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JOHANN LANGEMETS

Geometrical structure in diameter 2 Banach spaces

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Geometrical structure in diameter 2 Banach spaces

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Chapter 1

Introduction

1.1 Background

In 2001, O. Nygaard and D. Werner [NW] showed that in an infinite-dimensional uniform algebra every nonempty relatively weakly open subset of its closed unit ball has diameter equal to 2. If a Banach space satisfies this condition, then it is said to have the *diameter* 2 *property* (D2P) (see, e.g., [ABGRP], [ALN1], and [BGRP]). Besides the infinite-dimensional uniform algebras, for example, Banach spaces with the Daugavet property (see [Shv]), infinite-dimensional C^*-algebras (see [BGLPRP]), and nonreflexive M-embedded spaces (see [LP]) all have the D2P.

In addition to the diameter 2 property, T. A. Abrahamsen, V. Lima, and O. Nygaard in [ALN1] considered two other formally different diameter 2 properties—the local diameter 2 property and the strong diameter 2 property.

According to the terminology in [ALN1], a Banach space has the *local diameter* 2 *property* (LD2P) if every slice of its closed unit ball has diameter equal to 2; and it has the *strong diameter* 2 *property* (SD2P) if every convex combination of slices of the unit ball has diameter equal to 2. For example, the classical Banach spaces c_0, ℓ_∞, $C[0,1]$, $L_1[0,1]$, and $L_\infty[0,1]$ all have the SD2P.

The D2P clearly implies the LD2P. The SD2P implies the D2P, this follows directly from Bourgain's lemma (see [GGMS, Lemma II.1 p. 26]), which asserts that every nonempty relatively weakly open subset of the unit ball contains some convex combination of slices.

In [ALN1], it was conjectured that these three diameter 2 properties are different. Recently, J. Becerra Guerrero, G. López Pérez, and A. Rueda Zoca have shown that c_0 can be equivalently renormed to enjoy the LD2P, but failing the D2P (see [BGLPRZ2, Theorem 2.4]). The difference of the D2P and the SD2P was obtained independently in [Lan] (see also [HL1]) and in [ABGLP]. The key observation is that the ℓ_p-sum of two Banach spaces never has the SD2P for any p with $1 < p < \infty$. However, the D2P is stable by forming the ℓ_p-sum.

In 1989, G. Godefroy (see [God]) introduced the notion of an octahedral norm in order to characterize Banach spaces containing an isomorphic copy of ℓ_1. A Banach space X is *octahedral* (OH) if, for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there is a norm one element y in X such that

$$\|x + y\| \geq (1 - \varepsilon)\big(\|x\| + \|y\|\big) \quad \text{for all } x \in E.$$

G. Godefroy (see [God, Remark II.5, 2)], see also [Dev, Remark (c), p. 119]) remarks that a Banach space is OH if and only if every convex combination of weak* slices of the dual unit ball is of diameter 2. From this, it follows that a Banach space has the SD2P if and only if its dual space is OH. Similarly, one can characterize the dual space of a Banach space with the LD2P (cf. [DGZ, Proposition I.1.11]).

In order to characterize the dual of Banach spaces with the D2P, the concept of a weakly octahedral norm was introduced in [HLP]. A Banach space X is *weakly octahedral* (WOH) if, for every finite-dimensional subspace E of X, every element x^* in the dual unit ball, and every $\varepsilon > 0$, there is a norm one element y in X such that

$$\|x + y\| \geq (1 - \varepsilon)\big(|x^*(x)| + \|y\|\big) \quad \text{for all } x \in E.$$

Clearly, every OH Banach space is WOH.

Recently, almost square Banach spaces were introduced and studied (see [ALL]). A Banach space X is *almost square* (ASQ) if, for every natural number n and norm one elements $x_1, \ldots, x_n$ in X there exists a sequence (y_k) of norm one elements in X such that

$$\|x_i \pm y_k\| \to 1 \quad \text{for all } i \in \{1, \ldots, n\},$$

as $k \to \infty$. Our main motivation to study ASQ Banach spaces is their connection to diameter 2 properties. If a Banach space is ASQ, then it has the SD2P, and thus its dual is OH.

In [Whi], R. Whitley introduced the thickness index and the thinness index of a Banach space. It turns out that these indices are closely related to the aforementioned properties. Namely, a Banach space is OH if and only if its thickness index equals 2, and a Banach space is ASQ if and only if its thinness index equals 1.

1.2 Summary of the thesis

The main aim of this thesis is to investigate diameter 2 properties in Banach spaces, and related notions and properties such as the octahedrality of the norm, almost square Banach spaces, and thickness and thinness indices. We establish the dual connection between (weak*) diameter 2 properties and octahedralities. Almost square Banach spaces have the strong diameter 2 property. It turns out that octahedral Banach spaces are exactly the spaces whose thickness index is 2, and almost square Banach spaces are exactly the spaces whose thinness index is 1.

The thesis has been organized as follows.

Chapter 1 briefly introduces the historic background of diameter 2 properties, provides a summary of the thesis, and describes the notation used in the thesis.

In Chapter 2, we introduce three diameter 2 properties, the basic concepts of this thesis. We give a brief overview of the latest research showing the extreme differences of these properties. Also, the weak* versions of the diameter 2 properties are considered. A survey of some main classes of Banach spaces with diameter 2 properties is given. We present the stability results of diameter 2 properties when forming ℓ_p-sums. In particular, the (local) diameter 2 property is stable by taking the ℓ_p-sum for every p with $1 \leq p \leq \infty$. However, the ℓ_p-sum of two Banach spaces never has the strong diameter 2 property for any p with $1 < p < \infty$. We also show that diameter 2 properties carry over to the whole space from a nonzero M-ideal. These results are obtained in [Lan] and [HL1].

In Chapter 3, octahedral Banach spaces are introduced and studied. It is known that a Banach space has the strong diameter 2 property if and only if the norm on its dual space is octahedral. We consider two more versions of octahedrality, which we show are dual properties to the diameter 2 property and to the local diameter 2 property. We study stability properties of different types of octahedrality and provide alternative proofs of some known stability results of diameter 2 properties. Necessary and sufficient conditions for spaces of operators to be octahedral are also considered. This chapter is mainly based on [HLP].

In Chapter 4, we introduce and study almost square Banach spaces. These spaces have the strong diameter 2 property. We provide examples and characterizations of almost square Banach spaces. We prove that nonreflexive spaces which are M-ideals in their biduals are almost square. We show that every Banach space containing a complemented copy of c_0 can be renormed to be almost square. Furthermore, we study local and weak versions of almost squareness.

This chapter is based on [ALL].

In Chapter 5, we complement and extend some recent results on Whitley's indices of thickness and thinness in three directions. This is motivated by the fact that a Banach space is octahedral if and only if its thickness is 2, and a Banach space is almost square if and only if its thinness is 1. Firstly, we investigate both the indices when forming ℓ_p-sums of Banach spaces, and obtain estimations which show that they behave rather differently. Secondly, we examine the relation of the indices of the space and its subspace. Finally, every Banach space containing a complemented copy of c_0 can be equivalently renormed to have thickness and thinness 1. This chapter is based on [ALLN] and [HL2].

In the Appendix, we summarize some important examples of Banach spaces with the aforementioned properties and stability results in two cross tables.

1.3 Notation

Our notation is standard.

We consider only nontrivial Banach spaces over the field of real numbers. We usually assume that Banach spaces under consideration are infinite-dimensional.

In a Banach space X, we denote the unit sphere by S_X, the closed unit ball by B_X, and the closed ball with center at x and radius $r > 0$ by $B(x, r)$. By X^*, we denote the dual space of X. For a subset A of X, its diameter is denoted by $\operatorname{diam}(A)$, the set of its extreme points by $\operatorname{ext}(A)$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span}(A)$, and its convex hull by $\operatorname{conv}(A)$. The norm closures of the latter two sets are denoted by $\overline{\operatorname{span}}(A)$ and $\overline{\operatorname{conv}}(A)$, respectively. For closures with respect to other topologies, we mark the topology separately, such as $\overline{\operatorname{conv}}^{w^*}(A)$, etc.

A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding.

For Banach spaces X and Y, by $\mathcal{L}(X, Y)$ we will denote the Banach space of all bounded linear operators acting from X to Y. We denote by $\mathcal{F}(X, Y)$ the subspace of $\mathcal{L}(X, Y)$ of all finite rank operators. If T is an operator in $\mathcal{L}(X, Y)$, then its kernel is denoted by $\ker T$ and its range by $\operatorname{ran} T$. For a T in $\mathcal{L}(X, Y)$, the corresponding adjoint operator is denoted by T^*, and the restriction of T to a subset A of X will be denoted by $T|_A$.

We assume that the reader is familiar with well-known basic notions and theorems from the theory of Banach spaces and topological vector spaces (such as the Alaoglu theorem, the Goldstine theorem, the Krein–Milman theorem, the Principle of Local Reflexivity, the Sobczyk theorem, etc.), and we shall sometimes use them without proper references.

Chapter 2

Diameter 2 properties

Following [ALN1], we introduce the diameter 2 properties, the basic concept of this thesis. We give a brief overview of the latest research showing the extreme differences of these properties. Also, the weak* versions of the diameter 2 properties are considered. A survey of some main classes of Banach spaces with the diameter 2 properties is given. We present the known stability results of diameter 2 properties when forming ℓ_p-sums. In particular, the ℓ_p-sum of two Banach spaces, where p is such that $1 < p < \infty$, never has the strong diameter 2 property. We also show that the diameter 2 properties carry over to the whole space from a nonzero M-ideal. These results are obtained in [Lan] and [HL1].

2.1 Preliminaries

We start with an essential concept of this thesis—the notion of a slice.

Definition 2.1. Let X be a Banach space and let B be a nonempty bounded subset of X. A *slice* of B is a set of the form

$$S(B, x^*, \alpha) = \{x \in B \colon x^*(x) > \sup_{y \in B} x^*(y) - \alpha\},$$

where $x^* \in X^*$ and $\alpha > 0$. We usually assume that the defining functional x^* is in S_{X^*}.

If X is a dual space, then slices of B whose defining functional comes from (the canonical image of the) predual of X are called *weak* slices* of B.

A slice $S(B, x^*, \alpha)$ is clearly a nonempty intersection of B with an open half-space $\{x \in X : x^*(x) > \sup_{y\in B} x^*(y) - \alpha\}$. Therefore, a slice is always relatively weakly open, and, a weak* slice is always relatively weak* open.

A *convex combination of slices* of B_X is a set of the form

$$\sum_{i=1}^{n} \lambda_i S(B_X, x_i^*, \alpha_i),$$

where $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$.

In the thesis, we sometimes refer to the Radon–Nikodým property. Although it has many equivalent formulations, it can also be characterized by slices.

Definition 2.2 (see, e.g., [FHHMZ, Theorem 11.15]). A Banach space X is said to have the *Radon–Nikodým property*, if every nonempty bounded subset of X has slices of arbitrarily small diameter, that is, for every bounded subset B of X and for every $\varepsilon > 0$, there is an $x^* \in X^*$ and $\alpha > 0$ such that $S(B, x^*, \alpha)$ has diameter less than ε.

We remark that all reflexive Banach spaces enjoy the Radon–Nikodým property (see, e.g., [FHHMZ, Corollary 11.10]).

Lemma 2.3 (Choquet, 1969; see, e.g., [FHHMZ, Lemma 3.69]). *Let X be a Banach space. Let C be a weakly compact convex set in X and e an extreme point of C. Then slices of C containing e form a neighborhood basis at e in the relative weak topology of C.*

Remark 2.1. Observe that, for every $c \in C$, slices of C that contain c form a neighborhood subbasis for the relative weak topology of C at c; finite intersections of slices therefore form a basis for the relative weak topology. Choquet's lemma says that under the assumptions for C at any extreme point $e \in C$ the latter subbasis is, in fact, a basis.

Similarly, we have the result for the relative weak* topology.

Lemma 2.4 (Choquet). *Let X be a Banach space. Let C be a weak* compact convex set in X^* and e^* an extreme point of C. Then weak* slices of C containing e^* form a neighborhood basis at e^* in the relative weak* topology of C.*

Remark 2.2. Observe that, for every $c^* \in C$, weak* slices of C that contain c^* form a neighborhood subbasis for the relative weak* topology of C at c^*; finite intersections of weak* slices therefore form a basis for the relative weak* topology. Choquet's lemma says that under the assumptions for C at any extreme point $e^* \in C$ the latter subbasis is, in fact, a basis.

Lemma 2.5 (Bourgain, 1979; cf. [GGMS, Lemma II.1 p. 26])**.** *Let X be a Banach space. Let C be a bounded convex set in X^* and let U be a nonempty relatively weak* open subset of C. Then there exists $n \in \mathbb{N}$, weak* slices $S_1^*, \dots, S_n^*$ of C, and scalars $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that*

$$\sum_{i=1}^n \lambda_i S_i^* \subset U.$$

Proof. Let U be a relatively weak* open subset of C containing an element x^*. Find a weak* convex neighbourhood V of zero such that $(x^* + 2V) \cap C \subset U$.

By the Banach–Alaoglu theorem (see, e.g., [Meg, Theorem 2.6.18]), $\overline{C}^{w^*}$ is weak* compact. Therefore, by the Krein–Milman theorem (see, e.g., [FHHMZ, Theorem 3.65]), we have that $\overline{C}^{w^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{ext}(\overline{C}^{w^*}))$.

Denote by $E = \operatorname{ext}(\overline{C}^{w^*})$. Then clearly $x^* \in \overline{\operatorname{conv}}^{w^*}(E)$. Thus, there are $n \in \mathbb{N}$, $e_1^*, \dots, e_n^* \in E$, and scalars $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\sum_{i=1}^n \lambda_i e_i^* \in x^* + V.$$

By Lemma 2.4, there is a weak* slice $\tilde{S}_i^*$ of $\overline{C}^{w^*}$ with $\tilde{S}_i^* \subset e_i^* + V$ for every $i \in \{1, \dots, n\}$. We take $S_i^* = \tilde{S}_i^* \cap C$ for every $i \in \{1, \dots, n\}$. Then $S_1^*, \dots, S_n^*$ are weak* slices of C satisfying

$$\sum_{i=1}^n \lambda_i S_i^* \subset \sum_{i=1}^n \lambda_i (e_i^* + V) \cap C \subset (x^* + 2V) \cap C \subset U.$$

□

Lemma 2.6 (Bourgain)**.** *Let X be a Banach space. If U is a nonempty relatively weakly open subset of B_X, then there exists $n \in \mathbb{N}$, slices $S_1, \dots, S_n$ of B_X, and scalars $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that*

$$\sum_{i=1}^n \lambda_i S_i \subset U.$$

Proof. Let U be a nonempty relatively weakly open subset of B_X. Observe that U is a relatively weak* open subset of $B_X \subset X^{**}$. By Lemma 2.5, there exists $n \in \mathbb{N}$, weak* slices $S_1^*, \dots, S_n^*$ of B_X, and scalars $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i S_i^* \subset U.$$

Notice that the weak* slices of B_X are precisely the weak slices of B_X. This proves the result. □

2.2 Definitions and basic results

The Radon–Nikodým property, point of continuity property, and strong regularity are well-known and purely isomorphic properties. In Banach spaces with the Radon–Nikodým property one can always find arbitrarily small slices in any nonempty bounded set. Similarly, if a Banach space has the point of continuity property, then the unit ball has relatively weakly open subsets with diameter arbitrarily small. In a strongly regular Banach space, the unit ball has convex combinations of slices with diameter arbitrarily small.

In this chapter, we study the "extremely opposite" properties to these three classical properties, when all slices or all nonempty relatively weakly open subsets or all convex combinations of slices of the unit ball of a Banach space have diameter 2. For example, the classical Banach spaces c_0, ℓ_∞, $C[0,1]$, $L_1[0,1]$, and $L_\infty[0,1]$ all have the property that every convex combinations of slices of their unit ball have diameter 2.

The investigation of Banach spaces with this phenomena probably started in the paper [NW] by O. Nygaard and D. Werner, where it is shown that in an infinite-dimensional uniform algebra every nonempty relatively weakly open subset of the unit ball has diameter 2. By now it is known that in an infinite-dimensional uniform algebra even all convex combinations of slices of the unit ball are of diameter 2 (see [ALN1]).

A systematic treatment of all three diameter 2 properties was started by T. A. Abrahamsen, V. Lima, and O. Nygaard in [ALN1]. This paper is also a starting point for this thesis.

Definition 2.7 (see [ALN1])**.** A Banach space X has the

(i) *local diameter 2 property* (LD2P) if every slice of B_X has diameter equal to 2;

(ii) *diameter 2 property* (D2P) if every nonempty relatively weakly open subset of B_X has diameter equal to 2;

(iii) *strong diameter 2 property* (SD2P) if every convex combination of slices of B_X has diameter equal to 2.

We remark that the diameter 2 properties are isometric properties that might be ruined by passing to an equivalent norm.

Notice that in (an infinite-dimensional) Banach space a nonempty relatively weakly open subset of its unit ball always intersects with the unit sphere since weak neighborhoods contain an intersection of a finite number of open half-spaces.

A convex combination of slices need not be relatively weakly open (it might be contained in the open unit ball or even in some ball $B(0, r)$, where $r < 1$ (see, e.g., [GGMS, Remark IV.5 p. 48] or the proof of Theorem 2.33 and the remark after that)).

Remark 2.3.

(a) A Banach space X has the D2P if and only if every nonempty intersection of a finite number of slices of B_X has diameter 2 since slices form a subbasis of the relative weak topology.

(b) By Bourgain's lemma (see Lemma 2.6), a Banach space X has the SD2P if and only if every nonempty convex combination of relatively weakly open subsets of B_X is of diameter 2.

For a Banach space to have the SD2P it is enough to assume that every average of a finite number of slices is of diameter 2.

Proposition 2.8. *A Banach space X has the SD2P if and only if every average of a finite number of slices of B_X has diameter* 2*, that is,* $\operatorname{diam}(\frac{1}{n}\sum_{i=1}^{n} S_i) = 2$*, whenever $n \in \mathbb{N}$ and $S_1, \ldots, S_n$ are slices of B_X.*

For a Banach space the following implications hold:

$$\text{SD2P} \Rightarrow \text{D2P} \Rightarrow \text{LD2P}.$$

The first implication follows directly from Bourgain's lemma (see Lemma 2.6). The second implication is clear since any slice of the unit ball is also relatively weakly open.

In [ALN1], it was conjectured that these three diameter 2 properties are different. Recently, J. Becerra Guerrero, G. López Pérez, and A. Rueda Zoca have shown that c_0 can be equivalently renormed to enjoy the LD2P, but failing the D2P (see [BGLPRZ2]).

Theorem 2.9 (see [BGLPRZ2, Theorem 2.4]). *Let X be a Banach space. If X contains an isomorphic copy of c_0, then X can be equivalently renormed so that X has the LD2P and X contains nonempty relatively weakly open subsets with arbitrarily small diameter.*

The difference of the D2P and the SD2P was obtained independently in [Lan] (see also [HL1]) and in [ABGLP]. The key observation is that the ℓ_p-sum of two Banach spaces, where p is such that $1 < p < \infty$, never has the SD2P. However, the D2P is stable by forming the ℓ_p-sum.

Example 2.10 (see [Lan, Theorem 3.23], [HL1, Theorem 1] or [ABGLP, Theorem 3.2]). The Banach space $c_0 \oplus_2 c_0$ has the D2P but fails to have the SD2P. Indeed, by Theorem 2.31, $c_0 \oplus_2 c_0$ has the D2P and, by Theorem 2.33, it cannot have the SD2P.

Every convex combination of slices in $c_0 \oplus_2 c_0$ has diameter at least 1 (see [BGLPRZ3, Proposition 2.1]). The next result tells us that the D2P and the SD2P can be extremely different too.

Theorem 2.11 (see [BGLPRZ3, Theorem 2.5]). *Let X be a Banach space. If X contains an isomorphic copy of c_0, then X can be equivalently renormed so that X has the D2P and X contains convex combinations of slices with arbitrarily small diameter.*

A natural question to ask is whether diameter 2 properties of a dual space remain the same properties if, instead of all slices or relatively weakly open subsets, one considers only weak* slices or relatively weak* open subsets. The following example shows us that in a dual space one might have that every convex combination of weak* slices is of diameter 2, however, it has slices with arbitrarily small diameter.

Example 2.12 (see [BGLPRZ3] and [HLP, Example 1.1]). Every convex combination of weak* slices of $B_{C[0,1]^*}$ has diameter 2 (this follows by observing that every weak* slice of $B_{C[0,1]^*}$ contains infinitely many different functionals arising via integrating against a measure supported at a singleton); however, $B_{C[0,1]^*}$ has slices with arbitrarily small diameter (to see this, observe that $C[0,1]^* \cong \ell_1([0,1]) \oplus_1 C[0,1]^*$, and $\ell_1([0,1])$ has the Radon–Nikodým property).

Example 2.12 suggests that it makes sense to consider also the weak* versions of the diameter 2 properties.

Definition 2.13 (see [BGLPRZ3] and [HLP]). Let X be a Banach space. We say that X^* has the

(i) *weak* local diameter* 2 *property* (w*-LD2P) if every weak* slice of B_{X^*} has diameter equal to 2;

(ii) *weak* diameter* 2 *property* (w*-D2P) if every nonempty relatively weak* open subset of B_{X^*} has diameter equal to 2;

(iii) *weak* strong diameter* 2 *property* (w*-SD2P) if every convex combination of weak* slices of B_{X^*} has diameter equal to 2.

We first observe that for a dual Banach space the following implications hold:

$$\text{w}^*\text{-SD2P} \Rightarrow \text{w}^*\text{-D2P} \Rightarrow \text{w}^*\text{-LD2P}.$$

The first implication, similarly to the diameter 2 properties, follows directly from the weak* version of Bourgain's lemma (see Lemma 2.5). The second implication is clear since any weak* slice of the unit ball is also relatively weak* open.

Remark 2.4 (cf. Remark 2.3). Let X be a Banach space.

(a) Then X^* has the w*-D2P if and only if every nonempty intersection of a finite number of weak* slices of B_{X^*} has diameter 2 since weak* slices form a subbasis of the relative weak* topology.

(b) By the weak* version of Bourgain's lemma (see Lemma 2.5), X^* has the w*-SD2P if and only if every nonempty convex combination of relatively weak* open subsets of B_{X^*} is of diameter 2.

(c) Similarly to Proposition 2.8, one can show that X^* has the w*-SD2P if and only if every average of a finite number of weak* slices of B_{X^*} is of diameter 2.

The diameter 2 properties and the corresponding weak* diameter 2 properties are connected in a natural way.

Proposition 2.14 (see [Lan, Proposition 3.3]). *A Banach space X has the LD2P (resp. D2P, SD2P) if and only if X^{**} has the w*-LD2P (resp. w*-D2P, w*-SD2P).*

Proof. We will only prove it for the LD2P case. The other cases are similar.

Assume first that X has the LD2P. Let $S(B_{X^{**}}, x^*, \alpha)$ be a weak* slice of $B_{X^{**}}$. It is clear that $S(B_X, x^*, \alpha) \subset S(B_{X^{**}}, x^*, \alpha)$. By the assumption,

$$2 = \operatorname{diam}(S(B_X, x^*, \alpha)) \leq \operatorname{diam}(S(B_{X^{**}}, x^*, \alpha)) \leq 2.$$

Thus X^{**} has the w*-LD2P.

Assume now that X^{**} has the w*-LD2P. Let $S(B_X, x^*, \alpha)$ be a slice of B_X. Then $S(B_X, x^*, \alpha)$ is weak* dense in the corresponding weak* slice $S(B_{X^{**}}, x^*, \alpha)$ of $B_{X^{**}}$. Indeed, fix $x^{**} \in S(B_{X^{**}}, x^*, \alpha)$. By Goldstine's theorem (see, e.g., [FHHMZ, Theorem 3.96]), there is a net (x_α) in B_X which converges to x^{**} in the weak* topology. Since

$$1 - \alpha < x^{**}(x^*) = \lim_\alpha x^*(x_\alpha),$$

there is an index α_0 such that $x_\alpha \in S(B_X, x^*, \alpha)$ whenever $\alpha \geq \alpha_0$. This proves our claim.

Let $\varepsilon > 0$. By the assumption, there exist $x^{**}, \tilde{x}^{**} \in S(B_{X^{**}}, x^*, \alpha)$ such that $\|x^{**} - \tilde{x}^{**}\| > 2 - \varepsilon$. Since $S(B_X, x^*, \alpha)$ is weak* dense in $S(B_{X^{**}}, x^*, \alpha)$, there are nets (x_α) and $(\tilde{x}_\alpha)$ in $S(B_X, x^*, \alpha)$ such that the net $(x_\alpha - \tilde{x}_\alpha)$ converges to $x^{**} - \tilde{x}^{**}$ in the weak* toplogy. We have

$$2 - \varepsilon < \|x^{**} - \tilde{x}^{**}\| \leq \liminf_\alpha \|x_\alpha - \tilde{x}_\alpha\|,$$

because the norm on X^{**} is weak* lower semicontinuous. Thus, the diameter of $S(B_X, x^*, \alpha)$ is equal to 2 and X has the LD2P. □

Propositon 2.14 helps us to construct a dual space which has the w*-LD2P (resp. w*-D2P) but fails to have the w*-D2P (resp. w*-SD2P). Indeed, by Proposition 2.14, it is enough to consider the bidual of the space in Theorem 2.9 (resp. Example 2.10).

Note that if a dual space has some diameter 2 property, then it will also have the corresponding weak* diameter 2 property. Thus, from Proposition 2.14 we conclude that X inherits all three diameter 2 properties from its bidual X^{**}.

Corollary 2.15 (see [Lan, Corollary 3.4]). *If X^{**} has the LD2P (resp. D2P, SD2P), then X has the LD2P (resp. D2P, SD2P).*

The converses of Corollary 2.15 fail for each diameter 2 property.

Example 2.16. The Banach space $L_1[0, 1]$ has the SD2P (see Proposition 2.25), but its bidual $L_1[0, 1]^{**}$ contains slices with arbitrarily small diameter, thus it fails the LD2P. Indeed, since $L_1[0,1]^{**}$ is a dual space, then by the Krein–Milman theorem the unit ball of $L_1[0,1]^{**}$ has extreme points. Now, by [BGM1] and [BGM2], the unit ball even has strongly exposed points, thus it contains slices with arbitrarily small diameter.

Recently, T. A. Abrahamsen, V. Lima, and O. Nygaard have generalized Corollary 2.15 to include almost isometric ideals. The notion of an almost isometric ideal was introduced and studied in [ALN2].

Definition 2.17 (see [ALN2])**.** Let X be a Banach space and Y a subspace. Y is called an *almost isometric ideal* (ai-ideal) in X if for every $\varepsilon > 0$ and every finite-dimensional subspace $E \subset X$ there exists $T : E \to Y$ such that

(i) $Te = e$ for all $e \in E \cap Y$;

(ii) $(1+\varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1+\varepsilon)\|e\|$ for all $e \in E$.

Note that, by the Principle of Local Reflexivity (see, e.g., [FHHMZ, Theorem 6.3]), any Banach space X is an ai-ideal in its bidual X^{**}.

Proposition 2.18 (see [ALN2, Propositions 3.2, 3.3, and Corollary 3.4])**.** *Let X be a Banach space and Y an ai-ideal in X. If X has the LD2P (resp. D2P, SD2P), then Y has the LD2P (resp. D2P, SD2P).*

2.3 Examples

Examples of classes of Banach spaces with diameter 2 properties include infinite-dimensional M-embedded spaces, infinite-dimensional uniform algebras, and Banach spaces with the Daugavet property.

Proposition 2.19. *The Banach space c_0 has the SD2P.*

Proof. Let $\sum_{i=1}^{n} \frac{1}{n} S(B_{c_0}, x_i^*, \alpha_i)$ be a convex combination of slices of B_{c_0}, where $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in S_{\ell_1} = S_{c_0^*}$, and $\alpha_1, \ldots, \alpha_n > 0$. We will show that $\operatorname{diam}(\frac{1}{n}\sum_{i=1}^{n} S(B_{c_0}, x_i^*, \alpha_i)) = 2$.

We take $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}$. The dual space c_0^* is identified with ℓ_1 in the usual way. Thus, for every $i \in \{1, \ldots, n\}$, we identify the functional x_i^* with an element $(\beta_{i,k})$ from ℓ_1, where

$$x_i^*(x) = \sum_{k=1}^{\infty} \beta_{i,k} x_k \qquad \text{for all } x = (x_k) \in c_0.$$

Let $x_i = (x_{i,k}) \in S(B_{c_0}, x_i^*, \alpha)$, $i \in \{1, \dots, n\}$, be chosen. Choose a $K \in \mathbb{N}$ such that $\sum_{k=1}^{K} \beta_{i,k} x_{i,k} > 1 - \alpha/2$ for all $i \in \{1, \dots, n\}$. Thus $\sum_{k>K}^{\infty} |\beta_{i,k}| < \alpha/2$ for all $i \in \{1, \dots, n\}$.

Let $P_{K+1} \colon c_0 \to c_0$ denote the projection onto the $(K+1)$-coordinate, that is, if $x = (x_k)$, then $P_{K+1}(x) = (0, \dots, 0, x_{K+1}, 0, \dots)$.

To show that c_0 has the SD2P we will pick elements x and $\tilde{x}$ of $\frac{1}{n}\sum_{i=1}^{n} S(B_{c_0}, x_i^*, \alpha_i)$ such that $\|x - \tilde{x}\| = 2$ as follows.

By setting

$$x = \frac{1}{n}\sum_{i=1}^{n}(x_i - P_{K+1}(x_i) + e_{K+1})$$

and

$$\tilde{x} = \frac{1}{n}\sum_{i=1}^{n}(x_i - P_{K+1}(x_i) - e_{K+1}),$$

where e_k is the usual unit vector basis in c_0. It is clear that x and $\tilde{x}$ are in B_{c_0}. For $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} x_i^*(x_i - P_{K+1}(x_i) \pm e_{K+1}) &= \sum_{k=1}^{K} \beta_{i,k} x_{i,k} \pm \beta_{K+1} \cdot 1 + \sum_{k>K+1}^{\infty} \beta_{i,k} x_{i,k} \\ &> 1 - \alpha/2 - \alpha/2 = 1 - \alpha \geq 1 - \alpha_i. \end{aligned}$$

Therefore x and $\tilde{x}$ are in $\frac{1}{n}\sum_{i=1}^{n} S(B_{c_0}, x_i^*, \alpha_i)$. Obviously, $\|x - \tilde{x}\| = 2$. Consequently, c_0 has the SD2P.

□

Remark 2.5. Later on, we will see that c_0 enjoys an even stronger property (see Example 4.2).

A closed subspace Y of a Banach space X is called an *M-ideal* in X (see, e.g., [HWW]) if there exists a norm one projection P on X^* with $\ker P = Y^{\perp} = \{x^* \in X^* \colon x^*(y) = 0 \quad \text{for all } y \in Y\}$ and

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \qquad \text{for all } x^* \in X^*.$$

If, in addition, the range $\operatorname{ran} P$ of P is 1-*norming*, that is,

$$\|x\| = \sup\big\{|x^*(x)| \colon\ x^* \in \operatorname{ran} P,\ \|x^*\| \leq 1\big\} \quad \text{for all } x \in X,$$

then Y is called a *strict M-ideal.*

If a Banach space X is an M-ideal in its bidual X^{**}, then X is said to be *M-embedded*.

Example 2.20. (see, e.g., [HWW, Example III.1.4, (a)]) The Banach space c_0 is an M-ideal in its bidual ℓ_∞, that is, c_0 is M-embedded.

The next theorem tells us that there is a whole class of spaces with the SD2P and where c_0 is a particular example of such spaces.

Theorem 2.21 (see [ALN1, Theorem 4.10])**.** *Let X be a Banach space and let a proper subspace Y be a strict M-ideal in X. Then both Y and X have the SD2P. In particular, if X is a nonreflexive M-embedded Banach space, then both X and X^{**} have the SD2P.*

Now we will provide an example of a Banach space X such that X and X^{**} have the SD2P, however, X is not an M-ideal in X^{**}. Recall that a *Lindenstrauss space* is a Banach space such that the dual is an $L_1(\mu)$-space for some (positive) measure μ (see, e.g., [Lac]).

Example 2.22. The Banach space $C[0,1]$ is a Lindenstrauss space and thus, by Proposition 4.6 in [ALN2], the bidual $C[0,1]^{**}$ has the SD2P. From Corollary 2.15 or Theorem 2.28 we deduce that $C[0,1]$ has the SD2P. However, $C[0,1]$ is not an M-ideal in its bidual. Indeed, suppose to the contrary that $C[0,1]$ is an M-ideal in its bidual, then all subspaces of $C[0,1]$ would also be M-embedded (see [HWW, Theorem III.1.6]). Since $C[0,1]$ is universal (see, e.g., [FHHMZ, Theorem 5.8]), this would imply that, for example, ℓ_1 is M-embedded, which is a contradiction (see [HaLi, Theorem 3.5]).

In Example 2.22, we saw that the Banach space $C[0,1]$ has the SD2P. More generally, Banach spaces which are uniform algebras have the SD2P.

Definition 2.23. Let K be a nonempty compact Hausdorff space. A *uniform algebra* A is a subalgebra of $C(K)$ that is closed in the norm topology, contains the constant functions, and that separates the points of K.

Theorem 2.24 (see [ALN1, Theorem 4.2])**.** *Infinite-dimensional uniform algebras have the SD2P.*

Proposition 2.25. *The Banach space $L_1[0,1]$ has the SD2P.*

In order to prove this proposition, we use the following lemma.

Lemma 2.26. *Let $\alpha > 0$, $n \in \mathbb{N}$, and $f_1, \ldots, f_n \in S_{L_\infty[0,1]}$. Then there are pairwise disjoint subsets $E_1, \ldots, E_n$ of $[0,1]$ with positive measure such that, for every $i \in \{1, \ldots, n\}$,*

$$|f_i(t)| \geq 1 - \alpha \qquad \textit{for all } t \in E_i.$$

Proof. We will show the existence of such subsets $E_1, \ldots, E_n$ by induction. The existence of a set E_1 is immediate from $\|f_1\| = 1$.

Suppose that, for some $m \in \mathbb{N}$, where $m < n$, we can find sets $E_1, \ldots, E_m$ as needed. We will show the existence of a suitable E_{m+1}. Denote by

$$D_{m+1} = \{t \in [0,1] \colon |f_{m+1}(t)| \geq 1 - \alpha\}.$$

If $\mu(D_{m+1} \setminus \bigcup_{i=1}^m E_i) > 0$, then we may take $E_{m+1} = D_{m+1} \setminus \bigcup_{i=1}^m E_i$. Otherwise, there is an index $i_0 \in \{1, \ldots, m\}$ such that $\mu(D_{m+1} \cap E_{i_0}) > 0$. Choose disjoint subsets $\tilde{E}_{i_0}, E_{m+1} \subset D_{m+1} \cap E_{i_0}$ with positive measure (see, e.g., [AB, Theorem 10.52]) and redefine $E_{i_0} = \tilde{E}_{i_0}$. □

Proof of Proposition 2.25. Let $\frac{1}{n}\sum_{i=1}^n S(B_{L_1[0,1]}, f_i, \alpha_i)$ be a convex combination of slices of $B_{L_1[0,1]}$, where $n \in \mathbb{N}$, $f_1, \ldots, f_n \in S_{L_\infty[0,1]} = S_{L_1[0,1]^*}$, and $\alpha_1, \ldots, \alpha_n > 0$. We will show that $\operatorname{diam}(\frac{1}{n}\sum_{i=1}^n S(B_{L_1[0,1]}, f_i, \alpha_i)) = 2$.

We take $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}/2$. By Lemma 2.26, there are pairwise disjoint subsets $E_1, \ldots, E_n \subset [0,1]$ with positive measure such that, for every $i \in \{1, \ldots, n\}$,

$$|f_i(t)| \geq 1 - \alpha \qquad \text{for all } t \in E_i.$$

We shall split every E_i, $i \in \{1, \ldots, n\}$, further into two disjoint subsets F_i and G_i such that $E_i = F_i \cup G_i$ and $\mu(F_i) = \mu(G_i)$ (see, e.g., [AB, Theorem 10.52]).

We take

$$x = \frac{1}{n}\sum_{i=1}^n \frac{\operatorname{sgn} f_i \cdot \chi_{F_i}}{\mu(F_i)} \qquad \text{and} \qquad \tilde{x} = \frac{1}{n}\sum_{i=1}^n \frac{\operatorname{sgn} f_i \cdot \chi_{G_i}}{\mu(G_i)}.$$

Notice that $\|x\| = 1$, because

$$\int_0^1 |x(t)| \, dt = \frac{1}{n}\sum_{i=1}^n \frac{1}{\mu(F_i)}\mu(F_i) = \frac{1}{n}\sum_{i=1}^n 1 = 1,$$

and, for every $i \in \{1, \ldots, n\}$, one has

$$\int_0^1 f_i(t)\frac{\operatorname{sgn} f_i(t) \cdot \chi_{F_i}(t)}{\mu(F_i)} \, dt = \int_{F_i} \frac{|f_i(t)|}{\mu(F_i)} \, dt \geq 1 - \alpha > 1 - \alpha_i.$$

Thus, x is an element in $\frac{1}{n}\sum_{i=1}^{n} S(B_{L_1[0,1]}, f_i, \alpha_i)$. Similarly one can show that $\|\tilde{x}\| = 1$ and $\tilde{x}$ is an element in $\frac{1}{n}\sum_{i=1}^{n} S(B_{L_1[0,1]}, f_i, \alpha_i)$.

Therefore,

$$\operatorname{diam}\left(\frac{1}{n}\sum_{i=1}^{n} S(B_{L_1[0,1]}, f_i, \alpha_i)\right) \geq \|x - \tilde{x}\| = \|x\| + \|\tilde{x}\| = 2.$$

□

More generally, Banach spaces with the Daugavet property have the SD2P.

Definition 2.27 (see, e.g., [Wer2]). A Banach space X has the *Daugavet property* (DP) if

$$\|I_X + T\| = 1 + \|T\|$$

for every rank one operator $T : X \to X$.

The definition of the DP modestly involves only rank one operators, but it is well-known that then the latter norm identity also holds for all compact and even for all weakly compact operators.

The class of Banach spaces with the DP include the spaces $C(K)$, whenever K is a compact Hausdorff space without isolated points, and the spaces $L_1(\mu)$ and $L_\infty(\mu)$, when μ is a nonatomic measure. For more details, we refer the interested reader to a survey paper [Wer2] by D. Werner.

Theorem 2.28 (see [ALN1, Theorem 4.4]). *Let X be a Banach space. If X has the DP, then X has the SD2P.*

2.4 Stability results of diameter 2 properties

In this section, we study how diameter 2 properties are preserved by taking ℓ_p-sums of Banach spaces. We will also show that diameter 2 properties lift from M-ideals to the corresponding superspace.

The paper [ALN1] is probably the first investigation in this direction. It was shown that the LD2P and the D2P are stable by forming ℓ_p-sums, where p is such that $1 \leq p \leq \infty$. Some further development was carried out in [ABGLP] where, instead of ℓ_p-sums, product spaces with an absolute norm were considered. The SD2P is stable by forming the ℓ_1-sum or the ℓ_∞-sum (see also [BGLP, Lemma 2.1]). Whether the SD2P is stable by forming the ℓ_p-sums, where

p satisfies $1 < p < \infty$, was posed as a question in [ALN1]. In Theorem 2.33, we will show that the SD2P is not stable by forming the ℓ_p-sums for any p with $1 < p < \infty$.

In Chapter 3, we will present a dual space approach to these stability results. In order to compare different approaches, we include here the original proofs.

The following theorem lists the known stability results for the LD2P.

Theorem 2.29. *Let X and Y be Banach spaces.*

(a) *If X and Y have the LD2P, and p is such that $1 \leq p < \infty$, then $X \oplus_p Y$ has the LD2P (cf. [ALN1, Theorem 3.2], see also [ABGLP, Theorem 2.4]).*

(b) *If $X \oplus_p Y$ has the LD2P, where p is such that $1 \leq p < \infty$, then X has the LD2P (see [ABGLP, Proposition 2.5]).*

(c) *If X has the LD2P, then $X \oplus_\infty Y$ has the LD2P (cf. [ALN1, Theorem 3.2], see also [ABGLP, Theorem 2.4]).*

Proof. (a) Assume that X and Y have the LD2P, and p is such that $1 \leq p < \infty$. Denote by $Z = X \oplus_p Y$. Let q be such that $1/p + 1/q = 1$, if $p > 1$; and $q = \infty$, if $p = 1$. Consider a slice $S(B_Z, z^*, \alpha)$, where $z^* = (x^*, y^*) \in S_{Z^*} = S_{X^* \oplus_q Y^*}$ and $\alpha > 0$. Without loss of generality we may assume that $\alpha \leq 1$. Fix an arbitrary $\varepsilon > 0$. We will show the existence of elements in $S(B_Z, z^*, \alpha)$ with distance arbitrarily close to 2.

Assume that $x^* = 0$ or $y^* = 0$. To be more specific, suppose that $y^* = 0$, then $x^* \in S_{X^*}$. The other case is similar. Consider the slice $S(B_X, x^*, \alpha)$. By the assumption, we can find $x, \tilde{x} \in S(B_X, x^*, \alpha)$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$. We take $z = (x, 0)$ and $\tilde{z} = (\tilde{x}, 0)$. Clearly, z and $\tilde{z}$ are in $S(B_Z, z^*, \alpha)$ with $\|z - \tilde{z}\| \geq 2 - \varepsilon$.

Consider now the case $x^* \neq 0$ and $y^* \neq 0$. Find an element $z_0 = (x_0, y_0) \in S(B_Z, z^*, \alpha/4)$ with $\|z_0\| = 1$.

Choose $x, \tilde{x} \in S(B_X, x^*/\|x^*\|, \alpha/2)$ and $y, \tilde{y} \in S(B_Y, y^*/\|y^*\|, \alpha/2)$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$ and $\|y - \tilde{y}\| \geq 2 - \varepsilon$. We take $z = (\|x_0\| x, \|y_0\| y)$ and $\tilde{z} = (\|x_0\| \tilde{x}, \|y_0\| \tilde{y})$. Observe that $z, \tilde{z} \in S(B_Z, z^*, \alpha)$. In fact,

$$\|z\|^p = \|x_0\|^p \|x\|^p + \|y_0\|^p \|y\|^p \leq \|x_0\|^p + \|y_0\|^p = 1,$$

and

$$\begin{aligned} z^*(z) &= \|x_0\| x^*(x) + \|y_0\| y^*(y) > (\|x_0\| \|x^*\| + \|y_0\| \|y^*\|)(1 - \alpha/2) \\ &\geq z^*(z_0)(1 - \alpha/2) > (1 - \alpha/4)(1 - \alpha/2) > 1 - \alpha. \end{aligned}$$

Thus, $z \in S(B_Z, z^*, \alpha)$. Similarly we have $\tilde{z} \in S(B_Z, z^*, \alpha)$. Finally,

$$\begin{aligned} \|z - \tilde{z}\|^p &= \|x_0\|^p \|x - \tilde{x}\|^p + \|y_0\|^p \|y - \tilde{y}\|^p \\ &\geq (2 - \varepsilon)^p (\|x_0\|^p + \|y_0\|^p) = (2 - \varepsilon)^p. \end{aligned}$$

(b) Assume that $X \oplus_p Y$ has the LD2P, where p is such that $1 \leq p < \infty$. Denote by $Z = X \oplus_p Y$. Let $S(B_X, x^*, \alpha)$ be a slice and ε be such that $\varepsilon \in (0, \alpha)$. Consider the slice $S(B_Z, z^*, \varepsilon)$, where $z^* = (x^*, 0)$. By the assumption there are $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in $S(B_Z, z^*, \varepsilon)$ such that

$$(2 - \varepsilon)^p < \|z_1 - z_2\|^p = \|x_1 - x_2\|^p + \|y_1 - y_2\|^p.$$

Since $z^*(z_1) > 1 - \varepsilon$, we deduce that $x^*(x_1) > 1 - \varepsilon$. Thus $x_1 \in S(B_X, x^*, \alpha)$. Similarly we have that $x_2 \in S(B_X, x^*, \alpha)$.

From

$$(1 - \varepsilon)^p + \|y_1\|^p < \|x_1\|^p + \|y_1\|^p \leq 1$$

we deduce that $\|y_1\|^p \leq 1 - (1 - \varepsilon)^p = \delta(\varepsilon)$. Similarly one has that $\|y_2\|^p \leq \delta(\varepsilon)$. Finally,

$$\|x_1 - x_2\|^p > (2 - \varepsilon)^p - \|y_1\|^p - \|y_2\|^p \geq (2 - \varepsilon)^p - 2\delta(\varepsilon),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus X has the LD2P.

(c) First, observe that if $Z = X \oplus_\infty Y$, then for every slice $S(B_Z, z^*, \alpha)$ there exists a slice S of B_X and $y \in B_Y$ such that

$$S(B_Z, z^*, \alpha) \supset S \times \{y\}.$$

Indeed, let $z^* = (x^*, y^*) \in S_{Z^*} = S_{X^* \oplus_1 Y^*}$ and let $\alpha > 0$. If $x^* = 0$, then $S(B_Z, z^*, \alpha) \supset B_X \times \{y\}$ for any y in the slice $S(B_Y, y^*, \alpha)$. This proves our result since B_X can also be considered as a slice.

Assume now that $x^* \neq 0$. Choose an $y \in B_Y$ such that $y^*(y) > \|y^*\| - \alpha/2$. It is straightforward to verify that

$$S(B_Z, z^*, \alpha) \supset S(B_X, x^*/\|x^*\|, \beta/\|x^*\|) \times \{y\},$$

where $\beta = \alpha + \|x^*\| + y^*(y) - 1$.

This observation clearly implies that, if X has the LD2P, then also $X \oplus_\infty Y$ has the LD2P. □

Lemma 2.30 (see [ALN1, Lemma 4.5]). *Let X and Y be Banach spaces, W a nonempty weakly open subset in $Z = X \oplus_\infty Y$, and $(x_0, y_0) \in W$. There exist weakly open subsets U of X and V of Y such that $(x_0, y_0) \in U \times V \subset W$. Moreover, if W is a relatively weakly open subset of B_Z, then U and V can be chosen to be relatively weakly open subsets of B_X and B_Y, respectively.*

Proof. We may assume that

$$W_0 = \{(x, y) \in Z\colon\ |z_i^*(x, y) - z_i^*(x_0, y_0)| < 1, \quad i \in \{1, \dots, n\}\} \subset W$$

for some $n \in \mathbb{N}$ and $z_1^* = (x_1^*, y_1^*), \dots, z_n^* = (x_n^*, y_n^*) \in X^* \oplus_1 Y^*$.

Set

$$U = \{x \in X\colon\ |x_i^*(x) - x_i^*(x_0)| < \frac{1}{2}, \quad i \in \{1, \dots, n\}\}$$

and

$$V = \{y \in Y\colon\ |y_i^*(y) - y_i^*(y_0)| < \frac{1}{2}, \quad i \in \{1, \dots, n\}\}.$$

Then U and V are weakly open in X and Y, respectively, and $(x_0, y_0) \in U \times V \subset W_0$. In this part of the proof, we have not used the fact that Z is equipped with the supremum norm.

For the last part, notice that because of the supremum norm $B_Z = B_X \times B_Y$, and just redefine $U = U \cap B_X$ and $V = V \cap B_Y$. □

The following theorem lists the known stability results for the D2P.

Theorem 2.31. *Let X and Y be Banach spaces.*

(a) *If X and Y have the D2P, and p is such that $1 \leq p < \infty$, then $X \oplus_p Y$ has the D2P (see [ALN1, Theorem 3.2], see also [ABGLP, Theorem 2.4]).*

(b) *If $X \oplus_p Y$ has the D2P, where p is such that $1 \leq p < \infty$, then X has the D2P (see [ABGLP, Proposition 2.5]).*

(c) *If X has the D2P, then $X \oplus_\infty Y$ has the D2P (see [LP, Lemma 2.1], see also [ALN1, Theorem 2.7 (ii), and Theorem 3.2] and [ABGLP, Theorem 2.4]).*

Proof. (a) Assume that X and Y have the D2P, and p is such that $1 \leq p < \infty$. Denote by $Z = X \oplus_p Y$. Let W be a nonempty relatively weakly open subset of B_Z. Fix $z_0 = (x_0, y_0) \in W \cap S_Z$. We may assume that

$$W \supset \{z \in B_Z\colon\ |z_i^*(z - z_0)| < 1, \quad i \in \{1, \dots, n\}\}$$

for some $n \in \mathbb{N}$, $z_1^* = (x_1^*, y_1^*), \dots, z_n^* = (x_n^*, y_n^*) \in Z^*$. Fix an arbitrary $\varepsilon > 0$. We will show the existence of elements in W with distance arbitrarily close to 2.

Assume that $x_0 = 0$ or $y_0 = 0$. To be more specific, suppose that $y_0 = 0$, then $x_0 \in S_X$. The other case is similar. The set

$$U = \{x \in B_X \colon |x_i^*(x - x_0)| < 1, \quad i \in \{1, \dots, n\}\}$$

is a nonempty relatively weakly open subset of B_X. By the assumption, we can find $x, \tilde{x} \in U$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$. We take $z = (x, 0)$ and $\tilde{z} = (\tilde{x}, 0)$. Clearly, z and $\tilde{z}$ are elements in W with $\|z - \tilde{z}\| \geq 2 - \varepsilon$.

Suppose now that $x_0 \neq 0$ and $y_0 \neq 0$. Consider the sets

$$U = \{x \in B_X \colon |x_i^*(x - \frac{x_0}{\|x_0\|})| < \frac{1}{2\,\|x_0\|}, \quad i \in \{1, \dots, n\}\},$$

and

$$V = \{y \in B_Y \colon |y_i^*(y - \frac{y_0}{\|y_0\|})| < \frac{1}{2\,\|y_0\|}, \quad i \in \{1, \dots, n\}\}.$$

Clearly, U and V are nonempty relatively weakly open subsets of B_X and B_Y, respectively.

By the assumption, we can find $x, \tilde{x} \in U$ and $y, \tilde{y} \in V$ such that $\|x - \tilde{x}\| \geq 2 - \varepsilon$ and $\|y - \tilde{y}\| \geq 2 - \varepsilon$. We take $z = (\|x_0\|\, x, \|y_0\|\, y)$ and $\tilde{z} = (\|x_0\|\, \tilde{x}, \|y_0\|\, \tilde{y})$. Observe that $z, \tilde{z} \in W$. In fact

$$\|z\|^p = \|x_0\|^p \|x\|^p + \|y_0\|^p \|y\|^p \leq \|x_0\|^p + \|y_0\|^p = 1,$$

and

$$\begin{aligned}
|z_i^*(z - z_0)| &= |x_i^*(\|x_0\|\, x - x_0) + y_i^*(\|y_0\|\, y - y_0)| \\
&\leq \|x_0\|\, |x_i^*(x - \frac{x_0}{\|x_0\|})| + \|y_0\|\, |y_i^*(y - \frac{y_0}{\|y_0\|})| \\
&< \|x_0\| \frac{1}{2\,\|x_0\|} + \|y_0\| \frac{1}{2\,\|y_0\|} = 1,
\end{aligned}$$

for fixed $i \in \{1, \dots, n\}$. Thus, $z \in W$. Similarly, $\tilde{z} \in W$. Finally,

$$\begin{aligned}
\|z - \tilde{z}\|^p &= \|x_0\|^p \|x - \tilde{x}\|^p + \|y_0\|^p \|y - \tilde{y}\|^p \\
&\geq (2 - \varepsilon)^p (\|x_0\|^p + \|y_0\|^p) = (2 - \varepsilon)^p.
\end{aligned}$$

(b) The proof is similar to the proof of (b) in Theorem 2.29. Assume that $X \oplus_p Y$ has the D2P, where p is such that $1 \leq p < \infty$. Denote by $Z = X \oplus_p Y$. Note that, by Remark 2.3, it suffices to show that every nonempty intersection of a finite number of slices of B_X has diameter 2. Let $S = \bigcap_{i=1}^n S_i(B_X, x_i^*, \alpha_i)$ be a nonempty intersection of slices of B_X, $\alpha = \min\{\alpha_1, \dots, \alpha_n\}$, and ε be such that $\varepsilon \in (0, \alpha)$. Consider the intersection of slices $T = \bigcap_{i=1}^n S_i(B_Z, z_i^*, \varepsilon)$, where $z_i^* = (x_i^*, 0)$ for every $i \in \{1, \dots, n\}$. By the assumption, there are $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in $S_i(B_Z, z_i^*, \varepsilon)$ for every $i \in \{1, \dots, n\}$ such that

$$(2-\varepsilon)^p < \|z_1 - z_2\|^p = \|x_1 - x_2\|^p + \|y_1 - y_2\|^p.$$

Since $z_i^*(z_1) > 1 - \varepsilon$, we deduce that $x_i^*(x_1) > 1 - \varepsilon$ for every $i \in \{1, \dots, n\}$. Thus $x_1 \in S$. Similarly we have that $x_2 \in S$.

From

$$(1-\varepsilon)^p + \|y_1\|^p < \|x_1\|^p + \|y_1\|^p \leq 1$$

we deduce that $\|y_1\|^p \leq 1 - (1-\varepsilon)^p = \delta(\varepsilon)$. Similarly one has that $\|y_2\|^p \leq \delta(\varepsilon)$. Finally,

$$\|x_1 - x_2\|^p > (2-\varepsilon)^p - \|y_1\|^p - \|y_2\|^p \geq (2-\varepsilon)^p - 2\delta(\varepsilon),$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus X has the D2P.

(c) It is immediate from Lemma 2.30 that $Z = X \oplus_\infty Y$ has the D2P, whenever X or Y has the D2P. □

The following theorem lists the known stability results for the SD2P.

Theorem 2.32. *Let X and Y be Banach spaces.*

(a) *If X and Y have the SD2P, then $X \oplus_1 Y$ has the SD2P (see [ALN1, Theorem 2.7 (iii)], see also [BGLP, Lemma 2.1], and [ABGLP, Proposition 3.1]).*

(b) *If $X \oplus_1 Y$ has the SD2P, then X has the SD2P (see [ABGLP, Proposition 3.1]).*

(c) *If X has the SD2P, then $X \oplus_\infty Y$ has the SD2P (see [ALN1, Proposition 4.6]).*

Proof. (a) Assume that X and Y have the SD2P. Denote by $Z = X \oplus_1 Y$. Let $S = \sum_{i=1}^n \lambda_i S(B_Z, z_i^*, \alpha_i)$ be a convex combination of slices, where $n \in \mathbb{N}$, $z_1^* = (x_1^*, y_1^*), \dots, z_n^* = (x_n^*, y_n^*) \in S_{Z^*}$, $\alpha_1, \dots, \alpha_n > 0$, and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. We will show that the diameter of S is arbitrarily close to 2.

Split the set $\{1, \dots, n\}$ into two disjoint subsets I and J, such that $\|x_i^*\| = 1$ for every $i \in I$ and $\left\|y_j^*\right\| = 1$ for every $j \in J$. For every $i \in I$ consider the slice

$S(B_X, x_i^*, \alpha_i)$ and for every $j \in J$ consider the slice $S(B_Y, y_j^*, \alpha_j)$. Observe that $S(B_X, x_i^*, \alpha_i) \times \{0\} \subset S(B_Z, z_i^*, \alpha_i)$ for every $i \in I$ and $\{0\} \times S(B_Y, y_j^*, \alpha_j) \subset S(B_Z, z_j^*, \alpha_j)$ for every $j \in J$.

Denote by $\lambda_I = \sum_{i\in I} \lambda_i$ and $\lambda_J = \sum_{j\in J} \lambda_j$. Assume first that $\lambda_I = 0$ or $\lambda_J = 0$. To be more specific, suppose that $\lambda_J = 0$, then $\lambda_I = 1$. Let $\varepsilon > 0$. Since X has the SD2P, there are elements $x, \tilde{x} \in \sum_{i\in I} \lambda_i S(B_X, x_i^*, \alpha_i)$ such that $\|x - \tilde{x}\| > 2-\varepsilon$. Note that $(x, 0), (\tilde{x}, 0)$ are elements in S. Finally,

$$\operatorname{diam}(S) \geq \|x - \tilde{x}\| > 2 - \varepsilon.$$

Suppose now that $\lambda_I \neq 0$ and $\lambda_J \neq 0$. We have that

$$\sum_{i\in I} \frac{\lambda_i}{\lambda_I} S(B_X, x_i^*, \alpha_i) \times \{0\} \subset \sum_{i\in I} \frac{\lambda_i}{\lambda_I} S(B_Z, z_i^*, \alpha_i)$$

and

$$\{0\} \times \sum_{j\in J} \frac{\lambda_j}{\lambda_J} S(B_Y, y_j^*, \alpha_j) \subset \sum_{j\in J} \frac{\lambda_j}{\lambda_J} S(B_Z, z_j^*, \alpha_j).$$

Let $\varepsilon > 0$. Since X and Y both have the SD2P, there are elements $x, \tilde{x} \in \sum_{i\in I} \frac{\lambda_i}{\lambda_I} S(B_X, x_i^*, \alpha_i)$ and $y, \tilde{y} \in \sum_{j\in J} \frac{\lambda_j}{\lambda_J} S(B_Y, y_j^*, \alpha_j)$ such that $\|x - \tilde{x}\| > 2 - \varepsilon$ and $\|y - \tilde{y}\| > 2 - \varepsilon$. Note that $(\lambda_I x, \lambda_J y)$ is an element in S, because

$$\begin{aligned}(\lambda_I x, \lambda_J y) &= (\lambda_I x, 0) + (0, \lambda_J y) \\ &\in \sum_{i\in I} \lambda_i S(B_X, x_i^*, \alpha_i) \times \{0\} + \{0\} \times \sum_{j\in J} \lambda_j S(B_Y, y_j^*, \alpha_j) \\ &\subset \sum_{i\in I} \lambda_i S(B_Z, z_i^*, \alpha_i) + \sum_{j\in J} \lambda_j S(B_Z, z_j^*, \alpha_j) = S.\end{aligned}$$

Similarly, $(\lambda_I \tilde{x}, \lambda_J \tilde{y})$ is in S. Finally,

$$\operatorname{diam}(S) \geq \lambda_I \|x - \tilde{x}\| + \lambda_J \|y - \tilde{y}\| > 2 - \varepsilon.$$

(b) Assume that $X \oplus_1 Y$ has the SD2P. Denote by $Z = X \oplus_1 Y$. Let $S = \frac{1}{n}\sum_{i=1}^n S_i(B_X, x_i^*, \alpha_i)$ be a nonempty convex combination of slices of B_X, $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}$, and ε be such that $\varepsilon \in (0, \alpha)$. Consider the convex combination of slices $T = \frac{1}{n}\sum_{i=1}^n S_i(B_Z, z_i^*, \varepsilon)$, where $z_i^* = (x_i^*, 0)$ for every $i \in \{1, \ldots, n\}$. Taking $p = 1$ in the proof of Theorem 2.29, (b), we see that

$$S_i(B_Z, z_i^*, \varepsilon) \subset S_i(B_X, x_i^*, \alpha_i) \times \varepsilon B_Y$$

for every $i \in \{1, \dots, n\}$. Thus $T \subset S \times \varepsilon B_Y$ and therefore $\operatorname{diam}(T) \leq \operatorname{diam}(S) + 2\varepsilon$.

By the assumption, T is of diameter 2. Thus $\operatorname{diam}(S) \geq 2 - 2\varepsilon$ and X has the SD2P.

(c) Our proof is a slight modification of the proof in [ALN1]. Assume that X has the SD2P. Let $Z = X \oplus_\infty Y$ and let $P\colon Z \to X$ be the natural projection onto X. Let $S = \frac{1}{n}\sum_{i=1}^n S_i$, where $n \in \mathbb{N}$ and $S_1, \dots, S_n$ are slices of B_Z.

We recall that slices $S_1, \dots, S_n$ are relatively weakly open in B_Z. By Lemma 2.30, it follows that for every $i \in \{1, \dots, n\}$ one can find relatively weakly open subsets U_i of B_X and V_i of B_Y such that $U_i \times V_i \subset S_i$.

We have now

$$P(S) = \frac{1}{n}\sum_{i=1}^n P(S_i) \supset \frac{1}{n}\sum_{i=1}^n U_i.$$

By Remark 2.3, $\operatorname{diam}(\frac{1}{n}\sum_{i=1}^n U_i) = 2$. Since $\|P\| = 1$, we must have

$$\operatorname{diam}(P(S)) = \operatorname{diam}(S) = 2.$$

□

We will now show that if X and Y are Banach spaces, then the Banach space $X \oplus_p Y$, where p is such that $1 < p < \infty$, can never have the SD2P. This gives a negative answer to question (c) in [ALN1].

Theorem 2.33 is a joint result with M. Põldvere. The lack of the SD2P in ℓ_p-sums of Banach spaces, where p is such that $1 < p < \infty$, was obtained independently also in [ABGLP, Theorem 3.2] (see also [Oja]).

Theorem 2.33 (see [Lan, Theorem 3.23] or [HL1, Theorem 1])**.** *Let X and Y be Banach spaces and let p be such that $1 < p < \infty$. The Banach space $Z = X \oplus_p Y$ does not have the SD2P.*

To prove this theorem, we will need the following lemma.

Lemma 2.34 (see [Lan, Lemma 3.24] or [HL1, Lemma 2])**.** *Let q be such that $1 < q < \infty$ and $1/p + 1/q = 1$. If $z^* = (x^*, y^*)$ is an element in $S_{Z^*} = S_{X^* \oplus_q Y^*}$, then for every $\varepsilon > 0$ there exists $\alpha > 0$ such that*

$$\Big| \|x\| - \|x^*\|^{q-1} \Big| + \Big| \|y\| - \|y^*\|^{q-1} \Big| < \varepsilon,$$

whenever $z = (x, y)$ is an element in $S(z^*, \alpha)$.

Proof. Note that if $z = (x, y)$ is an element in $S(B_Z, z^*, \alpha)$, then $(\|x\|, \|y\|)$ and $(\|x^*\|^{q-1}, \|y^*\|^{q-1})$ are both elements of the slice $S(B_{\ell_p^2}, (\|x^*\|, \|y^*\|), \alpha)$. Obviously, when α tends to 0, then $\operatorname{diam}(S(B_{\ell_p^2}, (\|x^*\|, \|y^*\|), \alpha))$ tends to 0 as well. This proves the result. □

Proof of Theorem 2.33. We will show that, for every $\lambda \in (0, 1)$, there exists $\alpha > 0$ and $\beta > 0$ such that

$$\lambda S(B_Z, z^*, \alpha) + (1 - \lambda) S(B_Z, \tilde{z}^*, \alpha) \subset (1 - \beta) B_Z,$$

where $S(B_Z, z^*, \alpha)$ and $S(B_z, \tilde{z}^*, \alpha)$ are two suitable slices of B_Z.

Let $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$. We take $z^* = (x^*, 0)$ and $\tilde{z}^* = (0, y^*)$. Then z and $\tilde{z}$ are elements in S_{Z^*}. Fix $\lambda \in (0, 1)$. Denote by

$$\varepsilon = 1 - \Big(\lambda^p + (1 - \lambda)^p\Big)^{1/p}.$$

Clearly, $\varepsilon > 0$. By Lemma 2.34, there exists $\alpha > 0$ such that

$$\begin{aligned}
&\left(\Big(\lambda \|x\| + (1 - \lambda) \|\tilde{x}\|\Big)^p + \Big(\lambda \|y\| + (1 - \lambda) \|\tilde{y}\|\Big)^p\right)^{1/p} \\
&\leq \left(\Big(\lambda \cdot 1 + (1 - \lambda) \cdot 0\Big)^p + \Big(\lambda \cdot 0 + (1 - \lambda) \cdot 1\Big)^p\right)^{1/p} + \frac{\varepsilon}{2} \\
&= \Big(\lambda^p + (1 - \lambda)^p\Big)^{1/p} + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2},
\end{aligned}$$

whenever $z = (x, y) \in S(B_Z, z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(B_Z, \tilde{z}^*, \alpha)$.

One may take $\beta = \varepsilon/2$. Indeed, for $z = (x, y) \in S(B_Z, z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(B_Z, \tilde{z}^*, \alpha)$, we have

$$\begin{aligned}
&\|\lambda z + (1 - \lambda)\tilde{z}\| = \Big(\|\lambda x + (1 - \lambda)\tilde{x}\|^p + \|\lambda y + (1 - \lambda)\tilde{y}\|^p\Big)^{1/p} \\
&\leq \left(\Big(\lambda \|x\| + (1 - \lambda) \|\tilde{x}\|\Big)^p + \Big(\lambda \|y\| + (1 - \lambda) \|\tilde{y}\|\Big)^p\right)^{1/p} \\
&\leq 1 - \frac{\varepsilon}{2}.
\end{aligned}$$

□

Relations between the M-ideal structure and diameter 2 properties were first considered in [LP], where it was proven that if a proper subspace Y of a Banach space X is a strict M-ideal in X, then both Y and X have the D2P (see [LP, Theorem 2.4]). In [ALN1, Theorem 4.10], it is shown that, under the same assumptions, one can conclude that both Y and X even have the SD2P. An immediate corollary of this is that if a nonreflexive Banach space X is an M-ideal in its bidual X^{**}, then both X and X^{**} have the SD2P. In Proposition 3.35, we shall present a simple proof of this result.

Example 2.35. One can not omit here the assumption that an M-ideal Y in X is a strict M-ideal. Indeed, let Y be any Banach space and let $X = Y \oplus_\infty c_0$. Then, by [ALN1, Proposition 4.6] (or Proposition 2.36), X has the SD2P and Y is an M-ideal in X.

In the following we will show that if a nonzero M-ideal Y in X has some diameter 2 property, then X has the same diameter 2 property without the assumption that the range of the projection is 1-norming. This also generalizes [ALN1, Theorem 3.2] (the case of $p = \infty$) and [ALN1, Proposition 4.6].

Proposition 2.36 (see [Lan, Proposition 3.28] or [HL1, Proposition 3])**.** *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the SD2P, then X has the SD2P.*

Proof. Let $\frac{1}{n}\sum_{i=1}^n S(B_X, x_i^*, \alpha_i)$ be a convex combination of slices of B_X, where $n \in \mathbb{N}$. By Proposition 2.8, it suffices to show the diameter of $\frac{1}{n}\sum_{i=1}^n S(B_X, x_i^*, \alpha_i)$ is 2. Let $\alpha = \min\{\alpha_1, \dots, \alpha_n\}$ and ε be such that $\varepsilon \in (0, \alpha/3)$.

We will show the existence of $x_{1,1}, \dots, x_{1,n}$ and $x_{2,1}, \dots, x_{2,n}$ in B_X such that $x_{1,i}, x_{2,i} \in S(B_X, x_i^*, \alpha_i)$ for every $i \in \{1, \dots, n\}$ and

$$\left\|\frac{1}{n}\sum_{i=1}^n (x_{1,i} - x_{2,i})\right\| > \frac{2-\varepsilon}{1+\varepsilon}.$$

Denote by P the M-ideal projection on X^* with $\ker P = Y^\perp$. For all $i \in \{1, \dots, n\}$, we take

$$y_i^* = \frac{Px_i^*}{\|Px_i^*\|} \qquad \text{and} \qquad \beta_i = \frac{\varepsilon - \varepsilon\,\|Px_i^*\| + \varepsilon^2}{\|Px_i^*\|}.$$

Note that, if $Px_i^* \neq 0$, then $\beta_i > 0$. If $Px_i^* = 0$, we can take $y_i^* \in S_{Y^*}$ and $\beta_i > 0$ to be arbitrary. Observe that $\frac{1}{n}\sum_{i=1}^n S(B_Y, y_i^*, \beta_i)$ is a convex combination of slices of B_Y. Since Y has the SD2P, we can find $y_{1,1}, \dots, y_{1,n}$ and $y_{2,1}, \dots, y_{2,n}$ in B_Y such that

$$Px_i^*(y_{j,i}) > (\|Px_i^*\| - \varepsilon)(1+\varepsilon)$$

and

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (y_{1,i} - y_{2,i}) \right\| > 2 - \varepsilon.$$

There are $x_1, \dots, x_n \in B_X$ such that

$$(x_i^* - Px_i^*)(x_i) > (\|x_i^* - Px_i^*\| - \varepsilon)(1 + \varepsilon)$$

for all $i \in \{1, \dots, n\}$.

Since Y is an M-ideal in X, then by [Wer1, Proposition 2.3], we can, for every $i \in \{1, \dots, n\}$, choose $z_i \in B_Y$ such that

$$\|y_{j,i} + x_i - z_i\| < 1 + \varepsilon$$

and

$$|Px_i^*(x_i - z_i)| < \varepsilon.$$

We take

$$x_{j,i} = \frac{y_{j,i} + x_i - z_i}{1 + \varepsilon}.$$

Now, for every $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$, $x_{j,i}$ is an element in $S(B_X, x_i^*, \alpha_i)$, because

$$\begin{aligned} x_i^*(x_{j,i}) &= \frac{x_i^*(y_{j,i} + x_i - z_i)}{1 + \varepsilon} \\ &= \frac{Px_i^*(y_{j,i}) + (x_i^* - Px_i^*)(x_i) + Px_i^*(x_i - z_i)}{1 + \varepsilon} \\ &> \|Px_i^*\| - \varepsilon + \|x_i^* - Px_i^*\| - \varepsilon - \varepsilon \\ &= \|x_i^*\| - 3\varepsilon > 1 - \alpha_i. \end{aligned}$$

Finally, observe that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (x_{1,i} - x_{2,i}) \right\| = \frac{1}{1 + \varepsilon} \left\| \frac{1}{n} \sum_{i=1}^{n} (y_{1,i} - y_{2,i}) \right\| > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

□

We conclude with the LD2P and the D2P versions of Proposition 2.36.

Proposition 2.37 (see [Lan, Proposition 3.29] or [HL1, Proposition 4]). *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the LD2P, then X has the LD2P.*

Proof. Take $n = 1$ in the proof of Proposition 2.36. □

The next result is obtained in the proof of [LP, Theorem 2.4], but not stated explicitly. We will give a direct proof of this result.

Proposition 2.38 (see [Lan, Proposition 3.30] or [HL1, Proposition 5])**.** *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the D2P, then X has the D2P.*

Proof. The proof is similar to the proof of Proposition 2.36.

Let U be a nonempty relatively weakly open subset of B_X containing an element x_0. We may assume that

$$\{x \in B_X \colon |x_i^*(x - x_0)| < \gamma, \quad i \in \{1, \dots, n\}\} \subset U,$$

for some $n \in \mathbb{N}$, $x_1^*, \dots, x_n^* \in S_{X^*}$, and $\gamma > 0$.

Denote by P the M-ideal projection on X^* with $\ker P = Y^\perp$, and let $\delta = \max\{\|Px_i^*\| : i \in \{1, \dots, n\}\}$. Let $\varepsilon > 0$ be such that $\varepsilon(4 + \delta) < \gamma$. We will show the existence of elements x and $\tilde{x}$ in U such that

$$\|x - \tilde{x}\| > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

Since B_Y is dense in B_X in the weak topology $\sigma(X, \operatorname{ran} P)$, we can find an element $y_0 \in B_Y$ such that

$$|Px_i^*(x_0 - y_0)| < \varepsilon$$

for every $i \in \{1, \dots, n\}$. Consider the set

$$V = \{y \in B_Y \colon |Px_i^*(y - y_0)| < \varepsilon(\delta + 1), \quad i \in \{1, \dots, n\}\}.$$

Clearly, V is a nonempty relatively weakly open subset of B_Y. By the assumption, there are $y_1, y_2 \in V$ with $\|y_1 - y_2\| > 2 - \varepsilon$.

Since Y is an M-ideal in X, by [Wer1, Proposition 2.3], there is an element $z_0 \in B_Y$ such that

$$\left\|y_j + x_0 - z_0\right\| < 1 + \varepsilon$$

and

$$|Px_i^*(x_0 - z_0)| < \varepsilon$$

for every $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$.

We take
$$x_1 = \frac{y_1 + x_0 - z_0}{1+\varepsilon} \qquad \text{and} \qquad x_2 = \frac{y_2 + x_0 - z_0}{1+\varepsilon}.$$

Now, for every $i \in \{1, \dots, n\}$, we have

$$\begin{aligned}
|x_i^*(x_1 - x_0)| &= \frac{1}{1+\varepsilon}\,|x_i^*(y_1 - \varepsilon x_0 - z_0) \pm Px_i^*(x_0) \pm Px_i^*(y_0)| \\
&\leq \frac{1}{1+\varepsilon}\Big(\,|Px_i^*(y_1 - y_0)| + |Px_i^*(x_0 - z_0)| + \varepsilon\,|x_i^*(x_0)| + |Px_i^*(y_0 - x_0)|\,\Big) \\
&< \frac{1}{1+\varepsilon}(\varepsilon\delta + 4\varepsilon) < \gamma.
\end{aligned}$$

Thus, $x_1 \in U$. Similarly, one can show that $x_2 \in U$. Finally, observe that
$$\|x_1 - x_2\| = \frac{1}{1+\varepsilon}\,\|y_1 - y_2\| > \frac{2-\varepsilon}{1+\varepsilon}.$$

□

In Section 3.3, we will give a dual space approach to Propositions 2.36–2.38.

Chapter 3

Octahedral Banach spaces

It is known that a Banach space has the SD2P if and only if the norm on its dual space is octahedral. We consider two more versions of octahedrality, which we show are dual properties to the D2P and to the LD2P. We study stability properties of different types of octahedrality and provide alternative proofs of some known stability results of diameter 2 properties. Necessary and sufficient conditions for spaces of operators to be octahedral are also considered. This chapter is mainly based on [HLP].

3.1 Definitions and basic results

Definition 3.1 (see [God] and [DGZ], cf. [Dev]). Let X be a Banach space. The norm on X is *octahedral* (OH) if, for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there is a $y \in S_X$ such that

$$\|x + y\| \geq (1 - \varepsilon)\big(\|x\| + \|y\|\big) \quad \text{for all } x \in E.$$

Whenever it makes no confusion, spaces whose norm is OH will also be called OH for simplicity.

Octahedral norms were introduced by G. Godefroy and B. Maurey [GM] (see also [God]) in order to characterize Banach spaces containing an isomorphic copy of ℓ_1.

Theorem 3.2 (see [DGZ, Theorem III.2.5]). *A Banach space X can be equivalently renormed to be OH if and only if X contains an isomorphic copy of ℓ_1.*

We now point out some equivalent, but sometimes more convenient formulations of octahedrality.

Proposition 3.3 (see [HLP, Proposition 2.1]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is OH;*

(ii) *whenever E is a finite-dimensional subspace of X and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x + ty\| \geq (1-\varepsilon)\big(\|x\| + t\big) \quad \text{for all } x \in S_E \text{ and } t > 0; \tag{3.1}$$

(ii') *whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x_i + ty\| \geq (1-\varepsilon)\big(\|x_i\| + t\big) \quad \text{for all } i \in \{1, \ldots, n\} \text{ and } t > 0; \tag{3.2}$$

(iii) *whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x_i + y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \ldots, n\}.$$

Proof. (i)⇔(ii)⇒(ii')⇒(iii) is obvious.

(iii)⇒(ii'). Assume that (iii) holds. Let $x_1, \ldots, x_n \in S_X$ and let $\varepsilon > 0$. By (iii), pick any $y \in S_X$ with

$$\|x_i + y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \ldots, n\}.$$

We show that y satisfies (3.2). Suppose that $i \in \{1, \ldots, n\}$, $t > 0$, and $\varepsilon > 0$. Then

$$\begin{aligned}
\|x_i + ty\| &\geq \max\{1,t\}\|x_i + y\| - \big(\max\{1,t\} - \min\{1,t\}\big)\\
&\geq (1-\varepsilon)\max\{1,t\} + \min\{1,t\}\\
&= 1 + t - \varepsilon \cdot \max\{1,t\}\\
&\geq (1-\varepsilon)(1+t).
\end{aligned}$$

(ii')⇒(ii). Assume that (ii') holds. Let E be a nontrivial finite-dimensional subspace of X and let $\varepsilon > 0$. We shall show that there is a $y \in S_X$ satisfying (3.1). Let $A \subset S_X$ be a finite $\varepsilon/2$-net for S_E. By (ii'), there is a $y \in S_X$ satisfying

$$\|z + ty\| \geq (1 - \frac{\varepsilon}{2})(\|z\| + t) \quad \text{for all } z \in A \text{ and } t > 0.$$

Let $x \in S_E$ and $t > 0$ be arbitrary. Letting $z \in A$ be such that $\|x - z\| < \varepsilon/2$, one has

$$\begin{aligned}\|x + ty\| &\geq \|z + ty\| - \|x - z\| \\ &\geq (1 - \frac{\varepsilon}{2})(1 + t) - \frac{\varepsilon}{2} \\ &\geq (1 - \varepsilon)(1 + t).\end{aligned}$$

□

Example 3.4. The Banach space ℓ_1 is OH. Indeed, let $n \in \mathbb{N}$, $x_1 = (\alpha_{1,k}), \dots, x_n = (\alpha_{n,k}) \in S_{\ell_1}$ and let $\varepsilon > 0$. By Proposition 3.3, it suffices to find a $y \in S_{\ell_1}$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for all $i \in \{1, \dots, n\}$. Find an $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{N} |\alpha_{i,k}| > 1 - \varepsilon/2 \quad \text{for all } i \in \{1, \dots, n\}.$$

Define $y = e_{N+1}$, where (e_k) is the standard vector basis of ℓ_1. Now we have that

$$\begin{aligned}\|x_i + y\| &= \sum_{k=1}^{N} |\alpha_{i,k}| + |1 + \alpha_{i,N+1}| + \sum_{k=N+2}^{\infty} |\alpha_{i,k}| \\ &\geq \sum_{k=1}^{N} |\alpha_{i,k}| + 1 - |\alpha_{i,N+1}| - \sum_{k=N+2}^{\infty} |\alpha_{i,k}| \\ &\geq 1 - \varepsilon/2 + 1 - \varepsilon/2 = 2 - \varepsilon\end{aligned}$$

for all $i \in \{1, \dots, n\}$. Thus ℓ_1 is OH.

Similarly to Example 3.4, one can show that given a sequence of nontrivial Banach spaces (X_k) the ℓ_1-sum $(X_1 \oplus X_2 \oplus \dots)_{\ell_1}$ is always OH.

In order to show that spaces with the DP are OH, we recall a geometric characterization of the DP.

Lemma 3.5 (see [KSSW, Lemma 2.2]). *Let X be a Banach space. The following assertions are equivalent.*

(i) *X has the DP;*

(ii) *for every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$ there is a $y \in S_X$ such that $x^*(y) > 1 - \varepsilon$ and $\|x + y\| > 2 - \varepsilon$;*

(iii) *for every* $x \in S_X$, $x^* \in S_{X^*}$, *and* $\varepsilon > 0$ *there is a* $y^* \in S_{X^*}$ *such that* $y^*(x) > 1 - \varepsilon$ *and* $\|x^* + y^*\| > 2 - \varepsilon$.

Repeatedly using Lemma 3.5 gives us that X and X^* are OH whenever X has the DP. One can derive this result also from the duality of the SD2P and OH as it is done in [BGLPRZ1].

Theorem 3.6 (see [BGLPRZ1, Corollary 2.5])**.** *If* X *has the DP, then* X *and* X^* *are OH.*

The following example of a Banach space X such that X and X^* are OH, but X fails to have the DP is given in [BGLPRZ1, Remark 2.6].

Example 3.7. Let $X = L_1[0,1] \oplus_\infty \ell_1$. We know that $L_1[0,1]$ is OH, because it has the DP and, by Example 3.4, we have that ℓ_1 is OH. Now, by Proposition 3.32, (c), we get that X is OH. Since $X^* = L_\infty[0,1] \oplus_1 \ell_\infty$ and $L_\infty[0,1]$ has the DP, it follows, by Proposition 3.32, that X^* is also OH. Finally, X fails the DP, because ℓ_1 fails the DP.

In order to characterize the dual of Banach spaces with the D2P, we introduce the following octahedrality-type property of the norm.

Definition 3.8 (see [HLP, Definition 2.2])**.** Let X be a Banach space. We say that (the norm on) X is *weakly octahedral* (WOH) if, for every finite-dimensional subspace E of X, every $x^* \in B_{X^*}$, and every $\varepsilon > 0$, there is a $y \in S_X$ such that

$$\|x + y\| \geq (1 - \varepsilon)\big(|x^*(x)| + \|y\|\big) \quad \text{for all } x \in E.$$

It is clear that if a Banach space is OH, then it is WOH.

Next we point out some equivalent, but sometimes more convenient formulations of weak octahedrality.

Proposition 3.9 (see [HLP, Proposition 2.4])**.** *Let* X *be a Banach space. The following assertions are equivalent:*

(i) X *is WOH;*

(ii) *whenever* E *is a finite-dimensional subspace of* X, $x^* \in B_{X^*}$, *and* $\varepsilon > 0$, *there is a* $y \in S_X$ *such that*

$$\|x + ty\| \geq (1 - \varepsilon)\big(|x^*(x)| + t\big) \quad \textit{for all } x \in S_E \textit{ and } t > 0;$$

(ii') *whenever $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, $x^* \in B_{X^*}$, and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x_i + ty\| \geq (1-\varepsilon)\big(|x^*(x_i)| + t\big) \quad \textit{for all } i \in \{1, \dots, n\} \textit{ and } t > 0;$$

(iii) *whenever $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, $x^* \in B_{X^*}$, and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x_i + ty\| \geq (1-\varepsilon)\big(|x^*(x_i)| + t\big) \quad \textit{for all } i \in \{1, \dots, n\} \textit{ and } t \geq \varepsilon.$$

Proof. (i)⇔(ii)⇒(ii')⇒(iii) is obvious.

(iii)⇒(ii). Assume that (iii) holds. Let E be a nontrivial finite-dimensional subspace of X, let $x^* \in B_{X^*}$, and let ε be such that $\varepsilon \in (0,1)$. Pick $\delta > 0$ satisfying $\varepsilon \geq (2-\varepsilon)\delta$, and $\gamma > 0$ satisfying $\gamma(2-\delta) \leq \delta^2$. Let $A \subset S_X$ be a finite γ-net for S_E. By (iii), there is a $y \in S_X$ satisfying

$$\|z + ty\| \geq (1-\delta)\big(|x^*(z)| + t\big) \quad \text{for all } z \in A \text{ and all } t \geq \delta.$$

Let $x \in S_E$ and $t > 0$ be arbitrary. First suppose that $t \leq \delta$. In this case, observing that $-\delta \geq -\varepsilon + \delta - \varepsilon\delta$, i.e. $1 - \delta \geq (1-\varepsilon)(1+\delta)$, and thus also $1 - \delta \geq (1-\varepsilon)(1+t)$,

$$\|x + ty\| \geq 1 - \delta \geq (1-\varepsilon)(1+t) \geq (1-\varepsilon)\big(|x^*(x)| + t\big).$$

Now consider the case $t \geq \delta$. Letting $z \in A$ be such that $\|x - z\| < \gamma$, one has

$$\begin{aligned} \|x + ty\| &\geq \|z + ty\| - \gamma \geq (1-\delta)\big(|x^*(z)| + t\big) - \gamma \\ &\geq (1-\delta)\big(|x^*(x)| + t\big) - \gamma(1-\delta) - \gamma. \end{aligned}$$

Since $t \geq \delta$, one has

$$\gamma(1-\delta) + \gamma = \gamma(2-\delta) \leq \delta^2 \leq \delta\big(|x^*(x)| + t\big),$$

and it follows that

$$\|x + ty\| \geq (1-2\delta)\big(|x^*(x)| + t\big) \geq (1-2\varepsilon)\big(|x^*(x)| + t\big).$$

□

Remark 3.1. We do not know whether the equivalences in Proposition 3.9 remain the same by replacing the condition $t > 0$ with $t = 1$ (see also the remark after [HL2, Lemma 3.5]).

Proposition 3.10 (see [HLP, Proposition 2.5]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X^* is WOH;*

(ii) *whenever E is a finite-dimensional subspace of X^*, $x \in B_X$, and $\varepsilon > 0$, there is a $y^* \in S_{X^*}$ such that*

$$\|x^* + y^*\| \geq (1-\varepsilon)\big(|x^*(x)| + \|y^*\|\big) \quad \text{for all } x^* \in E;$$

(iii) *whenever $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in S_{X^*}$, $x \in B_X$, and $\varepsilon > 0$, there is a $y^* \in S_{X^*}$ such that*

$$\|x_i^* + ty^*\| \geq (1-\varepsilon)\big(|x_i^*(x)| + t\big) \quad \text{for all } i \in \{1, \ldots, n\} \text{ and } t \geq \varepsilon.$$

Proof. (i)⇒(ii)⇒(iii) is obvious.

(iii)⇒(ii) is similar to (iii)⇒(ii) in the proof of Proposition 3.9.

(ii)⇒(i). This is a standard use of the Principle of Local Reflexivity. Assume that (ii) holds. Let E be a finite-dimensional subspace of X^*, let $x^{**} \in B_{X^{**}}$, and let ε be such that $\varepsilon \in (0,1)$. Let $F = \operatorname{span}\{x^{**}\}$. By the Principle of Local Reflexivity, there is an ε-isometry $T\colon F \to X$ such that $1-\varepsilon \leq \|Tf\| \leq 1+\varepsilon$ for all $f \in S_F$ and $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^* \in E$. It is clear that $Tx^{**}/(1+\varepsilon) \in B_X$. By (ii), there is a $y^* \in S_{X^*}$ such that

$$\begin{aligned}
\|x^* + y^*\| &\geq (1-\varepsilon^2)\left(\left|x^*\left(\frac{Tx^{**}}{1+\varepsilon}\right)\right| + \|y^*\|\right) \\
&= \left(\frac{1-\varepsilon^2}{1+\varepsilon}\right)\left(|x^{**}(x^*)| + (1+\varepsilon)\|y^*\|\right) \\
&> (1-\varepsilon)(|x^{**}(x^*)| + \|y^*\|)
\end{aligned}$$

for all $x^* \in E$. □

In order to characterize the dual of Banach spaces with the LD2P (similarly to the D2P and the SD2P), we introduce the following octahedrality-type property of the norm.

Definition 3.11 (see the comment after [HLP, Lemma 3.1]). Let X be a Banach space. We say that (the norm on) X is *locally octahedral* (LOH) if, for every $x \in X$ and every $\varepsilon > 0$, there is a $y \in S_X$ such that

$$\|sx + y\| \geq (1-\varepsilon)\big(|s|\|x\| + \|y\|\big) \quad \text{for all } s \in \mathbb{R}.$$

Clearly, every WOH Banach space is LOH.

In the following we point out some equivalent, but sometimes more convenient formulations of LOH.

Proposition 3.12 (see [HLP, Lemma 3.1]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is LOH;*

(ii) *whenever $x \in S_X$ and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x \pm ty\| \geq (1-\varepsilon)\big(\|x\| + t\big) \quad \text{for all } t > 0; \tag{3.3}$$

(iii) *whenever $x \in S_X$ and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x \pm y\| \geq 2 - \varepsilon.$$

(iv) *whenever $x \in S_X$ then*

$$\limsup_{\|h\| \to 0} \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|h\|} = 2;$$

Proof. (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii) is obvious. (iv)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iv) are straightforward.

(iii)$\Rightarrow$(ii). Assume that (iii) holds. Let $x \in S_X$ and let $\varepsilon > 0$. By (iii), pick any $y \in S_X$ with $\|x \pm y\| \geq 2 - \varepsilon$. We show that y satisfies (3.3). Suppose that $t > 0$. Then

$$\begin{aligned}
\|x \pm ty\| &\geq \max\{1, t\}\|x \pm y\| - \big(\max\{1, t\} - \min\{1, t\}\big) \\
&\geq \max\{1, t\}(1-\varepsilon) + \min\{1, t\} \\
&= 1 + t - \max\{1, t\}\varepsilon \\
&\geq (1+t)(1-\varepsilon).
\end{aligned}$$

Thus y satisfies (3.3). □

Remark 3.2. Condition (iv) in Proposition 3.12 means that the norm on X is 2-rough. Thus X is LOH if and only if the norm on X is 2-rough. In order to provide a unified octahedrality-based approach, we prefer to use the notion LOH instead of 2-rough.

Definition 3.13 (see, e.g., [DGZ]). Let X be a Banach space and let $\varepsilon > 0$. We say that a norm $\|\cdot\|$ on X is *ε-rough* if

$$\limsup_{\|h\|\to 0} \frac{\|x+h\|+\|x-h\|-2\|x\|}{\|h\|} \geq \varepsilon \quad \text{for all } x \in X.$$

We say that X is *rough*, if it is ε-rough for some $\varepsilon > 0$, and we say that X is *nonrough* otherwise.

For a Banach space the following implications hold:

$$\text{OH} \Rightarrow \text{WOH} \Rightarrow \text{LOH}.$$

In general, these implications are not reversible. There is a Banach space which is WOH, but fails to be OH.

Example 3.14. The Banach space $\ell_1 \oplus_2 \ell_1$ is WOH, however it is not OH. Indeed, by Proposition 3.30, $\ell_1 \oplus_2 \ell_1$ is WOH and, by Proposition 3.32, $\ell_1 \oplus_2 \ell_1$ cannot be OH.

There is a Banach space which is LOH, but fails to be WOH.

Example 3.15. By Theorem 2.9, we can equivalently renorm c_0 such that it has the LD2P and fails the D2P. Thus, by Theorems 3.27 and 3.24, its dual space is LOH, but fails to be WOH.

Similarly as with the diameter 2 properties (see Proposition 2.18), almost isometric ideals inherit all three octahedralities.

Proposition 3.16 (see [ALLN, Proposition 3.1]). *Let X be a Banach space and Y an ai-ideal in X. If X is OH, then Y is OH.*

Proof. Assume that X is OH. Let $y_1, \ldots, y_n \in S_Y$ and $\varepsilon > 0$. Since X is OH, there is an $x \in S_X$ such that $\|y_i + x\| \geq 2 - \varepsilon$ for all $i \in \{1, \ldots, n\}$. Let $E = \operatorname{span}\{x, y_1, \ldots, y_n\}$. Since Y is an ai-ideal in X, there is a $T\colon E \to Y$ such that

(i) $Te = e$ for all $e \in E \cap Y$;

(ii) $(1+\varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1+\varepsilon)\|e\|$ for all $e \in E$.

Take $z = Tx/\|Tx\|$. Observe that $z \in S_Y$ and $\|z - Tx\| \leq \varepsilon$ because $(1+\varepsilon)^{-1} \leq \|Tx\| \leq 1+\varepsilon$. We have that

$$\|y_i + z\| \geq \|T(y_i + x)\| - \|Tx - z\| \geq \frac{\|y_i + x\|}{1+\varepsilon} - \varepsilon \geq \frac{2-\varepsilon}{1+\varepsilon} - \varepsilon$$

for all $i \in \{1, \dots, n\}$. Since $\varepsilon > 0$ was arbitrary, we deduce that Y is OH. □

Proposition 3.17. *Let X be a Banach space and Y an ai-ideal in X. If X is LOH, then Y is LOH.*

Proof. Take $n = 2$ and $y_2 = -y_1$ in the proof of Proposition 3.16. □

Proposition 3.18. *Let X be a Banach space and Y an ai-ideal in X. If X is WOH, then Y is WOH.*

Proof. Assume that X is WOH. Let $y_1, \dots, y_n \in S_Y$, $y^* \in B_{Y^*}$, and let ε be such that $\varepsilon \in (0,1)$. We will show that there is an element $z \in S_Y$ such that

$$\|y_i + z\| \geq (1-\varepsilon)(|y^*(y_i)| + t) \qquad \text{for all } i \in \{1, \dots, n\} \text{ and } t \geq \varepsilon.$$

Since X is WOH, there is an $x \in S_X$ such that

$$\|y_i + x\| \geq (1-\varepsilon^4/4)(|y^*(y_i)| + t) \qquad \text{for all } i \in \{1, \dots, n\} \text{ and } t > 0.$$

Let $E = \operatorname{span}\{x, y_1, \dots, y_n\}$. Since Y is an ai-ideal in X, there is a $T\colon E \to Y$ such that

(i) $Te = e$ for all $e \in E \cap Y$;

(ii) $(1+\varepsilon^2/2)^{-1}\|e\| \leq \|Te\| \leq (1+\varepsilon^2/2)\|e\|$ for all $e \in E$.

Take $z = Tx/\|Tx\|$. Observe that $z \in S_Y$ and $\|z - Tx\| \leq \varepsilon^2/2$ because $(1+\varepsilon^2/2)^{-1} \leq \|Tx\| \leq 1+\varepsilon^2/2$. For all $i \in \{1, \dots, n\}$ and $t \geq \varepsilon$, we have

$$\begin{aligned}
\|y_i + z\| &\geq \|T(y_i + x)\| - \|Tx - z\| \\
&\geq \frac{\|y_i + x\|}{1+\varepsilon^2/2} - \varepsilon^2/2 \geq \frac{(1-\varepsilon^4/4)(|y^*(y_i)| + t)}{1+\varepsilon^2/2} - \varepsilon^2/2 \\
&\geq (1-\varepsilon^2/2)(|y^*(y_i)| + t) - \varepsilon^2/2 \\
&\geq (1-\varepsilon^2/2)(|y^*(y_i)| + t) - t\varepsilon/2 \\
&\geq (1-\varepsilon/2)(|y^*(y_i)| + t) - t\varepsilon/2 \\
&\geq (1-\varepsilon)(|y^*(y_i)| + t).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we deduce that Y is WOH. □

Recall that any Banach space is an ai-ideal in its bidual.

Corollary 3.19. *If X^{**} is OH (resp. WOH, LOH), then X is OH (resp. WOH, LOH).*

The converse of Corollary 3.19 fails for each level of octahedrality.

Example 3.20. The Banach space $C[0,1]$ is OH, but its bidual $C[0,1]^{**}$ even fails to be LOH. By Example 2.12, we know that $C[0,1]^*$ fails to have the LD2P, thus, by Theorem 3.27, the bidual X^{**} cannot be LOH.

3.2 Duality of diameter 2 properties and octahedrality

The goal of this section is to establish the duality between diameter 2 properties and octahedrality. We also show that one can think of diameter 2 properties as sort of extension properties.

It is well-known that a Banach space X is ε-rough if and only if the diameter of every weak* slice of B_{X^*} is greater or equal to ε (see [DGZ, Proposition I.1.11]). Thus, by Remark 3.2, X is LOH if and only if X^* has the w*-LD2P. In Theorem 3.25, we give an implicit proof for this result.

G. Godefroy (see [God, Remark II.5, 2)], see also [Dev, Remark (c), p. 119]) remarks that a Banach space is OH if and only if every convex combination of weak* slices of B_{X^*} is of diameter 2. A proof of this result can also be found in [BGLPRZ1, Theorem 2.1].

Theorem 3.21 (see [HLP, Theorem 2.2] and [God, Remark II.5, 2)], see also [BGLPRZ1, Theorem 2.1])**.** *Let X be a Banach space. The following assertions are equivalent:*

(i) *X^* has the w*-SD2P;*

(ii) *X is OH;*

(iii) *whenever E is a finite-dimensional subspace of X, $n \in \mathbb{N}$, $x_1^*, \dots, x_n^* \in B_{X^*}$, $\varepsilon > 0$, and $\varepsilon_0 \in (0, \varepsilon)$, there is a $y \in S_X$ such that, whenever $|\gamma_i| \leq 1 + \varepsilon_0$, there are $y_i^* \in X^*$ satisfying*

$$y_i^*|_E = x_i^*|_E, \quad y_i^*(y) = \gamma_i, \quad \text{and} \quad \|y_i^*\| \leq 1 + \varepsilon \quad \text{for all } i \in \{1, \dots, n\};$$

(iv) *whenever E is a finite-dimensional subspace of X, $n \in \mathbb{N}$, $x_1^*, \dots, x_n^* \in B_{X^*}$, and $\varepsilon > 0$, there are $y \in S_X$ and $x_{1,i}^*, x_{2,i}^* \in X^*$, satisfying*

$$x_{1,i}^*|_E = x_{2,i}^*|_E = x_i^*|_E, \qquad \|x_{1,i}^*\|, \|x_{2,i}^*\| \leq 1 + \varepsilon, \tag{3.4}$$

and $x_{1,i}^*(y) - x_{2,i}^*(y) > 2 - \varepsilon$ for all $i \in \{1, \dots, n\}$.

Proof. The equivalence of (i)⇔(ii) was pointed out in [God, Remark II.5, 2)]. Since no details of the proof were given in [God], we include the proof for completeness.

(i)⇒(ii). Assume that (i) holds. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, and let $\varepsilon > 0$. By (i), for every $i \in \{1, \dots, n\}$, there are $x_{1,i}^*, x_{2,i}^* \in B_{X^*}$ and $y \in S_X$ such that

$$x_{1,i}^*(x_i), x_{2,i}^*(x_i) > 1 - \varepsilon \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}\left(x_{1,i}^*(y) - x_{2,i}^*(y)\right) > 2 - \frac{\varepsilon}{n}.$$

For every $i \in \{1, \dots, n\}$, since $x_{1,i}^*(y) > 1 - \varepsilon$, one has

$$\|x_i + y\| \geq x_{1,i}^*(x_i + y) > 2 - 2\varepsilon,$$

and X is OH by (the equivalence (i)⇔(iii) of) Proposition 3.3.

(ii)⇒(i). Assume that (ii) holds. Let $S_1 = S(B_{X^*}, x_1, \alpha_1), \dots, S_n = S(B_{X^*}, x_n, \alpha_n)$ be weak* slices of B_{X^*} and $S = 1/n\sum_{i=1}^n S_i$. Let $\alpha = \min\{\alpha_1, \dots, \alpha_n\}$ and ε be such that $\varepsilon \in (0, \alpha)$. By (ii), there is an $y \in S_X$ such that

$$\|x_i \pm y\| > 2 - \varepsilon/2 \quad \text{for all } i \in \{1, \dots, n\}.$$

For every $i \in \{1, \dots, n\}$ find $y_i^*, z_i^* \in S_{X^*}$ such that

$$y_i^*(x_i + y) = \|x_i + y\| \quad \text{and} \quad z_i^*(x_i - y) = \|x_i - y\|.$$

Observe that $y_i^*(x_i), z^*(x_i) > 1 - \varepsilon/2$ and $y_i^*(y), z_i^*(-y) > 1 - \varepsilon/2$. Thus $y_i^*, z_i^* \in S_i$ for every $i \in \{1, \dots, n\}$. Define $y^* = 1/n\sum_{i=1}^n y_i^*$ and $z^* = 1/n\sum_{i=1}^n z_i^*$. Thus $y^*, z^* \in S$ and

$$\|y^* - z^*\| \geq \frac{1}{n}\sum_{i=1}^{n}(y_i^* - z_i^*)(y) > \frac{1}{n}\sum_{i=1}^{n}(2 - \varepsilon) = 2 - \varepsilon.$$

Therefore X^* has the w*-SD2P.

(ii)⇒(iii). Assume that (ii) holds. Let E be a finite-dimensional subspace of X, let $n \in \mathbb{N}$, $x_1^*, \dots, x_n^* \in B_{X^*}$, and let $\varepsilon_0 \in (0, \varepsilon)$. Choose $y \in S_X$ to satisfy

$$\|x\| + |t| \leq \frac{1 + \varepsilon}{1 + \varepsilon_0}\|x + ty\| \quad \text{for all } x \in E,\ t \in \mathbb{R}, \text{ and } i \in \{1, \dots, n\}.$$

Letting $\gamma_i \in [-1-\varepsilon_0, 1+\varepsilon_0]$, $i = \{1, \dots, n\}$, and defining $g_i \in \big(\operatorname{span}(E \cup \{y\})\big)^*$ by $g_i|_E = x_i^*|_E$ and $g_i(y) = \gamma_i$, it suffices to show that $\|g_i\| \leq 1 + \varepsilon$ (because, in this case, one may define the desired $y_i^* \in X^*$ to be any norm-preserving extension of g_i). To this end, it remains to observe that, whenever $x \in E$, $t \in \mathbb{R}$, and $i \in \{1, \dots, n\}$,

$$\begin{aligned} |g_i(x + ty)| &\leq |x_i^*(x)| + |t|\,|\gamma_i| \leq (1 + \varepsilon_0)(\|x\| + |t|) \\ &\leq (1 + \varepsilon)\|x + ty\|. \end{aligned}$$

(iii)⇒(iv) is obvious.

(iv)⇒(i). Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, and let $\varepsilon > 0$. For every $i \in \{1, \dots, n\}$, choose $x_i^* \in B_{X^*}$, so that $x_i^*(x_i) > 1 - \varepsilon$, and let $y \in S_X$ and $x_{1,1}^*, x_{2,1}^*, \dots, x_{1,n}^*, x_{2,n}^* \in X^*$ be as in (iv), where $E = \operatorname{span}\{x_1, \dots, x_n\}$. It suffices to observe that, for all $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$, one has $\frac{x_{j,i}^*}{1+\varepsilon} \in B_{X^*}$,

$$\frac{x_{j,i}^*}{1 + \varepsilon}(x_i) = \frac{x_i^*(x_i)}{1 + \varepsilon} > \frac{1 - \varepsilon}{1 + \varepsilon},$$

and

$$\left\| \sum_{i=1}^{n} \frac{1}{n} \frac{x_{1,i}^*}{1 + \varepsilon} - \sum_{i=1}^{n} \frac{1}{n} \frac{x_{2,i}^*}{1 + \varepsilon} \right\| > \frac{\left| \sum_{i=1}^{n} \frac{1}{n}(x_{1,i}^*(y) - x_{2,i}^*(y)) \right|}{1 + \varepsilon} > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

□

The following theorem is an obvious consequence of Proposition 2.14 and Theorem 3.21.

Theorem 3.22 (see [BGLPRZ1, Corollary 2.2] and [HLP, Theorem 2.3]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the SD2P;*

(ii) *X^* is OH.*

The dual characterization of the D2P was established in [HLP].

Theorem 3.23 (see [HLP, Theorem 2.6]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X^* has the w^*-D2P;*

(ii) *X is WOH;*

(iii) *for every finite-dimensional subspace E of X, every $x^* \in B_{X^*}$, every $\varepsilon > 0$, and every $\varepsilon_0 \in (0, \varepsilon)$, there is a $y \in S_X$ such that, whenever $|\gamma| \leq 1 + \varepsilon_0$, there is a $y^* \in X^*$ satisfying*

$$y^*|_E = x^*|_E, \quad y^*(y) = \gamma, \quad \text{and} \quad \|y^*\| \leq 1 + \varepsilon;$$

(iv) *for every finite-dimensional subspace E of X, every $x^* \in B_{X^*}$, and every $\varepsilon > 0$, there are $y \in S_X$ and $x_1^*, x_2^* \in X^*$ satisfying*

$$x_1^*|_E = x_2^*|_E = x^*|_E, \quad x_1^*(y) - x_2^*(y) > 2 - \varepsilon,$$

and $\|x_1^*\|, \|x_2^*\| \leq 1 + \varepsilon$;

(v) *whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there are $y \in B_X$ and $x_1^*, x_2^* \in B_{X^*}$ such that*

$$\left|x_1^*(x_i) - x^*(x_i)\right| < \varepsilon \ \text{ and } \ \left|x_2^*(x_i) - x^*(x_i)\right| < \varepsilon \ \text{ for all } i \in \{1, \ldots, n\},$$

and

$$x_1^*(y) - x_2^*(y) > 2 - \varepsilon.$$

Proof. (i)⇒(ii). Assume that (i) holds. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, let $x^* \in B_{X^*}$, and let ε be such that $\varepsilon \in (0, 1)$. Pick $\delta \in (0, \varepsilon^2)$ satisfying $\delta < \varepsilon\,|x^*(x_i)|$ for all $i \in \{1, \ldots, n\}$ with $|x^*(x_i)| \neq 0$. By (i), there are $u^*, v^* \in B_{X^*}$ and $y \in S_X$ such that

$$\left|u^*(x_i) - x^*(x_i)\right| < \delta \quad \text{and} \quad \left|v^*(x_i) - x^*(x_i)\right| < \delta \quad \text{for all } i \in \{1, \ldots, n\},$$

and

$$v^*(y) - u^*(y) > 2 - \varepsilon.$$

Since $v^*(y) \leq 1$ and $u^*(y) \geq -1$, it follows that $v^*(y) > 1 - \varepsilon$ and $u^*(y) < -1 + \varepsilon$. Let $i \in \{1, \ldots, n\}$ and $t \geq \varepsilon$ be arbitrary. If $x^*(x_i) \neq 0$, then, choosing $z^* \in \{u^*, v^*\}$ so that $x^*(x_i)$ and $z^*(y)$ (and thus also $z^*(x_i)$ and $z^*(y)$) have the same sign, one has

$$\begin{aligned}
\|x_i + ty\| &\geq \left|z^*(x_i) + tz^*(y)\right| = |z^*(x_i)| + t|z^*(y)| \\
&\geq |x^*(x_i)| - |x^*(x_i) - z^*(x_i)| + t|z^*(y)| \\
&\geq |x^*(x_i)| - \varepsilon|x^*(x_i)| + (1 - \varepsilon)t \\
&= (1 - \varepsilon)\big(|x^*(x_i)| + t\big).
\end{aligned}$$

If $x^*(x_i) = 0$, then

$$\begin{aligned}\|x_i + ty\| &\geq \big|u^*(x_i) + tu^*(y)\big| \geq t|u^*(y)| - |u^*(x_i)| \\ &\geq (1-\varepsilon)t - \varepsilon^2 \geq (1-\varepsilon)t - t\varepsilon = (1-2\varepsilon)\big(|x^*(x_i)| + t\big),\end{aligned}$$

and it follows that X is WOH.

(ii)⇒(iii). Assume that (ii) holds. Let E be a finite-dimensional subspace of X, let $x^* \in B_{X^*}$, and let $\varepsilon_0 \in (0, \varepsilon)$. Choose $y \in S_X$ to satisfy

$$|x^*(x)| + |t| \leq \frac{1+\varepsilon}{1+\varepsilon_0}\|x + ty\| \quad \text{for all } x \in E \text{ and } t \in \mathbb{R}.$$

Letting $\gamma \in [-1-\varepsilon_0, 1+\varepsilon_0]$, and defining $g \in \big(\operatorname{span}(E \cup \{y\})\big)^*$ by $g|_E = x^*|_E$ and $g(y) = \gamma$, it suffices to show that $\|g\| \leq 1 + \varepsilon$ (because, in this case, one may define the desired $y^* \in X^*$ to be any norm-preserving extension of g). To this end, it remains to observe that, whenever $x \in E$ and $t \in \mathbb{R}$,

$$\begin{aligned}|g(x + ty)| \leq |x^*(x)| + |t|\,|\gamma| &\leq (1+\varepsilon_0)(|x^*(x)| + |t|) \\ &\leq (1+\varepsilon)\|x + ty\|.\end{aligned}$$

(iii)⇒(iv)⇒(v)⇒(i) is obvious. □

The following theorem is an obvious consequence of Proposition 2.14 and Theorem 3.23.

Theorem 3.24 (see [HLP, Theorem 2.7]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the D2P;*

(ii) *X^* is WOH.*

We conclude this section with the dual characterization of the LD2P.

Theorem 3.25 (cf. [DGZ, Proposition I.1.11]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X^* has the w^*-LD2P;*

(ii) *X is LOH;*

(iii) *for every* $x \in S_X$, *every* $\alpha \in [-1, 1]$, *every* $\varepsilon > 0$, *and every* $\varepsilon_0 \in (0, \varepsilon)$, *there is a* $y \in S_X$ *such that, whenever* $|\gamma| \leq 1 + \varepsilon_0$, *there is a* $y^* \in X^*$ *satisfying*

$$y^*(x) = \alpha, \quad y^*(y) = \gamma, \quad \textit{and} \quad \|y^*\| \leq 1 + \varepsilon;$$

(iv) *for every* $x \in S_X$, *every* $\alpha \in [-1, 1]$, *and every* $\varepsilon > 0$, *there are* $y \in S_X$ *and* $x_1^*, x_2^* \in X^*$ *satisfying*

$$x_1^*(x) = x_2^*(x) = \alpha, \qquad \|x_1^*\|, \|x_2^*\| \leq 1 + \varepsilon,$$

and $x_1^*(y) - x_2^*(y) > 2 - \varepsilon$.

Proof. (i)⇒(ii). Assume that (i) holds. Let $x \in S_X$ and let $\varepsilon > 0$. By (i), there are $x_1^*, x_2^* \in B_{X^*}$ and $y \in S_X$ such that

$$x_1^*(x), x_2^*(x) > 1 - \varepsilon/2 \quad \text{and} \quad (x_1^*(y) - x_2^*(y)) > 2 - \varepsilon/2.$$

Since $x_1^*(y), x_2^*(-y) > 1 - \varepsilon/2$, one has

$$\|x + y\| \geq x_1^*(x + y) > 2 - \varepsilon,$$

and

$$\|x - y\| \geq x_2^*(x - y) > 2 - \varepsilon.$$

Thus X is LOH by Proposition 3.12.

(ii)⇒(i). Assume that (ii) holds. Let $S(B_{X^*}, x, \alpha)$ be a weak* slice of B_{X^*}. Let $\varepsilon \in (0, \alpha)$. By (ii), there is a $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon/2$. Find $y^*, z^* \in S_{X^*}$ such that

$$y^*(x + y) = \|x + y\| \quad \text{and} \quad z^*(x - y) = \|x - y\|.$$

Observe that $y^*(x), z^*(x) > 1 - \varepsilon/2$ and $y^*(y), z^*(-y) > 1 - \varepsilon/2$. Therefore $y^*, z^* \in S(B_{X^*}, x, \alpha)$ and

$$\|y^* - z^*\| \geq (y^* - z^*)(y) > 2 - \varepsilon.$$

Thus X^* has the w*-LD2P.

(ii)⇒(iii). Assume that (ii) holds. Let $x \in S_X$, $\alpha \in [-1, 1]$, $\varepsilon > 0$, and let ε_0 be such that $\varepsilon_0 \in (0, \varepsilon)$. Choose $y \in S_X$ to satisfy

$$\|sx + y\| \geq \frac{1 + \varepsilon_0}{1 + \varepsilon}(|s| + \|y\|) \quad \text{for all } s \in \mathbb{R}.$$

Now let $|\gamma| \leq 1+\varepsilon_0$. Defining $g \in (\operatorname{span}\{x,y\})^*$ by

$$g(x)=\alpha, \quad g(y)=\gamma,$$

one has, for all $s \in \mathbb{R}$,

$$|g(sx+y)| \leq |s||\alpha| + |\gamma| \leq (1+\varepsilon_0)(|s|+\|y\|) \leq (1+\varepsilon)\|sx+y\|,$$

hence $\|g\| \leq 1+\varepsilon$. The desired y^* can be defined to be any norm preserving extension to X of g.

(iii)⇒(iv) is obvious.

(iv)⇒(i). Let $x \in S_X$, $\alpha = 1$, and $\varepsilon > 0$ be arbitrary, and let $y \in S_X$ and $x_1^*, x_2^* \in X^*$ be as in (iv). It remains to observe that $\frac{x_1^*}{1+\varepsilon}, \frac{x_2^*}{1+\varepsilon} \in B_{X^*}$,

$$\left\| \frac{x_1^*}{1+\varepsilon} - \frac{x_2^*}{1+\varepsilon} \right\| > \frac{|x_1^*(y) - x_2^*(y)|}{1+\varepsilon} > \frac{2-\varepsilon}{1+\varepsilon},$$

and, for all $i \in \{1,2\}$,

$$\frac{x_i^*}{1+\varepsilon}(x) = \frac{1}{1+\varepsilon}.$$

□

Corollary 3.26. *The following assertions are equivalent:*

(i) *X is nonrough;*

(ii) *the dual unit ball B_{X^*} has weak* slices of arbitrarily small diameter.*

The next theorem is an obvious consequence of Proposition 2.14 and Theorem 3.25.

Theorem 3.27 (see [HLP, Theorem 3.3]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the LD2P;*

(ii) *X^* is LOH.*

3.3 Stability results of octahedral norms

In this section, we study how octahedralities are preserved by taking ℓ_p-sums of Banach spaces. The obtained results are applied to provide a unified octahedrality based approach to derive stability results of diameter 2 properties. We conclude this section by characterizing octahedral spaces in terms of separable subspaces.

The following proposition is our main stability result for LOH spaces.

Proposition 3.28 (see [HLP, Proposition 3.4]). *Let X and Y be Banach spaces.*

(a) *If X is LOH, then $X \oplus_1 Y$ is LOH.*

(b) *If X and Y are LOH, and p is such that $1 < p \leq \infty$, then $X \oplus_p Y$ is LOH.*

(c) *If $X \oplus_p Y$ is LOH, where p is such that $1 < p \leq \infty$, then X is LOH.*

Remark 3.3. Note that Proposition 3.28, (c), fails if we take p = 1. This is clear by part (a).

Proof. (a). Assume that X is LOH. Fix $(x, y) \in S_{X\oplus_1 Y}$ and $\varepsilon > 0$. By our assumption, there exists a $u \in S_X$ such that

$$\|x \pm u\| \geq (1-\varepsilon)\big(\|x\|+1\big).$$

Hence,

$$\big\|(x, y) \pm (u, 0)\big\|_1 \geq (1-\varepsilon)\big(\|x\|+1\big) + \|y\| \geq 2 - 2\varepsilon.$$

Thus $X \oplus_1 Y$ is LOH.

(b). Assume that X and Y are LOH, and let p be such that $1 < p \leq \infty$. Let $(x, y) \in S_{X\oplus_p Y}$ and let ε be such that $\varepsilon \in (0, 1)$. It suffices to find a $(u, v) \in S_{X\oplus_p Y}$ such that

$$\|(x, y) \pm (u, v)\|_p \geq 2 - \varepsilon.$$

We may (and do) assume that $x \neq 0$ and $y \neq 0$. By our assumption, there exist $\tilde{u} \in S_X$ and $\tilde{v} \in S_Y$ such that

$$\left\|\frac{x}{\|x\|} \pm \tilde{u}\right\| \geq 2 - \varepsilon \text{ and } \left\|\frac{y}{\|y\|} \pm \tilde{v}\right\| \geq 2 - \varepsilon.$$

If $1 < p < \infty$, it follows that

$$\Big\|x \pm \|x\|\tilde{u}\Big\|^p + \Big\|y \pm \|y\|\tilde{v}\Big\|^p \geq (2-\varepsilon)^p.$$

This completes the proof for $1 < p < \infty$, because one may take $u = \|x\|\tilde{u}$ and $v = \|y\|\tilde{v}$.

If $p = \infty$, one may take $u = \tilde{u}$ and $v = \tilde{v}$ because

$$\big\|(x, y) \pm (\tilde{u}, \tilde{v})\big\|_\infty = \max\big\{\|x \pm \tilde{u}\|, \|y \pm \tilde{v}\|\big\} \geq 2 - \varepsilon.$$

(c). Assume that $X \oplus_p Y$ is LOH, where p is such that $1 < p \leq \infty$. Let $x \in S_X$ and let ε be such that $\varepsilon \in (0, 1)$. Since $\big\|(x, 0)\big\|_p = 1$, whenever $\delta > 0$, there exists a $(u, v) \in S_{X\oplus_p Y}$ such that

$$\big\|(x \pm u, v)\big\|_p \geq 2 - \delta. \tag{3.5}$$

It suffices to show that (3.5), with δ small enough, implies that

$$\|x \pm u\| \geq 2 - \varepsilon, \tag{3.6}$$

because, in this case, $\|u\| \geq 1 - \varepsilon$, thus

$$\left\|x \pm \frac{u}{\|u\|}\right\| \geq \|x \pm u\| - \big(1 - \|u\|\big) \geq 2 - 2\varepsilon,$$

and it follows that X is LOH.

If $p = \infty$, (3.5) means that $\max\big\{\|x \pm u\|, \|v\|\big\} \geq 2 - \delta$. Since $\|v\| \leq 1$, taking $\delta = \varepsilon$ implies (3.6).

If $1 < p < \infty$, (3.5) means that $\|x \pm u\|^p + \|v\|^p \geq (2 - \delta)^p$. Since $\|u\|^p + \|v\|^p = 1$, this implies that

$$\|x \pm u\|^p \geq (2 - \delta)^p - \big(1 - \|u\|^p\big), \tag{3.7}$$

thus it suffices to show that $\|u\|$ is as close to 1 as we want whenever δ is small enough. The latter is true because, by (3.7),

$$(1 + \|u\|)^p - \|u\|^p \geq (2 - \delta)^p - 1,$$

and the function $f \colon [0, 1] \to \mathbb{R}$, $f(t) = (1 + t)^p - t^p$, is strictly increasing with $\lim_{t\to 1} f(t) = 2^p - 1$. □

Proposition 3.28 combined, respectively, with Theorems 3.27 and 3.25 immediately gives the corresponding stability results for the LD2P (see Theorem 2.29) and for the w*-LD2P.

Corollary 3.29 (see [HLP, Corollary 3.6]). *Let X and Y be Banach spaces.*

(a) *If X^* has the w*-LD2P, then $(X \oplus_1 Y)^*$ has the w*-LD2P.*

(b) *If X^* and Y^* have the w^*-LD2P, and p is such that $1 < p \leq \infty$, then $(X \oplus_p Y)^*$ has the w^*-LD2P.*

(c) *If $(X \oplus_p Y)^*$ has the w^*-LD2P, where p is such that $1 < p \leq \infty$, then X^* has the w^*-LD2P.*

The following proposition is our main stability result for WOH spaces.

Proposition 3.30 (see [HLP, Proposition 3.7]). *Let X and Y be Banach spaces.*

(a) *If X is WOH, then $X \oplus_1 Y$ is WOH.*

(b) *If X and Y are WOH, and p is such that $1 < p \leq \infty$, then $X \oplus_p Y$ is WOH.*

(c) *If $X \oplus_p Y$ is WOH, where p is such that $1 < p \leq \infty$, then X is WOH.*

Proof. (a). Assume that X is WOH. Let E and F be finite-dimensional subspaces of X and Y, respectively, let $(x^*, y^*) \in B_{X^* \oplus_\infty Y^*}$, and let $\varepsilon > 0$. It suffices to show that there exists a $(u, v) \in S_{X \oplus_1 Y}$ such that, for all $x \in E$, all $y \in F$, and all $t \in \mathbb{R}$, one has

$$\big\|(x, y) + t(u, v)\big\|_1 \geq (1 - \varepsilon)\big(|x^*(x) + y^*(y)| + |t|\big).$$

By our assumption, there exists a $u \in S_X$ such that

$$\|x + tu\| \geq (1 - \varepsilon)\big(|x^*(x)| + |t|\big) \quad \text{for all } x \in E \text{ and } t \in \mathbb{R},$$

One has, for all $x \in E$, all $y \in F$, and all $t \in \mathbb{R}$,

$$\begin{aligned}\big\|(x, y) + t(u, 0)\big\|_1 &= \|x + tu\| + \|y\| \\ &\geq (1 - \varepsilon)\big(|x^*(x)| + |t|\big) + \|y\| \\ &\geq (1 - \varepsilon)\big(|x^*(x) + y^*(y)| + |t|\big).\end{aligned}$$

Thus $X \oplus_1 Y$ is WOH.

(b). Assume that X and Y are WOH, and let p be such that $1 < p \leq \infty$. Let E and F be finite-dimensional subspaces of X and Y, respectively, let $(x^*, y^*) \in S_{X^* \oplus_q Y^*}$, where q is the conjugate exponent of p (i.e., $1/p + 1/q = 1$ if $1 < p < \infty$, and $q = 1$ if $p = \infty$), and let $\varepsilon \in (0, 1)$. It suffices to find a $(u, v) \in S_{X \oplus_p Y}$ such that for all $x \in E$, all $y \in F$, and all $t \in \mathbb{R}$, one has

$$\big\|(x, y) + t\,(u, v)\big\|_p \geq (1 - \varepsilon)\big(|x^*(x) + y^*(y)| + |t|\big).$$

We may (and do) assume that $x^* \neq 0$ and $y^* \neq 0$.

By our assumption, there exist $\tilde{u} \in S_X$ and $\tilde{v} \in S_Y$ such that

$$\|x + t\tilde{u}\| \geq (1-\varepsilon)\left(\frac{|x^*(x)|}{\|x^*\|} + |t|\right) \quad \text{for all } x \in E \text{ and } t \in \mathbb{R},$$

and

$$\|y + t\tilde{v}\| \geq (1-\varepsilon)\left(\frac{|y^*(y)|}{\|y^*\|} + |t|\right) \quad \text{for all } y \in F \text{ and } t \in \mathbb{R}.$$

If $1 < p < \infty$, take $u = \|x^*\|^{q-1}\tilde{u}$ and $v = \|y^*\|^{q-1}\tilde{v}$, and observe that, for all $x \in E$, all $y \in F$, and all $t \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{(1-\varepsilon)^p}&\big(\|x+tu\|^p + \|y+tv\|^p\big) \\
&\geq \left(\frac{|x^*(x)|}{\|x^*\|} + \|x^*\|^{q-1}|t|\right)^p + \left(\frac{|y^*(y)|}{\|y^*\|} + \|y^*\|^{q-1}|t|\right)^p \\
&= \|x^*\|^q\left(\frac{|x^*(x)|}{\|x^*\|^q} + |t|\right)^p + \|y^*\|^q\left(\frac{|y^*(y)|}{\|y^*\|^q} + |t|\right)^p \\
&\geq \left(\|x^*\|^q\frac{|x^*(x)|}{\|x^*\|^q} + \|y^*\|^q\frac{|y^*(y)|}{\|y^*\|^q} + |t|\right)^p \\
&= \big(|x^*(x)| + |y^*(y)| + |t|\big)^p.
\end{aligned}$$

The last inequality holds because the function $[0,\infty) \to \mathbb{R}$, $s \mapsto (s+|t|)^p$, is convex for any fixed $t \in \mathbb{R}$.

If $p = \infty$, take $u = \tilde{u}$ and $v = \tilde{v}$, and observe that, for all $x \in E$, all $y \in F$, and all $t \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{(1-\varepsilon)}&\max\big\{\|x+tu\|, \|y+tv\|\big\} \\
&\geq \frac{1}{(1-\varepsilon)}\Big(\|x^*\|\,\|x+tu\| + \|y^*\|\,\|y+tv\|\Big) \\
&\geq \|x^*\|\left(\frac{|x^*(x)|}{\|x^*\|} + |t|\right) + \|y^*\|\left(\frac{|y^*(y)|}{\|y^*\|} + |t|\right) \\
&= |x^*(x)| + |y^*(y)| + |t|.
\end{aligned}$$

(c). Assume that $X \oplus_p Y$ is WOH, where p is such that $1 < p \leq \infty$. Let E be a finite-dimensional subspace of X, let $x^* \in S_{X^*}$, and let $\varepsilon \in (0,1)$. Choose $\delta > 0$ to satisfy $(1+\delta)^q - (1-\delta)^q < \varepsilon^q$, where q is the conjugate exponent of p. By enlarging E if necessary, we may assume that $\|x^*|_E\| \geq 1 - \delta$ (notice that X must be infinite-dimensional because X is WOH). By (the equivalence (ii)⇔(iii) of) Theorem 3.23, there are $z \in X$, $y \in Y$, with $\|(z, y)\|_p = 1$, and $z_i^* \in X^*$, $y_i^* \in Y^*$, with $\|(z_i^*, y_i^*)\|_q \leq 1 + \delta$, satisfying

$$z_i^*|_E = x^*|_E \quad \text{and} \quad z_i^*(z) + y_i^*(y) = (-1)^i \quad \text{for all } i \in \{1, 2\}.$$

Since

$$\|y_i^*\|^q \leq (1 + \delta)^q - \|z_i^*\|^q \leq (1 + \delta)^q - (1 - \delta)^q < \varepsilon^q,$$

one has $|y_i^*(y)| < \varepsilon$ for all $i \in \{1, 2\}$, and thus $z_2^*(z) > 1-\varepsilon$ and $z_1^*(z) < -1+\varepsilon$. Now let $x \in E$ be arbitrary. Choosing $i \in \{1, 2\}$ so that $x^*(x)$ and $z_i^*(z)$ have the same sign, one has

$$\begin{aligned}\left\| x + \frac{z}{\|z\|} \right\| &\geq \frac{1}{1+\varepsilon} \left| z_i^*(x) + \frac{z_i^*(z)}{\|z\|} \right| = \frac{1}{1+\varepsilon} \left(|x^*(x)| + \frac{|z_i^*(z)|}{\|z\|} \right) \\ &\geq \frac{1}{1+\varepsilon} \big(|x^*(x)| + 1 - \varepsilon \big) \geq \frac{1-\varepsilon}{1+\varepsilon} \big(|x^*(x)| + 1 \big),\end{aligned}$$

and it follows that X is WOH. □

Proposition 3.30 combined, respectively, with Theorems 3.24 and 3.23 immediately gives the corresponding stability results for the D2P (see Theorem 2.31) and for the w*-D2P.

Corollary 3.31 (see [HLP, Corollary 3.9]). *Let X and Y be Banach spaces.*

(a) *If X^* has the w*-D2P, then $(X \oplus_1 Y)^*$ has the w*-D2P.*

(b) *If X^* and Y^* have the w*-D2P, and p is such that $1 < p \leq \infty$, then $(X \oplus_p Y)^*$ has the w*-D2P.*

(c) *If $(X \oplus_p Y)^*$ has the w*-D2P, and p is such that $1 < p \leq \infty$, then X^* has the w*-D2P.*

The following proposition is our main stability result for OH spaces. It turns out that OH spaces are stable under ℓ_p-sums only if $p = 1$ or $p = \infty$.

Proposition 3.32 (see [HLP, Proposition 3.10]). *Let X and Y be Banach spaces.*

(a) *If X is OH, then $X \oplus_1 Y$ is OH.*

(b) *If p is such that $1 < p < \infty$, then $X \oplus_p Y$ is not OH.*

(c) *If X and Y are OH, then $X \oplus_\infty Y$ is OH.*

(d) *If $X \oplus_\infty Y$ is OH, then X is OH.*

Proof. (a). Assume that X is OH. Let $n \in \mathbb{N}$, $(x_1, y_1), \dots, (x_n, y_n) \in S_{X\oplus_1 Y}$, and let $\varepsilon > 0$. By our assumption, there exists a $u \in S_X$ such that

$$\|x_i + u\| \geq (1-\varepsilon)\big(\|x_i\| + 1\big) \quad \text{for all } i \in \{1, \dots, n\}.$$

Hence, for all $i \in \{1, \dots, n\}$,

$$\|(x_i, y_i) + (u, 0)\|_1 \geq (1-\varepsilon)\big(\|x_i\| + 1\big) + \|y_i\| \geq 2 - 2\varepsilon.$$

(b). Let $x \in S_X$, $y \in S_Y$, and let p be such that $1 < p < \infty$. We shall show that, for sufficiently small $\varepsilon > 0$, there is no $(u, v) \in S_{X\oplus_p Y}$ such that

$$\big\|(x, 0) + (u, v)\big\|_p \geq 2 - \varepsilon \qquad \text{and} \qquad \big\|(0, y) + (u, v)\big\|_p \geq 2 - \varepsilon.$$

If such an element (u, v) existed, then

$$\big\|(x + u, y + v)\big\|_p^p = \|x + u\|^p + \|y + v\|^p \geq 2(2-\varepsilon)^p - 1.$$

On the other hand,

$$\big\|(x + u, y + v)\big\|_p^p \leq \big(\|(x, y)\|_p + \|(u, v)\|_p\big)^p = (2^{1/p} + 1)^p.$$

For small ε, we would have a contradiction because

$$2^{p+1} - 1 > (2^{1/p} + 1)^p.$$

The last inequality is easily obtained from the Minkowski's inequality by considering $(2^{1/p}, 0), (1, 1) \in \mathbb{R}^2$.

(c). Assume that X and Y are OH. Let $n \in \mathbb{N}$, $(x_1, y_1), \dots, (x_n, y_n) \in S_{X\oplus_\infty Y}$, and let $\varepsilon > 0$. By our assumption, there are $u \in S_X$ and $v \in S_X$ such that

$$\|x_i + u\| \geq (1-\varepsilon)\big(\|x_i\| + 1\big) \quad \text{for all } i \in \{1, \dots, n\},$$

and

$$\|y_i + v\| \geq (1-\varepsilon)\big(\|y_i\| + 1\big) \quad \text{for all } i \in \{1, \dots, n\}.$$

Consequently, for all $i \in \{1, \dots, n\}$,

$$\begin{aligned}\big\|(x_i, y_i) + (u, v)\big\|_\infty &= \max\big\{\|x_i + u\|, \|y_i + v\|\big\} \\ &\geq (1-\varepsilon)\big(\max\{\|x_i\|, \|y_i\|\} + 1\big) \\ &= (1-\varepsilon)\big(\|(x_i, y_i)\|_\infty + 1\big).\end{aligned}$$

(d). Assume that $X \oplus_\infty Y$ is OH. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and let ε be such that $\varepsilon \in (0,1)$. By our assumption, there exists a $(u,v) \in S_{X\oplus_\infty Y}$ such that

$$\max\big\{\|x_i + u\|, \|v\|\big\} \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \ldots, n\}.$$

Since $\|v\| \leq 1$, we have

$$\|x_i + u\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \ldots, n\}.$$

It follows that $\|u\| \geq 1 - \varepsilon$. Therefore, for all $i \in \{1, \ldots, n\}$,

$$\left\| x_i + \frac{u}{\|u\|} \right\| \geq \|x_i + u\| - \big(1 - \|u\|\big) \geq 2 - 2\varepsilon.$$

□

Proposition 3.32 combined, respectively, with Theorems 3.22 and 3.21 immediately gives the corresponding stability results for the SD2P (see Theorems 2.32 and 2.33) and for the w*-SD2P.

Corollary 3.33 (see [HLP, Corollary 3.12]). *Let X and Y be Banach spaces.*

(a) *If X^* has the w*-SD2P, then $(X \oplus_1 Y)^*$ has the w*-SD2P.*

(b) *If p is such that $1 < p < \infty$, then $(X \oplus_p Y)^*$ does not have the w*-SD2P.*

(c) *If X^* and Y^* have the w*-SD2P, then $(X \oplus_\infty Y)^*$ has the w*-SD2P.*

(d) *If $(X \oplus_\infty Y)^*$ has the w*-SD2P, then X^* has the w*-SD2P.*

In Chapter 2, we saw that if an M-ideal Y in X has some diameter 2 property, then X has the same diameter 2 property. Using the duality between diameter 2 properties and octahedralities, we can give an alternative proof to this result.

Proposition 3.34 (see Propositions 2.36–2.38 and [HLP, Proposition 3.13]). *Let X be a Banach space and let Y be an M-ideal in X.*

(a) *If Y has the LD2P, then also X has the LD2P.*

(b) *If Y has the D2P, then also X has the D2P.*

(c) *If Y has the SD2P, then also X has the SD2P.*

Proof. Since Y is an M-ideal in X, one has $X^* = \operatorname{ran} P \oplus_1 \ker P$, where $P\colon X^* \to X^*$ is the M-ideal projection. Since $\operatorname{ran} P$ is isometrically isomorphic to Y^*, the assertions (a), (b), and (c) follow, respectively, from Theorem 3.27 combined with Proposition 3.28, (a), from Theorem 3.24 combined with Proposition 3.30, (a), and from Theorem 3.22 combined with Proposition 3.32, (a). □

Proposition 3.35 (see Theorem 2.21 and [HLP, Proposition 3.14]). *Let X be a Banach space and let a proper subspace Y be a strict M-ideal in X. Then both Y and X have the SD2P.*

Proof. We give an alternative octahedrality based proof to the original one given in [ALN1, Theorem 4.10].

Letting $P\colon X^* \to X^*$ be the M-ideal projection, throughout the proof, for convenience, we identify $\operatorname{ran} P$ and Y^* in the usual way.

By Proposition 3.34, it suffices to show that Y has the SD2P. To this end, letting $n \in \mathbb{N}$, $y_1^*, \dots, y_n^* \in S_{Y^*}$, and $\varepsilon > 0$ be arbitrary, by Theorem 3.22 and Proposition 3.3, it suffices to find a $y^* \in S_{Y^*}$ such that

$$(1+\varepsilon)\|y_i^* + y^*\| \geq 2 - 7\varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Choose an $x \in S_X$ so that $d(x, Y) > 1 - \varepsilon$, and $y_1, \dots, y_n \in S_Y$ so that $y_i^*(y_i) > 1 - \varepsilon$. By [Wer1, Proposition 2.3], there is a $z \in B_Y$ such that

$$|y_i^*(x - z)| < \varepsilon \quad \text{and} \quad \|\pm y_i + x - z\| < 1 + \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Let $y^* \in S_{Y^*}$ be such that

$$y^*(x - z) > \|x - z\| - \varepsilon \geq d(x, Y) - \varepsilon > 1 - 2\varepsilon.$$

Whenever $i \in \{1, \dots, n\}$, one has $y^*(y_i) > -3\varepsilon$ because

$$1 + \varepsilon > y^*(-y_i + x - z) > -y^*(y_i) + 1 - 2\varepsilon,$$

thus

$$\begin{aligned}
(1+\varepsilon)\|y_i^* + y^*\| &\geq (y_i^* + y^*)(y_i + x - z) \\
&= y_i^*(y_i) + y_i^*(x - z) + y^*(y_i) + y^*(x - z) \\
&> 1 - \varepsilon - \varepsilon - 3\varepsilon + 1 - 2\varepsilon \\
&= 2 - 7\varepsilon.
\end{aligned}$$

□

The inspiration for Propositions 3.36–3.38 came from the studying done in [Dev]. R. Deville (cf. [Dev, Proposition 5]) essentially showed that if a Banach space X is OH, then there is a separable closed subspace Y in X which is also OH. We will show that a slightly stronger version of this result holds.

Proposition 3.36. *A Banach space X is OH if and only if for every separable subspace Y of X, there is a separable OH subspace Z of X such that $Y \subset Z \subset X$.*

Proof. Necessity. Assume that X is OH. Let Y be a separable subspace of X, let $\{u_m\colon\ m \in \mathbb{N}\}$ be a dense subset in Y, and, for every $m \in \mathbb{N}$, let $\varepsilon_m > 0$ be such that $\varepsilon_m \to 0$ as $m \to \infty$. Put $Y_1 = \operatorname{span}\{u_1\}$. For $m \in \mathbb{N}$, choose Y_{m+1} as follows: find a $y_m \in S_X$ so that

$$\|x + y_m\| \geq 2 - \varepsilon_m \quad \text{for all } x \in S_{Y_m},$$

and put $Y_{m+1} = \operatorname{span}\big(Y_m \cup \{y_m\} \cup \{u_{m+1}\}\big)$.

Take $Z = \overline{\bigcup_{m=1}^{\infty} Y_m}$. To see that Z is OH, let $n \in \mathbb{N}$, $z_1, \ldots, z_n \in S_Z$, and let $\varepsilon > 0$. It suffices to find a $y \in S_Z$ such that

$$\|z_i + y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \ldots, n\}.$$

Pick $m \in \mathbb{N}$ so that $\varepsilon_m \leq \varepsilon/2$ and, for every $i \in \{1, \ldots, n\}$, there is an $x_i \in S_{Y_m}$ satisfying $\|z_i - x_i\| < \varepsilon/2$. For every $i \in \{1, \ldots, n\}$, one has

$$\|z_i + y_m\| \geq \|x_i + y_m\| - \|z_i - x_i\| \geq 2 - \varepsilon_m - \frac{\varepsilon}{2} \geq 2 - \varepsilon.$$

Sufficiency. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and let $\varepsilon > 0$. Take $Y = \operatorname{span}\{x_1, \ldots, x_n\}$. By our assumption, there is a separable subspace Z of X such that $Y \subset Z \subset X$ and Z is OH. Therefore there is a $z \in S_Z \subset S_X$ such that $\|x_i + z\| \geq 2 - \varepsilon$ for all $i \in \{1, \ldots, n\}$. Thus X is OH. ☐

Proposition 3.37. *A Banach space X is WOH if and only if for every separable subspace Y of X, there is a separable WOH subspace Z of X such that $Y \subset Z \subset X$.*

Proof. Necessity. Assume that X is WOH. Let Y be a separable subspace of X, let $\{u_m\colon\ m \in \mathbb{N}\}$ be a dense subset in Y, and, for every $m \in \mathbb{N}$, let $\varepsilon_m > 0$ be such that $\varepsilon_m \to 0$ as $m \to \infty$. Put $Y_1 = \operatorname{span}\{u_1\}$. For $m \in \mathbb{N}$, choose Y_{m+1} as follows: letting A_m be a finite ε_m-net in B_{X^*} for $B_{Y_m^*}$, for every $g \in A_m$, choose a $y_g \in S_X$ so that

$$\|x + t y_g\| \geq (1 - \varepsilon_m)\big(|g(x)| + t\big) \quad \text{for all } x \in S_{Y_m} \text{ and } t > 0,$$

and put $Y_{m+1} = \operatorname{span}\big(Y_m \cup \{y_g\colon\ g \in A_m\} \cup \{u_{m+1}\}\big)$.

Take $Z = \overline{\bigcup_{m=1}^{\infty} Y_m}$. To see that Z is WOH, let $n \in \mathbb{N}$, $z_1, \dots, z_n \in S_Z$, let $z^* \in B_{Z^*}$, and let ε be such that $\varepsilon \in (0,1)$. It suffices to show that there is a $y \in S_Z$ such that

$$\|z_i + ty\| \geq (1-\varepsilon)\big(|z^*(z_i)| + t\big) \quad \text{for all } i \in \{1,\dots,n\} \text{ and } t \geq \varepsilon,$$

Pick $m \in \mathbb{N}$ so that $\varepsilon_m \leq \varepsilon^2/4$ and, for every $i \in \{1,\dots,n\}$, there is an $x_i \in S_{Y_m}$ satisfying $\|z_i - x_i\| \leq \varepsilon^2/4$. Let $g \in A_m$ be such that $\|z^*|_{Y_m} - g\| < \varepsilon_m < \varepsilon^2/4$. One has, for every $i \in \{1,\dots,n\}$ and $t \geq \varepsilon$,

$$\begin{aligned}
\|z_i + ty_g\| &\geq \|x_i + ty_g\| - \|z_i - x_i\| \\
&\geq (1-\varepsilon_m)\big(|g(x_i)| + t\big) - \frac{\varepsilon^2}{4} \\
&\geq (1-\varepsilon_m)\big(|z^*(z_i)| - |z^*(z_i - x_i)| - |z^*(x_i) - g(x_i)| + t\big) - \frac{\varepsilon^2}{4} \\
&\geq (1-\varepsilon_m)\big(|z^*(z_i)| + t\big) - \frac{3\varepsilon^2}{4} \\
&\geq \left(1 - \frac{\varepsilon}{4}\right)\big(|z^*(z_i)| + t\big) - \frac{3\varepsilon}{4}t \\
&\geq (1-\varepsilon)\big(|z^*(z_i)| + t\big).
\end{aligned}$$

Sufficiency. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, let $x^* \in B_{X^*}$, and let $\varepsilon > 0$. Take $Y = \operatorname{span}\{x_1, \dots, x_n\}$. By our assumption, there is a separable subspace Z of X such that $Y \subset Z \subset X$ and Z is WOH. Therefore there is a $z \in S_Z \subset S_X$ such that

$$\|x_i + tz\| \geq (1-\varepsilon)(|x^*(x_i)| + t)$$

for all $i \in \{1,\dots,n\}$ and $t > 0$. Thus X is WOH. □

Proposition 3.38. *A Banach space X is LOH if and only if for every separable subspace Y of X, there is a separable LOH subspace Z of X such that $Y \subset Z \subset X$.*

Proof. The proof is similar to the proof of Proposition 3.36. □

3.4 Octahedral norms in spaces of operators

In this section, we study necessary and sufficient conditions for the space of bounded linear operators to be LOH, WOH or OH.

We start with the sufficient conditions.

Theorem 3.39 (see [BGLPRZ4, Theorem 3.5]). *Let X and Y be Banach spaces, and let H be a closed subspace of $\mathcal{L}(X, Y)$ containing the finite rank operators.*

(a) *If X^* and Y are OH, then H is OH.*

(b) *If X^* is LOH, then H is LOH.*

Proof. Our proof differs from the one in [BGLPRZ4] by not using the duality between diameter 2 properties and octahedrality.
(a) Assume that X^* and Y are OH. Let $n \in \mathbb{N}$, $S_1, \dots, S_n \in S_H$, and let $\varepsilon > 0$. Choose $x_i \in S_X$ to satisfy

$$\|S_i x_i\| \geq 1 - \varepsilon.$$

Since Y is OH, there is a $y \in S_Y$ such that, for every $i \in \{1, \dots, n\}$,

$$\|S_i x_i + y\| \geq (1-\varepsilon)\big(\|S_i x_i\| + \|y\|\big) \geq (1-\varepsilon)(2-\varepsilon).$$

Choose $y_i^* \in S_{Y^*}$ to satisfy

$$y_i^*(S_i x_i + y) = \|S_i x_i + y\|.$$

Since X^* is OH, by Theorem 3.21, (iii), there are $x^* \in S_{X^*}$ and $x_1^{**}, \dots, x_n^{**} \in X^{**}$ such that, for every $i \in \{1, \dots, n\}$,

$$x_i^{**}(S_i^* y_i^*) = (S_i^* y_i^*)(x_i), \quad x_i^{**}(x^*) = 1, \quad \text{and} \quad \|x_i^{**}\| \leq 1 + \varepsilon.$$

Now, for $T = x^* \otimes y$, one has $T \in S_H$ and, for every $i \in \{1, \dots, n\}$,

$$\begin{aligned}(1+\varepsilon)\|S_i + T\| &= (1+\varepsilon)\|S_i^{**} + T^{**}\| \geq \|(S_i^{**} + T^{**})(x_i^{**})\| \\ &\geq \|S_i^{**} x_i^{**} + x_i^{**}(x^*)\, y\| = \|S_i^{**} x_i^{**} + y\| \\ &\geq (S_i^{**} x_i^{**} + y)(y_i^*) = y_i^*(S_i x_i + y) \\ &\geq (1-\varepsilon)(2-\varepsilon).\end{aligned}$$

By Proposition 3.3, H is OH.

(b) Assume that X^* is LOH. Let $S \in S_H$ and let $\varepsilon > 0$. Choose $x \in S_X$ and $y^* \in S_{Y^*}$ to satisfy

$$y^*(Sx) \geq 1 - \varepsilon.$$

Since X^* is LOH, by Theorem 3.25, (iii), there are $x^* \in S_{X^*}$ and $x_1^{**}, x_2^{**} \in X^{**}$ such that, for $i \in \{1, 2\}$,

$$x_i^{**}(S^* y^*) = \|S^* y^*\|, \quad x_i^{**}(x^*) = (-1)^i, \quad \text{and} \quad \|x_i^{**}\| \leq 1 + \varepsilon.$$

Now, for $T = x^* \otimes Sx$, one has $T \in B_H$ and, for $i \in \{1, 2\}$,

$$\begin{aligned}(1+\varepsilon)\|S + (-1)^i T\| &= (1+\varepsilon)\|S^{**} + (-1)^i T^{**}\| \geq \|(S^{**} + (-1)^i T^{**})(x_i^{**})\| \\ &\geq \|S^{**}x_i^{**} + (-1)^i x_i^{**}(x^*)\, Sx\| = \|T^{**}x_i^{**} + Sx\| \\ &\geq (S^{**}x_i^{**} + Sx)(y^*) = y^*(Sx + Sx) \\ &\geq 2 - 2\varepsilon.\end{aligned}$$

By Proposition 3.12, H is LOH. □

Remark 3.4. A similar result to Theorem 3.39 for WOH Banach spaces seems to be unknown.

As an application of Theorem 3.39, one obtains stability results of diameter 2 properties for projective tensor products.

Corollary 3.40 (see [BGLPRZ4, Corollary 3.6]). *Let X and Y be Banach spaces.*

(a) *If X and Y have the SD2P, then $X\hat{\otimes}_\pi Y$ has the SD2P.*

(b) *If X has the LD2P, then $X\hat{\otimes}_\pi Y$ has the LD2P.*

Proof. The dual $(X\hat{\otimes}_\pi Y)^*$ is isometric to $\mathcal{L}(X, Y^*)$ (see [Rya, Theorem 2.9]). Thus, the assertions (a) and (b) follow, by combining Theorem 3.22, respectively Theorem 3.27, with Theorem 3.39. □

It seems to be unknown whether the projective tensor product of two Banach spaces always has the SD2P when one assumes that only one of the spaces has the SD2P. In [BGLPRZ4], it is proposed that to answer this question negatively a possible candidate could be $X\hat{\otimes}_\pi \ell_2^2$, for some Banach space X with the SD2P. We show in the forthcoming Proposition 3.42 that $c_0\hat{\otimes}_\pi \ell_2^2$ has the SD2P, that is, $\mathcal{L}(c_0, \ell_2^2)$ is OH. Thus $\mathcal{L}(X, Y)$ might be OH, when X^* is OH and Y is not OH.

Theorem 3.41 (see [BGLPRZ4, Theorem 3.1 and 3.2]). *Let X and Y be Banach spaces, and let H be a closed subspace of $\mathcal{L}(X, Y)$ containing the finite rank operators.*

(a) *If X^* is OH and there is a u in S_Y such that $\{y^* \in B_{Y^*} : y^*(u) = 1\}$ is norming for Y, then H is OH.*

(b) *If Y is OH and there is a u^* in S_{X^*} such that $\{x^{**} \in B_{X^{**}} : x^{**}(u^*) = 1\}$ is norming for X^*, then H is OH.*

Remark 3.5. A similar result to Theorem 3.41 for WOH Banach spaces seems to be unknown.

As a consequence of Theorem 3.41, the Banach spaces $\mathcal{L}(c_0, \ell_1^n)$, $\mathcal{L}(c_0, \ell_1)$, $\mathcal{L}(c_0, \ell_\infty^n)$, and $\mathcal{L}(c_0, \ell_\infty)$ are all OH. However, Theorem 3.41 can not be applied to $\mathcal{L}(c_0, \ell_2^2)$.

We remark that $\mathcal{L}(c_0, Y)$ is WOH for any Banach space Y. This can be proved similarly to [BGLPRZ4, Proposition 4.1], which says that $\mathcal{L}(c_0 \oplus_p c_0, Y)$ is WOH for every $p \geq 1$ and any Banach space Y.

Proposition 3.42. *Let p be such that $1 \leq p \leq \infty$. The Banach space $\mathcal{L}(c_0, \ell_p^2)$ is OH.*

The case when $p = 1$ or $p = \infty$ is contained in Theorem 3.41, but one can also show directly that $\mathcal{L}(c_0, \ell_1^2)$ and $\mathcal{L}(c_0, \ell_\infty^2)$ are OH. To prove the case, where p is such that $1 < p < \infty$, we use the following lemma.

Lemma 3.43. *Let $n \in \mathbb{N}$ and let p be such that $1 < p < \infty$. Let $a_1 = (\alpha_1, \beta_1), \ldots, a_n = (\alpha_n, \beta_n) \in S_{\ell_p^2}$ be such that $\alpha_i \geq 0$ for all $i \in \{1, \ldots, n\}$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. Then*

$$\theta_1 \cdot \frac{a_1 + a_n}{2} + \theta_2 \cdot \frac{a_2 - a_1}{2} + \cdots + \theta_n \cdot \frac{a_n - a_{n-1}}{2} \in B_{\ell_p^2}$$

for all $\theta_1, \ldots, \theta_n \in \{-1, 1\}$.

Proof. Assume that $\theta_1, \ldots, \theta_n \in \{-1, 1\}$. Denote by

$$x = \theta_1 \cdot \frac{a_1 + a_n}{2} + \theta_2 \cdot \frac{a_2 - a_1}{2} + \cdots + \theta_n \cdot \frac{a_n - a_{n-1}}{2}.$$

We will show that $x \in B_{\ell_p^2}$. Without loss of generality we may assume that $\theta_1 = 1$. Observe that

$$\frac{a_n}{2} = \frac{a_1}{2} + \frac{a_2 - a_1}{2} + \cdots + \frac{a_n - a_{n-1}}{2}.$$

Therefore

$$x = a_1 + \frac{a_2 - a_1}{2} + \cdots + \frac{a_n - a_{n-1}}{2} + \\ + \theta_2 \cdot \frac{a_2 - a_1}{2} + \cdots + \theta_n \cdot \frac{a_n - a_{n-1}}{2}.$$

Hence there is an odd number of increasing indices $k_1, \ldots, k_{2l+1}$ such that x is representable as

$$x = a_{k_1} - a_{k_2} + a_{k_3} - \cdots - a_{k_{2l}} + a_{k_{2l+1}}. \tag{3.8}$$

To show that $x \in B_{\ell_p^2}$, we use the following geometric properties of ℓ_p^2.

Fact. *For $a, b \in S_{\ell_p^2}$, denote by $\Theta_{a,b} = B_{\ell_p^2} \cap (B_{\ell_p^2} + (a+b))$.*

(a) *If $a, b \in S_{\ell_p^2}$ and $y \in \Theta_{a,b}$, then $a - y + b \in \Theta_{a,b}$.*

(b) *If a, b, and c are pairwise different elements of $S_{\ell_p^2}$ and $b \in \Theta_{a,c}$, then $\Theta_{a,b} \subset \Theta_{a,c}$.*

Since $a_{k_{l+1}} \in \Theta_{a_{k_l}, a_{k_{l+2}}}$, we have that $z = a_{k_l} - a_{k_{l+1}} + a_{k_{l+2}} \in \Theta_{a_{k_l}, a_{k_{l+2}}}$ by part (a) of the Fact above. We can write the middle part of the right hand side of (3.8) as

$$\dots a_{k_{l-1}} - (a_{k_l} - a_{k_{l+1}} + a_{k_{l+2}}) + a_{k_{l+3}} \dots$$

By part (b) of the Fact, $z \in \Theta_{a_{k_l}, a_{k_{l+2}}} \subset \Theta_{a_{k_{l-1}}, a_{k_{l+3}}}$. Applying part (a) of the Fact we have that $a_{k_{l-1}} - z + a_{k_{l+3}} \in \Theta_{a_{k_{l-1}}, a_{k_{l+3}}}$. Continuing in this way, we will finally have $x \in \Theta_{a_{k_1}, a_{k_{2l+1}}} \subset B_{\ell_p^2}$. □

Proof of Proposition 3.42. Let $n \in \mathbb{N}$, $S_1, \dots, S_n \in S_{\mathcal{L}(c_0, \ell_p^2)}$, and let ε be such that $\varepsilon \in (0,1)$. It suffices to show that there is a $T \in S_{\mathcal{L}(c_0, \ell_p^2)}$ such that $\|S_i + T\| \geq 2 - 3\varepsilon$ for all $i \in \{1, \dots, n\}$.

Choose $x_i \in S_{c_0}$ such that $\|S_i x_i\| \geq 1 - \varepsilon$. Without loss of generality we may assume that $x_1, \dots, x_n$ are finitely supported, that is, there is a $N_1 \in \mathbb{N}$ such that $x_1, \dots, x_n \in \operatorname{span}\{e_1, \dots, e_{N_1}\}$.

Since $S_1, \dots, S_n \in \mathcal{F}(c_0, \ell_p^2)$ and (e_k) is a weakly null sequence in c_0, there is a $N_2 \in \mathbb{N}$ such that $\|S_i e_k\| \leq \varepsilon / n$ for all $i \in \{1, \dots, n\}$ and $k \geq N_2$. Take $N = \max\{N_1, N_2\}$.

For all $i \in \{1, \dots, n\}$, denote by $a_i = S_i x_i / \|S_i x_i\|$. By reordering $a_1, \dots, a_n$ and by replacing a_i with $-a_i$ if necessary, we may assume that $a_1, \dots, a_n$ satisfy the assumptions of Lemma 3.43.

Define $T \colon c_0 \to \ell_p^2$ by

$$Te_{N+1} = \frac{a_1 + a_n}{2}, \quad Te_{N+2} = \frac{a_2 - a_1}{2}, \quad \dots, \quad Te_{N+n} = \frac{a_n - a_{n-1}}{2},$$

and $Te_k = 0$, if $k \in \mathbb{N} \setminus \{N+1, \dots, N+n\}$.

By Lemma 3.43, $\|T\| \leq 1$. On the other hand $\|T\| \geq 1$, because $T(e_{N+1} + \dots + e_{N+n}) = a_n$. Thus $\|T\| = 1$.

Fix $i \in \{1, \dots, n\}$. Find $\theta_1, \dots, \theta_n \in \{-1, 1\}$ such that

$$\theta_1 \cdot \frac{a_1 + a_n}{2} + \theta_2 \cdot \frac{a_2 - a_1}{2} + \dots + \theta_n \cdot \frac{a_n - a_{n-1}}{2} = a_i.$$

Denote by $y_i = \theta_1 e_{N+1} + \cdots + \theta_n e_{N+n}$. Thus $Ty_i = S_i x_i / \|S_i x_i\|$ and

$$\begin{aligned}\|S_i + T\| &\geq \|(S_i + T)(x_i + y_i)\| \\ &= \|S_i x_i + S_i y_i + T y_i\| \\ &\geq \|S_i x_i + T y_i\| - \|S_i y_i\| \\ &\geq 2\|S_i x_i\| - \varepsilon \geq 2 - 3\varepsilon.\end{aligned}$$

□

We conclude this section with the necessary conditions for the space of bounded linear operators to be LOH, WOH or OH. J. Becerra Guerrero, G. López Pérez, and A. Rueda Zoca showed in [BGLPRZ4] that if $\mathcal{L}(X, Y)$ is OH (resp. LOH) and X^* is nonrough, then Y is OH (resp. LOH). We will show that a similar statement is also true for WOH Banach spaces.

Theorem 3.44 (see [BGLPRZ4, Proposition 3.9, Corollary 3.10])**.** *Let X and Y be Banach spaces, and let H be a closed subspace of $\mathcal{L}(X, Y)$ containing the finite rank operators. Assume that H is OH.*

(a) *If X^* is nonrough, then Y is OH.*

(b) *If Y is nonrough, then X^* is OH.*

Proof. Our proof differs from the one in [BGLPRZ4] by not using the duality between diameter 2 properties and octahedrality.

(a). Assume that X^* is nonrough. We will prove that Y is OH. Let $n \in \mathbb{N}$, $y_1, \ldots, y_n \in S_Y$, and let $\varepsilon > 0$. It suffices to show that there is a $y \in B_Y$ such that $\|y_i + y\| \geq 2 - 2\varepsilon$ for all $i \in \{1, \ldots, n\}$.

Since X^* is nonrough, there is a slice $S(B_X, x^*, \alpha)$ with diameter less than ε. One may assume that $\alpha \leq \varepsilon$.

For every $i \in \{1, \ldots, n\}$, put $S_i = x^* \otimes y_i \in S_H$. Since H is OH, there is a $T \in S_H$ such that

$$\|S_i + T\| > 2 - \alpha \quad \text{for all } i \in \{1, \ldots, n\}.$$

Choose $x_i \in S_X$ and $y_i^* \in S_{Y^*}$ such that

$$y_i^*(S_i x_i + T x_i) > 2 - \alpha,$$

that is,

$$x^*(x_i)\, y_i^*(y_i) + y_i^*(T x_i) > 2 - \alpha.$$

One may assume that both $x^*(x_i) > 0$ and $y_i^*(y_i) > 0$, and thus $x_i \in S(B_X, x^*, \alpha)$. It follows that, for all $i \in \{1, \dots, n\}$,

$$\begin{aligned}\|y_i + Tx_1\| &\geq \|y_i + Tx_i\| - \|Tx_i - Tx_1\| \\ &\geq y_i^*(y_i + Tx_i) - \|x_i - x_1\| \\ &\geq x^*(x_i)y_i^*(y_i) + y^*(Tx_i) - \|x_i - x_1\| \\ &> 2 - \alpha - \varepsilon \geq 2 - 2\varepsilon.\end{aligned}$$

(b). The proof is similar to the proof of (a). □

Theorem 3.45. *Let X and Y be Banach spaces, and let H be a closed subspace of $\mathcal{L}(X, Y)$ containing the finite rank operators. Assume that H is LOH.*

(a) *If X^* is nonrough, then Y is LOH.*

(b) *If Y is nonrough, then X^* is LOH.*

Proof. (a). Take $n = 2$ and $y_2 = -y_1$ in the proof of part (a) of Theorem 3.44.

(b). The proof is similar to the proof of (a). □

Theorem 3.46 is a joint result with M. Põldvere.

Theorem 3.46. *Let X and Y be Banach spaces, and let H be a closed subspace of $\mathcal{L}(X, Y)$ containing the finite rank operators. Assume that H is WOH.*

(a) *If X^* is nonrough, then Y is WOH.*

(b) *If Y is nonrough, then X^* is WOH.*

Proof. (a). Assume that X^* is nonrough. We will show that Y is WOH. Let $n \in \mathbb{N}$, $y_1, \dots, y_n \in S_Y$, $y^* \in S_{Y^*}$, and let ε be such that $\varepsilon \in (0, 1)$. By Theorem 3.23, (v), it suffices to show that there are $y \in B_Y$ and $y_1^*, y_2^* \in B_{Y^*}$ such that

$$\left|y_1^*(y_i) - y^*(y_i)\right| < 6\varepsilon \quad \text{and} \quad \left|y_2^*(y_i) - y^*(y_i)\right| < 6\varepsilon \quad \text{for all } i \in \{1, \dots, n\},$$

and

$$y_1^*(y) - y_2^*(y) > 2 - 13\varepsilon.$$

Since X^* is nonrough, there is a slice $S(B_X, x^*, \alpha)$ with diameter less than ε. One may assume that $\alpha \leq \varepsilon$.

Choose $x^{**} \in S_{X^{**}}$ and $y_0 \in S_Y$ so that $x^{**}(x^*) = 1$ and $y^*(y_0) > 1 - \alpha^2$. For all $i \in \{0, 1, \ldots, n\}$, put $S_i = x^* \otimes y_i \in S_H$ and $\phi = x^{**} \otimes y^* \in H^*$. Since H is WOH, by Theorem 3.23, (iv), there are $T \in S_H$ and $\phi_1, \phi_2 \in H^*$, $\|\phi_1\|, \|\phi_2\| < 1 + \alpha^2$, such that

$$\phi_1(S_i) = \phi_2(S_i) = \phi(S_i) = y^*(y_i) \quad \text{for all } i \in \{0, 1, \ldots, n\},$$

and

$$\phi_1(T) - \phi_2(T) > 2 - \varepsilon.$$

Observe that $\left\| \frac{1}{\|\phi_j\|} \phi_j - \phi_j \right\| < \alpha^2$ for all $j \in \{1, 2\}$. Therefore, since $\mathrm{conv}(B_X \otimes B_{Y^*})$ is weak* dense in B_{H^*}, there are

$$\psi_j = \sum_{k=1}^{m_j} \lambda_{j,k} x_{j,k} \otimes y^*_{j,k} \in \mathrm{conv}(B_X \otimes B_{Y^*})$$

such that $\left|\psi_j(T) - \phi_j(T)\right| < \alpha^2$ and, for all $i \in \{0, 1, \ldots, n\}$,

$$\left|\psi_j(S_i) - \phi_j(S_i)\right| < \alpha^2,$$

that is,

$$\left|\sum_{k=1}^{m_j} \lambda_{j,k} x^*(x_{j,k}) y^*_{j,k}(y_i) - y^*(y_i)\right| < \alpha^2.$$

For all $j \in \{1, 2\}$, denote by,

$$M_j = \left\{k \in \{1, \ldots, m_j\} \colon\ x^*(x_{j,k}) > 1 - \alpha\right\} \quad \text{and} \quad \lambda_j = \sum_{k \notin M_j} \lambda_{j,k}.$$

One has $\lambda_j < 2\alpha$, because

$$\begin{aligned}
1 - 2\alpha^2 < y^*(y_0) - \alpha^2 &< \sum_{k=1}^{m_j} \lambda_{j,k} x^*(x_{j,k}) y^*_{j,k}(y_0) \\
&= \sum_{k \in M_j} \lambda_{j,k} x^*(x_{j,k}) y^*_{j,k}(y_0) + \sum_{k \notin M_j} \lambda_{j,k} x^*(x_{j,k}) y^*_{j,k}(y_0) \\
&\leq \sum_{k \in M_j} \lambda_{j,k} + \sum_{k \notin M_j} \lambda_{j,k} x^*(x_{j,k}) \\
&\leq 1 - \lambda_j + (1 - \alpha)\, \lambda_j \\
&= 1 - \alpha\, \lambda_j.
\end{aligned}$$

For all $j \in \{1,2\}$, denote by $y_j^* = \sum_{k=1}^{m_j} \lambda_{j,k} y_{j,k}^*$. We have that

$$\begin{aligned}
\left|y_j^*(y_i) - y^*(y_i)\right| &= \left|\sum_{k=1}^{m_j} \lambda_{j,k} y_{j,k}^*(y_i) - y^*(y_i)\right| \\
&\leq \left|\sum_{k=1}^{m_j} \lambda_{j,k} x^*(x_{j,k}) y_{j,k}^*(y_i) - y^*(y_i)\right| \\
&\quad + \sum_{k\in M_j} \lambda_{j,k} \left|1 - x^*(x_{j,k})\right| \left|y_{j,k}^*(y_i)\right| \\
&\quad + \sum_{k\notin M_j} \lambda_{j,k} \left|1 - x^*(x_{j,k})\right| \left|y_{j,k}^*(y_i)\right| \\
&< \alpha^2 + \alpha + 2\lambda_j < 6\alpha \leq 6\varepsilon.
\end{aligned}$$

Fix any $x \in S(B_X, x^*, \alpha)$. For all $j \in \{1,2\}$, we have that

$$\|x - x_{j,k}\| < \varepsilon \quad \text{for all } k \in M_j,$$

thus

$$\begin{aligned}
\left|y_j^*(Tx) - \psi_j(T)\right| &= \left|\sum_{k=1}^{m_j} \lambda_{j,k} (T^* y_{j,k}^*)(x - x_{j,k})\right| \\
&\leq \sum_{k=1}^{m_j} \lambda_{j,k} \|x - x_{j,k}\| \\
&= \sum_{k\in M_j} \lambda_{j,k} \|x - x_{j,k}\| + \sum_{k\notin M_j} \lambda_{j,k} \|x - x_{j,k}\| \\
&< \varepsilon + 2\lambda_j < \varepsilon + 4\alpha \leq 5\varepsilon.
\end{aligned}$$

It follows that

$$\begin{aligned}
y_1^*(Tx) - y_2^*(Tx) &= y_1^*(Tx) - \psi_1(T) + \psi_1(T) - \phi_1(T) + \phi_1(T) - \phi_2(T) \\
&\quad + \phi_2(T) - \psi_2(T) + \psi_2(T) - y_2^*(Tx) \\
&\geq \phi_1(T) - \phi_2(T) - \left|y_1^*(Tx) - \psi_1(T)\right| - \left|\psi_1(T) - \phi_1(T)\right| \\
&\quad - \left|\phi_2(T) - \psi_2(T)\right| - \left|\psi_2(T) - y_2^*(Tx)\right| \\
&> 2 - \varepsilon - 5\varepsilon - \alpha^2 - \alpha^2 - 5\varepsilon > 2 - 13\varepsilon.
\end{aligned}$$

(b). The proof is similar to the proof of (a). □

Chapter 4

Almost square Banach spaces

In this chapter, we introduce and study almost square Banach spaces. These spaces have the SD2P and their duals are OH. We provide several examples and characterizations of almost square spaces. We prove that nonreflexive Banach spaces which are M-ideals in their biduals are almost square. We show that every Banach space containing a complemented copy of c_0 can be renormed to be almost square. A local and a weak version of almost square spaces are also studied. This chapter is based on [ALL].

4.1 Definitions and basic results

Definition 4.1 (see [ALL]). Let X be a Banach space. We say that X is

(i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence (y_k) in B_X such that $\|x \pm y_k\| \to 1$ and $\|y_k\| \to 1$ as $k \to \infty$;

(ii) *weakly almost square* (WASQ) if for every $x \in S_X$ there exists a sequence (y_k) in B_X such that $\|x \pm y_k\| \to 1$, $\|y_k\| \to 1$ and $y_k \to 0$ weakly as $k \to \infty$;

(iii) *almost square* (ASQ) if for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$ there exists a sequence (y_k) in B_X such that $\|x_i \pm y_k\| \to 1$ for every $i \in \{1, \ldots, n\}$ and $\|y_k\| \to 1$ as $k \to \infty$.

Remark 4.1. In the definitions above, one may choose the sequence (y_k) from S_X.

For a Banach space the following implications hold:

$$\text{ASQ} \Rightarrow \text{WASQ} \Rightarrow \text{LASQ}.$$

The first implication will be shown in Theorem 4.14. The second implication is clear.

The prototype of an ASQ space is c_0.

Example 4.2 (see [ALL, Example 2.1])**.** The Banach space c_0 is ASQ. Indeed, let $n \in \mathbb{N}$, $x_1, \dots, x_n$ in S_{c_0}, and let e_k be the k'th standard basis vector in c_0. Then it is clear that $\|x_i \pm e_k\| \to 1$ for all $i \in \{1, \dots, n\}$ as $k \to \infty$. Thus c_0 is ASQ.

Similarly one can show that $c_0(X)$ is ASQ for any Banach space X. More generally, given a sequence of nontrivial Banach spaces (X_k), the c_0-sum $(X_1 \oplus X_2 \oplus \dots)_{c_0}$ is ASQ.

On the other hand, the Banach spaces ℓ_∞, $C[0,1]$, and $L_\infty[0,1]$ are not LASQ, thus also not WASQ nor ASQ. Recall that $C[0,1]$ and $L_\infty[0,1]$ have the DP. With the help of the previous example, we can construct a Banach space which is ASQ and has the DP, thus it is also OH by Theorem 3.6.

Example 4.3. The Banach space $c_0(L_1[0,1])$ is ASQ and has the DP. Indeed, by Example 4.2, $c_0(L_1[0,1])$ is ASQ and, by [BKSW, Theorem 5.1], $c_0(L_1[0,1])$ has the DP.

J. Gao and K.-S. Lau have shown in [GL] that $L_1[0,1]$ is LASQ. We will show that $L_1[0,1]$ is WASQ.

Example 4.4 (see [ALL, Example 2.4])**.** The Banach space $L_1[0,1]$ is WASQ. Let $f \in S_{L_1}$ and define $f_k(t) = f(t) \cdot r_k(t)$ for all $t \in [0,1]$, where (r_k) are the Rademacher functions. The sequence (f_k) in S_{L_1} is weakly null (see, e.g., [AK, Lemma 6.3.2]) and we have that

$$\|f \pm f_k\| = \int_0^1 |f(t)|(1 \pm r_k(t))dt = \int_0^1 |f(t)|dt \pm \int_0^1 |f(t)|r_k(t)dt \to 1$$

as $k \to \infty$.

The Banach space $L_1[0,1]$ is a Cesàro function space. In [ALL], it is shown that all Cesàro function spaces are WASQ and fail to be ASQ. Thus ASQ is strictly stronger than WASQ.

Question 4.5 (see [ALL, Question 3.12])**.** *Is WASQ strictly stronger than LASQ?*

The following two examples are from the class of Lindenstrauss spaces. We remark that Lindenstrauss spaces whose unit ball contains an extreme point are not LASQ (see [Lin, Theorem 6.1, (14)]). Although $C[0,1]$ is not LASQ, these two examples are codimension one subspaces of $C[0,1]$, which are ASQ and WASQ, respectively.

Example 4.6. Let $X = \{f \in C[0,1] \colon f(0) = 0\}$. By [Lac, p. 140], X is a Lindenstrauss space. We will show that X is ASQ and OH. Let $n \in \mathbb{N}$ and $f_1, \dots, f_n \in S_X$.

First, let us show that X is ASQ. For all $k \in \mathbb{N}$, find $s_k \in (0,1)$ such that $|f_i(t)| < 1/k$ for all $t \in (0, s_k)$ and $i \in \{1, \dots, n\}$. Choose g_k in S_X such that $g_k(t) = 0$ if $t \notin (0, s_k)$. Then $1 - 1/k \leq \|f_i \pm g_k\| \leq 1 + 1/k$ for all $i \in \{1, \dots, n\}$. Hence $\|f_i \pm g_k\| \to 1$ for all $i \in \{1, \dots, n\}$ as $k \to \infty$ and X is ASQ.

Now we will show that X is OH. Let $\varepsilon > 0$. Choose pairwise different $t_1, \dots, t_n \in [0,1]$ such that $|f_i(t_i)| > 1 - \varepsilon/2$. Let $h \in S_X$ be such that $h(t_i) = f_i(t_i)$ for all $i \in \{1, \dots, n\}$. Then $\|f_i + h\| \geq 2|f_i(t_i)| > 2 - \varepsilon$ for every $i \in \{1, \dots, n\}$ and thus X is OH.

Example 4.7. Let $X = \{f \in C[0,1] \colon f(0) = -f(1)\}$. By [Lac, p. 140], X is a Lindenstrauss space. We will show that X is WASQ but not ASQ.

Let us show that X is WASQ. Let $f \in S_X$. Since f has a zero in $[0,1]$ we can always find a point $s \in (0,1]$ such that $f(s) = 0$. For all $k \in \mathbb{N}$, let $s_k \in (0, s)$ be such that $s_k \to s$ as $k \to \infty$ and $|f(t)| < 1/k$ if $t \in (s_k, s)$. Choose g_k in S_X such that $g_k(t) = 0$ if $t \notin (s_k, s)$. Then $1 - 1/k \leq \|f \pm g_k\| \leq 1 + 1/k$. Hence $\|f \pm g_k\| \to 1$ as $k \to \infty$ and X is LASQ. Finally, observe that g_k converges pointwise to 0 which in turn implies that g_k converges weakly to 0 (see, e.g., [Die1, p. 66]). Thus X is WASQ.

To see that X is not ASQ, let f_1 be any function in S_X which is equal to 1 on $[0, \frac{1}{2}]$ and let $f_2(t) = f_1(1-t)$. Then $\max_i \|f_i \pm g\| = 2$ for any $g \in S_X$.

We now point out some equivalent, but sometimes more convenient formulations of LASQ and ASQ spaces.

Proposition 4.8 (see [ALL, Proposition 3.3])**.** *Let X be a Banach space.*

(a) *X is LASQ if and only if whenever $x \in S_X$ and $\varepsilon > 0$ there is a $y \in S_X$ such that*
$$\|x \pm y\| \leq 1 + \varepsilon.$$

(b) *X is ASQ if and only if whenever $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, and $\varepsilon > 0$, there is a $y \in S_X$ such that*
$$\|x_i + y\| \leq 1 + \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

To prove Proposition 4.8 we use the following elementary lemma, but we include its proof for completeness.

Lemma 4.9. *If $x, y \in S_X$ and $\varepsilon > 0$ are such that $\|x \pm y\| \leq 1 + \varepsilon$, then*

$$(1-\varepsilon)\max(|\alpha|, |\beta|) \leq \|\alpha x + \beta y\| \leq (1+\varepsilon)\max(|\alpha|, |\beta|)$$

for all scalars α and β.

Proof. Let $x, y \in S_X$ and let $\varepsilon > 0$. Assume that $\|x \pm y\| \leq 1 + \varepsilon$. Observe first that $\|x \pm y\| \geq 1 - \varepsilon$. Indeed, suppose on the contrary that $\|x - y\| < 1 - \varepsilon$. Then

$$1 = \|x\| \leq \frac{1}{2}\|x + y\| + \frac{1}{2}\|x - y\| < \frac{1}{2}(1 + \varepsilon + 1 - \varepsilon) = 1,$$

which is a contradiction. Therefore $\|x - y\| \geq 1 - \varepsilon$. Similarly, one can show that $\|x + y\| \geq 1 - \varepsilon$.

Fix $\lambda \in \mathbb{R}$ such that $\lambda \in (0, 1]$. To prove Lemma 4.9, it suffices to see that

$$(1-\varepsilon) \leq \|\lambda x + y\| \leq (1+\varepsilon).$$

Since $\|x - y\| \leq 1 + \varepsilon$, we have

$$\|\lambda^{-1}y + x\| = \|(1 + \lambda^{-1})y - (y - x)\| \geq (1 + \lambda^{-1}) - \|x - y\| \geq \lambda^{-1} - \varepsilon.$$

Hence $\|\lambda x + y\| \geq 1 - \varepsilon\lambda \geq 1 - \varepsilon$.

Also

$$\|\lambda^{-1}y + x\| = \|(\lambda^{-1} - 1)y + (y + x)\| \leq (\lambda^{-1} - 1) + 1 + \varepsilon = \lambda^{-1} + \varepsilon,$$

hence $\|\lambda x + y\| \leq 1 + \varepsilon\lambda \leq 1 + \varepsilon$. □

Proof of Proposition 4.8. (a). Assume that X is LASQ. Let $x \in S_X$ and let $\varepsilon > 0$. By the definition there is a sequence (y_k) in S_X such that $\|x \pm y_k\| \to 1$ as $k \to \infty$. Find $k \in \mathbb{N}$ such that $\|x \pm y_k\| \leq 1 + \varepsilon$.

For the converse, let $x \in S_X$. By the assumption, we can find a sequence (y_k) in S_X such that $\|x \pm y_k\| \leq 1 + 1/k$. By Lemma 4.9, we also have that $1 - 1/k \leq \|x \pm y_k\|$. Thus $\|x \pm y_k\| \to 1$ as $k \to \infty$, hence X is LASQ.

(b). The proof is similar to the proof of (a). □

Corollary 4.10 (see [ALL, Corollary 3.5])**.** *If X is LASQ, then X contains an almost isometric copy of ℓ_∞^2.*

Proof. Assume that X is LASQ. Let $x \in S_X$ and let $\varepsilon > 0$. By the assumption, we can find an element $y \in S_X$ such that $1 - \varepsilon \leq \|x \pm y\| \leq 1 + \varepsilon$.

Let $E = \operatorname{span}\{x, y\}$. Define $T \colon \ell_\infty^2 \to E$ by $T(1, 0) = x$ and $T(0, 1) = y$. By Lemma 4.9, T is an ε-isometry. □

We will now show that ASQ spaces contain an almost isometric copy of c_0.

Theorem 4.11 (see [ALL, Theorem 3.6]). *Let X be a Banach space. If X is ASQ, then for every finite-dimensional subspace E of X and $\varepsilon > 0$ there exists a $y \in S_X$ such that*

$$(1 - \varepsilon) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E$.

Moreover, given a finite-dimensional subspace F of X^* we may choose the above y so that $|f(y)| \leq \varepsilon\|f\|$ for every $f \in F$.

It is clear from Proposition 4.8, that the above theorem is actually a characterization of ASQ.

Proof. Assume that X is ASQ. Let E be a finite-dimensional subspace of X and let $\varepsilon > 0$. Find an $\varepsilon/2$-net $\{x_1, \ldots, x_n\}$ in S_X for S_E. Choose a $y \in S_X$ such that $1 - \varepsilon/2 < \|x_i \pm y\| < 1 + \varepsilon/2$ for all $i \in \{1, \ldots, n\}$.

Let $x \in S_E$. Find i such that $\|x_i - x\| < \varepsilon/2$. Then

$$\|x \pm y\| \leq \|x_i \pm y\| + \|x - x_i\| < 1 + \varepsilon.$$

Hence, by using Lemma 4.9, we get

$$(1 - \varepsilon) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E$.

For the moreover part, let F be a finite dimensional subspace of X^* and let $\{f_1, \ldots, f_m\}$ be an $\varepsilon/2$-net in S_{X^*} for S_F. For each i choose $z_i \in S_X$ with $f_i(z_i) > 1 - \varepsilon/4$. Let $E' = \operatorname{span}(E \cup \{z_1, \ldots, z_m\})$ and use the first part of the proof to find a $y \in S_X$ such that

$$(1 - \varepsilon/4) \max(\|x\|, |\lambda|) \leq \|x + \lambda y\| \leq (1 + \varepsilon/4) \max(\|x\|, |\lambda|)$$

for all scalars λ and all $x \in E'$.

Since $|f_i(z_i \pm y)| \leq \|z_i \pm y\| \leq 1 + \varepsilon/4$, we get

$$-\varepsilon/2 = 1 - \varepsilon/4 - (1 + \varepsilon/4) \leq f_i(z_i) - f_i(z_i - y) = f_i(y)$$
$$= f_i(z_i + y) - f_i(z_i) \leq 1 + \varepsilon/4 - 1 + \varepsilon/4 = \varepsilon/2.$$

Therefore $|f_i(y)| \leq \varepsilon/2$. Thus, for every $f \in S_F$ and for some i, we have $|f(y)| \leq |(f - f_i)(y)| + |f_i(y)| \leq \varepsilon$. □

Repeated use of Theorem 4.11 gives the following lemma.

Lemma 4.12 (see [ALL, Lemma 3.9])**.** *If X is ASQ, then for every finite-dimensional subspace E of X and every $\varepsilon > 0$ there exists a subspace Y of X which is ε-isometric to c_0 such that $E \oplus Y$ is ε-isometric to $E \oplus_\infty c_0$.*

Proof. Assume that X is ASQ. Let E be a finite-dimensional subspace of X and let $\varepsilon > 0$. Find a sequence (ε_k) in $\mathbb{R}^+$ such that $\prod_{k=1}^{\infty}(1 + \varepsilon_k) < 1 + \varepsilon$ and $\prod_{k=1}^{\infty}(1 - \varepsilon_k) > 1 - \varepsilon$. Using Theorem 4.11, we inductively choose a sequence (y_k) in S_X such that

$$(1 - \varepsilon_k)\max\{\|e\|, |\lambda|\} \leq \|e + \lambda y_k\| \leq (1 + \varepsilon_k)\max\{\|e\|, |\lambda|\}$$

for every $e \in \operatorname{span}(E \cup \{y_1, \ldots, y_{k-1}\})$ and every $\lambda \in \mathbb{R}$. Denote by $Y = \overline{\operatorname{span}}\{y_1, y_2, \ldots\}$. Note that $E \cap Y = \{0\}$ and $T\colon c_0 \to Y$, $Te_k = y_k$, is an ε-isometry. Define $S : E \oplus_\infty c_0 \to E \oplus Y$ by $S(e, a) = e + Ta$.

We have

$$\begin{aligned}\|S(e, \sum_{k=1}^{n} \alpha_k e_k)\| &= \|e + \sum_{k=1}^{n} \alpha_k y_k\| \leq (1 + \varepsilon_n)\max\{\|e + \sum_{k=1}^{n-1} \alpha_k y_k\|, |a_n|\}\\ &\leq \cdots \leq \prod_{k=1}^{n}(1 + \varepsilon_k)\max\{\|e\|, |\alpha_1|, \ldots, |\alpha_n|\}\\ &< (1 + \varepsilon)\|(e, \sum_{k=1}^{n} \alpha_k e_k)\|,\end{aligned}$$

and similarly, $\|S(e, \sum_{k=1}^{n} \alpha_k e_k)\| > (1 - \varepsilon)\|(e, \sum_{k=1}^{n} \alpha_k e_k)\|$. Thus S is an ε-isometry onto $E \oplus Y$. □

Corollary 4.13. *If X is ASQ, then X contains an almost isometric copy of c_0.*

A consequence of Lemma 4.12 is that the sequence (y_k) in the definition of ASQ may be chosen to be weakly null. This enables us to connect the properties ASQ and WASQ.

Theorem 4.14 (see [ALL, Theorem 3.10]). *Let X be a Banach space. If X is ASQ, then for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$ there exists a sequence (y_k) in B_X such that*

$$\|x_i \pm y_k\| \to 1 \quad \text{for all } i \in \{1, \dots, n\},$$

$y_k \to 0$ weakly, and $\|y_k\| \to 1$ as $k \to \infty$.

In particular, ASQ implies WASQ.

Proof. Assume that X is ASQ. Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in S_X$. Put $E = \operatorname{span}\{x_1, \dots, x_n\}$ and choose sequences (ε_k) in $\mathbb{R}^+$ and (y_k) in S_X as in the proof of Lemma 4.12. Denote by $Y = \overline{\operatorname{span}}\{y_1, y_2, \dots\}$. Since the standard basis (e_k) in c_0 is weakly null, so is the sequence $((0, e_k))$ weakly null in $E \oplus_\infty c_0$. Let $S\colon E \oplus_\infty c_0 \to E \oplus Y$ be the ε-isometry from the proof of Lemma 4.12. The weak-weak continuity of S shows that $y_k \to 0$ weakly in $E \oplus Y$ as $k \to \infty$, and hence also in X.

By the definition, $S(e, \pm e_k) = e \pm y_k$ for every $e \in E$. Since

$$(1 - \varepsilon_k) \max\{\|e\|, 1\} \leq \|e \pm y_k\| \leq (1 + \varepsilon_k) \max\{\|e\|, 1\}$$

for every $e \in E$, we in particular have $(1 - \varepsilon_k) \leq \|x_i \pm y_k\| \leq (1 + \varepsilon_k)$, so $\|x_i \pm y_k\| \to 1$ for all $i \in \{1, \dots, n\}$. □

Corollary 4.15 (see [ALL, Corollary 3.11]). *ASQ is strictly stronger than WASQ.*

Proof. From Theorem 4.14 we have that all ASQ spaces are WASQ. By Example 4.4, $L_1[0, 1]$ is WASQ, but $L_1[0, 1]$ does not contain c_0 (see [AK, Corollary 5.2.11]) so it is not ASQ. □

Our main interest in studying the properties LASQ, WASQ, and ASQ come from their relation to diameter 2 properties. D. Kubiak observed that if a Banach space is LASQ, then it has the LD2P, and similarly, if it is WASQ, then it has the D2P (see [Kub, Propositions 2.5 and 2.6]). The same idea from his proof works also for ASQ, but we will prove it using the dual characterization of the SD2P.

Proposition 4.16 (cf. [ALL, Proposition 1.3]). *Let X be a Banach space. If X is ASQ, then X^* is OH.*

Proof. Assume that X is ASQ. Let $n \in \mathbb{N}$, $x_1^*, \dots, x_n^* \in S_{X^*}$, and let $\varepsilon > 0$. Find $x_1, \dots, x_n \in S_X$ such that $x_i^*(x_i) > 1 - \varepsilon$. Using Theorem 4.11, find a $y \in S_X$ such that $\|x_i \pm y\| \leq 1 + \varepsilon$ and $|x_i^*(y)| < \varepsilon$ for all $i \in \{1, \dots, n\}$.

Find $y^* \in S_{X^*}$ such that $y^*(y) = 1$. Then

$$1 + \varepsilon \geq \|x_i \pm y\| \geq \pm y^*(x_i) + y^*(y) = \pm y^*(x_i) + 1$$

and thus $|y^*(x_i)| \leq \varepsilon$. Now

$$\begin{aligned} \|x_i + y\| \|x_i^* + y^*\| &\geq x_i^*(x_i) + x_i^*(y) + y^*(x_i) + y^*(y) \\ &> 1 - \varepsilon - 2\varepsilon + 1 \end{aligned}$$

and hence

$$\|x_i^* + y^*\| > \frac{2 - 3\varepsilon}{1 + \varepsilon},$$

which shows that X^* is OH by Proposition 3.3. □

Corollary 4.17. *If X is ASQ, then X has the SD2P.*

Proof. If X is ASQ, then, by Proposition 4.16, X^* is OH. Thus, X has the SD2P by Theorem 3.22. □

We know that every ASQ Banach space contains a copy of c_0. Recall that, by [ALN1, Proposition 4.7], any Banach space containing c_0 can be equivalently renormed to have the SD2P. In [ALL, Theorem 3.14], it is claimed that any Banach space containing c_0 can be equivalently renormed to be ASQ. Unfortunately, there is a gap in that proof and we do not know whether the claim holds. However, every Banach space which contains a complemented copy of c_0 can be equivalently renormed to be ASQ.

Proposition 4.18 (see also Proposition 4.22)**.** *Every Banach space which contains a complemented copy of an ASQ space can be equivalently renormed to be ASQ.*

Proof. Let $(X, \|\cdot\|)$ be a Banach space which contains a complemented copy of an ASQ space Y. We will show that X can be equivalently renormed to be ASQ. We may (and do) assume that X contains Y isometrically (see [DGZ, Lemma 8.1]). Let $P\colon X \to X$ be a bounded linear projection onto Y.

Define an equivalent norm on X by

$$|||x||| = \max\{\|Px\|, \|x - Px\|\} \quad \text{for all } x \in X.$$

Note that $|||\cdot|||$ agrees with $\|\cdot\|$ on Y.

Let us show that $(X, |||\cdot|||)$ is ASQ. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_{(X,|||\cdot|||)}$, and let $\varepsilon > 0$. Clearly $\|Px_i\| \leq 1$ and $\|x_i - Px_i\| \leq 1$ for all $i \in \{1, \dots, n\}$. Since Y is

ASQ, there is a norm one element $y \in Y$ such that $\|Px_i + y\| \leq 1 + \varepsilon$ for all $i \in \{1, \ldots, n\}$. Thus we have

$$\begin{aligned} |||x_i + y||| &= \max\{\|P(x_i + y)\|, \|x_i + y - P(x_i + y)\|\} \\ &= \max\{\|Px_i + y\|, \|x_i - Px_i\|\} \leq 1 + \varepsilon \end{aligned}$$

for all $i \in \{1, \ldots, n\}$. Therefore X is ASQ. □

Corollary 4.19. *Every separable Banach space which contains a copy of c_0 can be equivalently renormed to be ASQ.*

Proof. Let X be a separable Banach space which contains a copy of c_0. By Sobczyk's theorem (see, e.g. [AK, Theorem 2.5.8]), X contains a complemented copy of c_0, and thus the conclusion follows from Proposition 4.18. □

4.2 Stability results of almost square Banach spaces

In this section, we study stability results for almost square spaces by taking ℓ_p-sums of Banach spaces. We show that LASQ and WASQ are stable by forming ℓ_p-sums. It turns out that, for every p with $1 \leq p < \infty$, the ℓ_p-sum of two Banach spaces is never ASQ. Further, we show that nonreflexive Banach spaces which are M-ideals in their biduals are ASQ. This improves Theorem 4.10 in [ALN1], where it is shown that such spaces have the SD2P.

The following proposition is our main stability result for LASQ spaces.

Proposition 4.20. *Let X and Y be Banach spaces.*

(a) *If X and Y are LASQ, and p is such that $1 \leq p < \infty$, then $X \oplus_p Y$ is LASQ (see [ALL, Proposition 5.3]).*

(b) *If $X \oplus_p Y$ is LASQ, where p is such that $1 \leq p < \infty$, then X is LASQ (see [ALL, Proposition 5.7]).*

(c) *If X is LASQ, then $X \oplus_\infty Y$ is LASQ (see [ALL, Proposition 5.8]).*

(d) *If $X \oplus_\infty Y$ is LASQ, then either X or Y is LASQ (see [ALL, Proposition 5.8]).*

Proof. (a). Assume that X and Y are LASQ, and let p be such that $1 \leq p < \infty$. Let $(x, y) \in S_{X \oplus_p Y}$ and let $\varepsilon > 0$. It suffices to find a $(u, v) \in S_{X \oplus_p Y}$ such that

$$\|(x, y) \pm (u, v)\|_p \leq 1 + \varepsilon.$$

We may (and do) assume that $x \neq 0$ and $y \neq 0$. By our assumption, there exist $\tilde{u} \in S_X$ and $\tilde{v} \in S_Y$ such that

$$\left\| \frac{x}{\|x\|} \pm \tilde{u} \right\| \leq 1 + \varepsilon \text{ and } \left\| \frac{y}{\|y\|} \pm \tilde{v} \right\| \leq 1 + \varepsilon.$$

It follows that

$$\left\| x \pm \|x\|\tilde{u} \right\|^p + \left\| y \pm \|y\|\tilde{v} \right\|^p \leq (1+\varepsilon)^p.$$

This completes the proof, because one may take $u = \|x\|\tilde{u}$ and $v = \|y\|\tilde{v}$.

(b). The function $f(t) = t^{1/p}$ is uniformly continuous on $[0, 2]$, so given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(t)-f(s)| \leq \varepsilon$ whenever $|t-s| \leq \delta$. The function $g(t) = t^p$ is continuous at $t = 1$, so there exists $\eta > 0$ such that $|g(1)-g(s)| \leq \delta$ whenever $|1 - s| \leq \eta$.

Assume that $X \oplus_p Y$ is LASQ, where p is such that $1 \leq p < \infty$. We will show that X is LASQ. Let $x \in S_X$ and let ε be such that $\varepsilon \in (0, 2^{1/p} - 1)$. Find a δ and a η as above such that $\eta < \delta < \varepsilon$. By our assumption, there is a $(u, v) \in S_{X\oplus_p Y}$ such that

$$\|(x, 0) \pm (u, v)\|_p = (\|x \pm u\|^p + \|v\|^p)^{1/p} \leq 1 + \eta.$$

(Note that $u \neq 0$, otherwise $\|(x, v)\| = 2^{1/p} > 1 + \varepsilon$.) Then

$$\|x \pm u\|^p + \|v\|^p \leq (1 + \eta)^p.$$

Since $|1 - (1 + \eta)| \leq \eta$, we have

$$(1 + \eta)^p = g(1 + \eta) \leq g(1) + \delta = 1 + \delta.$$

Hence

$$\|x \pm u\|^p \leq 1 + \delta - \|v\|^p = \|u\|^p + \|v\|^p + \delta - \|v\|^p = \|u\|^p + \delta.$$

Take p-th roots to obtain

$$\|x \pm u\| \leq (\|u\|^p + \delta)^{1/p}.$$

Since $|(\|u\|^p + \delta) - \|u\|^p| \leq \delta$, we have

$$(\|u\|^p + \delta)^{1/p} = f(\|u\|^p + \delta) \leq f(\|u\|^p) + \varepsilon = \|u\| + \varepsilon.$$

Consequently,

$$\|x \pm u\| \leq \|u\| + \varepsilon.$$

Denote by $z = u/\|u\|$. Then

$$\|x \pm z\| \leq \|x \pm u\| + \|z - u\| \leq \|u\| + \varepsilon + 1 - \|u\| = 1 + \varepsilon.$$

Thus X is LASQ.

(c). Assume that X is LASQ. Denote by $Z = X \oplus_\infty Y$. Let $z = (x, y) \in S_Z$ and let $\varepsilon > 0$. We may assume that $x \neq 0$. By our assumption, we can find a $u \in S_X$ such that $\|\frac{x}{\|x\|} \pm u\| \leq 1 + \varepsilon$. Therefore

$$\begin{aligned} \|x \pm u\| &= \left\| \|x\|(\frac{x}{\|x\|} \pm u) \pm (1 - \|x\|)u \right\| \\ &\leq \|x\|(1 + \varepsilon) + (1 - \|x\|)\|u\| \leq 1 + \varepsilon. \end{aligned}$$

Take $w = (u, 0) \in S_Z$. Thus

$$\|z \pm w\| = \max\{\|x \pm u\|, \|y\|\} \leq 1 + \varepsilon$$

and Z is LASQ.

(d). Assume that $Z = X \oplus_\infty Y$ is LASQ. We will show that X or Y is LASQ. Suppose to the contrary that neither X nor Y is LASQ. Thus there are $x \in S_X$, $y \in S_Y$, and $\varepsilon > 0$ such that

$$\|x + u\| > 1 + \varepsilon \quad \text{or} \quad \|x - u\| > 1 + \varepsilon \quad \text{for all } u \in S_X$$

and

$$\|y + v\| > 1 + \varepsilon \quad \text{or} \quad \|y - v\| > 1 + \varepsilon \quad \text{for all } v \in S_Y.$$

Take $z = (x, y) \in S_Z$. By our assumption, there is a $w = (u_0, v_0) \in S_Z$ such that $\|z \pm w\| = \max\{\|x \pm u_0\|, \|y \pm v_0\|\} \leq 1 + \varepsilon$, a contradiction. Thus X or Y must be LASQ. □

The following proposition is our main stability result for WASQ spaces.

Proposition 4.21. *Let X and Y be Banach spaces.*

(a) *If X and Y are WASQ, and p is such that $1 \leq p < \infty$, then $X \oplus_p Y$ is WASQ (see [ALL, Proposition 5.4]).*

(b) *If $X \oplus_p Y$ is WASQ, where p is such that $1 \leq p < \infty$, then X is WASQ (see [ALL, Proposition 5.7]).*

(c) *If X is WASQ, then $X \oplus_\infty Y$ is WASQ (see [ALL, Proposition 5.8]).*

(d) *If $X \oplus_\infty Y$ is WASQ, then either X or Y is WASQ (see [ALL, Proposition 5.8]).*

Proof. (a). The proof is similar to the proof of Proposition 4.20, (a). Assume that X and Y are WASQ, and let p be such that $1 \leq p < \infty$. Let $(x, y) \in S_{X \oplus_p Y}$. It suffices to find a sequence $(u_k, v_k) \in S_{X \oplus_p Y}$, $k \in \mathbb{N}$, such that

$$\|(x, y) \pm (u_k, v_k)\|_p \leq 1 + \frac{1}{k} \quad \text{for all } k \in \mathbb{N}$$

and $(u_k, v_k) \to 0$ weakly as $k \to \infty$.

We may (and do) assume that $x \neq 0$ and $y \neq 0$. By our assumption, there exist sequences $(\tilde{u}_k)$ in S_X and $(\tilde{v}_k)$ in S_Y such that

$$\left\|\frac{x}{\|x\|} \pm \tilde{u}_k\right\| \leq 1 + \frac{1}{k} \text{ and } \left\|\frac{y}{\|y\|} \pm \tilde{v}_k\right\| \leq 1 + \frac{1}{k}$$

and $\tilde{u}_k \to 0$, $\tilde{v}_k \to 0$ weakly as $k \to \infty$.

It follows that

$$\left\|x \pm \|x\|\tilde{u_k}\right\|^p + \left\|y \pm \|y\|\tilde{v_k}\right\|^p \leq (1 + \frac{1}{k})^p$$

and $(\|x\|\tilde{u_k}, \|y\|\tilde{v_k}) \to 0$ weakly as $k \to \infty$. This completes the proof, because one may take $u_k = \|x\|\tilde{u}_k$ and $v_k = \|y\|\tilde{v}_k$.

The proof of (b), (c), and (d) is similar to the proof of Proposition 4.20, (b), (c), and (d), respectively. □

The following proposition is our main stability result for ASQ spaces.

Proposition 4.22. *Let X and Y be Banach spaces.*

(a) *If p is such that $1 \leq p < \infty$, then $X \oplus_p Y$ is not ASQ (see [ALL, Lemma 5.6]).*

(b) *If X is ASQ, then $X \oplus_\infty Y$ is ASQ (see [ALL, Proposition 5.8]).*

(c) *If $X \oplus_\infty Y$ is ASQ, then either X or Y is ASQ (see [ALL, Proposition 5.8]).*

Proof. (a). If p is such that $1 < p < \infty$, then $X \oplus_p Y$ is not ASQ, because it fails the SD2P (see Theorem 2.33). Recall that the SD2P is stable by forming the ℓ_1-sum (see Theorem 2.32), however we will show that $X \oplus_1 Y$ is not ASQ.

Denote by $Z = X \oplus_1 Y$. Let $x \in S_X$ and $y \in S_Y$. Consider norm one elements $z_1 = (-x/3, 2y/3)$ and $z_2 = (2x/3, -y/3)$ in Z. Suppose to the contrary that Z is ASQ. Then there is a $w = (u, v) \in S_Z$ with $\|z_i \pm w\| \leq 1 + 1/9$. We have that

$$\|u\| + \|\frac{2}{3}y\| \leq \frac{1}{2}\Big(\|-\frac{1}{3}x + u\| + \|\frac{1}{3}x + u\| + \|\frac{2}{3}y + v\| + \|\frac{2}{3}y - v\|\Big)$$
$$= \max\{\|z_1 + w\|, \|z_1 - w\|\} \leq 1 + \frac{1}{9},$$

therefore $\|u\| \leq 1/3 + 1/9 = 4/9$. Similarly, $\|v\| \leq 4/9$. Thus we have $\|w\| = \|u\| + \|v\| \leq 8/9 < 1$, a contradiction.

The proof of (b) and (c) is similar to the proof of Proposition 4.20, (b) and (c), respectively. ☐

Example 4.23. The Banach space $c_0(L_1[0, 1]) \oplus_\infty \ell_1$ is ASQ and OH, but fails to have the DP. Indeed, by Example 4.3, $c_0(L_1[0, 1])$ is ASQ and OH. Thus, by Propositions 3.32 and 4.22, the Banach space $c_0(L_1[0, 1]) \oplus_\infty \ell_1$ is also both ASQ and OH. It fails the DP, because ℓ_1 fails the DP.

The next lemma shows that LASQ and ASQ pass down from a Banach space to its ai-ideal.

Proposition 4.24 (see [ALL, Lemma 4.5])**.** *Let X be a Banach space. If X is ASQ (resp. LASQ) and Y is an ai-ideal in X, then Y is ASQ (resp. LASQ).*

Proof. We only show the ASQ case, the other case is similar. Assume that X is ASQ and Y is an ai-ideal in X. Let $n \in \mathbb{N}$, $y_1, \ldots, y_n \in S_Y$ and let ε be such that $\varepsilon \in (0, 1)$. By our assumption, we can find a $x \in S_X$ such that $\|y_i - x\| \leq 1 + \varepsilon/4$ for all $i \in \{1, \ldots, n\}$. Since Y is an ai-ideal in X, we can choose an $\varepsilon/4$-isometry $T : E \to Y$ such that T is the identity on $E \cap Y$, where $E = \operatorname{span}\{x, y_1, \ldots, y_n\}$. Define $z = Tx/\|Tx\|$. Then $z \in S_Y$ and $\|Tx - z\| = |\|Tx\| - 1| \leq \varepsilon/4$, and

$$\|y_i - z\| \leq \|T(y_i - x)\| + \|Tx - z\| \leq (1 + \frac{\varepsilon}{4})(1 + \frac{\varepsilon}{4}) + \frac{\varepsilon}{4} \leq 1 + \varepsilon$$

for all $i \in \{1, \ldots, n\}$. Thus Y is ASQ by Proposition 4.8. ☐

Remark 4.2. A similar result to Proposition 4.24 for WASQ spaces seems to be unknown.

For M-ideals we often get ASQ for free.

Theorem 4.25 (see [ALL, Theorem 4.6]). *Let X be a nonreflexive Banach space and Y be a proper subspace of X. If Y is both an M-ideal and an ai-ideal in X, then Y is ASQ.*

Proof. Assume that Y is both an M-ideal and an ai-ideal in X. We will show that Y is ASQ. Let $n \in \mathbb{N}$, $y_1, \ldots, y_n \in S_Y$ and let $\varepsilon > 0$. Since Y is an M-ideal in X, we have that $X^* = P(X^*) \oplus_1 Y^\perp$, where P denotes here the M-ideal projection on X^*. Then $X^{**} = (P(X^*))^\perp \oplus_\infty Y^{\perp\perp}$.

Choose a $\delta \in (0,1)$ with $(1+\delta)^2(1+3\delta(1+\delta)^2) < 1+\varepsilon$. Let $z \in S_{(P(X^*))^\perp}$. Note that $\|y_i - z\| = \max\{\|y_i\|, \|z\|\} \leq 1$ for all $i \in \{1, \ldots, n\}$. Put $E = \operatorname{span}\{y_1, \ldots, y_n, z\} \subset X^{**}$. Use the Principle of Local Reflexivity to find a δ-isometry $S : E \to X$ which is the identity on $E \cap X$. Further, put $F = S(E) \subset X$ and use that Y is an ai-ideal in X to find a δ-isometry $T : F \to Y$ which is the identity on $F \cap Y$. Now with $y = T(Sz)/\|T(Sz)\| \in S_Y$ we use $T(Sy_i) = y_i$ to get

$$\begin{aligned}\|y_i - y\| = \|y_i - \frac{T(Sz)}{\|T(Sz)\|}\| &\leq (1+\delta)^2 \|y_i - \frac{z}{\|T(Sz)\|}\| \\ &\leq (1+\delta)^2 \Big(\|y_i - z\| + \|z - \frac{z}{\|T(Sz)\|}\| \Big) < 1+\varepsilon,\end{aligned}$$

since

$$\begin{aligned}\|z - \frac{z}{\|T(Sz)\|}\| &= \frac{1}{\|T(Sz)\|}\Big|1 - \|T(Sz)\|\Big| \\ &\leq (1+\delta)^2 \Big(\Big|1 - \|Sz\|\Big| + \Big|\|Sz\| - \|T(Sz)\|\Big| \Big) \\ &\leq (1+\delta)^2(\delta + \delta(1+\delta)) \leq 3\delta(1+\delta)^2.\end{aligned}$$

By Proposition 4.8, Y is ASQ. □

Since every Banach space is an ai-ideal in its bidual, we immediately have the following corollary.

Corollary 4.26 (see [ALL, Corollary 4.7]). *Nonreflexive M-embedded Banach spaces are ASQ.*

For example, c_0 and the Banach space $\mathcal{K}(H)$ of compact operators on a Hilbert space H are M-embedded. (For more examples see Chapter III in [HWW].) By Example 4.2, the Banach space $c_0(\ell_1)$ is ASQ. However, this space contains a copy of ℓ_1 and therefore can not be M-embedded ([HaLi, Theorems 3.4.a and 3.5]). Thus the class of ASQ spaces properly contains the class of M-embedded spaces.

4.3 Connection with the intersection property

In this section, we explore the connection between ASQ spaces and the intersection property introduced in [BH] (see also [HWW, Chapter II.4]).

Definition 4.27 (see [BH]). A Banach space X has the *intersection property* (IP), if for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$, $x_1, \dots, x_n$ in X with $\|x_i\| < 1$ for all $i \in \{1, \dots, n\}$, such that if $y \in X$ with $\|x_i - y\| \leq 1$ for all $i \in \{1, \dots, n\}$, then $\|y\| \leq \varepsilon$.

For a subset I of $[0, 1]$, we will use the notation $B_I = \{x \in X : \|x\| \in I\}$. For example $B_X = B_{[0,1]}$, $S_X = B_{\{1\}}$, and $B_X \setminus S_X = B_{[0,1)}$.

If X fails the IP, then for some $\varepsilon \in (0, 1)$ we have $\gamma(\varepsilon) \leq 1$, where

$$\gamma(\varepsilon) = \sup_{\substack{x_1,\dots,x_n \in B_{[0,1)} \\ n \in \mathbb{N}}} \inf_{y \in B_{(\varepsilon,1]}} \max_{1 \leq i \leq n} \|x_i - y\|.$$

On the other hand, $\gamma(\varepsilon) \geq 1$ for all $\varepsilon > 0$, because $\max\{\|x+y\|, \|x-y\|\} \geq \|x\|$ for all $x, y \in X$. Thus, if X fails the IP, then $\gamma(\varepsilon) = 1$ for some $\varepsilon \in (0, 1)$.

We will say that X *ε-fails the IP* if $\gamma(\varepsilon) = 1$. The index $\gamma(\cdot)$ is very similar to the index $\alpha(\cdot)$ defined by E. Maluta and P. L. Papini in [MP]. Here are two equivalent definitions of $\alpha(\varepsilon)$, where $\varepsilon \in [0, 1]$ (see Proposition 3.3 in [MP])

$$\begin{aligned}
\alpha(\varepsilon) &= \sup_{\substack{x_1,\dots,x_n \in S_X \\ n \in \mathbb{N}}} \inf_{y \in B_{[\varepsilon,1]}} \max_{1 \leq i \leq n} \|x_i - y\| \\
&= \sup_{\substack{x_1,\dots,x_n \in B_{[0,1)} \\ n \in \mathbb{N}}} \inf_{y \in B_{[\varepsilon,1]}} \max_{1 \leq i \leq n} \|x_i - y\|.
\end{aligned}$$

It is clear that $\alpha(\cdot)$ is monotone and $\alpha(0) = 1$. A straightforward argument shows that $\alpha(1) = 1$ if and only if X is ASQ.

In [GL], J. Gao and K.-S. Lau considered, for a Banach space X, the parameter

$$G(X) = \sup_{x \in S_X} \inf_{y \in S_X} \max\{\|x + y\|, \|x - y\|\}.$$

Observe that X is LASQ if and only if $G(X) = 1$.

Proposition 4.28 (see [ALL, Theorem 6.1]). *A Banach space X is ASQ if and only if X ε-fails the IP for all $\varepsilon \in (0, 1)$.*

Proof. Assume that X is ASQ and let $\varepsilon \in (0, 1)$. Since

$$1 = \alpha(1) \geq \gamma(\varepsilon) \geq \alpha(\varepsilon) \geq \alpha(0) = 1,$$

we get $\gamma(\varepsilon) = 1$.

Assume now that X ε-fails the IP for all $\varepsilon \in (0, 1)$. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$ and let ε be such that $\varepsilon \in (0, 1)$. Let $z_i = (1 + \varepsilon)^{-1} x_i$. Since X $(1 - \varepsilon)$-fails the IP, there is a $y \in B_{(1-\varepsilon,1]}$ with $\max_i \|z_i - y\| \leq 1 + \varepsilon$. Then

$$\|x_i - \frac{y}{\|y\|}\| \leq \|x_i - z_i\| + \|z_i - y\| + (1 - \|y\|) \leq 1 + 3\varepsilon.$$

By Proposition 4.8, we conclude that X is ASQ. □

In Proposition 4.28, we saw that a Banach space X is ASQ if and only if X ε-fails the IP for all $\varepsilon \in (0, 1)$. We will now provide an example of a Banach space, which is not LASQ (thus not ASQ), but ε-fails the IP for all $\varepsilon \in (0, 1/2]$.

Example 4.29 (see [ALL, Example 6.4])**.** Let

$$X = \{f \in C[0, 1] : f(0) = 2f(1)\}.$$

We will show that X is a non-LASQ space which ε-fails the IP for all $\varepsilon \in (0, 1/2]$, but does not ε-fail the IP for any $\varepsilon \in (1/2, 1)$.

Let us first show that X is not LASQ. Let $f(t) = 1 - t/2$. It is clear that $f \in S_X$. Let $g \in S_X$ be arbitrary. Find $t_0 \in [0, 1]$ such that $|g(t_0)| = 1$. Then

$$\frac{1}{2} + 1 \leq \max_{\pm} |f(t_0) \pm g(t_0)| \leq \max_{\pm} \|f \pm g\|.$$

This shows that X is not LASQ since $\max_{\pm} \|f \pm g\|$ is bounded away from 1.

Let us now show that X ε-fails IP for all $\varepsilon \in (0, 1/2]$. Observe that for $f \in X$ with $\|f\| < 1$ we have $|f(1)| < 1/2$ (since $2|f(1)| = |f(0)| < 1$).

Let $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X$ with $\|f_i\| < 1$. Find an interval $(s, 1)$ such that $|f_i(t)| < 1/2$ for $t \in (s, 1)$. Let $g \in X$ with $\operatorname{supp} g \subset (s, 1)$. Then $\|f_i + g\| < 1/2 + \|g\|$, hence there exists a $\delta > 0$ such that $\|f_i + g\| + \delta \leq 1/2 + \|g\|$. If we choose g as above with $\|g\| = 1/2 + \delta$, then $\max_i \|f_i + g\| \leq 1$ and $\|g\| > 1/2$. Thus X ε-fails IP for all $\varepsilon \in (0, 1/2]$.

We will now provide an example of a Banach space which is ASQ, but which is not a c_0-sum of some Banach spaces nor M-embedded.

Example 4.30 (see [ALL, Example 6.3]). For all $m \in \mathbb{N}$, denote by $X_m = C_\Sigma(S^m)$. The Banach space $X = (X_1 \oplus X_2 \oplus \dots)_{\ell_\infty}$ is ASQ. Here S^m is the Euclidean sphere in $\mathbb{R}^{m+1}$ and

$$C_\Sigma(S^m) = \{f \in C(S^m) : f(s) = -f(-s) \text{ for all } s \in S^m\}.$$

The Banach space X is not a c_0-sum of ASQ-spaces nor M-embedded (see [HWW, Example II.4.6, p. 78]), but a small adjustment to the proof of [HWW, Proposition II.4.2 (h), p. 76] shows that X ε-fails the IP for every $\varepsilon \in (0, 1)$. Thus X is ASQ by Proposition 4.28.

Let us first prove that for every $f_1, \dots, f_m$ in the open unit ball of $C_\Sigma(S^m)$ and for every $\varepsilon \in (0, 1)$ there is a $g \in C_\Sigma(S^m)$ with $\|f_i \pm g\| \leq 1$ for every $i \in \{1, \dots, m\}$, but $\varepsilon < \|g\| \leq 1$.

Indeed, let $f_1, \dots, f_m \in C_\Sigma(S^m)$ be such that $\|f_i\| < 1$ for every $i \in \{1, \dots, m\}$. Fix a $s_0 \in S^m$ such that $f_1(s_0) = \dots = f_m(s_0) = 0$. Such a s_0 exists by a corollary of the Borsuk–Ulam theorem (see [AH, p. 485, Satz VIII]), according to which any m functions in $C_\Sigma(S^m)$ have a common zero. Let ε and δ be such that $\varepsilon \in (0, 1)$ and $\delta \in (\varepsilon, 1)$. To construct the desired g, choose a neighbourhood U of s_0 in S^m such that $|f_i(s)| < 1 - \delta$ for all $i \in \{1, \dots, m\}$ and $s \in U$, and so for $s \in -U$ too. We may (and do) assume that $U \cap -U = \emptyset$.

Let $h\colon S^m \to [0, 1]$ be a continuous function vanishing outside U with $h(s_0) = 1$. Define $g(s) = \delta(h(s) - h(-s))$. Then $g \in C_\Sigma(S^m)$, $\|g\| = \delta$, and g vanishes outside $U \cap -U$. It follows that $\|f_i \pm g\| \leq 1$ for all $i \in \{1, \dots, m\}$.

Let us now show that X ε-fails the IP for all $\varepsilon \in (0, 1)$. Suppose to the contrary that for some $\varepsilon \in (0, 1)$ there are $n \in \mathbb{N}$ and $x_1 = (x_{1,m}), \dots, x_n = (x_{n,m}) \in X$ such that $\|x_i\| < 1$ for every $i \in \{1, \dots, n\}$ and if $y \in X$ with $\|x_i - y\| \leq 1$ for all $i \in \{1, \dots, n\}$, then $\|y\| \leq \varepsilon$.

For an arbitrary m, we have that $x_{1,m}, \dots, x_{n,m}$ are in the open unit ball of $C_\Sigma(S^m)$. If $n \leq m$, then by the argument above there is a $y_m \in C_\Sigma(S^m)$ such that $\|x_{i,m} - y_m\| \leq 1$ for all $i \in \{1, \dots, n\}$, but $\|y_m\| > \varepsilon$. Thus for $y = (0, \dots, 0, y_m, 0, \dots)$ we have $\|x_i - y\| \leq 1$ for all $i \in \{1, \dots, n\}$, but $\|y\| > \varepsilon$. Therefore, we must have that $n > m$ for all m, which is a contradiction.

Next we will show that every ASQ space contains a separable subspace which is ASQ. The basic idea for the next proof goes back to Theorem 4.4 in Lindenstrauss' memoir [Lin].

Proposition 4.31 (see [ALL, Proposition 6.5]). *A Banach space X is ASQ if and only if for every separable subspace Y of X, there exists a separable ASQ subspace Z of X such that $Y \subset Z \subset X$.*

Proof. Necessity. Assume that X is ASQ. Let Y be a separable subspace of X, let $\{u_m\colon m \in \mathbb{N}\}$ be a dense subset in Y, and, for every $m \in \mathbb{N}$, let $\varepsilon_m > 0$ be such that $\varepsilon_m \to 0$ as $m \to \infty$. Put $Y_1 = \operatorname{span}\{u_1\}$. For $m \in \mathbb{N}$, choose Y_{m+1} as follows: find a $y_m \in S_X$ so that

$$\|x + y_m\| \leq 1 + \varepsilon_m \quad \text{for all } x \in S_{Y_m},$$

and put $Y_{m+1} = \operatorname{span}\big(Y_m \cup \{y_m\} \cup \{u_{m+1}\}\big)$.

Take $Z = \overline{\bigcup_{m=1}^{\infty} Y_m}$. To see that Z is ASQ, let $n \in \mathbb{N}$, $z_1, \ldots, z_n \in S_Z$, and let $\varepsilon > 0$. It suffices to find a $y \in S_Z$ such that

$$\|z_i + y\| \leq 1 + \varepsilon \quad \text{for all } i \in \{1, \ldots, n\}.$$

Pick $m \in \mathbb{N}$ so that $\varepsilon_m \leq \varepsilon/2$ and, for every $i \in \{1, \ldots, n\}$, there is an $x_i \in S_{Y_m}$ satisfying $\|z_i - x_i\| < \varepsilon/2$. For every $i \in \{1, \ldots, n\}$, one has

$$\|z_i + y_m\| \leq \|z_i - x_i\| + \|x_i + y_m\| \leq \frac{\varepsilon}{2} + 1 + \varepsilon_m \leq 1 + \varepsilon.$$

Thus Z is ASQ.

Sufficiency. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and let $\varepsilon > 0$. Denote by $Y = \operatorname{span}\{x_1, \ldots, x_n\}$. By our assumption, there is a separable subspace Z of X such that $Y \subset Z \subset X$ and Z is ASQ. Therefore there is a $z \in S_Z \subset S_X$ such that $\|x_i + z\| \leq 1 + \varepsilon$ for all $i \in \{1, \ldots, n\}$. Thus X is ASQ. □

In [BH], it is asked whether all dual Banach spaces have the IP. Similarly one can ask the following.

Question 4.32 (see [ALL, Question 6.6]). *Is there a dual Banach space which is ASQ?*

Recall that a Banach space X is said to be *weakly compactly generated* if X contains a weakly compact absolutely convex set whose linear span is dense in X. For example, reflexive Banach spaces are weakly compactly generated, because their unit ball is weakly compact. More examples can be found in, e.g., [FHHMZ].

Remark 4.3 (see [ALL, Remark 6.7]). In Remark 2a in [HR, p. 289], P. Harmand and T. S. S. R. K. Rao noted the following partial answer to the question about the IP: If X^* is such that for any separable subspace Y of X^* there is separable subspace Z with $Y \subset Z \subset X^*$ and Z complemented in X^*, then X^* has the IP. The assumption is satisfied if, for example, X^* is weakly compactly generated (see [Die2, p. 149]).

Their argument works also for ASQ spaces and show that an ASQ space can never be a subspace of a weakly compactly generated dual space. Indeed, suppose to the contrary that a Banach space X is ASQ and a subspace of a weakly compactly generated dual space W^*. By Proposition 4.31, there is a separable closed subspace Y of X which is also ASQ. From [Die2, p. 149] we conclude that there is a separable closed subspace Z of W^* such that $Y \subset Z \subset W^*$ and Z is complemented in W^*. By Corollary 4.13, Y contains an isomorphic copy of c_0. Therefore Z contains an isomorphic copy of ℓ_∞ (see [Ros, Corollary 1.5]), which gives the contradiction, because Z is separable.

Proposition 4.33 (see [ALL, Proposition 6.8]). *A Banach space X is LASQ if and only if for every separable subspace Y of X, there exists a separable LASQ subspace Z of X such that $Y \subset Z \subset X$.*

Proof. The proof is similar to the proof of Proposition 4.31. □

Remark 4.4. A similar result to Proposition 4.33 for WASQ Banach spaces seems to be unknown.

Chapter 5

Thickness and thinness of Banach spaces

In this chapter, we complement and extend some recent results on Whitley's indices of thickness and thinness in three directions. This is motivated by the fact that a Banach space is OH if and only if its thickness is 2, and a Banach space is ASQ if and only if its thinness is 1. Firstly, we investigate both the indices when forming ℓ_p-sums of Banach spaces, and obtain formulas which show that they behave rather differently. Secondly, we consider the relation of the indices of the space and a subspace. Finally, we show that every Banach space X containing a complemented copy of c_0 can be equivalently renormed so that in the new norm both the thickness and thinness index of X equal to 1. This chapter is based on [ALLN] and [HL2].

5.1 Definitions and basic results

Let X be a Banach space. R. Whitley introduced in [Whi] the *index of thickness*,

$$T_W(X) = \inf\left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exists } n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in S_X \\ \text{such that } \;S_X \subset \bigcup_{i=1}^n B(x_i, r) \end{array} \right\},$$

and the *index of thinness*,

$$t(X) = \inf\left\{ r > 0 \;\middle|\; \begin{array}{l} \text{for all } n \in \mathbb{N},\, x_1, \ldots, x_n \in S_X, \text{ and } \varepsilon > 0 \text{ there} \\ \text{exists } x \in S_X \text{ with } \;\max_{1\le i\le n}\|x_i - x\| < r + \varepsilon \end{array} \right\}.$$

The subscript W in $T_W(X)$ is to indicate that this is Whitley's original definition. As is easily observed, if $\dim X < \infty$, then $T_W(X) = 0$ and $t(X) = 2$, and if

$\dim X = \infty$, then $T_W(X), t(X) \in [1, 2]$. In [Whi], it is proved that

$$T_W(\ell_p) = 2^{1/p} = t(\ell_p) \quad \text{for all } 1 \le p < \infty.$$

Together with Whitley's observations that $T_W(c_0) = 1$, $T_W(\ell_\infty) = 1$, $t(c_0) = 1$, and $t(\ell_\infty) = 2$, it is clear that the whole range $[1, 2]$ of values is possible and that $(1, 2)$ is covered by indices of reflexive Banach spaces. We will see that, by choosing appropriate reflexive Banach spaces X and Y, we have $T_W(X) = 1$ and $t(Y) = 2$, but, for a reflexive Banach space X one never has $T_W(X) = 2$ nor $t(X) = 1$.

Before proceeding, let us just mention that in [CPS] it is noted that if $\dim X = \infty$ and $S_X \subset \bigcup_{i=1}^n B(x_i, r)$, where $x_1, \dots, x_n$ are in S_X and $r > 0$, then $B_X \subset \bigcup_{i=1}^n B(x_i, r)$. Thus, for $\dim X = \infty$, the index

$$T(X) = \inf \left\{ r > 0 \;\middle|\; \begin{array}{l} \text{there exists } n \in \mathbb{N} \text{ and } x_1, \dots, x_n \in S_X \\ \text{with } \; B_X \subset \bigcup_{i=1}^n B(x_i, r) \end{array} \right\}$$

is equal to $T_W(X)$. Note that when $\dim X < \infty$, we always have $T(X) = 1$ (while $T_W(X) = 0$). From now on we are only interested in calculating the index for infinite-dimensional Banach spaces and will thus take the freedom to use $T(X)$ in what follows to denote also $T_W(X)$.

We observe now that the octahedrality of the norm on X is characterized by the condition $T(X) = 2$. In fact, this is a direct consequence of the following observation made by G. Godefroy in [God].

Proposition 5.1 (cf. [God, p. 12], see [HL2, Proposition 2.3]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is OH;*

(ii) *if $x_1, \dots, x_n \in S_X$ and $r_1, \dots, r_n > 0$ are such that*

$$S_X \subset \bigcup_{i=1}^n B(x_i, r_i),$$

then $S_X \subset B(x_i, r_i)$ for some i in $\{1, \dots, n\}$;

(iii) $T(X) = 2$.

Proof. (i) $\Rightarrow$ (ii). Assume that X is OH, and consider a finite number of closed balls $B(x_1, r_1), \dots, B(x_n, r_n)$ in X, where $x_1 \dots, x_n \in S_X$ and $r_1, \dots, r_n > 0$ with

$$S_X \subset \bigcup_{i=1}^n B(x_i, r_i).$$

Since X is OH (see Proposition 3.3), for every $\varepsilon > 0$ there are i in $\{1, \dots, n\}$ and norm one y in $B(x_i, r_i)$ with

$$\|x_i - y\| \geq 2 - \varepsilon,$$

which yields $r_i \geq 2 - \varepsilon$. Consequently, $r_i \geq 2$ for at least one i in $\{1, \dots, n\}$. Thus $S_X \subset B(x_i, r_i)$ for some i in $\{1, \dots, n\}$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Assume that $T(X) = 2$ holds. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, and let $\varepsilon > 0$. By Proposition 3.3, it suffices to find a $y \in S_X$ such that

$$\|x_i - y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Suppose to the contrary that, for every $y \in S_X$ there is an x_i such that $\|x_i - y\| < 2 - \varepsilon$. Then

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, 2 - \varepsilon).$$

Thus $T(X) < 2$, which is a contradiction. □

Proposition 5.1, (ii), characterizes OH Banach spaces in terms of covering of the unit ball. We have similar characterizations also for LOH and WOH Banach spaces.

Proposition 5.2 (see [HL2, Proposition 3.2, Corollary 3.3]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is LOH;*

(ii) *if $S_X \subset B(x, r) \cup B(-x, r)$ for some $x \in S_X$ and $r > 0$, then $S_X \subset B(x, r)$;*

(iii) *$g'(X) = 2$, where*

$$g'(X) = \inf\{r > 0 \colon S_X \subset B(x, r) \cup B(-x, r) \text{ for some } x \text{ in } S_X\}.$$

Remark 5.1. The interested reader can find more about this constant $g'(X)$ in [Pap], where P. L. Papini has compared it with R. Whitley's thickness constant.

Proposition 5.3 (see [HL2, Proposition 3.4]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is WOH;*

(ii) *if* $x_1, \dots, x_n \in X$ *and* $r_1, \dots, r_n > 0$ *are such that*

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i),$$

then for every $x^* \in S_{X^*}$ *one has* $S_X \subset \{x \in X : |x^*(x - x_i)| \le r_i\}$ *for some* i *in* $\{1, \dots, n\}$.

Proof. (i) $\Rightarrow$ (ii). Assume that X is WOH, and $S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i)$ for some $x_1, \dots, x_n \in X$ and $r_1, \dots, r_n > 0$. Let $x^* \in S_{X^*}$. We have to show that for some $i \in \{1, \dots, n\}$, one has

$$S_X \subset \{x \in X : |x^*(x - x_i)| \le r_i\}.$$

Suppose to the contrary that, for every $i \in \{1, \dots, n\}$ there is an $x \in S_X$ such that $|x^*(x - x_i)| > r_i$. Pick $\varepsilon > 0$ satisfying

$$r_i < (1 - \varepsilon)(1 + |x^*(x_i)|) \quad \text{for all } i \in \{1, \dots, n\}.$$

Since X is WOH, there is a $y \in S_X$ such that

$$(1 - \varepsilon)(1 + |x^*(x_i)|) \le \|x_i - y\| \quad \text{for all } i \in \{1, \dots, n\}.$$

This yields $r_i < \|x_i - y\|$ for every $i \in \{1, \dots, n\}$, which is a contradiction.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. Let E be a finite-dimensional subspace of X. Let $x^* \in S_{X^*}$ and let ε be such that $\varepsilon \in (0, 1]$. Suppose that, for every $y \in S_X$ there is an $x \in E$ such that

$$\|x - y\| < (1 - \varepsilon)(|x^*(x)| + 1). \tag{5.1}$$

Then $\|x\| < \frac{2-\varepsilon}{\varepsilon}$. Denote by $\delta = \varepsilon/2$. Consider now a finite δ-net $\{x_1, \dots, x_n\}$ in X for $\frac{2-\varepsilon}{\varepsilon} B_E$. If $y \in S_X$, then find a corresponding $x \in E$ such that (5.1) holds, and choose x_i such that $\|x - x_i\| < \delta$. By (5.1), we have

$$\begin{aligned}
\|x_i - y\| &\le \|x - y\| + \delta \\
&< (1 - \varepsilon)(|x^*(x)| + 1) + \delta \\
&\le (1 - \varepsilon)(|x^*(x_i)| + \delta + 1) + \delta \\
&= (1 - \varepsilon)|x^*(x_i)| + 1 - \varepsilon\delta \\
&\le (1 - \varepsilon^2/2)(|x^*(x_i)| + 1).
\end{aligned}$$

Thus $S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i)$, where $r_i = (1 - \varepsilon^2/2)(|x^*(x_i)| + 1)$. On the other hand,

$$S_X \not\subset \{x \in X : |x^*(x - x_i)| \le r_i\} \quad \text{for all } i \in \{1, \dots, n\}.$$

This contradicts (ii). $\square$

It is easily seen that ASQ Banach spaces X are characterized by the condition $t(X) = 1$.

Proposition 5.4. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is ASQ;*

(ii) *for all $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$ we have that*

$$S_X \cap \bigcap_{i=1}^{n} B(x_i, 1+\varepsilon) \neq \emptyset;$$

(iii) *$t(X) = 1$.*

Proof. The equivalences follow immediately from the definitions. □

In Proposition 4.16, we saw that the dual of an ASQ Banach space is OH. In terms of the thickness and thinness index, we thus get the following.

Corollary 5.5 (see [ALLN, Proposition 2.1])**.** *If $t(X) = 1$, then $T(X^*) = 2$.*

Remark 5.2. The converse of Corollary 5.5 is in general not true. As an example, consider the Banach space $C[0,1]$. Since $C[0,1]$ has the DP, we have that $C[0,1]^*$ is OH (see Theorem 3.6). Thus $T(C[0,1]^*) = 2$ by Proposition 5.1. However, by considering the constant one function in $C[0,1]$, it is clear that $t(C[0,1]) = 2$ (see also [Whi, Lemma 8]).

In Proposition 4.18, we saw that if X contains a complemented copy of c_0, then X can be equivalently renormed such that X in this new norm is an ASQ space, that is, $t(X) = 1$ (see Proposition 5.4). We will now show that with respect to this new norm $T(X) = 1$.

Proposition 5.6. *Every Banach space X which contains a complemented copy of an ASQ space Y can be equivalently renormed such that $T(X) \leq T(Y)$, $t(X) = 1$, and X contains Y isometrically.*

Proof. Assume that a Banach space $(X, \|\cdot\|)$ contains a complemented ASQ subspace Y. Let $P\colon X \to X$ be the bounded linear projection onto Y. Recall the equivalent norm $|||\cdot|||$ from the proof of Proposition 4.18, where

$$|||x||| = \max\{\|Px\|, \|x - Px\|\} \quad \text{for all } x \in X.$$

Note that $|||\cdot|||$ agrees with $\|\cdot\|$ on Y.

By Proposition 4.18, we know that $(X, |||\cdot|||)$ is ASQ, that is, $t(X) = 1$. We will now show that $T(X) \leq T(Y)$. Let $\varepsilon > 0$. It suffices to find $x_1, \ldots, x_n \in S_{(X,|||\cdot|||)}$ such that

$$B_{(X,|||\cdot|||)} \subset \bigcup_{i=1}^{n} B_{(X,|||\cdot|||)}(x_i, T(Y) + \varepsilon).$$

By the definition of $T(Y)$, there are $y_1, \ldots, y_n \in S_Y$ such that

$$B_Y \subset \bigcup_{i=1}^{n} B_Y(y_i, T(Y) + \varepsilon).$$

Let $x \in B_{(X,|||\cdot|||)}$. Then $Px \in B_Y$ and hence, for some $i \in \{1, \ldots, n\}$, we have that $\|Px - y_i\| \leq T(Y) + \varepsilon$. Therefore

$$\begin{aligned} |||x - y_i||| &= \max\{\|P(x - y_i)\|, \|x - y_i - P(x - y_i)\|\} \\ &= \max\{\|Px - y_i\|, \|x - Px\|\} \\ &\leq \max\{\|Px - y_i\|, 1\} \\ &\leq T(Y) + \varepsilon. \end{aligned}$$

□

Remark 5.3. In [ALLN, Theorem 4.1], it is claimed that if a Banach space X contains a copy of c_0, then it can be equivalently renormed such that $T(X) = t(X) = 1$. This claim relies on the same renorming technique as in [ALL, Theorem 3.14] and it also has a gap (see the comment preceding Proposition 4.18).

Corollary 5.7. *Every separable Banach space X which contains a copy of c_0 can be equivalently renormed such that in this new norm $T(X) = t(X) = 1$.*

Proof. Let X be a separable Banach space which contains a copy of c_0. By Sobczyk's theorem, X contains a complemented copy of c_0, and thus the conclusion follows from Proposition 5.6. □

Dual to Proposition 5.6, using the norm suggested by W. B. Johnson (see [KSW, p. 11]), one can show that a Banach space X containing ℓ_1 may be equivalently renormed to have $T(X) = t(X) = 2$.

Theorem 5.8 (see [KSW, p. 11]). *Let X be a Banach space. If X contains an isomorphic copy of ℓ_1, then X can be equivalently renormed so that $T(X) = t(X) = 2$.*

Proof. Let Y be a subspace of X isomorphic to ℓ_1, and let (e_k) be the canonical basis in ℓ_1. To begin with, we can renorm X in such a way that Y is isometric to ℓ_1 (see, e.g., [DGZ, Lemma II.8.1]), thus (e_k) is an isometric ℓ_1-basis in X.

We denote by (r_k) the sequence of Rademacher functions in $L_\infty[0,1]$. Define $S\colon Y \to L_\infty[0,1]$ by $S(e_k) = r_k$ for all $k \in \mathbb{N}$. Thus $\|Sy\| = \|y\|$ for all $y \in Y$. Since $L_\infty[0,1]$ is 1-injective (see, e.g., [FHHMZ, Exercise 5.91]), the operator can be extended to a norm one operator $S\colon X \to L_\infty[0,1]$.

Define

$$|||x||| = \|Sx\| + \|[x]\|_{X/Y} \quad \text{for all } x \in X.$$

We will show that $|||\cdot|||$ is an equivalent norm on X such that $T(X) = t(X) = 2$ with respect to $|||\cdot|||$.

Let us first show that $\frac{1}{3}\|x\| \leq |||x||| \leq 2\|x\|$ for all $x \in X$. For the upper estimate we have

$$|||x||| = \|Sx\| + \|[x]\|_{X/Y} \leq \|Sx\| + \|x\| \leq 2\|x\|$$

for all $x \in X$.

For the lower estimate, fix $x \in X$ with $\|x\| = 1$. If $\|[x]\|_{X/Y} \geq 1/3$, there is nothing to prove. If $\|[x]\|_{X/Y} < 1/3$, then there is a $y \in Y$ such that $\|x - y\| < 1/3$. Thus $\|y\| \geq \|x\| - \|x - y\| > 2/3$. We now have that

$$|||x||| \geq \|Sx\| \geq \|Sy\| - \|S(x-y)\| \geq \|y\| - \|x - y\| > \frac{1}{3}.$$

Let us now show that $T(X) = 2$. By Proposition 5.1, it suffices to show that X is OH. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ with $|||x_i||| = 1$, and let $\varepsilon > 0$. Find an $N \in \mathbb{N}$ such that $\|Sx_i + r_N\| \geq \|Sx_i\| + 1 - \varepsilon$ for all $i \in \{1, \ldots, n\}$. We have that

$$\begin{aligned} |||x_i + e_N||| &= \|S(x_i + e_N)\| + \|[x_i + e_N]\|_{X/Y} \\ &= \|Sx_i + r_N\| + \|[x_i]\|_{X/Y} \\ &\geq \|Sx_i\| + 1 - \varepsilon + \|[x_i]\|_{X/Y} \\ &= |||x_i||| + 1 - \varepsilon = 2 - \varepsilon \end{aligned}$$

for all $i \in \{1, \ldots, n\}$. Thus X is OH by Proposition 3.3.

Finally, let us show that $t(X) = 2$. Let $x \in X$ with $|||x||| = 1$. It suffices to see that $\max_\pm |||x \pm e_1||| = 2$. Indeed,

$$\begin{aligned} \max_\pm |||x \pm e_1||| &= \max_\pm \left(\|S(x \pm e_1)\| + \|[x \pm e_1]\|_{X/Y} \right) \\ &= \max_\pm \|Sx \pm r_1\| + \|[x]\|_{X/Y} \\ &= \|Sx\| + 1 + \|[x]\|_{X/Y} \\ &= |||x||| + 1 = 2. \end{aligned}$$

Thus, by the definition of the thinness index, $t(X) = 2$. □

5.2 Stability results of thickness and thinness

We start this section by studying the behaviour of the thickness and thinness indices when forming c_0-sums and ℓ_p-sums of Banach spaces.

Next we address the problem of the relation between the thickness and thinness indices of the space and a subspace. Our generalization of the observation $T(X) \geq T(X^{**})$ is that the thickness of an ai-ideal is at least as big as the thickness of the space itself. On the other hand, the thinness of an ai-ideal cannot exceed the thinness of the space itself.

It is well-known that a Banach space is a Lindenstrauss space if and only if it is an ideal in every superspace. Some spaces are even ai-ideals in every superspace; these spaces are exactly the *Gurariĭ spaces* (see [ALN2, Theorem 4.3]). Being an ai-ideal in every superspace will imply that every Gurariĭ space has thickness index 2 and thinness index 1.

Let us first observe that given a sequence of nontrivial Banach spaces (X_k) the c_0-sum $(X_1 \oplus X_2 \oplus \ldots)_{c_0}$ has thinness 1.

Proposition 5.9 (see [ALLN, Lemma 2.2])**.** *If (X_k) is a sequence of nontrivial Banach spaces, then $t((X_1 \oplus X_2 \oplus \ldots)_{c_0}) = 1$.*

Proof. From Example 4.2 we know that the c_0-sum $(X_1 \oplus X_2 \oplus \ldots)_{c_0}$ is always ASQ. Thus, by Proposition 5.4, $t((X_1 \oplus X_2 \oplus \ldots)_{c_0}) = 1$. □

Remark 5.4. The thinness of a subspace may be greater than the thinness of the space itself. For example, $t(\ell_1) = 2$, however $t(c_0(\ell_1)) = 1$ by Proposition 5.9.

With a little more work, we derive the corresponding result on the thickness index.

Proposition 5.10 (see [ALLN, Lemma 2.3])**.** *If (X_k) is a sequence of nontrivial Banach spaces, then $T((X_1 \oplus X_2 \oplus \ldots)_{c_0}) = \inf_k T(X_k)$.*

Proof. The proof follows the idea in the proof of [CPS, Theorem 2, (3)].

Denote by $Z = (X_1 \oplus X_2 \oplus \ldots)_{c_0}$. First we show that $T(Z) \leq \inf_k T(X_k)$. Fix $\varepsilon > 0$. Choose $j \in \mathbb{N}$ such that $T(X_j) \leq \inf_k T(X_k) + \varepsilon$. Find $x_1, \ldots, x_n$ in S_{X_j} such that

$$B_{X_j} \subset \bigcup_{i=1}^{n} B(x_i, T(X_j) + \varepsilon).$$

Then

$$B_Z \subset \bigcup_{i=1}^{n} B_Z((0,0,\ldots,x_i,0,\ldots),T(X_j)+2\varepsilon)$$

Thus $T(Z) \leq \inf_k T(X_k)$.

Now we will show that $T(Z) \geq \inf_k T(X_k)$. Suppose to the contrary that $T(Z) < \inf_k T(X_k)$. Then we can find $\alpha > 0$ and $\varepsilon > 0$ such that

$$T(Z) < \alpha - \varepsilon < \alpha < \alpha + \varepsilon < \inf_k T(x_k).$$

By the definiton of the thickness index, there exists an α-net $\{z_1,\ldots,z_m\}$ in S_Z for B_Z. For every k define

$$I_k = \{i \in \{1,\ldots,m\} : \|z_i(k)\|_{X_k}= 1\}.$$

Note that for each i with $1 \leq i \leq m$ there is a k with $i \in I_k$.

If $I_k = \emptyset$, then let $x_k = 0$. If $I_k \neq \emptyset$, then we know that $\{z_i(k)\}_{i\in I_k}$ is not an $(\alpha+\varepsilon)$-net for B_{X_k}, and thus there exists $x_k \in B_{X_k}$ such that

$$\|z_i(k) - x_k\|> \alpha + \varepsilon$$

for all $i \in I_k$. Define $z = (x_k) \in B_Z$. Then

$$\|z_i - z\|= \max\|z_i(k) - z(k)\|> \alpha + \varepsilon > \alpha$$

for all $i \in \{1,\ldots,m\}$, which gives us a contradiction.

Thus $T(Z) = \inf_k T(X_k)$. □

Remark 5.5. Observe that Proposition 5.10 implies that there is in fact an equality in [BJ, Proposition 2.14, (1)].

Proposition 5.11 (see [ALLN, Proposition 2.4]). *For every $\alpha \in [1,2]$, there is a Banach space X with $T(X) = \alpha$ while $t(X) = 1$ and $T(X^*) = 2$.*

Proof. The statement "and $T(X^*) = 2$" is Corollary 5.5. From Whitley's paper ([Whi, Lemma 4]) we know that $T(\ell_p) = 2^{1/p}$ for every p with $1 \leq p < \infty$. From Proposition 5.10, we get that also $T(c_0(\ell_p)) = 2^{1/p}$. From Proposition 5.9, we know that $t(c_0(\ell_p)) = 1$. Thus the result has been proved for all $\alpha \in (1,2]$. For $\alpha = 1$ consider $X = c_0$. Indeed, from [Whi], we know that $T(c_0) = 1$ and, by Example 4.2, we have that $t(c_0) = 1$. □

It is clear that $t(X) > 1$ and $T(X) < 2$ for all reflexive Banach spaces X, this follows from Propositions 5.1 and 5.4. The next proposition shows that all other possible values of $t(X)$ and $T(X)$ are covered by infinite-dimensional reflexive spaces.

Proposition 5.12 (see [ALLN, Proposition 2.5]). *For every $\alpha \in [1, 2)$, there is an infinite-dimensional reflexive Banach space X with $T(X) = \alpha$, and for every $\alpha \in (1, 2]$, there is an infinite-dimensional reflexive Banach space X with $t(X) = \alpha$.*

Proof. R. Whitley showed that $T(\ell_p) = 2^{1/p} = t(\ell_p)$ for every p with $1 < p < \infty$, and this covers the interval $(1, 2)$.

Let Y be any infinite-dimensional reflexive Banach space. If we let $X = Y \oplus_\infty \mathbb{R}$, then it follows easily from the proof of [CPS, Lemma 3] that $T(X) = 1$. Thus X is a reflexive Banach space with $T(X) = 1$.

On the other hand, if we let $X = Y \oplus_1 \mathbb{R}$, then $t(X) = 2$ by Corollary 5.14 below, since $t(\mathbb{R}) = 2$. Thus X is a reflexive Banach space with $t(X) = 2$. $\square$

For ℓ_p-sums we have the following result.

Proposition 5.13 (see [ALLN, Proposition 2.6]). *Let X and Y be Banach spaces and let p be such that $1 \leq p < \infty$. Then $t(Z) \geq ((t(X) - 1)^p + 1)^{1/p}$, where $Z = X \oplus_p Y$.*

Proof. In Proposition 4.22, we saw that for $1 \leq p < \infty$, $X \oplus_p Y$ is never ASQ, i.e. $t(X \oplus_p Y) > 1$. Since $((t(X) - 1)^p + 1)^{1/p} = 1$ when $t(X) = 1$ we may assume that $t(X) > 1$.

Fix an α with $\alpha \in (1, t(X))$. There exist $x_1, \ldots, x_n$ in S_X and $\varepsilon > 0$ such that $\max_i \|x - x_i\| \geq \alpha + \varepsilon$ for all $x \in S_X$. Define $z_i = (x_i, 0)$ for all $i \in \{1, \ldots, n\}$.

Let $z = (x, y) \in S_Z$. If $x = 0$, then it is clear that $\max_i \|z_i - z\|^p = 2 \geq (\alpha + \varepsilon - 1)^p + 1$. If $x \neq 0$, then

$$\begin{aligned}
\max_{1 \leq i \leq n} \|z_i - z\|^p &= \max_{1 \leq i \leq n} \|x_i - x\|^p + \|y\|^p \\
&\geq \Big(\max_{1 \leq i \leq n} \|x_i - \frac{x}{\|x\|}\| - \|\frac{x}{\|x\|} - x\| \Big)^p + \|y\|^p \\
&\geq (\alpha + \varepsilon - 1 + \|x\|)^p + \|y\|^p \\
&\geq (\alpha + \varepsilon - 1)^p + \|x\|^p + \|y\|^p \\
&= (\alpha + \varepsilon - 1)^p + 1.
\end{aligned}$$

Since $\alpha < t(X)$ and $z \in S_Z$ are arbitrary, we obtain $t(Z) \geq ((t(X)-1)^p + 1)^{1/p}$. ☐

Remark 5.6. As a general lower bound this is the best possible since $t(\ell_p \oplus_p X) = 2^{1/p}$, by [BJ, Proposition 4.3], for any space with $t(X) = 2$, for example, $X = \ell_1$. On the other hand, Proposition 5.13 gives us that $t(\ell_1 \oplus_2 \ell_1) \geq \sqrt{2}$, however $t(\ell_1 \oplus_2 \ell_1) = 2$ (see [ALLN, Proposition 2.9]).

Corollary 5.14 (see [ALLN, Corollary 2.7])**.**

(a) *If X and Y are Banach spaces, then $t(X \oplus_1 Y) \geq \max\{t(X), t(Y)\}$.*

(b) *Let (X_k) be a sequence of nontrivial Banach spaces and $Z = (X_1 \oplus X_2 \oplus \dots)_{\ell_p}$. Then $t(Z) \geq \sup_k((t(X_k)-1)^p + 1)^{1/p}$. Moreover, if $\sup_k t(X_k) = 2$, then $t(Z) = 2^{1/p}$.*

Proof. It is clear that (a) holds. For the moreover part in (b), it suffices to observe that the upper bound is proved in [BJ, Lemma 4.1]. ☐

Remark 5.7. Note that there appears to be a misprint in [BJ, Lemma 4.1]. The authors state that $t((X_1 \oplus X_2 \oplus \dots)_{\ell_\infty}) = 1$, but with $X_k = \mathbb{R}$, we have $t(\ell_\infty) = 2$ (see [Whi, Lemma 8]).

Proposition 5.15 (see [ALLN, Proposition 2.8])**.** *Let X and Y be a Banach spaces. Then $t(X \oplus_\infty Y) = \min\{t(X), t(Y)\}$.*

Proof. Let α and β be such that $\alpha < t(X)$ and $\beta < t(Y)$. Then there exist $x_1, \dots, x_n$ in S_X, $y_1, \dots, y_m$ in S_Y and $\varepsilon > 0$ such that $\max_i \|x_i - x\| \geq \alpha + \varepsilon$ for all $x \in S_X$ and $\max_j \|y_j - y\| \geq \beta + \varepsilon$ for all $y \in S_Y$.

Without loss of generality we may assume that $m = n$ by just repeating some vectors. Define $z_i = (x_i, y_i)$ for all $i \in \{1, \dots, n\}$. Let $z = (x, y) \in X \oplus_\infty Y$ with $\|z\| = 1$. Then either $\|x\| = 1$ or $\|y\| = 1$ and hence

$$\max_{1 \leq i \leq n} \|z_i - z\| = \max_{1 \leq i \leq n} \{\|x_i - x\|, \|y_i - y\|\} \geq \min\{\alpha, \beta\} + \varepsilon.$$

Thus $t(X \oplus_\infty Y) \geq \min\{t(X), t(Y)\}$.

Observe that the following holds in every Banach space: If two norm one elements x and x' are such that $\|x - x'\| < a$, where $a \geq 1$, then for all $0 \leq r \leq 1$ we have $\|rx - x'\| < a$. Indeed,

$$\|rx - x'\| = \|rx - rx' + rx' - x'\| \leq r\|x - x'\| + 1 - r < a.$$

Now, suppose that $\min\{t(X), t(Y)\} = t(X)$ and let $\varepsilon > 0$. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a finite number of elements in the unit sphere of $X \oplus_\infty Y$. Let $u_i = x_i/\|x_i\|$, if $x_i \neq 0$. Then there is an element $x \in S_X$ such that $\max_i \|u_i - x\| < t(X) + \varepsilon$. Consider the element $(x, 0)$ from the unit sphere of $X \oplus_\infty Y$. By the observation above, we get

$$\max_{1 \leq i \leq n} \|(x_i, y_i) - (x, 0)\| = \max_{1 \leq i \leq n} \{\| \|x_i\| u_i - x\|, \|y_i\|\} < t(X) + \varepsilon.$$

Finally, if $x_i = 0$ for every i, then for any $x \in S_X$ we have

$$\|(0, y_i) - (x, 0)\| = 1 \leq t(X).$$

□

The behavior of Whitley's thickness index with respect to ℓ_p-sums has been recently studied in [BJ] and [CPS]. If X and Y are Banach spaces and p is such that $1 < p < \infty$, then $T(X \oplus_p Y) \leq \max\{T(X), T(Y)\}$ (see [CPS, Theorem 2]), and for finite-dimensional X one has $T(X \oplus_p Y) \leq 2^{1/p}$ (see [BJ, Lemma 2.3]). We have the following estimation for $X \oplus_p Y$.

Proposition 5.16 (see [HL2, Proposition 2.7]). *If X and Y are Banach spaces and $1 < p < \infty$, then*

$$T(X \oplus_p Y) \leq \left(\frac{(2^{1/p} + 1)^p + 1}{2} \right)^{1/p}.$$

Thus

$$T(X \oplus_2 Y) \leq \sqrt{2 + \sqrt{2}}.$$

Remark 5.8. This estimation is sharp since $T(\ell_1 \oplus_2 \ell_1) = \sqrt{2 + \sqrt{2}}$ (see [CPS, Lemma 2]). On the other hand, $T(\ell_p \oplus_p Y) = 2^{1/p}$ (see [CPS, Proposition 1]) which is strictly less than our estimation shows.

Proof. Denote by $Z = X \oplus_p Y$ and $r = \left(\frac{(2^{1/p}+1)^p+1}{2} \right)^{1/p}$. Fix arbitrarily $x_1 \in S_X$ and $y_1 \in S_Y$. We will show that

$$S_Z \subset B((x_1, 0), r) \cup B((0, y_1), r).$$

Let $(x, y) \in S_Z$ and set

$$m = \min\{\|(x_1, 0) - (x, y)\|_p, \|(0, y_1) - (x, y)\|_p\}.$$

Then

$$\begin{aligned} m^p &= \min\{\|x_1 - x\|^p + \|y\|^p, \|x\|^p + \|y_1 - y\|^p\} \\ &\le \frac{\|x_1 - x\|^p + \|y\|^p + \|x\|^p + \|y_1 - y\|^p}{2} \\ &= \frac{\|(x_1 - x, y_1 - y)\|^p + 1}{2}. \end{aligned}$$

Since

$$\|(x_1 - x, y_1 - y)\| \le \|(x_1, y_1)\| + \|(x, y)\| = 2^{1/p} + 1,$$

we get

$$m^p \le \frac{(2^{1/p} + 1)^p + 1}{2}.$$

□

Recall that $T(X^{**}) \le T(X)$. This inequality may be strict. For example, $T(C[0,1]) = 2$ while $T(C[0,1]^{**}) = 1$. Indeed, $T(C[0,1]) = 2$ by Theorem 3.6 or by [Whi, Lemma 3]. Since $C[0,1]^{**}$ can be viewed as a $C(K)$ space (see, e.g., [AK, Theorems 4.3.7 and 4.3.8]), we have that $T(C[0,1]^{**})$ is either 1 or 2 (see [Whi, Lemma 3]). By Example 2.12, we know that $C[0,1]^{**}$ is not OH, thus $T(C[0,1]^{**}) \ne 2$. Therefore $T(C[0,1]^{**}) = 1$. Note that this answers a question in [CP] whether we always have $T(X) = T(X^{**})$. We will now put these observations into a broader perspective. Since a Banach space X is always an ai-ideal in X^{**}, the observation above that $T(X^{**}) \le T(X)$ is a very particular case of the following proposition.

Proposition 5.17 (see [ALLN, Proposition 3.1])**.** *Let X be a Banach space. If Y is an ai-ideal in X, then $T(X) \le T(Y)$ and $t(Y) \le t(X)$.*

Proof. Let Y be an ai-ideal in X. Let $y_1, \dots, y_n$ be a finite r-net for S_Y in S_Y for some $r > 0$. Let $\varepsilon > 0$. Let $x \in S_X$ and $E = \operatorname{span}\{x, y_1, \dots, y_n\}$. Find an ε-isometry $T : E \to Y$ such that $Ty = y$ for all $y \in E \cap Y$. Let $z = Tx/\|Tx\|$. Then $\|Tx - z\| \le \varepsilon$ since $(1+\varepsilon)^{-1} \le \|Tx\| \le 1 + \varepsilon$. Now find j such that $\|y_j - z\| \le r$, then

$$\begin{aligned} \|x - y_j\| &\le (1+\varepsilon)\|Tx - Ty_j\| \\ &= (1+\varepsilon)\|Tx - y_j\| \\ &\le (1+\varepsilon)(\|Tx - z\| + \|y_j - z\|) \\ &\le (1+\varepsilon)(r + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $T(X) \le T(Y)$.

For $t(Y) \leq t(X)$, let $y_1, \dots, y_n \in S_Y$ and let $\varepsilon > 0$. Find $x \in S_X$ such that $\max_i \|x - y_i\| < t(X) + \varepsilon$. Let $x \in S_X$ and $E = \operatorname{span}\{x, y_1, \dots, y_n\}$. Find an ε-isometry $T : E \to Y$ such that $Ty = y$ for all $y \in E \cap Y$. Let $z = Tx/\|Tx\|$. Then, as above, $\|Tx - z\| \leq \varepsilon$. Now

$$\begin{aligned}\|y_i - z\| &\leq \|Tx - y_i\| + \varepsilon \\ &= \|Tx - Ty_i\| + \varepsilon \\ &\leq (1+\varepsilon)\|x - y_i\| + \varepsilon \\ &\leq (1+\varepsilon)(t(X) + \varepsilon)\end{aligned}$$

for all $i \in \{1, \dots, n\}$. Since $\varepsilon > 0$ is arbitrary, we have shown that $t(Y) \leq t(X)$. □

Remark 5.9. An ai-ideal may well have strictly less thinness than its super-space. Note that $t(c_0) = 1$ while $t(\ell_\infty) = 2$ (see [Whi]), so we have that $t(c_0) < t(c_0^{**})$.

Proposition 5.17 will turn out to provide us with a class of spaces which are both OH and ASQ at the same time, namely the Gurariĭ spaces. Selected known results of Gurariĭ spaces can be found in [GK]. We will, however, use the alternative description of Gurariĭ spaces (see [ALN2, Theorem 4.3]): The Gurariĭ spaces is exactly the class of Banach spaces with the property that they form an ai-ideal in every super-space.

Proposition 5.18 (see [ALLN, Proposition 3.2]). *Let X be a Banach space. If X is a Gurariĭ space, then $T(X) = 2$ and $t(X) = 1$, that is, X is OH and ASQ.*

Proof. Let X be a Gurariĭ space. Then, by [ALN2, Theorem 4.3], X is an ai-ideal in any super-space. Since every Banach space is isometrically isomorphic to a closed subspace of $Z = C(B_{X^*}, w^*)$ we have that X is an ai-ideal in Z. Note that (B_{X^*}, w^*) does not have any isolated points. Thus, by [Whi, Lemma 3] and Proposition 5.17 above, $T(X) \geq T(Z) = 2$.

Finally, let us show that $t(X) = 1$. Note that X is isometrically isomorphic to $(X \oplus \{0\} \oplus \{0\} \dots)_{c_0}$ and since X is a Gurariĭ space, by [ALN2, Theorem 4.3], it is an ai-ideal in $Z = c_0(X)$. By Example 4.2, we know that $t(Z) = 1$. Thus, by Proposition 5.17, $t(X) \leq t(Z) = 1$. □

Note that all Gurariĭ spaces are Lindenstrauss spaces. Lindenstrauss proves in his memoir (see [Lin, Theorem 6.1]) that when a Banach space X is such that $X^* = L_1(\mu)$ (i.e. X is a Lindenstrauss space) and B_X has extreme points, then X is isometric to a subspace of some $C(K)$ space that contains the constant one function. It follows that $t(X) = 2$. Thus we have derived the following result.

Proposition 5.19 (see [ALLN, Proposition 3.3]). *If X is a Gurariĭ space, then* $\text{ext}(B_X) = \emptyset$.

Proposition 5.20 (see [ALLN, Proposition 3.4]). *For every $\alpha \in [1, 2]$, there is a Lindenstrauss space X with $t(X) = \alpha$. For any Lindenstrauss space X we have $T(X^*) = 2$ and $t(X^{**}) = 2$.*

Proof. Let us first show that every Lindenstrauss space X has $T(X^*) = 2$ and $t(X^{**}) = 2$. By [ALN2, Proposition 4.6], X has the SD2P, thus $T(X^*) = 2$ by Theorem 3.22. Because X^{**} is also a Lindenstrauss space and $\text{ext}(B_{X^{**}}) \neq \emptyset$, we have $t(X^{**}) = 2$. Thus, if X is a Lindenstrauss space, then $T(X^*) = 2$ and $t(X^{**}) = 2$.

Let us show that, for every $\alpha \in [1, 2]$, there is a Lindenstrauss space X with $t(X) = \alpha$. For $\alpha = 1$, by Proposition 5.18, we can take X to be any Gurariĭ space. For $\alpha = 2$, we can take $X = C[0, 1]$.

We now consider the case where $\alpha \in (1, 2)$. For this, let $r > 1$ and let $X_r = \{f \in C[0, 1] : f(0) = rf(1)\}$. Then the space X_r is a Lindenstrauss space (see, e.g., [HWW, p. 83]). We are going to show that $t(X_r) = 1 + 1/r$. Note that for all $f \in B_{X_r}$, we have $|f(1)| \leq 1/r$.

To see that $t(X_r) \geq 1 + 1/r$, consider $f_1, f_2 \in S_{X_r}$, where $f_1(t) = (1 - t) + t/r$ and $f_2(t) = -f_1(t)$. If $g \in S_{X_r}$, then $|g(s_0)| = 1$ for some $s_0 \in [0, 1]$. Then $1/r + 1 \leq \max_i |f_i(s_0) - g(s_0)| \leq \max_i \|f_i - g\|$. Hence $t(X_r) \geq 1 + 1/r$.

To see that $t(X_r) \leq 1 + 1/r$, let $f_1, \ldots, f_n \in S_{X_r}$ and let $\varepsilon > 0$. Find an interval $(s, 1)$, where $|f_i(t)| < 1/r + \varepsilon$ for all $i \in \{1, \ldots, n\}$. Now choose any $g \in S_{X_r}$ with support on $(s, 1)$. Then $\|f_i - g\| < 1 + 1/r + \varepsilon$ for all $i \in \{1, \ldots, n\}$, hence $t(X) \leq 1 + 1/r$. $\square$

Appendix

In the following two tables, we summarize some important examples and stability results obtained in the thesis.

Examples

	c_0	Example 2.10 $c_0 \oplus_2 c_0$	ℓ_1	Example 3.14 $\ell_1 \oplus_2 \ell_1$	ℓ_∞	$C[0,1]$	$L_1[0,1]$	Example 4.3 $c_0(L_1[0,1])$	Example 4.23 $c_0(L_1[0,1]) \oplus_\infty \ell_1$
LD2P	yes	yes	no	no	yes	yes	yes	yes	yes
D2P	yes	yes	no	no	yes	yes	yes	yes	yes
SD2P	yes	no	no	no	yes	yes	yes	yes	yes
LASQ	yes	yes	no	no	no	no	yes	yes	yes
WASQ	yes	yes	no	no	no	no	yes	yes	yes
ASQ	yes	no	no	no	no	no	no	yes	yes
LOH	no	no	yes	yes	no	yes	yes	yes	yes
WOH	no	no	yes	yes	no	yes	yes	yes	yes
OH	no	no	yes	no	no	yes	yes	yes	yes
DP	no	no	no	no	no	yes	yes	yes	no
$t(X)$	1	> 1	2	2	2	2	2	1	1
$T(X)$	1	1	2	$\sqrt{2+\sqrt{2}}$	1	2	2	2	2

Stability on ℓ_p-sums

$X \oplus_p Y$	X, Y	p	Related reference
LD2P	X and Y LD2P	$1 \leq p < \infty$	Theorem 2.29
	X or Y LD2P	$p = \infty$	
D2P	X and Y D2P	$1 \leq p < \infty$	Theorem 2.31
	X or Y D2P	$p = \infty$	
SD2P	X and Y SD2P	$p = 1$	Theorem 2.32
	X or Y SD2P	$p = \infty$	
LASQ	X and Y LASQ	$1 \leq p < \infty$	Proposition 4.20
	X or Y LASQ	$p = \infty$	
WASQ	X and Y WASQ	$1 \leq p < \infty$	Proposition 4.21
	X or Y WASQ	$p = \infty$	
ASQ	X or Y ASQ	$p = \infty$	Proposition 4.22
LOH	X or Y LOH	$p = 1$	Proposition 3.28
	X and Y LOH	$1 < p \leq \infty$	
WOH	X or Y WOH	$p = 1$	Proposition 3.30
	X and Y WOH	$1 < p \leq \infty$	
OH	X or Y OH	$p = 1$	Proposition 3.32
	X and Y OH	$p = \infty$	
DP	X and Y DP	$p = 1$	[KSSW]
	X and Y DP	$p = \infty$	
$t(X \oplus_p Y)$	$\geq ((\max\{t(X), t(Y)\} - 1)^p + 1)^{1/p}$	$1 \leq p < \infty$	Proposition 5.13
	$= \min\{t(X), t(Y)\}$	$p = \infty$	Proposition 5.15
$T(X \oplus_p Y)$	$\leq (((2^{1/p} + 1)^p + 1)/2)^{1/p}$	$1 \leq p < \infty$	Proposition 5.16
	$= \min\{T(X), T(Y)\}$	$p = \infty$	[CPS]

From the table above, for example, it reads $X \oplus_p Y$ has the LD2P whenever X and Y have the LD2P and p is such that $1 \leq p < \infty$.

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Diameeter-2 omadusega Banachi ruumide geomeetriline struktuur

Kokkuvõte

2001. aastal näitasid O. Nygaard ja D. Werner, et mis tahes lõpmatumõõtmelises ühtlases algebras on ühikkera iga mittetühja suhteliselt nõrgalt lahtise alamhulga diameeter kaks. Kui Banachi ruumil on selline omadus, siis öeldakse, et tal on *diameeter-2 omadus* (vt. [ABGRP], [ALN1] ja [BGRP]). Diameeter-2 omadusega on näiteks Daugaveti omadusega Banachi ruumid (vt. [Shv]), lõpmatumõõtmelised C^*-algebrad (vt [BGLPRP]) ja mitterefleksiivsed Banachi ruumid, mis on M-ideaalid oma teises kaasruumis (vt. [LP]).

Suhteliselt nõrgalt lahtise alamhulga erijuhuks on viil, kusjuures on teada, et ühikkera iga mittetühi suhteliselt nõrgalt lahtine alamhulk sisaldab teatud viilude kumerat kombinatsiooni. Seda asjaolu silmas pidades vaatlevad T. A. Abrahamsen, V. Lima ja O. Nygaard artiklis [ALN1] diameeter-2 omaduse kõrval selle kahte erinevat versiooni – *tugevat diameeter-2 omadust* ja *lokaalset diameeter-2 omadust*. Klassikalistest Banachi ruumidest on tugeva diameeter-2 omadusega näiteks c_0, ℓ_∞, $C[0,1]$, $L_1[0,1]$ ja $L_\infty[0,1]$, aga näiteks $c_0 \oplus_2 c_0$ on diameeter-2 omadusega, kuid tal ei ole tugevat diameeter-2 omadust.

Käesoleva väitekirja põhieesmärk on uurida diameeter-2 omadusi ning nendega seotud mõisteid ja omadusi, nagu näiteks oktaeedrilised normid, peaaegu ruudu omadusega Banachi ruumid ning tihkuse ja peenuse indeks. Töös näidatakse, kuidas diameeter-2 omadused on duaalselt seotud normi oktaeedrilisusega. Peaegu ruudu omadusega Banachi ruumid on ka tugeva diameeter-2 omadusega, seega nende kaasruum on oktaeedriline. Uuritakse R. Whitley poolt sisse toodud Banachi ruumi tihkuse ja peenuse indeksit. Osutub, et Banachi ruum on oktaeedriline parajasti siis, kui tema tihkuse indeks on kaks ja Banachi ruum on peaaegu ruudu omadusega parajasti siis, kui tema peenuse indeks on üks.

Väitekiri koosneb viiest peatükist ja ühest lisast. Esimeses peatükis tutvustatakse lühidalt diameeter-2 omaduste ajaloolist tausta, esitatakse väitekirja ülevaade ja kirjeldatakse töös kasutatavaid tähistusi.

Teises peatükis tuuakse sisse kolm diameeter-2 omadust ja selgitatakse nende ekstremaalset erinevust. Vaadeldakse kaasruumi $*$-nõrkasid diameeter-2 omadusi ja näidatakse, et üldiselt erinevad need tavalistest diameeter-2 omadustest. Antakse lühiülevaade Banachi ruumide klassidest, millel on diameeter-2 omadus. T. A. Abrahamsen, V. Lima ja O. Nygaard näitasid, et (lokaalne) diameeter-2 omadus kandub liidetavatelt üle ℓ_p-summale iga $1 \leq p \leq \infty$ korral (vt. [ALN1]). Teisalt, kui $1 < p < \infty$, siis Banachi ruumide ℓ_p-summal ei ole kunagi tugevat diameeter-2 omadust. Kõik kolm diameeter-2 omadust kanduvad alamruumilt, mis on M-ideaal üle kogu ruumile. See peatükk tugineb magistritööle [Lan] (vt. ka [HL1]).

Väitekirja kolmandas peatükis uurime oktaeedrilise normiga Banachi ruume. Varasemast on teada, et Banachi ruumil on tugev diameeter-2 omadus parajasti siis, kui tema kaasruumi norm on oktaeedriline. Vaadeldakse oktaeedrilisuse kõrval selle kahte erinevat versiooni, mis vastavad diameeter-2 omadusele ja lokaalsele diameeter-2 omadusele. Uuritakse oktaeedrilisuse ülekandumist ℓ_p-summadele ja saadud stabiilsustulemusi rakendatakse diameeter-2 omaduste uurimisel. Antakse nii tarvilikke kui ka piisavaid tingimusi operaatorite ruumi oktaeedrilisuseks. See peatükk põhineb peamiselt artiklil [HLP].

Neljandas peatükis tuuakse sisse peaaegu ruudu mõiste. Kui Banachi ruum on peaegu ruudu omadusega, siis tal on tugev diameeter-2 omadus. Esitatakse peaaegu ruudu omadusega ruumide näiteid ja kirjeldusi. Näidatakse, et mitterefleksiivsed Banachi ruumid, mis on M-ideaalid oma teises kaasruumis, on peaaegu ruudu omadusega. Tõestatakse, et iga Banachi ruumi, milles c_0 on täiendatav, on võimalik ekvivalentselt ümber normeerida nii, et tal on peaaegu ruudu omadus. Uuritakse peaaegu ruudu omaduse lokaalset ja nõrka versiooni. See peatükk tugineb artiklile [ALL].

Väitekirja viiendas peatükis täiendatakse mõningaid hiljutisi tulemusi Whitley' tihkuse ja peenuse indeksi kohta. Osutub, et Banachi ruum on oktaeedriline parajasti siis, kui tema tihkus on kaks ja Banachi ruum on peaaegu ruudu omadusega parajasti siis, kui tema peenus on üks. Näidatakse, et need indeksid käituvad ℓ_p-summadel erinevalt. Uuritakse Banachi ruumi ja tema alamruumi tihkuse (peenuse) omavahelist seost. Tõestatakse, et iga Banachi ruumi, milles c_0 on täiendatav, on võimalik ekvivalentselt ümber normeerida nii, et tema tihkus ja peenus on mõlemad võrdsed ühega. See peatükk põhineb artiklitel [ALLN] ja [HL2].

Väitekirja lisa kahte tabelisse on koondatud eelmainitud omaduste olulisemad näited ja stabiilsustulemused.

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