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COLLOCATION AND PROJECTION METHODS FOR INTEGRAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

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CONVERGENCE RATE OF A MODIFIED CUBATURE FORMULA METHOD FOR MULTIDIMENSIONAL WEAKLY SINGULAR INTEGRAL EQUATIONS

Gennadi Vainikko and Arvet Pedas

A class of weakly singular integral equations of the second kind on a bounded open set $G \subset \mathbb{R}^n$ *is considered. Discretisations of the problem are constructed by means of three methods: a piecewise constant collocation method, a classical cubature formula method (using a second order cubature formula constructed in the paper), a modified cubature formula method introduced by Kantorovich and Krylov* [8]. *First two methods are investigated in other papers. The aim of this paper is to investigate the convergence rate of the third method and to compare it with first two ones (theoretically and numerically).*

1.The problem. Let $G \subset \mathbb{R}^n$ $(n \ge 2)$ be an open bounded set with a piecewise smooth boundary ∂G. Consider the integal equation

$$u(x) = \int_G K(x,y)u(y)dy + f(x), \quad x \in G. \tag{1}$$

We assume that the kernel $K(x,y)$ is twice continuously differentiable on $(G\times G)\setminus\{x=y\}$ and that there exists a real number $\nu\in(-\infty,n)$ such that the following estimates hold:

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial x_n}+\frac{\partial}{\partial y_n}\right)^{\beta_n}K(x,y)\right| \le$$

$$\le b\cdot\begin{cases} 1+|x-y|^{-\nu-|\alpha|} & \text{if} \quad \nu+|\alpha|\ne 0 \\ 1+|\ln|x-y|| & \text{if} \quad \nu+|\alpha|=0 \end{cases} \quad (|\alpha|+|\beta|\le 2). \tag{2}$$

Here $b=\text{const}$, $|\alpha|=\alpha_1+\ldots+\alpha_n$ for a multiindex $\alpha=(\alpha_1,\ldots,\alpha_n)$ with non negative $\alpha_k\in Z$ and $|x|=(x_1^2+\ldots+x_n^2)^{1/2}$ for $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$. Typical examples of kernels satifying (2) are given by $K(x,y)=k(x,y)|x-y|^{-\nu}$

$(0<\nu<n)$ and $K(x,y)=k(x,y)\ln|x-y|$ $(\nu=0)$ where $k(x,y)$ is a twice continuously differentiable function on $(G\times G)\setminus\{x=y\}$ such that the derivatives $D_x^\alpha k(x,y)$ $(|\alpha|\le 2)$ are bounded or, e.g. in case $0<\nu<n$,

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial x_n}+\frac{\partial}{\partial y_n}\right)^{\beta_n}k(x,y)\right|\le b_1|x-y|^{-|\alpha|}\quad(|\alpha|+|\beta|\le 2).$$

A further example is given by the kernel of the Peierls integral equations ($n=3,\nu=2$; see [9]).

Denote by $\rho(x)=\inf_{y\in\partial G}|x-y|$ the distance from x to ∂G and introduce the weight space $C^{2,\mu}(G)$, $\mu<n$, consisting of twice continuously differentiable functions u on G such that

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}u(x)\right|\le c\begin{cases}1 & ,\ |\alpha|<n-\mu\\ 1+|\ln\rho(x)| & ,\ |\alpha|=n-\mu\\ \rho(x)^{n-\mu-|\alpha|} & ,\ |\alpha|>n-\mu\end{cases}\qquad(|\alpha|\le 2, x\in G),$$

where $c=c_u$ is a constant.We assume that

$$f\in C^{2,\mu}(G),\ \nu\le\mu<n. \tag{3}$$

Our last assumtion about equation (1) is that,for any pair of points $x^1,x^2\in G$,

$$|f(x^1)-f(x^2)|\le \text{const}\begin{cases}d_G(x^1,x^2) & ,\ \mu<n-1\\ d_G(x^1,x^2)\left[1+|\ln d_G(x^1,x^2)|\right] & ,\ \mu=n-1\\ d_G(x^1,x^2)^{n-\mu} & ,\ \mu>n-1\end{cases} \tag{4}$$

where $d_G(x^1,x^2)$, the "inner distance " between x^1 and x^2, is the infimum of the lengths of polygons in G joining x^1 and x^2; if x^1 and x^2 belong to different domains of connectivity of G we put $d_G(x^1,x^2)=d_\infty$ where $d_\infty\le\infty$ is greater than d_G-diameter of any connectivity component.

Actually, (4) is a consequence from (3) in case $\mu<n-1$.Thus assumtion (4) is needed only in case $\mu\ge n-1$.

<u>**2.Cubature formula.**</u> For a $h>0$, introduce an "approximate subdivision" of G into open sets ("cells ") $G_{jh}\subset\mathbb{R}^n$ $(j=1,\dots,l_h)$ such that

$$\left\{\begin{array}{c}G_{jh}\cap G\ne\emptyset,\ G_{ih}\cap G_{jh}=\emptyset\ (i\ne j)\\ \operatorname{diam}G_{jh}\le h,\ d_G\text{-}\operatorname{diam}(G_{jh}\cap G)\le c_1h\ \ (j=1,\dots,l_h),\\ (\bar G\setminus\bar G_h)\cup(\bar G_h\setminus\bar G)\subset\{x\in\mathbb{R}^n:\rho(x)\le c_2h^2\}\end{array}\right\} \tag{5}$$

where

$$G_h=\bigcup_{j=1}^{l_h}G_{jh}$$

and the constants c_1 and c_2 do not depend on h. Choose points $\xi_{jh} \in G_{jh} \cap G$ $(j=1,\dots,l_n)$ as follows:

$$\begin{cases} \xi_{jh} = (\operatorname{mes} G_{jh})^{-1} \int\limits_{G_{jh}} y\,dy \ (=\text{centroid of } G_{jh}) & \text{if } \operatorname{dist}(\operatorname{co}G_{jh}, \partial G) \ge h, \\ \xi_{jh} \in G_{jh} \cap G \quad \text{is an arbitrary point} & \text{if } \operatorname{dist}(\operatorname{co}G_{jh}, \partial G) < h. \end{cases} \tag{6}$$

Here coA is the convex hull of a set $A \subset \mathbb{R}^n$ and $\operatorname{dist}(A_1,A_2) = \inf_{x^1 \in A_1, x^2 \in A_2} |x^1 - x^2|$. Now we can introduce a cubature formula:

$$\int\limits_G u(y)dy \approx \sum_{j=1}^{l_h} w_{jh} u(\xi_{jh}), \qquad w_{jh} = \operatorname{mes} G_{jh}. \tag{7}$$

3.Discretisation of the integral equation. We introduce the following three discretisations of equation (1):

$$u_{ih} = \sum_{j=1}^{l_h} t_{ijh} u_{jh} + f(\xi_{ih}) \quad (i=1,\dots,l_h), \quad t_{ijh} = \int\limits_{G_{jh}} K(\xi_{ih},y)dy; \tag{8}$$

$$u_{ih} = \sum_{j=1}^{l_h} t'_{ijh} u_{jh} + f(\xi_{ih}) \quad (i=1,\dots,l_h), \quad t'_{ijh} = \begin{cases} K(\xi_{ih},\xi_{jh})w_{jh}, & i \ne j \\ 0 & , i=j; \end{cases} \tag{8'}$$

$$u_{ih} = \sum_{j=1}^{l_h} t''_{ijh} u_{jh} + f(\xi_{ih}) \quad (i=1,\dots,l_h),$$

$$t''_{ijh} = \begin{cases} K(\xi_{ih},\xi_{jh})w_{jh}, & i \ne j \\ \int\limits_{G_h} K(\xi_{ih},y)dy - \sum\limits_{\substack{k=1 \\ k \ne i}}^{l_h} K(\xi_{ih},\xi_{kh})w_{kh}, & i=j. \end{cases} \tag{8''}$$

In methods (8) and (8") it may happen that the kernel K(x,y) must be integrated over $G_{jh} \not\subset G$ or $G_h \equiv \bigcup_{1 \le j \le l_h} G_{jh} \not\subset G$. We assume that K(x,y) is extended with respect to y on G_h so that estimate (2) remains valid for $|\alpha| = |\beta| = 0$:

$$|K(x,y)| \le b \begin{cases} 1 + |x-y|^{-\nu}, & \nu \ne 0, \\ 1 + |\ln|x-y||, & \nu = 0, \end{cases} \qquad (x \in G, y \in G_h).$$

System (8) corresponds to a collocation method: denoting by φ_{jh} the characteristic function of G_{jh}, approximating the solution u of equation (1) by a piecewise constant function

$$\bar{u}_h = \sum_{j=1}^{l_h} u_{jh} \varphi_{jh}$$

and collocating the equation in the points ξ_{ih} $(i=1, \dots ,l_h)$ with G_h as the domain of integration instead of G, we obtain system (8).

System (8') corresponds to a classical cubature formula method: approximating the integral in (1) by means of cubature formula (7) and collocating in the points $\xi_{ih}(i=1, \dots ,i_h)$ we obtain system (8') if we reject the terms where $i=j$ (the kernel $K(x,y)$ is not defined for $x=y$).On the other hand, (8') may be considered as an approximation to (8):

$$t_{ijh}=\int_{G_{jh}} K(\xi_{ih},y)dy \approx K(\xi_{ih},\xi_{jh})w_{jh}=t'_{ijh} \quad (i\neq j).$$

System (8") corresponds to the Kantorovich-Krylov modification of the cubature formula method [8].Namely,rewrite equation (1) as follows:

$$u(x)=\int_G K(x,y)[u(y)-u(x)]dy+u(x)\int_G K(x,y)dy+f(x).$$

The second integral replace by

$$\int_{G_h} K(x,y)dy$$

and calculate it, for $x=\xi_{ih}$ $(i=1, \dots ,l_h)$, exactly (or,say,at least with h^2-accuracy).For the first integral with $x=\xi_{ih}$, use cubature formula (7).As the result one then obtains system (8").It may be considered as a refinement of system (8') - a new formula instead of $t'_{iih}=0$ is given, not changing the elements t'_{ijh} with $i\neq j$.

Methods (8) and (8') are investigated in [3,4]; we shall refer the main results and some technical details needed for method (8"). One dimensional case $(n=1)$ is analysed in [5,6,10].

4.Error estimates. Theorem 1. *Let conditions* (2)-(6) *be fulfilled and let* 1 *be a non-characteristic value of integral equation* (1).*Then there exists a* $h_0>0$ *such that, for* $0<h<h_0$, *system* (8) *has a unique solution* (u_{ih}), *and*

$$\max_{1\le i\le l_h} |u_{ih}-u(\xi_{ih})| \le \text{const}\, \varepsilon_{\nu h}\varepsilon_{\mu h}, \qquad \varepsilon_{\nu h}=\begin{cases} h & ,\nu<n-1,\\ h(1+|\ln h|) & ,\nu=n-1,\\ h^{n-\nu} & ,\nu>n-1, \end{cases} \quad (9)$$

where u *is the (unique) solution to* (1), $u\in C^{2,\mu}(G)$.

A **proof** of Theorem 1 is given in [3] and, more fully, in [4]. Here we represent some blocks of the proof that will be used in the sequal.Denote by $\Xi_h=\{\xi_{jh}: j=1, \dots ,l_h\}$ the grid and by $E_h=C(\Xi_h)$

the space of grid functions $u_h : \Xi_h \to \mathbb{R}$ (or $\mathbb{C}$),

$$\| u_h \| = \| u_h \|_{E_h} = \max_{1 \le j \le l_h} | u_h(\xi_{jh}) | \quad \text{for} \quad u_h \in E_h .$$

A $l_h \times l_h$–matrix $T_h = (t_{ijh})$ may be considered as a linear operator in E_h, thereby

$$\| T_h \| = \| T_h \|_{L(E_h, E_h)} = \max_{1 \le i \le l_h} \sum_{j=1}^{l_h} | t_{ijh} | .$$

Now consider (8) as an equation $u_h = T_h u_h + p_h f$ in E_h, defining $p_h \in L(E, E_h)$ as a restriction operator:

$$(p_h u)(\xi_{jh}) = u(\xi_{jh}) \quad (j = 1, \dots, l_h), \qquad u \in E = BC(G), \ \|u\|_E = \sup_{x \in G} |u(x)| .$$

The proof of Theorem 1 consists of two traditional blocks. Firstly, one shows the stability of the method: for sufficiently small h (say $0 < h < h_0$) the operator $I_h - T_h$ (where I_h is the identity in E_h) is invertible and

$$\| (I_h - T_h)^{-1} \|_{L(E_h, E_h)} \le \text{const} \quad (0 < h < h_0). \tag{10}$$

Secondly, for the solution $u \in C^{2,\mu}(G)$ (see [2]) of equation (1) one establishes the approximation property

$$\| p_h T u - T_h p_h u \| \le \text{const}\, \varepsilon_{\nu h} \varepsilon_{\mu h} , \tag{11}$$

where $T \in L(E, E)$ is the integral oprator of equation (1),

$$(Tu)(x) = \int_G K(x, y) u(y) dy .$$

Theorem 2. *Let the conditions of Theorem 1 be fulfilled and let*

$$| \xi_{ih} - \xi_{jh} | \ge c_0 h \quad (i \ne j) \tag{12}$$

with a constant $c_0 > 0$ *not depending on* h. *Then there exists a* $h'_0 > 0$ *such that, for* $0 < h < h'_0$, *system* (8') *has a unique solution* (u_{ih}), *and*

$$\max_{1 \le i \le l_h} | u_{ih} - u(\xi_{ih}) | \le \text{const}(\varepsilon_{\nu h} \varepsilon_{\mu h} + \varepsilon'_{\nu h}), \quad \varepsilon'_{\nu h} = \begin{cases} h^2 & , \nu < n-2, \\ h^2(1 + |\ln h|) & , \nu = n-2, \\ h^{n-\nu} & , \nu > n-2. \end{cases} \tag{9'}$$

The proof is based on (10),(11) and the inequality

$$\| T'_h - T_h \|_{L(E_h, E_h)} \le \text{const}\, \varepsilon'_{\nu h} \tag{13}$$

where $T_h = (t_{ijh})$ and $T'_h = (t'_{ijh})$ are the matrices with elements defined in (8) and (8').For details, see [4].

Theorem 3. *Let the conditions of Theorem 2 be fulfilled and let*

$$d_G(x^1,x^2) \le \text{const}\,|x^1-x^2| \quad (x^1,x^2 \in G). \tag{14}$$

Then there exists a h_0'' *such that , for* $0<h<h_0''$, *system* (8") *has a unique solution* (u_{ih}), *and error estimate* (9) *holds.*

The **proof** is given Section 5.

We see that methods (8) and (8") are of the same accuracy. In general, one has less work computing t''_{ijh} $(i,j=1,\dots,l_h)$ instead of t_{ijh} $(i,j=1,\dots,l_h)$. On the other hand , the possibilities to apply method (8") is restricted by supplementary condition (14) , e.g. an inner boundary of G is not allowed.

For $\nu \ge n-2$, method (8") is more precise than (8').For $\nu<n-2$, all three methods (8),(8'),(8") are of the same accurary $O(h^2)$.

Remark 1. Using the solution (u_{ih}) of system (8"), define

$$u_h(x) = \sum_{\substack{j=1\\ |x-\xi_{jh}|\ge c_0 h}}^{l_h} K(x,\xi_{jh})w_{jh}u_{jh} + f(x), \quad x\in G. \tag{15}$$

Then, under assumtions of Theorem 3

$$\sup_{x\in G} |u_h(x)-u(x)| \le \text{const}\,\varepsilon_{\nu h}\varepsilon_{\mu h}. \tag{16}$$

Remark 2. Condition (12) may be omitted if we slightly modify the construction of systems (8) and (8"):

$$t'_{ijh} = \begin{cases} K(\xi_{ih},\xi_{jh})w_{jh} & \text{if } |\xi_{ih}-\xi_{jh}| \ge c_0 h, \\ 0 & \text{if } |\xi_{ih}-\xi_{jh}| < c_0 h, \end{cases}$$

$$t''_{ijh} = \begin{cases} K(\xi_{ih},\xi_{jh})w_{jh} & \text{if } |\xi_{ih}-\xi_{jh}| \ge c_0 h, \\ 0 & \text{if } |\xi_{ih}-\xi_{jh}| < c_0 h \text{ but } i \ne j, \\ \int_{G_h} K(\xi_{ih},y)\,dy - \sum_{\substack{k=1\\ k\ne i}}^{l_h} K(\xi_{ih},\xi_{kh})w_{kh} & \text{if } i=j. \end{cases}$$

5.Proof of Theorem 3. (i) It sufficies to show that

$$\|T''_h - T'_h\| \le \text{const}\,\varepsilon_{\nu h} \tag{17}$$

and

$$\|(T''_h - T_h)p_h u\| \le \text{const}\,\varepsilon_{\nu h}\varepsilon_{\mu h} \tag{18}$$

where u is the solution to equation (1) and T_h, T'_h, T''_h are the matrices with elements $t_{ijh}, t'_{ijh}, t''_{ijh}$ defined in (8),(8'),(8").From

(17),(13) and (10), the stability of method (8") follows:

$$\|(I_h - T''_h)^{-1}\|_{L(E_h, E_h)} \le \text{const} \quad (0 < h < h''_0). \tag{19}$$

From (18) and (11) we obtain the approximation property

$$\|p_h Tu - T''_h p_h u\| \le \text{const}\, \varepsilon_{\nu h} \varepsilon_{\mu h}. \tag{20}$$

For solutions to (1) and (8") we have

$$(I_h - T''_h)(u_h - p_h u) = T''_h p_h u - p_h Tu$$

and, together with (19) and (20),

$$\|u_h - p_h u\| \le \text{const}\, \varepsilon_{\nu h} \varepsilon_{\mu h}.$$

This is estimate (9) for the solution $u_h = (u_{ih})$ of system (8").

(ii) To prove (17) and (18), we need some inequalities for weakly singular integrals. First two ones are elementary:

$$\int_{|x-y| \le h} |x-y|^{-\lambda}\, dy \le \text{const}\, h^{n-\lambda} \quad (0 \le \lambda < n), \tag{21}$$

$$\int_{h < |x-y| < 1} |x-y|^{-\lambda}\, dy \le \text{const} \begin{cases} 1, & 0 \le \lambda < n, \\ 1 + |\ln h|, & \lambda = n, \\ h^{n-\lambda}, & \lambda > n. \end{cases} \tag{22}$$

The next two inequalities need the piecewise smoothness of ∂G:

$$\int_{\{y \in G : \rho(y) < h\}} |x-y|^{-\lambda}\, dy \le \text{const}\, \varepsilon_{\lambda h} \quad (0 \le \lambda < n), \tag{23}$$

$$\int_{\{y \in G : \rho(y) < h, |x-y| > h\}} |x-y|^{-n} \le \text{const}. \tag{24}$$

Finally, from (2) it follows, that for any $u \in L_\infty(G)$ and $x^1, x^2 \in G$,

$$|(Tu)(x^1) - (Tu)(x^2)| \le$$

$$\le \text{const} \sup_{y \in G} |u(y)| \begin{cases} d_G(x^1,x^2), & \nu < n-1, \\ d_G(x^1,x^2)\left[1 + |\ln d_G(x^1,x^2)|\right], & \nu = n-1, \\ d_G(x^1,x^2)^{n-\nu}, & \nu > n-1. \end{cases} \tag{25}$$

For detailed proofs of (23)-(25) see [4].

(iii) Now we prove (17). We have

$$\|T''_h - T'_h\| = \max_{1 \le i \le l_h} \left| \int_{G_h} K(\xi_{ih}, y)\, dy - \sum_{\substack{j=1 \\ j \ne i}}^{l_h} K(\xi_{ih}, \xi_{jh}) w_{jh} \right| \le \delta_h^{(1)} + \delta_h^{(2)} + \delta_h^{(3)}$$

where, with constant c_1 from (5),

$$\delta_h^{(1)} = \max_{1\le i\le l_h} \sum_{\substack{j=1\\ j\ne i,\,\mathrm{dist}(\xi_{ih},G_{jh})\le 2c_1 h}}^{l_h} \int_{G_{jh}} \left[|K(\xi_{ih},y)| + |K(\xi_{ih},\xi_{jh})|\right]dy,$$

$$\delta_h^{(2)} = \max_{1\le i\le l_h} \sum_{\substack{j=1\\ \mathrm{dist}(\xi_{ih},G_{jh}) > 2c_1 h}}^{l_h} \int_{G_{jh}\setminus G} \left[|K(\xi_{ih},y)| + |K(\xi_{ih},\xi_{jh})|\right]dy,$$

$$\delta_h^{(3)} = \max_{1\le i\le l_h} \sum_{\substack{j=1\\ \mathrm{dist}(\xi_{ih},G_{jh}) > 2c_1 h}}^{l_h} \int_{G_{jh}\cap G} |K(\xi_{ih},y) - K(\xi_{ih},\xi_{jh})|\,dy.$$

Consider the most complicated case when $\nu>0$ in (2): $|K(\xi_{ih},y)|\le b|\xi_{ih}-y|^{-\nu}$ and, by virtue of (12), $|K(\xi_{ih},\xi_{jh})|\le b|\xi_{ih}-\xi_{jh}|^{-\nu} \le b'h^{-\nu}$. Using (21) we obtain

$$\delta_h^{(1)} \le \max_{1\le i\le l_h} \left\{ b \int_{|y-\xi_{ih}|\le(2c_1+1)h} |y-\xi_{ih}|^{-\nu}dy + b'h^{-\nu} \int_{|y-\xi_{ih}|\le(2c_1+1)h} dy \right\} \le$$

$$\le \mathrm{const}\; h^{n-\nu} \le \mathrm{const}\; \varepsilon_{\nu h}.$$

Further, $|\xi_{ih}-\xi_{jh}| \asymp |\xi_{ih}-y|$ for $y\in G_{jh}$ when $\mathrm{dist}(\xi_{ih},G_{jh})>2c_1h$, therefore

$$\delta_h^{(2)} \le c \max_{1\le i\le l_h} \int_{\{y\in G:\, \rho(y)\le c_2h^2\}} |\xi_{ih}-y|^{-\nu}dy \le c'\varepsilon_{\nu h^2} \le c''(\varepsilon_{\nu h})^2$$

(see (5) and (23)). Connecting points $y\in G_{jh}\cap G$ and $\xi_{jh}\in G_{jh}\cap G$ with a polygon in G of length $\le(3/2)c_1h$ (see(5)) and, using on its any link the Lagrange mean value theorem we see that

$$|K(\xi_{ih},y)-K(\xi_{ih},\xi_{jh})| \le \mathrm{const}\; h|\xi_{ih}-y|^{-\nu-1} \qquad (y\in G_{jh}\cap G).$$

Here $\mathrm{const}|\xi_{ih}-y|^{-\nu-1}$ is an estimate to first derivatives of $K(\xi_{ih},y^1)$ with y^1 from the polygon; note that it follows from inequalities $\mathrm{dist}(\xi_{ih},G_{jh})>2c_1h$ and $|y^1-y|\le(3/2)c_1h$ that $|\xi_{ih}-y^1| \asymp |\xi_{ih}-y|$. We obtain (see (22))

$$\delta_h^{(3)} \le c'h \max_{1\le i\le l_h} \int_{2c_1h<|\xi_{ih}-y|<1} |\xi_{ih}-y|^{-\nu-1}dy \le c''h \begin{cases} 1 & ,\nu<n-1, \\ 1+|\ln h|, & \nu=n-1, \\ h^{n-\nu-1}, & \nu>n-1, \end{cases}$$

and therefore $\delta_h^{(3)} \le \mathrm{const}\,\varepsilon_{\nu h}$. The proof of (17) is completed.

(iv) To prove (18) we first remark that

$$((T_h-T''_h)p_h u)(\xi_{ih}) = \sum_{\substack{j=1\\ j\neq i}}^{l_h} \int_{G_{jh}} [K(\xi_{ih},y) - K(\xi_{ih},\xi_{jh})][u(\xi_{jh}) - u(\xi_{ih})]dy.$$

From (4) and (25) it follows that a similar inequality holds for u=f+Tu.Thanks condition (14) , it takes the form

$$|u(x^1) - u(x^2)| \le const \begin{cases} |x^1-x^2| & ,\mu<n-1, \\ |x^1-x^2|(1+|\ln|x^1-x^2||), & \mu=n-1, \\ |x^1-x^2|^{n-\mu} & ,\mu>n-1. \end{cases} \qquad (x^1,x^2\in G)$$

Further,according to (6), for "inner cells", we have

$$\int_{G_{jh}} (y-\xi_{jh})dy = 0$$

and therefore

$$\int_{G_{jh}} \frac{\partial K(\xi_{ih},\xi_{jh})}{\partial y}(y-\xi_{jh}) = 0$$

where $\partial K(x,y)/\partial y$ is the Frechet derivative of K(x,y) with respect to y.Using (21) -(24) and some arguments from Section (iii) we obtain

$$\|(T_h-T''_h)p_h u\| \le const((\varepsilon_{\nu h})^2 + \delta_h^{(4)} + \delta_h^{(5)} + \delta_h^{(6)})$$

where

$$\delta_h^{(4)} = \max_{1\le i\le l_h} \sum_{\substack{j=1\\ j\neq i\\ dist(\xi_{ih},G_{jh})\le\\ \le 2c_1 h}}^{l_h} \int_{G_{jh}} [|K(\xi_{ih},y)|+|K(\xi_{ih},\xi_{jh})|]dy \begin{Bmatrix} h & ,\mu<n-1 \\ h(1+|\ln h|), & \mu=n-1 \\ h^{n-\mu} & ,\mu>n-1 \end{Bmatrix} \le$$

$$\le const\, \varepsilon_{\nu h}\varepsilon_{\mu h},$$

$$\delta_h^{(5)} = \max_{1\le i\le l_h} \sum_{\substack{j=1\\ dist(\xi_{ih},G_{jh})>2c_1 h\\ dist(coG_{jh},\partial G)<h}}^{l_h} \int_{G\cap G_{jh}} |K(\xi_{ih},y)-K(\xi_{ih},\xi_{jh})|dy \times$$

$$\times \begin{Bmatrix} |\xi_{ih}-\xi_{jh}| & ,\mu<n-1 \\ |\xi_{ih}-\xi_{jh}|(1+|\ln|\xi_{ih}-\xi_{jh}||), & \mu=n-1 \\ |\xi_{ih}-\xi_{jh}|^{n-\mu} & ,\mu>n-1 \end{Bmatrix} \le$$

$$\le ch \max_{1\le i\le l_h} \int_{\{y\in G:\rho(y)<2h,\ |y-\xi_{ih}|>2c_1h\}} \left\{ \begin{array}{ll} |y-\xi_{ih}|^{-\nu} & ,\mu<n-1 \\ |y-\xi_{ih}|^{-\nu}(1+|\ln|y-\xi_{ih}||) & ,\mu=n-1 \\ |y-\xi_{ih}|^{-\nu-1+n-\mu} & ,\mu>n-1 \end{array} \right\} dy \le$$

$$\le c'h \left\{ \begin{array}{ll} h & ,\mu<n-1 \\ h(1+|\ln h|) & ,\nu<n-1,\mu=n-1 \\ h(1+|\ln h|)^2 & ,\nu=\mu=n-1 \\ h & ,\mu>n-1,\nu+\mu<2(n-1) \\ h^{2n-\nu-\mu-1} & ,\mu>n-1,\nu+\mu\ge 2(n-1) \end{array} \right\} \le c''\varepsilon_{\nu h}\varepsilon_{\mu h},$$

$$\delta_h^{(6)} = \max_{1\le i\le l_h} \sum_{\substack{j=1 \\ \operatorname{dist}(\xi_{ih},G_{jh})>2c_1h \\ \operatorname{dist}(\operatorname{co}G_{jh},\partial G)\ge h}}^{l_h} \int_{G_{jh}} \Big| K(\xi_{ih},y) - K(\xi_{ih},\xi_{jh}) -$$

$$- \frac{\partial K(\xi_{ih},\xi_{jh})}{\partial y}(y-\xi_{jh})\Big| \left\{ \begin{array}{ll} |\xi_{jh}-\xi_{ih}| & ,\mu<n-1 \\ |\xi_{jh}-\xi_{ih}|(1+|\ln|\xi_{jh}-\xi_{ih}||) & ,\mu=n-1 \\ |\xi_{jh}-\xi_{ih}|^{n-\mu} & ,\mu>n-1 \end{array} \right\} \le$$

$$\le ch^2 \max_{1\le i\le l_h} \int_{\{y\in G:\ |y-\xi_{ih}|\ge 2c_1h\}} \left\{ \begin{array}{ll} |y-\xi_{ih}|^{-\nu-1} & ,\mu<n-1 \\ |y-\xi_{ih}|^{-\nu-1}(1+|\ln|y-\xi_{ih}||) & ,\mu=n-1 \\ |y-\xi_{ih}|^{-\nu-2+n-\mu} & ,\mu>n-1 \end{array} \right\} dy \le$$

$$\le c'h^2 \left\{ \begin{array}{ll} 1 & ,\mu<n-1 \\ 1+|\ln h| & ,\nu<n-1,\mu=n-1 \\ (1+|\ln h|)^2 & ,\nu=\mu=n-1 \\ 1 & ,\mu>n-1,\nu+\mu<2(n-1) \\ 1+|\ln h| & ,\mu>n-1,\nu+\mu=2(n-1) \\ h^{2n-\nu-\mu-2} & ,\mu>n-1,\nu+\mu>2(n-1) \end{array} \right\} \le c''\varepsilon_{\nu h}\varepsilon_{\mu h}.$$

Therefore $\|(T_h - T''_h)p_h u\| \le \text{const}\ \varepsilon_{\nu h}\varepsilon_{\mu h}$ and, in case $\nu>0$, the proof of Theorem 3 is completed.

If (2) is fulfilled with a $\nu\le 0$ then (2) is fulfilled also with a $\nu\in(0,n-1)$, and (9) gives $\|u_h - p_h u\| \le \text{const}\ h\varepsilon_{\mu h}$, q.e.d.

6.Numerical results. In this section we solve numerically the integral equation

$$u(x_1,x_2)=\int_0^1\int_0^1 K(x_1,x_2,y_1,y_2)\,u(x_1,x_2)dy_1dy_2+f(x_1,x_2),\ (x_1,x_2)\in G, \qquad (26)$$

where

$$K(x_1,x_2,y_1,y_2)=\ln[(x_1-y_1)^2+(x_2-y_2)^2]^{1/2}, \qquad (27)$$

$$f(x_1,x_2)=x_1+x_2^2$$

and

$$G=\{(x_1,x_2): 0<x_1<1,\ 0<x_2<1\}.$$

In this case conditions (2) (4) and (14) are fulfilled($n=2,\nu=0,\mu=0$). Let $N\geq 3$ be a integer,

$$h=1/N,\ x_1^0=x_2^0=0, x_1^1=x_2^1=h/2,\ x_1^N=x_2^N=1-(h/2), x_1^{N+1}=x_2^{N+1}=1,$$

$$x_1^i=x_2^i=(h/2)+(i-1)h,\ i=2,\ \dots,N-1.$$

Let us devide the unit square G into rectangles

$$G_h^{ij}=\{(x_1,x_2): x_1^{i-1}<x_1<x_1^i,\ x_2^{j-1}<x_2<x_2^j\},\quad i,j=1,\dots,N+1,$$

and define the points (ξ_1^i,ξ_2^j), $i,j=1,\dots,N+1$, as the mid-points of these rectangles:

$$\xi_1^i=(x_1^{i-1}+x_1^i)/2,\quad \xi_2^j=(x_2^{j-1}+x_2^j)/2,\quad i,j=1,\dots,N+1.$$

For such selection of G_h^{ij} and (ξ_1^i,ξ_2^j) conditions (5),(6) and (12) are fulfilled (diam $G_h^{ij}\leq\sqrt{2}\,h$).

Using this selection of G_h^{ij} and (ξ_1^i,ξ_2^j) we solved (26) numerically by methods (8), (8') and (8"). For approximate values u_h^{ij} to exact solution $u(x_1,x_2)$ of (26) at the points (ξ_1^i,ξ_2^j), these methods yield,correspondingly,the following linear algebraic systems:

$$u_h^{ij}=\sum_{k=1}^{N+1}\sum_{l=1}^{N+1}\Big[\int_{x_1^{k-1}}^{x_1^k}\int_{x_2^{l-1}}^{x_2^l} K(\xi_1^i,\xi_2^j,y_1,y_2)dy_1dy_2\Big]u_h^{kl}+$$

$$+f(\xi_1^i,\xi_2^j),\quad i,j=1,\dots,N+1; \qquad (28)$$

$$u_h^{ij} = h^2 \sum_{\substack{k=1 \\ (i,j)\neq(k,l)}}^{N+1} \sum_{l=1}^{N+1} K(\xi_1^i,\xi_2^j,\xi_1^k,\xi_2^l)\, u_h^{kl} + f(\xi_1^i,\xi_2^j), \qquad i,j=1,\dots,N+1; \qquad (29)$$

$$u_h^{ij} = h^2 \sum_{\substack{k=1 \\ (i,j)\neq(k,l)}}^{N+1} \sum_{l=1}^{N+1} K(\xi_1^i,\xi_2^j,\xi_1^k,\xi_2^l)\left[u_h^{kl} - u_h^{ij}\right] +$$

$$+ u_h^{ij} \int_0^1 \int_0^1 K(\xi_1^i,\xi_2^j,y_1,y_2)\, dy_1 dy_2 + f(\xi_1^i,\xi_2^j), \qquad i,j=1,\dots,N+1. \qquad (30)$$

The numerical calculation of the integrals in systems (28) and (30) is facilitated by the fact that, due to the relative simplicity of the logarithmic kernel (27), the necessary two-dimensional integrals can be found analytically:

$$\int_a^b \int_c^d \ln\left[(x_1-y_1)^2+(x_2-y_2)^2\right]^{1/2} dy_1 dy_2 = g(b-x_1,d-x_2)+$$

$$+g(x_1-a,d-x_2)+g(b-x_1,x_2-c)+g(x_1-a,x_2-c),$$

where

$$g(x_1,x_2)=x_1x_2\ln(x_1^2+x_2^2)+2x_1^2\operatorname{arctg}(x_2/x_1)+$$

$$+(x_1^2+x_2^2)\operatorname{arctg}(x_1/x_2)-3x_1x_2.$$

Systems (28),(29) and (30) were solved in the Computing Centre of Tartu University on the computer EC-1060. Some results for N=3, N=6, N=12 are given in the Table 1. In Table 1, the approximate solutions to $u(x_1,x_2)$ at points (0.33,0.33),(0.33,0.67) and (0.67,0.67) are given. For comparasion, we give also some numerical results from [1](see also [7]) which were obtained by means of the linear spline-collocation method supposedly with accuracy $1.1\cdot 10^{-5}$; if we take those values as exact ones then we obtain "guessed" error which is given as a special column in Table 1.

Table 1

Approximate values of $u(x_1,x_2)$

	At point (0.33,0.33)	Error	At point (0.33,0.67)	Error	At point (0.67,0.67)	Error
Method [1]:	0.1311		0.3524		0.5736	
Method (28):						
N=3	0.1287	2.4 E-3	0.3567	4.3 E-3	0.5846	1.1 E-2
N=6	0.1304	7.0 E-4	0.3530	6.0 E-4	0.5757	2.1 E-3
N=12	0.1310	1.0 E-4	0.3525	1.0 E-4	0.5741	5.0 E-4
Method (29):						
N=3	0.1281	3.0 E-3	0.4053	5.3 E-2	0.6826	1.1 E-1
N=6	0.1299	1.2 E-4	0.3654	1.1 E-2	0.6008	2.7 E-2
N=12	0.1308	3.0 E-4	0.3560	3.6 E-3	0.5812	7.6 E-3
Method (30):						
N=3	0.1283	2.8 E-3	0.3560	3.6 E-3	0.5835	1.0 E-2
N=6	0.1303	8.0 E-4	0.3530	6.0 E-4	0.5756	2.0 E-3
N=12	0.1310	1.0 E-4	0.3525	1.0 E-4	0.5741	5.0 E-4

According to Theorem 1 and 3 we have for methods (28) and (30) the estimation

$$\max_{1\le i,j\le N+1} \left| u_h^{ij} - u(\xi_1^i,\xi_2^j) \right| \le \text{const}\, h^2. \tag{31}$$

According to Theorem 2 we have for method (29) the estimation

$$\max_{1\le i,j\le N+1} \left| u_h^{ij} - u(\xi_1^i,\xi_2^j) \right| \le \text{const}\, h^2 (1+\ln h). \tag{32}$$

From Table 1 we can see that the numerical results are consistent with these theoretical estimations (31) and (32): decreasing the step $h=1/N$ 2 times, the error decreases approximately 4 times.

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БЫСТРОТА СХОДИМОСТИ МОДИФИЦИРОВАННОГО МЕТОДА МЕХАНИЧЕСКИХ КВАДРАТУР ДЛЯ МНОГОМЕРНЫХ СЛАБО СИНГУЛЯРНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ

Геннадий Вайникко и Арвет Педас

Резюме

Рассматривается интегральное уравнение (1), где $G \subset \mathbb{R}^n$ - откытое ограниченное множество с кусочно гладкой границей. На ядро и свободный член налагаются условия (2)-(4).Строится кубатурная формула (5)-(7) и на ее основе три метода решения уравнения (1)-методы (8),(8') и (8"). Теоремы 1 и 2, касающиеся методов (8) и (8'), доказаны в других работах.Целью данной работы является установление теоремы 3, описывающей быстроту сходимости модифицированного по Канторовичу-Крылову метода механических кубатур (метода (8")). Проводится его сравнение с методами (8) и (8'), в частности,численные.

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МЕТОД СПЛАЙН-КОЛЛОКАЦИИ ДЛЯ РЕШЕНИЯ ДВУМЕРНОГО ИНТЕГРАЛЬНОГО УРАВНЕНИЯ С ЛОГАРИФМИЧЕСКИМ ЯДРОМ

Урве Кангро

В статье рассматривается двумерное интегральное уравнение с логарифмическим ядром в прямоугольной области. Для решения этого уравнения исследуется метод коллокации с билинейными сплайнами на равномерной сетке. Получена оценка скорости сходимости этого метода и приведены несколько результатов вычислений.

1. Гладкость решения. Рассматриваем интегральное уравнение

$$u(x)=\int_{\Omega} a(x,y)\ln|x-y|u(y)dy+f(x),\quad x\in\Omega, \tag{1}$$

где $\Omega\subset\mathbb{R}^2$ – прямоугольник с параллельными осям сторонами. В статье [4] доказан следующий результат.

Теорема 1. Пусть $\Omega\subset\mathbb{R}^2$ – открытое ограниченное множество с кусочно-ляпуновской границей Γ. Предположим, что $f\in C^2(\bar{\Omega})$, $a\in C^2(\bar{\Omega}\times\bar{\Omega})$ и пусть интегральное уравнение (1) имеет решение $u\in L(\Omega)$. Тогда $u\in C^1(\bar{\Omega})\cap C^2(\Omega)$, причем в точках гладкости Γ вторые производные от u непрерывны вплоть до границы, а в окрестности любой однократной угловой точки $y^*\in\Gamma$ они имеют представление

$$\frac{\partial^2u(x)}{\partial x_j^2}=(-1)^ju(y^*)a(y^*,y^*)\Big[(\omega_1\omega_2-\omega_1'\omega_2')\ln|x-y^*|+$$
$$+(\omega_1^2-\omega_1'^2)\arg_*(x)\Big]+v_j(x),\ j=1,2,$$

$$\frac{\partial^2u(x)}{\partial x_1\partial x_2}=u(y^*)a(y^*,y^*)\Big[(\omega_1^2-\omega_1'^2)\ln|x-y^*|+$$
$$+(\omega_1'\omega_2'-\omega_1\omega_2)\arg_*(x)\Big]+v(x),$$

где v и v_j – непрерывны в пересечении $\bar{\Omega}$ с окрестностью точки y^*, $\omega=(\omega_1,\omega_2)$ и $\omega'=(\omega_1',\omega_2')$ – пределы единичных внутренних нормалей к Γ при приближении к y^* вдоль Γ соответственно в положительном и отрицательном направлении, $\arg_*(x)=\mathrm{Arg}\left[x_1-y_1^*+i(x_2-y_2^*)\right]$, причем однозначность этой функции в окрестности y^* достигается разрезом плоскости вдоль кривой в $\mathbb{R}^2\setminus\Omega$, начинающейся в точке y^*.

Заметим, что если Ω – прямоугольник с параллельными осям сторонами, то в угловых точках $\omega_1\omega_2-\omega'_1\omega'_2=0$ и значит, $\frac{\partial^2 u}{\partial x_1^2}$ и $\frac{\partial^2 u}{\partial x_2^2}$ являются ограниченными.

2. Метод сплайн-коллокации. Пусть $\Omega=(0,b_1)\times(0,b_2)$. В этой области определяем равномерную сетку с узлами

$$x^{ij}=(x_1^i,x_2^j)=\left(\frac{i}{N_1}b_1,\frac{j}{N_2}b_2\right),\quad i=0,\dots,N_1,\ j=0,\dots,N_2.$$

Обозначим $h_k=\frac{b_k}{N_k}$, $N=(N_1,N_2)$. Приближенное решение u_N уравнения (1) ищем в виде

$$u_N(x)=\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}u_{ij}\varphi_{ij}(x),\tag{2}$$

где φ_{ij} – билинейные базисные сплайны:

$$\varphi_{ij}(x)=\begin{cases}\left(1-\frac{|x_1-x_1^i|}{h_1}\right)\left(1-\frac{|x_2-x_2^j|}{h_2}\right), & x\in[x_1^{i-1},x_1^{i+1}]\times[x_2^{j-1},x_2^{j+1}],\\ 0 & \text{в остальных случаях.}\end{cases}$$

Коэффициенты u_{ij} определим из условия, чтобы u_N удовлетворяла уравнению (1) в узлах x^{ij}:

$$u_N(x^{ij})=\int_\Omega a(x^{ij},y)\ln|x^{ij}-y|\,u_N(y)\,dy+f(x^{ij}),\ i=0,\dots,N_1,\ j=0,\dots,N_2,$$

или, используя выражение (2),

$$u_{ij}-\sum_{k=0}^{N_1}\sum_{l=0}^{N_2}u_{kl}\int_\Omega a(x^{ij},y)\ln|x^{ij}-y|\,\varphi_{kl}(y)\,dy=f(x^{ij}),\tag{3}$$
$$i=0,\dots,N_1,\ j=0,\dots,N_2.$$

Мы получили линеиную систему уравнений для определения коэффициентов u_{ij}.

3. Сходимость метода.

Теорема 2. Пусть однородное уравнение, соответствующее уравнению (1), имеет лишь тривиальное решение. Предположим, что $f\in C^2(\bar\Omega)$ и $a\in C^2(\bar\Omega\times\bar\Omega)$. Тогда найдутса N_1' и N_2' такие, что если $N_1\ge N_1'$ и $N_2\ge N_2'$, то система уравнении (3) однозначно разрешима и имеет место оценка

$$\max_{x\in\bar\Omega}|u_N(x)-u_*(x)|\le C_1h_1^2+C_2h_2^2,$$

где u_* – решение уравнения (1).

Для доказательства нам понадобятся следующие результаты. Рассматриваем уравнение

$$u = Tu + f, \qquad (4)$$

где $T \in \mathscr{L}(E)$, E - банахово пространство. Вместо уравнения (4) решаем уравнение

$$u_N = P_N T u_N + P_N f, \qquad (5)$$

где $P_N \in \mathscr{L}(E)$ - некоторые проекторы. Тогда имеет место следующий резултат ([5], теорема 15.3 и лемма 15.4).

Лемма 1. Пусть T вполне непрерывен в E, пусть $I-T$ обратим. Предположим, что $\|P_N u - u\| \to 0$ при $N \to \infty$ для каждого $u \in E$. Тогда для достаточно больших N приближенное уравнение (5) имеет единственное решение u_N, имеет место оценка

$$C_1 \|u_* - P_N u_*\| \le \|u_N - u_*\| \le C_2 \|u_* - P_N u_*\|, \qquad (6)$$

где u - решение уравнения (4).

Обозначим

$$(P_N u)(x) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} u(x^{ij}) \varphi_{ij}(x).$$

Лемма 2. Пусть $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ и $\left|\frac{\partial^2 u(x)}{\partial x_k^2}\right| \le M_k$, $x \in \Omega$, $k = 1,2$. Тогда

$$\max_{x \in \bar{\Omega}} |u(x) - P_N u(x)| \le \frac{1}{8}(M_1 h_1^2 + M_2 h_2^2). \qquad (7)$$

Доказательство. Пусть $x \in [x_1^i, x_1^{i+1}] \times [x_2^j, x_2^{j+1}]$. Обозначим $t_1 = \frac{x_1 - x_1^i}{h_1}$, $t_2 = \frac{x_2 - x_2^j}{h_2}$. Тогда

$$(P_N u)(x) = (1-t_1)(1-t_2)u(x^{ij}) + (1-t_1)t_2 u(x^{i,j+1}) + t_1(1-t_2)u(x^{i+1,j}) + \\ + t_1 t_2 u(x^{i+1,j+1}).$$

Итак,

$$u(x) - (P_N u)(x) = (1-t_1)\left[u(x_1^i, x_2) - (1-t_2)u(x^{ij}) - t_2 u(x^{i,j+1})\right] + \\ + t_1\left[u(x_1^{i+1}, x_2) - (1-t_2)u(x^{i+1,j}) - t_2 u(x^{i+1,j+1})\right] + \\ + \left[u(x_1, x_2) - (1-t_1)u(x_1^i, x_2) - t_1 u(x_1^{i+1}, x_2)\right].$$

Выражения в квадратных скобках представляют погрешность одномерной линейной интерполяции при фиксированной второй переменной. Так как при линейной интерполяции функции одной переменной $v \in W_\infty^2$ погрешность оценивается через $\frac{h^2}{8}\|v''\|_\infty$ ([3], теорема 2.1), то

$$|u(x) - P_N u(x)| \le (1-t_1)\frac{1}{8}h_1^2 M_1 + t_1 \frac{1}{8}h_1^2 M_1 + \frac{1}{8}h_2^2 M_2 = \\ = \frac{1}{8}(M_1 h_1^2 + M_2 h_2^2).$$

Лемма доказана.

<u>Доказательство теоремы.</u> Пусть $E=C(\bar{\Omega})$. Обозначим

$$(Tu)(x)=\int\limits_{\Omega}a(x,y)\ln|x-y|\,u(y)\,dy.$$

Тогда $T\in\mathcal{L}(C(\Omega))$ вполне непрерывен в $C(\Omega)$ как слабо сингулярный интегральный оператор (см. [6], теорема 7.4.1). Для каждой $u\in C(\bar{\Omega})$ выполнено и условие $\|P_N u-u\|\to 0$ при $N_1,N_2\to\infty$. Поэтому справедлива оценка (6). Из теоремы 1 вытекает, что для решения уравнения (1) выполнены предположения леммы 2. Следовательно, имеет место оценка (7). Теорема доказана.

<u>Замечание.</u> Из работы [1] для этого уравнения вытекает скорость сходимости $O(h^2|\ln h|)$.

4. Применение метода сплайн-коллокации. При практических вычислениях возникают некоторые трудности. В общем случае не удается точно вычислить матрицу системы (3). Но если функция $a(x,y)$ зависит только от переменной x, как это в некоторых практических задачах, то надо вычислять интегралы

$$\int\limits_{\Omega}\ln|x^{ij}-y|\,\varphi_{kl}(y)\,dy.$$

Это можно сделать точно, пользуясь таблицами интегралов.

Вторая трудность связана с решением системы. Размерность системы быстро растет, поэтому обычные методы решения не годятся. Можно использовать специальный итерационный метод ([2]):

$$v_N^{(k)}=P_N f-(I-P_N T)u_N^{(k)},$$
$$u_N^{(k+1)}=u_N^{(k)}+v_N^{(k)}+P_N(I-P_\nu T)^{-1}(P_\nu T)P_\nu v_N^{(k)},$$

где $\nu=(\nu_1,\nu_2)$, $\nu\ll N$. Тогда на каждом шаге решается система размерности $(\nu_1+1)(\nu_2+1)$. Эта система однозначно разрешима и метод сходится со скоростью геометрической прогрессии при каждом начальном приближении, если N и ν достаточно большие.

5. Примеры. Пусть $\Omega=(0,1)\times(0,1)$, $a\equiv 1$. Чтобы исследовать сходимость метода, я задала решение, по ему вычислила свободный член и затем применила метод коллокации. При решении системы в итерационном методе было достаточно выбирать $\nu_1=\nu_2=3$. Результаты вычислений следующие (здесь $N=N_1=N_2$):

Пример 1. $u(x_1,x_2)=x_1+x_2^2$

N	Погрешность Рунге в узлах	Максимальная погрешность в узлах	Погрешность в норме $C(\overline{\Omega})$
3	——	0,0099	0,023
6	0,0025	0,0026	0,0058
12	0.00066	0.00065	0,0015
24	0.00016	0.00016	0,00037
5	——	0,0037	0,0083
10	0,00093	0,00094	0,0021
20	0,00024	0,00024	0,00053

Пример 2. $u(x_1,x_2)=x_1x_2\ln(x_1^2+x_2^2)+(x_2^2-x_1^2)\operatorname{arctg}\frac{x_1}{x_2}+2+(\frac{\pi}{2}-2)x_1-$
$-2x_2+(2-\frac{\pi}{2}-\ln 2)x_1x_2$

N	Погрешность Рунге в узлах	Максимальная погрешность в узлах	Погрешность в норме $C(\overline{\Omega})$
3	——	0,0011	0,044
6	0,00033	0,00011	0,0109
12	0.000032	0,000012	0,0028
24	0.0000029	0,0000089	0,00069
5	——	0,00020	0,0157
10	0,000060	0,000020	0,0040
20	0,0000055	0,0000059	0,00099

Пример 3. $f(x_1,x_2)=x_1+x_2^2$

N	Погрешность Рунге в узлах
6	0.0071
12	0,0018
24	0,00044
48	0,00011
10	0,0025
20	0,00063
40	0,00016

Во втором примере задано такое решение, чтобы оно имело особенности, описанные в теореме 1 (в точке (0,0) решение имеет особенность, в остальных углах области $u(x)=0$, поэтому там особенностей не возникает). Можно показать ([4], замечание 1), что тогда свободный член является гладким ($f \in C^2(\bar{\Omega})$). В третьем примере уравнение (1) решено при гладком свободном члене (точное решение неизвестно). Как видно, оценка погрешности теоремы 2 хорошо подверждается численно.

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COLLOCATION METHOD WITH BILINEAR SPLINES FOR TWO-DIMENSIONAL INTEGRAL EQUATION WITH LOGARITHMIC KERNEL

U. Kangro

Summary

Consider integral equation (1) in rectangular domain. In [4] the smoothness of the solution is investigated. In this paper we study the collocation method with bilinear splines to solve the equation. Rate of convergence $O(h^2)$ is got (theorem 2). Some results of calculations are given.

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ON THE SMOOTHNESS OF SOLUTIONS TO AN INTEGRAL EQUATION WITH A KERNEL HAVING A SINGULARITY ON A CURVE

Raul Kangro

In this work we examine the smoothness of solutions to an one-dimensional linear integral equation of the second kind assuming the integral operator's kernel or its partial derivative of some order have a weak singularity on a smooth curve $t=\varphi(s)$.

1. Introduction. In tis paper we examine the smoothness of solutions to the integral equation

$$u(t)=\int_{a_1}^{a_2} g(t,s)\,\varkappa(t-\varphi(s))\,u(s)\,ds+f(t), \tag{1}$$

assuming that the following conditions are fulfilled:

$$\begin{cases} g\in C^m([a_1,a_2]\times[a_1,a_2]), \\ \varkappa\in C^{m-1}(\mathbb{R}\setminus\{0\})\cap C^p(\mathbb{R}),\ p\geq -1, \\ |\varkappa^{(m-1)}(t)|\leq c\,|t|^{-\beta},\ 0\leq\beta<m-1-p, \\ \varphi\in C^{m+1}(\mathbb{R}),\ \varphi'(s)\neq 0,\ s\in\mathbb{R} \end{cases} \tag{2}$$

(with $C^{-1}[a,b]$ we denote the class of functions that have at most a finite number of discontinuity points on the interval $[a,b]$).

In [3] a special case $\varphi(a_1)=a_1$, $\varphi(a_2)=a_2$, $f\in C^m[a_1,a_2]$ is considered. In this case all continuous solutions to (1) have continuous derivatives of order m in the interval (a_1,a_2) and

$$u^{(k)}(t)\leq C\begin{cases} 1, & m-\beta-k>0, \\ \big|\ln|t-a_1|\big|+\big|\ln|t-a_2|\big|, & m-\beta-k=0,\quad 0\leq k\leq m. \\ |t-a_1|^{m-\beta-k}+|t-a_2|^{m-\beta-k}, & m-\beta-k<0, \end{cases}$$

It occurs that if $\varphi(a_1)\neq a_1$ or $\varphi(a_2)\neq a_2$ then the singularities of the derivatives of solutions to (1) unlike the cases examined previously may lie inside the interval $[a_1,a_2]$ (see [2]), namely in points $\varphi(a_j)$, $\varphi(\varphi(a_j))$, ... $(j=1,2)$. Our aim is to characterise the order of singularities at those points.

2. The formulation of the main result. Define the function

$$h(t,\alpha)=\begin{cases} |t|^{-\alpha}, & \alpha\neq 0, \\ \big|\ln|t|\big|+1, & \alpha=0. \end{cases}$$

For a given set

$$S=\{(x_i,\beta_i,p_i)_{i=1}^{q}\}, \text{ with } x_i\in\mathbb{R},\ \beta_i\in\mathbb{R},\ p_i\in\mathbb{N},$$

denote

$$I_S=\{x_i\}_{i=1}^{q},$$

$$w_S(k,t)=1+\sum_{i=1}^{q} h(t-x_i,\beta_i+k-m).$$

If $p_i\le m-1-[\beta_i]$ ($[\alpha]=\max\{m\in\mathbb{Z}:\ m\le\alpha\}$ for all $\alpha\in\mathbb{R}$) then the space of functions

$$E_S^m(a,b)=\Big\{u\in C[a,b]\cap C^m([a,b]\setminus I_S)\cap\Big[\bigcap_{x_i\in[a,b]} C^{p_i}(x_i)\Big]:\ \sup_{t\in[a,b]\setminus I_S}\frac{|u^{(m)}(t)|}{w_S(m,t)}<\infty\Big\}$$

equipped with the norm

$$\|u\|_S=\|u\|_C+\max_{1\le k\le m}\ \sup_{t\in[a,b]\setminus I_S}\frac{|u^{(k)}(t)|}{w_S(k,t)}$$

is a Banach space. Here $C^{p_i}(x_i)$ denotes the set of funktions which are p_i times continuously differentiable on a neighborhood of x_i.

Remark 1. The norm

$$\|u\|_{(S)}=\|u\|_C+\sup_{t\in[a,b]\setminus I_S}\frac{|u^{(m)}(t)|}{w_S(m,t)}$$

is equivalent to the norm $\|u\|_S$.

Denote $(\varphi)^1(t)=\varphi(t)$, $(\varphi)^2(t)=\varphi(\varphi(t))$, $(\varphi)^3(t)=\varphi(\varphi(\varphi(t)))$ and so on.

Theorem 1. Assume (2). Let $q=\left[\frac{m}{m-\beta}\right]$, $p=m-2-[\beta]$,
$0<\varepsilon<\min\limits_{2\le i\le q+1}\left\{\frac{[i\beta]-i\beta+1}{q},\ \frac{qm-(q+1)\beta}{q-1}\right\}$, $p_{ij}=im-1-[i\beta]$, $\beta_{ij}=\beta-(i-1)(m-\beta-\varepsilon)$,
$x_{ij}=(\varphi)^i(a_j)$, $i=1,\dots,q$, $j=1,2$, $S=\{(x_{ij},\beta_{ij},p_{ij})_{i=1}^{q}{}_{j=1}^{2}\}$. If $f\in E_S^m(a_1,a_2)$ then all continuous solutions to the integral equation (1) belong to the space $E_S^m(a_1,a_2)$.

Theorem 2. Assume (2). Let $-1\le p<m-2-[\beta]$, $q=\left[\frac{m}{p+2}\right]$, $p_{ij}=i(p+2)-1$,
$\beta_{ij}=\beta-(i-1)(p+2)$, $x_{ij}=(\varphi)^i(a_j)$, $i=1,\dots,q$, $j=1,2$, $S=\{(x_{ij},\beta_{ij},p_{ij})_{i=1}^{q}{}_{j=1}^{2}\}$.
If $f\in E_S^m(a_1,a_2)$ then all continuous solutions to the integral equation (1) belong to the space $E_S^m(a_1,a_2)$.

Remark 2. Notice that if

$$|x^{(m-1)}(t)| \le c|t|^{-\beta}, \quad 0<\beta<m,$$

then

$$|x^{(k)}(t)| \le c_k\big(h(t,\beta-m+k)+1\big), \quad 0\le k\le m.$$

Hence for $0\le i\le m-2-[\beta]$ the function $x^{(i)}$ must be bounded. Theorem 1 deals with the case when $x^{(i)}$ is continuous for $0\le i\le m-2-[\beta]$. The case of the first kind discontinuous $x^{(i)}$ ($p<i\le m-2-[\beta]$) is observed in Theorem 2. Smoothness of continuous solutions to the class of integral equations (1), (2) is described by the set $S=\{ (x_{ij},\beta_{ij},p_{ij})_{i=1,j=1}^{q\ \ 2} \}$ as follows : the points $x_{ij}\in[a_1,a_2]$ are critical points of the solutions - at those points derivatives of the solutions may have singularities; β_{ij} gives us upper bound for the degree of singularity of the solution at the point x_{ij} i.e.

$$|u^{(m)}(t)| \le c\big(h(t-x_{ij},\beta_{ij})+1\big) \text{ as } t\to x_{ij};$$

p_{ij} characterizes the smoothness - solution to (1),(2) is at least p_{ij} times continuously differentiable at the point x_{ij}.

Remark 3. There are examples where Theorem 2 is exact: for a $f\in C^m[a_1,a_2]$, the solution u does not belong to a more narrow weight space as $E_S^m(a_1,a_2)$.

3. Lemmas. Let us give some auxiliary results needed for proving Theorems 1 and 2. Proofs are based on the following lemma (for proof see [1], p. 10 lemma 1.1).

Lemma 1. Let E and E_1 be Banach spaces, $E\subset E_1$ densely and continuously. Let $T_1: E_1\to E_1$ and $T: E\to E$ be linear compact operators, let T be the restriction of T_1 to the space E. If for $f\in E$ the equation $u_1=T_1u_1+f$ is solvable in E_1 then its solutions belong to E.

Further we need the following estimation.

Lemma 2. Assume that

$$-M\le a(x,y)\le x\le b(x,y)\le M, \quad -M\le y\le M \quad (0<M<\infty),$$

$$\operatorname{dist}\big(y,[a(x,y),b(x,y)]\big)\ge c_1|x-y| \text{ with some } c_1=\text{const.}>0.$$

Then for every $\alpha\in[0,1)$, $\beta\ge0$, $\varepsilon>0$ there exists a constant $c=c(\varepsilon,\alpha,\beta,c_1,M)$ such that

$$\left|\int_{a(x,y)}^{b(x,y)} h(s-x,\alpha)h(s-y,\beta)\,ds\right| \le c\big(h(x-y,\alpha+\beta+\varepsilon-1)+1\big). \tag{3}$$

<u>Proof.</u> Denote $a=a(x,y)$, $b=b(x,y)$. Let $a\le x\le b<y$. Define

$$z=z(x,y)=\max\big(a(x,y),\ x-c_1|x-y|\big).$$

We have

$$\int_a^z h(s-x,\alpha)h(s-y,\beta)ds \le c\int_a^z h(s-x,\alpha+\beta+\varepsilon)ds \le$$

$$\le c\max_{s=a,z} h(s-x,\alpha+\beta+\varepsilon-1) \le c\big(h(x-y,\alpha+\beta+\varepsilon-1)+1\big),$$

$$\int_z^b h(s-x,\alpha)h(s-y,\beta)ds \le h(b-y,\beta)\int_z^b h(s-x,\alpha)ds \le$$

$$\le ch\big(c_1(y-x),\beta\big)h(b-z,\alpha-1)\big|\ln|b-z|\big| \le$$

$$\le ch(y-x,\beta)\,h(y-x,\alpha-1)\big|\ln|x-y|\big| \le c\big(h(x-y,\alpha+\beta+\varepsilon-1)+1\big)$$

and hence we have proved that the inequality (3) holds. The case $y<a\le x\le b$ can be treated in a similar way.■

Define

$$(Tu)(t)=\int_a^b k(t,s)\varkappa(t-s)u(s)ds,$$

$$k_j(t,s)=\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^j k(t,s),$$

$$(T^{i,j}u)(t)=\int_a^b \frac{\partial^i}{\partial t^i}\big(k_j(t,s)\,\varkappa(t-s)\big)u(s)ds,$$

$$\left(\Lambda_a^{l,k+1}u\right)(t)=\sum_{i=0}^{k}\frac{d^{k+l-i}}{dt^{k+l-i}}\Big(\sum_{j=0}^{l}C_l^j k_j(t,a)\varkappa(t-a)u^{(l-j)}(a)\Big).$$

Lemma 3. Assume that

$$k\in C^1([c,d]\times[a,b]) \quad (a<b,\ c<d),$$

$$\varkappa\in C([c-b,d-a]\setminus\{0\}),\ |\varkappa(t)|\le c|t|^{-\alpha},\ 0\le\alpha<1,$$

$$u\in C^1((a,b)\setminus\{y_i\}_{i=0}^{q})\cap C[a,b],$$

$$|u'(s)|\le c\Big(|s-a|^{-\mu}+\sum_{i=1}^{q}|s-y_i|^{-\gamma}+|s-b|^{-\nu}\Big),\ 0<\gamma<1-\alpha,\ \ 0<\mu,\nu<1.$$

Then the function $Tu\in C^1([c,d]\setminus\{a,b\})\cap C[c,d]$ and

$$\frac{d}{dt}(Tu)(t)=(Tu')(t)+(T^{0,1}u)(t)+\left(\Lambda_a^{0,1}u\right)(t)-\left(\Lambda_b^{0,1}u\right)(t). \tag{4}$$

Proof. Let

$$\delta=\frac{1}{3}\min_{0\le i<j\le k}\big\{|y_i-a|,|y_i-b|,|y_i-y_j|\big\}.$$

Define

$$\varkappa_k(t)=\begin{cases}\varkappa(t), & t\in\left[-\frac{\delta}{k},\frac{\delta}{k}\right],\\ \frac{\delta-kt}{2\delta}\varkappa\left(-\frac{\delta}{k}\right)+\frac{\delta+kt}{2\delta}\varkappa\left(\frac{\delta}{k}\right), & t\in\left[-\frac{\delta}{k},\frac{\delta}{k}\right],\end{cases}$$

$$u_k(s)=u(s_1)+\int_{s_1}^{s}u_k'(t)dt, \qquad s\in[a,b],$$

where $s_1\in[a,b]$ is such that $\operatorname{dist}\big(s_1,\{a,b,y_1,\dots,y_q\}\big)\ge\delta$ and the function u_k' is defined by

$$u_k'(t)=\begin{cases}u'\left(a+\frac{\delta}{k}\right), & t\in\left[a,a+\frac{\delta}{k}\right],\\ \frac{\delta-k(t-y_i)}{2\delta}u'\left(y_i-\frac{\delta}{k}\right)+\frac{\delta+k(t-y_i)}{2\delta}u'\left(y_i+\frac{d}{k}\right), & t\in\left[y_i-\frac{\delta}{k},y_i+\frac{\delta}{k}\right],\ 1\le i\le q,\\ u'\left(b-\frac{\delta}{k}\right), & t\in\left[b-\frac{\delta}{k},b\right],\\ u'(t) & \text{otherwise.}\end{cases}$$

Obviously $u_k \in C^1[a,b]$, $\varkappa_k \in C[c-b,d-a]$ and

$$|u_k'(s)| \le c\Big(|s-a|^{-\mu} + \sum_{l=1}^{q} |s-y_l|^{-\gamma} + |s-b|^{-\nu}\Big), \quad |\varkappa_k(t)| \le c|t|^{-\alpha},$$

where the constants c do not depend on k. Denote

$$v_k(t) = \int_a^b k(t,s)\varkappa_k(t-s)u_k(s)\,ds.$$

We have

$$\frac{d}{dt} v_k(t) = -\frac{d}{dt} \int_{t-a}^{t-b} k(t,t-x)\varkappa_k(x)u_k(t-x)\,dx =$$

$$= -\int_{t-a}^{t-b} k_1(t,t-x)\varkappa_k(x)u_k(t-x)\,dx - \int_{t-a}^{t-b} k(t,t-x)\varkappa_k(x)u_k'(t-x)\,dx$$

$$-k(t,b)\varkappa_k(t-b)u_k(b) + k(t,a)\varkappa_k(t-a)u_k(a) =$$

$$= \int_a^b k(t,s)\varkappa_k(t-s)u_k'(s)\,ds + \int_a^b k_1(t,s)\varkappa_k(t-s)u_k(s)\,ds +$$

$$+ k(t,a)\varkappa_k(t-a)u_k(a) - k(t,b)\varkappa_k(t-b)u_k(b).$$

It's not difficult to see that if $k \to \infty$ and $[c_1,d_1] \subset [c,d] \setminus \{a,b\}$, then

$$\|v_k(t) - (Tu)(t)\|_{C[c,d]} \to 0,$$

$$\Big\|\int_a^b k_1(t,s)\varkappa_k(t-s)u_k(s)\,ds - (T^{0,1}u)(t)\Big\|_{C[c,d]} \to 0,$$

$$\|k(t,a)\varkappa_k(t-a)u_k(a) - (\Lambda_a^{0,1}u)(t)\|_{C[c_1,d_1]} \to 0,$$

$$\|k(t,b)\varkappa_k(t-b)u_k(b) - (\Lambda_b^{0,1}u)(t)\|_{C[c_1,d_1]} \to 0.$$

To prove the Lemma we must show that

$$\Big\|\int_a^b k(t,s)\varkappa_k(t-s)u_k'(s)\,ds - (Tu')(t)\Big\|_{C[c_1,d_1]} \to 0 \quad \text{if } k \to \infty.$$

We have

$$\Big|\int_a^b k(t,s)\varkappa_k(t-s)u_k'(s)\,ds - (Tu')(t)\Big| =$$

$$= \Big|\int_a^b k(t,s)\big(\varkappa_k(t-s)u_k'(s) - \varkappa(t-s)u'(s)\big)ds\Big| \le$$

$$\le \|k\|_C \int_a^b \Big(\big|\varkappa_k(t-s) - \varkappa(t-s)\big|\,|u_k'(s)| + |\varkappa(t-s)|\,|u_k'(s) - u'(s)|\Big)ds \le$$

$$\le c\Big(\int_{t-\frac{\delta}{k}}^{t+\frac{\delta}{k}} |t-s|^{-\alpha}\Big(|s-a|^{-\mu} + \sum_{l=1}^{q}|s-y_l|^{-\gamma} + |s-b|^{-\nu}\Big)ds +$$

$$+\int_a^{a+\frac{\delta}{k}} |t-s|^{-\alpha}|s-a|^{-\mu}\,ds + \sum_{l=1}^{q}\int_{y_l-\frac{\delta}{k}}^{y_l+\frac{\delta}{k}} |t-s|^{-\alpha}|s-y_l|^{-\gamma}\,ds +$$

$$+\int_{b-\frac{\delta}{k}}^{b} |t-s|^{-\alpha}|s-b|^{-\nu}\,ds\Big).$$

Note that for every $a,b>0$, $x,y\le 0$ the inequality $a^x b^y \le a^{x+y}+b^{x+y}$ is valid and hence

$$\int_{z-\frac{\delta}{k}}^{z+\frac{\delta}{k}} |t-s|^{-\alpha}|s-y_i|^{-\gamma}\,ds \le 2\int_{-\frac{\delta}{k}}^{\frac{\delta}{k}} |s|^{-\alpha-\gamma}ds < ck^{\alpha+\gamma-1} \text{ for every } z\in\mathbb{R}.$$

Let $e=\operatorname{dist}\big([c_1,d_1],\{a,b\}\big)$. Then for each $k>\frac{2\delta}{e}$ we have

$$\int_{t-\frac{\delta}{k}}^{t+\frac{\delta}{k}} |t-s|^{-\alpha}|s-a|^{-\mu}ds \le \left|e-\frac{\delta}{k}\right|^{-\mu}\int_{-\frac{\delta}{k}}^{\frac{\delta}{k}} |s|^{-\alpha}ds \le c\left|\frac{e}{2}\right|^{-\mu}k^{\alpha-1},$$

$$\int_{t-\frac{\delta}{k}}^{t+\frac{\delta}{k}} |t-s|^{-\alpha}|s-b|^{-\nu}ds \le c\left|\frac{e}{2}\right|^{-\nu}k^{\alpha-1},$$

$$\int_{a}^{a+\frac{\delta}{k}} |t-s|^{-\alpha}|s-a|^{-\mu}ds \le \left|e-\frac{\delta}{k}\right|^{-\alpha}\int_{a}^{a+\frac{\delta}{k}} |s-a|^{-\mu}ds \le c\left|\frac{e}{2}\right|^{-\alpha}k^{\mu-1},$$

$$\int_{b-\frac{\delta}{k}}^{b} |t-s|^{-\alpha}|s-b|^{-\nu}ds \le c\left|\frac{e}{2}\right|^{-\alpha}k^{\nu-1}$$

and therefore

$$\left\|\int_a^b k(t,s)\varkappa_k(t-s)u'_k(s)\,ds-(Tu')(t)\right\|_{C[c_1,d_1]} \le ck^{\max\{\mu,\nu,\alpha+\gamma\}-1}\to 0, \text{ if } k\to\infty.$$

Consequently we have constructed a sequence $v_k\in C^1[c,d]$ such that it converges to $(Tu)(t)$ in $C[c,d]$ and the sequence of derivatives v'_k converges to the right-hand side of the formula (4) in $C[c_1,d_1]$. Since the interval $[c_1,d_1]\subset[c,d]\setminus\{a,b\}$ was arbitrary Lemma 3 is proved.■

Corollary 1. Assume that

$k\in C^{l+n}([c,d]\times[a,b])$ $(a<b,\ c<d)$,

$\varkappa\in C^{l+n-1}([c-b,d-a]\setminus\{0\})\cap C^{l-1}[c-b,d-a]$, $|\varkappa^{(l)}(t)|\le c|t|^{-\alpha}$, $0\le\alpha<1$,

$u\in C^n((a,b)\setminus\{y_i\}_{i=0}^q)\cap C^{n-1}[a,b]$,

$|u^{(n)}(s)|\le c\Big(|s-a|^{-\mu}+\sum_{i=1}^{q}|s-y_i|^{-\gamma}+|s-b|^{-\nu}\Big)$, $\gamma<1-\alpha$, $0<\mu,\nu<1$.

Then the function $Tu\in C^{l+n}([c,d]\setminus\{a,b\})\cap C^{l+n-1}[c,d]$.

$$\frac{d^{l+n}}{dt^{l+n}}(Tu)(t)=\sum_{i=0}^{n}C_n^{l}\big(T^{l,l}u^{(n-l)}\big)(t)+\big(\Lambda_a^{l,n}u\big)(t)-\big(\Lambda_b^{l,n}u\big)(t). \qquad (5)$$

Proof is based on matematical induction and is trivial.■

4. The smoothness of (Tu)(t) and evaluation of $\frac{d^m}{dt^m}$ (Tu)(t).

In this section we assume that

$k\in C^m([c,d]\times[a,b])$ $(a<b,\ c<d)$,

$\varkappa\in C^{m-1}([c-b,d-a]\setminus\{0\})\cap C^p[c-b,d-a]$, $p\ge -1$,

$|\varkappa^{(m-1)}(t)|\le c|t|^{-\beta}$, $0\le\beta<m-1-p$.

Let $S=\{(x_i,\beta_i,p_i)_{i=1}^{q}\}$ be such that

$$x_i\in[a,b],\ 0\le p_i<m,\ \beta_i<m-p_i \qquad i=1,\dots,q,$$
$$p_i+p+2-H\big((p-m+2+\beta)+(p_i-m+1+\beta_i)-1\big)<m, \qquad i=1,\dots,r,$$
$$p_i+p+2-H\big((p-m+2+\beta)+(p_i-m+1+\beta_i)-1\big)\ge m, \quad i=r+1,\dots,q,$$

where $H(t)$ is the Heaviside function $\left(H(t)=\begin{cases}1, & t\ge 0,\\ 0, & t<0\end{cases}\right)$. Let

$$x_1<\dots<x_r.$$

Denote

$$y_0=a,\ \ y_i=x_i \ \ i=1,\dots,r,\ \ y_{r+1}=b,$$
$$p'_0=p'_{r+1}=p+1,\ \ p'_i=p_i+p+2-H\big((p-m+2+\beta)+(p_i-m+1+\beta_i)-1\big),\ \ i=1,\dots,r.$$

Lemma 4. Let $u\in E_S^m(a,b)$. Then

$$(Tu)(t)\in C[c,d]\cap C^m([c,d]\setminus\{y_0,\dots,y_{r+1}\})\cap\Big(\bigcap_{i=0}^{r+1}C^{p'_i}(y_i\cap[c,d])\Big).$$

<u>Proof.</u> It's obvious that $(Tu)(t)\in C[c,d]$. Let $z\in[c,d]\setminus\{y_0,\dots,y_{r+1}\}$. Then there exists $\delta>0$ such that

$$[z-2\delta,z+2\delta]\in[c,d]\setminus\{y_0,\dots,y_{r+1}\}.$$

Denote $K=[z-2\delta,z+2\delta]\cap[a,b]$. Note that

$$u\in C^{m-p-2}(K)\cap C^{m-p-1}(K\setminus\{x_{r+1},\dots,x_q\})$$

and

$$|u^{(m-p-1)}(t)|\le\sum_{i=r+1}^{q}|t-x_i|^{-\gamma},\ \ t\in K \text{ for some } \gamma<\min\{1,1-(p-m+2+\beta)\}.$$

We have

$$(Tu)(t)=\Big(\int_{[a,b]\setminus K}+\int_K\Big)k(t,s)\chi(t-s)u(s)ds. \tag{6}$$

The first integral on the right-hand side of the formula (6) satisfies the conditions of Corollary 1 with $c=z-\delta$, $d=z+\delta$, $l=m-1$, $\alpha=0$, $n=1$ and the second integral satisfies them with $c=z-\delta$, $d=z+\delta$, $l=p+1$, $n=m-p-1$, $\alpha=\max\{0,p-m+2+\beta\}$, hence $Tu\in C^m[z-\delta,z+\delta]$.

Since we can differentiate the function $(Tu)(t)$ $p+1$ times under the integral sign, we have $Tu\in C^{p'_0}(a\cap[c,d])\cap C^{p'_{r+1}}(b\cap[c,d])$.

Let $z=y_i\in[c,d]\setminus\{a,b\}$. Then there exists $\delta>0$ such that

$$[z-2\delta,z+2\delta]\cap\{a,b,x_1,\dots,x_q\}=\{y_i\},$$

therefore

$$u\in C^{p_i}[z-2\delta,z+2\delta]\cap C^m([z-2\delta,z+2\delta]\setminus\{y_i\}).$$

Let $K=[z-2\delta,z+2\delta]$. Then, as previously, the first integral on the right-hand side of the formula (6) is m times continuously differentiable on the interval $[z-\delta,z+\delta]$ and for the differentiation of the second integral we can use Corollary 1 with $l=p+1$, $n=p_i+1-H\big((p-m+2+\beta)+(p_i-m+1+\beta_i)-1\big)$, $c=z-\delta$, $d=z+\delta$ and consequently $Tu\in C^{p'_i}[y-\delta,y+\delta]$ for $0\le i\le r+1$. Lemma 4 is proved.■

Next we shall find a formula for $\frac{d^m}{dt^m}(Tu)(t)$. Denote

$$\bar{y}_i = \frac{y_i + y_{i+1}}{2}, \qquad i=0,\dots,r,$$

$$T_i = \int_{y_i}^{y_{i+1}} k(t,s)\varkappa(t-s)u(s)ds, \qquad i=0,\dots,r,$$

$$T_{ij} = (-1)^j \int_{y_{i+j}}^{\bar{y}_i} k(t,s)\varkappa(t-s)u(s)ds, \qquad i=0,\dots,r, \quad j=0,1,$$

$$J(t)-\{i:\ 0\le i\le r,\ t\notin[y_i, y_{i+1}]\},$$

$$n_0 = n_{r+1} = 1, \quad n_i = \min\{m-[\beta_i],\ p_i+2\} \quad i=1,\dots,r,$$

$$\beta_0 = 0,\ \beta_{r+1} = 0.$$

Let $u\in E_S^m(a,b)$. We have

$$\frac{d^m}{dt^m}(Tu)(t) = \sum_{i=0}^{r} \frac{d^m}{dt^m}(T_i u)(t).$$

If $i\in J(t)$, then by formula (5) we get

$$\frac{d^m}{dt^m}(T_i u)(t) = \sum_{j=0}^{1} \frac{d^m}{dt^m}(T_{ij}u)(t) =$$

$$= \sum_{j=0}^{1} \sum_{k=0}^{n_{i+j}} C_{n_{i+j}}^k \left(T_{ij}^{m-n_{i+j},k} u^{(n_{i+j}-k)}\right)(t) + \left(\Lambda_{y_i+0}^{m-n_i,n_i} u\right)(t) -$$

$$-\left(\Lambda_{\bar{y}_i}^{m-n_i,n_i} u\right)(t) + \left(\Lambda_{\bar{y}_i}^{m-n_{i+1},n_{i+1}} u\right)(t) - \left(\Lambda_{y_{i+1}-0}^{m-n_{i+1},n_{i+1}} u\right)(t).$$

In the case $t\in(y_i, y_{i+1})$ we denote

$$\bar{y}_{i1} = \frac{1}{2}(y_i+t),\ \bar{y}_{i2} = \frac{1}{2}(y_{i+1}+t),$$

$$(T_{iI}u)(z) = \int_{y_i}^{\bar{y}_{i1}} k(z,s)\varkappa(z-s)u(s)ds,$$

$$(T_{iII}u)(z) = \int_{\bar{y}_{i1}}^{\bar{y}_{i2}} k(z,s)\varkappa(z-s)u(s)ds,$$

$$(T_{iIII}u)(z) = \int_{y_{i2}}^{y_{i+1}} k(z,s)\varkappa(z-s)u(s)ds.$$

Then

$$\frac{d^m}{dz^m}(T_i u)(z)\Big|_{z=t} = \left(\frac{d^m}{dz^m}(T_{iI}u)(z) + \frac{d^m}{dz^m}(T_{iII}u)(z) + \frac{d^m}{dz^m}(T_{iIII}u)(z)\right)\Big|_{z=t} =$$

$$= \sum_{k=0}^{n_i} C_{n_i}^k \left(T_{iI}^{m-n_i,k} u^{(n_i-k)}\right)(t) + \sum_{k=0}^{m-p-1} C_{m-p-1}^k \left(T_{iII}^{p+1,k} u^{(m-p-1-k)}\right)(t) +$$

$$+ \sum_{k=0}^{n_{i+1}} C_{n_{i+1}}^k \left(T_{iIII}^{m-n_{i+1},k} u^{(n_{i+1}-k)}\right)(t) + \left(\Lambda_{y_i+0}^{m-n_i,n_i} u\right)(t) -$$

$$-\left(\Lambda_{\bar{y}_{i1}}^{m-n_i,n_i} u\right)(t) + \left(\Lambda_{\bar{y}_{i1}}^{p+1,m-p-1} u\right)(t) - \left(\Lambda_{\bar{y}_{i2}}^{p+1,m-p-1} u\right)(t) +$$

$$+ \left(\Lambda_{\bar{y}_{i2}}^{m-n_{i+1},n_{i+1}} u\right)(t) - \left(\Lambda_{y_{i+1}-0}^{m-n_{i+1},n_{i+1}} u\right)(t).$$

Define

$$(Ku)(t)=\sum_{i\in J(t)}\sum_{j=0}^{1}\sum_{k=0}^{n_{i+j}}C_{n_{i+j}}^{k}\left(T_{ij}^{m-n_{i+j},k}u^{(n_{i+j}-k)}\right)(t)+$$

$$+\sum_{\substack{0\le i\le r\\ i\notin J(t)}}\left(\sum_{k=0}^{1}C_{n_i}^{k}\left(T_{iI}^{m-n_i,k}u^{(n_i-k)}\right)(t)+\sum_{k=0}^{m-p-1}C_{m-p-1}^{k}\left(T_{iII}^{p+1,k}u^{(m-p-1-k)}\right)(t)+\right.$$

$$+\sum_{k=0}^{n_{i+1}}C_{n_{i+1}}^{k}\left(T_{iIII}^{m-n_{i+1},k}u^{(n_{i+1}-k)}\right)(t)-\sum_{j=1}^{2}(-1)^{j}\Big(\Lambda_{\bar{y}_{ij}}^{p+1,m-p-1}u-$$

$$\left.-\Lambda_{\bar{y}_{ij}}^{m-n_{i+j-1},n_{i+j-1}}u\Big)(t)\right),$$

$$(Vu)(t)=\frac{d^{m-1}}{dt^{m-1}}\big(k(t,a)\chi(t-a)u(a)-k(t,b)\chi(t-b)u(b)\big)\cdot$$

$$+\sum_{i=1}^{r}\left(\Lambda_{y_i+0}^{m-n_i,n_i}u-\Lambda_{y_i-0}^{m-n_i,n_i}u\right)(t)+$$

$$+\sum_{i\in J(t)}\left(\Lambda_{\bar{y}_i}^{m-n_{i+1},n_{i+1}}u-\Lambda_{\bar{y}_i}^{m-n_i,n_i}u\right)(t).$$

Now we can write a short formula for $\frac{d^m}{dt^m}(Tu)(t)$:

$$\frac{d^m}{dt^m}(Tu)(t)=(Ku)(t)+(Vu)(t)\qquad \text{for } t\in[c,d]\setminus\{y_0,\dots,y_{r+1}\}. \tag{7}$$

The operator K has a quality, which is very important in the proofs of Theorems 1 and 2.

Lemma 5. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence of functions in the space $E_S^m(a,b)$. Then the sequence $(Ku_k)_{k\in\mathbb{N}}$ has a subsequence $(Ku_k)_{k\in\mathbb{N}}$, which converges uniformly on each set

$$G_\nu=[c,d]\setminus\Big(\bigcup_{i=0}^{r+1}(y_i-\nu,\ y_i+\nu)\Big),\qquad \nu>0.$$

<u>Proof.</u> Let $\nu>0$ be fixed. We want to use the Arzela Theorem. So we have to show that the sequence $(Ku_k)_{k\in\mathbb{N}}$ is equi-continuous and equi-bounded on the set G_ν.

Using the estimation (3) we obtain that for some sufficiently small $\varepsilon>0$

$$\left|\left(T_{ij}^{m-n_{i+j},k}u^{(n_{i+j}-k)}\right)(t)\right|\le$$

$$\le c\|u\|_S\left|\int_{y_{i+j}}^{\bar{y}_i}h(t-s,\beta+1-n_{i+j})h(s-y_{i+j},\beta_{i+j}-m+n_{i+j})ds\right|\le$$

$$\le c\|u\|_S\big(h(t-y_{i+j},\beta_{i+j}+\beta-m+\varepsilon)+1\big)\quad\text{for } i\in J(t).$$

$$\left|\left(T_{iI}^{m-n_i,k}u^{(n_i-k)}\right)(t)\right|\le c\|u\|_S\int_{y_i}^{\bar{y}_{i1}}h(t-s,\beta+1-n_i)h(s-y_i,\beta_i-m+n_i)ds\le$$

$$\le c\|u\|_S\big(h(t-y_i,\beta_i+\beta-m+\varepsilon)+1\big)\,.\qquad t\in[y_i,y_{i+1}].$$

$$\left|\left(T_{iII}^{p+1,k}u^{(m-p-1-k)}\right)(t)\right|\le c\,\|u\|_S\big(h(t-y_i,\beta_i+\beta-m+\varepsilon)+$$
$$+h(t-y_{i+1},\beta_{i+1}+\beta-m+\varepsilon)\big)\quad t\in[y_i,y_{i+1}],$$
$$\left|\left(T_{iIII}^{m-n_{i+1},k}u^{(n_{i+1}-k)}\right)(t)\right|\le c\|u\|_S\big(h(t-y_{i+1},\beta_{i+1}+\beta-m+\varepsilon)+1\big),\ t\in[y_i,y_{i+1}],$$
$$\left|\left(\Lambda_{\bar y_{ij}}^{p+1,m-p-1}u-\Lambda_{\bar y_{ij}}^{m-n_{i+j-1},n_{i+j-1}}u\right)(t)\right|=$$
$$=\left|\sum_{k=n_{i+j-1}}^{m-p-2}\frac{d^{m-1-k}}{dt^{m-1-k}}\Big(\sum_{l=0}^{k}C_k^l k_l(t,\bar y_{ij})\varkappa(t-\bar y_{ij})u^{(k-l)}(\bar y_{ij})\Big)\right|\le$$
$$\le c\|u\|_S\sum_{k=n_{i+j-1}}^{m-p-2}h(t-\bar y_{ij},\beta-k)h(y_{i+j-1}-\bar y_{ij},\beta_{i+j-1}-m+k)\le$$
$$\le c\|u\|_S\big(h(t-y_{i+j-1},\beta_{i+j-1}+\beta-m+\varepsilon)+1\big),\ j=1,2,\ t\in[y_i,y_{i+1}].$$

We have got the estimation

$$|(Ku)(t)|\le c\|u\|_S\sum_{i=1}^{r}h(t-y_i,\beta_i+\beta-m+\varepsilon),\tag{8}$$

hence the sequence $(Ku_k)_{k\in\mathbb{N}}$ is equi-bounded on the set G_ν.

The equi-continuity of the sequence $(Ku_k)_{k\in\mathbb{N}}$ we can prove by using the following facts :

1.the functions $\frac{\partial^i}{\partial t^i}\big(k_j(t,s)\varkappa(t-s)\big)$, $i\le m-1$, $i+j\le m$ are equi-continuous on every set $[c,d]\times[a,b]\setminus\{(t,s):|t-s|\ge\delta,\ \delta>0\}$,

2.the rate of convergence of the integral

$$\int_B\left|\frac{\partial}{\partial t^i}\big(k_j(t,s)\varkappa(t-s)\big)u^{(k)}(s)\right|ds,\ B\subseteq G_\nu\cap[a,b],\quad k\le m-p-1,\ i\le p+1,\ i+j\le m,\quad t\in G_\nu$$

to zero with $\mathrm{mes}(B)\to 0$ depends only on $\|u\|_S$,

3. $u\in C^{m-p-2}(y_i,y_{i+1})$, $i=0,\dots,r$,

$$|u^{(m-p-2)}(t_1)-u^{(m-p-2)}(t_2)|\le c\|u\|_S|t_1-t_2|^\mu \text{ for some } \mu>0,\quad t_1,t_2\in G_\nu .$$

Then according to the Arzela theorem there exists an infinitive subsequence $(Ku_k)_{k\in\mathbb{N}'}$ which converges uniformly on the set G_ν.

Let we have a sequence $\nu_j\to 0$. Then we have

$(Ku_k)_{k\in\mathbb{N}_1}$, $\mathbb{N}_1\subseteq\mathbb{N}$, converges uniformly on the set G_{ν_1},

$(Ku_k)_{k\in\mathbb{N}_2}$, $\mathbb{N}_2\subseteq\mathbb{N}_1$, converges uniformly on the set G_{ν_2},

……………………………………………………

$(Ku_k)_{k\in\mathbb{N}_j}$, $\mathbb{N}_j\subseteq\mathbb{N}_{j-1}$, converges uniformly on the set G_{ν_j},

……………………………………………………

By the diagonal process we get the subsequence of $(Ku_k)_{k\in\mathbb{N}}$ which converges on the each set G_ν. Lemma 5 is proved.■.

5. The proofs of Theorems 1 and 2. Now we have all what we need for proving Theorems 1 and 2. Assume the conditions (2). Denote

$$(T_1u)(t)=\int_{a_1}^{a_2} g(t,s)\,\varkappa(t-\varphi(s))\,u(s)\,ds.$$

Then we have

$$(T_1u)(t)=\int_{\varphi(a_1)}^{\varphi(a_2)} g(t,\psi(s))\,\varkappa(t-s)\,u(\psi(s))\,\psi'(s)\,ds,$$

where ψ is the inverse function to φ. Define the operators T and Ψ by

$$(\Psi u)(t)=u(\psi(t)),$$

$$(Tu)(t)=\int_{\varphi(a_1)}^{\varphi(a_2)} k(t,s)\,\varkappa(t-s)\,u(s)\,ds,$$

where $k(t,s)=g(t,\psi(s))\,\psi'(s)$. Then

$$(T_1u)(t)=(T\Psi u)(t).$$

Proof of Theorem 1. We have the set $S=\{(x_{ij},\beta_{ij},p_{ij})_{i=1}^{q}{}_{j=1}^{2}\}$, where

$$p_{ij}=im-1-[i\beta],\ \beta_{ij}=\beta-(i-1)(m-\beta-\varepsilon),\ x_{ij}=(\varphi)^{-1}(a_j),\ i=1,\dots,q,\ j=1,2,$$

$$q=\left[\frac{m}{m-\beta}\right],\ 0<\varepsilon<\min_{2\le i\le q+1}\left\{\frac{[i\beta]-i\beta+1}{q},\ \frac{qm-(q+1)\beta}{q-1}\right\}.$$

We want to use Lemma 1, therefore we have to prove that the operator T_1 is completely continuous from $E^m_S(a_1,a_2)$ into $E^m_S(a_1,a_2)$.

Let us define the set S^* by

$$S^*=\{(\varphi(x_{ij}),\beta_{ij},p_{ij}):(x_{ij},\beta_{ij},p_{ij})\in S\}.$$

It is obvious that the operator Ψ is linear and bounded from $E^m_S(a_1,a_2)$ into $E^m_{S^*}(\varphi(a_1),\varphi(a_2))$ and hence to prove Theorem 1 we have to prove that T is completely continuous from $E^m_{S^*}(\varphi(a_1),\varphi(a_2))$ into $E^m_S(a_1,a_2)$.

First let us check that for $u\in E^m_{S^*}(\varphi(a_1),\varphi(a_2))$ the function Tu belongs to the space $E^m_S(a_1,a_2)$. Since $p=m-2-[\beta]$ and

$$\begin{aligned}
p_{ij}+p+2-H((p-m+2+\beta)+(p_{ij}-m+1+\beta_{ij})-1) &= im-1-[i\beta]+m-2-[\beta]+2-\\
&-H(m-2-[\beta]-m+2+\beta+im-1-[i\beta]-m+1+\beta-(i-1)(m-\beta-\varepsilon)-1)=\\
&=(i+1)m-[i\beta]-[\beta]-1-H((i+1)\beta-[i\beta]-[\beta]-1+(i-1)\varepsilon)=\\
&=(i+1)m-1-[i\beta]-[\beta]-[(i+1)\beta-[i\beta]-[\beta]+(i-1)\varepsilon]=\\
&=(i+1)m-1-[(i+1)\beta+(i-1)\varepsilon]=(i+1)m-1-[(i+1)\beta]=\\
&=\begin{cases} p_{(i+1)j}, & 1\le i\le q-1,\\ -[(q+1)(\beta-m)]-1\ge m, & i=q,\end{cases}
\end{aligned}$$

Lemma 4 ensures, that

$$Tu\in C[a_1,a_2]\cap C^m([a_1,a_2]\setminus I_S)\cap\left[\bigcap_{j=1}^{2}\bigcap_{i=1}^{q}C^{p_{ij}}(x_{ij})\right].$$

We want to show, that

$$\sup_{t\in[a_1,a_2]\setminus I_S}\frac{(Tu)^{(m)}(t)}{w(m,t)}<\infty.$$

We have (see formula (7) and estimation (8))

$$|(Tu)^{(m)}(t)| \le |(Vu)(t)| + |(Ku)(t)| \le$$
$$\le |(Vu)(t)| + c\,\|u\|_{S^*} \sum_{j=1}^{2}\sum_{i=2}^{q} h(t-x_{ij}, \beta_{(i-1)j}+\beta-m+e),$$

where e is an arbitrary small positive real number. Since $p_{ij}=m-1-[\beta_{ij}]$, we have

$$|(Vu)(t)| \le c\,\|u\|_{S^*}\big(h(t-\varphi(a_1),\beta) + h(t-\varphi(a_2),\beta)\big)=$$
$$= c\,\|u\|_{S^*}\big(h(t-x_{11},\beta_{11}) + h(t-x_{12},\beta_{12})\big).$$

If $e<\varepsilon$, then $\beta_{(i-1)j}+\beta-m+e<\beta_{(i-1)j}+\beta-m+\varepsilon=\beta_{ij}$ and therefore

$$|(Tu)^{(m)}(t)| \le c\,\|u\|_{S^*} \sum_{j=1}^{2}\sum_{i=1}^{q} h(t-x_{ij},\beta_{ij}) = c\,\|u\|_{S^*}\, w_S(m,t).$$

Consequently we have proved, that $T\in\mathcal{L}\big(E^m_{S^*}(\varphi(a_1,\varphi(a_2)),\ E^m_S(a_1,a_2)\big)$.

Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $E^m_{S^*}(\varphi(a_1,\varphi(a_2))$, $\|u_k\|_{S^*}\le M$. We know that the operator T is completely continuous from $C[\varphi(a_1),\varphi(a_2)]$ into $C[a_1,a_2]$. It is not difficult to show (by using Lemma 5), that there exists an infinite subset $\mathbb{N}'\subset\mathbb{N}$ such that

i) the sequence $(Tu_k)_{k\in\mathbb{N}'}$ is convergent in $C[a_1,a_2]$,

ii) the sequence $\left(\frac{(Vu_k)(t)}{w_S(m,t)}\right)_{k\in\mathbb{N}'}$ converges uniformly on the set $[a_1,a_2]\setminus I_S$,

iii) the sequence $(Ku_k)_{k\in\mathbb{N}'}$ converges uniformly on each set $G_\nu=[a_1,a_2]\setminus\big(\bigcup_{x\in I_S}(x-\nu,x+\nu)\big)$.

Let us prove, that the sequence $\left(\frac{(Ku_k)(t)}{w_S(m,t)}\right)_{k\in\mathbb{N}'}$ converges uniformly on the set $[a_1,a_2]\setminus I_S$. For every sufficiently small $\nu>0$ we have

$$\sup_{t\in[a_1,a_2]\setminus I_S}\left|\frac{(Ku_k)(t)}{w_S(m,t)}-\frac{(Ku_l)(t)}{w_S(m,t)}\right|\le$$
$$\le\left\|\frac{1}{w_S(m,t)}\right\|_C \sup_{t\in G_\nu}|(Ku_k-Ku_l)(t)| +$$
$$+2Mc\sup_{t\in[a_1,a_2]\setminus G_\nu}\frac{\sum_{j=1}^{2}\sum_{i=1}^{q} h(t-x_{ij},\beta_{ij}-\varepsilon+e)}{\sum_{j=1}^{2}\sum_{i=1}^{q} h(t-x_{ij},\beta_{ij})}.$$

Let $e<\varepsilon$. Then for every $\delta>0$ there exists $\nu>0$ and $k_0\in\mathbb{N}$ such that

$$\sup_{t\in[a_1,a_2]\setminus G_\nu}\frac{\sum_{j=1}^{2}\sum_{i=1}^{q} h(t-x_{ij},\beta_{ij}-\varepsilon+e)}{\sum_{j=1}^{2}\sum_{i=1}^{q} h(t-x_{ij},\beta_{ij})}\le\frac{\delta}{4Mc},$$

$$\left\|\frac{1}{w_S(m,t)}\right\|_C \sup_{t\in G_\nu}|(Ku_k-Ku_l)(t)|\le\frac{\delta}{2},\quad k,l\in\mathbb{N}',\ k,l\ge k_0$$

and therefore

$$\sup_{t\in[a_1,a_2]\setminus I_S}\left|\frac{(Ku_k)(t)}{w_S(m,t)}-\frac{(Ku_l)(t)}{w_S(m,t)}\right|\le\delta,\quad \text{if}\quad k,l\in N',\ k,l\ge k_0.$$

Hence the sequence $\left(\frac{(Ku_k)(t)}{w_S(m,t)}\right)_{k\in N'}$ converges uniformly on the set $[a_1,a_2]\setminus I_S$. This with i) and ii) gives us, that the sequence $(Tu_k)_{k\in N'}$ is convergent in the space $E_S^m(a_1,a_2)$. Hence the operator T is completely continuous from $E_S^m(\varphi(a_1),\varphi(a_2))$ into $E_S^m(a_1,a_2)$ and consequently the operator T_1 is completely continuous from $E_S^m(a_1,a_2)$ into $E_S^m(a_1,a_2)$. Now Lemma 1 completes the proof. ■

Proof of Theorem 2. We have the set $S=\{(x_{ij},\beta_{ij},p_{ij})_{i=1}^{q}{}_{j=1}^{2}\}$, where $q=\left[\frac{m}{p+2}\right]$, $p_{ij}=i(p+2)-1$, $\beta_{ij}=\beta-(i-1)(p+2)$, $x_{ij}=(\varphi)^i(a_j)$, $i=1,\dots,q$, $j=1,2$. Denote $\alpha_{ij}=\max\{\beta_{ij},\check{Z}\}$, $i=1,\dots,q$, $j=1,2$. Let us define the sets $S_\circledast$ and S^* by $S_\circledast=\{(x_{ij},\alpha_{ij},p_{ij})_{i=1}^{q}{}_{j=1}^{2}\}$ and $S^*=\{(\varphi(x_{ij}),\alpha_{ij},p_{ij})_{i=1}^{q}{}_{j=1}^{2}\}$. It is obvious that $E_S^m(a_1,a_2)\subset E_{S_\circledast}^m(a_1,a_2)$.

The idea of the proof is as follows. We prove that the operator T_1 is completely continuous from $E_{S_\circledast}^m(a_1,a_2)$ into $E_{S_\circledast}^m(a_1,a_2)$. Now by Lemma 1 we obtain, that if $f\in E_{S_\circledast}^m(a_1,a_2)$, then all continuous solutions to (1) belong to $E_{S_\circledast}^m(a_1,a_2)$. In conclusion we show, that if $f\in E_S^m(a_1,a_2)$, then all solutions to (1) in $E_{S_\circledast}^m(a_1,a_2)$ belong to $E_S^m(a_1,a_2)$.

It is obvious that the operator Ψ is linear and bounded from $E_{S_\circledast}^m(a_1,a_2)$ into $E_{S^*}^m(\varphi(a_1),\varphi(a_2))$. Let us prove that T is completely continuous from $E_{S^*}^m(\varphi(a_1),\varphi(a_2))$ into $E_{S_\circledast}^m(a_1,a_2)$.

Let $u\in E_{S^*}^m(\varphi(a_1),\varphi(a_2))$. Since $p<m-2-[\beta]$, we have

$$p_{ij}+p+2-H((p-m+2+\beta)+(p_{ij}-m+1+\beta_{ij})-1)-(i+1)(p+2)-1-H(2(p-m+2+\beta))=$$

$$=\begin{cases}p_{(i+1)j}, & 1\le i\le q-1,\\ (q+1)(p+2)-1\ge m, & i=q,\end{cases}$$

and consequently Lemma 4 gives us, that

$$Tu\in C[a_1,a_2]\cap C^m([a_1,a_2]\setminus I_S)\cap\left[\bigcap_{j=1}^{2}\bigcap_{i=1}^{q}C^{p_{ij}}(x_{ij})\right].$$

From (7) and (8) we get

$$|(Tu)^{(m)}(t)|\le|(Vu)(t)|+|(Ku)(t)|\le$$

$$<|(Vu)(t)|+c\|u\|_{S^*}\sum_{j=1}^{2}\sum_{i=2}^{q}h(t-x_{ij},\alpha_{(i-1)j}+\beta-m+e),$$

where e is an arbitrary small positive real number. We have

$$|(Vu)(t)|\le c\|u\|_{S^*}\sum_{j=1}^{2}\sum_{i=1}^{q}h(t-x_{ij},\beta_{ij}).$$

Let $e<1+[\beta]-\beta$ and $\varepsilon=1+[\beta]-\beta-e$. Since $p\le m-3-[\beta]$, we have

$$\alpha_{(i-1)j}+\beta-m+e=\alpha_{(i-1)j}+[\beta]-m+1-\varepsilon\le\alpha_{(i-1)j}-p-2-\varepsilon\le$$
$$\le\begin{cases}\beta_{ij}-\varepsilon, & \text{if } \alpha_{(i-1)j}>\frac{1}{2},\\ -\frac{1}{2} & \alpha_{(i-1)j}=\frac{1}{2}\end{cases}$$

and therefore

$$|(Ku)(t)|\le c\,\|u\|_{S*}\sum_{j=1}^{2}\sum_{i=2}^{q}h(t-x_{ij},\alpha_{(i-1)j}+\beta-m+e)\le$$
$$\le c\,\|u\|_{S*}\sum_{j=1}^{2}\sum_{i=2}^{q}h(t-x_{ij},\beta_{ij}-\varepsilon).$$

Hence

$$|(Tu)^{(m)}(t)|\le c\,\|u\|_{S*}\,w_S(m,t)\le c\,\|u\|_{S*}\,w_{S\circledast}(m,t). \tag{9}$$

and consequently $Tu\in E^m_{S\circledast}(a_1,a_2)$.

The proof of the compactness of the operator T from $E^m_{S*}(\varphi(a_1),\varphi(a_2))$ into $E^m_{S\circledast}(a_1,a_2)$ is similar to that on the proof of Theorem 1. So we have, that the operator T_1 is a compact operator in $E^m_{S\circledast}(a_1,a_2)$ and therefore, according to Lemma 1, in case $f\in E^m_{S\circledast}(a_1,a_2)$ all continuous solutions to (1) belong to $E^m_{S\circledast}(a_1,a_2)$.

Let $f\in E^m_S(a_1,a_2)$ and let $u\in E^m_{S\circledast}(a_1,a_2)$ be a solution to (1). Then the first inequality of the estimation (9) gives us, that $T_1u=T\Psi u\in E^m_S(a_1,a_2)$. Since $u=T_1u+f$, we have $u\in E^m_S(a_1,a_2)$. This proves the Theorem.

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ГЛАДКОСТЬ РЕШЕНИЯ ИНТЕГРАЛЬНОГО УРАВНЕНИЯ, ЯДРО КОТОРОГО ИМЕЕТ ОСОБЕННОСТЬ НА КРИВОЙ

Р. Кангро

Резюме

В работе доказаны две теоремы о гладкости решений одного класса интегральных уравнений второго рода в случае, когда ядро интегрального оператора или его некоторая производная имеет особенность на гладкой кривой $t=\varphi(s)$.

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РЕШЕНИЕ ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ МЕТОДОМ ПОДОБЛАСТЕЙ

Энн Тамме

В работе построен алгоритм для решения параболического уравнения методом подобластей, причем решение задачи аппроксимируется квадратическими сплайнами. Исследуется устойчивость и сходимость метода. При этом пользуется идеями из работ [2, 3]. В первой из них исследуется метод коллокации для параболического уравнения, во второй методы коллокации и подобластей для краевых задач.

1. Оценки для квадратического сплайна. Пусть функция $f(x)$ интегрируема на отрезке $[a, b]$, где $a < b$. Рассмотрим квадратический сплайн $\mathcal{S}(x)$ класса C^1 на сетке узлов $x_i = a + ih$, $i = 0, 1, \dots, N$, где $h = (b - a)/N$. В дальнейшем, при исследовании сходимости метода подобластей пользуемся квадратическим сплайном $\mathcal{S} \in C^1[a, b]$, аппроксимирующим функцию f в следующем смысле:

$$\int_{x_{i-1}}^{x_i} \mathcal{S}(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\,dx,\ i = 1, \dots, N,\ \mathcal{S}(a) = f(a),\ \mathcal{S}(b) = f(b). \tag{1.1}$$

Обычной техникой (см. [1, 3]) получаются для такого сплайна следующие оценки. Если $f \in C^3[a, b]$, то

$$\max_{a \le x \le b} \left| \mathcal{S}^{(r)}(x) - f^{(r)}(x) \right| \le c_r h^{3-r} M_3, \quad r = 0, 1, 2, \tag{1.2}$$

где $M_3 = \max\limits_{a \le x \le b} |f'''(x)|$ и $c_0 < 1$, $c_1 < 2{,}4$, $c_2 < 2{,}8$. Если $f \in C^4[a, b]$, то

$$\max_{1 \le i \le N} \left| \mathcal{S}''(\bar{x}_i) - f''(\bar{x}_i) \right| \le 0{,}9 h^2 M_4, \tag{1.3}$$

где $M_4 = \max\limits_{a \le x \le b} |f^{(4)}(x)|$ и $\bar{x}_i = (x_{i-1} + x_i)/2$.

Квадратический сплайн представим в виде

$$\mathcal{S}(x) = \sum_{j=0}^{N+1} \alpha_j \mathcal{B}_j(x), \tag{1.4}$$

где $\mathcal{B}_j$ - квадратические $\mathcal{B}$-сплайны:

$$\mathcal{B}_j(x) = \frac{1}{2h^2} \begin{cases} (x - x_{j-2})^2 & \text{при } x \in [x_{j-2}, x_{j-1}], \\ h^2 + 2(x - x_{j-1})(x_j - x) & \text{при } x \in [x_{j-1}, x_j], \\ (x - x_{j+1})^2 & \text{при } x \in [x_j, x_{j+1}], \\ 0 & \text{при } x \notin (x_{j-2}, x_{j+1}). \end{cases}$$

При этом пользуемся дополнительными узлами $x_i = a + ih$, $i = -2, -1, N+1, N+2$.

Если подставим выражение (1.4) в условия (1.1), то для определения коэффициентов α_j получаем систему

$$\alpha_{i-1} + 4\alpha_i + \alpha_{i+1} = \frac{6}{h}\int_{x_{i-1}}^{x_i} f(x)\,dx,\quad i = 1,\ldots,N,$$

$$\alpha_0 + \alpha_1 = 2f(a),\quad \alpha_N + \alpha_{N+1} = 2f(b).$$

2. Метод подобластей. Рассмотрим в прямоугольнике

$$\mathcal{D} = \{(x,t): a \le x \le b,\ 0 \le t \le T\}$$

решение задачи

$$\begin{gathered}\frac{\partial u}{\partial t} = \mathcal{L}u + f(x,t),\\ u(x,0) = u^0(x),\quad u(a,t) = \varphi(t),\quad u(b,t) = \psi(t),\end{gathered}\tag{2.1}$$

где

$$\mathcal{L}u = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) - q(x)\,u.$$

Пусть удовлетворены условия

$$k(x) \ge \varkappa > 0,\quad q(x) \ge 0 \ \text{при}\ x \in [a,b].\tag{2.2}$$

Считаем, что k, k', q, u^0 непрерывны на отрезке $[a, b]$, φ, ψ непрерывны на $[0, T]$ и f непрепывен на $\mathcal{D}$. Предположим, что задача (2.1) имеет непрерывное на $\mathcal{D}$ решение u.

Приближенное решение задачи (2.1) ищем в виде

$$\mathcal{S}(x,t) = \sum_{j=0}^{N+1}\alpha_j(t)\,\mathcal{B}_j(x),$$

где $\mathcal{B}_j$ - квадратические $\mathcal{B}$-сплайны. В методе подобластей коэффициенты α_j вычисляются используя условие

$$\int_{x_{i-1}}^{x_i}\left(\frac{\partial\mathcal{S}}{\partial t} - \mathcal{L}\mathcal{S} - f\right)dx = 0,\quad i = 1,\ldots,N,\quad t \in [0, T]$$

и начальные и граничные условия задачи (2.1). Таким образом получаем для определения этих коэффициентов систему обыкновенных дифференциальных уравнений, которую решаем разностным методом:

$$\begin{gathered}\int_{x_{i-1}}^{x_i}\left[\frac{\mathcal{S}^n - \mathcal{S}^{n-1}}{\tau} - \sigma\mathcal{L}\mathcal{S}^n - (1-\sigma)\mathcal{L}\mathcal{S}^{n-1} - f^n\right]dx = 0,\\ \mathcal{S}^n(a) = \varphi_n,\quad \mathcal{S}^n(b) = \psi_n,\quad n = 1,\ldots,N_0 = [T/\tau],\end{gathered}\tag{2.3}$$

где $0 < \sigma \le 1$, $\tau > 0$, $t_n = n\tau$, $\varphi_n = \varphi(t_n)$, $\psi_n = \psi(t_n)$, $f^n = f^n(x) = \sigma f(x, t_n) + (1-\sigma) f(x, t_{n-1})$ и $\mathcal{S}^n = \mathcal{S}^n(x)$ – аппроксимация решения $u(x, t_n)$. Функцию $\mathcal{S}^n$ ищем в виде

$$\mathcal{S}^n(x) = \sum_{j=0}^{N+1}\alpha_j^n\,\mathcal{B}_j(x).\tag{2.4}$$

Коэффициенты α_j^0 найдем как коэффициенты квадратичного сплайна, аппроксимирующего функцию u^0 в смысле (1.1), т. е. из системе линейных уравнений

$$\begin{gathered}\alpha_{i-1}^0 + 4\alpha_i^0 + \alpha_{i+1}^0 = \frac{6}{h}\int\limits_{x_{i-1}}^{x_i} u^0(x)\,dx, \quad i = 1, \dots, N, \\ \alpha_0^0 + \alpha_1^0 = 2u^0(a), \quad \alpha_N^0 + \alpha_{N+1}^0 = 2u^0(b).\end{gathered} \tag{2.5}$$

Если подставим в (2.3) выражение (2.4) и используем формулу трапеции

$$\int\limits_{x_{i-1}}^{x_i} q(x)\,\mathcal{Y}^n(x)\,dx \approx \frac{h}{2}\left[q_{i-1}\mathcal{Y}^n(x_{i-1}) + q_i\mathcal{Y}^n(x_i)\right],$$

то получаем для определения α_j^n систему уравнений

$$\begin{aligned}&\frac{h}{6}(\alpha_{i-1}^n + 4\alpha_i^n + \alpha_{i+1}^n) - \frac{\tau\sigma}{h}[k_{i-1}\alpha_{i-1}^n - (k_{i-1}+k_i)\alpha_i^n + k_i\alpha_{i+1}^n] + \\ &+ \frac{h\tau\sigma}{4}[q_{i-1}\alpha_{i-1}^n + (q_{i-1}+q_i)\alpha_i^n + q_i\alpha_{i+1}^n] = \frac{h}{6}(\alpha_{i-1}^{n-1} + 4\alpha_i^{n-1} + \alpha_{i+1}^{n-1}) + \\ &+ \frac{\tau(1-\sigma)}{h}[k_{i-1}\alpha_{i-1}^{n-1} - (k_{i-1}+k_i)\alpha_i^{n-1} + k_i\alpha_{i+1}^{n-1}] - \frac{h\tau(1-\sigma)}{4}[q_{i-1}\alpha_{i-1}^{n-1} + \\ &+ (q_{i-1}+q_i)\alpha_i^{n-1} + q_i\alpha_{i+1}^{n-1}] + h\tau f_i^n, \qquad i = 1, \dots, N,\end{aligned} \tag{2.6}$$

$$\alpha_0^n + \alpha_1^n = 2\varphi_n, \quad \alpha_N^n + \alpha_{N+1}^n = 2\psi_n, \tag{2.7}$$

где $q_i = q(x_i)$, $k_i = k(x_i)$ и $f_i^n = \frac{1}{h}\int\limits_{x_{i-1}}^{x_i} \mathscr{f}^n(x)\,dx$. Исключая из (2.6) α_0^n, α_{N+1}^n, α_0^{n-1} и α_{N+1}^{n-1}, преобразуем эту систему к виду

$$C\alpha^n + \tau\sigma\mathcal{A}\alpha^n = C\alpha^{n-1} - \tau(1-\sigma)\mathcal{A}\alpha^{n-1} + \tau\mathcal{F}^n, \tag{2.8}$$

где $\alpha^n = (\alpha_1^n, \dots, \alpha_N^n)^T$, $\mathcal{F}^n = \Big(f_1^n - \frac{1}{3\tau}(\varphi_n - \varphi_{n-1}) + 2\Big(\frac{k_0}{h^2} - \frac{q_0}{4}\Big)[\sigma\varphi_n + (1-\sigma)\varphi_{n-1}]$, $f_2^n, \dots, f_{N-1}^n$, $f_N^n - \frac{1}{3\tau}(\psi_n - \psi_{n-1}) + 2\Big(\frac{k_N}{h^2}\ \frac{q_N}{4}\Big)[\sigma\psi_n + (1-\sigma)\psi_{n-1}]\Big)^T$, $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ и трехдиагональные матрицы порядка N

$$C = \frac{1}{6}\begin{bmatrix} 3 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 3 \end{bmatrix},$$

$$\mathcal{A}_1 = \frac{1}{h^2}\begin{bmatrix} 2k_0 + k_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{N-2} & k_{N-2} + k_{N-1} & -k_{N-1} \\ & & & -k_{N-1} & k_{N-1} + 2k_N \end{bmatrix},$$

$$\mathcal{A}_2 = \frac{1}{4}\begin{bmatrix} q_1 & q_1 & & & \\ q_1 & q_1 + q_2 & q_2 & & \\ & \cdot & \cdot & \cdot & \\ & & q_{N-2} & q_{N-2} + q_{N-1} & q_{N-1} \\ & & & q_{N-1} & q_{N-1} \end{bmatrix}.$$

Таким образом получим для решения задачи (2.1) следующий алгоритм. Коэффициенты α_j^0 найдем из системы (2.5), коэффициенты α_j^n при $n = 1, \ldots, N_0$ из уравнений (2.6) и (2.7). Затем приближение $y^n(x)$ решения $u(x, t_n)$ задачи (2.1) получаем в виде (2.4).

Отметим, что такой метод не является "чистым" методом подобластей, а реализацией этого метода, в которой интегралы, связанные с членом qu, вычисляются формулой трапеции. Теорема, доказанная в следующем, остается верной и тогда, когда также интегралы в системе (2.5) и интегралы f_i^n вычисляются формулой трапеции или формулой средних прямоугольников.

3. Сходимость метода. Система (2.8) является двухслойной разностной схемой. Преобразуем эту схему к каноническому виду

$$(C + \tau\sigma\mathcal{A})\frac{\alpha^n - \alpha^{n-1}}{\tau} + \mathcal{A}\alpha^{n-1} = \mathcal{F}^n \tag{3.1}$$

и исследуем ее устойчивость, пользуясь результатами из [4]. Матрицы $\mathcal{A}$ и C являются симметрическими. Покажем, что они положительно определены, если выполнены условия (2.2).

Определим скалярное произведение векторов $\alpha = (\alpha_1, \ldots, \alpha_N)^T$ и $\beta = (\beta_1, \ldots, \beta_N)^T$:

$$(\alpha, \beta) = \sum_{i=1}^{N} \alpha_i \beta_i h.$$

Этому скалярному произведению соответствует норма $\|\alpha\| = (\alpha, \alpha)^{1/2}$. Несложными преобразованиями получаются соотношения

$$(\mathcal{A}_1\alpha, \alpha) = \frac{1}{h}\sum_{i=2}^{N} k_{i-1}(\alpha_i - \alpha_{i-1})^2 + \frac{2}{h}(k_0\alpha_1^2 + k_N\alpha_N^2) \geq$$

$$\geq \frac{\varkappa}{h}\left[\sum_{i=2}^{N}(\alpha_i - \alpha_{i-1})^2 + 2(\alpha_1^2 + \alpha_N^2)\right] \geq \frac{4\varkappa}{(b-a)^2}\|\alpha\|^2,$$

$$(\mathcal{A}_2\alpha, \alpha) = \frac{h}{4}\sum_{i=2}^{N} q_{i-1}(\alpha_{i-1} + \alpha_i)^2 \geq 0,$$

$$(C\alpha, \alpha) \geq \frac{1}{3}\|\alpha\|^2.$$

Таким образом, при $\sigma \geq 1/2$ имеет место оценка

$$((C + \tau\sigma\mathcal{A})\alpha, \alpha) \geq \frac{1}{3}(\alpha, \alpha) + \frac{\tau}{2}(\mathcal{A}\alpha, \alpha).$$

Из этой следует оценка устойчивости для решения схемы (3.1) (см. [4], с. 211):

$$\|\alpha^n\|^2_{\mathcal{A}} < \|\alpha^0\|^2_{\mathcal{A}} + \frac{3}{2}\tau \sum_{j=1}^{n} \|\mathcal{F}^j\|^2, \tag{3.2}$$

где $\|\alpha\|_{\mathcal{A}} = (\mathcal{A}\alpha, \alpha)^{1/2}$. При помощи этой оценки докажем следующий результат о сходимости метода подобластей.

Теорема. Пусть $1/2 \le \sigma \le 1$, выполнены условия (2.2), k, $q \in C^2[a, b]$ и решение задачи (2.1) $u \in C^4(\mathcal{D})$. Тогда рассматриваемый метод сходится со скоростью

$$\max_{a \le x \le b} |\varphi^n(x) - u(x, t_n)| \le c(h^2 + \tau^p), \quad n = 0, 1, \dots, N_0, \tag{3.3}$$

где c – постаянная, $p = 2$ при $\sigma = 1/2$ и $p = 1$ при $\sigma > 1/2$, а квадратический сплайн $\varphi^n(x)$ найден при помощи (2.4) – (2.7).

Доказательство. Пусть

$$\bar{\varphi}^n(x) = \sum_{j=0}^{N+1} \bar{\alpha}_j^n \mathcal{B}_j(x) \tag{3.4}$$

квадратический сплайн, аппроксимирующий решение u задачи (2.1) в следующем смысле:

$$\int_{x_{i-1}}^{x_i} \bar{\varphi}^n(x)\,dx = \int_{x_{i-1}}^{x_i} u(x, t_n)\,dx, \quad i = 1, \dots, N,$$

$$\bar{\varphi}^n(a) = \varphi_n, \quad \bar{\varphi}^n(b) = \psi_n.$$

Обозначим

$$\varepsilon_i^n = \int_{x_{i-1}}^{x_i} \left[\frac{\bar{\varphi}^n - \bar{\varphi}^{n-1}}{\tau} - \sigma\frac{\partial}{\partial x}\left(k\frac{\partial\bar{\varphi}^n}{\partial x}\right) - (1-\sigma)\frac{\partial}{\partial x}\left(k\frac{\partial\bar{\varphi}^{n-1}}{\partial x}\right) - \tilde{f}^n\right]dx +$$
$$+ \frac{h\sigma}{2}\left[q_{i-1}\bar{\varphi}^n(x_{i-1}) + q_i\bar{\varphi}^n(x_i)\right] +$$
$$+ \frac{h(1-\sigma)}{2}\left[q_{i-1}\bar{\varphi}^{n-1}(x_{i-1}) + q_i\bar{\varphi}^{n-1}(x_i)\right]. \tag{3.5}$$

Используя оценки (1.2), (1.3) и остаточный член формулы трапеции, получаем

$$|\varepsilon_i^n| \le c_1 h\,(h^2 + \tau^p), \quad i = 1, \dots, N, \quad n = 1, \dots, N_0. \tag{3.6}$$

Подставляя выражение (3.4) в (3.5) и учитывыя граничные условия для $\varphi^n(x)$, получаем

$$(C + \tau\sigma\mathcal{A})\frac{\bar{\alpha}^n - \bar{\alpha}^{n-1}}{\tau} + \mathcal{A}\bar{\alpha}^{n-1} = \mathcal{F}^n + \frac{1}{h}\varepsilon^n,$$

где $\bar{\alpha}^n = (\bar{\alpha}_1^n, \dots, \bar{\alpha}_N^n)$ и $\varepsilon^n = (\varepsilon_1^n, \dots, \varepsilon_N^n)$. Вычитая из последнего равенства равенство (3.1), получаем

$$(C + \tau\sigma\mathcal{A})\frac{z^n - z^{n-1}}{\tau} + \mathcal{A}z^{n-1} = \frac{1}{h}\varepsilon^n,$$

где $z^n = \bar{\alpha}^n - \alpha^n$ ($z^0 = 0$). Из неравенств (3.2) и (3.6) следует

$$\|z^n\|_A \le \left(\frac{3\tau}{2h^2}\sum_{j=1}^{n}\|\varepsilon^j\|^2\right)^{1/2} \le c_2(h^2 + \tau^n), \quad n = 1, \dots, N_0 .$$

При помощи оценки

$$\max_{1\le i\le N}|z_i^n| \le \left(\frac{b-a}{4\varkappa}\right)^{1/2}\|z^n\|_A$$

получаем, что

$$\max_{a\le x\le b}|\bar{\varphi}^n(x) - \varphi^n(x)| = \max_{a\le x\le b}\left|\sum_{j=0}^{N+1}(\bar{\alpha}_j^n - \alpha_j^n)B_j(x)\right| \le$$

$$\le \max_{0\le j\le N+1}|\bar{\alpha}_j^n - \alpha_j^n| \le c_3(h^2 + \tau^n), \quad n = 1, \dots, N_0 .$$

Отсюда и из оценки (1.2) следует

$$\max_{a\le x\le b}|\varphi^n(x) - u(x, t_n)| \le \max_{a\le x\le b}|\varphi^n(x) - \bar{\varphi}^n(x)| +$$

$$+ \max_{a\le x\le b}|\bar{\varphi}^n(x) - u(x, t_n)| \le c(h^2 + \tau^n),$$

т.е. оценка погрешности (3.3). Теорема доказана.

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SUBREGIONS METHOD FOR SOLVING PARABOLIC EQUATIONS

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Summary

We construct an algorithm for solving parabolic equation (2.1) using the subregions method with quadratic splines. The stability and the convergence of the method is proved.

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О РЕШЕНИИ КРАЕВЫХ ЗАДАЧ КУБИЧЕСКИМИ СПЛАЙНАМИ

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Изучают методы коллокации и подобластей кубическими сплайнами для краевых задач обыкновенных дифференциальных уравнений второго порядка, где в краевых условиях присутствуют производные искомой функции. На равномерной сетке в методе коллокации приведена оценка погрешности порядка $O(h^2)$ с главными членами, в методе подобластей установлен более высокий порядок сходимости, при достаточно гладком решении до $O(h^4)$. Данная работа является продолжением исследований [2] и [3].

1. Оценка погрешности метода коллокации. Рассмотрим краевую задачу

$$\begin{aligned} &(Lu)(x)\equiv p(x)u''(x)+q(x)u'(x)+r(x)u(x)=f(x),\ x\in(a,b),\\ &\alpha_1u(a)+\beta_1u'(a)=\gamma_1, \qquad\qquad (1)\\ &\alpha_2u(b)+\beta_2u'(b)=\gamma_2. \end{aligned}$$

Пусть функции p, q и r достаточно гладкие, $p(x)\geq p_0>0$, $r(x)\leq r_0<0$, $x\in(a,b)$, $\alpha_1, \alpha_2\geq 0$, $\beta_1\leq 0$, $\beta_2\geq 0$, $|\alpha_i|+|\beta_i|\neq 0$, $i=1, 2$. Предположим, что задача (1) имеет решение $u\in C^4[a,b]$. Традиционной техникой теории краевых задач можно показать, что тогда решение задачи (1) единственное.

На отрезке $[a, b]$ введем равномерную сетку $x_i=a+ih$, $i=0,\dots,n$, $h=(b-a)/n$. В методе коллокации приближенное решение $\tilde u(x)$ задачи (1) как кубический сплайн из класса C^2 с узлами x_i определим условиями

$$\begin{aligned} &(L\tilde u)(x_i)=f(x_i),\ i=0,\dots,n,\\ &\alpha_1\tilde u(a)+\beta_1\tilde u'(a)=\gamma_1, \qquad\qquad (2)\\ &\alpha_2\tilde u(b)+\beta_2\tilde u'(b)=\gamma_2. \end{aligned}$$

Введем кубические B-сплайны

$$B_i(x)=\frac{1}{h^3}\begin{cases} (x-x_{i-2})^3, & x\in[x_{i-2},x_{i-1}],\\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2-3(x-x_{i-1})^3, & x\in[x_{i-1},x_i],\\ 4h^3-6h(x-x_i)^2+3(x-x_i)^3, & x\in[x_i,x_{i+1}],\\ (x_{i+2}-x)^3, & x\in[x_{i+1},x_{i+2}], \end{cases}$$

вне отрезка $[x_{i-2}, x_{i+2}]$ положим $B_i(x)=0$. Если дополнить сетку узлами $x_i=a+ih$ вне отрезка $[a, b]$ (нам достаточно взять еще $i=-3, -2, -1, n+1, n+2, n+3$), можно определить $B_{-1}(x),\dots,B_{n+1}(x)$, они образуют базис в пространстве кубических сплайнов класса C^2 на отрезке $[a, b]$ с узлами $x_0,\dots,x_n$. При таком выборе B - сплайнов имеем

$$\sum_{i=-1}^{n+1} B_i(x) = 6, \ x \in [a,b].$$

Функция $\tilde{u}(x)$ как кубический сплайн имеет представление

$$\tilde{u}(x) = \sum_{i=-1}^{n+1} c_i B_i(x),$$

ее коэффициенты c_i определяются на основании условий (2) из системы

$$\begin{cases} \left(\alpha_1 - \frac{3}{h}\beta_1\right)c_{-1} + 4\alpha_1 c_0 + \left(\alpha_1 + \frac{3}{h}\beta_1\right)c_1 = \gamma_1, \\ \left(\frac{6p(x_i)}{h^2} - \frac{3q(x_i)}{h} + r(x_i)\right)c_{i-1} + \left(-\frac{12p(x_i)}{h^2} + 4r(x_i)\right)c_i + \\ + \left(\frac{6p(x_i)}{h^2} + \frac{3q(x_i)}{h} + r(x_i)\right)c_{i+1} = f(x_i), \ i=0,\dots,n, \\ \left(\alpha_2 - \frac{3}{h}\beta_2\right)c_{n-1} + 4\alpha_2 c_n + \left(\alpha_2 - \frac{3}{h}\beta_2\right)c_{n+1} = \gamma_2. \end{cases} \qquad (3)$$

При малых h система (3) однозначно разрешима.

Решению $u(x)$ задачи (1) построим интерполирующий кубический сплайн $\bar{u}(x)$ (т.е. $\bar{u}(x_i)=u(x_i)$, $i=0,\dots,n$), удовлетворяющий краевым условиям

$$\bar{u}''(a)=u''(a)-\tfrac{1}{12}h^2u^{IV}(a), \quad \bar{u}''(b)=u''(b)-\tfrac{1}{12}h^2u^{IV}(b).$$

Тогда (см. [2], ср. также [1])

$$\bar{u}''(x_i)=u''(x_i)-\tfrac{1}{12}h^2u^{IV}(x_i)+o(h^2),$$

причем остаточный член имеет порядок $O(h^{2+\alpha})$, если $u^{IV} \in \mathrm{Lip}\ \alpha$, $0<\alpha\le 1$. Можно показать (например, при помощи оценок из [2], а также [1]), что

$$\bar{u}'(x_i)=u'(x_i)+o(h^3),$$

причем в остаточном члене есть $O(h^{3+\alpha})$, если $u^{IV} \in \mathrm{Lip}\ \alpha$, $0<\alpha\le 1$. При рассматриваемых краевых условиях сохраняются общеизвестные порядки приближения $\|\bar{u}-u\|_\infty = O(h^4)$ и $\|\bar{u}'-u'\|_\infty = O(h^3)$, здесь и в дальнейшем мы имеем ввиду максимум-норму на отрезке $[a, b]$. Поскольку $\|\bar{u}-u\|_\infty = O(h^4)$, то достаточно оценивать $\|\tilde{u}-\bar{u}\|_\infty$.

Кубический сплайн $\tilde{u}-\bar{u}$ удовлетворяет условиям

$$L(\tilde{u}-\bar{u})(x_i)=\frac{h^2}{12}p(x_i)u^{IV}(x_i)+o(h^2),\quad i=0,\dots,n,$$

$$\alpha_1(\tilde{u}(a)-\bar{u}(a))+\beta_1(\tilde{u}'(a)-\bar{u}'(a))=\beta_1(u'(a)-\bar{u}'(a)),$$

$$\alpha_2(\tilde{u}(b)-\bar{u}(b))+\beta_2(\tilde{u}'(b)-\bar{u}'(b))=\beta_2(u'(b)-\bar{u}'(b)).$$

Поэтому, если использовать прéдставление

$$\bar{u}(x)=\sum_{i=-1}^{n+1}\bar{c}_i B_i(x),$$

то $c_i-\bar{c}_i$, $i=-1,\dots,n+1$, определяются из системы с матрицей системы (3), но с правыми частями $\beta_1(u'(a)-\bar{u}'(a))$, $\frac{r}{12}h^2p(x_i)u^{IV}(x_i)+o(h^2)$, $i=0,\dots,n$, $\beta_2(u'(b)-\bar{u}'(b))$. Во внутренних уравнениях главная диагональ преобладает с разностью $6|r(x_i)|$, они дают в оценку решения компоненту $\frac{h^2}{72}\frac{p(x_i)}{|r(x_i)|}u^{IV}(x_i)+o(h^2)$. Обычный прием исключения $c_{-1}-\bar{c}_{-1}$ и $c_{n+1}-\bar{c}_{n+1}$ из системы приведет к уравнениям с разностью преобладания главной диагонали

$$\frac{6\alpha_1 h}{\alpha_1 h-3\beta_1}\left(\frac{6p(x_0)}{h^2}-\frac{3q(x_0)}{h}+r(x_0)\right)+6|r(x_0)|,$$

$$\frac{6\alpha_2 h}{\alpha_2 h+3\beta_2}\left(\frac{6p(x_n)}{h^2}+\frac{3q(x_n)}{h}+r(x_n)\right)+6|r(x_n)|$$

и правыми частями

$$\frac{h^2}{12}p(x_0)u^{IV}(x_0)-\frac{\beta_1 h(u'(a)-\bar{u}'(a))}{\alpha_1 h-3\beta_1}\left(\frac{6p(x_0)}{h^2}-\frac{3q(x_0)}{h}+r(x_0)\right)+o(h^2),$$
$$\tag{4}$$

$$\frac{h^2}{12}p(x_n)u^{IV}(x_n)-\frac{\beta_2 h(u'(b)-\bar{u}'(b))}{\alpha_2 h+3\beta_2}\left(\frac{6p(x_n)}{h^2}+\frac{3q(x_n)}{h}+r(x_n)\right)+o(h^2)$$

соответственно. Если, например, $\alpha_1=0$, то разность преобладания в соответствующем уравнении есть $6|r(x_0)|$, если $\alpha_1\neq 0$, $\beta_1\neq 0$, то $\frac{1}{h}\left(12\frac{\alpha_1}{|\beta_1|}p(x_0)+O(h)\right)$, а если $\alpha_1\neq 0$, $\beta_1=0$, то даже $\frac{36}{h^2}p(x_0)+O(\frac{1}{h})$. Поскольку в выражениях (4) вторые слагаемые имеют порядок $o(h^2)$, то только при $\alpha_1=0$ в оценку входит компонента $\frac{h^2}{72}\frac{p(x_0)}{|r(x_0)|}u^{IV}(x_0)+o(h^2)$, в остальных случаях $o(h^2)$. Этим у нас доказана, что

$$\max_{0\le i\le n}|c_i-\bar{c}_i|\le\frac{h^2}{72}\max_{i\in I}\left|\frac{pu^{IV}}{r}(x_i)\right|+o(h^2),\tag{5}$$

где $I=\{1,\dots,n-1\}$, если $\alpha_i\neq 0$, $i=1,2$; $I=\{0,\dots,n\}$, если $\alpha_i=0$, $\beta_i\neq 0$, $i=1,2$, и, например, $I=\{0,\dots,n-1\}$, если $\alpha_1=0$, $\beta_1\neq 0$, $\alpha_2\neq 0$, $\beta_2\neq 0$. Теперь можно применять неравенство

$$\max_{x_1 \le x \le x_{n-1}} |\tilde{u}(x) - \bar{u}(x)| \le 6 \max_{0 \le i \le n} |c_i - \bar{c}_i|.$$

Поставим вопрос об оценке $\tilde{u}(x) - \bar{u}(x)$ на отрезке $[x_0, x_1]$. Пусть $x = a + th$, обозначим еще $\xi_i = c_i - \bar{c}_i$. Тогда разложение Тейлора $\tilde{u}(x) - \bar{u}(x)$ в точке a, а затем подстановка в полученное выражение ξ_{-1} из условия

$$\left(\alpha_1 - \frac{3}{h}\beta_1\right)\xi_{-1} + 4\alpha_1\xi_0 + \left(\alpha_1 + \frac{3}{h}\beta_1\right)\xi_1 = \beta_1(u'(a) - \bar{u}'(a))$$

приводит к равенству

$$\tilde{u}(x) - \bar{u}(x) = \xi_0\left(4 - 6t^2 + 3t^3 - \frac{4\alpha_1 h}{\alpha_1 h - 3\beta_1}(1-t)^3\right) +$$

$$+ \xi_1\left((1+t)^3 - 4t^3 - \frac{\alpha_1 h + 3\beta_1}{\alpha_1 h - 3\beta_1}(1-t)^3\right) +$$

$$+ \xi_2 t^3 + (1-t)^3 \frac{\beta_1 h}{\alpha_1 h - 3\beta_1}(u'(a) - \bar{u}'(a)).$$

Непосредственным анализом устанавливается, что коэффициенты при ξ_0, ξ_1 и ξ_2 неотрицательны, если $t \in [0, 1]$, а последнее слагаемое имеет порядок $o(h^4)$. Тогда при $x \in [x_0, x_1]$ имеем

$$|\tilde{u}(x) - \bar{u}(x)| \le 6 \max_{0 \le i \le 2} |\xi_i| \frac{\alpha_1 h(3t - 3t^2 + t^3) - 3\beta_1}{\alpha_1 h - 3\beta_1} + o(h^4).$$

Из этого, в свою очередь, получаем, что

$$\max_{x_0 \le x \le x_1} |\tilde{u}(x) - \bar{u}(x)| \le 6 \max_{0 \le i \le 2} |c_i - \bar{c}_i| + o(h^4). \qquad (6)$$

Из соображении симметрии ясно, что аналогичная оценка имеет место и на отрезке $[x_{n-1}, x_n]$. Этим у нас доказана, что

$$\|\tilde{u} - u\|_\infty < \frac{h^2}{12} \max_{i \in I} \left|\frac{pu^{IV}}{r}(x_i)\right| + o(h^2), \qquad (7)$$

где $I = \{0, \ldots, n\}$, а если $\alpha_i \ne 0$, то в соответсвующем конце из I можно упустить один индекс. В остаточном члене можно писать $O(h^{2+\alpha})$, если $u^{IV} \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$. Отметим, что случай $\beta_1 = \beta_2 = 0$ был нами изучен в [2].

2. Оценка погрешности метода подобластей. Рассмотрим краевую задачу (1) при тех же ограничениях на входные данные и решение. Кроме точек сетки $x_i = a + ih$ введем еще $y_i = (x_{i-1} + x_i)/2$, $i = 1, \ldots, n$. В методе подобластей приближенное решение $\tilde{u}(x)$ определим условиями

$$\int_{x_0}^{y_1}(L\tilde{u}-f)(x)dx=0,\ \int_{y_i}^{y_{i+1}}(L\tilde{u}-f)(x)dx=0, i=1,\dots,n-1,$$
$$\int_{y_n}^{x_n}(L\tilde{u}-f)(x)dx=0. \qquad (8)$$
$$\alpha_1\tilde{u}(a)+\beta_1\tilde{u}'(a)=\gamma_1,\ \alpha_2\tilde{u}(b)+\beta_2\tilde{u}'(b)=\gamma_2.$$

Для определения коэффициентов $\tilde{u}(x)=\sum_{i=-1}^{n+1}c_iB_i(x)$ получаем систему

$$\begin{cases}(\alpha_1-\frac{3}{h}\beta_1)c_{-1}+4\alpha_1c_0+(\alpha_1+\frac{3}{h}\beta_1)c_1=\gamma_1,\\ A_ic_{i-2}+B_ic_{i-1}+C_ic_i+D_ic_{i+1}+E_ic_{i+2}=\\ \qquad =\frac{1}{h}\int_{y_i}^{y_{i+1}}f(x)dx,\ i=0,\dots,n,\\ (\alpha_2-\frac{3}{h}\beta_2)c_{n-1}+4\alpha_2c_n+(\alpha_2+\frac{3}{h}\beta_2)c_{n+1}=\gamma_2,\end{cases} \qquad (9)$$

где условно считаем, что $A_0=E_n=0$, $y_0=x_0$, $y_{n+1}=x_n$, а другие коэффициенты A_i, B_i, C_i, D_i, E_i в разложенном виде приведены в [2], из-за громоздкости мы не будем их воспроизводить здесь. Отметим только, что после обычного исключения c_{-1} и c_{n+1} из системы (9) при малых h главная диагональ будет преобладать, значит, система (9) однозначно разрешима.

Построим решению $u(x)$ задачи (1) интерполянту $\bar{u}(x)$ как в пункте 1. Тогда коэффициенты $\xi_i=c_i-\bar{c}_i$ в разложении $\tilde{u}-\bar{u}$ определяются из системы

$$\begin{cases}(\alpha_1-\frac{3}{h}\beta_1)\xi_{-1}+4\alpha_1\xi_0+(\alpha_1+\frac{3}{h}\beta_1)\xi_1=\beta_1(u'(a)-\bar{u}'(a)),\\ B_0\xi_{-1}+C_0\xi_0+D_0\xi_1+E_0\xi_2=\frac{1}{h}\int_{x_0}^{y_1}L(u-\bar{u})(x)dx,\\ A_i\xi_{i-2}+B_i\xi_{i-1}+C_i\xi_i+D_i\xi_{i+1}+E_i\xi_{i+2}=\\ \qquad =\frac{1}{h}\int_{y_i}^{y_{i+1}}L(u-\bar{u})(x)dx,\ i=1,\dots,n-1,\\ A_n\xi_{n-2}+B_n\xi_{n-1}+C_n\xi_n+D_n\xi_{n+1}=\frac{1}{h}\int_{y_n}^{x_n}L(u-\bar{u})(x)dx,\\ (\alpha_2-\frac{3}{h}\beta_2)\xi_{n-1}+4\alpha_2\xi_n+(\alpha_2+\frac{3}{h}\beta_2)\xi_{n+1}=\beta_2(u'(b)-\bar{u}'(b)).\end{cases} \qquad (10)$$

Точно так, как в [2], показываем, что

$$\frac{1}{h}\int_{y_i}^{y_{i+1}}L(\tilde{u}-\bar{u})(x)dx=o(h^2),\ i=0,\dots,n,$$

где $o(h^2)$ можно заменить на $O(h^{2+\alpha})$, если $u^{IV} \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$. В системе (10) во внутренних (при $i = 1,\dots,n-1$) уравнениях главная диагональ преобладает с разностью $6\,|r(x_i)| + O(h)$, значит, вклад этих уравнений в оценку решения составляет $o(h^2)$ или $O(h^{2+\alpha})$. Если обычным способом исключить из системы (10) ξ_{-1} и ξ_{n+1}, то, например, в первом уравнении получаемой системы разность преобладания главной диагонали будет равняться

$$\frac{1}{\alpha_1 h - 3\beta_1}\left(-3\beta_1(3\,|r(x_0)| + O(h)) + \alpha_1 h\left(\frac{27}{2}\,\frac{p(x_0)}{h^2} + O\left(\frac{1}{h}\right)\right)\right),$$

а правая часть принимает вид

$$\frac{1}{h}\int_{x_0}^{y_1} L(\tilde{u} - \bar{u})(x)\,dx - \frac{\beta_1 h(u'(a) - \bar{u}'(a))}{\alpha_1 h - 3\beta_1}\left(\frac{9}{4}\,\frac{p(x_0)}{h^2} + O\left(\frac{1}{h}\right)\right)$$

и имеет порядок $o(h^2)$ или $O(h^{2+\alpha})$. Значит, и это уравнение не ухудшает оценку $\max\limits_{0 \le i \le n} |c_i - \bar{c}_i|$ по сравнению с внутренними уравнениями. Во втором уравнении разность преобладания главной диагонали будет

$$\frac{1}{\alpha_1 h - 3\beta_1}\left(-3\beta_1(6\,|r(x_1)| + O(h)) + \alpha_1 h\left(\frac{9}{2}\,\frac{p(x_1)}{h^2} + O\left(\frac{1}{h}\right)\right)\right),$$

а правая часть равняется

$$\frac{1}{h}\int_{y_1}^{y_2} L(u - \bar{u})(x)\,dx - \frac{\beta_1 h(u'(a) - \bar{u}'(a))}{\alpha_1 h - 3\beta_1}\left(\frac{3}{4}\,\frac{p(x_1)}{h^2} + O\left(\frac{1}{h}\right)\right)$$

и имеет также порядок $o(h^2)$ или $O(h^{2+\alpha})$, следовательно, и отсюда не получаем ухудшения оценки. Из условия

$$\left(\alpha_1 - \frac{3}{h}\beta_1\right)(c_{-1} - \bar{c}_{-1}) + 4\alpha_1(c_0 - \bar{c}_0) + \\ + \left(\alpha_1 + \frac{3}{h}\beta_1\right)(c_1 - \bar{c}_1) = \beta_1(u'(a) - \bar{u}'(a))$$

получаем, что

$$|c_{-1} - \bar{c}_{-1}| \le 5 \max_{i=0,1} |c_i - \bar{c}_i| + o(h^3), \tag{11}$$

где в остаточном члене можно писать $O(h^{3+\alpha})$, если $u^{IV} \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$. По симметрии ясно, что подобные оценки имеют место и в другом конце системы, значит, нами установлена оценка $\max\limits_{-1 \le i \le n+1} |c_i - \bar{c}_i| = o(h^2)$, следовательно, погрешность метода подобластей $\|\tilde{u} - u\|_\infty$ оценивается как $o(h^2)$ или $O(h^{2+\alpha})$, если $u^{IV} \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$. Этим мы обобщили результат работы [2], где рассматривают случай краевых условий $\beta_1 = \beta_2 = 0$.

3. Метод подобластей при более гладком решении. В [3] показано, что если решение задачи (1) при $\beta_1 = \beta_2 = 0$ имеет бóльшую гладкость, то метод подобластей кубическими сплайнами (8) имеет порядок $O(h^4)$. Поставим вопрос о распространении такой оценки на более общие краевые условия.

Предположим, что задача (1) имеет решение $u \in C^6[a, b]$. Интерполянту $\bar{u}(x)$ решению $u(x)$ построим по краевым условиям

$$\bar{u}''(a) = u''(a) - \frac{1}{12} h^2 u^{IV}(a) + \frac{1}{360} h^4 u^{VI}(a),$$

$$\bar{u}''(b) = u''(b) - \frac{1}{12} h^2 u^{IV}(b) + \frac{1}{360} h^4 u^{VI}(b).$$

В [3] мы показали, что

$$\frac{1}{h} \int_{y_i}^{y_{i+1}} L(u - \bar{u})(x)\,dx = O(h^4), \quad i = 1, \dots, n-1.$$

Опираясь на равенства $\bar{u}(x_i) = u(x_i)$, $i = 0, \dots, n$, а также на оценки (см. [1] и ср. с [3])

$$\bar{u}'(x_i) = u'(x_i) - \frac{1}{180} h^4 u^{V}(x_i) + O(h^5), \quad i = 0, \dots, n,$$

можно показать, что

$$\frac{1}{h} \int_{x_0}^{y_1} L(u - \bar{u})(x)\,dx = O(h^3), \quad \frac{1}{h} \int_{y_n}^{x_n} L(u - \bar{u})(x)\,dx = O(h^3).$$

Дальнейшее изучение проведем раздельно в различных случаях.

1) Пусть сначала $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. Исключаем из системы (10) ξ_{-1} и ξ_{n+1} простой заменой их из первой и последней уравнений в другие. В получаемой системе в первых двух уравнениях разности преобладания главной диагонали будут в случае $\beta_1 \neq 0$ соответственно $\frac{\alpha_1}{-3\beta_1} \cdot \frac{27}{2} \frac{p(x_0)}{h} + O(1)$ и $\frac{\alpha_1}{-3\beta_1} \cdot \frac{9}{2} \frac{p(x_1)}{h} + O(1)$ (в случае $\beta_1 = 0$ даже $\frac{21}{2} \frac{p(x_0)}{h^2} + O(\frac{1}{h})$ и $\frac{9}{4} \frac{p(x_1)}{h^2} + O(\frac{1}{h})$ соответственно), а правые части имеют порядок $O(h^3)$. Внутренние уравнения нами просмотрены в [3] и этим установлена оценка $\max\limits_{0 \le i \le n} |c_i - \bar{c}_i| = O(h^4)$. Применяя еще неравенство (11) или опираясь на оценку (6), можно считать, что в данном случае доказана оценка $\|\tilde{u} - u\|_\infty = O(h^4)$.

2) Рассмотрим случай $\alpha_1 = \alpha_2 = 0$, $\beta_1 \neq 0$, $\beta_2 \neq 0$. В системе (10) умножим все внутренние уравнения на число -1, тогда в полученных уравнениях при малых h на главной диагонали стоят положительные элементы, а все недиагональные ненулевые элементы отрицательны. Крайние

уравнения запишем в виде

$$\xi_{-1} - \xi_1 = -\frac{h}{3}(u'(a) - \bar{u}'(a)),$$

$$-\xi_{n-1} + \xi_{n+1} = \frac{h}{3}(u'(b) - \bar{u}'(b)).$$

Ищем для решения этой системы мажоранту в виде $\eta_i = Kh^4 - K_1 h^4 (x_n - x_i)(x_i - x_0)$, $i = -1, \ldots, n+1$, $K, K_1 > 0$. Во внутренних уравнениях (при $i = 1, \ldots, n-1$) имеем

$$A_i \eta_{i-2} + B_i \eta_{i-1} + C_i \eta_i + D_i \eta_{i+1} + E_i \eta_{i+2} =$$

$$= (A_i + B_i + C_i + D_i + E_i) K h^4 -$$

$$- (A_i + B_i + C_i + D_i + E_i) K_1 h^4 (x_n - x_i)(x_i - x_0) -$$

$$- K_1 h^5 (b-a)(-2A_i - B_i + D_i + 2E_i) -$$

$$- K_1 h^6 (4A_i + B_i + D_i + 4E_i) > \left| \frac{1}{h} \int_{y_i}^{y_{i+1}} L(u - \bar{u})(x)\,dx \right|,$$

если учитывать, что $A_i + B_i + C_i + D_i + E_i = 6\,|r(x_i)| + O(h)$, $-2A_i - B_i + D_i + 2E_i = O\left(\frac{1}{h}\right)$, $4A_i + B_i + D_i + 4E_i = O\left(\frac{1}{h^2}\right)$, $(x_n - x_i)(x_i - x_0) \le \frac{1}{4}(b-a)^2$ и K взять достаточно большое. Кроме того, например,

$$B_0 \eta_{-1} + C_0 \eta_0 + D_0 \eta_1 + E_0 \eta_2 =$$

$$= Kh^4 (6\,|r(x_0)| + O(h)) + K_1 (b-a) h^3 \left(\frac{15}{4} p(x_0) + O(h) \right) \ge$$

$$\ge \left| \frac{1}{h} \int_{x_0}^{y_1} L(u - \bar{u})(x)\,dx \right|,$$

если сначала выбирать подходящее число $K_1 > 0$, а затем нужное $K > 0$. Еще получаем, что

$$\eta_{-1} - \eta_1 = \eta_{n+1} - \eta_{n-1} = 2K_1 (b-a) h^5 \ge$$

$$\ge \frac{h}{3} \max\{ |u'(a) - \bar{u}'(a)|,\ |u'(b) - \bar{u}'(b)| \}$$

для некоторого $K_1 > 0$. Таким образом, $|\xi_i| \le |\eta_i| \le (K + K_1 (b-a)^2) h^4$, $i = -1, \ldots, n+1$, и нами установлено, что и в этом случае метод подобластей (8) допускает оценку погрешности $\|\tilde{u} - u\|_\infty = O(h^4)$.

3) Пусть еще, например, $\alpha_1 = 0$, $\beta_1 \ne 0$, $\alpha_2 \ne 0$, $\beta_2 \ne 0$. Тогда систе-

му (10) разложим на две следующие:

$$\begin{cases} \xi_{-1} - \xi_1 = 0, \\ A_i\xi_{i-2} + B_i\xi_{i-1} + C_i\xi_i + D_i\xi_{i+1} + E_i\xi_{i+2} = 0, \\ \left(\alpha_2 - \frac{3}{h}\beta_2\right)\xi_{n-1} + 4\alpha_2\xi_n + \left(\alpha_2 + \frac{3}{h}\beta_2\right)\xi_{n+1} = \beta_2(u'(b) - \bar{u}'(b)), \end{cases}$$

$$\begin{cases} \xi_{-1} - \xi_1 = -\frac{h}{3}(u'(a) - \bar{u}'(a)), \\ A_i\xi_{i-2} + B_i\xi_{i-1} + C_i\xi_i + D_i\xi_{i+1} + E_i\xi_{i+2} = \\ \qquad = \frac{1}{h}\int_{y_i}^{y_{i+1}} L(u - \bar{u})(x)dx, \quad i = 0,\dots, n, \\ \left(\alpha_2 - \frac{3}{h}\beta_2\right)\xi_{n-1} + 4\alpha_2\xi_n + \left(\alpha_2 + \frac{3}{h}\beta_2\right)\xi_{n+1} = 0. \end{cases}$$

В первой из них исключаем ξ_1 и ξ_{n+1}, а решение полученной системы будем оценивать как в случае 1) на основе преобладания главной диагонали. Во второй системе исключаем ξ_{n+1}, умножим все уравнения (кроме первого) полученной системы на -1, а затем построим ее решению мажоранту как в случае 2). В итоге получаем и здесь оценку погрешности метода подобластей $\|\tilde{u} - u\|_\infty = O(h^4)$.

Замечание. Если $u^{V} \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$, то при подходящем выборе краевых условий для интерполянты $\bar{u}$ можно установить оценки

$$\frac{1}{h}\int_{y_i}^{y_{i+1}} L(u - \bar{u})(x)dx = O(h^{3+\alpha}), \quad i = 1,\dots, n-1,$$

причем оценки

$$\frac{1}{h}\int_{x_0}^{y_1} L(u - \bar{u})(x)dx = O(h^3), \quad \frac{1}{h}\int_{y_1}^{x_n} L(u - \bar{u})(x)dx = O(h^3)$$

имеют место уже при $u^{V} \in C\,[a, b]$; ясно, что ключевым моментом здесь является установление оценок разностей $u - \bar{u}$ и $u' - \bar{u}'$ в точках x_0, y_i, $i = 1,\dots, n$, x_n. Тогда решение системы (10) оценивается через $O(h^{3+\alpha})$, значит, погрешность метода подобластей оценивается как $\|\tilde{u} - u\|_\infty = O(h^{3+\alpha})$.

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ON THE SOLUTION OF BOUNDARY VALUE PROBLEMS WITH CUBIC SPLINES

Peeter Oja, Ülle Pettai

Summary

We consider the numerical solution of boundary value problem

$$p(x)u''(x)+q(x)u'(x)+r(x)u(x)=f(x),\ x\in(a,b),$$
$$\alpha_1 u(a)+\beta_1 u'(a)=\gamma_1,\ \alpha_2 u(b)+\beta_2 u'(b)=\gamma_2$$

using the collocation and subregions methods with cubic splines on a uniform partition of the interval $[a, b]$ with step $h=(b-a)/n$. We assume the functions p, q, r and f are smooth enough, $p(x)\ge p_0>0$, $r(x)\le r_0<0$, $\alpha_1, \alpha_2\ge 0$, $\beta_1\le 0$, $\beta_2\ge 0$, $|\alpha_i|+|\beta_i|\ne 0$, $i=1,2$, so that we have a unique solution $u\in C^4[a,b]$ of the boundary value problem. We show the dependence (7) on p, r, u and α_i, β_i of the main terms of the bounds of error for the collocation method. The rate of convergence for the subregions method is from $o(h^2)$ to $O(h^4)$ corresponding to the initial data and solution of the boundary value problem.

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THE CONVERGENCE RATE OF APPROXIMATE EIGENVALUES IN PROJECTION-LIKE METHODS

Otto Karma

The approximation of the eigenvalue problem $A(\lambda)u=0$ with holomorphic Fredholm operator-function $A(\cdot)$ is studied. A discrete approximation scheme for spaces and a regular approximation scheme for operator-functions are used. We approximate $A(\cdot)$ by the operator-functions $B_n(\cdot)=q_nA(\cdot)r_n$ (where q_n are "projectors" and r_n are "embedding" operators). The convergence rate of approximate eigenvalues to the exact one λ_0 is estimated by fractional powers of ε_n - the product of approximation errors of the unit spheres of the generalized eigenspaces of $A(\lambda_0)$ and $A^*(\lambda_0)$. Three estimations are given - estimations (2.1) and (2.2) for any eigenvalue of approximating problem individually and estimation (2.3) for the weighted arithmetic mean of eigenvalues converging to λ_0. The constructions to prove these estimations follow the corresponding ones in [13,15,3].

For linear $A(\cdot)$ estimation (2.1) is proved in [5] and (2.3) in [1] . For holomorphic $A(\cdot)$ the case of compact approximation in subspaces is discussed in [13].The estimation of the convergence rate in a frame of a regular approximation is discussed in [2,3,6,8,9,15]. (See survey in [7] on this topic, too.)

1. General Assumptions and Basic Notions

1.1 (The exact problem). Let U,V be complex Banach spaces and $A(\cdot)$ be an operator-function from a region (open connected set) $\Lambda\in\mathbb{C}$ to $\mathcal{B}(U,V)$, Banach space of bounded linear operators . Consider a generalized eigenvalue problem

$$A(\lambda)u=0,\ u\neq 0\ . \tag{1.1}$$

The solutions λ_0, u_0 of (1.1) are called <u>eigenvalues</u> and <u>eigenelements</u> of $A(\cdot)$, correspondingly. We assume that:

a1) $A(\cdot):\Lambda\to\mathcal{B}(U,V)$ is a holomorphic function of λ on Λ ;

a2) for every λ in Λ , operator $A(\lambda)\in\mathcal{B}(U,V)$ is a Fredholm operator;

a3) the <u>resolvent set</u> $\varrho(A)=\{\ \lambda\in\Lambda:\ \exists A(\lambda)^{-1}\in\mathcal{B}(V,U)\ \}$ is not empty.

Denote by U^*, V^* the conjugate spaces $\mathcal{B}(U,\mathbb{C})$, $\mathcal{B}(V,\mathbb{C})$ to U,V, correspondingly, and by $A^*\in\mathcal{B}(V^*,U^*)$ the conjugate operator to $A\in\mathcal{B}(U,V)$. From a1) a3) it follows (see, e.g., [11,10]) that: 1) the operator-function $A^*(\cdot):\Lambda\to\mathcal{B}(V^*,U^*)$, defined on Λ by the equality $A^*(\lambda)=[A(\lambda)]^*$, also satisfies all the assumptions a1)-a3); 2) $\varrho(A^*)=\varrho(A)$; 3) the *spectrum* $\sigma(A)=\Lambda\setminus\varrho(A)$ of $A(\cdot)$ does not have any cluster point in Λ and consists of eigenvalues of $A(\cdot)$ only; 4) the operator-function $A^{-1}(\cdot)$, defined on $\varrho(A)$ by the equality $A^{-1}(\lambda)=A(\lambda)^{-1}$ is holomorphic on $\varrho(A)$ and has in Λ poles $\lambda_0\in\sigma(A)=\sigma(A^*)$ of finite order $\varkappa(\lambda_0,A)=\varkappa(\lambda_0,A^*)$ only.

1.2 (The approximation of the spaces). Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two sequences of complex Banach spaces and let $\{p_n: U\to X_n\}_{n\in\mathbb{N}}$, $\{q_n\in\mathcal{B}(V,Y_n)\}_{n\in\mathbb{N}}$ be two sequences of connecting operators so that:

$$\|p_n u\|_{X_n}\to\|u\|_U \quad (n\in\mathbb{N}) \quad \forall u\in U; \qquad \|q_n v\|_{Y_n}\to\|v\|_V \quad (n\in\mathbb{N}) \quad \forall v\in V;$$

$$\|p_n(\alpha u+\alpha' u')-(\alpha p_n u+\alpha' p_n u')\|_{X_n}\to 0 \quad (n\in\mathbb{N}) \quad \forall u,u'\in U,\ \alpha,\alpha'\in\mathbb{C}.$$

We shall denote infinite subsequences of $\mathbb{N}$ by $\mathbb{N}'$, $\mathbb{N}''$, ... By the definition, a sequence $\{x_n\}_{n\in\mathbb{N}'}$ of elements x_n with $x_n\in X_n$ is:

a) *converging* to $u\in U$ if $\|p_n u-x_n\|\to 0$ $(n\in\mathbb{N}')$; we write $x_n\to u$ $(n\in\mathbb{N}')$;

b) *compact* if, for every subsequence $\{x_n\}_{n\in\mathbb{N}''}$ with $\mathbb{N}''\subseteq\mathbb{N}'$, there exist $\mathbb{N}'''\subseteq\mathbb{N}''$ and $u\in U$ so that $x_n\to u$ $(n\in\mathbb{N}''')$.

1.3 (The approximation of the eigenvalue problem). Let there be given the operators $r_n\in\mathcal{B}(X_n,U)$ with $\|r_n\|\le C$, $n\in\mathbb{N}$. Let us approximate problem (1.1) by a sequence of problems

$$B_n(\lambda)x_n := [q_n A(\lambda) r_n]x_n=0, \quad x_n\neq 0 \quad (n\in\mathbb{N}). \tag{1.2}$$

Note that here:

b0) on every compact $\Lambda_0\subset\Lambda$ the norms $\|B_n(\lambda)\|$ are uniformly bounded:

$$\|B_n(\lambda)\|\le c(\Lambda_0)<\infty, \ \forall n\in\mathbb{N},\ \lambda\in\Lambda_0;$$

b1) all $B_n(\cdot)$, $n\in\mathbb{N}$ are holomorphic operator-functions from the region $\Lambda\subseteq\mathbb{C}$ to $\mathcal{B}(X_n,Y_n)$.

We assume that:

b2) for $\forall\, n\in\mathbb{N}$, $\lambda\in\Lambda$, the operator $B_n(\lambda)$ is a Fredholm operator with index 0;

b3) for $\forall\, n\in\mathbb{N}$, $\lambda\in\Lambda$, the sequence $\{B_n(\lambda)\}_{n\in\mathbb{N}}$ *converges regularly* to $A(\lambda)$, i.e.:

b3.1) the sequence of operators $\{B_n(\lambda)\}_{n\in\mathbb{N}}$ *converges* to $A(\lambda)$:

$$x_n\to u \ (n\in\mathbb{N}) \ \Rightarrow \ B_n(\lambda)x_n\to A(\lambda)u \ (n\in\mathbb{N}),$$

b3.2) the sequence of operators $\{B_n(\lambda)\}_{n\in\mathbb{N}}$ is *regular*:

$$\|x_n\|\le c,\ \{B_n(\lambda)x_n\}_{n\in\mathbb{N}} \text{ is compact } \Rightarrow \ \{x_n\}_{n\in\mathbb{N}} \text{ is compact.}$$

These conditions are fulfilled in a special case of projection methods for an operator-function $A(\lambda)=A_0+K(\lambda)$ with A_0 Fredholm operator with index 0 and $K(\lambda)$ compact for every λ and holomorphic on Λ, $X_n\subset U$, $Y_n\subset V$, $r_nx_n=x_n$, whereby p_n, q_n are linear projectors which converge strongly to unity and such that

$$\underline{\lim}\ \inf\{\|q_nA_0x_n\|/\|A_0x_n\| \mid x_n\in X_n\}>0\ (n\in\mathbb{N}).$$

2. The Convergence Rate of Eigenvalues

2.1 (The convergence of eigenvalues). From a1)-b3) it follows that:

1) for every compact $\Lambda_0\subset\varrho(A)$, there exist an index $n(\Lambda_0)$ and a constant $c(\Lambda_0)$ so that
$$\Lambda_0\subset\varrho(B_n),\ \|B_n(\lambda)^{-1}\|\le c(\Lambda_0),\ \forall\lambda\in\Lambda_0,\ n\ge n(\Lambda_0);$$
2) $\lambda_0\in\sigma(A)\Rightarrow\exists n_0=n(\lambda_0),\ \{\lambda_n\}_{n\in\mathbb{N}}:\ \lambda_n\in\sigma(B_n),\ \lambda_n\to\lambda_0\ (n\in\mathbb{N},\ n\ge n_0);$

3) $\lambda_n\in\sigma(B_n),\ \lambda_n\to\lambda_0\in\Lambda\ (n\in N')\ \Rightarrow\ \lambda_0\in\sigma(A).$

2.2 (A generalized eigenspace). Let a1)-a3) hold. Denote by $A^{(j)}(\lambda_0)$, $j=1,2,\ldots$ the j-th derivative of $A(\cdot)$ at λ_0 and by A_j the operator $\frac{1}{j!}A^{(j)}(\lambda_0)$, $j=1,2,\ldots$. Vector $(u^0,u^1,\ldots,u^k)$ with $u^0\neq 0$ is called a chain of generalized eigenelements, or a Jordan chain, of the length $(k+1)$ for $A(\cdot)$ at λ_0, if

$$A_0u^0=0,\ A_0u^1+A_1u^0=0,\ \ldots,\ A_0u^k+A_1u^{k-1}+\ldots+A_ku^0=0.$$

A generalized eigenspace $\mathcal{J}(A,\lambda_0)$ of $A(\cdot)$ at λ_0 is a closed linear hull of all generalized eigenelements of $A(\cdot)$ at λ_0. The order $\nu(u^0)$ of the eigenelement u^0 is the maximal length of the Jordan chains beginning with u^0. From a1)-a3) it follows (see, e.g.,[10,17]) that $\max\{\nu(u)\mid u\in\mathcal{N}(A(\lambda_0)),\ u\neq 0\}=\varkappa(\lambda_0,A)$ and that $\dim\mathcal{J}(A.\lambda_0)<\infty$.

Let $\dim\mathcal{N}(A(\lambda_0))=m$. A basis $(u_1^0,\ldots,u_m^0)$ of $\mathcal{N}(A(\lambda_0))$ is called a canonical system of eigenelements for $A(\cdot)$ at $\lambda_0\in\sigma(A)$ if

$$\nu(u_i^0)=\max\{\nu(u)\mid u\in\mathcal{N}(A(\lambda_0))\setminus\mathcal{L}(u_1^0,\ldots,u_{i-1}^0)\},\ i=1,\ldots,m,$$

with $\mathcal{L}(u_1^0,\ldots,u_{i-1}^0)$ the linear hull of the elements $u_1^0,\ldots,u_{i-1}^0$.

It occurs that the vector $(\nu_1,\ldots,\nu_m)$ with $\nu_i=\nu(u_i^0)$, $i=1,\ldots,m$ does not depend on the concrete choice of a canonical system of eigen elements. The number $\nu(\lambda_0,A)=\nu_1+\nu_2+\ldots+\nu_m$ is called the algebraic multiplicity of the eigenvalue λ_0 of the operator-function $A(\cdot)$.

From a1)-b3) we get (see, e.g., [4 ,31]) the following

Theorem 2.2. For every compact $\Lambda_0\subset\Lambda$ with boundary $\Gamma\subset\varrho(A)$, there exists an index $n(\Lambda_0)$ so that for all $n\ge n(\Lambda_0)$:

$$\Gamma\subset\varrho(B_n),\quad \sum_{\lambda\in\sigma(A)\cap\Lambda_0}\nu(\lambda,A)=\sum_{\lambda\in\sigma(B_n)\cap\Lambda_0}\nu(\lambda,B_n)\quad (\text{here }\sum_{\lambda\in\emptyset}\ldots=0).$$

2.3 (The asymptotic estimation of the convergence rate). For $u \in U$ and $W \subset U$, by definition, $d(u, W) = \inf\{\|u-w\| \mid w \in W\}$. We define

$$d_n = \max\{d(u, r_n X_n) \mid u \in \mathcal{J}(A, \lambda_0), \|u\|=1\},$$
$$d_n^* = \max\{d(h, q_n^* Y_n^*) \mid h \in \mathcal{J}(A^*, \lambda_0), \|f\|=1\}.$$

Theorem 2.3. Let a1) - b3) hold, $\lambda_0 \in \sigma(A)$, and $\Lambda_0 \subset \Lambda$ be a compact with boundary $\Gamma \subset \varrho(A)$ so that $\Lambda_0 \cap \sigma(A) = \{\lambda_0\}$. Let $r_n \in \mathcal{B}(X_n, U)$, $n \in \mathbb{N}$ be such that $\langle r_n p_n u, f\rangle = f(r_n p_n u) \to \langle u, f\rangle$ $(n \in \mathbb{N})$ $\forall u \in \mathcal{N}(A(\lambda_0))$, $f \in U^*$. Let $d_n d_n^* \to 0$ $(n \in \mathbb{N})$. Then, for almost all $n \in \mathbb{N}$, the following estimations hold:

$$1)\ |\lambda_n - \lambda_0| \le c\,\varepsilon_n^{1/\varkappa(\lambda_0, A)}, \qquad \forall \lambda_n \in \sigma(B_n) \cap \Lambda_0; \tag{2.1}$$

$$2)\ |\lambda_n - \lambda_0| \le c\,\varepsilon_n^{1/k_n}, \qquad \forall \lambda_n \in \sigma(B_n) \cap \Lambda_0, \tag{2.2}$$

with k_n equal to a number of different $\lambda_n \in \sigma(B_n)$ in Λ_0;

$$3)\ |\bar{\lambda}_n - \lambda_0| \le c\,\varepsilon_n, \tag{2.3}$$

with $\bar{\lambda}_n = \sum_{\lambda \in \sigma(B_n) \cap \Lambda_0} \mu_\lambda \cdot \lambda$, $\mu_\lambda = \nu(\lambda, B_n)/\nu(\lambda_0, A)$.

Here ε_n are defined in (3.6) and can be estimated as follows:

$$\varepsilon_n \le c\, d_n d_n^* . \tag{2.4}$$

Proof. It follows from Section 2.1 that it is sufficient to consider a compact Λ_0 in some quite small neighborhood of the eigenvalue λ_0. For such Λ_0 estimation (2.4) will be established in Corollary 3.6 and estimations (2.1)-(2.3) in Section 4.2.∎

3. Reducing the Problem to the Case of Matrix-Functions

3.1. Let $\lambda_0 \in \sigma(A)$, $\dim \mathcal{N}(A(\lambda_0)) = m$, $(u_1^0, \dots, u_m^0)$ be a canonical system of eigenelements for $A(\cdot)$ at λ_0 and $(u_i^0, \dots, u_i^{k_i})$, $i=1,\dots,m$ be Jordan chains of the length $k_i + 1 = \nu(u_i^0)$. A *canonical system of Jordan polynomials* for $A(\cdot)$ at λ_0 is a system of polynomials $u_i(\cdot)$, $i=1,\dots,m$ with $u_i(\lambda) = u_i^0 + (\lambda-\lambda_0)u_i^1 + \dots + (\lambda-\lambda_0)^{k_i} u_i^{k_i}$. Note that for such polynomials $u_i(\lambda) \in \mathcal{J}(A, \lambda_0)$, $\forall \lambda \in \Lambda$.

3.2 (Some auxiliary notions). Let $\lambda_0 \in \sigma(A)$ and let:

1°) $(u_1(\cdot), u_2(\cdot), \dots, u_m(\cdot))$ be a canonical system of Jordan polynomials for $A(\cdot)$ at λ_0; we denote by u_i^0 the element $u_i(\lambda_0)$, $i=1,\dots,m$;

2°) $(h_1(\cdot), h_2(\cdot), \dots, h_m(\cdot))$ be a canonical system of Jordan polynomials for $A^*(\cdot)$ at λ_0; we denote by h_i^0 the element $h_i(\lambda_0)$, $i=1,\dots,m$;

3°) $v_i(\lambda) = (\lambda-\lambda_0)^{-\nu(u_i^0)} A(\lambda) u_i(\lambda)$, $i=1,\dots,m$; at that (see, e.g., [3], Lemma 3.2) the elements $v_i^0 = v_i(\lambda_0)$, $i=1,\dots,m$ are linearly independent and V is direct sum of $A(\lambda_0)U$ and $\mathcal{L}(v_1^0, \dots, v_m^0)$;

4°) $f_i^*(\lambda)=(\lambda-\lambda_0)^{-v(h_i^0)}A^*(\lambda)h_i(\lambda)$, $i=1,\dots,m$; at that, analogically to the $v_i(\lambda_0)$ in 3°), the elements $f_i^*(\lambda_0)$, $i=1,\dots,m$ are linearly independent and U^* is direct sum of $A^*(\lambda_0)V^*$ and $\mathcal{L}(f_1^*(\lambda_0),\dots,f_m^*(\lambda_0))$;

5°) $\{f_k(\cdot), k=1,\dots,m\}$ be a system of linear combinations of functions $f_i^*(\cdot)$, $i=1,\dots,m$ so that for $f_k^0=f_k(\lambda_0)$ it holds:

$$\langle u_i^0, f_k^0\rangle = \delta_{ik}, \quad i,k=1,\dots,m . \tag{3.1}$$

(Since $u_1^0,\dots,u_m^0$ are linearly independent and $\langle u_i^0,f\rangle=0$, $\forall f\in A^*(\lambda_0)V^*$, we can find linear combinations $f_k^0=\sum_{j=1}^{j=m}\alpha_{kj}f_j^*(\lambda_0)$ so that $\langle u_i^0,f_k^0\rangle=\delta_{ik}$, $i,k=1,\dots,m$. We denote by $f_k(\cdot)$ the sum $\sum_{j=1}^{j=m}\alpha_{kj}f_j^*(\cdot)$.)

From (3.1) it follows that the elements f_k^0, $k=1,\dots,m$ are linearly independent and so U^* is direct sum of $A^*(\lambda_0)V^*$ and $\mathcal{L}(f_1^0,\dots,f_m^0)$, too.

3.3 (The auxiliary operator-functions). Let $\langle r_n p_n u, f\rangle \to \langle u, f\rangle$, $\forall u\in\mathcal{N}(A(\lambda_0))$, $f\in U^*$ (as in Theorem 2.3). We define operator-functions $K(\cdot)$, $R(\cdot): \Lambda\to\mathcal{B}(U,V)$ and $L_n(\cdot)$, $S_n(\cdot): \Lambda\to\mathcal{B}(X_n,Y_n)$, $n\in\mathbb{N}$ as follows :

$$K(\lambda)u=\sum_{i=1}^{i=m}\langle u, f_i(\lambda)\rangle v_i(\lambda), \quad R(\lambda)=A(\lambda)+K(\lambda), \tag{3.3}$$

$$L_n(\lambda)x_n=\sum_{i=1}^{i=m}\langle x_n, g_n^i(\lambda)\rangle q_n v_i(\lambda), \quad S_n(\lambda)=B_n(\lambda)+L_n(\lambda), \tag{3.4}$$

with $g_n^i(\cdot)=r_n^*f_i(\cdot)$. At that :

1) the elements $g_n^i=g_n^i(\lambda_0)=r_n^*f_i(\lambda_0)$ and the functions $g_n^i(\cdot)=r_n^*f_i(\cdot)$, $i=1,\dots,m$ satisfy all the requirements to the g_n^i and $g_n(\cdot)$ in [3];
2) $L_n(\lambda)=\sum_{i=1}^{i=m}\langle r_n\cdot, f_i(\lambda)\rangle q_n v_i(\lambda)=q_n K(\lambda)r_n$, $S_n(\lambda)=q_n R(\lambda)r_n$;
3) all these operator functions are holomorphic on Λ and uniformly bounded on every compact $\Lambda_0\subset\Lambda$ (and, hence, equicontinuous on every compact $\Lambda_0\subset\Lambda$);
4) for every $\lambda\in\Lambda$, $n\in\mathbb{N}$ the operators $K(\lambda)$ and $L_n(\lambda)$ are finite-dimensional and thus the operators $R(\lambda)$ and $S_n(\lambda)$ are Fredholm operators with index 0.

Moreover, the following lemma holds :

Lemma 3.3. In some neighborhood $\Lambda_1\subset\Lambda$ of λ_0, for almost all indices n, say for $n\in\mathbb{N}_1$, all the operators $R(\lambda)$, $S_n(\lambda)$ are continuously invertible and the inverses $R(\lambda)^{-1}$, $S_n(\lambda)^{-1}$ are uniformly bounded:

$$\sup\{\|R(\lambda)^{-1}\|, \|S_n(\lambda)^{-1}\| \mid \lambda\in\Lambda_1, n\in\mathbb{N}_1\} \le c .$$

Proof. a) The operator $R(\lambda_0)$ is continuously invertible by the construction. (Since $R(\lambda_0)$ is Fredholm operator with index 0, it is sufficient to verify that $\mathcal{N}(R(\lambda_0))=\{0\}$. Let $R(\lambda_0)u_0=0$. Then $A(\lambda_0)u_0=0$ and $K(\lambda_0)u_0=0$, and $\langle u_0, f_i^0\rangle=0$, $i=1,\dots,m$ since $v_i(\lambda_0)$, $i=1,\dots,m$ are linearly independent. Therefore $\langle u_0, f\rangle=0$, $\forall f\in U^*$ because U^* is direct sum of $A^*(\lambda_0)V^*$ and $\mathcal{L}(f_1^0, \dots, f_m^0)$.) Consequently, $R(\lambda)$ will be continuously invertible in some neighbourhood of λ_0, too.

b) For $S_n(\cdot)$ Lemma 3.3 is proved in [3](Lemma 3.3). ■

3.4 (The introduction of the matrix functions). Let us, further, consider the equations

$$A(\lambda)u=0,\ B_n(\lambda)x_n=0 \tag{3.5}$$

in such neighborhood $\Lambda_1\subset\Lambda$ of $\lambda_0\in\sigma(A)$ and for such indices $n\in N_1$ that the operators $R(\lambda), S_n(\lambda)$ are continuously invertible and the inverses are uniformly bounded on Λ_1 (see Lemma 3.3). Note that on Λ_1 the operator-functions $R^{-1}(\cdot)$ and $S_n^{-1}(\cdot)$ are holomorphic.

On Λ_1 the equation $A(\lambda)u=0$ can be written as follows:

$$R(\lambda)(I-R^{-1}\lambda)K(\lambda))u=0,\quad R(\lambda)[I-\Sigma_{i=1}^{i=m}\langle u, f_i(\lambda)\rangle R^{-1}(\lambda)v_i(\lambda)]=0,$$

$$R(\lambda)[I-D(\lambda)C(\lambda)]u=0,$$

where the operator-functions $C(\cdot):\Lambda_1\to\mathcal{B}(U,\mathbb{C}^m)$ and $D(\cdot):\Lambda_1\to\mathcal{B}(\mathbb{C}^m,U)$ are defined by the following equalities.

$$C(\lambda)u=(\langle u,f_1(\lambda)\rangle,\dots,\langle u,f_m(\lambda)\rangle),\quad D(\lambda)(\alpha_1,\dots,\alpha_m)=\Sigma_{i=1}^{i=m}\alpha_i R^{-1}(\lambda)v_i(\lambda).$$

Let us define the operator-function $M(\cdot):\Lambda_1\to\mathcal{B}(\mathbb{C}^m,\mathbb{C}^m)$ by the equality $M(\cdot)=I-C(\cdot)D(\cdot)$. Then

$$(M(\lambda)(\alpha_1,\dots,\alpha_m))_r=\alpha_r-\langle\Sigma_{i=1}^{i=m}\alpha_i R^{-1}(\lambda)v_i(\lambda), f_r(\lambda)\rangle=$$

$$=\alpha_r-\Sigma_{i=1}^{i=m}\alpha_i\langle R^{-1}(\lambda)v_i(\lambda), f_r(\lambda)\rangle,$$

i.e., $M(\lambda)$ can be represented as a matrix with the elements

$$m_{rs}(\lambda)=\delta_{rs}-\langle R^{-1}(\lambda)v_s(\lambda), f_r(\lambda)\rangle.$$

Note that here all the functions $C(\cdot), D(\cdot), M(\cdot)$ and $m_{rs}(\cdot)$ are holomorphic on Λ_1, uniformly bounded, and equicontinuous on every compact $\Lambda_0\subset\Lambda_1$. (It is not important for us which norm is fixed in $\mathbb{C}^m$.)

3.5 (The reduction of the problem to the case of matrix functions). From Lemma 3.5 and Corollary 3.5 in [3], it follows

Corollary 3.5. For the operator-functions $A(\cdot), B_n(\cdot), M(\cdot)$ and $T_n(\cdot)$, $n\in N_1$, with $T_n(\lambda)\in\mathcal{B}(\mathbb{C}^m,\mathbb{C}^m)$ the matrices with the elements

$$t_{rs}^n(\lambda)=\delta_{rs}-\langle S_n^{-1}(\lambda)q_n v_s(\lambda), g_n^r(\lambda)\rangle,$$

it holds

$$\Lambda_1\cap\sigma(A)=\Lambda_1\cap\sigma(M)=\{\lambda_0\},\quad \Lambda_1\cap\sigma(B_n)=\Lambda_1\cap\sigma(T_n),$$

and for $\lambda_0\in\sigma(A)$ and every $\lambda_n\in\Lambda_1\cap\sigma(B_n)$:

$$\nu(\lambda_0,A)=\nu(\lambda_0,M),\ \nu(\lambda_n,B_n)=\nu(\lambda_n,T_n),\ \varkappa(\lambda_0,A)=\varkappa(\lambda_0,M),\ \varkappa(\lambda_n,B_n)=\varkappa(\lambda_n,T_n).$$

3.6 (The estimation of the difference $M(\cdot)-T_n(\cdot)$). For us it will be important to estimate $\|M(\lambda)-T_n(\lambda)\|$ and $|\det M_n(\lambda)-\det T_n(\lambda)|$ on some arbitrary fixed compact $\Lambda_0 \subset \Lambda_1$. We denote by $e^n_{rs}(\lambda)$ the element $m_{rs}(\lambda)-t^n_{rs}(\lambda)$ of the matrix $M(\lambda)-T_n(\lambda)$ and define

$$\varepsilon_n=\varepsilon_n(\Lambda_0)=\max\{\,|e^n_{rs}(\lambda)|\;\big|\;\lambda\in\Lambda_0,\ r,s=1,\dots,m\}. \tag{3.6}$$

Since all the elements $m_{rs}(\lambda)$ and $t^n_{rs}(\lambda)$ of the matrices $M(\lambda)$ and $T_n(\lambda)$ are uniformly bounded on Λ_0, the following estimations hold :

$$\|M(\lambda)-T_n(\lambda)\|\le c\,\varepsilon_n,\ |\det M(\lambda)\ \det T_n(\lambda)|\le c\,\varepsilon_n,\qquad \forall\,\lambda\in\Lambda_0,\ n\in N_1. \tag{3.7}$$

To get the estimation for $e^n_{rs}(\lambda)$, we note that

$$\begin{aligned} e^n_{rs}(\lambda)&=m_{rs}(\lambda)-t^n_{rs}(\lambda)=\langle R^{-1}(\lambda)v_s(\lambda), f_r(\lambda)\rangle-\langle S_n^{-1}(\lambda)q_n(\lambda)v_s(\lambda), r_n^* f_r(\lambda)\rangle=\\ &=\langle (R^{-1}(\lambda)-r_nS_n^{-1}(\lambda)q_n(\lambda))\ v_s(\lambda),\ f_r(\lambda)\rangle=\\ &=\langle [\,I_U-r_nS_n^{-1}(\lambda)q_n(\lambda)R(\lambda)]\,R^{-1}(\lambda)\,v_s(\lambda),\ f_r(\lambda)\rangle. \end{aligned} \tag{3.8}$$

Let us recall now that $S_n(\lambda)=q_nR(\lambda)r_n$. Thus

$$S_n^{-1}(\lambda)[q_nR(\lambda)r_n]=I_{X_n},\ [q_nR(\lambda)r_n]\,S_n^{-1}(\lambda)=I_{Y_n},\ (S_n^{-1}(\lambda))^*r_n^*R(\lambda)^*q_n^*=I_{Y_n^*}.$$

and , for every $x_n\in X_n$:

$$[I_U-r_nS_n^{-1}(\lambda)q_nR(\lambda)]r_nx_n=r_n\,[I_{X_n}-S_n^{-1}(\lambda)q_nR(\lambda)r_n]x_n=0.$$

$$[I_U-r_nS_n^{-1}(\lambda)q_nR(\lambda)]u=[I_U-r_nS_n^{-1}(\lambda)q_nR(\lambda)](u-r_nx_n)\ . \tag{3.9}$$

Further ,

$$[I_U-r_nS_n^{-1}(\lambda)q_nR(\lambda)]=R^{-1}(\lambda)\,[I_V-R(\lambda)\,r_nS_n^{-1}(\lambda)q_n]\,R(\lambda)\ ,$$

and, for every $h_n\in Y_n^*$:

$$[I_{V^*}-q_n^*(S_n^{-1}(\lambda))^*r_n^*R(\lambda)^*]q_n^*\,h_n=q_n^*\,[I_{Y_n^*}-(S_n^{-1}(\lambda))^*r_n^*R(\lambda)^*q_n^*\,]\,h_n=0.$$

Therefore, for every $u\in U$, $f\in U^*$, $x_n\in X_n$ and $h_n\in Y_n^*$:

$$\begin{aligned} &\langle\,[I_U-r_nS_n^{-1}(\lambda)q_nR(\lambda)]u,\ f\,\rangle=\\ &=\langle R(\lambda)(u-r_nx_n),\ [I_{V^*}-q_n^*(S_n^{-1}(\lambda))^*\,r_n^*R(\lambda)^*\,](R^{-1}(\lambda)^*f-q_n^*h_n)\,\rangle=\\ &=\langle\ [I_V-R(\lambda)\,r_nS_n^{-1}(\lambda)q_n]\,R(\lambda)(u-r_nx_n),\,R^{-1}(\lambda)^*f-q_n^*h_n\rangle\ . \end{aligned} \tag{3.10}$$

Finally, making use of (3.8) and (3.10), we have for $e^n_{rs}(\lambda)$:

$$e^n_{rs}(\lambda)=\langle\Phi_n(\lambda)\,[R^{-1}(\lambda)v_s(\lambda)-r_nx_n],\ R^{-1}(\lambda)^*f_r(\lambda)-q_n^*h_n\rangle\ , \tag{3.11}$$

with auxiliary $x_n\in X_n$, $h_n\in Y_n^*$ and with

$$\Phi_n(\lambda)=R(\lambda)-R(\lambda)\,r_nS_n^{-1}(\lambda)q_n\,R(\lambda) \tag{3.12}$$

Here $\Phi_n(\lambda)\in\mathcal{L}(U,U)$, $q_n\Phi_n(\lambda)=0_{U\to X_n}$, $\Phi_n(\lambda)r_n=0_{X_n\to U}$, $\forall\lambda\in\Lambda_1, n\in N_1$, and, for every fixed compact $\Lambda_0\subset\Lambda_1$:

$$\|\,\Phi_n(\lambda)\|\le c,\ \ \forall\lambda\in\Lambda_0,\ n\in N_1\ . \tag{3.13}$$

Lemma 3.6. $R^{-1}(\lambda)v_i(\lambda)\in\mathcal{J}(A,\lambda_0)$, $(R^{-1}(\lambda))^*f_r(\lambda)\in\mathcal{J}(A^*,\lambda_0)$, $\forall\lambda\in\Lambda_0$.

Proof. 1) For $\lambda\in\Lambda_0$, $\lambda\neq\lambda_0$, the following identities hold for $R^{-1}(\lambda)$:

$$R^{-1}(\lambda)=A^{-1}(\lambda)[A(\lambda)+K(\lambda)-K(\lambda)]R^{-1}(\lambda)=A^{-1}(\lambda)[I-K(\lambda)R^{-1}(\lambda)]. \quad (3.14)$$

At the same time for $\lambda\neq\lambda_0$:

$$A^{-1}(\lambda)v_i(\lambda)=(\lambda-\lambda_0)^{-\nu(u_i^0)}u_i(\lambda)\in\mathcal{J}(A,\lambda_0),\ \ i=1,\dots,m, \quad (3.15)$$

$$A^{-1}(\lambda)K(\lambda)u=\Sigma_{i=1}^{i=m}\langle u,f_i(\lambda)\rangle A^{-1}(\lambda)v_i(\lambda)\in\mathcal{J}(A,\lambda_0). \quad (3.16)$$

Hence, $R^{-1}(\lambda)v_i(\lambda)\in\mathcal{J}(A,\lambda_0)$ for $\lambda\in\Lambda_0$, $\lambda\neq\lambda_0$, and from continuity of $R^{-1}(\cdot)v_i(\cdot)$ we conclude that $R^{-1}(\lambda)v_i(\lambda)\in\mathcal{J}(A,\lambda_0)$, $\forall\lambda\in\Lambda_0$.

2) Recall now that every $f_r(\lambda)$, $r=1,\dots,m$ is a linear combination of $f_k^*(\lambda)$, $k=1,\dots,m$. So, for $\lambda\neq\lambda_0$, from

$$[A^*(\lambda)]^{-1}f_k^*(\lambda)=(\lambda-\lambda_0)^{-\nu(h_k^0)}h_k(\lambda)\in\mathcal{J}(A^*,\lambda_0),\ k=1,\dots,m, \quad (3.17)$$

it follows that $[A^*(\lambda)]^{-1}f_r(\lambda)\in\mathcal{J}(A^*,\lambda_0)$, $r=1,\dots,m$, also.

Further, the following identities hold for $\lambda\in\Lambda_0$, $\lambda\neq\lambda_0$:

$$[R^{-1}(\lambda)]^*=[A^*(\lambda)]^{-1}[A^*(\lambda)+K^*(\lambda)-K^*(\lambda)][R^{-1}(\lambda)]^*=[A^*(\lambda)]^{-1}[I-K^*(\lambda)[R^{-1}(\lambda)]^*],$$

$$[A^*(\lambda)]^{-1}K^*(\lambda)f=\Sigma_{i=1}^{i=m}\langle v_i(\lambda),f\rangle[A^*(\lambda)]^{-1}f_i(\lambda). \quad (3.18)$$

Therefore, $(R^{-1}(\lambda))^*f_r(\lambda)\in\mathcal{J}(A^*,\lambda_0)$ for $\lambda\in\Lambda_0$, $\lambda\neq\lambda_0$, and from continuity of $(R^{-1}(\cdot))^*f_r(\cdot)$ we conclude that $(R^{-1}(\lambda))^*f_r(\lambda)\in\mathcal{J}(A^*,\lambda_0)$, $\forall\lambda\in\Lambda_0$. ∎

Corollary 3.6. On every compact $\Lambda_0\subset\Lambda_1$, for almost all $n\in\mathbb{N}$, the following estimation holds:

$$\varepsilon_n=\varepsilon_n(\Lambda_0)\leq c(\Lambda_0)\cdot d_n d_n^* .$$

Proof. This estimation follows from (3.11),(3.13) and Lemma 3.6 since $R^{-1}(\lambda)$, $v_i(\lambda)$ and $f_r(\lambda)$ are bounded on Λ_0. ∎

4. The Proof of Theorem 2.3

4.1 (Some auxiliary results).

Lemma 4.1 (a special case of Lemma 4.3 in [3] or Theorem 2 in [13]). Let W be a complex Banach space, and $M(\cdot)$, $T_n(\cdot)$ $(n\in\mathbb{N})$ be holomorphic in some region $\Lambda_1\subset\mathbb{C}$ operator-functions with values in $\mathcal{B}(W,W)$. Let $\lambda_0\in\sigma(M)$, $\Lambda_0=\{\lambda\in\mathbb{C}\mid|\lambda-\lambda_0|\leq\delta\}\subset\Lambda_1$, Γ be the boundary of Λ_0 and let the following assumptions hold:

1) $\sigma(M)\cap\Lambda_1=\{\lambda_0\}$,

2) $\eta_n=\eta_n(\Gamma)=\sup_{\lambda\in\Gamma}\|M(\lambda)-T_n(\lambda)\|\to 0$ $(n\in\mathbb{N})$.

Then, for almost all $n\in\mathbb{N}$, the following estimation holds:

$$\max_{\lambda\in\sigma(T_n)\cap\Lambda_0}|\lambda-\lambda_0|\leq c(\eta_n)^{1/\varkappa},$$

with $\varkappa=\varkappa(\lambda_0,M)$ the order of λ_0 as a pole of $M^{-1}(\cdot)$.

Theorem 4.1 (see [12,16]). Let $M(\cdot)$ be holomorphic in some neighborhood of $\lambda_0 \in \mathbb{C}$ matrix-function. Let $\lambda_0 \in \sigma(M)$. Then the algebraic multiplicity $\nu(\lambda_0, M)$ of λ_0 is equal to multiplicity of λ_0 as a zero of the holomorphic function $\det M(\cdot)$.

Proposition 4.1 (a special case of Propositions 2.1, 3.1 in [14]). Let $f(\cdot)$ and $\{h_n(\cdot)\}_{n \in \mathbb{N}}$ be holomorphic in some region $\Lambda_1 \subseteq \mathbb{C}$ functions. Let λ_0 be zero of multiplicity ν for $f(\cdot)$, $\Lambda_0 = \{\lambda \,|\, |\lambda - \lambda_0| \le \delta\} \subset \Lambda_1$, Γ be the boundary of Λ_0, and let the following assertions hold:

1) function $f(\cdot)$ has in Λ_0 only one zero $\lambda = \lambda_0$;

2) $\eta_n = \eta_n(\Gamma) = \sup\limits_{\lambda \in \Gamma} |f(\lambda) - h_n(\lambda)| \to 0 \quad (n \in \mathbb{N})$.

Denote by $\sigma(h_n)$ the set of zeros of $h_n(\cdot)$, by $\nu(\lambda, h_n)$ the multiplicity of the zero λ of $h_n(\cdot)$, and by $\bar{\lambda}$ the sum $\sum\limits_{\lambda \in \Lambda_0 \cap \sigma(h_n)} [\nu(\lambda, h_n)/\nu] \cdot \lambda$.

Then, for almost all $n \in \mathbb{N}$, we have $\Lambda_0 \cap \sigma(h_n) \neq \emptyset$, $|\bar{\lambda} - \lambda_0| \le c\eta_n$ and $\max\{|\lambda_n - \lambda_0| \,|\, \lambda_n \in \Lambda_0 \cap \sigma(h_n)\} \le c\eta_n^{1/k_n}$, with k_n equal to the number of different λ_n in $\Lambda_0 \cap \sigma(h_n)$.

4.2 (The proof of Theorem 2.3). As we already mentioned, it is sufficient to consider Λ_0 in some quite small neighborhood of λ_0. For such Λ_0 we established estimation (2.4) in Corollary 3.6. The remained estimations (2.1), (2.2) and (2.3) follow, by Corollary 3.5 and Theorem 4.1, from Lemma 4.1 and Proposition 4.1. (Note that $\eta_n \le c\varepsilon_n$ for η_n in Lemma 4.1 and Proposition 4.1, and $\varepsilon_n \to 0$ $(n \in \mathbb{N})$ under assumptions of Theorem 2.3). ∎

5. An Complementary Remark

5.1. Lemma 5.1. Let $\lambda_0 \in \sigma(A)$ and let the functions $u_i(\cdot)$, $h_i(\cdot)$, $v_i(\cdot)$ and $f_i(\cdot)$, $i=1,\dots,m$ be determined as in Section 3.2. Let Λ_1 be fixed as in Section 3.4 and $\Lambda_0 \subset \Lambda_1$ be a compact with boundary $\Gamma \subset \varrho(A)$. Then the following estimations hold:

$$\|R^{-1}(\lambda) v_i(\lambda)\| \le c \cdot \max\{\|u_i(\lambda)\| \,|\, \lambda \in \Lambda_0,\ i=1,\dots,m\}, \tag{5.1}$$

$$\|[R^{-1}(\lambda)]^* f_i(\lambda)\| \le c \cdot \max\{\|h_i(\lambda)\| \,|\, \lambda \in \Lambda_0,\ i=1,\dots,m\}. \tag{5.2}$$

Proof. 1) For holomorphic functions $R^{-1}(\lambda) v_i(\lambda)$, $i=1,\dots,m$ we have, for every $\lambda \in \Lambda_0$:

$$\|R^{-1}(\lambda) v_i(\lambda)\| \le \max\{\|R^{-1}(\lambda) v_i(\lambda)\| \,|\, \lambda \in \Gamma\}.$$

We denote by r the distance $d(\lambda_0, \Gamma) = \inf\{|\lambda - \lambda_0| \,|\, \lambda \in \Gamma\}$. Then it follows from (3.14)-(3.16) that, for every $\lambda \in \Gamma$,

$$\|R^{-1}(\lambda) v_i(\lambda)\| \le c r^{-\varkappa(\lambda_0, A)} \cdot \max\{\|u_i(\lambda)\| \,|\, i=1,\dots,m\}.$$

(Recall that $\nu(u_i^0) \le \varkappa(\lambda_0, A)$, $i=1,\dots,m$.)

2) Estimation (5.2) follows analogically from (3.17)-(3.18) since $f_k(\cdot)$ are linear combinations of $f_i^*(\cdot)$. ∎

5.2 Proposition 5.2. The quantities ε_n in Theorem 2.3 can be estimated as follows :

$$\varepsilon_n \le c \cdot \varrho_n \varrho_n^{*}, \tag{5.3}$$

with

$$\varrho_n = \max \{ d(u_i(\lambda), r_n X_n) \mid u_i(\cdot),\ i=1,\dots,m \text{ -some canonical system of Jordan polynomials for } A(\cdot) \text{ at } \lambda_0,\ \lambda \in \Gamma \},$$

$$\varrho_n^{*} = \max \{ d(h_i(\lambda), q_n^{*} Y_n^{*}) \mid h_i(\cdot),\ i=1,\dots,m \text{ -some canonical system of Jordan polynomials for } A^{*}(\cdot) \text{ at } \lambda_0,\ \lambda \in \Gamma \}.$$

Proof. Estimation (5.3) follows from (3.11),(3.13) and Lemma 5.1 ■

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О СКОРОСТИ СХОДИМОСТИ ПРИБЛИЖЕННЫХ СОБСТВЕННЫХ ЗНАЧЕНИЙ В МЕТОДАХ ПРОЕКЦИОННОГО ТИПА

О. Карма

Резюме

В статье исследуется аппроксимация проблемы собственных значений $A(\lambda)u=0$ с голоморфной фредгольмовой оператор-функцией $A(\cdot)$. Используется схема дискретной аппроксимации пространств. Предполагается существование линейных операторов "проектирования" q_n и "вложения" r_n и в качестве приближенных задач рассматриваются задачи $[q_n A(\lambda) r_n] x_n = 0$, $n \in \mathbb{N}$. Скорость сходимости собственных значений приближенных задач к собственным значениям задачи $A(\lambda)u=0$ оценивается через произведение ошибок аппроксимации обобщенных собственных подпространств (корневых подпространств) оператор-функций $A(\cdot)$ и $A^*(\cdot)$. Оценка дается как для каждого приближенного собственного значения в отдельности (оценки (2.1),(2.2)), так и для их взвешенного среднего (оценка (2.3)) , причем последняя оценка лучше по порядку.

В случае линейной зависимости от параметра оценка (2.1) имеется в [5] и оценка (2.3) в [1], для голоморфной $A(\cdot)$ случай компактной аппроксимации рассмотрен в [13]. В общей схеме регулярной аппроксимации скорость сходимости собственных значений рассмотрен в [2, 3, 6, 8, 9, 15] (см. также обзор по этим вопросам в [7]).

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ON THE SELF-REGULARIZATION BY SOLVING ILL-POSED PROBLEMS BY PROJECTION METHODS

Uno Hämarik

Applications to self-regularization conditions [11] *in projection methods for ill-posed problems are given, generalizing results of* [8,10]..

1. Introduction. Let H,F be Hilbert spaces. We consider the equation

$$Au=f \tag{1}$$

with the operator $A\in\mathcal{L}(H,F)$. Let $f\in\mathcal{R}(A)$; $\mathcal{R}(A)$ may be non-closed. Assume, that instead of A and $f\in F$ only approximations $A_\eta\in\mathcal{L}(H,F)$ and $f_\delta\in F$ are available with $\|A_\eta-A\|\le\eta$, $\|f_\delta-f\|\le\delta$. We solve equation (1) by a projection method. Let $H_n\subseteq H$ and $F_n\subseteq F$ be subspaces in H and F with orthoprojectors P_n and Q_n respectively. Let $\dim H_n=\dim F_n=n$. As approximation to the solution u_* (with minimal norm if $\mathcal{N}(A)\neq\{0\}$) of equation (1) we search a solution $u_n\in H_n$ of the projected equation

$$Q_nA_\eta u_n=Q_nf_\delta,\quad u_n\in H_n. \tag{2}$$

Projection methods for ill-posed problems are studied in [1,4,6-11]. This paper gives some applications to self-regularization conditions in [11]. The self-regulation is a situation, where by appropriate choice of $n=n(\delta,\eta)$ we get

$$u_{n(\delta,\eta)}\to u_*\quad\text{as}\quad \delta\to0,\ \eta\to0. \tag{3}$$

An example of projection methods is the least squares method where $F_n=A_\eta H_n$. Some conditions for self-regulation in least squares method are collected in Lemma 1 which follows from Theorems 1,2 and Lemmas 4,5 in [11] (see also [10] and Theorems 1.3 ,1.3' in [8]).

Lemma 1. *Let* $\mathcal{N}(A)=\{0\}$,

$$\|u-P_nu\|\to0\quad(\forall u\in H,\ n\to\infty), \tag{4}$$

$$\exists\,\alpha\in R,\ \alpha>0:\quad \varkappa_n^\alpha\,\|(I-P_n)|A|^\alpha\|\le\text{const}\ (n=1,2,\ldots), \tag{5}$$

where

$$\varkappa_n=\sup_{v_n\in H_n}\frac{\|v_n\|}{\|Av_n\|},\qquad |A|=(A^*A)^{1/2}.$$

Then equation (2) has in case $\eta\varkappa_n<1$ *a unique solution* $u_n\in H_n$.

If $n=n(\delta,\eta)$ *is chosen so that* $n(\delta,\eta)\to\infty$ $(\delta,\eta\to 0)$,

$$(\delta+\eta)\,\varkappa_{n(\delta,\eta)}\to 0 \quad \text{as} \quad \delta,\eta\to 0, \tag{6}$$

then (3) holds. If $\eta\varkappa_n\le c'<1$, *it holds the error estimation*

$$\|u_n-u_*\|\le c\,[\|(I-P_n)u_*\|+\varkappa_n(\delta+\eta)]. \tag{7}$$

If there is given an increasing sequence of natural numbers $\{n_i\}\in N$ (*for example* $n_i=i$ or $n_i=k^i$, $k\in N$) *and instead of (5) it holds*

$$\exists\alpha\in R,\ \alpha>0:\ (\varkappa_{n_i}+\varkappa_{n_{i+1}})^{\alpha}\,\|(I-P_n)|A|^{\alpha}\|\le \mathrm{const}\quad (i=1,2,\ldots), \tag{8}$$

then by sufficiently small δ,η *we may choose* $n=n(\delta,\eta)$ *as the first term in* $\{n_i\}$ *such that it holds*

$$\|A_\eta u_{n_i}-f_\delta\|\le b\,(\delta+\|u_{n_i}\|\eta),\quad b=\mathrm{const}>1. \tag{9}$$

By this choice of $n=n(\delta,\eta)$ *(3),(6) holds.*

If $\{\varphi_1,\ldots,\varphi_n\}$ is a base of H_n, then equation (2) in least squares method is equivalent to the system of linear algebraic equations

$$\mathbf{A}_{n,\eta}\mathbf{x}=\mathbf{g} \tag{10}$$

for coefficients x_i in $u_n=\sum_{i=1}^{n}x_i\varphi_i$, where $(\mathbf{A}_{n,\eta})_{ij}=(A_\eta\varphi_i,A_\eta\varphi_j)$, $(\mathbf{g}^{\tau})_i=(f_\delta,A_\eta\varphi_i)$, $\mathbf{x}=(x_1,\ldots,x_n)^{\tau}$. On the condition number $\mathrm{cond}(\mathbf{A}_{n,\eta})$ of the matrix $\mathbf{A}_{n,\eta}$ (quotient of the largest and smallest singular value of $\mathbf{A}_{n,\eta}$) the following result [9] holds.

Lemma 2. *If* $\eta\varkappa_n\le c<1$, *then* $\mathrm{cond}(\mathbf{A}_{n,\eta})\le c'\,\mathrm{cond}(\mathbf{D}_n)\,\varkappa_n^2$, where $(\mathbf{D}_n)_{ij}=(\varphi_i,\varphi_j)$, $i,j=1,\ldots,n$.

2. Equation and projection subspaces. Let $A\in\mathscr{L}(L_2(\Omega_1),L_2(\Omega_2))$, where Ω_1 and Ω_2 are bounded domains in R^m with sufficiently regular boundaries. Denote by $A^{\tau}\in\mathscr{L}(L_2(\Omega_2),L_2(\Omega_1))$ the transposed operator:

$$(Au,w)_{L_2(\Omega_2)}=(u,A^{\tau}w)_{L_2(\Omega_1)}\quad (\forall u\in L_2(\Omega_1),\forall w\in L_2(\Omega_2)).$$

Here $A^{\tau}=A^{*}$. In least squares method we assume that

$$\mathring{H}^{l}(\Omega_1)\subset A^{\tau}L_2(\Omega_2)\subset H^{l}(\Omega_1),\quad \mathscr{N}(A)=\{0\}, \tag{11}$$

$$\exists t\in R,\ t>0:\ \|\,|A|^{t/l}v\,\|_{H^{t}(\Omega_1)}\le c\,\|v\|_{L_2(\Omega_1)}\quad (\forall v\in L_2(\Omega_1)) \tag{12}$$

Here $H^{l}(\Omega)$ is a Sobolev space with $l\in R$, $l>0$, $\Omega\subset R^m$; $\mathring{H}^{l}(\Omega)$ is the closure of infinitely differentiable functions in $H^{l}(\Omega)$ with compact supporters in Ω.

Let $q,r\in R$, $0\le r<q$, $0\le l\le q$. We use projection subspaces $S_n^{q,r,l}(\Omega)\subset H^{r}(\Omega)$ (mostly we consider case $r=0$), satisfying following conditions:

$$\forall v\in H^{s}(\Omega),\ r\le s\le q\ \ \exists v_n\in S_n^{q,r,l}(\Omega):\ \|v-v_n\|_{H^{r}(\Omega)}\le c\,n^{(r-s)/m}\|v\|_{H^{s}(\Omega)}, \tag{13}$$

$$\forall v_n\in S_n^{q,r,l}(\Omega):\ \|v_n\|_{H^{r}(\Omega)}\le c\,n^{(l+r)/m}\,\|v_n\|_{H^{-l}(\Omega)} \tag{14}$$

Here $H^{-l}(\Omega)=(\mathring{H}^{l}(\Omega))^{*}$ is the conjugate space to $\mathring{H}^{l}(\Omega)$ with norm

$$\|v\|_{H^{-l}(\Omega)} = \sup_{w\in \overset{\circ}{H}{}^{l}(\Omega)} \frac{(v,w)_{L_2(\Omega)}}{\|w\|_{H^l(\Omega)}} .$$

The systems $S_n^{q,r,l}(\Omega)$ exists if domain $\Omega\subset R^m$ is sufficiently regular (see [2]). In case $m=1$ examples are trigonometric polynomials and splines, in case $m\ge 2$ there are many finite element systems (see [1–5,7]). The splines may be generalized: the spline function need not be on the subintervals an algebraic polynomial, but also as a trigonometric polynomial or a solution of a homogenous differential equation (see [5]).

3. Least squares method. Theorem 1. *Assume that conditions (11), (12) hold. Then by solving equation (1) by least squares method with* $H_n=S_n^{q,0,l}(\Omega_1)$ *the following assertions are valid :*

1) $\varkappa_n \le c\, n^{l/m}$; (15)

2) *if* $\eta\varkappa_n \le c'<1$ *and* $\mathrm{cond}(D_n)\le c$, *then* $\mathrm{cond}(A_{n,\eta})\le c''n^{2l/m}$;

3) *if* $\delta=\eta=0$, *then* $\|u_n-u_*\|_{L_2(\Omega_1)} \to 0$ *as* $n\to\infty$;

4) *if* $\eta\varkappa_n<1$, *then system (10) has a unique solution; if* $n=n(\delta,\eta)$ *is chosen so that* $n(\delta,\eta)\to\infty$ $(\delta,\eta\to 0)$,

$$(\delta+\eta)\,[n(\delta,\eta)]^{l/m} \to 0 \quad \textit{as} \quad \delta,\eta\to 0, \tag{16}$$

then

$$\|u_{n(\delta,\eta)} - u_*\|_{L_2(\Omega_1)} \to 0 \quad \textit{as} \quad \delta,\eta\to 0; \tag{17}$$

5) *if* δ,η *are sufficiently small and we have sequence* $\{n_i\}$ *such that* $n_{i+1}/n_i \le \mathrm{const}$ $(i=1,2,\dots)$, *then by choosing* $n=n(\delta,\eta)$ *as the first term in* $\{n_i\}$ *such that it holds (9) we have (16),(17) ;*

6) *if* $u_*\in H^s(\Omega_1)$, $s\le q$ *and* $\eta\varkappa_n\le c'<1$, *then it holds*

$$\|u_n - u_*\|_{L_2(\Omega_1)} \le c\,[n^{-s/m} + n^{l/m}(\delta+\eta)]$$

and by choice $n\asymp(\delta+\eta)^{-m/(s+l)}$ *also*

$$\|u_n-u_*\|_{L_2(\Omega_1)} \le c\,(\delta+\eta)^{s/(s+l)} ;$$

7) *if* $u_*\in H^s(\Omega_1)$, $H_n= S_n^{q,r,l}(\Omega_1)$, $0<r<s\le q$, $\eta\tilde{\varkappa}_n\le c'<1$, *where*

$$\tilde{\varkappa}_n = \sup_{v_n\in S_n^{q,r,l}(\Omega_1)} \frac{\|v_n\|_{H^r(\Omega_1)}}{\|Av_n\|_{L_2(\Omega_2)}} ,$$

and for some $t\in(r,q)$ $|\tilde{A}|^{(t-r)/(l+r)}$ *is the bounded operator* $H^r(\Omega_1)\to H^t(\Omega_1)$, *where* $\tilde{A}$ *is the restriction of the operator* A to $H^r(\Omega_1)$, *then it holds*

$$\|u_n-u_*\|_{H^r(\Omega_1)} \le c\,[n^{(r-s)/m} + n^{(l+r)/m}(\delta+\eta)] \tag{18}$$

and by choice $n\asymp(\delta+\eta)^{-m/(s+l)}$ *also*

$$\|u_n-u_*\|_{H^r(\Omega\ )} \le c\,(\delta+\eta)^{(s-r)/(s+l)}$$

Proof. For every $u \in L_2(\Omega_1)$ we have (see (11))

$$\left.\begin{aligned} \|u\|_{H^{-l}(\Omega_1)} &= \sup_{v \in \mathring{H}^l(\Omega_1)} \frac{(u,v)_{L_2(\Omega_1)}}{\|v\|_{H^l(\Omega_1)}} \le \sup_{v \in A^\top L_2(\Omega_1)} \frac{(u,v)_{L_2(\Omega_1)}}{\|v\|_{H^l(\Omega_1)}} = \\ &= \sup_{\substack{w \in L_2(\Omega_2), \\ w \perp \mathcal{N}(A^\top)}} \frac{(u,A^\top w)_{L_2(\Omega_1)}}{\|A^\top w\|_{H^l(\Omega_1)}} \asymp \sup_{\substack{w \in L_2(\Omega_2), \\ w \in \overline{\mathcal{R}(A)}}} \frac{(Au,w)_{L_2(\Omega_2)}}{\|w\|_{L_2(\Omega_2)}} = \|Au\|_{L_2(\Omega_2)}. \end{aligned}\right\} \quad (19)$$

Using for $u = v_n \in S_n^{q,0,l}(\Omega_1)$ (14) and (19), we obtain (15):

$$\|v_n\|_{L_2(\Omega_1)} \le c\, n^{l/m} \|v_n\|_{H^{-l}(\Omega_1)} \le c' n^{l/m} \|Av_n\|_{L_2(\Omega_2)} .$$

Assertion 2) follows from Lemma 2.

Using (12) and (13) with r=0, s=t, we have for $\forall v \in L_2(\Omega_1)$

$$\|(I-P_n)|A|^{t/l} v\|_{L_2(\Omega_1)} \le c\, n^{-t/m} \| |A|^{t/l} v \|_{H^t(\Omega_1)} \le c' n^{-t/m} \|v\|_{L_2(\Omega_1)} .$$

Hence

$$\|(I-P_n)|A|^{t/l}\|_{L_2(\Omega_1) \to L_2(\Omega_1)} \le c\, n^{-t/m} \quad (20)$$

and from (15) follows (5) with $\alpha = t/l$. So assumptions of Lemma 1 are fulfilled and from there we obtain assertions 3),4) and also 5), since (8) is fulfilled due to assumption $n_{i+1}/n_i \le c$. Assertion 6) follows from (7), since from (13) we have for $u_* \in H^s(\Omega_1)$, $s \le q$

$$\|(I-P_n)u_*\|_{L_2(\Omega_1)} \le c\, n^{-s/m} \|u_*\|_{H^s(\Omega_1)} . \quad (21)$$

To prove assertion 7), remark that in case $u_* \in H^s(\Omega_1)$, $s > r$ we may consider equation $\tilde{A}u = f$ instead of $Au = f$. To prove that assumptions of Lemma 1 are satisfied also for $\tilde{A}u = f$, we use for $u = v_n \in S_n^{q,r,l}(\Omega_1)$ relations (14),(19) and get $\tilde{\varkappa}_n \le c\, n^{(l+r)/m}$. From (13) we obtain for $\forall w \in H^r(\Omega_1)$

$$\begin{aligned} \|(I-P_n)|\tilde{A}|^{(t-r)/(l+r)} w\|_{H^r(\Omega_1)} &\le c\, n^{(r-t)/m} \| |\tilde{A}|^{(t-r)/(l+r)} w\|_{H^t(\Omega_1)} \le \\ &\le c\, n^{(r-t)/m} \|w\|_{H^r(\Omega_1)} , \end{aligned}$$

hence

$$\|(I-P_n)|\tilde{A}|^{(t-r)/(l+r)}\|_{H^r(\Omega_1) \to H^r(\Omega_1)} \le c\, n^{(r-t)/m}$$

and (5) is fulfilled (with $\alpha = (t-r)/(l+r)$ and with $\tilde{\varkappa}_n$ instead of $\varkappa_n$ and with $\tilde{A}$ instead of A). Since for $u_* \in H^s(\Omega_1)$, $r < s \le q$ we have by (13)

$$\|(I-P_n)u_*\|_{H^r(\Omega_1)} \le c\, n^{(r-s)/m} \|u_*\|_{H^s(\Omega_1)} ,$$

then (18) follows from (7). Theorem 1 is proved.

Remark 1. Presented in assertion 5) of Theorem 1 the rule for the selection of parameter n may be easily completed so, that it is guaranteed the selection of n for every $\delta, \eta > 0$. An example of a completed rule is the following. Let $d \in (0,1)$ and the function g(n) satisfies the assumption $\varkappa \le g(n) \le c\,\varkappa$. Choose the first $n \in \{n\}$ such that $\eta\, g(n) \ge d$ or (9) is satisfied. For small δ,η this complimenting does not influence the choice of $n=n(\delta,\eta)$.

Remark 2. Consider the equation $Au=f$, $A\in\mathscr{L}(L_2(\Gamma),L_2(\Gamma))$ where Γ is sufficiently regular m dimensional manifold in R^{m+1}. Let $A^{\tau}L_2(\Gamma)=H^1(\Gamma)$, $\mathcal{N}(A)=\{0\}$. If equation $Au=f$ is solved by least squares method with $H_n=S_n^{q,0,1}(\Gamma)$, satisying (13),(14) with Γ instead of Ω, Theorem 1 is valid (with Γ instead of Ω). Similar remarks are true for Theorems 2 and 3 following below.

4.Galerkin method. Let $H=F$.

$$A=A^*>0,\quad A_\eta=A_\eta^*\ge 0, \tag{22}$$

In Galerkin method $H_n=F_n$ and the method is equivalent to the system of linear algebraic equations (10) for coefficients x_i $(i=1,\dots,n)$ in approximation $u_n=\sum_{i=1}^{n}x_i\varphi_i$, where $(A_{n,\eta})_{ij}=(A_\eta\varphi_i,\varphi_j)$, $(g^\eta)_i=(f_\delta,\varphi_i)$.

The following Lemma 3 is a conclusion of Theorems 1.2 and Lemmas 4,5 in [11] (see also [10] and Theorems 1.5, 1.5' in [8]).

Lemma 3. *Let (4), (5) and (22) holds. Then in Galerkin method equation (10) has in case* $\eta\gamma_n<1$ *a unique solution, where*

$$\gamma_n=\sup_{v_n\in H_n}\frac{\|v_n\|^2}{(Av_n,v_n)}$$

Then assertions of Lemma 1 are valid (with sufficiently large b in (9)).

Theorem 2. *Let* $A\in\mathscr{L}(L_2(\Omega),L_2(\Omega))$, *(22), (12) holds and*

$$\overset{\circ}{H}{}^1(\Omega)\subseteq AL_2(\Omega)\subseteq H^1(\Omega) \tag{23}$$

Then in Galerkin method for (1) with $H_n=S_n^{q,0,1/2}(\Omega)$ *the following assertions are valid:*

1) $\gamma_n\le c\,n^{1/m}$; (24)

2) *if* $\eta\gamma_n\le c'<1$ *and* $\operatorname{cond}(D_n)\le c$, *then* $\operatorname{cond}(A_{n,\eta})\le c''n^{1/m}$;

3), 4), 5), 6) *corresponding assertions of theorem 1, where in (9)* b *is sufficiently large and in* 4), 6) γ_n *is used instead of* $\varkappa_n$.

Proof. It can be shown, that from (23) follows $\overset{\circ}{H}{}^{1/2}(\Omega)\subseteq A^{1/2}L_2(\Omega)\subseteq H^{1/2}(\Omega)$. Since assumptions of Theorem 1 are satisfied by replacing A to $A^{1/2}$, 1 to 1/2, we have (20) and from analogue of (15) for $A^{1/2}$ (24), so as $[\varkappa_n(A^{1/2})]^2=\gamma_n(A)$. Considering (20), (24) and inequality $\varkappa_n\le\gamma_n$, condition (5) is satisfied. From Lemma 3 we have assertions 3),4), 5),6) (see (21)). Assertion 2) follows from analogue of Lemma 2 for matrix $(A_{n,\eta})_{ij}=(A_\eta\varphi_i,\varphi_j)$ in Galerkin method. Theorem 2 is proved.

5. Least error method. In the least error method $H_n=A_\eta^*F_n$. If $\{\psi_1,\dots,\psi_n\}$ is a base of F_n, then this method is equivalent to the system of linear algebraic equations $\mathbf{A}_{n,n}\,\mathbf{x}=\mathbf{g}$ for coefficients x_i in $u_n=\sum_{i=1}^{n} x_i\, A_n^*\,\psi_i$, where $(\mathbf{A}_{n,n})_{ij} = (A_n^*\psi_i,\, A_n^*\psi_j)$, $(\mathbf{g}^T)_i = (f_\delta, \psi_i)$.

Lemma 4. (The conclusion of Theorems 1,2 and Lemmas 4,6 in [11] (see also [10] and Theorems 1.4,1.4' in [8])). *Let* $\mathcal{N}(A^*)=\{0\}$ *and*

$$\| z-Q_n z\| \to 0 \text{ as } n\to\infty \quad (\forall z \perp \mathcal{N}(A)). \tag{25}$$

Then the assertions of Lemma 1 are for least error method true, replacing $\varkappa_n$ *by*

$$\varkappa_n^* = \sup_{z_n \in F_n} \frac{\|z_n\|}{\|A^* z_n\|},$$

(5) by

$$\exists \alpha \in R,\ \alpha>0 : (\varkappa_{n_i}^* + \varkappa_{n_{i+1}}^*)^\alpha \, \|(I-Q_{n_i})\,|A^*|^\alpha\| \le \text{const}\ (i=1,2,\dots) \tag{26}$$

and with sufficiently large b in (9).

Theorem 3. *Let* $\mathcal{N}(A^*)=\{0\}$,

$$A \in \mathcal{L}(L_2(\Omega_1), L_2(\Omega_2)), \qquad \overset{\circ}{H}{}^1(\Omega_2) \subseteq A L_2(\Omega_1) \subseteq H^1(\Omega_2) , \tag{27}$$

$$\exists t \in R,\ t>0 : \|\,|A^*|^{t/l} w\|_{H^t(\Omega_2)} \le c\, \|w\|_{L_2(\Omega_2)} \quad (\forall w \in L_2(\Omega_2)). \tag{28}$$

Then in the Least error method for (1) with $F_n = S_n^{q,0,l}(\Omega_2)$ *the assertions 1)-6) of Theorem 1 are valid with replacement of* $\varkappa_n$ *by* $\varkappa_n^*$, *with (26) instead of (8) and with sufficiently large b in (9).*

Proof. For every $w \in L_2(\Omega_2)$ we have (see (27))

$$\|w\|_{H^{-1}(\Omega_2)} = \sup_{v \in \overset{\circ}{H}{}^1(\Omega_2)} \frac{(w,v)_{L_2(\Omega_2)}}{\|v\|_{H^1(\Omega_2)}} \le \sup_{v \in \mathcal{R}(A)} \frac{(w,v)_{L_2(\Omega_2)}}{\|v\|_{H^1(\Omega_2)}} \asymp$$

$$\asymp \sup_{\substack{u \in L_2(\Omega_1),\\ u \perp \mathcal{N}(A)}} \frac{(w, Au)_{L_2(\Omega_2)}}{\|u\|_{L_2(\Omega_1)}} = \sup_{\substack{u \in L_2(\Omega_1),\\ u \in \overline{\mathcal{R}(A^*)}}} \frac{(A^*w, u)_{L_2(\Omega_1)}}{\|u\|_{L_2(\Omega_1)}} = \|A^* w\|_{L_2(\Omega_1)}.$$

Using for $w=z_n \in S_n^{q,0,l}(\Omega_2)$ this relation and (14), we have

$$\|z_n\|_{L_2(\Omega_2)} \le c\, n^{1/m}\, \|z_n\|_{H^{-1}(\Omega_2)} \le c\, n^{1/m}\, \|A^* z_n\|_{L_2(\Omega_1)}$$

and so $\varkappa_n^* \le c\, n^{1/m}$. Using (13),(28) with $r=0$, $s=t$, we get for $\forall w \in L_2(\Omega_2)$

$$\begin{aligned} \|(I-Q_n)\,|A^*|^{t/l} w\|_{L_2(\Omega_2)} &\le c n^{-t/m}\, \|\,|A^*|^{t/l} w\|_{H^t(\Omega_2)} \le \\ &\le c n^{-t/m}\, \|w\|_{L_2(\Omega_2)} , \end{aligned} \tag{29}$$

hence

$$\|(I-Q_n)\,|A^*|^{t/l}\|_{L_2(\Omega_2)\to L_2(\Omega_2)} \le c\, n^{-t/m} \tag{30}$$

To prove (25), we use Banach-Steinhaus theorem : $\|I-Q_n\|_{L_2(\Omega_2)\to L_2(\Omega_2)} = 1$, but for $z=|A^*|^{t/l} w$ $(w \in L_2(\Omega_2))$ we have by (29) $\|(I-Q_n) z\| \to 0$ as $n\to\infty$; elements in form $|A^*|^{t/l} w$ generate dense subset in $\mathcal{N}(A)^\perp$.

If we have sequence $\{n_i\}$ with $n_{i+1}/n_i \le$ const $(i=1,2,\dots)$, then from estimate $\varkappa_n^* \le c\, n^{1/m}$ and (30) follows (26) with $\alpha=t/l$. So the assumptions of Lemma 4 are fulfilled and from there follows Theorem 3.

6. Applications (see [8 ,10] and references there). The operators, satisfying conditions of type (11),(23),(27) are for example integral operators with kernels of the following type: kernel of Green function type. Volterra kernel. difference kernel of special type. logarithmic kernel (defined on the contour), kernel of the operator of Radon transform.

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О САМОРЕГУЛЯРИЗАЦИИ ПРИ РЕШЕНИИ НЕКОРРЕКТНЫХ ЗАДАЧ ПРОЕКЦИОННЫМИ МЕТОДАМИ

У.Хямарик

Резюме

Само-регуляризация в проекционных методах для некорректных задач является ситуацией, когда при правильном выборе размерностей проекционных подпространств в соответствии с уровнями погрешностей η и δ оператора и правой части соответственно удается достигнуть того, что решение проекционного уравнения сходится к решению первоначального уравнения, если $\delta,\eta\to 0$.

В статье рассматриваються уравнения с операторами, являющимися гомеоморфизмами из одного пространства Соболева в другое. Для решения их применяются методы наименьших квадратов, Галеркина и наименьшей ошибки. В качестве проекционных подпространств используются в одномерном случае сплайны (любого порядка) или тригонометрические полиномы, в многомерном случае конечные элементы. Указаны правила выбора размерностей проекционных подпространств априорном способом и также по невязке, обеспечивающие саморегуляризацию. Приведены оценки погрешностей как в L_2-норме так и в норме Соболевских пространств, если решение достаточно гладкое. Даны оценки для чисел обусловленности матриц систем линейных алгебраических уравнении, возникающих в этих методах после выбора базиса проекционных подпространств.

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AN A APOSTERIORI CHOICE OF THE REGULARIZATION PARAMETER IN CASE OF APPROXIMATELY GIVEN ERROR BOUND OF DATA

Toomas Raus

In this paper a class of regularization methods for ill-posed problems is regarded. We consider the equation Au=f with self-adjoint operator A and approximately given right-hand side f. We suppose that the error bound is unknown and only some guess about the error is available. Under this condition we propose an a posteriori parameter choice and give the theorem of convergence. The error estimation of the approximate solution is deduced only in the case of properly determined error bound. Numerical examples confirm the theoretical results.

1. Introduction

Let H be a Hilbert space and $A \in \mathcal{L}(H,H)$, i. e. $A: H \to H$ is a continuous linear operator. Suppose that $A = A^* \geq 0$ and consider the equation

$$Au = f. \tag{1.1}$$

We do not suppose that range R(A) is closed and kernel N(A) is trivial. We assume that $f \in R(A)$, but only an approximation $\tilde{f} \in H$ to f is available. At that the level of the error of the right-hand term f is given approximately. It means that the supposed error level $\delta > 0$ is given, but we do not know exactly if $\|\tilde{f} - f\| \leq \delta$ or not.

To determine an approximation to the solution (1.1), we consider a class of regularization methods studied in [4]. Let $g_r: [0, a] \to \mathbb{R}$, $r \geq 0$, $\|A\| \leq a$, be Borel measurable functions. We assume that the functions g_r satisfy the following conditions:

$$\sup_{0 \leq \lambda \leq a} g_r(\lambda) \leq \gamma r, \quad r \geq 0, \tag{1.2}$$

$$\sup_{0 \leq \lambda \leq a} \lambda^p (1 - \lambda g_r(\lambda)) \leq \gamma_p r^{-p}, \ r > 0, \ 0 \leq p \leq p_0, \tag{1.3}$$

where p_0, γ and γ_p are positive constants. The greatest value p_0, for which the inequality (1.3) holds, is named a qualification of method (see [4]).

Let $u_0 \in H$ be an initial approximation and u_* the solution of equation (1.1), which is the nearest to u_0. We find the approximation u_r to u_* by the formula

$$u_r = (I - Ag_r(A))u_0 + g_r(A)\tilde{f}. \tag{1.4}$$

We denote

$$\beta_r(\lambda) := \begin{cases} 1, & \text{if } p_0 = \infty, \\ (1 - \lambda g_r(\lambda))^{1/p_0}, & \text{if } p_0 < \infty, \end{cases}$$

$$\tau(s) := \begin{cases} 1, & \text{if } p_0 = \infty, \\ 1 + (s+1)/p_0, & \text{if } p_0 < \infty, \end{cases}$$

$$\tilde{\gamma}_p^s := (\gamma_{p/\tau(s)})^{\tau(s)}$$

and suppose in this paper that in addition to inequalities (1.2) and (1.3) the functions g_r satisfy the conditions 1) – 5):

1) the function $r \to g_r(\lambda)$ is continuous for each $\lambda > 0$;

2) for $0 \le \lambda \le a$, $0 \le r_2 \le r_1$

$$0 \le 1 - \lambda g_{r_1}(\lambda) \le 1 - \lambda g_{r_2}(\lambda) \le 1; \tag{1.5}$$

3) for $0 \le \lambda \le a$, $r > 0$

$$\left.\frac{\partial(g_s(\lambda))}{\partial s}\right|_{s=r} \le \bar{\gamma}\gamma\beta_r(\lambda)(1 - \lambda g_r(\lambda)), \quad \bar{\gamma} = \text{const}; \tag{1.6}$$

4) for $0 \le r_2 < r_1$, $p \ge 0$

$$\sup_{0 \le \lambda \le a} (\lambda\beta_{r_1}(\lambda))^p (1 - \lambda g_{r_1}(\lambda))(1 - \lambda g_{r_2}(\lambda))^{-1} \le \tilde{\gamma}_p^{p-1}(r_1 - r_2)^{-p}; \tag{1.7}$$

5) for each $p > 0$ there exists a constant $R_0(p)$, so that, for $r \ge R_0(p)$ and ε, $0 < \varepsilon < \tilde{\gamma}_p^p$, we can find a constant $Q(\varepsilon, p)$ such that

$$(r\lambda)^p (\beta_r(\lambda))^{p+1}(1 - \lambda g_r(\lambda)) \le \varepsilon, \tag{1.8}$$

if $\lambda \ge Q(\varepsilon, p)/r$.

Some examples of regularization methods, for which (1.2),(1.3) and the conditions 1) – 5) are satisfied, are presented in Section 2. In Section 3 and 4 we propose two rules for a posteriori parameter choice and prove the convergence theorem. The error estimation of the approximate solution is deduced under the condition that our guess about error $\|\tilde{f} - f\|$ is true. Finally, in Section 5 some numerical examples are presented.

2. Examples of methods

1. The method of Lavrentiev. We determine the approximation u_r as the solution of the equation

$$(A + r^{-1}I)u = \tilde{f}.$$

The method is of the form (1.4) with $u_0 = 0$ and $g_r(\lambda) = (\lambda + r^{-1})^{-1}$. Conditions (1.2), (1.3), (1.6) and (1.8) hold with $p_0 = 1$, $\gamma = 1$, $\gamma_p = p^p(1 - p)^{1-p}$, $\bar{\gamma} = 1$, $R_0(p) = 0$, $Q(\varepsilon,p) = 1/\sqrt{\varepsilon} - 1$.

2. The iterative variant of the method of Lavrentiev. Let $m \in N$, $m \geq 1$. We determine consequtively the solutions $u_{1,r}, u_{2,r}, \ldots, u_{m,r}$ of the equations

$$r^{-1}u_{n,r} + Au_{n,r} = r^{-1}u_{n-1,r} + \tilde{f}, \quad n = 1,2,\ldots,m.$$

As the approximation u_r to the solution u_* we take an element $u_{m,r}$. The method is of the form (1.4) with $g_r(\lambda) = (1 - (1 + r\lambda)^{-m})/\lambda$. Conditions (1.2), (1.3), (1.6) and (1.8) hold with $p_0 = m$, $\gamma = m$, $\gamma_p = (p/m)^p(1 - p/m)^{m-p}$, $\bar{\gamma} = 1$, $R_0(p) = 0$, $Q(\varepsilon,p) = \varepsilon^{-1/(m+1)} - 1$.

3. The method of successive approximation (explicit scheme). Suppose that $0 < \mu < 1/\|A\|$ and compute consequtively

$$u_r = u_{r-1} - \mu(Au_{r-1} - \tilde{f}), \quad r = 1,2,\ldots .$$

The method is of the form (1.4) with $g_r(\lambda) = (1 - (1 - \mu\lambda)^r)/\lambda$. Conditions (1.2), (1.3), (1.6) and (1.8) hold with $p_0 = \infty$, $\gamma = \mu$, $\gamma_p = (p/\mu e)^p$, $\bar{\gamma} = 1/(1 - \mu\|A\|)$, $R_0(p) = 0$ and $Q(\varepsilon,p) = x_0$, where x_0 is the greatest solution of the equation $x^p e^{-\mu x} = \varepsilon$.

4. Implicit scheme. Let $\alpha > 0$ be a constant. We determine consequtively u_r as the solutions of the equations

$$\alpha u_r + Au_r = \alpha u_{r-1} + \tilde{f}, \quad r = 1,2,\ldots.$$

The method is of the form (1.4) with $g_r(\lambda) = (1 - (\alpha/(\alpha + \lambda))^r)/\lambda$. Conditions (1.2), (1.3), (1.6) and (1.8) hold with $p_0 = \infty$, $\gamma = 1/\alpha$, $\gamma_p = (\alpha p)^p$, $\bar{\gamma} = 1$, $R_0(p) = p + 1$ and $Q(\varepsilon,p) = x_0$, where x_0 is the greatest solution of the equation $x^p(1 + x/(\alpha(p + 1)))^{-(p+1)} = \varepsilon$.

5. The method of Cauchy problem (a continuous analog of iterative methods). We take an approximation u_r to the solution (1.1) the solution of Cauchy problem

$$u'(r) + Au(r) = \tilde{f}, \quad u(0) = u_0.$$

The method is of the form (1.4) with $g_r(\lambda) = (1 - e^{-r\lambda})/\lambda$. Conditions (1.2), (1.3), (1.6) and (1.8) hold with $p_0 = \infty$, $\gamma = 1$, $\gamma_p = (p/e)^p$, $\bar{\gamma} = 1$, $R_0(p) = 0$ and $Q(\varepsilon,p) = x_0$, where x_0 is the greatest solution of the equation $x^p e^{-x} = \varepsilon$.

In the case of the known error bound ($\|\tilde{f} - f\| \le \delta$) a discrepancy principle is well known and effective a posteriori parameter choice (see, e.g. [4 - 7]). But if $\|\tilde{f} - f\| > b_1\delta$ ($b_1 > 1$ – a constant used in the discrepancy principle; usually $b_1 \in [1.5, 2]$), then the choice of the parameter by discrepancy principle is unstable in this sense that the error of approximate solution may be arbitrarily great independently of the value of relation $\|\tilde{f} - f\|/\delta$. Therefore, such a choice of the parameter in the case of approximately given error bound is not reasonable.

In this Section we present two rules for stable parameter choice which guarantee the convergence of the approximate solution to exact solution if only relation $\|\tilde{f} - f\|/\delta$ is bounded in the process $\delta \to 0$.

Let us define operator B_r depending on qualification of method p_0, as follows:

$$B_r = \int_0^{\|A\|} \beta_r(\lambda)\,dP(\lambda) = \begin{cases} I, & \text{if } p_0 = \infty, \\ (I - Ag_r(A))^{1/p_0}, & \text{if } p_0 < \infty. \end{cases}$$

Let $k \ge 1/2$, $2k \in \mathbb{N}$ and denote

$$\varphi_k(r) := r^k \|A^k B_r^{k+1}(Au_r - \tilde{f})\|.$$

We consider the following two rules for the choice of the regularization parameter r.

Rule R1$_k$. Let b_1 and b_2 be the constants such that $b_2 \ge b_1 > \tilde{\gamma}_k^k$. If $\varphi_k(1) \le b_2\delta$, then choose $r(\delta) = 1$. In the contrary case choose $r = r(\delta) > 1$ such that

$$\varphi_k(r(\delta)) \le b_2\delta, \tag{3.1}$$

$$\varphi_k(r(\delta)) \ge b_1\delta, \qquad r \in [1, r(\delta)] \tag{3.2}$$

The next parameter selection rule may be applied to iteration procedures.

Rule R2$_k$. Let $b > \tilde{\gamma}_k^k$. We choose the least integer parameter $r = r(\delta) \ge 1$ such that

$$\varphi_k(r(\delta)) \le b\,\delta, \tag{3.3}$$

To show that our parameter selection rules are practicable, we shall prove some properties of functions $\varphi_k(r)$.

Lemma 3.1. Suppose that the condition (1.3) holds for functions g_r. Then for each $\tilde{f} \in H$ we have

$$\lim_{r \to \infty} \varphi_k(r) = 0 \qquad (k \ge 1/2,\ 2k \in \mathbb{N}). \tag{3.4}$$

Proof. Let P be an orthogonal projection in H onto $\overline{R(A)}$ and

$$G_r := I - Ag_r(A), \qquad D_r := A^k B_r^{k+1} G_r. \tag{3.5}$$

Then we have

$$\varphi_k(r) = r^k\|A^k B_r^{k+1} P(Au_r - \tilde{f})\|, \quad Au_r - \tilde{f} = G_r(Au_0 - \tilde{f}),$$

which implies that

$$\varphi_k(r) = r^k\|D_r P(Au_0 - \tilde{f})\|.$$

Applying the inequalities (1.3) and $k/\tau(k) < p_0$ we obtain

$$r^k\|D_r\| \le r^k \max_{0\le\lambda\le a} \lambda^k(1 - \lambda g_r(\lambda))^{\tau(k)} = \\ = r^k \max_{0\le\lambda\le a} [\lambda^{k/\tau(k)}(1 - \lambda g_r(\lambda))]^{\tau(k)} \le \tilde{\gamma}_k^k = \text{const} \tag{3.6}$$

and similarly

$$r^k\|AD_r\| \le \tilde{\gamma}_{k+1}^k r^{-1} \to 0 \quad (r \to \infty). \tag{3.7}$$

Since $P(Au_0 - \tilde{f}) \in \overline{R(A)}$, then using now the Banach-Steinhaus theorem, we get the convergence (3.4). □

Lemma 3.2. Let $\|\tilde{f} - f\| \le \delta$, $\|u_0 - u_*\| \le M$, $b > \tilde{\gamma}_k^k$. Suppose that the condition (1.3) holds for functions g_r. Then for each r, $r \ge R_{M,\delta} = \tilde{\gamma}_{k+1}^k M/((b - \tilde{\gamma}_k^k)\delta)$, we have

$$\varphi_k(r) \le b\delta. \tag{3.8}$$

Proof. We have (see (3.5))

$$A^k B_r^{k+1}(Au_r - \tilde{f}) = AD_r(u_0 - u_*) - D_r(\tilde{f} - f). \tag{3.9}$$

Using now (3.6),(3.7) and (3.9) we get the assertion of Lemma:

$$\varphi_k(r) \le r^k\|AD_r(u_0 - u_*)\| + r^k\|D_r(\tilde{f} - f)\| \le \\ \le \tilde{\gamma}_{k+1}^k M/r + \tilde{\gamma}_k^k\delta \le b\delta \qquad (r \ge R_{M,\delta}) . \; \square$$

From condition 1) it follows that the function $r \to \varphi_k(r)$ is continuous and then Lemma 3.1 yields that the choice of parameter $r(\delta)$ according to Rule R1$_k$ or Rule R2$_k$ is possible. If we know a constant $M > 0$ such that $\|u_0 - u_*\| \le M$, then it is sufficient to search the parameter $r(\delta)$ in the finite interval $[1, R_{M,\delta}]$. If $\varphi_k(r) > b_2\delta$ for $r \in [1, R_{M,\delta}]$, then from Lemma 3.2 it follows that $\|\tilde{f} - f\| \ge \delta$ and, consequently, instead of δ it is necessary to take some $\delta' > \delta$. Note that the functions $\varphi_k(r)$ are non-monotone and therefore in Rule R1$_k$ we must use the conditions (3.1) and (3.2) instead of inequalities $b_1\delta \le \varphi_k(r) \le b_2\delta$.

Remark 3.3. If $k = n + 1/2$, $n \in \mathbb{N}$, then we may compute the function $\varphi_k(r)$ by the formula

$$\varphi_k(r) = r^{n+1/2}\left([A^n B_r^{n+1}(Au_r - \tilde{f}), A^{n+1} B_r^{n+2}(Au_r - \tilde{f})]_H\right)^{1/2},$$

where $[\cdot,\cdot]_H$ is a scalar product.

4. Convergence and error estimation of the approximate solution

Theorem 4.1. Let $A \in \mathscr{L}(H,H)$, $\|A\| \le a$ and $f \in R(A)$. Suppose that conditions (1.2),(1.3),(1.5) and (1.8) hold for functions g_r and the parameter $r = r(\delta)$ is chosen according to Rule $R1_k$ or Rule $R2_k$. If in the process $\delta \to 0$

$$\|\tilde{f} - f\| \le C\delta, \qquad C = \text{const},$$

then

$$\|u_{r(\delta)} - u_*\| \to 0 \qquad (\delta \to 0). \qquad (4.1)$$

Proof. 1. At first we give some auxiliary results. We have (see (3.5))

$$u_r - u_* = G_r(u_0 - u_*) + g_r(A)(\tilde{f} - f), \qquad (4.2)$$

$$A^k B_r^{k+1}(Au_r - \tilde{f}) = AD_r(u_0 - u_*) - D_r(\tilde{f} - f). \qquad (4.3)$$

From (1.2) and (1.3) we obtain (see also (3.6))

$$\|g_r(A)(\tilde{f} - f)\| \le \gamma r \|\tilde{f} - f\| \le \gamma C r \delta, \qquad (4.4)$$

$$\|D_r\| \le \tilde{\gamma}_k^k r^{-k}. \qquad (4.5)$$

If the parameter $r = r(\delta)$ is chosen according to Rule $R1_k$ or Rule $R2_k$, then from (3.1) - (3.3), (4.3) and (4.5) it follows that

$$\begin{aligned}(r(\delta))^k \|AD_{r(\delta)}(u_0 - u_*)\| &\le (r(\delta))^k \|A^k B_{r(\delta)}^{k+1}(Au_{r(\delta)} - \tilde{f})\| + \\ &+ (r(\delta))^k \|D_{r(\delta)}(\tilde{f} - f)\| \le (b_2 + \tilde{\gamma}_k^k C)\delta,\end{aligned} \qquad (4.6)$$

where $b_2 = b$ in case of Rule $R2_k$. From (4.2) and (4.4) we get

$$\|u_{r(\delta)} - u_*\| \le \|G_{r(\delta)}(u_0 - u_*)\| + \gamma r(\delta) C \delta. \qquad (4.7)$$

To prove the theorem, it suffices to show the convergence of the right-hand side of (4.7).

2. First we show the convergence of the first term of (4.7). From (4.6) we get (as $r(\delta) \ge 1$)

$$\|AD_{r(\delta)}(u_0 - u_*)\| \to 0 \qquad (\delta \to 0).$$

In [4] (p.43,66) is proved that if $AG_{r_n}(u_0 - u_*) \to 0$ $(n \to \infty)$, then $G_{r_n}(u_0 - u_*) \to 0$. Similarly we can show that if $AD_{r_n}(u_0 - u_*) = A^{k+1} B_{r_n}^{k+1} G_{r_n}(u_0 - u_*) \to 0$ $(n \to \infty)$ then $G_{r_n}(u_0 - u_*) \to 0$. Hence, $\|G_{r(\delta)}(u_0 - u_*)\| \to 0$ $(\delta \to 0)$ and the convergence of the first term of (4.7) is proved.

3. To show the convergence of the second term of (4.7), we prove at first the convergence

$$r_\delta \, \delta \to 0 \qquad (\delta \to 0), \qquad (4.8)$$

where r_δ is the greatest parameter, for which

$$r_\delta^k \|AD_{r_\delta}(u_0 - u_*)\| = (b_1 - \tilde{\gamma}_k^k)\delta \qquad (4.9)$$

(in case of Rule $R2_k$ $b_1 = b$). If $r_\delta \le \bar{r} = \text{const}$ in the process $\delta \to 0$, then (4.8) is obvious. If $r_\delta \to \infty$ $(\delta \to 0)$, then applying the Banach-Steinhaus theorem we can prove similarly as in Lemma 3.1 that

$$r_\delta^{k+1} \|AD_{r_\delta}(u_0 - u_*)\| \to 0 \qquad (r_\delta \to \infty) \qquad (4.10)$$

and then the convergence (4.8) follows from (4.9) and (4.10).

To complete the proof, we will show that

$$r(\delta) - q \le d_{k,c} r_0 , \tag{4.11}$$

where

$$r_0 = \max\{r_\delta, 1, R_0(k)\}, \quad d_{k,c} = (Q_0/\varepsilon_0)^m = \mathrm{const},$$

$$m = \mathrm{entier}(3C^2 + 1), \quad \varepsilon_0 = (\tilde{\gamma}_k^k/\sqrt{3}\, C)^{1/k}, \quad Q_0 = \max\{Q(\varepsilon_0^k, k), \varepsilon_0 + 1\}.$$

Here the constants $Q(\varepsilon,k)$ and $R_0(k)$ are defined by condition 5), $q = 0$ in case of Rule $R1_k$ and $q = 1$ in case of Rule $R2_k$. As $d_{k,c} > 1$ then in case of $r(\delta) - q \le r_0$ the inequality (4.11) holds. In the following we consider the case $r(\delta) - q > r_0$.

If the parameter $r = r(\delta)$ is chosen according to Rule $R1_k$ or Rule $R2_k$, then

$$r^k \| A^k B_r^{k+1} (Au_r - \tilde{f}) \| \ge b_1 \delta, \quad 1 \le r \le r(\delta) - q. \tag{4.12}$$

As r_δ is the greatest parameter, for which (4.9) holds, then we have for r, $r \ge r_0 \ge r_\delta$, that

$$r^k \| AD_r(u_0 - u_*) \| \le (b_1 - \tilde{\gamma}_k^k)\delta. \tag{4.13}$$

Now applying (4.3),(4.12) and (4.13) we obtain for r, $r_0 \le r \le r(\delta) - q$, that

$$r^k \| D_r(\tilde{f} - f) \| \ge r^k \| A^k B_r^{k+1} (Au_r - \tilde{f}) \| - r^k \| AD_r(u_0 - u_*) \| \ge \tilde{\gamma}_k^k \delta. \tag{4.14}$$

Further we give the upper bound for $r^k \| D_r(\tilde{f} - f) \|$. From (1.3), (1.5) and (1.8) we get that

$$T(r,\lambda) := (r\lambda)^k (\beta_r(\lambda))^{k+1} (1 - \lambda g_r(\lambda)) \le \varepsilon_0^k \quad (\lambda \le \varepsilon_0/r),$$

$$T(r,\lambda) \le \varepsilon_0^k \quad (\lambda \ge Q_0/r),$$

$$\sup_{0 \le \lambda \le a} T(r,\lambda) \le \tilde{\gamma}_k^k .$$

Then we can estimate for $r \ge r_0$

$$\begin{aligned} (r^k \| D_r(\tilde{f} - f) \|)^2 &\le 2\varepsilon_0^{2k} \| \tilde{f} - f \|^2 + \int_{\varepsilon_0/r}^{Q_0/r} T^2(r,\lambda)\, d\langle P(\lambda)(\tilde{f} - f), \tilde{f} - f \rangle \le \\ &\le 2(\tilde{\gamma}_k^k)^2 \delta^2/3 + (\tilde{\gamma}_k^k)^2 \| (P(Q_0/r) - P(\varepsilon_0/r))(\tilde{f} - f) \|^2, \end{aligned} \tag{4.15}$$

where $P(\lambda)$ is the spectral family of the projectors of operator A. Now from (4.14) and (4.15) it follows that

$$\| (P(Q_0/r) - P(\varepsilon_0/r))(\tilde{f} - f) \| \ge \delta/\sqrt{3}, \quad r_0 \le r \le r(\delta) - q. \tag{4.16}$$

Let $r_j := r_0 (Q_0/\varepsilon_0)^j$, $j = 0,1,2,\dots,m$. Then $\varepsilon_0/r_j = Q_0/r_{j+1}$, $j = 0,1,\dots,m-1$ and hence,

$$(P(Q_0/r_j)H \ominus P(\varepsilon_0/r_j)H) \cap (P(Q_0/r_i)H \ominus P(\varepsilon_0/r_i)H) = \emptyset, \quad i \ne j. \tag{4.17}$$

If we now suppose in contradiction to (4.11) that $r(\delta) - q > d_{k,c} r_0$, then with regard (4.16),(4.17) we have

$$\begin{aligned} \| \tilde{f} - f \|^2 &\ge \sum_{j=0}^{m} \| (P(Q_0/r_j) - P(\varepsilon_0/r_j))(\tilde{f} - f) \|^2 \ge \\ &\ge (m+1)\delta^2/3 = (1 + \mathrm{entier}(3C^2 + 1))\delta^2/3 > C^2\delta^2; \end{aligned} \tag{4.18}$$

what contradicts to the assumption of the theorem. Hence (4.11) holds, which together with (4.8) and (4.12) proves the convergence of the second term of (4.7). □

In the next theorem we give an estimate for the error of the approximate solution in case, if our supposition about error $\|\tilde{f} - f\|$ is found to be true. The error of the approximate solution is estimated by the least error $M(\delta)$ of the method (see [7]), where

$$M(\delta) = \sup_{\tilde{f},\ \|\tilde{f}-f\| \le \delta} \ \inf_{r,\ r \ge 0} \|\bar{u}_r - u_*\| \tag{4.19}$$

and

$$\bar{u}_r = (I - Ag_r(A))u_0 + g_r(A)\tilde{f}.$$

Theorem 4.2. Let $A \in \mathscr{L}(H,H)$, $\|A\| \le a$, $f \in R(A)$ and $\|\tilde{f} - f\| \le \delta$. Suppose that conditions (1.2),(1.3),(1.5)-(1.8) hold for functions g_r and parameter $r = r(\delta)$ is chosen according to Rule $R1_k$ or Rule $R2_k$. Then

$$\|u_{r(\delta)} - u_*\| \le C_k(b_1,b_*)\ M(\delta) + \gamma\delta, \tag{4.20}$$

where

$$C_k(b_1,b_*) = \max\{c_{b_1}, c_{b_*}\}, \tag{4.21}$$

$$c_{b_1} = \max_{x,\ 0 \le x \le x_0} \left[(a_1(1 - x)^{k+1} + 1)/(a_1^2(1 - x)^{2k+2} + x^2)^{1/2}\right], \tag{4.22}$$

x_0 is the greatest solution of the equation $x^2 = a_1(1 - x)^{k+1}$,

$$a_1 = (b_1 - \tilde{\gamma}_k^k)/(\tilde{\gamma}_{k+1}^k\gamma), \tag{4.23}$$

$$c_{b_*} = \max_{x,\ x \ge 0} \left[((x^{1/(k+1)} + a_2)^{k+1} + 1)/(x^2 + 1)^{1/2}\right], \tag{4.24}$$

$$a_2 = \bar{\gamma}(\gamma^k(b_* + \tilde{\gamma}_k^k))^{1/(k+1)} \tag{4.25}$$

and

$$b_* = \begin{cases} b_2\ , & \text{if } r(\delta) \ge R(\delta), \\ \max\limits_{r(\delta) \le r \le R(\delta)} \varphi_k(r)/\delta, & \text{if } r(\delta) < R(\delta), \end{cases} \tag{4.26}$$

where $R(\delta)$ is the greatest parameter, for which $\varphi_k(r) = b_2\delta$.

Remark 4.3. For c_{b_1} and c_{b_*} hold the upper bounds

$$c_{b_1} \le \begin{cases} \sqrt{2}\,(1 + 1/a_1), & \text{if } a_1 < 1/k, \\ \sqrt{2}\,(1 + 1/k)/(1 + 1/k - (ka_1)^{-1/(k+1)}), & \text{if } a_1 \ge 1/k, \end{cases}$$

$$c_{b_*} \le 1 + a_2(1 + a_2^{-2(k+1)/(2k+1)})^{k+1/2}.$$

It is easy to show that $\gamma\delta \le M(\delta)$ if $\|u_0 - u_*\| \ge \gamma\delta$. Now we can say by estimate (4.20)-(4.26) that the Rules $R1_k$ and $R2_k$ give the quazioptimal choices of the parameter if $\|u_0 - u_*\| \ge \gamma\delta$ and $\|\tilde{f} - f\| \le \delta$. But the coefficient of the quazioptimum $C_k(b_1,b_*)$ depends on function $\varphi_k(r)$ and may be computed separately for every problem. Nevertheless, the numerical examples show (see Section 5) that in most cases $r(\delta) \geq R(\delta)$ and $b_* = b_2$. In Table 1 the least values of the coefficients $C_{1/2}(b_1,b_*)$, $C_1(b_1,b_*)$ and corresponding value $b_{\min} = b_1 = b_*$ are presented for methods 1 - 5 (see Section 2).

Table 1.

	$C_{1/2}(b_1,b_*)$	$b_{\min}$	$C_1(b_1,b_*)$	$b_{\min}$
Method 1	2.50	0.46	2.67	0.29
Method 2 (m=2)	2.65	0.39	2.96	0.20
Method 3($\mu\|A\|=0.1$)	3.09	$1.56\,\tilde{\gamma}_k^k$	3.86	$1.62\tilde{\gamma}_k^k$
Method 4	3.95	$1.97\,\tilde{\gamma}_k^k$	6.06	$1.86\tilde{\gamma}_k^k$
Method 5	2.93	0.70	3.58	0.63

To prove Theorem 4.2 we need the following lemma.

Lemma 4.4. Let $c > 0$, $v \in H$ and parameter r_δ such that

$$r^k \|AD_r v\| \leq c\delta \quad \text{for each } r \geq r_\delta. \tag{4.27}$$

Then

$$r_\delta \geq r_0 ,$$

where r_0 is the parameter for which the function

$$\varrho(r) = \|G_r v\|^{1/(k+1)} + \gamma\bar{\gamma}(cr\delta)^{1/(k+1)}$$

has a global minimum.

Proof. If r_0 is the global minimum point of the function $\varrho(r)$, then

$$\begin{aligned} \varrho'(r_0) = (2k+2)^{-1}(\|G_{r_0}v\|^2)^{-(2k+1)/(2k+2)} \frac{\partial}{\partial r}(\|G_r v\|^2)\Big|_{r=r_0} + \\ + (\gamma\bar{\gamma}/(k+1))(c\delta)^{1/(k+1)}\, r_0^{-k/(k+1)} = 0. \end{aligned} \tag{4.28}$$

Using (1.6) and inequality

$$\frac{\partial((1-\lambda g_r(\lambda))^2)}{\partial r} = -2\lambda(1-\lambda g_r(\lambda))\,\frac{\partial(g_r(\lambda))}{\partial r}$$

we obtain

$$-\frac{\partial}{\partial r}(\|G_r v\|^2)\Big|_{r=r_0} \leq 2\gamma\bar{\gamma}\,\|B_{r_0}^{1/2}A^{1/2}G_{r_0}v\|^2. \tag{4.29}$$

Applying the inequality of the moments ($\|B^p v\| \leq \|B^q v\|^{p/q}\|v\|^{1-p/q}$, $0 < p \leq q$) we have

$$\|B_{r_0}^{1/2}A^{1/2}G_{r_0}v\|^2 \leq \|AD_{r_0}v\|^{1/(k+1)}\,\|G_{r_0}v\|^{(2k+1)/(2k+2)}. \tag{4.30}$$

Now using (4.28)-(4.30) we get

$$\begin{aligned} c\delta = \Big(-(2\gamma\bar{\gamma})^{-1}\frac{\partial}{\partial r}(\|G_r v\|^2)\Big|_{r=r_0}\Big)^{k+1}\,\|G_{r_0}v\|^{-(2k+1)}\,r_0^k \leq \\ \leq \|B_{r_0}^{1/2}A^{1/2}G_{r_0}v\|^{2(k+1)}\,\|G_{r_0}v\|^{-(2k+1)}\,r_0^k \leq r_0^k\,\|AD_{r_0}v\|, \end{aligned}$$

from which together with (4.27) follows the assertion of Lemma. □

Proof of Theorem 4.2. Similarly as in the proof of Theorem 4.1 we have

$$\|u_{r(\delta)} - u\ \| \leq \|G_{r(\delta)}(u_0 - u\ )\| + \gamma r(\delta)\delta, \tag{4.31}$$

$$r^k \| D_r(\tilde{f} - f) \| \le \tilde{\gamma}_k^k \delta, \tag{4.32}$$

$$A^k B_r^{k+1}(Au_r - \tilde{f}) = AD_r(u_0 - u_*) - D_r(\tilde{f} - f). \tag{4.33}$$

If the parameter $r(\delta)$ is chosen according to Rule $R1_k$ or Rule $R2_k$, then using (3.1) - (3.3), (4.26), (4.32) and (4.33) we get for $r \ge r(\delta)$

$$\begin{aligned} r^k \| AD_r(u_0 - u_*) \| &\le r^k \| A^k B_r^{k+1}(Au_r - \tilde{f}) \| + \\ &+ r^k \| D_r(\tilde{f} - f) \| \le (b_* + \tilde{\gamma}_k^k)\delta \end{aligned} \tag{4.34}$$

and for $r(\delta) > 1$

$$\begin{aligned} (r')^k \| AD_{r'}(u_0 - u_*) \| &\ge (r')^k \| A^k B_{r'}^{k+1}(Au_{r'} - \tilde{f}) \| - \\ &- (r')^k \| D_{r'}(\tilde{f} - f) \| \ge (b_1 - \tilde{\gamma}_k^k)\delta, \end{aligned} \tag{4.35}$$

where $r' = r(\delta)$ in case of Rule $R1_k$ and $r' = r(\delta) - 1$ in case of Rule $R2_k$.

Let r_* be a parameter for which the function

$$\omega(r) = (\| G_r(u_0 - u_*) \|^2 + (\gamma r \delta)^2)^{1/2} \tag{4.36}$$

has a global minimum. In [7] is proved that

$$\omega(r_*) \le M(\delta). \tag{4.37}$$

To prove the estimation (4.20) we consider separately three cases: 1) $r_* < r'$, 2) $r_* \ge r(\delta)$ and 3) $r' \le r_* < r(\delta)$.

1. Case $r_* < r'$. As the function $r \to \| G_r(u_0 - u_*) \|$ is a decreasing function (see condition 2)), then

$$\| G_{r(\delta)}(u_0 - u_*) \| \le \| G_{r_*}(u_0 - u_*) \|. \tag{4.38}$$

If $r(\delta) \le 1$, then from (4.31), (4.36)-(4.38) it follows that

$$\| u_{r(\delta)} - u_* \| \le \| G_{r_*}(u_0 - u_*) \| + \gamma\delta \le M(\delta) + \gamma\delta$$

and (4.20) holds.

If $r(\delta) > 1$, then we similarly have

$$\| u_{r(\delta)} - u_* \| \le T\, M(\delta) + \gamma\delta, \tag{4.39}$$

where

$$T = \frac{\| G_{r_*}(u_0 - u_*) \| + \gamma r' \delta}{(\| G_{r_*}(u_0 - u_*) \|^2 + (\gamma r_* \delta)^2)^{1/2}} \,. \tag{4.40}$$

To estimate the value of T we show at first that for $r(\delta) > 1$

$$\| G_{r_*}(u_0 - u_*) \| \ge a_1 \gamma (r' - r_*)^{k+1} (r')^{-k} \delta, \tag{4.41}$$

where the constant a_1 is defined by formula (4.23). Indeed, using (1.7) we get

$$\begin{aligned} \| AD_{r'}(u_0 - u_*) \| &\le \| AD_{r'} G_{r_*}^{-1} \| \| G_{r_*}(u_0 - u_*) \| \le \\ &\le \tilde{\gamma}_{k+1}^k (r' - r_*)^{-(k+1)} \| G_{r_*}(u_0 - u_*) \|, \end{aligned}$$

from which with regard (4.35) follows inequality (4.41).

Now we can estimate

$$T \le \max_{r_*,\, r_* < r'} \; \max_{y,\, y \ge \gamma a_1 (r' - r_*)^{k+1} (r')^{-k} \delta} (y + \gamma r' \delta)/(y^2 + (\gamma r_* \delta)^2)^{1/2} =$$

$$= \max_{x,\, 0 \le x \le 1} \ \max_{z,\, z \ge a_1(1-x)^{k+1}} (z+1)/(z^2+x^2)^{1/2} \quad .$$

The function $W(z) := (z+1)/(z^2+x^2)^{1/2}$, $z \in [0, a_1(1-x)^{k+1}]$, has a maximum at the point $z_{max} = \max[x^2, a_1(1-x)^{k+1}]$. It is easy to show that

$$\max_{x,\, x \ge x_0} W(x^2) \le \max_{x,\, 0 \le x \le x_0} W(a_1(1-x)^{k+1})$$

and hence,

$$T \le \max_{x,\, 0 \le x \le x_0} (a_1(1-x)^{k+1}+1)/(a_1^2(1-x)^{2k+2}+x^2)^{1/2}. \tag{4.42}$$

Now estimate (4.20) follows from (4.39) and (4.42).

2. Case $r_* \ge r(\delta)$. Using the Lemma 4.4 and inequality (4.34) we get

$$r(\delta) \ge r_0 , \tag{4.43}$$

where r_0 is the global minimum point of the function

$$\varrho(r) = \|G_r(u_0 - u_*)\|^{1/(k+1)} + \bar{\gamma}\gamma((b_* + \tilde{\gamma}_k^k)r\delta)^{1/(k+1)}.$$

From inequalities (1.5), (4.43) and $\varrho(r_0) \le \varrho(r_*)$ it follows that

$$\begin{aligned} \|G_{r(\delta)}(u_0 - u_*)\| &\le \|G_{r_0}(u_0 - u_*)\| \le \\ &\le (\|G_{r_*}(u_0 - u_*)\|^{1/(k+1)} + a_2(\gamma r_* \delta)^{1/(k+1)})^{k+1}, \end{aligned} \tag{4.44}$$

where the constant a_2 is defined by (4.25).

Now using (4.31), (4.36), (4.37) and (4.44) we get

$$\|u_{r(\delta)} - u_*\| \le \|G_{r(\delta)}(u_0 - u_*)\| + \gamma r_* \delta \le T\, M(\delta),$$

where

$$T \le \frac{(\|G_{r_*}(u_0 - u_*)\|^{1/(k+1)} + a_2(\gamma r_* \delta)^{1/(k+1)})^{k+1} + \gamma r_* \delta}{(\|G_{r_*}(u_0 - u_*)\|^2 + (\gamma r_* \delta)^2)^{1/2}} \le$$

$$\le \max_{x,\, x \ge 0} \left[((x^{1/(k+1)} + a_2)^{k+1} + 1)/(x^2+1)^{1/2}\right],$$

and (4.20) holds in case of $r_* \ge r(\delta)$ also.

3. Case $r' \le r_* < r(\delta)$. From (1.5), (4.31), (4.37) we get

$$\begin{aligned} \|u_{r(\delta)} - u_*\| &\le \|G_{r(\delta)}(u_0 - u_*)\| + \gamma r(\delta)\delta \le \\ &\le \|G_{r_*}(u_0 - u_*)\| + \gamma r_* \delta + \gamma\delta \le \sqrt{2}\, M(\delta) + \gamma\delta. \end{aligned}$$

As $\sqrt{2} \le \min_{b_1, b_*,\, b_1 \le b_*} C_k(b_1, b_*)$, then the estimate (4.20) holds. □

5. Numerical examples.

The following Fredholm integral equations of the first kind

$$\int_a^b \mathcal{K}(t,s)u(s)ds = f(t), \quad a \le t \le b, \tag{5.1}$$

with $\mathcal{K} \in L^2([a,b],[a,b])$, $u \in L^2[a,b]$ were solved by Lavrentiev method using the choice of the parameter by Rule $R1_k$, $k = 1$.

<u>Example 1</u> (see [2]). Kernel $\mathcal{K}(t,s) = (t+s)/2 + ts + 1/3$, exact

solution $u(s) = 1$, right-hand term $f(t) = t + 7/12$, $a = 0$, $b = 1$.

Example 2 ([8]).

$\mathcal{K}(t,s) = 1/(\pi((s-t)^2 + 1))$, $u(s) = (1 - s^2)^2$, $a = -1$, $b = 1$.

Example 3 ([3]).

$$\mathcal{K}(t,s) = \begin{cases} \pi^2 t(1-s), & \text{if } t \le s, \\ \pi^2 s(1-t), & \text{if } t > s, \end{cases}$$

$u(s) = 1$, $a = 0$, $b = 1$.

Example 4 ([1]).

$\mathcal{K}(t,s) = \exp(ts)$, $u(s) = 1$, $f(t) = (\exp(t) - 1)/t$, $a = 0$, $b = 1$.

Example 5 ([1]).

$\mathcal{K}(t,s) = ts$, $u(s) = s/2$, $f(t) = t/6$, $a = 0$, $b = 1$.

Example 6 ([1]).

$$\mathcal{K}(t,s) = \begin{cases} t(1-s), & \text{if } t \le s, \\ s(1-t), & \text{if } t > s, \end{cases}$$

$u(s) = s - 2s^3 + s^4$, $f(t) = (3t - 5t^3 + 3t^5 - t^6)/30$, $a = 0$, $b = 1$.

Example 7 ([1]).

$$\mathcal{K}(t,s) = \begin{cases} t(1-s)(2s - s^2 - t^2), & \text{if } t \le s, \\ s(1-t)(2s - s^2 - t^2), & \text{if } t > s, \end{cases}$$

$u(s) = 1$, $f(t) = (t - 2t^3 + t^4)$, $a = 0$, $b = 1$.

After discretization of equation (5.1) we get

$$h \sum_{j=1}^{n} K_{ij} u_j = f_i, \quad i = 1,2,\dots,n, \tag{5.2}$$

where $K_{ij} = \mathcal{K}(t_i, s_j)$, $u_j = u(s_j)$, $h = 1/n$, $s_i = t_i = h/2 + (i-1)h$, $i = 1,2,\dots,n$. In case of the given right-hand side of (5.1) we take $f_i = f(t_i)$; in the contrary case the numbers f_i, $i = 1,2,\dots,n$ are computed by formula (5.2). For obtaining the approximate right-hand side $\tilde{f} = \{\tilde{f}_i\}_1^n$ the vector $f = \{f_i\}_1^n$ was randomly perturbed so that $\|\tilde{f} - f\|_{R^n} = \delta_0$. A norm in the space R^n is defined by formula

$$\|v\|_{R^n} = \Big(h \sum_{i=1}^{n} v_i^2\Big)^{1/2}.$$

The approximate solution $\tilde{u}_r = \{\tilde{u}_i\}_1^n$ was computed as the solution of linear system

$$(r\mathcal{J} + \mathfrak{K})\tilde{u}_r = \tilde{f},$$

where matrix $\mathfrak{K} = (hK_{ij})$ and $\mathcal{J}$ is the unique matrix. Choosing the parameter $r(\delta)$ by Rule $R1_k$, $k = 1$, we use the function

$$\varphi_1(r) = r\|\mathfrak{K}(r\mathcal{J} + \mathfrak{K})^{-2}(\mathfrak{K}\tilde{u}_r - \tilde{f})\|_{R^n}.$$

For all examples we took the supposed error $\delta = 0.3 * 10^{-3}\|\tilde{f}\|_{R^n}$, $n = 20$ and $b_1 = 0.75$, $b_2 = 0.80$ (the numerical examples show that it is recommendable to take the constants b_1 and b_2 greater than b_{min} in Table 1). The actual error δ_0 of the right-hand side was $\delta_0 = \delta$, $\delta_0 = 3\delta$, $\delta_0 = 5\delta$, $\delta_0 = 10\delta$ and $\delta_0 = 15\delta$.

In case of $\delta = \delta_0$ we compared the Rule $R1_k$, $k = 1$, with a modification of discrepancy principle (see [7]). For this choice the regularization parameter r_δ is determined by the conditions

$$b_1' \delta \le \|B_r(Au_r - \tilde{f})\| \le b_2' \delta,$$

where $b_1' > 1$ and $b_2' \ge b_1'$. For our examples we took $b_1' = 1.35$ and $b_2' = 1.40$.

The results of the calculations are given in Table 2. There

$$q = \|\tilde{u}_{r(\delta)} - u_*\|_{R^n} / \|u_*\|_{R^n}, \quad u_* = \{u_j\}_1^n$$

and

$$C_* = \|\tilde{u}_{r(\delta)} - u_*\|_{R^n} / \omega(\delta_0),$$

$$\omega(\delta_0) = \inf_{r \ge 0} (\|(J + r\mathfrak{A})^{-1} u_*\|_{R^n}^2 + (\gamma r \delta_0)^2)^{1/2}.$$

As $\omega(\delta_0) \le M(\delta_0) \le \sqrt{2}\omega(\delta_0)$ (see [7]), then

$$C_* M(\delta_0)/\sqrt{2} \le \|\tilde{u}_{r(\delta)} - u_*\|_{R^n} \le C_* M(\delta_0)$$

and the numbers C_* characterizes the relation of the error of the approximate solution to the least error of method $M(\delta_0)$.

Table 2.

	$\delta_0 = \delta$, $R1_k$, k=1			$\delta_0 = \delta$, Discr. principle		
Ex.	q	C_*	$r(\delta)$	q	C_*	r_δ
1	0.161	1.21	96	0.124	0.93	208
2	0.029	0.86	88	0.031	0.92	72
3	0.081	0.83	192	0.090	0.93	164
4	0.032	0.89	64	0.032	0.90	58
5	0.024	1.00	192	0.024	0.98	176
6	0.020	0.82	640	0.023	0.96	488
7	0.282	0.99	3580	0.279	0.98	3840
	$\delta_0 = 3\delta$, $R1_k$, k=1			$\delta_0 = 5\delta$, $R1_k$, k=1		
Ex.	q	C_*	$r(\delta)$	q	C_*	$r(\delta)$
1	0.183	0.96	96	0.233	1.08	112
2	0.058	1.00	88	0.087	1.18	92
3	0.095	0.64	182	0.082	0.46	216
4	0.067	1.15	64	0.115	1.56	64
5	0.053	1.28	192	0.089	1.66	192
6	0.033	0.78	656	0.047	0.87	704
7	0.307	0.94	3710	0.359	1.03	4610
	$\delta_0 = 10\delta$, $R1_k$, k=1			$\delta_0 = 15\delta$, $R1_k$, k=1		
Ex.	q	C_*	$r(\delta)$	q	C_*	$r(\delta)$
1	0.629	2.60	192	0.676	2.66	144
2	0.229	2.26	104	0.745	6.10	192
3	0.203	0.93	288	0.349	1.43	352
4	0.250	2.51	68	0.385	3.26	64
5	0.192	2.55	192	0.288	3.14	192
6	0.199	2.64	2050	0.374	4.09	3200
7	0.800	2.11	5120	6.122	15.82	82000

On the bases of the results of Table 2 we may draw the following conclusions. In case of $\delta = \delta_0$ the choice of the parameter by Rule $R1_k$, $k = 1$, give approximately the same result as the modification of the discrepancy principle. If $\delta_0/10 \le \delta \le \delta_0$, then the actual error of the right-hand term of the equation has a relatively small effect on the choice of the parameter. For that reason the error of the approximate solution increases relatively slowly by the enlargement of the relation $\|\tilde{f} - f\|/\delta$ $(C_* \le (\|\tilde{f} - f\|/\delta)^{1/2})$. If $\delta_0 > 10\delta$, then the parameter of regularization and the error $\|u_{r(\delta)} - u_*\|$ may be in some cases spasmodically increase . To sum up, we can say that in case of the known order of the level of the error $\|\tilde{f} - f\|$ the choice of the parameter by Rule $R1_k$, $k = 1$, gave good results.

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АПОСТЕРИОРНЫЙ ВЫБОР ПАРАМЕТРА РЕГУЛЯРИЗАЦИИ В СЛУЧАЕ ПРИБЛИЖЕННО ЗАДАННОГО УРОВНЯ ПОГРЕШНОСТИ ИСХОДНЫХ ДАННЫХ

Т. Раус

Резюме

В статье рассматривается уравнение $Au = f$ в гильбертовом пространстве H. Предполагается, что оператор $A = A^* \geq 0$ задан точно, а вместо $f \in R(A)$ задано некоторое $\tilde{f} \in H$. Также предполагается, что нам известна некоторая предполагаемая уровень ошибки δ правой части уравнения, но мы не знаем, действительно ли $\|\tilde{f} - f\| \leq \delta$ или нет. При таких условиях даются для класса регуляризационных методов (1.4) правила выбопа параметра регуляризации. Доказывается, что приближенное решение сходится к точной, если только отношение $\|\tilde{f} - f\|/\delta$ остается ограниченным в процессе $\delta \to 0$. В частном случае $\|\tilde{f} - f\| \leq \delta$ погрешность приближенного решения оценивается через наименьшую погрешность метода $M(\delta)$ (см.(4.19)). В конце статьи приведены результаты вычислительных экспериментов.

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