{-72 m_W^2 + 182 m_Z^2 + \sum_f m_Z^2}{36 m_Z^2 s_W^2}\right. \\
&- \frac{1}{24(m_H^2 - m_W^2) m_W^2 m_Z^4 s_W^4} \\
&\times\Big(m_W^2(48 m_W^6 - 46 m_W^4 m_Z^2 + 17 m_W^2 m_Z^4 - m_Z^6) \\
&- m_H^2(48 m_W^6 - 39 m_W^4 m_Z^2 + 10 m_W^2 m_Z^4 - m_Z^6) + m_H^4 m_Z^4 s_W^2\Big) A_f(m_W)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{16m_W^6-51m_W^4m_Z^2+16m_W^2m_Z^4+m_Z^6}{24m_W^4m_Z^2s_W^4}A_f(m_Z)\\
&+(4m_W^2-m_Z^2)\frac{12m_W^4+20m_W^2m_Z^2+m_Z^4}{24m_W^4m_Z^2s_W^2}B_f(m_W^2;m_W,m_Z)\\
&+m_H^2\frac{11m_W^4-4m_W^2m_H^2+m_H^4}{24(m_H^2-m_W^2)m_W^4s_W^2}A_f(m_H)\\
&-\frac{12m_W^4-4m_W^2m_H^2+m_H^4}{24m_W^4s_W^2}B_f(m_W^2;m_W,m_H)+\frac{1}{12m_W^4s_W^2}\\
&\times\sum_i\Bigg(\Big((m_{\nu_i}^2-m_{\ell_i}^2)^2+(m_\mu^2+m_{\ell_i}^2)m_W^2-2m_W^4\Big)B_f(m_W^2;m_{\nu_i},m_{\ell_i})\\
&-\Big((m_{\nu_i}^2-m_{\ell_i}^2)^2+(m_{\nu_i}^2+m_{\ell_i}^2)m_W^2\Big)\frac{m_{\nu_i}^2A_f(m_{\nu_i})-m_{\ell_i}^2A_f(m_{\ell_i})}{m_{\nu_i}^2-m_{\ell_i}^2}\Bigg)\\
&+\frac{1}{12m_W^4s_W^2}\sum_{i,j}|V_{ij}|^2\Bigg(\Big((m_i^2-m_j^2)^2+(m_i^2+m_j^2)m_W^2-2m_W^4\Big)\\
&\times B_f(m_W^2;m_i,m_j)\\
&-\Big((m_i^2-m_j^2)^2+(m_i^2+m_j^2)m_W^2\Big)\frac{m_i^2A_f(m_i)-m_j^2A_f(m_j)}{m_i^2-m_j^2}\Bigg)\Bigg], \qquad (269)
\end{aligned}
$$

$$
\delta Z_e^f\Big|_{G_\mu} = \frac{2e^2}{s_W^2}\left(1-A_f(m_W)\right). \qquad (270)
$$

Together with

$$
\delta Z_{WW}^{Ts}=\frac{e^2}{12s_W^2\varepsilon}\left(19-\sum_f 1\right) \qquad (271)
$$

from above, one has $\delta Z_{WW}^{Ts}-\delta s_W^s/s_W+\delta Z_e^s|_{G_\mu}=-2e^2/(s_W^2\varepsilon)$ as for the other schemes, only that the first two UV singularities cancel each other and the only singularity that is left is the one from $\delta Z_e|_{G_\mu}$.

9.1 Observables for the decay process

Observables which are also measured in the experiment, are based on the differential decay rate given by

$$
\frac{d\Gamma}{d\cos\theta} = \Gamma_0 W(\theta) \qquad (272)
$$

where $W(\theta)$ is found in Eq. (1) and

$$
\Gamma_0=\frac{G_\mu m_W^3}{6\pi\sqrt{2}}N_c|V_{ij}|^2=\frac{\alpha m_W}{12}N_c|V_{ij}|^2 \qquad (273)
$$

Table 2: Numerical values for the three observables c_f, $A_{\rm FB}$ and $\cos\theta_{\rm extr}$ to leading order and for three next-to-leading order schemes with exact fermion masses and in the collinear limit

scheme,	case	c_f	$A_{\rm FB}$	$\cos\theta_{\rm extr}$
LO	$m_f \neq 0$	-0.792444	$+0.231632$	$+0.292300$
	$m_f = 0$	-0.795975	$+0.231975$	$+0.291435$
$\alpha(0)$	$m_f \neq 0$	-0.792196	$+0.231596$	$+0.292347$
	$m_f = 0$	-0.791380	$+0.231934$	$+0.293076$
$\alpha(m_W^2)$	$m_f \neq 0$	-0.792168	$+0.231592$	$+0.292352$
	$m_f = 0$	-0.790868	$+0.231930$	$+0.293260$
G_μ	$m_f \neq 0$	-0.792186	$+0.231595$	$+0.292349$
	$m_f = 0$	-0.791192	$+0.231933$	$+0.293143$

is the Born term rate in the collinear limit into quark flavours i and j. The normalised decay rate is given by $\hat\Gamma = (d\Gamma/d\cos\theta)/\Gamma$. In order to describe the angular distribution, we analysed three observables. The first observable is the convexity parameter, given by

$$c_f = \frac{d^2\hat\Gamma}{d(\cos\theta)^2} = \frac{3}{4}(\rho_{++} - 2\rho_{00} + \rho_{--})\left(\frac{H^{++} - 2H^{00} + H^{--}}{H^{++} + H^{00} + H^{--}}\right). \tag{274}$$

The second and third observables are the forward–backward asymmetry of the decay distribution defined by

$$A_{FB} = \frac{\hat\Gamma(F) - \hat\Gamma(B)}{\hat\Gamma(F) + \hat\Gamma(B)} = \frac{3}{4}(\rho_{++} - \rho_{--})\left(\frac{H^{++} - H^{--}}{H^{++} + H^{00} + H^{--}}\right) \tag{275}$$

with $\hat\Gamma(F) = \hat\Gamma(0 \leq \theta \leq \pi/2)$ and $\hat\Gamma(B) = \hat\Gamma(\pi/2 \leq \theta \leq \pi)$, and the position of the extremum

$$\cos\theta\Big|_{\rm extr} = -\frac{A_{FB}}{c_f} = -\frac{(\rho_{++} - \rho_{--})}{(\rho_{++} - 2\rho_{00} + \rho_{--})}\left(\frac{H^{++} - H^{--}}{H^{++} - 2H^{00} + H^{--}}\right). \tag{276}$$

The values for these observables are found in Table 2. At the same time, we see that in the collinear limit the electroweak correction is larger and amounts to 7.0%, 4.2% and 7.4% for the G_μ, $\alpha(m_W^2)$ and $\alpha(0)$ schemes, respectively.

9.2 Results

For interactions of the W boson (in particular, for the decay of the W boson), the G_μ scheme is the most appropriate, as the Fermi constant collects parameters in a

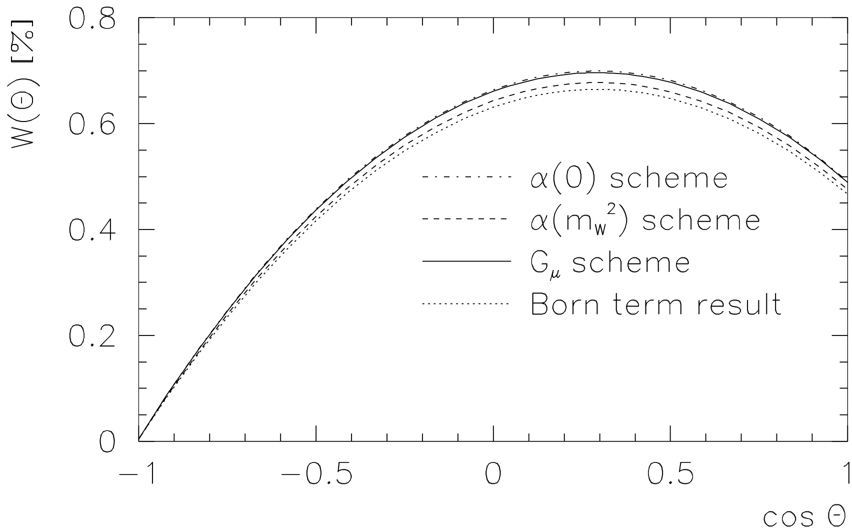


Figure 8: Angular distribution in dependence on the subtraction scheme employed

single parameters present in these interactions, see Ref. [23]. The EW corrections are assumed to be small. The effect of these schemes are displayed in the Fig. 8. With the $\alpha(m_W^2)$ scheme the EW correction is of the order of 1.9% while for the other schemes the corrections are larger, 4.7% and 5.2% for the G_μ and $\alpha(0)$ schemes, respectively.

We have compared the angular distribution for the LO Born term result and for the NLO result in the G_μ scheme with the results in the collinear limit in order to study mass effects. In the second column of Table 3 we give the helicity bilinears for the different schemes. We can see in this table compared to the mass zero case for the final quarks, i.e. the collinear limit, results with the quark masses taken into account will be smaller, as the phase space is shrunk by the masses, an effect that gives the main contribution.

In Figure 9 we have compared the angular distribution for the LO Born term result and for the NLO result in the G_μ scheme with the results in the collinear limit in order to study mass effects. While mass corrections for the Born term results are difficult to discern and are of the order -0.2%, mass effects in the G_μ scheme amount to -2.3%, with the mass effects for the $\alpha(m_W^2)$ and $\alpha(0)$ schemes given by -2.4% and -2.2%, respectively.

Table 3: helicity bilinears for the Born term contribution and the electroweak radiative corrections. Shown are numerical values for the exact fermion masses, and for the fermion masses except for the top quark mass set to zero, i.e., in the collinear limit.

scheme,	bilinear	m_f	$m_f = 0, f \neq t$
LO	H^{00}	0.00148	0
	H^{++}	7×10^{-7}	0
	H^{+-}	0	0
	H^{--}	0.99704	1
$\alpha(0)$	H^{00}	$+0.00019$	$+0.00206$
	H^{++}	$+0.00003$	-0.00094
	H^{+-}	$+0.00005$	$+0.00103$
	H^{--}	$+0.05195$	$+0.07333$
$\alpha(m_W^2)$	H^{00}	$+0.00015$	$+0.00222$
	H^{++}	$+0.00003$	-0.00101
	H^{+-}	$+0.00006$	$+0.00111$
	H^{--}	$+0.01911$	$+0.04208$
G_μ	H^{00}	$+0.00018$	$+0.00214$
	H^{++}	$+0.00003$	-0.00097
	H^{+-}	$+0.00006$	$+0.00107$
	H^{--}	$+0.04701$	$+0.06916$

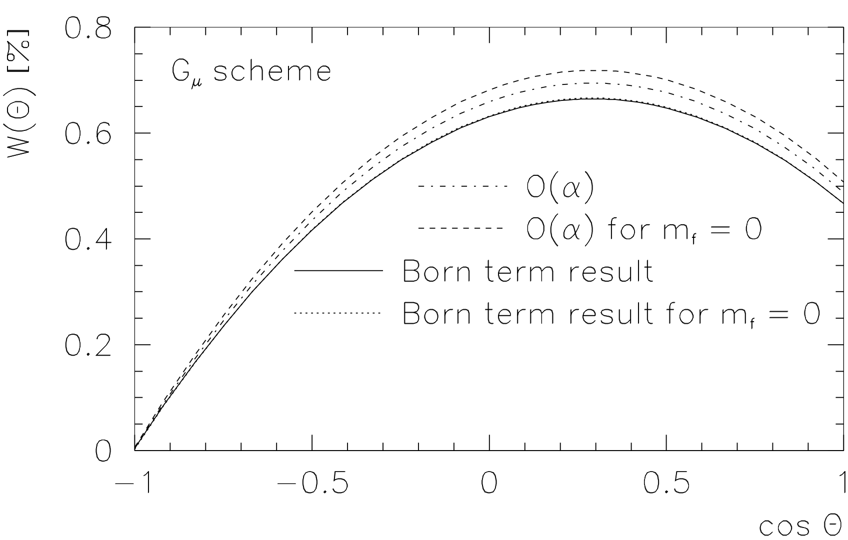


Figure 9: Mass dependence of the angular distribution in the G_μ scheme

10 Gauge dependence of the gauge boson projector

10.1 Introduction

In electroweak theory we can derive the propagator of a gauge boson in two different ways, like the semi-classical approach called Green function and the field theoretical approach in which we calculate the vacuum expectation value of the time-ordered product of the field operators, for the massless photon case or the massive vector bosons $W^{\pm}$ and Z. The gauge boson projector, the central tensorial object related to the completeness relation for the polarisation four-vectors, can be defined by comparing the semi-classical with the field theoretical approach. In the case of the photon (massless vector boson) there are two polarisation states, while there is an additional longitudional polarisation state for the massive vector $W^{\pm}$ or Z bosons. The time component of the polarisation state depends on the choice of the gauge. In order to fix the gauge to a general R_{ξ} gauge, we add the gauge fixing term to the Lagrange density,

$$\mathcal{L} = -\frac{1}{2}\partial_{\mu}V_{\nu}(\partial^{\mu}V^{\nu} - \partial^{\nu}V^{\mu}) + \frac{1}{2}m_V^2 V_{\mu}V^{\mu} - \frac{1}{2\xi_V}(\partial_{\mu}V^{\mu})^2, \tag{277}$$

and construct the vector boson propagator in a semi-classical way as Green function,

$$\begin{aligned} D_V^{\mu\nu}(x-y) &= \int \frac{d^4k}{(2\pi)^4}\frac{iP_V^{\mu\nu}(k)e^{-ik(x-y)}}{k^2 - m_V^2 + i\epsilon}, \\ P_V^{\mu\nu}(k) &:= -g^{\mu\nu} + (1-\xi_V)\frac{k^{\mu}k^{\nu}}{k^2 - \xi_V m_V^2} \end{aligned} \tag{278}$$

where $(g^{\mu\nu}) = \text{diag}(1;-1,-1,-1)$ is the Minkowski metric. For the simplest case of Feynman gauge ($\xi_V = 1$), the gauge boson projector reads $P_V^{\mu\nu}(k) = -g^{\mu\nu}$. For Landau gauge $\xi_V = 0$, we obtained the purely transverse projector reads $P_V^{\mu\nu}(k) = -g^{\mu\nu} + k^{\mu}k^{\nu}/k^2$, and for the unitary gauge $\xi_V \to \infty$ we obtained the $P_V^{\mu\nu}(k) = -g^{\mu\nu} + k^{\mu}k^{\nu}/m_V^2$ which is transverse only on the mass shell $k^2 = m_V^2$.

10.2 Methodology

In order to calculate the propagator from the correlator function, we start with the quantisation of the vector field $V^{\mu}(x)$ by introducing creation and annihilation operators

$$V^{\mu}(x) = \sum_{\lambda}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\vec{k}\,)}}\left[\varepsilon^{\mu}(\vec{k},\lambda)a(\vec{k},\lambda)e^{-ikx} + \varepsilon^{\mu *}(\vec{k},\lambda)a^{\dagger}(\vec{k},\lambda)e^{ikx}\right] \tag{279}$$

with $[a(\vec{k},\lambda),a^\dagger(\vec{k}',\lambda')]=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}')$ and $\omega^2(\vec{k}\,)=\vec{k}^2+m_V^2$. From this we can calculate the two-point correlator function being the propagator as

$$\begin{aligned}D_V^{\mu\nu}(x-y) &= \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega(\vec{k}\,)}\\ &\times\left[\varepsilon^\mu(\vec{k},\lambda)\varepsilon^{\nu *}(\vec{k},\lambda)e^{-ik(x-y)}-\varepsilon^\nu(\vec{k},\lambda)\varepsilon^{\mu *}(\vec{k},\lambda)e^{ik(x-y)}\right].\end{aligned}\qquad(280)$$

We want to obtain the sum over the polarisations which is the completeness relation for the polarisation four-vectors. Except for the final step, we reached up to Eq. (5).

$$P_{V3}^{\mu\nu}(k)=\sum_{\lambda=1}^{3}\varepsilon^\mu(k,\lambda)\varepsilon^{\nu *}(k,\lambda)=g^{\mu 0}g^{\nu 0}-g^{\mu\nu}.\qquad(281)$$

in order to reach up to the completeness relation, one could consider to interpret $g^{\mu 0}$ as the components of the polarisation vector in time direction, $\varepsilon^\mu(k,0)$. However, in moving this part to the other side, one obtains a relative minus sign. This corresponds to an expression by David Tong, accomplished in a lecture on“Quantum Field Theory” given at the University of Cambridge,

$$\sum_{\lambda,\lambda'=0}^{3}g_{\lambda\lambda'}\varepsilon^\mu(k,\lambda)\varepsilon^{\nu *}(k,\lambda')=g^{\mu\nu}.\qquad(282)$$

relating the metric $g_{\lambda,\lambda'}$ in the ”inner” polarisation spacetime to the metric $g^{\mu\nu}$ in the ”outer” spacetime in the same way as this is accomlished by tetrads in Gravitation Theory (in this case, the ”tetrads” are the polarisation vectors). However, this does not solve our problem, as the completeness relation does not include the Minkowskian metric. Instead, we propose a pragmatic approach, stating that the propagator can be derived from the Green function and, therefore, the completeness relation is identical to the photon projector,

$$P_V^{\mu\nu}(k)=\sum_\lambda\varepsilon^\mu(k,\lambda)\varepsilon^{\nu *}(k,\lambda)=-g^{\mu\nu}+(1-\xi_V)\frac{k^\mu k^\nu}{k^2-\xi_V m_V^2}.\qquad(283)$$

Before deriving this general formula, we start with the case of the (massless) photon.

10.3 Green function of the photon

We start with the Lagrange density of the photon in order to explore the connection between the completeness relation and the propagator in greater depth,

$$\mathcal{L}_A=\frac{1}{2}(\vec{E}^2-\vec{B}^2)=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu},\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\qquad(284)$$

contains only the self energy of the photon, by variation of the action integral $S_A = \int \mathcal{L}_A d^4x$ the Euler–Lagrange equations can be obtained. One obtains

$$\delta S_A \quad = \quad -\frac{1}{4}\int \delta F_{\mu\nu}F^{\mu\nu}d^4x \;=\; \int \delta A_\nu \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)d^4x, \quad (285)$$

In order to vanish for an arbitrary variation δA_ν of the gauge field, one has to claim that

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\mu\partial^\nu A_\mu = (\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\mu = 0. \quad (286)$$

At the same time, the Green function $D_A^{\mu\rho}(x)$ cannot be determined by the equation

$$(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)D_A^{\mu\rho}(x-y) = ig_\nu^\rho \delta^{(4)}(x-y) \quad (287)$$

because the operator is not invertible. As indicated by Faddeev and Popov in 1967, this problem turns out to be deeply related to the gauge degree of freedom [26]. So the solution for this is to add a gauge fixing term to the classical action integral, that mean the Lagrangian is given by,

$$\mathcal{L}_{A+} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi_A}(\partial_\mu A^\mu)^2, \quad (288)$$

In order to find the solution for the Green function for this case, we derive the differential equation for the gauge field as $(\partial^2 g^{\mu\nu} - (1-\xi_A^{-1})\partial^\mu\partial^\nu)A_\mu = 0$ and then instead of the field we use the Green function reads

$$\left(\partial^2 g_{\mu\nu} - \left(1 - \frac{1}{\xi_A}\right)\partial_\mu\partial_\nu\right) D_A^{\mu\rho}(x-y) = ig_\nu^\rho \delta^{(4)}(x) \quad (289)$$

by this we can derive the Euler–Lagrange equation. We can calculate the general solution by calculating the corresponding equation for the Greens function. In momentum space the equation reads

$$\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi_A}\right)k_\mu k_\nu\right)\tilde{D}_A^{\mu\rho}(k) = ig_\nu^\rho, \quad (290)$$

and inverting this algebraic equation by using the ansatz $\tilde{D}_V^{\mu\rho}(k) = \tilde{D}^g g^{\mu\rho} + \tilde{D}^k k^\mu k^\rho$ one obtains the result in the form of the Green function

$$D_A^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} P_A^{\mu\nu}(k)\frac{ie^{-ik(x-y)}}{k^2} \quad (291)$$

As the projector defined as

$$P_A^{\mu\nu}(k) = -g^{\mu\nu} + (1-\xi_A)\frac{k^\mu k^\nu}{k^2} \quad (292)$$

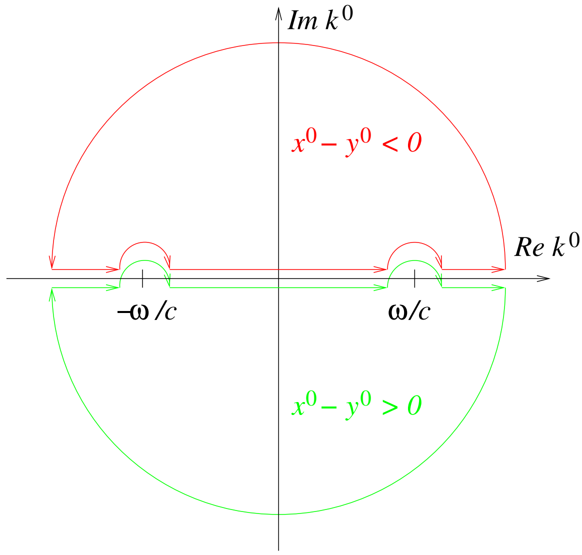


Figure 10: Conventions for the retarded propagator

Now in order to solve it, we have to make the connection for this pole that means for instance depending on how the convention for the poles at $k^2 = 0$

$$D_A^{\mu\nu}(x-y) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} P_A^{\mu\nu}(k) \frac{ie^{-ik(x-y)}}{k_0^2 - \omega^2(\vec{k})} \tag{293}$$

poles at $k_0 = \pm\omega(\vec{k})$, for the retarded propagator these have to be circumvented in the upper complex plane.

We are summing this poles by small circles, and for $x_0 > y_0$ we calculate and using the Cauchy residue theorem get,

$$D_A^{\mu\nu}(x-y) = -\int \frac{d^3k}{(2\pi)^3} \frac{2i\pi}{2\pi} \sum_{\pm} Res \left[P_A^{\mu\nu}(k) \frac{ie^{-ik(x-y)}}{k_0^2 - \omega^2(\vec{k})}; k_0 = \pm\omega(\vec{k}) \right]$$

$$D_A^{\mu\nu}(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(\vec{k})} P_A^{\mu\nu}(k) e^{-ik(x-y)} \Big|_{k_0=\omega(\vec{k})} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{-2\omega(\vec{k})} P_A^{\mu\nu}(k) e^{-ik(x-y)} \Big|_{k_0=-\omega(\vec{k})}$$

$$D_A^{\mu\nu}(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(\vec{k})} P_A^{\mu\nu}(k) \left(e^{-ik(x-y)} - e^{-ik(x-y)} \right) \Big|_{k_0=\omega(\vec{k})} \tag{294}$$

then we can compare the Eq. (4) and see that it will give us the projector and second part also give us the projector and this symmetric in μ and ν both are same and equal to the projector and we reached the goal as Eq. (278)

10.4 Green functions of massive gauge bosons

As we dealt with the photon case now we are including the mass and dealing with the massive gauge boson that means for this we replace $V^\mu(x) \to V^\mu(x) + \partial^\mu \lambda(x)$ and inserting this in the Euler–Lagrange equation of the Lagrange density without the gauge fixing term we get,

$$\mathcal{L}_V = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_V^2 V_\mu V^\mu \tag{295}$$

$$\partial_\mu F^{\mu\nu} + m_V^2 V^\nu = \Big((\partial^2 + m_V^2) g_{\mu\nu} - \partial_\mu \partial_\nu \Big) V^\mu = 0, \tag{296}$$

For the general case we can solve in this way as, for the V boson the Lagrange density contribution is given by

$$\mathcal{L}_V = -\frac{1}{2} \partial_\mu V_\nu (\partial^\mu V^\nu - \partial^\nu V^\mu) + \frac{1}{2} m_V^2 V_\mu V^\mu - \frac{1}{2\xi_V} (\partial_\mu V^\mu)^2 \tag{297}$$

and therefore the Green function for this Lagrange density reads as

$$\left(\partial^2 g_{\mu\nu} - \left(1 - \frac{1}{\xi_V} \right) \partial_\mu \partial_\nu + m_V^2 g_{\mu\nu} \right) D_V^{\mu\rho}(x) = i g_\nu^\rho \delta^{(4)}(x), \tag{298}$$

and this Proca equation is solved by

$$D_V^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} P_V^{\mu\nu}(k) \frac{i e^{-ik(x-y)}}{k^2 - m_V^2} \tag{299}$$

As the projector defined as

$$P_V^{\mu\nu}(k) = -g^{\mu\nu} + (1 - \xi_V) \frac{k^\mu k^\nu}{k^2 - \xi_V m_V^2}. \tag{300}$$

and then this result should be compared with our propagator, we can calculate from one side the greens function and from other hand side the propagator and then we compare the propagator with the greens function and from this we obtained this result for this projector (gauge boson propagator). We can defined this for Proca equation and for the Goldstone boson field h_Z. The final result will be the Eq. (278) defined, here we defined propogator with the special projector here $P_V^{\mu\nu}$.

10.5 Results

By using the pragmatic approach, we could construct the propagator as the Green function of the equation of motion. In this way, we can fix the sum over the polarisations to be this projector. This is essential for our third publication, where we use it in the usual Feynman gauge, as well as in the unitary gauge.

11 Identical particle and lepton mass effects in the decay $H \to Z^*(\to \tau^+\tau^-) + Z^*(\to \tau^+\tau^-)$

11.1 Introduction

Related to the decay of weak vector boson, in our case the Higgs boson decays into two off shell Z bosons, that again decay into a pair of charged leptons, $H \to Z^*(\to \ell^+\ell^-) + Z^*(\to \ell^+\ell^-)$. This channel offers a clean experimental signature and is important in the study of properties of the Higgs boson.

In 2021, the ATLAS and CMS collaborations at the LHC observed an excess of events in the 4ℓ final state around a mass of approximately $125\,\mathrm{GeV}$. The decay channel $H \to ZZ^{(*)} \to 4\ell$ was one of the main decay channels [27, 28]. Therefore, this particular channel is referred to as the golden Higgs channel. Identical particle final states such as $eeee$, $\mu\mu\mu\mu$ and $\tau\tau\tau\tau$ present a unique challenge due to the impossibility to distinguish the particles. However, some observables can still be extracted from such final states by carefully studying the corrections and patterns in the decay products. These observables provide a valuable information about the properties of the particles involved in the process.

In the related publication, a detail investigation is carried out for the decay $H \to \ell\ell\ell\ell$ of the Higgs boson into four leptons, and the identical particle effects in the decay distributions of $H \to Z^*Z^* \to \ell^+\ell^-\ell^+\ell^-$ are shown in detail see Refs. [29–34]. The importance of identical particle effects can be seen from the comparison of the branching ratios of a Higgs decaying into nonidentical and identical lepton pairs, collected from different original works in Ref. [35]. The branching ratios are listed in Ref. [35] are $B(H \to ee\mu\mu) = 5.93 \cdot 10^{-5}$ and $B(H \to eeee) = 3.27 \cdot 10^{-5}$. The approximate factor of two between the two rates reflects (i) the statistical factor of $1/4$ and (ii) the doubling of noninterference contributions in the identical particle case. The small deviation of the rate ratio from the exact value of 2 must be assigned to the contributions of the two interference terms.

In overall we intend to highlight the importance of considering the lepton masses in the phase space calculations and discuss about the observables that can be measured for the process involving the identical particles in the final state.

11.2 Methodolody

There are numerous Feynman diagrams that contribute to $H \to \ell\ell\ell\ell$ for massive leptons as for instance for $\ell = \tau^{\pm}$. These are divided up into the three classes I, II and III. In my thesis I concentrate on the first class containing the Feynman diagram that contribute to $H \to \ell\ell\ell\ell$ also in the zero lepton mass case. For the II class where $m \neq 0$ one has the additional contributions proportional to $g_{H\ell\ell}$ that is dealt with in the publication. The class III contributions $H \to \gamma^*\gamma^* \to \ell\ell\ell\ell$ stand for

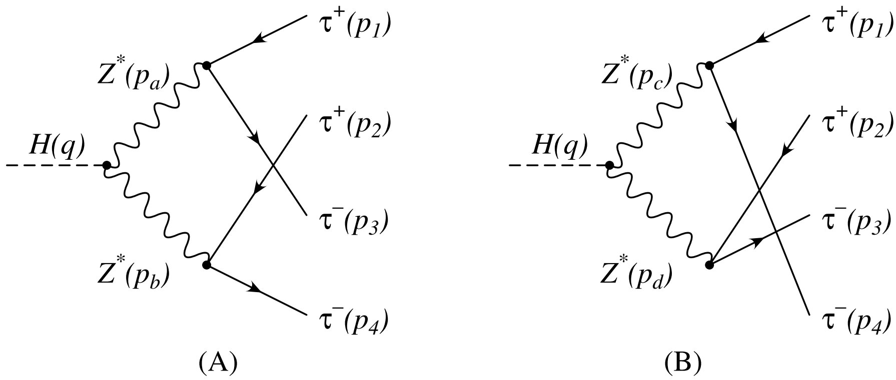


Figure 11: Feynman diagrams A and B contributing to $H \to Z^*(\to \tau^+\tau^-) + Z^*(\to \tau^+\tau^-)$

NLO radiative corrections or higher order loop contributions. These contributions involve virtual photons which then decay into charged lepton pairs. They can provide additional insights into the decay process. However, these contributions are generally smaller compared to the contributions in class I and class II and are not considered in the publication.

11.3 Contributing Feynman diagrams

In the four-body decay $H \to Z(\to \tau^+\tau^-) + Z^*(\to \tau^+\tau^-)$ we have two pairs of identical particles in the final state. Therefore, we have to take into account interference effects.

According to the two diagrams in Fig. 11 one has the two amplitudes

$$\begin{aligned} M_A &= M\left(\tau^+(p_1), \tau^-(p_3), \tau^+(p_2), \tau^-(p_4)\right) \\ M_B &= M\left(\tau^+(p_1), \tau^-(p_4), \tau^+(p_2), \tau^-(p_3)\right) \end{aligned} \tag{301}$$

By squaring the amplitudes one obtains

$$|M_A + M_B|^2 = |M_A|^2 + 2\,\mathrm{Re}(M_A M_B^*) + |M_B|^2. \tag{302}$$

The contribution rates from the two non-interference diagonal contributions, $|M_A|^2$ and $|M_B|^2$, are identical to each other, resulting in the absorptive part of a fermionic two-loop diagram. The nondiagonal interference contribution proportional to $2\,\mathrm{Re}(M_A M_B^*)$ corresponds to the absorptive part of a fermionic one-loop diagram. We have to include the extra minus sign for the nondiagonal contribution. The statistical factor $1/4$ has been taken into account in the identical fermion case. The loop diagrams prior to taking the absorptive part via the Cutkosky cut are shown in Fig. 12. We will calculate the decay rates for the two non-interference contributions $|M_A|^2 + |M_B|^2$ and the interference contributions $2\,\mathrm{Re}(M_A M_B^*)$ separately.

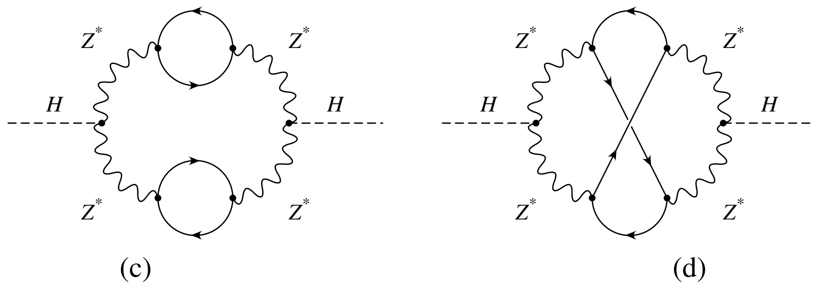


Figure 12: Contributions (a) $|M_A|^2 = |M_B|^2$ and (b) $\mathrm{Re}(M_A M_B^*) = \mathrm{Re}(M_B M_A^*)$ from $|M_A + M_B|^2$

11.4 The diagonal noninterference contribution $|M_A|^2 = |M_B|^2$

In order to get the absolute value of the squared matrix element of the process $|M_A|^2$ one has to take into account the HZZ vertex which is given by the factor

$$V_{\mu\nu}^{HZZ} = \frac{iem_Z g_{\mu\nu}}{\sin\theta_W \cos\theta_W}$$

The two Zff vertices are given by

$$V_{\mu}^{Zff} = \frac{ie}{4\sin\theta_W \cos\theta_W}(v_\ell \gamma_\mu - a_\ell \gamma_\mu \gamma_5)$$

In general, the four-particle phase space is given by

$$\Phi = \frac{1}{2^{14}\pi^6 m_H^2} \int v_a\, v_b \sqrt{\lambda(m_H^2, p_a^2, p_b^2)} dp_a^2 dp_b^2 d\cos\theta_a d\cos\theta_b d\phi$$

where v_a and v_b are velocities, defined as

$$v_a = \sqrt{1 - \frac{4m^2}{p_a^2}}, \qquad v_b = \sqrt{1 - \frac{4m^2}{p_b^2}}.$$

We obtain a two-fold trace like

$$\begin{aligned} |\mathcal{M}|^2_{AA} \sim\, & \mathrm{trace}(\gamma^\mu (v_\ell - a_\ell \gamma_5)(-\not{p}_1 + m)\gamma^\nu (v_\ell - a_\ell \gamma_5)(\not{p}_3 + m)) \\ & \times \mathrm{trace}(\gamma_\mu (v_\ell - a_\ell \gamma_5)(-\not{p}_2 + m)\gamma_\nu (v_\ell - a_\ell \gamma_5)(\not{p}_4 + m)) \end{aligned}$$

where $v_\ell = -1 + 4\sin^2\theta_W$ and $a_\ell = -1$ are the vector and axial vector couplings of the Z boson, respectively. After integration over $\cos\theta_a$, $\cos\theta_b$ and ϕ, we obtain the differential decay rate

$$|\mathcal{M}|^2_{AA} \sim \frac{1}{4p_a^2 p_b^2}(p_a^2 + 2m^2)(p_b^2 + 2m^2)\left(\lambda(m_H^2, p_a^2, p_b^2) + 12 p_a^2 p_b^2\right) v_\ell^4 + \ldots$$

Now the result has to be contracted with the Z boson propagators

$$\frac{-i}{p_a^2 - m_Z^2 + i\epsilon}\left(g^{\mu\nu} - \frac{p_a^\mu p_a^\nu}{m_Z^2}\right) = \frac{-i}{p_a^2 - m_Z^2 + i\epsilon}\left(P_1^{\mu\nu}(p_a) + F_S(p_a)P_0^{\mu\nu}(p_a)\right),$$

where the spin-1 and spin-0 projectors given by

$$P_1^{\mu\nu}(p_a) = g^{\mu\nu} - \frac{p_a^\mu p_a^\nu}{p_a^2}, \qquad P_0^{\mu\nu}(p_a) = \frac{p_a^\mu p_a^\nu}{p_a^2}.$$

The differential rate corresponding to the contribution of $|M_A|^2 + |M_B|^2 = 2|M_A|^2$ can be written in the form (including the identical particle factor of 1/4)

$$\begin{aligned}
&\frac{d\Gamma^{AA}}{dp_a^2 dp_b^2}(p_a^2, p_b^2) \\
&= \frac{1}{4}\cdot\frac{2\alpha^3}{9\pi^2 m_H^2}|\vec{p}_{Z^*}(p_a^2,p_b^2)|\frac{|\vec{p}_\ell(p_a)|}{\sqrt{p_a^2}}\frac{|\vec{p}_\ell(p_b)|}{\sqrt{p_b^2}}\frac{m_Z^2}{256\sin^6\theta_W\cos^6\theta_W} \\
&\quad\times\frac{1}{(p_a^2-m_Z^2)^2+m_Z^2\Gamma_Z^2}\,\frac{1}{(p_b^2-m_Z^2)^2+m_Z^2\Gamma_Z^2}\,p_a^2 p_b^2 \\
&\quad\times\left\{v_\ell^2(1+2\frac{m^2}{p_a^2})P_{1\mu\nu}(p_a) + a_\ell^2\Big((1-4\frac{m^2}{p_a^2})P_{1\mu\nu}(p_a)\right. \\
&\quad\left. -\,3a_\ell^2\cdot 2\frac{m^2}{p_a^2}F_S^2(p_a^2)P_{0\mu\nu}(p_a)\Big)\right\} \\
&\quad\times\left\{v_\ell^2(1+2\frac{m^2}{p_b^2})P_1^{\mu\nu}(p_b) + a_\ell^2\Big((1-4\frac{m^2}{p_b^2})P_1^{\mu\nu}(p_b)\right. \\
&\quad\left. -\,3a_\ell^2\cdot 2\frac{m^2}{p_b^2}F_S^2(p_b^2)P_0^{\mu\nu}(p_b)\Big)\right\}.
\end{aligned} \tag{303}$$

The general factors are coming from the phase space integration and vector boson propagators and projectors. $\alpha = e^2/(4\pi)$ is the fine structure constant for which we use the value $\alpha(m_H) \approx 1/120$. Singularities at $p_a^2 = m_Z^2$ and $p_b^2 = m_Z^2$ are defended by adding a Breit–Wigner contribution to the denominator factor.

11.5 Kinematics and the phase space for p_a^2 and p_b^2

The parameters of the phase space integration are p_a^2, p_b^2 , θ_a, θ_b and ϕ. The calculation shall be done in the rest frame of the Higgs boson.

For the kinematics we use $p_a = (E_a; \vec{p}_{Z^*})$ and $p_b = (E_b; -\vec{p}_{Z^*})$ with

$$E_a = \frac{m_H^2 + p_a^2 - p_b^2}{2m_H}, \quad E_b = \frac{m_H^2 - p_a^2 + p_b^2}{2m_H}, \quad |\vec{p}_{Z^*}| = \frac{1}{2m_H}\sqrt{\lambda(m_H^2, p_a^2, p_b^2)}, \tag{304}$$

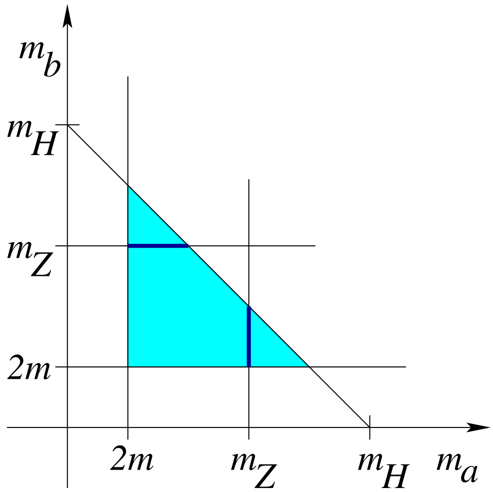


Figure 13: Phase space domain for the invariant masses m_a and m_b in dependence on the Higgs boson mass m_H, as restricted by the three straight lines $m_a = 2m$, $m_b = 2m$ and $m_a + m_b = m_H$ and painted in light blue.

and $p_a p_b = \frac{1}{2}(m_H^2 - p_a^2 - p_b^2)$, where $\lambda(a,b,c) := a^2+b^2+c^2-2ab-2ac-2bc$ is the Källén function. The remaining phase space factors $|\vec{p}_1| = |\vec{p}_3| = |\vec{p}_\ell(p_a)|$ and $|\vec{p}_2| = |\vec{p}_4| = |\vec{p}_\ell(p_b)|$ are calculated in the respective rest frames of the decaying vector bosons and read

$$|\vec{p}_\ell(p_a)| = \frac{1}{2}\sqrt{p_a^2 - 4m^2} =: \frac{1}{2}\sqrt{p_a^2}\, v_a, \qquad |\vec{p}_\ell(p_b)| = \frac{1}{2}\sqrt{p_b^2 - 4m^2} =: \frac{1}{2}\sqrt{p_b^2}\, v_b. \tag{305}$$

The rate is obtained from Eq. (303) by the integration over p_a^2 and p_b^2 according to

$$\Gamma^{AA} = \int_{4m^2}^{(m_H-2m)^2} dp_b^2 \int_{4m^2}^{(m_H-\sqrt{p_b^2})^2} dp_a^2 \quad \frac{d\Gamma^{AA}}{dp_a^2 dp_b^2}(p_a^2, p_b^2). \tag{306}$$

The boundary of the phase space domain can be found by demanding that the measure is real, i.e. all the radicands are positive. Claiming that

$$1 - \frac{4m^2}{p_a^2} \geq 0, \qquad 1 - \frac{4m^2}{p_b^2} \geq 0, \qquad \lambda(m_H^2, p_a^2, p_b^2) \geq 0, \tag{307}$$

the integration limits in terms of the p_a^2 and p_b^2 are given by

$$4m^2 \leq p_a^2 \leq (m_H - 2m)^2, \qquad 4m^2 \leq p_b^2 \leq (m_H - \sqrt{p_a^2})^2. \tag{308}$$

11.6 The nondiagonal interference contribution

In order to calculate the processes $H \to Z^*(\to \ell^+\ell^-)Z^*(\to \ell^+\ell^-)$, we take into account also the crossing contribution, that means the nondiagonal interference contribution. In this case the angular integration cannot be performed analytically because angular dependencies appear also in the denominator. In order to restore the angular integration we insert

$$\frac{1}{8\pi}\int d\cos\theta_a d\cos\theta_b d\phi \tag{309}$$

or, more symmetrically (but with dummy angle ϕ_b)

$$\frac{1}{16\pi^2}\int d\cos\theta_a d\cos\theta_b d\phi_a d\phi_b \tag{310}$$

to obtain (again without statistics factor $1/4$)

$$\begin{aligned}\frac{d\Gamma_{AB}}{dp_a^2 dp_b^2}(p_a^2,p_b^2) &= \frac{g^6}{2^{24}\pi^8\cos^6\theta_W}\frac{m_Z^2}{m_H^2}|\vec{p}_{Z^*}(p_a^2,p_b^2)|\frac{|\vec{p}_1|}{\sqrt{p_a^2}}\frac{|\vec{p}_2|}{\sqrt{p_b^2}}\\ &\times\int d\cos\theta_a d\cos\theta_b d\phi_a d\phi_b\\ &\times\frac{v_\ell^4 N_0+v_\ell^2 a_\ell^2 N_2+a_\ell^4 N_4}{(D_a^2+m_Z^2\Gamma_Z^2)(D_b^2+m_Z^2\Gamma_Z^2)(D_c^2+m_Z^2\Gamma_Z^2)(D_d^2+m_Z^2\Gamma_Z^2)},\end{aligned} \tag{311}$$

where N_i ($i=0,1,2,3,4$) are too long to be presented here.

Due to the crossing of momenta, for the nondiagonal interference contribution we cannot perform the calculation analytically, as we have angular dependencies in the denominator contained in D_c and D_d. That is why it is necessary to perform the calculation numerically. For this, we begin to set up a five dimensional phase space using VEGAS [36]. The Monte Carlo program VEGAS allows for up to 10 integrations at a time. The kinematics is expressed in terms of four-vectors in the rest frames of the decaying Z bosons with polar angles θ_a and θ_b, boosted to the rest frame of the Higgs boson via the rapidities

$$\lambda_a=\operatorname{arctanh}\left(\frac{\sqrt{\lambda(m_H^2,p_a^2,p_b^2)}}{m_H^2+p_a^2-p_b^2}\right),\quad \lambda_b=-\operatorname{arctanh}\left(\frac{\sqrt{\lambda(m_H^2,p_a^2,p_b^2)}}{m_H^2-p_a^2+p_b^2}\right), \tag{312}$$

and turned around the z axis through ϕ, resulting in

$$\begin{aligned}&p_{1/3}\\ &\quad=\frac{1}{2}\sqrt{p_a^2}\Big(\cosh\lambda_a\pm v_a\cos\theta_a\sinh\lambda_a;\pm v_a\sin\theta_a\cos\phi,\pm v_a\sin\theta_a\sin\phi,\\ &\qquad\qquad\sinh\lambda_a\pm v_a\cos\theta_a\cosh\lambda_a\Big),\\ &p_{2/4}\end{aligned}$$

$$= \frac{1}{2}\sqrt{p_b^2}\Big(\cosh\lambda_b \pm v_b\cos\theta_b\sinh\lambda_b; \pm v_b\sin\theta_b, 0,$$
$$\sinh\lambda_b \pm v_b\cos\theta_b\cosh\lambda_b\Big). \tag{313}$$

11.7 Results and Conclusion

We have dealt with the peculiarities of identical particle effects and worked on the dependence on the lepton mass. We have found that for increasing lepton mass the decay rates decrease for class-I contributions while class-II contributions lead to an increase of the decay rates. We have shown that nondiagonal class-I interference contributions are suppressed compared to diagonal class-I noninterference contributions by about a factor of 10. Lepton mass dependent class-II contributions correct the result by 6% but can be safely neglected for the lighter leptons. Mixed contributions between class-I and class-II processes can be neglected in all cases. In the publication we have also dealt with the narrow width approximation, and we have analysed in detail the angular dependence of the decay rate. For this and as a possible observable for future experiments, we worked on single angle decay distributions and identified the contributions of the diagonal and nondiagonal terms to two separate peaks of the distribution, indicating a clear assignment of the lepton momenta to the intermediate virtual Z bosons or a mixture of those, respectively.

Summary

First order electroweak radiative corrections to the decay of polarised W boson

According to common wisdom, our universe is made up of building blocks called particles. These particles are too small to be seen by human eye. Because of this the scientists constructed specialized detectors in order to observe these particles. The Large Hadron Collider (LHC) hosts a couple of these detectors. Located at CERN, the largest particle collider in the world operates at an energy of 13 TeV for proton-proton beam collisions. The most prominent result was the detection of the Higgs boson, reported in 2012.

From the side of theory, there are a couple of Gauge Theories that describe particle interactions, containing the electroweak and the strong interactions, by the exchange of vector bosons. Theses theories are collected in what we know nowadays as the Standard Model of Particle Physics. Our research in based on this Standard Model.

The thesis consists of three main topics. In the first topic, we deal with the propogator of vector bosons in the electoweak theory. We calculated the polarisation sum of the vector bosons, which is called the vector boson projector, and used a pragmatic approach by comparing the classical Green function with the propagator from Quantum Field Theory. We also tested that the unitary gauge is the most appropriate gauge for massive vector bosons. This is consistent with our expectation that the gauge theory should be independent of the choice of gauge.

The second topic focuses on identical particle effects and in particular on the dependence on the lepton mass. Here we studied in detail the decay channel $H \to Z^*(\to \ell^+\ell^-) + Z^*(\to \ell^+\ell^-)$, the cascade decay of the Higgs boson into two off-shell Z bosons, followed by the decay of each of those into a pair of tau lepton and antilepton. We also considered subordinate leading order decays with the same final state. We analysed the class-I diagrams, containing the Feynman diagrams that are also present for massless final states. By class-II, we encountered additional diagrams that are absent in the case of massless final states. We have found that for increasing lepton mass the decay rates decrease for class-I contributions while class-II contributions invert this trend. This is because the phase space shrink due to mass effects but we have additional diagrams, and therefore in this case the

cross section is increased. We calculated the decay rates and single angle decay distributions and compare them with the corresponding rates and distributions for the decay into electrons and muons.

In our third topic we work on the precision estimates of the Standard Model. As the Standard Model of particle physics is not completed up to now, it does not explain everything. Therefore, physicsts are looking for the physics beyond the Standard Model. In order to see physics beyond the Standard Model it is inevitable to improve the precision of the Standard Model prediction. We present next-to-leading order electroweak radiative corrections to the decay $W^+(\uparrow) \to c + \bar{b}$ of the polarised W boson into a pair of heavy quarks by looking at the angular distribution of the decay and by taking into account the polarisation and mass effects of the decay particles. Experiments have shown that decay rates increases if the final state contains massive particles. This demonstrates the importance of incorporating mass effects into the calculation. Incorporating the nonvanishing masses of the final state quarks and comparing with the limit of vanishing quark masses, known as the collinear limit, we found that the decay rate was reduced, contrary to our expectation. Therefore, our problem remains unsolved and is waiting for additional research.

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A very special place in my heart belongs to my son, who was born at the beginning of my PhD journey and has been with me throughout these years. Growing alongside my academic path, he has been a source of immense joy, motivation, and inspiration. His presence gave deeper meaning to my perseverance and strengthened my determination to overcome every challenge.

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I dedicate this thesis to my parents, my husband, and my son, with deepest love and gratitude.

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Kokkuvõte (in Estonian)

Esimest järku elektronõrgad kiirgusparandid polariseeritud W-bosoni lagunemisele

Üldlevinud arusaama kohaselt koosneb meie universum ehitusplokkidest, mida nimetatakse osakesteks. Need osakesed on liiga väikesed, et inimsilm neid näeks. Seetõttu on teadlased ehitanud spetsiaalsed detektorid nende osakeste vaatlemiseks. Suures Hadronite Põrguti (LHC) hõlmab paar sellist detektorit. CERNis asuv maailma suurim osakeste põrguti töötab prooton-prootonkiire kokkupõrgete energial 13 TeV. Kõige silmapaistvam tulemus oli Higgsi bosoni avastamine, millest teatati 2012. aastal.

Teoreetilisest küljest on mitmed kalibratsiooniteooriat, mis kirjeldavad osakeste vastastikmõjusid, sealhulgas elektronõrka ja tugevat vastastikmõju, vektorbosonite vahendamisel. Need teooriad on koondatud osakestefüüsika standardmudelisse. Meie uurimistöö põhineb sellel standardmudelil.

Väitekiri koosneb kolmest põhiteemast.

Esimeses teemas käsitleme vektorbosoni propagaatorit elektronõrga teoorias. Arvutasime vektorbosonite polarisatsioonisumma, mida nimetatakse vektorbosonite projektoriks, ja kasutasime pragmaatilist lähenemist, võrreldes klassikalist Greeni funktsiooni kvantväljateooria propagaatoriga. Samuti testisime, et unitaarne kalibratsioon on massiivsete vektorbosonite jaoks kõige sobivam kalibratsioon. See on kooskõlas meie ootusega, et kalibratsiooniteooria peaks olema kalibratsioonivalikust sõltumatu.

Teine teema keskendub identsetele osakeste efektidele ja eriti sõltuvusele leptoni massist. Siin uurisime üksikasjalikult lagunemiskanalit $H \to Z^*(\to \ell^+\ell^-) + Z^*(\to \ell^+\ell^-)$, Higgsi bosoni kaskaadlagunemist kaheks väliseks Z-bosoniks, millele järgneb mõlema lagunemine tau-leptoni ja antileptoni paariks. Samuti käsitlesime sama lõppseisundiga alluvaid juhtiva järgu lagunemisi. Analüüsisime I klassi diagramme, mis sisaldavad Feynmani diagramme, esinedes ka massita lõppseisundite puhul. II klassi puhul kohtasime täiendavaid diagramme, mis massita lõppseisundite puhul puuduvad. Leidsime, et leptoni massi suurenedes vähenevad I klassi panuste lagunemiskiirused, samas kui II klassi panused pööravad selle trendi ümber. See on tingitud faasiruumi kahanemisest massiefektide tõttu, kuid meil on täiendavaid diagramme ja seetõttu suureneb antud juhul ristlõige. Arvutasime lagunemismäärad ja uksiknurka lagunemisjaotused ning võrdlesime neid vastavate määrade ja jaotustega elektronideks ja müüoniteks lagunemise puhul.

Kolmandas teemas tegeleme standardmudeli täpsusennestustega. Kuna osakestefüüsika standardmudel pole täiuslik, ei seleta see kõike. Seetõttu otsivad füüsikud füüsikat, mis läheb standardmudelist kaugemale. Selleks, et näha standardmudelist kaugemat füüsikat, on vältimatu parandada standardmudeli ennustuse täpsust. Esitleme polariseeritud W bosoni lagunemise $W^+(\uparrow) \to c + \bar{b}$ esimest järku elektronõrga kiirgusparandusi raskete kvarkide paariks, vaadeldes lagunemise nurkjaotust ja võttes arvesse lagunevate osakeste polarisatsiooni ja massiefekte. Katsed on näidanud, et lagunemiskiirus suureneb, kui lõppolek sisaldab massiivseid osakesi. See näitab massiefektide arvutusse kaasamise olulisust. Lisades lõppoleku kvarkide mittekaduvad massid ja võrreldes neid kaduvate kvarkide masside piirväärtusega, mida tuntakse kollineaarse piirväärtusena, leidsime, et lagunemiskiirus vähenes, vastupidiselt meie ootustele. Seega jääb meie probleem lahendamata ja ootab täiendavaid uuringuid.

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