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TENSOR REPRESENTATIONS OF LINEAR GROUP $GL(2,\mathbb{R})$

Master Thesis

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Contents

Introduction

In this research the vector fields on the plane, flows of the vector fields and the transformations of tensor fields (including functions, vector fields and differential forms) in the flow are studied. In the part I we deal with Lie differentiating, derive necessary formulas and speak about derivation formulae in the case of natural basis. With vector field X associates one-form ω that annuls on the vector field X, i.e. $\omega(X) = 0$. If one-form is exact then it is locally differential of the invariant of vector field X: $\omega = dI$. Otherwise we multiply one-form with integrating factor f. In this case one-form $f\omega$ is exact and invariant is therefore found, $f\omega = dI$.

In the part II we deal with linear vector fields on the plane. In the case of linear vector field the flow is determined with exponential law

$$U' = CU \Rightarrow U_t = e^{Ct}U.$$

$U' = CU$ is system of ordinary linear differential equations (dynamic system) that corresponds to the vector field, $U_t = e^{Ct}U$ is general solution of the system, C is a constant matrix, U is fixed point on the plane and e^{Ct} is exponential of the matrix Ct.

Linear vector field is an operator of the linear group $GL(2, \mathbb{R})$. Four operator can be as basis operators. Then all other operators are their linear combinations with constant coefficients. Dilation operator commutate with other operators and therefore it is their infinitesimal symmetry. Linear vector fields can be classify as elliptical, hyperbolic and parabolic. Dilatation group influences these vector fields, i.e. dilatation or contraction of the plane takes place. Classification table is given on page 22, which is analogical with table in the book [2], page 68, but it is much more precise. It can be said that equi-affine transformations on the plane form a factor group of the group of centro-affine transformations with respect to the dilation subgroup.

For a linear vector field is characteristic that the straight lines in its flow remain straight and the parallelism of lines remain.

The main property of linear flow is the phase portrait of linear flow repeats itself in the moving frame if its initial point moves along some trajectory. In the non-linear flow it is not so. Non-linear flow can be linearized if we fix in some point Jacob matrix, which is composed of the partial derivatives of the vector field X components. In the case of linear flow the Jacob matrix is constant matrix C but in the case of non-linear flow the Jacob matrix is different at every point and therefore "linear appraisal" changes if we move from point to point.

Part III concentrates to the main goal of the paper which is to study tensor fields in the linear flow. Generally the tensor field is determined with components in the (non-holonomic) basis, i.e. it is determined in the reper and in the dual coreper. Therefore in the given flow the components of the tensor field and also the basis changes. It is possible to describe the change of the tensor field if we know how the basis changes. For that we have to know the derivation formulae of the reper and coreper. In the natural basis are derivation formulae determined with Jacob matrix of the vector field. In the case of linear vector field with constant matrix C.

If the flow is linear and the Jacob matrix is constant matrix C then reper and coreper change according to the exponential law. In this case change of tensor field is much easier to describe. It can be distinguished two main cases:

1) Components of the tensor field in the given flow do not change, i.e. they are constant on the trajectories of the vector field X. Change of the tensor field is completely determined with transformation of the basis. With help of the derivation formulae we calculate Lie derivations of the tensor field. It appears that Lie derivations of the tensor field are related linearly. Reveals ODE that solution determines change of the tensor field in the given flow.

2) Tensor field itself is invariant in the given flow. Obviously in this case the components of the tensor field are transformed. We are talking about the action of the linear group $GL(2,\mathbb{R})$ in the space $\mathbb{R}^N$, where coordinates are components of the tensor. Space dimension is equal to number of components N. This representation is linear in the space $\mathbb{R}^N$. Therefore every transformation is determined with $N \times N$ matrix. Operators of the group $GL(2,\mathbb{R})$ are linear vector fields, i.e. the linear vector field X is replaced with the vector field $\bar{X}$ and matrix C corresponds to the matrix $\bar{C}$. Matrix $\bar{C}$ is completely determined by the matrix C and its eigenvalues are linear combinations of the eigenvalues of matrix C. It appears that eigenvalues of the matrix $\bar{C}$ are placed in node points of the particular grid on the complex plane.

Explicitly tensor fields of type $(0,1)$, $(1,0)$, $(1,1)$, $(0,2)$, $(0,3)$ and $(1,2)$ have been studied. For these cases corresponding ODE's has been derived.
The correspondence $X \to \bar{X}$ can be interpreted as an extension of the action of the group $GL(2,\mathbb{R})$ into the tensor spaces. Therefore we reach the theory

of the algebraic invariants in the classical sense. We see particular cases of the extended operators $\bar{X}$ in the lectures of D. Hilbert (see [3], operators D and Δ on pages 27,30) and in the theory of polynomial dynamic systems of K. Sibirski (see [10]).

If a vector field is an infinitesimal version of the movement ("stop-shot" of the movement) then the operators are infinitesimal versions of the action of group on the orbits of the group. If a vector field X determines action of the linear group on the uv plane then the vector field $\bar{X}$ determines action of the linear group in tensor space.

During this research has been revealed thesis [5], article [6] and book [9].

Preliminary notes

To avoid misunderstandings that can arise during reading this paper we make the following conventions. We consider that all operations like differentiating of functions and of (tensor) fields are allowed. Also, we presume the existence of solutions of differential equations and the convergence of series.

1. Differentiating a function f (on a plane, in space, on a manifold) with respect to the vector field X, we presume that the function f is differentiable. We do not require the existence of continuous partial derivatives but only the "smoothness" of the function f on the trajectories of the vector field X. This is valid also when we differentiate the tensor fields, i.e. calculating the Lie derivatives.

2. Speaking about solutions of the system of ordinary differential equations

$$\begin{cases} u' = x(u, v) \\ v' = y(u, v) \end{cases},$$

we naturally presume that the functions x and y satisfy the conditions of the existence theorem, see [11], p. 266. From the theorem it follows that the general solution of the system determines a set of trajectories (integral curves) and the movement of the points along the trajectories. On the uv plane there arises a flow. We realize the flow of the vector field as one-parameter (pseudo) group of local transformations on differentiable manifold.

3. We describe the transformation of the tensor field f in the flow $a_t = \exp tX$ by the series of Lie-Maclaurin

$$f_t = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!},$$

where the coefficients $f^{(k)}$ are Lie derivatives with respect to the vector field X. We also presume that these coefficients exist, i.e. field f belongs to the class C^∞, and power series f_t converges in a neighbourhood. Detail analysis would be appropriate in this situation but it is not the goal of this paper.

Part I
Lie derivative of the tensor fields

1 Vector fields and flow

Let M be a n-dimensional (smooth) manifold[1] and $(u^1, \ldots, u^n)$ local coordinates. Suppose γ is a smooth curve on manifold M. At each point u of γ the curve has a *tangent vector*. The collection of all tangent vectors to all possible curves passing through the point u is called *tangent space* of M at u and is denoted by T_uM. T_uM is a vector space with basis $\{\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}\}$. The collection of all tangent spaces corresponding to all points $u \in M$ is called the *tangent bundle* of M and is denoted by $TM = \bigcup_{u\in M} T_uM$.

A *vector field* X on M assigns a tangent vector $X_u \in T_uM$ to each point $u \in M$. Hence vector field on the manifold M is differential operator

$$X = x^1 \frac{\partial}{\partial u^1} + \ldots + x^n \frac{\partial}{\partial u^n}, \tag{1}$$

where the components $x^1, \ldots, x^n$ are functions of coordinates $u^1, \ldots, u^n$ (see [8], p. 24). With the vector field (1) associates a system of ordinary differential equations

$$\begin{cases} \frac{du^1}{dt} = x^1(u^1, \ldots, u^n) \\ \cdots \\ \frac{du^1}{dt} = x^n(u^1, \ldots, u^n) \end{cases}. \tag{2}$$

In the following we consider the vector field X on the plane $\mathbb{R}^2$ with coordinates u and v, i.e. $u^1 = u$ and $u^2 = v$

$$X = x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}, \tag{3}$$

where the components $x, y \in C^\infty$ are functions of the u and v.

The finding of integral curves $u(t) = a_t(u)$ of the vector field X is equivalent to solve the system of equations (2). The map $a_t : \mathbb{R}\times M \to M$, $(a_t : u \to u_t)$ is called a *flow* generated by X (see [7], p. 150).

[1] In the following we presume that all (tensor) fields on the manifold M are differentiable of class C^k, where $k = 1, 2, \ldots$, or even of class C^ω.

2 Differential forms

While the tangent space to the manifold M at the point u is a vector space T_uM then there exists a dual vector space to T_uM, whose elements are linear functions from T_uM to $\mathbb{R}$. The dual space is called *cotangent space* and is denoted by T_u^*M. An element $\omega : T_uM \to \mathbb{R}$ of T_u^*M is called *cotangent vector* or *one-form* (see [7], p 145 also [8], p. 53). If $(u^1, \dots, u^n)$ are local coordinates and the tangent space has a basis $\{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}\}$ then the dual cotangent space T_u^*M has a dual basis, denoted $\{du^1, \dots, du^n\}$; thus

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \begin{cases} 0, if & i \neq j \\ 1, if & i = j \end{cases} \quad \forall i, j \in \{1, \dots, n\}. \tag{4}$$

3 Lie derivative of the function

Let X be a vector field on the uv plane. In the following we denote the partial derivatives of the function $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ with respect to the variables u and v with f_1 and f_2 respectively. The derivative of the function f with respect to the vector field X we denote Xf or f'

$$Xf = f' = x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}. \tag{5}$$

For brevity we write

$$f' = xf_1 + yf_2, \tag{6}$$

where

$$f_1 = \frac{\partial f}{\partial u} \quad \text{and} \quad f_2 = \frac{\partial f}{\partial v}. \tag{7}$$

n-th order Lie derivative of the function f with respect to the vector field X is denoted $X^n f$ or $f^{(n)}$

$$X^n f = f^{(n)} = x\frac{\partial(X^{n-1}f)}{\partial u} + y\frac{\partial(X^{n-1}f)}{\partial v}. \tag{8}$$

The function f changes under the influence of the flow a_t:

$$f \mapsto f_t = f \circ a_t \tag{9}$$

Differentiating $f_t = f \circ a_t$ with respect to t we get $(f \circ a_t)' = Xf = f'$. The function f_t can be developed in the Lie-Maclaurin series:

$$f_t = f + f't + f''\frac{t^2}{2} + \cdots = \sum_{k=0}^{\infty} f^{(k)}\frac{t^k}{k!}. \tag{10}$$

In the following we use also notations (see (6))

$$x_1 = \frac{\partial x}{\partial u}, \quad x_2 = \frac{\partial x}{\partial v}, \quad y_1 = \frac{\partial y}{\partial u}, \quad y_2 = \frac{\partial y}{\partial v}.$$

The matrix that consists of partial derivatives of components x and y

$$C = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \tag{11}$$

is called Jacob matrix. Later we refer components of the matrix C also as C_i^j, where i is column index and j is row index.

4 Derivation formulae

Natural base (operators of the partial derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ and derivatives of the coordinate functions du and dv) changes under influence of the flow a_t of the vector field X.

Proposition 4.1 *Lie derivatives of differentials of coordinate functions (coreper) in the matrix form are*

$$\begin{pmatrix} du \\ dv \end{pmatrix}' = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \cdot \begin{pmatrix} du \\ dv \end{pmatrix}, \tag{12}$$

or briefly $\Theta' = C\Theta$.

Proof. Applying formula (17) to the functions u and v we get
$(du)' = dx = x_1 du + x_2 dv$ and $(dv)' = dy = y_1 du + y_2 dv$.

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Proposition 4.2 *Lie derivatives of the operators of partial derivatives (reper) in the matrix form are*

$$\begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \end{pmatrix}' = -\begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \tag{13}$$

or briefly $R' = -RC$.

Proof. While

$$du\left(\frac{\partial}{\partial u}\right) = \frac{\partial u}{\partial u} = 1, \qquad dv\left(\frac{\partial}{\partial u}\right) = \frac{\partial v}{\partial u} = 0,$$

$$du\left(\frac{\partial}{\partial v}\right)=\frac{\partial u}{\partial v}=0,\qquad dv\left(\frac{\partial}{\partial v}\right)=\frac{\partial v}{\partial v}=1,$$

then differentiating these equations we get

$$du\left(\left(\frac{\partial}{\partial u}\right)'\right)+(du)'\left(\frac{\partial}{\partial u}\right)=0,\qquad dv\left(\left(\frac{\partial}{\partial u}\right)'\right)+(dv)'\left(\frac{\partial}{\partial u}\right)=0,$$

$$du\left(\left(\frac{\partial}{\partial v}\right)'\right)+(du)'\left(\frac{\partial}{\partial v}\right)=0,\qquad dv\left(\left(\frac{\partial}{\partial v}\right)'\right)+(dv)'\left(\frac{\partial}{\partial v}\right)=0.$$

Considering formula (12) we get components for the vector fields $\left(\frac{\partial}{\partial u}\right)'$ and $\left(\frac{\partial}{\partial v}\right)'$

$$du\left(\left(\frac{\partial}{\partial u}\right)'\right)=-x_1,\qquad dv\left(\left(\frac{\partial}{\partial u}\right)'\right)=-y_1,$$

$$du\left(\left(\frac{\partial}{\partial v}\right)'\right)=-x_2,\qquad dv\left(\left(\frac{\partial}{\partial v}\right)'\right)=-y_2.$$

And therefore

$$\left(\frac{\partial}{\partial u}\right)'=-x_1\frac{\partial}{\partial u}-y1\frac{\partial}{\partial v},\qquad \left(\frac{\partial}{\partial v}\right)'=-y_1\frac{\partial}{\partial u}-x_2\frac{\partial}{\partial v}.$$

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5 Lie derivative of the one-form

One-form Φ change under the influence of the flow a_t:

$$\Phi\rightarrow\Phi_t \tag{14}$$

Changing one-form Φ_t is uniquely determined with formulae

$$\Phi_t(Y_t)=(\Phi(Y))_t, \tag{15}$$

where Y is arbitrary vector field and Y_t is defined above, see (23). Differentiating this equation on left and right we get

$$\Phi'(Y)=X(\Phi(Y))-\Phi([XY]). \tag{16}$$

Last equation determines the derivative of one-form Φ with respect to the vector field X. One-form Φ_t can be developed to the Lie-Maclaurin series

$$\Phi_t=\Phi+\Phi' t+\Phi''\frac{t^2}{2}+\cdots=\sum_{k=0}^{\infty}\Phi^{(k)}\frac{t^k}{k!},$$

where the coefficients $\Phi^{(k)}$ are Lie derivatives.

Proposition 5.1 *Lie derivative of the differential of the function f is differential of the derivative f', i.e.*

$$(df)' = df'. \tag{17}$$

Proof. Let Y be arbitrary vector field. While df is one-form then we apply the formula (16) and get

$$\begin{aligned} (df)'(Y) &= X(df(Y)) - df([XY]) \\ &= XYf - [XY]f = YXf = Yf' = df'(Y), \quad \forall\, Y. \end{aligned} \tag{18}$$

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Proposition 5.2 *If one-form Φ is given in matrix form*

$$\Phi = \begin{pmatrix} \varphi & \psi \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}, \tag{19}$$

then the corresponding Lie derivative is

$$\Phi' = \left\{ \begin{pmatrix} \varphi' & \psi' \end{pmatrix} + \begin{pmatrix} \varphi & \psi \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \right\} \begin{pmatrix} du \\ dv \end{pmatrix}, \tag{20}$$

or

$$\Phi' = (X\varphi + x_1\varphi + y_1\psi)du + (X\xi + x_2\varphi + y_2\psi)dv. \tag{21}$$

Proof. To get formulae (20) and (21) we differentiate equation (19) and get directly corresponding formulae with help of Leibniz rule and derivation formula (12).

■

6 Lie derivative of the vector field

Change of the vector field Y in the flow a_t:

$$Y \to Y_t \tag{22}$$

is described by the formula

$$Y_t f_t = (Yf)_t, \tag{23}$$

where f is arbitrary function. Equation (23) determines uniquely the changing vector field Y_t. Differentiating this equation with respect to the parameter

t at $t = 0$ we get $Y'f + Yf' = (Yf)'$, i.e. $Y'f = (XY - YX)f$. The operator $[XY] = XY - YX$ is called bracket of the vector fields X and Y. The derivative of the vector field Y with respect to the vector field X is the bracket

$$Y' = [XY]. \tag{24}$$

The vector field Y_t can be developed in the Lie-Maclaurin series

$$Y_t = Y + Y't + Y''\frac{t^2}{2} + \cdots = \sum_{k=0}^{\infty} Y^{(k)}\frac{t^k}{k!}, \tag{25}$$

where the coefficients are the brackets $Y' = [XY]$, $Y'' = [X[XY]]$,....

Proposition 6.1 *If the vector field Y is given in matrix form*

$$Y = \begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{26}$$

then the corresponding Lie derivative is

$$Y' = \begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \end{pmatrix} \left\{ \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} - \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\}, \tag{27}$$

or

$$Y' = (X\xi - Yx)\frac{\partial}{\partial u} + (X\eta - Yy)\frac{\partial}{\partial v}. \tag{28}$$

Proof. To prove formulae (27) and (28) we differentiate the equation (26) and get directly corresponding formulae with help of Leibniz rule and derivation formula (13).

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7 Differential forms. Integrating factor

The vector field X is related with an one-form $\omega = -ydu + xdv$.

Proposition 7.1 *The one-form $\omega = -ydu + xdv$ nullifies on the vector field X, i.e. $\omega(X) = 0$. Lie derivative ω' and exterior derivative $d\omega$ are related with divergence of vector field X as follows:*

$$\omega' = \mathrm{div}X \cdot \omega, \quad d\omega = \mathrm{div}X \cdot (du \wedge dv). \tag{29}$$

Proof.

$$\begin{aligned}
\omega' &= -y'du - ydx + x'dv + xdy = \\
&= -(y_1x + y_2y)du - y(x_1du + x_2dv) + (x_1x + x_2y)dv + x(y_1du + y_2dv) = \\
&= (x_1 + y_2)(-ydu + xdv) = \text{div}X \cdot \omega, \\
d\omega &= -dy \wedge du + dx \wedge dv = (x_1du + x_2dv) \wedge dv - (y_1du + y_2dv) \wedge du = \\
&= (x_1 + x_2)(du \wedge dv) = \text{div}X \cdot (du \wedge dv).
\end{aligned}$$

■

Proposition 7.2 *If* $\text{div}X = 0$ *then the related one-form* ω *is (locally) exact and it is differential of an invariant of the vector field* X*:*

$$\text{div}X = 0 \Rightarrow \omega' = 0, \quad d\omega = 0 \Rightarrow \exists I, \quad \omega = dI \Rightarrow I' = 0. \tag{30}$$

Proof. If $\text{div}X = 0$ then one-form ω is invariant of the vector field X, i.e. $\omega' = 0$ and exterior derivative $d\omega = 0$, see (29). Therefore ω is (locally) differential of some function I, $\omega = dI$.

■

Proposition 7.3 *Let* f *be a function that satisfies the condition*

$$f' + f\text{div}X = 0. \tag{31}$$

Multiplying the vector field X *with function* f *we get a solenoid field,* $\text{div}(fX) = 0$*. Multiplying the one-form* ω *with function* f *we get an exact one-form,* $d(f\omega) = 0$*. One-form* $f\omega$ *is differential of an invariant of the vector field* X*,* $f\omega = dI$*,* $I' = 0$*.*

Proof. While $\text{div}X = x_1 + y_2$, see (29), then $\text{div}(fX) = (fx)_1 + (fy)_2 = f_1x + f_2y + f(x_1 + x_2) = f' + f\text{div}X$. From here we get the condition (31). Rest of the proposition is a conclusion of the proposition 7.2.

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The function f that satisfies the condition (31) is called *integrating factor*.

8 Lie derivative of the tensor field

In the following is considered a particular tensor field of type $(1, 2)$:

$$S = \frac{\partial}{\partial u^i} s^i_{jk} du^j \otimes du^k, \quad i, j, k = 1, 2. \tag{32}$$

The following formula shows how to calculate its Lie derivative with respect to the vector field X on the uv-plane. Using the derivation formulae (12) and (13) we get

$$S' = \frac{\partial}{\partial u^i} \left(X s^i_{jk} - C^i_l s^l_{jk} + s^i_{lk} C^l_j + s^i_{jl} C^l_k \right) du^j \otimes du^k, \tag{33}$$

where

$$C = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}. \tag{34}$$

In the case of general tensor field of type (p, q) the situation is analogous.

Part II
Linear vector fields

9 Linear vector field

Vector field is called *linear vector field* if it's components x and y are homogeneous linear functions of the coordinates u and v

$$X = (c_1 u + c_2 v)\frac{\partial}{\partial u} + (c_1 u + c_2 v)\frac{\partial}{\partial v}. \tag{35}$$

The system (2) in matrix form and the Jacob matrix (11) are repectively

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}. \tag{36}$$

The solution of characteristic equation

$$|C - \lambda E| = \lambda^2 - \operatorname{tr} C \cdot \lambda + \det C = 0 \tag{37}$$

depends on the discriminant

$$\Delta = \operatorname{tr}^2 C - 4\det C = \operatorname{tr} C^2 - 2\det C. \tag{38}$$

Eigenvalues of the Jacob matrix C can be

1)	conjugate complex numbers	$\lambda_{1,2} = \alpha \pm i\beta,$	if $\Delta < 0,$
2)	not equal real numbers	$\lambda_{1,2} = \alpha \pm \beta,$	if $\Delta > 0,$
3)	equal real numbers	$\lambda_1 = \lambda_2 = \alpha,$	if $\Delta = 0,$

where α, $\beta \in \mathbb{R}$.
The following equations are valid and can be derived from the definitions of the trace, determinant of matrix C and discriminant:

$$\begin{aligned} \operatorname{tr} C &= c_1 + c_4 = 2\alpha, \\ \det C &= c_1 c_4 - c_2 c_3 = \alpha^2 \pm \beta^2, \\ \operatorname{tr} C^2 &= c_1^2 + c_4^2 + 2c_2 c_3 = 2(\alpha^2 \mp \beta^2), \\ \Delta &= (c_1 - c_4)^2 + 4c_2 c_3 = \mp\beta^2. \end{aligned} \tag{39}$$

The upper sign corresponds to the case $\Delta < 0$ and the lower sign corresponds to the case $\Delta > 0$. In the case $\Delta = 0$ is $\beta = 0$.

10 Exponential law

The flow of the linear vector field (35) is determined by the exponential of the matrix Ct:

$$e^{Ct} = E + Ct + C^2\frac{t^2}{2} + \cdots = \sum_{k=0}^{\infty} C^k \frac{t^k}{k!}. \tag{40}$$

The system (36) is like differential equation of type

$$U' = CU. \tag{41}$$

General solution of the equation (41) is

$$U_t = e^{Ct}U, \tag{42}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $U_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix}$.

Proposition 10.1 *Exponential e^{Ct} and general solution U_t are*

1) $e^{Ct} = e^{\alpha t}\left[E\cos\beta t + (C-\alpha E)\frac{\sin\beta t}{\beta}\right], \quad U_t = e^{\alpha t}\left[U\cos\beta t + (U'-\alpha U)\frac{\sin\beta t}{\beta}\right],$
if $\Delta < 0$,

2) $e^{Ct} = e^{\alpha t}\left[E\,\mathrm{ch}\,\beta t + (C-\alpha E)\frac{\mathrm{sh}\,\beta t}{\beta}\right], \quad U_t = e^{\alpha t}\left[U\,\mathrm{ch}\,\beta t + (U'-\alpha U)\frac{\mathrm{sh}\,\beta t}{\beta}\right],$
if $\Delta > 0$,

3) $e^{Ct} = e^{\alpha t}\left[E + (C-\alpha E)t\right], \quad U_t = e^{\alpha t}\left[U + (U'-\alpha U)t\right],$
if $\Delta = 0$.

Proof. Considering that $e^{\alpha E t} = e^{\alpha t}E$ we get

$$e^{Ct} = e^{\alpha t}e^{(C-\alpha E)t}. \tag{43}$$

We have to calculate the exponential $e^{(C-\alpha E)t}$. Obviously

$$C^2 - \mathrm{tr}\,C\cdot C + \det\cdot E = 0, \tag{44}$$

where 0 is zero matrix. From the equation (44) can be derived that

$$C^2 - 2\alpha C + (\alpha^2 \pm \beta^2)E = 0, \quad (C-\alpha E)^2 \pm \beta^2 E = 0. \tag{45}$$

Using last equation in the Maclaurin series and considering signs in cases 1 and 2 we get

$$\begin{aligned} e^{(C-\alpha E)t} &= E + (C-\alpha E)t \mp E\frac{\beta^2 t^2}{2!} \mp (C-\alpha E)\frac{\beta^2 t^3}{3!} + E\frac{\beta^4 t^4}{4!} + (C-\alpha E)\frac{\beta^4 t^5}{5!} \mp \cdots \\ &= E(1 \mp \frac{\beta^2 t^2}{2} + \frac{\beta^4 t^4}{4!} \mp \cdots) + (C-\alpha E)\frac{1}{\beta}(\beta t \mp \frac{\beta^3 t^3}{3!} + \frac{\beta^5 t^5}{5!} \mp \cdots). \end{aligned}$$

In the case $\beta = 0$

$$e^{(C-\alpha E)t} = E + (C-\alpha E)t.$$

Considering equations

$$\begin{aligned} \cos\beta t = 1 - \frac{\beta^2 t^2}{2} + \frac{\beta^4 t^4}{4!} - \cdots, &\quad \sin\beta t = 1 - \frac{\beta^3 t^3}{3} + \frac{\beta^5 t^5}{5!} - \cdots, \\ ch\,\beta t = 1 + \frac{\beta^2 t^2}{2} + \frac{\beta^4 t^4}{4!} + \cdots, &\quad sh\,\beta t = 1 + \frac{\beta^3 t^3}{3} + \frac{\beta^5 t^5}{5!} + \cdots, \end{aligned}$$

we get corresponding formulae.

■

11 Main property of the linear flow

The peculiarity of the linear flow consists in the following.

Proposition 11.1 *A linear function f in the linear flow remains linear function. Straight lines remain straight, parallelism of the straight lines remains and every straight line envelops the trajectory whose tangent it is.*

Proof. Let a linear function be in matrix form $f = PU$, where $P = (p_1, p_2)$. The function $f = PU$ transforms in the flow, $U \to U_t = e^{Ct}U$, but it remains linear function $f_t = P_t U$, where $P_t = Pe^{Ct}$. It means that the tangent-line of the function f is also tangent-line of the function f_t. Therefore every tangent-line of the function f_t remains straight. Remains the parallelism of two parallel lines.
The characteristic point of the straight line $PU = K - const$ is determined with formula $PU' = 0$ and the velocity vector U' is directed along the straight line. Therefore the straight line envelopes trajectory whose tangent it is.

■

Example. In the Figure 1 we observe two parallel straight segments in the flow of linear vector field $X = (u-v)\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$. We see that these segments change but they remains straight and parallel.

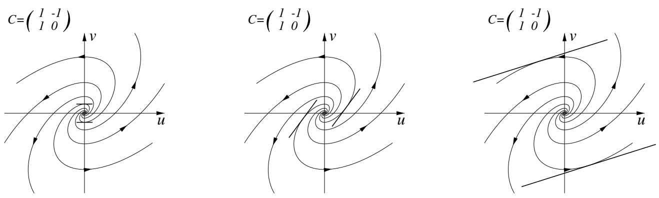

Figure 1: Two segment in the linear flow

12 Local phase portrait

The term *phase portrait* is defined and explained in [4], p. 37. Two phase portraits are similar if the matrices C of the corresponding vector fields are equal.

Proposition 12.1 *The phase portrait in the frame which moves along the trajectory $U_t = e^{Ct}U$ is similar to the phase portrait in the global frame.*

Proof. Point U and it's neighbour point $U + dU$ transform according to equation $U_t = e^{Ct}U$ and $(U + dU)_t = e^{Ct}(U + dU)$. Then $U_t + (dU)_t = e^{Ct}U + e^{Ct}dU$ and $(dU)_t = e^{Ct}dU$.

■

Example. Let's consider a hyperbolic flow of the vector field $X = u\frac{\partial}{\partial u} + (u-v)\frac{\partial}{\partial v}$. Observing one particular moving point we see that around the point global phase portrait repeats itself in the local frame. See Figure 2.

13 Linearization of nonlinear vector field

In the case of nonlinear flow the phase portrait does not repeat itself in the moving frame. To get the linear vector field at some point, we have to fix the elements in the Jacob matrix of the nonlinear vector field using the coordinates of the point.

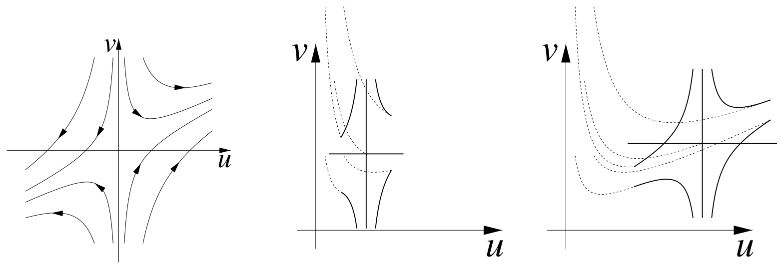

Figure 2: Linear phase portrait in the moving frame

Example. Let us consider a nonlinear system and corresponding Jacob matrix J:

$$\left\{ \begin{array}{l} u^{'} = u - v - u(u^2 + v^2) \\ v^{'} = u + v - v(u^2 + v^2) \end{array} \right. , \quad J = \begin{pmatrix} 1 - 3u^2 - v^2 & -1 - 2uv \\ 1 - 2uv & 1 - u^2 - 3v^2 \end{pmatrix}, \quad (46)$$

see [4], page 59. For every fixed point Jacob matrix J is constant and it determines linear flow around this point, see Figure 3, page 21.

The phase portrait of the corresponding linear vector field changes moving from point to point. At first we are inside the circle $u^2 + v^2 = 1$ (attractor). The divergence of the vector field, i.e. the trace of the matrix J, changes its sign on the circle $u^2 + v^2 = \frac{1}{2}$ (marked as dotted line). Moving away from the 0-point we see unstable focus. On the circle the focus is turned into elliptical rotation. Outside the circle and inside the attractor we see a stable focus. On the attractor the vector field is turned into stable parabolic node. Outside the attractor we see stable hyperbolic node. Therefore a nonlinear vector field, in this case 3th order polynomial vector field, produces different situations in different points.

14 Classification

The linear vector fields can be classified with respect to the value

$$\Delta = \mathrm{tr}^2 C - 4 \det C.$$

In following we consider the vector fields in the xy-frame, where $x = \mathrm{tr} C$ and $y = \det C$. The equation $\Delta = 0$ determines a parabola.

1. Inequality $\Delta > 0$ determines a region inside the parabola. In this case the phase portrait of the vector field at the 0-point is *focus.* If $\mathrm{tr} C < 0$ then the

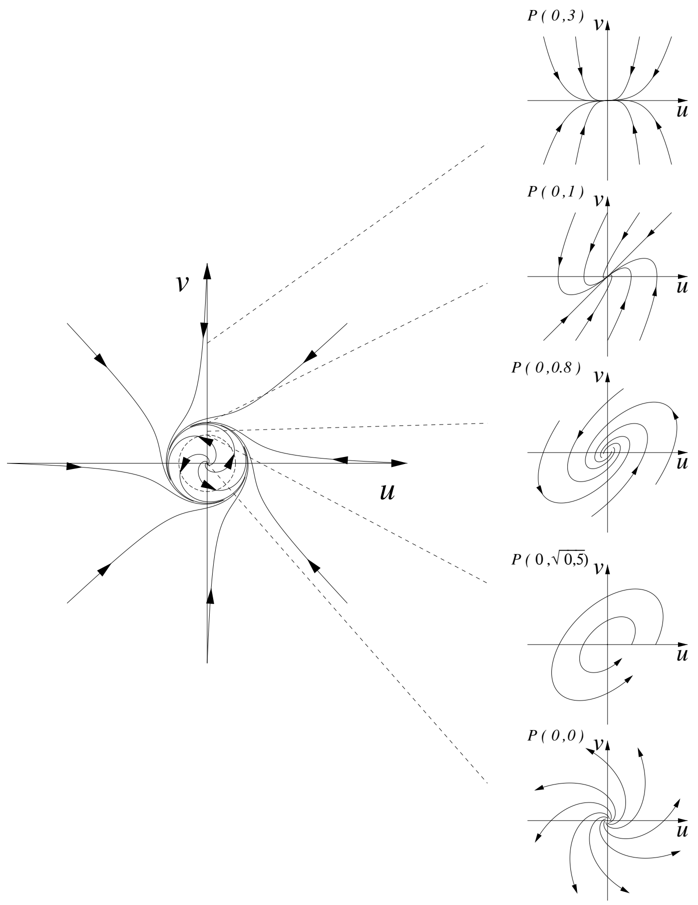

Figure 3: Linearization of the nonlinear flow

saddle is stable, if $\mathrm{tr}C > 0$ then the saddle is unstable. In case $\mathrm{tr}C = 0$ the vector field is source free.

2. The inequality $\Delta < 0$ determines a region outside the parabola. The phase portrait of the vector field at the 0-point is *saddle*, if $\det C < 0$, and *hyperbolic node*, if $\det C > 0$ (it might be stable if $\mathrm{tr}C < 0$ or unstable $\mathrm{tr}C > 0$). In the case $\det C = 0$ the trajectories are parallel straight lines.

3. In the case of equality $\Delta = 0$ the vector field is placed on the parabola. Corresponding phase portrait is *parabolic node*, stable (if $\mathrm{tr}C < 0$) or unstable (if $\mathrm{tr}C > 0$). In the case $\det C = 0$ vector field is degenerated.

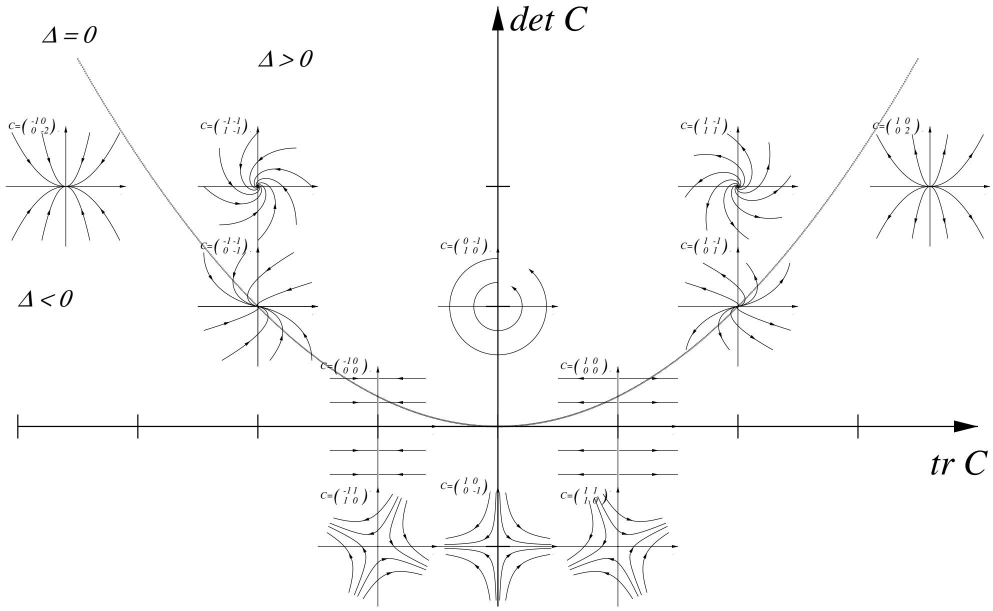

Figure 4: Classification of the linear vector fields

The schema Figure 4 comparing with the schema in the book [2] is far perfect. We distinguish hyperbolic and parabolic nodes. Latter is placed on the parabola $\Delta = 0$.

Part III

Tensor representations of the linear group

In the following we use terms that can be found in the bibliography. Terms like *Lie group* and *Lie algebra* can be found in [1], vol. III, p. 552 and [1], vol. III, p. 532 and term *one-parameter subgroup* can be found in [1], vol. IV, p. 218. We are particularly interested in general linear group $GL(2,\mathbb{R})$ - the set of all invertible 2×2 matrices with real entries.

15 Action of $GL(2,\mathbb{R})$ and $gl(2,\mathbb{R})$ on the uv-plane

A group G is said to act on set M when there is a map $\phi : G \times M \to M$ such that the following conditions hold for all elements $u \in M$:

1. $\phi(e,u) = u$, where e is identity element of G and
2. $\phi(g,\phi(h,u)) = \phi(gh,u)$, for all $g,h \in G$.

Here G is called *transformation group* and ϕ is called the group action. If G is Lie group and M is differentiable manifold then it is assumed that map ϕ is differentiable.

On the uv-plane an action of the linear group $GL(2,\mathbb{R})$ is determined, i.e. every regular matrix $A \in GL(2,\mathbb{R})$ determines a transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}, \tag{47}$$

$$U \to \tilde{U} = AU. \tag{48}$$

For more about Lie groups and their action see [8], pages 13-17.

Remark 15.1 *Generally the transformation* (47) *is centro-affine. If* $\operatorname{tr} A = 0$ *then the transformation is called equi-affine (the areas remain).*

A *representation* of a linear group G is a group action of G on a vector space V by invertible linear maps.

On the uv-plane the action of the Lie algebra $gl(2,\mathbb{R})$ is also determined, i.e. every (non-regular) matrix $C \in gl(2,\mathbb{R})$ determines a system of ODE's

$U' = CU$, i.e. linear vector field is determined

$$X = (c_1 u + c_2 v)\frac{\partial}{\partial u} + (c_3 u + c_4 v)\frac{\partial}{\partial v}, \tag{49}$$

where $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$. This is representation of the $gl(2,\mathbb{R})$ on the uv-plane. The exponential e^{Ct} of the matrix Ct determines an one-parametric subgroup in GL. The flow a_t of the vector field X on the uv-plane is given.

16 Operators of linear group $GL(2,\mathbb{R})$

The matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ can be presented as sum of four matrices

$$\begin{aligned} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} &= \frac{a_1+a_4}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a_1+a_3}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \\ &+ \frac{a_1-a_4}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{a_2-a_3}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{50}$$

Therefore the vector field X, see (35), can be present as a linear combination of four vector field:

$$X = p_0 X_0 + p_1 X_1 + p_2 X_2 + p_3 X_3, \tag{51}$$

where

$$p_0 = \frac{a_1+a_4}{2}, \quad p_1 = \frac{a_1+a_3}{2}, \quad p_2 = \frac{a_1-a_4}{2}, \quad p_3 = \frac{a_2-a_3}{2}, \tag{52}$$

$$\begin{aligned} X_0 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_1 = v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \\ X_2 &= u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad X_3 = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}. \end{aligned} \tag{53}$$

The vector field X_0 is called *dilation operator*. The vector field X_3 is called *rotation operator*. Vector fields (53) are the operators of the group GL on the uv plane and they form a basis for this group.

In the following is given the commutator table of the operators X_i, $i = 0, 1, 2, 3$, see Table 1. Also is given the commutator table of normalized operators $Y_i = \frac{1}{2}X_i$, $i = 0, 1, 2, 3$:

Commutator table is skew-symmetric since $[X_i, X_j] = -[X_j, X_i]$.

	X_0	X_1	X_2	X_3
X_0	0	0	0	0
X_1	0	0	$2X_3$	$-2X_2$
X_2	0	$-2X_3$	0	$2X_1$
X_3	0	$2X_2$	$-2X_1$	0

	Y_0	Y_1	Y_2	Y_3
Y_0	0	0	0	0
Y_1	0	0	Y_3	$-Y_2$
Y_2	0	$-Y_3$	0	Y_1
Y_3	0	Y_2	$-Y_1$	0

Table 1: Commutator tables of the operators X_i and $Y_i = \frac{1}{2}X_i$, $i = 0, 1, 2, 3$.

17 Natural basis in the linear flow

The flow transforms the natural basis $(R, \Theta) \to (\tilde{R}, \tilde{\theta})$, where

$$\begin{pmatrix} \frac{\partial}{\partial \tilde{u}} & \frac{\partial}{\partial \tilde{v}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{-1}, \quad \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} du \\ dv \end{pmatrix}$$

or simply

$$\tilde{R} = RA^{-1}, \quad \tilde{\theta} = A\theta. \tag{54}$$

If $A_t = e^{Ct}$ then formulae (54) are in the form

$$R_t = RA_t^{-1}, \quad \theta_t = A_t\theta, \tag{55}$$

or

$$R_t = Re^{-Ct}, \quad \theta_t = e^{Ct}\theta. \tag{56}$$

The formulae (56) demonstrate how the basis (R, θ) is transformed under subgroup $A_t = e^{Ct}$. Differentiating formulae (56) with respect to the parameter t at $t = 0$ we get an infinitesimal transformation in the flow $a_t : U \to U_t$

$$R' = -RC, \quad \theta' = C\theta. \tag{57}$$

Prime means, as before, Lie derivatives with respect to the vector field X.

18 Tensor fields in the linear flow

All kind of tensor fields on the uv-plane are transformed in the linear flow. A tensor S of type (p, q) in the 2-dimensional space is given with respect to the basis (R, Θ)

$$S = R_{i_1} \otimes \cdots \otimes R_{i_p} s^{i_1 \ldots i_p}_{j_1 \ldots j_q} \Theta^{j_1} \otimes \cdots \otimes \Theta^{j_p}. \tag{58}$$

Proposition 18.1 *Components s are transformed by the transformation* (47) *as follows*

$$\tilde{s}^{i_1\dots i_p}_{j_1\dots j_q} = A^{i_1}_{k_1}\cdots A^{i_p}_{k_p} s^{k_1\dots k_p}_{l_1\dots l_q} \bar{A}^{l_1}_{j_1}\cdots \bar{A}^{l_p}_{j_p}, \tag{59}$$

where $\bar{A} = A^{-1}$.

Proof. Applying formulae (54) to the tensor S we get

$$S = R_{i_1}\otimes\cdots\otimes R_{i_p}\bar{A}^{k_1}_{i_1}\cdots\bar{A}^{k_p}_{i_p}\tilde{s}^{i_1\dots i_p}_{j_1\dots j_q}A^{j_1}_{l_1}\cdots A^{j_q}_{l_q}\Theta^{j_1}\otimes\cdots\otimes\Theta^{j_p}. \tag{60}$$

Here

$$\bar{A}^{k_1}_{i_1}\cdots\bar{A}^{k_p}_{i_p}\tilde{s}^{i_1\dots i_p}_{j_1\dots j_q}A^{j_1}_{l_1}\cdots A^{j_q}_{l_q} = s^{i_1\dots i_p}_{j_1\dots j_q}. \tag{61}$$

which means that

$$\tilde{s}^{i_1\dots i_p}_{j_1\dots j_q} = A^{i_1}_{k_1}\cdots A^{i_p}_{k_p} s^{k_1\dots k_p}_{l_1\dots l_q} \bar{A}^{l_1}_{j_1}\cdots \bar{A}^{l_p}_{j_p}. \tag{62}$$

■

Example. A tensor field of type $(1, 2)$ is determined as

$$S = R_i s^i_{jk}\theta^j\theta^k. \tag{63}$$

The components s are transformed by the transformation (59) as follows

$$\tilde{s}^i_{jk} = A^i_p s^p_{qs}\bar{A}^q_j\bar{A}^s_k, \tag{64}$$

where $\bar{A} = A^{-1}$. Applying formulae (54) to the tensor S we get (without indexes)

$$S = RA^{-1}\tilde{s}AA\theta\theta. \tag{65}$$

Here $A^{-1}\tilde{s}AA = s$, which means that $\tilde{s}^i_{jk} = A^i_p s^p_{qs}\bar{A}^q_j\bar{A}^s_k$.

Our main idea is to use instead of representations of group GL the representations of Lie algebra gl. In case of representation of group GL, the matrix A and its inverse A^{-1} determine a transformation (64). The transformation formulae (64) are not linear with respect to the elements of A and A^{-1}. Presuming instead of A the matrix A_t and differentiating with respect to t at $t = 0$ we get

$$(s^i_{jk})' = C^i_p s^p_{jk} - s^i_{pk}C^p_j - s^i_{jp}C^p_k. \tag{66}$$

In the last equation the elements of C are presented linearly.

19 Tensor representations of linear group $GL(2,\mathbb{R})$ and it's Lie algebra $gl(2,\mathbb{R})$

The matrix C, element of Lie algebra $gl(2,\mathbb{R})$, determines an one-parametric subgroup e^{Ct} of the group $GL(2,\mathbb{R})$. Presuming instead of A the matrix $A_t = e^{Ct}$ in (59) we get a family of one-parametric tensors S_t. In the tensor space appear a linear flow and corresponding linear vector field $\bar{X}$. The components of the vector field $\bar{X}$ we get in the following way: instead of matrix A we presume the exponential e^{Ct} in (59) and differentiate with respect to the parameter t at $t = 0$. We get the equation

$$(s^{i_1 \dots i_p}_{j_1 \dots j_q})' = C^{i_1}_k s^{k \dots i_p}_{j_1 \dots j_q} + \dots + C^{i_p}_k s^{i_1 \dots k}_{j_1 \dots j_q} - s^{i_1 \dots i_p}_{k \dots j_q} C^k_{j_1} - \dots - s^{i_1 \dots i_p}_{j_1 \dots k} C^k_{j_p}. \tag{67}$$

If we compare the formula (67) with formula (59) then we see that the elements of the matrix C are presented linearly on the right side of (67). On the right side of (59) the elements of the matrix A are not presented linearly. The formula (67) can be presented in the form

$$s' = \bar{C}s, \tag{68}$$

where the matrix $\bar{C}$ is determined with the matrix C uniquely. Clearly to a linear vector field $\bar{X}$ corresponds the matrix $\bar{C}$ in the space $\mathbb{R}^N$, where N is number of tensor components. In this space the exponential law

$$s' = \bar{C}s \quad \Rightarrow \quad s_t = e^{\bar{C}t}s \tag{69}$$

is also valid. We say the correspondence $C \to \bar{C}$ (or $X \to \bar{X}$) determines a representation of Lie algebra $gl(2,\mathbb{R})$ in the tensor space $\mathbb{R}^N$.

Proposition 19.1 *Let the linear group $GL(2,\mathbb{R})$ act in (p,q) type tensor space $\mathbb{R}^N$. If the matrix C has eigenvalues λ_1 and λ_2 then the matrix $\bar{C}$ has a set of eigenvalues*

$$\lambda_{i_1} + \dots + \lambda_{i_p} - \lambda_{j_1} - \dots - \lambda_{j_q}, \tag{70}$$

where the indices $i_1, \dots, i_p$ and $j_1, \dots, j_q$ get the all combinations of values 1 *and* 2.

Proof. Let's take the basis (R, Θ) so that the matrix C is in the diagonal form $c^i_j = \delta^i_j \lambda_i$, where δ^i_j is Kronecker delta. Then the elements of matrix e^{Ct} are $\delta^i_j e^{\lambda_i t}$. Considering instead of A the matrix e^{Ct} in (59) we get

$$e^{\lambda_{i_1} t} \dots e^{\lambda_{i_p} t} s^{i_1 \dots i_p}_{j_1 \dots j_q} e^{-\lambda_{j_1} t} \dots e^{-\lambda_{j_q} t} = s^{i_1 \dots i_p}_{j_1 \dots j_q} e^{(\lambda_{i_1} + \dots + \lambda_{i_p} - \lambda_{j_1} - \dots - \lambda_{j_q})t}. \tag{71}$$

Differentiating last equation with respect to the t at $t = 0$ we get the components of the tensor S multiplied with factors (70). A diagonal matrix with the values (70) on the main diagonal appears, see $s' = \bar{C}s$, where the elements of the main diagonal are the eigenvalues of primary matrix $\bar{C}$.

■

If the eigenvalues of the matrix C are conjugate complex numbers λ_1 and λ_2 ($\Delta < 0$) then we set all the eigenvalues (70) of the matrix $\bar{C}$ onto the complex plane. Arises a particular grid. At the node points of the grid are complex numbers (70), see Figure 5, page 28. If $\lambda_{1,2} = \alpha \pm i\beta$ then the complex numbers have common real part $(p - q)\alpha$. It means that for the fixed (p, q), i.e. in the case of (p, q) type tensor, the eigenvalues (70) are placed in the same line which is parallel with imaginary axis.

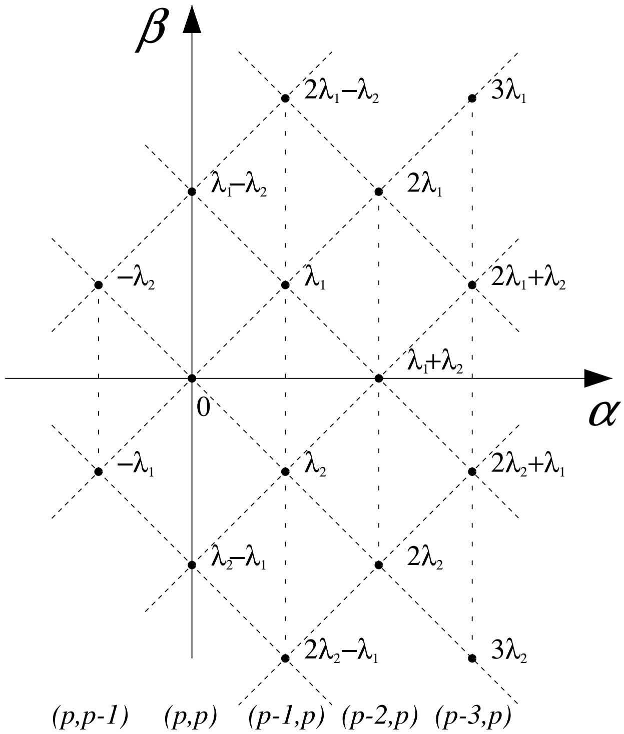

Figure 5: Eigenvalues of the matrix $\bar{C}$

20 Dual differential equations for the tensor fields

The linear differential equations with respect to the vector field $\bar{X}$ in the space $\mathbb{R}^N$ can be written using the Proposition 19.1 in following idea.

If the eigenvalues λ_1, λ_2, ..., λ_n of the matrix C are not equals in pairs then the matrix C satisfies the equation

$$(C - \lambda_1 E)(C - \lambda_2 E) \cdots (C - \lambda_n E) = 0.$$

Last equation can be rewrite in the (Hamilton-Cayley) form

$$\sigma_0 C^n + \sigma_1 C^{n-1} + \ldots + \sigma_n E = 0, \tag{72}$$

where σ_0, σ_1, ..., σ_n are symmetric polynomials on n variables λ_1, λ_2, $\cdots$, λ_n

$$\begin{aligned}
\sigma_0 &= 1, \\
\sigma_1 &= -(\lambda_1 + \lambda_2 + \ldots + \lambda_n), \\
\sigma_2 &= \lambda_1\lambda_2 + \ldots + \lambda_{n-1}\lambda_n, \\
&\ldots, \\
\sigma_n &= (-1)^n \lambda_1\lambda_2 \cdots \lambda_n.
\end{aligned}$$

Considering equations

$$U' = CU, \quad U'' = C^2U, \quad \ldots, \quad U^{(n)} = C^nU \tag{73}$$

we get from the equation (72) an linear ordinary differential equation in the form

$$U^{(n)} + \sigma_1 U^{(n-1)} + \sigma_2 U^{(n-2)} + \ldots + \sigma_n U = 0. \tag{74}$$

In the case of the tensor field S there can be two (dual) cases:

1) the components of the tensor field S are constant with respect to the flow, i.e. $s' = 0$. In this case we get differential equation for the tensor field. Then the tensor field S (i.e. it's Lie derivatives) satisfies an ODE that is determined by the eigenvalues λ_1 and λ_2 of the matrix C. The coefficients of differential equation are symmetric polynomials of the eigenvalues of the matrix $\bar{C}$ multiplied with -1;

2) the tensor field S is invariant with respect to the flow, i.e. $S' = 0$. In this case we get a differential equation for the components of tensor field. The components of tensor field satisfy similar (but dual) ODE. The coefficients of this differential equation are symmetric polynomials of the eigenvalues of the matrix $\bar{C}$ (with the sign +1).

21 Examples

In the following some particular tensor representations of the group $GL(2, \mathbb{R})$ are studied.

21.1 Action of vector field X on tensor of type (0,1)

The flow of the vector field $X = RCx$ transforms an one-form $\Phi = \varphi\Theta$. Lie derivative of Φ is

$$\Phi' = (\varphi' + \varphi C)\Theta. \tag{75}$$

The transformation of the one-form Φ in the flow a_t is described by Lie-Maclaurin series:

$$\Phi_t = \sum_{k=0}^{\infty} \Phi^{(k)} \frac{t^k}{k!}, \tag{76}$$

where the coefficients $\Phi^{(k)}$ are the Lie derivatives.

Proposition 21.1 *Let the components φ in formula $\Phi = \varphi\Theta$ be invariants of the vector field X, i.e. $\varphi' = 0$. Then the one-form Φ satisfies a differential equation*

$$\Phi'' - \mathrm{tr}\, C \cdot \Phi' + \det C \cdot \Phi = 0. \tag{77}$$

Proof. Considering the theorem of Hamilton-Cayley in case of matrix C

$$C^2 - \mathrm{tr}\, C \cdot C + \det C \cdot E = 0, \tag{78}$$

we get

$$\begin{aligned}
\Phi &= \varphi\Theta, \\
\Phi' &= \varphi\Theta' = \varphi C\Theta, \\
\Phi'' &= \varphi C\Phi' = \varphi C^2\Theta \\
&= \varphi\left[\mathrm{tr}\, C \cdot C - \det C \cdot E\right]\Theta \\
&= \mathrm{tr}\, C \cdot \Phi' - \det C \cdot \Phi.
\end{aligned}$$

■

Proposition 21.2 *Let the one-form $\Phi = \varphi\Theta$ be invariant with respect to the vector field X. Then its components φ satisfys a differential equation*

$$\varphi'' + \mathrm{tr} C \cdot \varphi' + \det C \cdot \varphi = 0. \tag{79}$$

Proof. While

$$\Phi' = (\varphi' + \varphi C)\Theta \tag{80}$$

then the equation $\Phi' = 0$ is equivalent to the equations $\varphi' = -\varphi C$. In condition $\Phi' = 0$ must all the components φ satisfy system of differential equations $\varphi' = -\varphi C$ and every component individually the differential equation (79).

■

21.2 Action of vector field X on tensor of type (1,0)

The flow of X transforms the vector field $Y = Ry$ as follows. Considering Lie derivatives we get:

$$Y' = R(y' - Cy). \tag{81}$$

The transformation of the vector field Y in the flow a_t is described by Lie-Maclaurin series:

$$Y_t = \sum_{k=0}^{\infty} Y^{(k)} \frac{t^k}{k!}, \tag{82}$$

where the coefficients $Y^{(k)}$ are Lie derivatives.

Proposition 21.3 *If the components y of the vector field Y are invariants of the vector field X, i.e. $y' = 0$, then we get that the vector field Y satisfies a differential equation*

$$Y'' + \mathrm{tr} C \cdot Y' + \det C \cdot Y = 0. \tag{83}$$

Proof. Considering equation (78) we get

$$\begin{aligned} Y &= Ry, \\ Y' &= R'y = -RCy, \\ Y'' &= -R'Cy = RC^2 y \\ &= R\left[\mathrm{tr} C \cdot C - \det C \cdot E\right] y \\ &= -\mathrm{tr} C \cdot Y' - \det C \cdot Y. \end{aligned}$$

■

Proposition 21.4 *Let the vector field $Y = Ry$ be invariant with respect to the vector field X. Then its components y satisfies a differential equation*

$$y'' - \mathrm{tr}C \cdot y' + \det C \cdot y = 0. \tag{84}$$

Proof. In condition $Y' = 0$ the components y satisfy the system of differential equations $y' = Cy$. Considering theorem of Hamilton-Cayley for the matix C we see, that every component individually satisfy the differential equation (84).

∎

Remark 21.5 *For the vector field $Y = Ry$ and for the one-form $\Phi = \varphi\Theta$ the following implications are valid*

$$Y' = 0 \quad \Rightarrow \quad y'' - \mathrm{tr}C \cdot y' + \det C \cdot y = 0, \tag{85}$$

$$y' = 0 \quad \Rightarrow \quad Y'' + \mathrm{tr}C \cdot Y' + \det C \cdot Y = 0, \tag{86}$$

$$\Phi' = 0 \quad \Rightarrow \quad \varphi'' + \mathrm{tr}C \cdot \varphi' + \det C \cdot \varphi = 0, \tag{87}$$

$$\varphi' = 0 \quad \Rightarrow \quad \Phi'' - \mathrm{tr}C \cdot \Phi' + \det C \cdot \Phi = 0. \tag{88}$$

If the vector field Y is invariant then it components satisfies equation (85). *If the components y of the vector field are invariant then the vector field Y itself satisfies equation* (86). *If one-form Φ is invariant then its components φ satisfies equation* (87). *If the components φ of the one-form Φ are invariant then one-form Φ itself satisfies equation* (88).

21.3 Action of vector field X on tensor of type (1,1)

Let us consider tensor field of type $(1, 1)$

$$S = R_i s^i_j \Theta^j, \tag{89}$$

where $s = (s^i_j)$ is a 2×2 matrix. The Lie derivative of the tensor field S is

$$S' = R(s' - Cs + sC)\Theta. \tag{90}$$

The transformation of the tensor field S in the flow a_t is described by Lie-Maclaurin series:

$$S_t = \sum_{k=0}^{\infty} S^{(k)} \frac{t^k}{k!}, \tag{91}$$

where the coefficients $S^{(k)}$ are Lie derivatives of S with respect to X.

Proposition 21.6 *If the components s of the tensor field S satisfy the condition $s' = 0$, i.e. elements of matrix s are invariant with respect to the vector field X, then the tensor field satisfies a differential equation*

$$S''' = \Delta \cdot S', \tag{92}$$

where $\Delta = \mathrm{tr}^2 C - 4 \det C$.

Proof. In case of affine tensor field the indices can be omit

$$\begin{aligned}
S' &= R' s\Theta + Rs\Theta' \\
S'' &= R'' s\Theta + Rs\Theta'' + 2R' s\Theta' = \\
&= -(\mathrm{tr}\, C \cdot R' + \det C \cdot R) s\Theta \\
&\quad + Rs(\mathrm{tr}\, C \cdot \Theta' - \det C \cdot \Theta) + 2R' s\Theta', \\
S'' + 2 \det C \cdot S' &= -\mathrm{tr}\, C \cdot (R' s\Theta - Rs\Theta') + 2R' s\Theta', \\
S''' + 2 \det C \cdot S'' &= (\mathrm{tr}^2 C - \det C) \cdot (R' s\Theta + Rs\Theta') \\
&= (\Delta + 2 \det C) \cdot S', \\
S''' &= \Delta \cdot S'.
\end{aligned}$$

■

Proposition 21.7 *If the tensor field S is invariant with respect to the vector field X, i.e. $S' = 0$, then its components s satisfies a differential equation*

$$s''' = \Delta \cdot s', \tag{93}$$

where $\Delta = \mathrm{tr}^2 C - 4 \det C$.

Proof. Considering that

$$C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \tag{94}$$

and correspondences

$$s^1_1 = s_1, \quad s^1_2 = s_2, \quad s^2_1 = s_3, \quad s^2_2 = s_4, \tag{95}$$

we can rewrite the system $s' = Cs - sC$, or with indices $(s^i_j)' = C^i_k s^k_j - s^i_k C^k_j$, in the form

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}' = \begin{pmatrix} 0 & -c_3 & c_2 & 0 \\ -c_2 & c_1 - c_4 & 0 & c_2 \\ c_3 & 0 & c_4 - c_1 & -c_3 \\ 0 & c_3 & -c_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}. \tag{96}$$

Eigenvalues of the 4×4 matrix $\bar{C}$ are $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_1$ and 0 (multiple eigenvalues). Now we can write differential equation for the components using Proposition 19.1 and schema that is given in section 20 on the page 29. Equation (74) is in the form

$$s''' + \sigma_1 s'' + \sigma_2 s' + \sigma_3 s = 0, \tag{97}$$

where

$$\begin{aligned}
\sigma_1 &= 0, \\
\sigma_2 &= (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1) = -\mathrm{tr}^2 C + 4 \det C = -\Delta, \\
\sigma_3 &= 0.
\end{aligned}$$

And therefore $s''' = \Delta \cdot s'$.

■

The vector field X from the uv-plane is carried to the 4-dimensional $s_1 s_2 s_3 s_4$-space into the vector field $\bar{X}$

$$\bar{X} = s_1' \frac{\partial}{\partial s_1} + s_2' \frac{\partial}{\partial s_2} + s_3' \frac{\partial}{\partial s_3} + s_4' \frac{\partial}{\partial s_4} \tag{98}$$

or in matrix form

$$\bar{X} = \left(\tfrac{\partial}{\partial s_1} \quad \tfrac{\partial}{\partial s_2} \quad \tfrac{\partial}{\partial s_3} \quad \tfrac{\partial}{\partial s_4} \right) \cdot \begin{pmatrix} 0 & -c_3 & c_2 & 0 \\ -c_2 & c_1 - c_4 & 0 & c_2 \\ c_3 & 0 & c_4 - c_1 & -c_3 \\ 0 & c_3 & -c_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}. \tag{99}$$

The matrix $\bar{C}$ in the formula (96) can be rewrite in the form:

$$\begin{pmatrix} 0 & -c_3 & c_2 & 0 \\ -c_2 & c_1 - c_4 & 0 & c_2 \\ c_3 & 0 & c_4 - c_1 & -c_3 \\ 0 & c_3 & -c_2 & 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \tag{100}$$

$$+c_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

And therefore vector field $\bar{X}$ can be rewrite in the form

$$\bar{X} = c_1 \bar{X}_1 + c_2 \bar{X}_2 + c_3 \bar{X}_3 + c_4 \bar{X}_4, \tag{101}$$

where

$$
\begin{aligned}
\bar{X}_1 &= s_2\frac{\partial}{\partial s_2} - s_3\frac{\partial}{\partial s_3},\\
\bar{X}_2 &= s_3\frac{\partial}{\partial s_1} - s_1\frac{\partial}{\partial s_2} - s_4\frac{\partial}{\partial s_3} - s_3\frac{\partial}{\partial s_4},\\
\bar{X}_3 &= -s_2\frac{\partial}{\partial s_1} + s_4\frac{\partial}{\partial s_2} + s_1\frac{\partial}{\partial s_3} + s_2\frac{\partial}{\partial s_4},\\
\bar{X}_4 &= -s_2'\frac{\partial}{\partial s_2} + s_3'\frac{\partial}{\partial s_3} = -\bar{X}_1.
\end{aligned}
$$

21.4 Action of vector field X on tensor of type (0,2)

Let us consider tensor field of type (0,2)

$$G = g_{ij}\Theta^i \otimes \Theta^j, \tag{102}$$

where the components g_{ij} are symmetric with respect to indices, i.e. $g_{ij} = g_{ji}$. The Lie derivative of G is

$$G' = (g_{ij}' + 2g_{ik}C_j^k)\Theta^i\Theta^j. \tag{103}$$

Proposition 21.8 *If the components g_{ij} of tensor field* (102) *are invariants of the vector field X, then G satisfies a differential equation*

$$G''' - 3\mathrm{tr}\,C\cdot G'' + 2(\mathrm{tr}^2 C + 2\det C)G' - 4\mathrm{tr}\,C\cdot\det C\cdot G = 0. \tag{104}$$

Proof. Here we can do also without indices because of symmetry with respect to indices i and j

$$
\begin{aligned}
&G' = 2g\Theta\Theta\\
&G'' = 2g\Theta\Theta'' + 2g\Theta'\Theta'\\
&\quad = 2g\Theta(\mathrm{tr}\,C\cdot\Theta' - \det C\cdot\Theta) + 2g\Theta'\Theta',\\
&G'' - \mathrm{tr}\,C\cdot G' + 2\det C\cdot G' = 2g\Theta'\Theta',\\
&G''' - \mathrm{tr}\,C\cdot G'' + 2\det C\cdot G' = 4g\Theta'(\mathrm{tr}\,C\cdot\Theta' - \det C\cdot\Theta)\\
&\quad = 2\mathrm{tr}\,C(G'' - \mathrm{tr}\,C\cdot G' + 2\det C\cdot G - 2\det C)G'.
\end{aligned}
$$

■

Proposition 21.9 *If the tensor field G is invariant with respect to the vector field X then it's components g satisfy the differential equation*

$$g''' + 3\mathrm{tr}\,Cg'' + 2(\mathrm{tr}^2 C + 2\det C)g' + 4\mathrm{tr}\,C\cdot\det C\cdot g = 0. \tag{105}$$

Proof. In case of $G' = 0$ we get that $g'_{ij} = -2g_{ik}C^k_j$ or without indices simply $g' = -2gC$. The latter equation can be rewrite in the form

$$\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix}' = -\begin{pmatrix} 2c_1 & 2c_3 & 0 \\ c_2 & c_1 + c_4 & c_3 \\ 0 & 2c_2 & 2c_4 \end{pmatrix} \cdot \begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix}. \tag{106}$$

Eigenvalues of the matrix $\bar{C}$ are $-2\lambda_1$, $-\lambda_1 - \lambda_2$ and $-2\lambda_2$. Equation (74) is in the form

$$g''' + \sigma_1 g'' + \sigma_2 g' + \sigma_3 g = 0, \tag{107}$$

where

$$\begin{aligned} \sigma_1 &= 2\lambda_1 + \lambda_1 + \lambda_2 + 2\lambda_2 = 3\mathrm{tr}\,C, \\ \sigma_2 &= 2\lambda_1(\lambda_1 + \lambda_2) + 4\lambda_1\lambda_2 + 2\lambda_2(\lambda_1 + \lambda_2) = 2(\mathrm{tr}^2 C + 2\det C), \\ \sigma_3 &= 4\lambda_1\lambda_2(\lambda_1 + \lambda_2) = 4\mathrm{tr}\,C \det C. \end{aligned}$$

And therefore $g''' + 3\mathrm{tr}\,C g'' + 2(\mathrm{tr}^2 C + 2\det C)g' + 4\mathrm{tr}\,C \cdot \det C \cdot g = 0$.

■

The vector field carries to the 3-dimensional $g_{11}g_{12}g_{22}$-space

$$\bar{X} = g'_{11}\frac{\partial}{\partial g_{11}} + g'_{12}\frac{\partial}{\partial g_{12}} + g'_{22}\frac{\partial}{\partial g_{22}}. \tag{108}$$

Formula (108) can be rewrite in the matrix form

$$\bar{X} = \begin{pmatrix} \frac{\partial}{\partial g_{11}} & \frac{\partial}{\partial g_{12}} & \frac{\partial}{\partial g_{22}} \end{pmatrix} \cdot \begin{pmatrix} 2c_1 & 2c_3 & 0 \\ c_2 & c_1 + c_4 & c_3 \\ 0 & 2c_2 & 2c_4 \end{pmatrix} \cdot \begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix}. \tag{109}$$

The matrix in (106) and (109) can be rewrited in the following way:

$$\begin{pmatrix} 2c_1 & 2c_3 & 0 \\ c_2 & c_1 + c_4 & c_3 \\ 0 & 2c_2 & 2c_4 \end{pmatrix} = \tag{110}$$

$$= c_1\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

From here we get

$$\bar{X} = c_1\bar{X}_1 + c_2\bar{X}_2 + c_3\bar{X}_3 + c_4\bar{X}_4, \tag{111}$$

where

$$
\begin{aligned}
\bar{X}_1 &= -2g_{11}\frac{\partial}{\partial g_{11}} - g_{12}\frac{\partial}{\partial g_{12}},\\
\bar{X}_2 &= -g_{11}\frac{\partial}{\partial g_{12}} - 2g_{12}\frac{\partial}{\partial g_{22}},\\
\bar{X}_3 &= -2g_{12}\frac{\partial}{\partial g_{11}} - g_{22}\frac{\partial}{\partial g_{12}},\\
\bar{X}_4 &= -g_{12}\frac{\partial}{\partial g_{12}} - 2g_{22}\frac{\partial}{\partial g_{22}}.
\end{aligned}
\tag{112}
$$

21.5 Action of vector field X on tensor of type (0,3)

The flow of the vector field $X = RCx$ transforms a cubic-form $H = h\Theta\Theta\Theta$. Lie derivative of H is

$$H' = (h' + 3hC)\Theta\Theta\Theta. \tag{113}$$

In the formula (113) we have omitted indices. Actually we have to understand formula (113) in the following way

$$H' = (h'_{ijk} + 3h_{ijl}C^l_k)\Theta^i\Theta^j\Theta^k, \tag{114}$$

where the components h_{ijk} are symmetric with respect to the indices. The transformation of H in the flow a_t is described by Lie-Maclaurin series:

$$H_t = \sum_{k=0}^{\infty} H^{(k)}\frac{t^k}{k!}, \tag{115}$$

where the coefficients $H^{(k)}$ are Lie derivatives.

Proposition 21.10 *Let the components h in formula $H = h\Theta\Theta\Theta$ be invariants of the vector field X, i.e. $h' = 0$. The cubic-form H satisfies a differential equation*

$$
\begin{aligned}
&H'''' - 6(\lambda_1 + \lambda_2)H''' +\\
&+(11\lambda_1^2 + 32\lambda_1\lambda_2 + 11\lambda_2^2)H'' -\\
&-6(\lambda_1 + \lambda_2)(\lambda_1^2 + 7\lambda_1\lambda_2 + \lambda_2^2)H' +\\
&+9\lambda_1\lambda_2(2\lambda_1^2 + 5\lambda_1\lambda_2 + 2\lambda_2^2)H = 0
\end{aligned}
\tag{116}
$$

Proposition 21.11 *Let the cubic-form $H = h\Theta\Theta\Theta$ be invariant with respect to the vector field X. Then its components h satisfies a differential equation*

$$
\begin{aligned}
&h'''' + 6(\lambda_1 + \lambda_2)h''' +\\
&+(11\lambda_1^2 + 32\lambda_1\lambda_2 + 11\lambda_2^2)h'' +\\
&+6(\lambda_1 + \lambda_2)(\lambda_1^2 + 7\lambda_1\lambda_2 + \lambda_2^2)h' +\\
&+9\lambda_1\lambda_2(2\lambda_1^2 + 5\lambda_1\lambda_2 + 2\lambda_2^2)h = 0.
\end{aligned}
\tag{117}
$$

In case of $H' = 0$ we get that $h' = -3hC$. The latter equation can be rewrited in the form

$$\begin{pmatrix} h_{111} \\ h_{112} \\ h_{122} \\ h_{222} \end{pmatrix}' = - \begin{pmatrix} c_1 & 3c_3 & 0 & 0 \\ c_2 & 2(c_1 + c_4) & 2c_3 & 0 \\ 0 & 2c_2 & 2(c_1 + c_4) & c_3 \\ 0 & 0 & 3c_2 & c_4 \end{pmatrix} \cdot \begin{pmatrix} h_{111} \\ h_{112} \\ h_{122} \\ h_{222} \end{pmatrix}. \quad (118)$$

The vector field X is carried to the 4-dimensional $h_{111}h_{112}h_{122}h_{222}$-space

$$\bar{X} = h_{111}^{'} \frac{\partial}{\partial h_{111}} + h_{112}^{'} \frac{\partial}{\partial h_{112}} + h_{122}^{'} \frac{\partial}{\partial h_{122}} + h_{222}^{'} \frac{\partial}{\partial h_{222}}. \quad (119)$$

Last equation can be rewrite in the form

$$\bar{X} = c_1 \bar{X}_1 + c_2 \bar{X}_2 + c_3 \bar{X}_3 + c_4 \bar{X}_4, \quad (120)$$

where

$$\begin{aligned} \bar{X}_1 &= -2h_{111} \frac{\partial}{\partial h_{111}} - 2h_{112} \frac{\partial}{\partial h_{112}} - h_{122} \frac{\partial}{\partial h_{122}}, \\ \bar{X}_2 &= -h_{111} \frac{\partial}{\partial h_{112}} - 2h_{112} \frac{\partial}{\partial h_{122}} - 3h_{122} \frac{\partial}{\partial h_{222}}, \\ \bar{X}_3 &= 3h_{112} \frac{\partial}{\partial h_{111}} - 2h_{122} \frac{\partial}{\partial h_{112}} - h_{222} \frac{\partial}{\partial h_{122}}, \\ \bar{X}_4 &= -2h_{112} \frac{\partial}{\partial h_{112}} - 2h_{122} \frac{\partial}{\partial h_{122}} - h_{222} \frac{\partial}{\partial h_{222}}. \end{aligned} \quad (121)$$

21.6 Action of vector field X on tensor of type (1,2)

Let us consider the tensor field of type (1,2)

$$K = R_i k^i_{jk} \Theta^j \otimes \Theta^k. \quad (122)$$

Proposition 21.12 *If the components k^i_{jk} of tensor field (122) are invariants of the vector field X, i.e. $(k^i_{jk})' = 0$, then K satisfies a differential equation*

$$\begin{aligned} & K^{IV} - 2\mathrm{tr} C K^{'''} - (\mathrm{tr}^2 C - 10 \det C) K^{''} + \\ & + \mathrm{tr} C (10 \det C - 2\mathrm{tr}^2 C) K^{'} - \det C (2\mathrm{tr}^2 C - 9 \det C) K = 0. \end{aligned} \quad (123)$$

Proof.

$$
\begin{aligned}
(Rs\theta\theta)' &= R' s\theta\theta + 2Rs\theta'\theta, \\
(R' s\theta\theta)' &= -\mathrm{tr}\, C R' s\theta\theta - \det C Rs\theta\theta + 2R' s\theta'\theta, \\
(Rs\theta'\theta)' &= R' s\theta'\theta + \mathrm{tr} C Rs\theta'\theta - \det C Rs\theta\theta + Rs\theta'\theta', \\
(Rs\theta'\theta')' &= R' s\theta'\theta' + 2\mathrm{tr} C Rs\theta'\theta' - 2\det C Rs\theta'\theta, \\
(R' s\theta'\theta)' &= -\det C Rs\theta'\theta - \det C R' s\theta\theta + R' s\theta'\theta', \\
(R' s\theta'\theta')' &= \mathrm{tr} C R' s\theta'\theta' - \det C Rs\theta'\theta' - 2\det C R' s\theta'\theta.
\end{aligned}
$$

Second derivative of the tensor field is

$$
\begin{aligned}
(Rs\theta\theta)'' &= -\mathrm{tr} C R' s\theta\theta - \det C Rs\theta\theta + 2R' s\theta'\theta \\
&\quad +2(R' s\theta'\theta + \mathrm{tr} C Rs\theta'\theta - \det C Rs\theta\theta + Rs\theta'\theta') \\
&= -3\det C(Rs\theta\theta) - \mathrm{tr} C(R' s\theta\theta) \\
&\quad +2\mathrm{tr} C(Rs\theta'\theta) + 2(Rs\theta'\theta') + 4(R' s\theta'\theta)
\end{aligned}
$$

and multiplied with coefficient $3\det C$

$$
\begin{aligned}
&-6\det C(Rs\theta'\theta') - 6\mathrm{tr} C \det C(Rs\theta'\theta) - 12\det C(R' s\theta'\theta) = \\
&= -3\det C(Rs\theta\theta)'' - 9\det{}^2 C(Rs\theta\theta) - 3\det C \mathrm{tr} C(R' s\theta\theta). \qquad (124)
\end{aligned}
$$

Third derivative of the tensor field is

$$
\begin{aligned}
(Rs\theta\theta)''' &= -3\det C(Rs\theta\theta)' - \mathrm{tr} C(R' s\theta\theta)' \\
&\quad +2\mathrm{tr} C(Rs\theta'\theta)' + 2(Rs\theta'\theta')' + 4(R' s\theta'\theta)' \\
&= -7\det C(Rs\theta\theta)' + \mathrm{tr}^2 C(Rs\theta\theta)' \\
&\quad -\mathrm{tr} A \det C(Rs\theta\theta) + 6\mathrm{tr} C(Rs\theta'\theta') + 6(R' s\theta'\theta')
\end{aligned}
$$

and multiplied with coefficient $2\mathrm{tr} C$

$$
\begin{aligned}
&12\mathrm{tr} C(R' s\theta'\theta') + 12\mathrm{tr}^2 C(Rs\theta'\theta') = \\
&= 2\mathrm{tr} C(Rs\theta\theta)''' + 14\det C \mathrm{tr} C(Rs\theta\theta)' \\
&\quad -2\mathrm{tr}^3 C(Rs\theta\theta)' + 2\mathrm{tr}^2 C \det C(Rs\theta\theta). \qquad (125)
\end{aligned}
$$

We get result by substituting equations (124) and (125) into the forth derivative of the tensor field

$$
\begin{aligned}
(Rs\theta\theta)^{IV} &= -7\det C(Rs\theta\theta)'' + \mathrm{tr}^2 C(Rs\theta\theta)'' - \mathrm{tr} C \det C(Rs\theta\theta)' \\
&\quad +6\mathrm{tr} C(Rs\theta'\theta')' + 6(R' s\theta'\theta')' \\
&= -7\det C(Rs\theta\theta)'' + \mathrm{tr}^2 C(Rs\theta\theta)'' - \mathrm{tr} C \det C(Rs\theta\theta)'
\end{aligned}
$$

$$
\begin{aligned}
& +12\mathrm{tr}C(R^{'}s\theta^{'}\theta^{'}) + 12\mathrm{tr}^2C(Rs\theta^{'}\theta^{'}) - \\
& -6\det C(Rs\theta^{'}\theta^{'}) - 6\mathrm{tr}C\det C(Rs\theta^{'}\theta) \\
& -12\det C(R^{'}s\theta^{'}\theta) - 6\mathrm{tr}C\det C(Rs\theta^{'}\theta) \\
= \; & 2\mathrm{tr}C(Rs\theta\theta)''' + (\mathrm{tr}^2C - 10\det C)(Rs\theta\theta)'' \\
& +\mathrm{tr}C(10\det C - 2\mathrm{tr}^2C)(Rs\theta\theta)' + \det C(2\mathrm{tr}^2C - 9\det C)Rs\theta\theta
\end{aligned}
$$

■

Proposition 21.13 *If tensor field K is invariant with respect to the vector field X then it's components k^i_{jk} satisfies a differential equation*

$$
\begin{aligned}
& k^{IV} + 2\mathrm{tr}Ck^{'''} - (\mathrm{tr}^2C - 10\det C)k^{''} - \\
& -\mathrm{tr}C(10\det C - 2\mathrm{tr}^2C)k^{'} - \det C(2\mathrm{tr}^2C - 9\det C)k = 0. \quad (126)
\end{aligned}
$$

Proof. In case $K' = 0$ we get that $k' = Ck - 2kC$. The last equation can be rewrite on the form

$$
\begin{pmatrix} k^1_{11} \\ k^1_{12} \\ k^1_{22} \\ k^2_{11} \\ k^2_{12} \\ k^2_{22} \end{pmatrix}' = - \begin{pmatrix} c_1 & 2c_3 & 0 & -c_2 & 0 & 0 \\ c_2 & c_4 & c_3 & 0 & -c_2 & 0 \\ 0 & 2c_2 & 2c_4 - c_1 & 0 & 0 & -c_2 \\ -c_3 & 0 & 0 & 2c_1 - c_4 & 2c_3 & 0 \\ 0 & -c_3 & 0 & c_2 & c_1 & c_3 \\ 0 & 0 & -c_3 & 0 & 2c_2 & c_4 \end{pmatrix} \cdot \begin{pmatrix} k^1_{11} \\ k^1_{12} \\ k^1_{22} \\ k^2_{11} \\ k^2_{12} \\ k^2_{22} \end{pmatrix}. \quad (127)
$$

Eigenvalues of the matrix $\bar{C}$ are $-2\lambda_1 + \lambda_2$, $-\lambda_1$, $-\lambda_2$, $-2\lambda_2 + \lambda_1$, where $-\lambda_1$ and $-\lambda_2$ are multiple eigenvalues. Equation (74) is in the form

$$
k^{IV} + \sigma_1 k''' + \sigma_2 k'' + \sigma_3 k' \sigma_4 k = 0, \quad (128)
$$

where

$$
\begin{aligned}
\sigma_1 &= 2\lambda_1 - \lambda_2 + \lambda_1 + \lambda_2 + \lambda_2 - \lambda_1 = 2\mathrm{tr}C, \\
\sigma_2 &= -\lambda_1(\lambda_2 - 2\lambda_1) - \lambda_2(\lambda_2 - 2\lambda_1)(\lambda_2 - 2\lambda_1)(\lambda_1 - 2\lambda_2) \\
&\quad +\lambda_1\lambda_2 - \lambda_1(\lambda_1 - 2\lambda_2) - \lambda_2(\lambda_1 - 2\lambda_2) = -(\mathrm{tr}^2C - 10\det C), \\
\sigma_3 &= \lambda_1\lambda_2(\lambda_2 - 2\lambda_1) - \lambda_1(\lambda_2 - 2\lambda_1)(\lambda_1 - 2\lambda_2) \\
&\quad -\lambda_2(\lambda_2 - 2\lambda_1)(\lambda_1 - 2\lambda_2) + \lambda_1\lambda_2(\lambda_1 - 2\lambda_2) = -\mathrm{tr}C(10\det C - 2\mathrm{tr}^2C), \\
\sigma_4 &= \lambda_1\lambda_2(\lambda_2 - 2\lambda_1)(\lambda_1 - 2\lambda_2) = -\det C(2\mathrm{tr}^2C - 9\det C).
\end{aligned}
$$

And therefore

$$k^{IV} + 2\mathrm{tr}\,Ck^{'''} - (\mathrm{tr}^2 C - 10\det C)k^{''} -$$
$$-\mathrm{tr}\,C(10\det C - 2\mathrm{tr}^2 C)k^{'} - \det C(2\mathrm{tr}^2 C - 9\det C)k = 0.$$

■

The operator of linear group $GL(2,\mathbb{R})$ has an action in the 6-dimensional $k^1_{11}k^1_{12}k^1_{22}k^2_{11}k^2_{12}k^2_{22}$-space. In this space acts vector field $\bar{X}$

$$\bar{X} = -c_1\bar{X}_1 - c_2\bar{X}_2 - c_3\bar{X}_3 - c_4\bar{X}_4, \tag{129}$$

where

$$\begin{aligned}
\bar{X}_1 &= k^1_{11}\frac{\partial}{\partial k^1_{11}} - k^1_{22}\frac{\partial}{\partial k^1_{22}} + 2k^2_{11}\frac{\partial}{\partial k^2_{11}} + k^2_{12}\frac{\partial}{\partial k^2_{12}},\\
\bar{X}_2 &= -k^2_{11}\frac{\partial}{\partial k^1_{11}} + (k^1_{11} - k^2_{12})\frac{\partial}{\partial k^1_{12}} + (2k^1_{12} - k^2_{22})\frac{\partial}{\partial k^1_{22}} + k^2_{11}\frac{\partial}{\partial k^2_{12}} + 2k^2_{22}\frac{\partial}{\partial k^2_{22}},\\
\bar{X}_3 &= 2k^1_{12}\frac{\partial}{\partial k^1_{11}} + k^1_{22}\frac{\partial}{\partial k^1_{12}} + (2k^2_{12} - k^1_{11})\frac{\partial}{\partial k^2_{11}} + (k^2_{22} - k^1_{12})\frac{\partial}{\partial k^2_{12}} - k^2_{22}\frac{\partial}{\partial k^2_{22}},\\
\bar{X}_4 &= k^1_{12}\frac{\partial}{\partial k^1_{12}} + 2k^1_{22}\frac{\partial}{\partial k^1_{22}} - k^2_{11}\frac{\partial}{\partial k^2_{11}} + k^2_{22}\frac{\partial}{\partial k^2_{22}}.
\end{aligned}$$

LINEAARRÜHMA $GL(2,\mathbb{R})$ TENSORESITUSED
(Magistritöö lühikokkuvõte)

Hannes Lepp

1. Üldiselt on teada, kuidas tasandil vektorväli X määrab voo $a_t = \exp tX$. Voo mõjul punktid liiguvad piki trajektoore

$$a_t : u \to u_t = a_t(u)$$

ja funktsioonid tesienevad vastavalt kompositsioonile

$$f \to f_t = f \circ a_t.$$

Voos teisenevad tensorväljad, sh. vektorväljad ja diferentsiaalvormid.

2. Funktsiooni f_t on võimalik (teatud tingimustel) arendada astemeritta

$$f_t = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!},$$

kus kordajaiks $f^{(k)}$ on funktsiooni f Lie tuletised vektorvälja X suhtes. Samamoodi kehtib Lie-Maclaurini rida suvalise tensorvälja puhul, kus kordajaiks on tensorvälja Lie tuletised. Juhul, kui Lie tuletised on mingil moel seotud ja moodustavad hariliku diferentsiaalvõrrandi, siis selle lahend määrab tensorvälja teisenemise antud voos.

3. Kuna tensorväli on antud (holonoomses või mitteholonoomses) baasis, siis voos a_t teiseneb ka baas ja me peame teadma reeperi ning koreeperi derivatsioonivalemeid

$$R' = -RC, \quad \Theta' = C\Theta.$$

Priim tähistab Lie tuletist vektorvälja X suhtes ja C on vektorvälja komponenetide Jacobi maatriks (holonoomses baasis osatuletistest koosnev).

4. Lineaarse vektorvälja puhul on voog a_t määratud eksponentsiaalseadusega

$$U' = CU \quad \Rightarrow \quad U_t = e^{tC}U,$$

kus C on konstantne maatriks. Sama eksponentsiaalseadus määrab baasi (reeperi ja koreeperi) teisenemise:

$$
\begin{aligned}
R' = -RC \quad &\Rightarrow \quad R_t = Re^{-tC}, \\
\Theta' = C\Theta \quad &\Rightarrow \quad \Theta_t = e^{tC}\Theta.
\end{aligned}
$$

5. Lineaarses voos sirgjooned jäävad sirgjoonteks ja säilub sirgete paralleelsus. Pindalad kahanevad või paisuvad olenevalt divergentsist, kas $\mathrm{div}X < 0$ või vastavalt $\mathrm{div}X > 0$.

6. Kui lineaarses voos liikuda koos teljestikuga mööda mingit trajektoori, siis lokaalses (liikuvas) teljestikus kordub globaalne faasiportree.

7. Tasandil on määratud rühma $GL(2,\mathbb{R})$ toime ning lineaarne vektorväli X on selle rühma operaator. Konstantne maatriks C on tõlgendatav nagu Lie algebra $gl(2,\mathbb{R})$ element ja eksponentsiaal e^{tC} nagu rühma $GL(2,\mathbb{R})$ 1-parameetriline alamrühm. Rühma $GL(2,\mathbb{R})$ toime on esitatud tasandil lineaarsete vektorväljadega.

   Homoteetiate operaator kommuteerub kõikide lineaarsete vektorväljadega ja on seega nende infinitesimaalne sümmeetria. Lineaarsed vood klassifitseeruvad elliptilisteks, hüperboolseteks ja paraboolseteks (nulldivergentsiga ekviafiinsed teisendused), mida mõjutavad homoteetiad.

8. Kui tensorväli S on lineaarse voo a_t suhtes invariantne, s.t. $S_t = S$, siis tema komponendid baasis (R, Θ) muutuvad vastavalt eksponentsiaalseadusele
$$
s' = \bar{C}s \quad \Rightarrow s_t = e^{t\bar{C}}s,
$$
   kus maatriks $\bar{C}$ on maatriksi C elementide poolt (lineaarselt) määratud. Seega tensorruumis (ruumis $\mathbb{R}^N$, kus koordinaatideks on tensori S komponendid ja dimensioon N võrdub komponentide arvuga) on määratud lineaarrühma $GL(2,\mathbb{R})$ ja Lie algebra $gl(2,\mathbb{R})$ toime. Rühma $GL(2,\mathbb{R})$ operaatoreiks on vastavad lineaarsed vektorväljad ruumis $\mathbb{R}^N$. Taolised operaatorid on tuntud algebraliste invariantide teoorias (vt. D. Hilbert, K.Sibirski, D.Boularas jt.).

9. Eksponentsiaal e^{tC} (eksponentsiaal $e^{t\bar{C}}$) on taastatav maatriksi C (maatriksi $\bar{C}$) omaväärtuste kaudu. Kui λ_1 ja λ_2 on maatriksi C omaväärtused, siis maatriksi $\bar{C}$ omaväärtusteks (p,q)-tüüpi tensori korral on
$$
\lambda_{i_1} + \ldots + \lambda_{i_p} - \lambda_{j_1} - \ldots - \lambda_{j_q},
$$
   kus indeksite $i_1, \ldots, i_p, j_1, \ldots, j_q$ väärtusteks on 1 ja 2. Neil omaväärtustel on ühine reaalosa $(p-q)\alpha$, kus α on kaaskompleksarvude $\lambda_{1,2} = \alpha \pm i\beta$ reaalosa.

10. Maatriksi C puhul kehtib Hamilton-Cayley valem:

$$C^2 - \operatorname{tr} C \cdot C + \det C \cdot E = 0,$$

kus $\operatorname{tr} C = \lambda_1 + \lambda_2 = 2\alpha$, $\det C = \lambda_1\lambda_2 = \alpha^2 + \beta^2$ ja 0 tähistab null-vektorit. Kuna $U' = CU$ ja $U'' = C^2U$, siis siit järeldub, et kehtib lineaarne diferentsiaalvõrrand

$$U'' - \operatorname{tr} C \cdot U' + \det C \cdot U = 0,$$

kus 0 tähistab null-maatriksit. Samamoodi, kuna $\Theta' = C\Theta$, $\Theta'' = C^2\Theta$ ja $R' = -RC$, $R'' = RC^2$, siis kehtivad (Lie tuletistega) kaks differentsiaalvõrrandit

$$\begin{aligned} \Theta'' - \operatorname{tr} C \cdot \Theta' + \det C \cdot \Theta &= 0, \\ R'' + \operatorname{tr} C \cdot R' + \det C \cdot R &= 0. \end{aligned}$$

Esimene koreeperi, teine reeperi jaoks.

11. Vektorvälja $Y = Ry$ ja 1-vormi $\Phi = \varphi\Theta$ puhul kehtivad järgmised implikatsioonid

$$Y' = 0 \quad \Rightarrow \quad y'' - \operatorname{tr} C \cdot y' + \det C \cdot y = 0, \tag{130}$$

$$y' = 0 \quad \Rightarrow \quad Y'' + \operatorname{tr} C \cdot Y' + \det C \cdot Y = 0, \tag{131}$$

$$\Phi' = 0 \quad \Rightarrow \quad \varphi'' + \operatorname{tr} C \cdot \varphi' + \det C \cdot \varphi = 0, \tag{132}$$

$$\varphi' = 0 \quad \Rightarrow \quad \Phi'' - \operatorname{tr} C \cdot \Phi' + \det C \cdot \Phi = 0. \tag{133}$$

Kui vektorväli Y on invariantne, siis tema komponendid rahuldavad võrrandit (130). Kui tema komponendid y on invariantsed, siis vektorväli Y ise rahuldab võrrandit (131). Kui 1-vorm Φ on invariantne, siis tema komponendid φ rahuldavad võrrandit (132). Kui tema komponendid φ on invariantsed, siis 1-vorm Φ ise rahuldab võrrandit (133).

Võrrandid vektorvälja Y (1-vormi Φ) ja selle komponentide jaoks, s.t. (130) ja (131) ning (132) ja (133), on duaalses vastavuses: nende karakteristlike (ruut) võrrandite lahendid on vastasmärgilised. Võrrandid (130) ja (133) ning (131) ja (132) on ühesugused.

12. Maatriksi $\bar{C}$ puhul kehtib samuti Hamilton-Cayley valem. Sellele vastab omakorda lineaarne diferentsiaalvõrrand, mille kordajad on maatriksi $\bar{C}$ omaväärtuste (vt. p. 9) sümmeetrilised funktsioonid. Kui võtta need omaväärtused vastasmärkidega, saame duaalse diferentsiaalvõrrandi.

Vastavad diferentsiaalvõrrandid on tuletatud konkreetsetel juhtudel: $(p, q) = (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2)$. Juhtudel $(0, 2)$ ja $(0, 3)$ on tegu ruut- ja kuupvormiga, juhul $(1, 1)$ afiinorväljaga, juhul $(1, 2)$ dünaamilise süsteemiga e. tensorväljaga, mille komponendid on homogeensed ruutfunktsioonid. Juhud $(0, 1)$ ja $(1, 0)$, vastavalt 1-vormile ja vektorväljale, on eespool mainitud.

13. Maatriks $\bar{C}$ on määratud tensori tüübiga e. valentsusega (p, q). Maatriksite $\bar{C}$ omaväärtused kõikide valentsuste puhul moodustavad komplekstasandil korrapärase võrestiku. Konreetse valentsuse (p, q) korral on võimalik maatriksi $\bar{C}$ omaväärtused ära näidata.

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