

UNIVERSITY OF TARTU
FACULTY OF SCIENCE AND TECHNOLOGY
INSTITUTE OF MATHEMATICS AND STATISTICS

Nikolai Sovetnikov
Transposed Poisson 3-Lie Superalgebras

Mathematics and Statistics Curriculum

Mathematics

Master's thesis (30 ECTS)

Supervisor: prof. emeritus Viktor Abramov

TARTU 2026

TRANSPONEERITUD POISSONI 3-LIE SUPERALGEBRAD

Magistritöö

Nikolai Sovetnikov

Lühikokkuvõte

Käesolev magistritöö on pühendatud Lie algebrate ja Poissoni algebrate mõiste üldistustele. Töös uuritakse transponeeritud Poissoni superalgebraid ja nende seost 3-Lie superalgebratega. Sisse tuuakse ja uuritakse transponeeritud 3-Lie Poissoni superalgebra mõistet. Esitatakse meetod transponeeritud 3-Lie Poissoni superalgebrate konstrueerimiseks transponeeritud Poissoni superalgebrate paarisderivatsioonide abil. Selle meetodi abil konstrueeritakse lai klass transponeeritud 3-Lie Poissoni superalgebraid kommutatiivse superalgebra ning selle kahe kommuteeruva paarisderivatsiooni korral.

CERCS teaduseriala: P120 Arvuteooria, väljateooria, algebraline geomeetria, algebra, rühmateooria. P150 Geomeetria, algebraline topoloogia.

Märksõnad: Superalgebra, Lie algebra, Poissoni algebra, algebra derivatsioon, transponeeritud Poissoni algebra, 3-Lie algebra, transponeeritud Poissoni superalgebra, 3-Lie superalgebra.

TRANSPOSED POISSON 3-LIE SUPERALGEBRAS

Master thesis

Nikolai Sovetnikov

Abstract

This master thesis is devoted to generalizations of the concepts of Lie algebras and Poisson algebras. The thesis studies transposed Poisson superalgebras and their connection with 3-Lie superalgebras. The notion of a transposed 3-Lie Poisson superalgebra is introduced and investigated. The thesis proposes

a method for constructing transposed 3-Lie Poisson superalgebras by means of even derivations of transposed Poisson superalgebras. Using this method, a wide class of transposed 3-Lie Poisson superalgebras is constructed for the case of a commutative superalgebra and its two commuting even derivations.

CERCS research specialisation: P120 Number theory, field theory, algebraic geometry, algebra, group theory. P150 Geometry, algebraic topology.

Key Words: Superalgebra, Lie algebra, Poisson algebra, derivation of algebra, transposed Poisson algebra, 3-Lie algebra, transposed Poisson superalgebra, 3-Lie superalgebra.

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Introduction

Lie algebras play an important role in many areas of modern geometry and theoretical physics. Lie algebras and their various generalizations continue to be actively studied. One of the directions of development in this area is the generalization of Lie algebras to the case of an n -ary Lie bracket, where n is an integer and $n > 2$. In the paper [10], the definition of an n -Lie algebra was first introduced and studied. At the core of this definition lies a generalization of the Jacobi identity, which was later termed the Filippov–Jacobi identity. Subsequently, such generalizations of Lie algebras gained wide popularity.

Lie algebras are widely used in theoretical mechanics. In theoretical mechanics, a Lie algebra is constructed by means of the Poisson bracket, which is defined on the algebra of smooth functions [12]. Moreover, the Poisson bracket satisfies the Leibniz rule with respect to the multiplication of functions. Thus, a Poisson algebra is a vector space with two operations, one of which is associative and commutative pointwise multiplication of functions, while the other is a Lie bracket. These two operations are related by a compatibility condition, which can be expressed as follows: the Lie bracket is a derivation of the commutative associative multiplication of functions. Since in this work we consider structures with several operations, we will use the notation (V, φ, ψ) to denote algebraic structures, where V denotes a vector space and φ, ψ are operations on it. Poisson algebras are extremely important in geometry and theoretical physics, and their various generalizations are actively studied [15]. The n -ary generalization of a Poisson algebra is called a Nambu–Poisson manifold, and this concept was introduced in [22].

Recently, a new concept of a transposed Poisson algebra was proposed [4]. A transposed Poisson algebra is, in a certain sense, a dual structure to a Poisson algebra. A transposed Poisson algebra is also a vector space with two operations, one of which is associative and commutative, and the other is a Lie bracket. However, in a transposed Poisson algebra, the roles of these two operations are reversed:

the commutative associative operation is a derivation of the Lie bracket. That is, the compatibility condition of the two operations in a transposed Poisson algebra has a different form than in the case of a Poisson algebra. The paper [4] presents many similarities between these two structures as well as their main differences. Moreover, the same paper shows a close connection between the new structure of a transposed Poisson algebra and already known algebras widely used in various areas of mathematics and physics, such as n -Lie algebras, Novikov-Poisson algebras, Hom-Lie algebras and others. Transposed Poisson algebras are an actively developing area of modern algebra [5].

In the paper [4], a construction of a 3-Lie algebra based on a transposed Poisson algebra and its derivation is given as the following theorem:

Theorem. Let $(\mathcal{L}, \cdot, [,])$ be a transposed Poisson algebra and D its derivation. Define a ternary bracket by the formula

$$[x, y, z] := D(x)[y, z] + D(y)[z, x] + D(z)[x, y], \quad x, y, z \in \mathcal{L}. \quad (0.1)$$

Then $(\mathcal{L}, [, ,])$ is a 3-Lie algebra.

Moreover, in the same paper it is proved that the ternary bracket (0.1) satisfies the identity

$$3u \cdot [x, y, z] = [u \cdot x, y, z] + [x, u \cdot y, z] + [x, y, u \cdot z]. \quad (0.2)$$

This identity was used to introduce a new structure, namely a transposed Poisson 3-Lie algebra [4].

The main goal of this thesis is to introduce and study a generalization of a transposed Poisson 3-Lie algebra, namely, a transposed Poisson 3-Lie superalgebra. For this purpose, we generalize the bracket (0.1) to the case of an even derivation of a transposed Poisson superalgebra

$$[x, y, z] := D(x) \cdot [y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})} D(y) \cdot [z, x] + (-1)^{(\bar{x}+\bar{y})\bar{z}} D(z) \cdot [x, y]. \quad (0.3)$$

In the Theorem 6 we prove that the ternary bracket (0.3) satisfies the following identity:

$$3u[x, y, z] = [ux, y, z] + (-1)^{\bar{x}\bar{u}}[x, uy, z] + (-1)^{\bar{u}(\bar{x}+\bar{y})}[x, y, uz], \quad (0.4)$$

which can be considered as an analogue of the identity (0.2) for superalgebras. We use this identity to introduce a new structure, namely a transposed Poisson 3-Lie superalgebra. As a consequence of Theorem 6, we present a wide class of transposed Poisson 3-Lie superalgebras constructed on a commutative superalgebra by means of its two commuting even derivations.

Throughout this work, all vector spaces are considered over a number field denoted by \mathbb{K} , and it is assumed that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. That is, we consider all vector spaces over the field of real or complex numbers. In what follows, we will use the term algebra exclusively for associative algebras with unity. In the proofs of the main results, the multiplication sign will be omitted to simplify formulas and computations.

The thesis is divided into four chapters. The first chapter is devoted to Lie algebras. In the first section of this chapter, the definition of a Lie algebra is given, along with the main concepts and the most well-known examples of this structure. In the second section, we demonstrate the close connection between Lie groups and Lie algebras using the example of matrix groups and Lie algebras. The third section of this chapter is devoted to the concept of a derivation of an algebra. The notion of derivation plays a key role throughout the work. This section presents all the main results related to derivations. The final section of this chapter is devoted to one generalization of Lie algebras, namely 3-Lie algebras.

The second chapter is devoted to Poisson algebras and transposed Poisson algebras. This chapter is divided into two sections. The first provides an introduction to Poisson algebras, defining all the main concepts related to this structure. It also includes a brief survey of the application of the Poisson bracket in theoretical physics to demonstrate the importance of this structure in applications. The second

section contains all the necessary information about transposed Poisson algebras. It demonstrates the close relationship between Poisson algebras and transposed Poisson algebras, as well as explains their fundamental differences. It should be noted that Poisson algebras and transposed Poisson algebras are algebras with two operations. Therefore, when we speak of derivations of these structures, we mean that a derivation of a Poisson algebra or a transposed Poisson algebra is a derivation with respect to both operations.

The third chapter is devoted to the concept of a superalgebra, that is, a \mathbb{Z}_2 -graded algebra. It is assumed that the reader is familiar with the structure of the ring \mathbb{Z}_2 . All other notions are described in detail. For brevity of definitions, in the first section of the third chapter we introduce the notion of a vector superspace, on the basis of which the notions of a superalgebra, commutative superalgebra, Lie superalgebra, and 3-Lie superalgebra are defined. The main difference between a superalgebra and an ordinary algebra is that certain elements, called homogeneous, are assigned a parity. Any element of a superalgebra can be uniquely represented as a sum of an even and an odd element. It is important to note that linear transformations of an algebra may preserve parity or change it to the opposite. In this connection, linear transformations of a superalgebra, in particular derivations, also acquire parity. This construction is given as a separate definition, since the distinction between even and odd derivations is of fundamental importance in the theory of superalgebras. As mentioned above, in the final chapter of this work, when formulating and proving the main results, we restrict ourselves to considering even derivations. The second section of the third chapter is devoted to the concept of a transposed Poisson superalgebra. This section is a survey exposition of the paper [1], where the notion of a transposed Poisson superalgebra was first introduced and studied. The most important result from that paper for the present work is a set of identities, which are actively used in computations and proofs involving transposed Poisson superalgebras.

The final chapter of the work is devoted to the concept of a transposed Poisson

3-Lie superalgebra. This chapter contains all the original results of this work. The central result is the aforementioned Theorem 6 on the construction of a transposed 3-Lie Poisson superalgebra based on a transposed Poisson superalgebra by means of its even derivation. The compatibility condition for a transposed Poisson 3-Lie superalgebra, given in the definition, is based precisely on this construction. As a corollary of the theorem, a construction of a wide class of transposed Poisson 3-Lie superalgebras built on a commutative superalgebra by means of its two commuting even derivations is presented.

These results are a continuation of paper [2], in which it was shown that the bracket (0.3) defines a structure of a 3-Lie superalgebra. The results of that paper show that the condition for the derivation to be even is necessary for this construction to be compatible with the structure of a superalgebra. Moreover, the paper [2] proposes a new approach to transposed Poisson superalgebras, allowing the construction of compatible structures by means of an odd derivation. The consideration of such structures would significantly deviate from the main results and constructions presented earlier. Therefore, they are not considered within the scope of the present thesis.

1 Lie Algebras and Their Generalizations

The aim of this chapter is to give an understanding of the structure of a Lie algebra. Lie algebra is the most important structure for the present thesis. In the first section, the definition of a Lie algebra and the main related concepts are given. In addition, a broad class of examples of Lie algebras is presented, constructed on an algebra using the commutator.

One of the important aspects of the theory of Lie algebras is their connection with Lie groups. Each Lie group is canonically associated with a Lie algebra. This Lie algebra is constructed using the exponential map, which plays an important role in the theory of Lie algebras. In the second section of this chapter, we present this construction using the example of matrix Lie groups and Lie algebras.

The third section is devoted to the notion of a derivation of an associative algebra and the connection between derivations and Lie algebras. In it, we present an example of constructing a Lie algebra on an associative algebra, where the commutator is defined using a derivation.

In the last section of this chapter, we consider a generalization of Lie algebras called 3-Lie algebras, which are a special case of n -Lie algebras, where n is an integer greater than two. This generalization is based on the idea of increasing the number of arguments in the commutator. In this paragraph, we present well-known examples of these structures.

1.1 Lie Algebras

In this section we consider a Lie algebra as an independent structure, without relating it to any Lie group.

Definition 1. A *Lie algebra* is a vector space \mathcal{L} with a bilinear map $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$

that satisfies the following conditions

$$[x, y] = -[y, x], \quad (\text{skew-symmetry})$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad (\text{Jacobi identity})$$

where $x, y, z \in \mathcal{L}$. The bilinear map $[\cdot, \cdot]$ is called the Lie bracket.

A Lie algebra in which the bracket is identically zero is called *commutative* or *trivial*. Let $(\mathcal{L}, [\cdot, \cdot])$ be a Lie algebra. A vector subspace $L \subset \mathcal{L}$ is called a *subalgebra* of the Lie algebra \mathcal{L} if for any elements $x, y \in L$ one has $[x, y] \in L$. A vector subspace $I \subset \mathcal{L}$ is called an *ideal* of the Lie algebra if for any $x \in \mathcal{L}$ and $y \in I$ one has $[x, y] \in I$.

Let $(\mathcal{L}_1, [\cdot, \cdot]_1)$ and $(\mathcal{L}_2, [\cdot, \cdot]_2)$ be two Lie algebras. A linear map $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is called a *homomorphism* if $f([x, y]_1) = [f(x), f(y)]_2$ for all $x, y \in \mathcal{L}_1$. In the special case when the map f is bijective, it is called an *isomorphism*. If there exists an isomorphism between two Lie algebras, then the Lie algebras \mathcal{L}_1 and \mathcal{L}_2 are called *isomorphic*. We consider isomorphic Lie algebras to be equivalent and indistinguishable.

Let $(\mathcal{L}, [\cdot, \cdot])$ be a finite-dimensional Lie algebra with a basis $\{e_1, \dots, e_n\}$ ($n \in \mathbb{N}$). Thus any element $x \in \mathcal{L}$ can be represented in coordinates as

$$x = x^1 e_1 + \dots + x^n e_n = \sum_{i=1}^n x^i e_i.$$

For simplicity, we will use Einstein summation convention, that is, we will not write the summation sign every time. If the same index appears twice in an expression, once as an upper index and once as a lower index, then summation over this index is implied, that is $x^i e_i = x^1 e_1 + \dots + x^n e_n$. Thus the vector x can be written in coordinates as $x = x^i e_i$. If we fix another vector $y = y^i e_i$, then since a Lie bracket

$[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ is bilinear we obtain

$$[x, y] = [x^i e_i, y^j e_j] = x^i y^j [e_i, e_j].$$

It follows that a Lie bracket is uniquely determined once the products (brackets) of basis vectors are defined. These products $[e_i, e_j]$ also have coordinates with respect to the chosen basis:

$$[e_i, e_j] = c_{ij}^k e_k. \quad (1.1)$$

The scalars c_{ij}^k are called *structure constants*. Let us determine what conditions the structure constants must satisfy in order for the vector space \mathcal{L} with bilinear map $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ to be a Lie algebra. From skew-symmetry one immediately obtains the condition

$$c_{ij}^k = -c_{ji}^k. \quad (1.2)$$

It can be shown that the Jacobi identity implies additional conditions

$$c_{ij}^m c_{mk}^l + c_{jk}^m c_{mi}^l + c_{ki}^m c_{mj}^l = 0 \quad (1.3)$$

for all combinations of indices $i, j, k, l \in \{1, \dots, n\}$. Thus any set of structure constants satisfying conditions (1.2) and (1.3) defines a Lie algebra. The relations (1.1) for a specific Lie algebra \mathcal{L} are called *commutation relations* of a Lie algebra. Remark that from the skew-symmetry it easily follows that $[e_i, e_i] = 0$ for any $i \in \{1, \dots, n\}$. Usually, for each specific Lie algebra, only non-trivial commutation relations are given.

We now present some important examples of Lie algebras.

Example 1. The first example is the three-dimensional Euclidean space \mathbb{R}^3 with the cross product $[\cdot, \cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$. It is well-known that this operation is skew-symmetric and satisfies the Jacobi identity. Thus $(\mathbb{R}^3, [\cdot, \cdot])$ is a Lie algebra.

If we fix an orthonormal basis $\{e_1, e_2, e_3\} \subset \mathbb{R}^3$, then commutation relations of a

Lie algebra are as follows

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \quad (1.4)$$

Other non-zero products $[e_i, e_j]$ can be derived from (1.4) due to skew-symmetry of Lie bracket. The cross product of two vectors $x, y \in \mathbb{R}^3$ with coordinates $x = x^i e_i$ and $y = y^i e_i$ respectfully can be written by the formula

$$[x, y] = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ e_1 & e_2 & e_3 \end{vmatrix}. \quad (1.5)$$

Later we use this example as a basis for a ternary generalization of a Lie algebra.

Using structure constants and commutation relations, one can classify low-dimensional Lie algebras. The classification of three-dimensional Lie algebras was first obtained by L. Bianchi [7]. This classification contains 11 classes of Lie algebras, 9 of which contain a single Lie algebra and two of which contain a continuum-sized family of Lie algebras. It played an important role in the development of the theory of Lie algebras and its applications in differential geometry. A detailed description of the Bianchi classification can be found in [9], [13].

Example 2. Let (\mathcal{A}, \cdot) be an algebra. Define a new operation $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ as follows: for each pair of elements $x, y \in \mathcal{A}$ define $[x, y] = x \cdot y - y \cdot x$. This operation is called the *commutator* of two elements of the algebra \mathcal{A} . It is clear that the commutator is skew-symmetric. Let us show that the commutator also satisfies the Jacobi identity. We have

$$[x, [y, z]] = x[y, z] - [y, z]x = x(yz - zy) - (yz - zy)x = xyz - xzy - yzx + zyx.$$

Similarly, we compute the other two terms

$$[y, [z, x]] = yzx - yxz - zxy + xzy, \quad [z, [x, y]] = zxy - zyx - xyz + yxz.$$

Adding the three double brackets, we obtain the Jacobi identity. Thus we have shown that the commutator is a Lie bracket and $(\mathcal{A}, [,])$ is a Lie algebra.

In the special case when the algebra (\mathcal{A}, \cdot) is commutative, then for all $x, y \in \mathcal{A}$ one has $[x, y] = 0$. That is, a corresponding Lie algebra $(\mathcal{A}, [,])$ is trivial.

The structure described above makes it possible to construct a broad class of Lie algebras, in particular matrix Lie algebras. By this we mean that, given an algebra of matrices, equipping it with the commutator yields a matrix Lie algebra.

A natural question then arises: if an abstract finite-dimensional Lie algebra is given, can it be realized as a matrix Lie algebra? The answer is provided by the fundamental Ado's theorem [3], which can be stated as follows: for every finite-dimensional Lie algebra \mathcal{L} , there exists a vector space V such that \mathcal{L} is isomorphic to a subalgebra of the Lie algebra of endomorphisms of V . We will discuss matrix Lie algebras in more detail in the next section.

1.2 Matrix Lie Groups and Lie Algebras

The founder of the theory of Lie algebras is considered to be the Norwegian mathematician M. S. Lie, who primarily studied continuous groups. His original idea was to apply the methods of Galois theory to the theory of differential equations [21], [25]. It was precisely in connection with his work on continuous groups that the concept of a Lie algebra appeared. The definition of an abstract Lie group is based on the notion of a smooth manifold. A Lie group is a set that is a group and at the same time a smooth manifold. Moreover, it is required that the group operation and the assignment of an inverse to each element are smooth maps. A

description of these structures would significantly increase the volume of this dissertation, therefore we restrict ourselves to considering matrix Lie groups, which constitute the most important class of Lie groups. Our exposition of matrix Lie groups follows the books [12],[11].

Before giving the general definition of a matrix Lie group, we introduce the necessary notation. The set of all $n \times n$ matrices over the field \mathbb{K} will be denoted by $M(n; \mathbb{K})$. The identity matrix of order n will be denoted by I_n . The zero matrix of order n will be denoted by Θ_n . The set $M(n; \mathbb{K})$ is an associative algebra with unity with respect to addition, multiplication by elements of the field \mathbb{K} , and matrix multiplication. The group of all invertible $n \times n$ matrices will be denoted in the standard way by $GL(n; \mathbb{K})$.

Definition 2. *A **matrix Lie group** is any subgroup G of the group $GL(n; \mathbb{K})$ such that if a sequence of matrices $A_m \in G$ converges to a matrix $A \in GL(n; \mathbb{K})$, then the matrix A belongs to this subgroup, $A \in G$.*

In this definition, convergence is understood entrywise. It can be shown that a matrix Lie group is a Lie group [11]. Examples of matrix Lie groups include the general linear group $GL(n; \mathbb{K})$, the special linear group

$$SL(n; \mathbb{K}) = \{A \in GL(n; \mathbb{K}) \mid \det(A) = 1\},$$

the orthogonal group

$$O(n; \mathbb{K}) = \{A \in GL(n; \mathbb{K}) \mid A^t = A^{-1}\},$$

where A^t denotes the transpose of the matrix A . The group of unitary matrices is defined only over the field of complex numbers. For a matrix $A \in GL(n; \mathbb{C})$ denote the Hermitian conjugate matrix as A^\dagger , that is, $A^\dagger = \overline{A}^t$. The group of unitary

matrices is defined as

$$U(n) = \{A \in GL(n; \mathbb{C}) \mid A^\dagger = A^{-1}\},$$

Moreover, important matrix Lie groups include the special orthogonal groups and special unitary groups, which can be defined as

$$SO(n; \mathbb{K}) = O(n; \mathbb{K}) \cap SL(n; \mathbb{K}), \quad SU(n) = U(n) \cap SL(n; \mathbb{C}).$$

To each Lie group one can associate a Lie algebra. In the case of a matrix Lie group, this Lie algebra is constructed using the notion of the matrix exponential. Let $A \in M(n; \mathbb{K})$ be a matrix. The exponential of the matrix A is defined by the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

It is known that for any matrix this series converges. The properties of the matrix exponential are formulated in the following proposition. Proofs of the listed properties can be found in [12], [25].

Proposition 1. *For any matrices $A, B \in M(n; \mathbb{K})$ the following properties hold*

1. $e^{\Theta_n} = I_n$,
2. $(e^A)^\dagger = e^{A^\dagger}$, $(e^A)^t = e^{A^t}$,
3. e^A is always invertible and $(e^A)^{-1} = e^{-A}$,
4. $e^{(\alpha+\beta)A} = e^{\alpha A} e^{\beta A}$ for any $\alpha, \beta \in \mathbb{K}$,
5. if $AB = BA$, then $e^{A+B} = e^A e^B = e^B e^A$,
6. if B is an invertible matrix, then $e^{BAB^{-1}} = B e^A B^{-1}$,
7. $\det(e^A) = e^{\text{Tr}(A)}$.

It should be noted that in general $e^{A+B} \neq e^A e^B$.

Let G be a matrix Lie group and let L be the set of matrices $A \in M(n; \mathbb{K})$ such that for any real number $t \in \mathbb{R}$ one has $e^{tA} \in G$. It can be shown [12] that L is a subspace of the vector space $M(n; \mathbb{K})$. Moreover, it can be shown that for any matrices $A, B \in L$ their commutator $[A, B]$ also belongs to L . It follows that $(L, [,]) is a Lie algebra. Such a Lie algebra is called the *Lie algebra of the matrix Lie group* G .$

For example, since for any matrix $A \in M(n; \mathbb{K})$ its exponential e^A is an invertible matrix (property 3), that is $e^A \in GL(n; \mathbb{K})$, the Lie algebra of the matrix Lie group $GL(n; \mathbb{K})$ is the vector space of all $n \times n$ matrices equipped with the commutator. Denote this Lie algebra by $\mathfrak{gl}(n; \mathbb{K})$.

To find the Lie algebra for the special linear group $SL(n; \mathbb{K})$, one needs to use property 7 of the matrix exponential: for each matrix $A \in M(n; \mathbb{K})$ we have $\det(e^A) = e^{\text{Tr}(A)}$. Therefore, for a matrix A to satisfy $e^A \in SL(n; \mathbb{K})$ it is necessary and sufficient that $\text{Tr}(A) = 0$. Then the Lie algebra corresponding to the Lie group $SL(n; \mathbb{K})$ is

$$\mathfrak{sl}(n; \mathbb{K}) = \{A \in M(n; \mathbb{K}) \mid \text{Tr}(A) = 0\}.$$

Similarly, using properties of the matrix exponential, one can show that

$$\begin{aligned} \mathfrak{o}(n; \mathbb{K}) = \mathfrak{so}(n; \mathbb{K}) &= \{A \in M(n; \mathbb{K}) \mid A^t = -A\}, \\ \bar{\mathfrak{u}}(n) &= \{A \in M(n; \mathbb{K}) \mid A^\dagger = -A\}, \\ \mathfrak{su}(n) &= \{A \in M(n; \mathbb{K}) \mid A^\dagger = -A, \text{Tr}(A) = 0\}, \end{aligned}$$

where $\mathfrak{o}(n; \mathbb{K})$, $\mathfrak{so}(n; \mathbb{K})$, $\bar{\mathfrak{u}}(n)$ and $\mathfrak{su}(n)$ are Lie algebras corresponding to the Lie groups $O(n; \mathbb{K})$, $SO(n; \mathbb{K})$, $U(n)$, $SU(n)$, respectively.

Matrix Lie algebras play an important role in theory of Lie algebras. Mentioned above Ado's theorem states that a finite-dimensional Lie algebra \mathcal{L} has a linear rep-

representation on a finite-dimensional vector space V , that is a faithful representation, making \mathcal{L} isomorphic to a subalgebra of the Lie algebra of endomorphisms of V [3]. As an illustration of Ado's theorem, let us consider the three-dimensional Euclidean space equipped with the cross product $(\mathbb{R}^3, [,])$. In the previous Example 1, it was shown that this is a three-dimensional Lie algebra when the Lie bracket is given by the cross product. Recall that the commutation relations of this Lie algebra are of the form

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

A matrix realization of these commutation relations can be obtained using matrices

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1.6)$$

Indeed, it is straightforward to verify that the commutation relations

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2 \quad (1.7)$$

hold, where the Lie bracket denotes the commutator of two matrices. Thus, we obtain a matrix realization of the Lie algebra $(\mathbb{R}^3, [,])$. It is well known in the theory of Lie algebras that the matrices (1.6) form a basis of $\mathfrak{so}(3; \mathbb{R})$. From this, in particular, it follows that the three-dimensional Lie algebra $(\mathbb{R}^3, [,])$ is isomorphic to the Lie algebra of the rotation group, that is, $(\mathbb{R}^3, [,]) \cong \mathfrak{so}(3; \mathbb{R})$.

1.3 Derivations of Algebras and Lie Algebras

In this section we give the definition of a derivation of an algebra and show that derivations of an algebra form a Lie algebra. We present an example of constructing a Lie algebra on an associative algebra using a derivation. Then, by analogy with

associative algebras, we define derivations of a Lie algebra and present the class of inner derivations of Lie algebras as an example.

Throughout this paragraph let (\mathcal{A}, \cdot) be an algebra.

Definition 3. A *derivation* of an algebra \mathcal{A} is a linear map $D : \mathcal{A} \rightarrow \mathcal{A}$ such that for any elements $x, y \in \mathcal{A}$ the following condition holds

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y). \quad (1.8)$$

Property (1.8) is called the *Leibniz rule*.

Two derivations $D_1, D_2 : \mathcal{A} \rightarrow \mathcal{A}$ are called *commuting* if for any element $x \in \mathcal{A}$ one has $(D_1 \circ D_2)(x) = (D_2 \circ D_1)(x)$, where \circ denotes composition of maps. Examples of derivations of the algebra of smooth functions defined on \mathbb{R}^n include first-order partial derivatives of these functions. Denote the set of all derivations of the algebra (\mathcal{A}, \cdot) by $Der(\mathcal{A})$. It is easy to verify that $Der(\mathcal{A})$ is a vector space with operations defined as follows

$$(D_1 + D_2)(x) = D_1(x) + D_2(x), \quad (\alpha D_1)(x) = \alpha D_1(x),$$

for any $D_1, D_2 \in Der(\mathcal{A})$, $x \in \mathcal{A}$, $\alpha \in \mathbb{K}$. Consider the commutator of two derivations $D_1, D_2 \in Der(\mathcal{A})$

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1. \quad (1.9)$$

We show that the commutator of two derivations is again a derivation.

Proposition 2. If $D_1, D_2 \in Der(\mathcal{A})$, then $[D_1, D_2] \in Der(\mathcal{A})$

Proof. It is not difficult to show that $[D_1, D_2]$ is a linear map on the algebra \mathcal{A} . Let us show that it satisfies the Leibniz rule (1.8). Let $x, y \in \mathcal{A}$ be arbitrary elements.

Then using the Leibniz rule and linearity for the derivations D_1, D_2 we obtain

$$\begin{aligned}
[D_1, D_2](x \cdot y) &= D_1(D_2(x \cdot y)) - D_2(D_1(x \cdot y)) \\
&= D_1(D_2(x) \cdot y) + D_1(x \cdot D_2(y)) - D_2(D_1(x) \cdot y) - D_2(x \cdot D_1(y)) \\
&= D_1(D_2(x)) \cdot y + D_2(x) \cdot D_1(y) + D_1(x) \cdot D_2(y) + x \cdot D_1(D_2(y)) \\
&\quad - D_2(D_1(x)) \cdot y - D_1(x) \cdot D_2(y) - D_2(x) \cdot D_1(y) - x \cdot D_2(D_1(y)) \\
&= (D_1(D_2(x)) - D_2(D_1(x))) \cdot y + x \cdot (D_1(D_2(y)) - D_2(D_1(y))) \\
&= [D_1, D_2](x) \cdot y + x \cdot [D_1, D_2](y).
\end{aligned}$$

□

Hence $Der(\mathcal{A})$ is closed under the commutator of derivations. Since (1.9) is a commutator, it is easy to verify that skew-symmetry and the Jacobi identity hold. Thus $(Der(\mathcal{A}), [,])$ is a Lie algebra.

For this work, a central construction is the following class of Lie algebras constructed on a commutative algebra using a derivation. Let D be a derivation of algebra \mathcal{A} . Define a bracket on the algebra \mathcal{A} as follows:

$$[x, y] = D(x) \cdot y - x \cdot D(y). \quad (1.10)$$

Proposition 3. *If (\mathcal{A}, \cdot) is a commutative algebra, then $(\mathcal{A}, [,])$ is a Lie algebra.*

Proof. It is easy to see that the operation defined in this way is bilinear and skew-symmetric. Let us show that it satisfies the Jacobi identity. We have

$$\begin{aligned}
[x, [y, z]] &= D(x)[y, z] - xD([y, z]) \\
&= D(x)(D(y)z - yD(z)) - xD(D(y)z - yD(z)) \\
&= D(x)D(y)z - D(x)yD(z) - xD^2(y)z + xyD^2(z).
\end{aligned}$$

Similarly,

$$\begin{aligned} [y, [z, x]] &= D(y)D(z)x - D(y)zD(x) - yD^2(z)x + yzD^2(x) \\ &= xD(y)D(z) - D(x)D(y)z - xyD^2(z) + D^2(x)yz, \end{aligned}$$

$$\begin{aligned} [z, [x, y]] &= D(z)D(x)y - D(z)xD(y) - zD^2(x)y + zxD^2(y) \\ &= D(x)yD(z) - xD(y)D(z) - D^2(x)yz + xD^2(y)z. \end{aligned}$$

Adding all three terms yields the Jacobi identity. Thus $(\mathcal{A}, [,]) is a Lie algebra. $\square$$

The notion of a derivation of an algebra naturally extends to Lie algebras.

Definition 4. Let $(\mathcal{L}, [,]) be a Lie algebra. A *derivation* of a Lie algebra \mathcal{L} is a linear map $D : \mathcal{L} \rightarrow \mathcal{L}$ that satisfies the Leibniz rule$

$$D([x, y]) = [D(x), y] + [x, D(y)]. \quad (1.11)$$

Fix an element $x \in \mathcal{L}$ and define a linear map

$$ad_x : \mathcal{L} \rightarrow \mathcal{L}, \quad ad_x(y) := [x, y].$$

From the Jacobi identity it follows that

$$[x, [y, z]] = -[y, [z, x]] - [z, [x, y]] = [[x, y], z] + [y, [x, z]],$$

which can be written as

$$ad_x([y, z]) = [ad_x(y), z] + [y, ad_x(z)].$$

Thus we see that for any element $x \in \mathcal{L}$ the map ad_x is a derivation of the Lie algebra $(\mathcal{L}, [,]) . Such derivations are called *inner*. Inner derivations of Lie algebras are important for the representation theory of Lie algebras.$

1.4 3-Lie Algebras

Lie algebras play an exceptionally important role both in modern geometry and in theoretical physics. The theory of Lie algebras is continuously developing, and various generalizations of the concept of Lie algebras are being proposed. One such generalization is the notion of an n -Lie algebra. This structure was introduced and studied by V. T. Filippov [10]. Independently of Filippov, a similar structure was introduced by Nambu [17]. This structure is called a Nambu–Poisson algebra. In this paragraph we restrict ourselves to the description of 3-Lie algebras, which are defined as follows:

Definition 5. A **3-Lie algebra** is a vector space \mathcal{L} with a trilinear mapping $[\cdot, \cdot, \cdot] : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$, which satisfies the following conditions

$$\begin{aligned} [y, x, z] &= -[x, y, z] = [x, z, y], \\ [[x, y, z], u, v] &= [[x, u, v], y, z] + [x, [y, u, v], z] + [x, y, [z, u, v]], \end{aligned}$$

where $x, y, z, u, v \in \mathcal{L}$. The trilinear map $[\cdot, \cdot, \cdot]$ is called the *ternary Lie bracket*.

We will call the first condition skew-symmetry, as in the case of Lie algebras. The second condition has several names in the literature. We will call this condition the Filippov–Jacobi identity, as suggested in [8].

Let $(\mathcal{L}, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra. A vector subspace $L \subset \mathcal{L}$ is called a *subalgebra* of the 3-Lie algebra if for any elements $x, y, z \in \mathcal{L}$ one has $[x, y, z] \in L$. A vector subspace $I \subset \mathcal{L}$ is called an *ideal* of the 3-Lie algebra if for any elements $x, y \in \mathcal{L}$ and $z \in I$ one has $[x, y, z] \in I$. By analogy with Lie algebras, a linear map $D : \mathcal{L} \rightarrow \mathcal{L}$ is called a *derivation* of a 3-Lie algebra if for any elements $x, y, z \in \mathcal{L}$ the condition

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)].$$

holds. This property is also called the *Leibniz rule*, as in the case of Lie algebras. From the Filippov–Jacobi identity it is easy to see that for any fixed elements

$u, v \in \mathcal{L}$ the map $ad_{u,v} : \mathcal{L} \rightarrow \mathcal{L}$, defined by $ad_{u,v}(x) = [x, u, v]$, is a derivation. Such derivations are called *inner*, as in the case of Lie algebras. The notions of homomorphism and isomorphism of algebras also naturally extend to 3-Lie algebras. As in the case of Lie algebras, isomorphic 3-Lie algebras are considered equivalent and indistinguishable.

One of the most well-known examples of a 3-Lie algebra is an analogue of the vector product in four-dimensional space. Consider a four-dimensional Euclidean space \mathbb{R}^4 with an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ and let $x = x^i e_i$, $y = y^i e_i$ and $z = z^i e_i$ be arbitrary vectors. Define a ternary operation $[\cdot, \cdot, \cdot] : \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$ analogously to formula (1.5)

$$[x, y, z] = \begin{vmatrix} x^1 & x^2 & x^3 & x^4 \\ y^1 & y^2 & y^3 & y^4 \\ z^1 & z^2 & z^3 & z^4 \\ e_1 & e_2 & e_3 & e_4 \end{vmatrix}. \quad (1.12)$$

It can be shown that the space \mathbb{R}^4 , equipped with the operation (1.12), is a 3-Lie algebra [10]. Another remarkable example of a 3-Lie algebra can be constructed using derivations of the algebra of smooth functions. Let $C^\infty(\mathbb{R}^n)$ be the algebra of smooth functions on \mathbb{R}^n and let D_1, D_2, D_3 be its pairwise commuting derivations. Define a ternary operation $\{ \cdot, \cdot, \cdot \}$ for any $f, g, h \in C^\infty(\mathbb{R}^n)$ by the formula

$$\{f, g, h\} = \begin{vmatrix} D_1(f) & D_1(g) & D_1(h) \\ D_2(f) & D_2(g) & D_2(h) \\ D_3(f) & D_3(g) & D_3(h) \end{vmatrix}.$$

A similar structure was first studied in [17]. In [10] it was shown (in a more general setting) that $(C^\infty(\mathbb{R}^n), \{ \cdot, \cdot, \cdot \})$ is a 3-Lie algebra. This example and its generalizations are of great interest in theoretical physics, in particular in connection with the generalization of Hamiltonian mechanics [22] and M-brane theory [18].

2 Poisson Algebras and Transposed Poisson Algebras

The main aim of this chapter is to give the reader an understanding of the structure of a transposed Poisson algebra. The transposed Poisson algebra is one of the main objects of study in this work. In a certain sense, a transposed Poisson algebra is a structure dual to a Poisson algebra. To explain what we mean, it is necessary to give the definition of a Poisson algebra, which will be done in the first paragraph of this chapter. In addition to the definition of a Poisson algebra, the first paragraph will present the most important examples of this structure.

The study of Poisson algebras originated in theoretical physics, namely in classical mechanics. This structure has numerous applications in various areas of geometry and physics, such as the theory of Poisson manifolds, algebraic geometry, quantization theory, as well as classical and quantum mechanics. More detailed information about Poisson structures can be found in the book [15]. Many generalizations of Poisson algebras find wide application in theoretical physics [12]. The development of Hamiltonian mechanics led to the appearance of a generalization of the Poisson bracket, called the Nambu–Poisson bracket [17]. In the theory of Poisson algebras, noncommutative or nonassociative Poisson algebras are also considered [19], [23]. Poisson algebras are also closely related to other algebraic structures, for example, to Novikov algebras [24].

The second paragraph is devoted to the notion of a transposed Poisson algebra. The transposed Poisson algebra was first defined in 2023 in the paper [4]. In this fundamental work, in addition to the definition of a new structure, a close connection between Poisson algebras and transposed Poisson algebras, as well as other structures of mathematical physics such as Novikov–Poisson algebras, pre-Lie algebras, Hom-Lie algebras, and n -Lie algebras, was demonstrated. After the publication of [4], the study of the structure of transposed Poisson algebras continued actively,

and remarkable results and connections with other algebraic structures were discovered [5]. The paper [6] presents a classification of transposed Poisson algebras in low dimensions. In the second paragraph of this work, we will give the definition of a transposed Poisson algebra and explain its connection with Poisson algebras. Here we will also present a broad class of examples of transposed Poisson algebras constructed using derivations of a commutative algebra. At the end of this chapter, we present a construction of a 3-Lie algebra built on a transposed Poisson algebra. Moreover, in [4] a new structure is proposed, which the authors call a transposed Poisson 3-Lie algebra. In this work, we propose and study a generalization of a transposed 3-Lie Poisson algebra, which we call a transposed Poisson 3-Lie superalgebra.

2.1 Poisson Algebras

In this paragraph we give the definition and present examples of a Poisson algebra. The notion of a Poisson algebra arose in connection with the development of Hamiltonian mechanics, which is a branch of classical mechanics. One of the main concepts of Hamiltonian mechanics is the so-called Poisson bracket of two differentiable functions defined on phase space. This example of a Poisson bracket is one of the most important in applications. In this paragraph we give a precise and detailed description of it. Recall that the space of differentiable functions is a commutative algebra with respect to pointwise multiplication of functions. The Poisson bracket of two differentiable functions has all the properties of a Lie bracket. However, it also has an additional property. Namely, the Poisson bracket is a derivation of the product of functions. Thus, a Poisson algebra is an algebra with two operations, one of which is commutative, and the other is a Lie bracket, and these two operations are compatible with each other by a certain condition. We now give the definition of a Poisson algebra.

Definition 6. *A vector space \mathcal{P} with two bilinear maps $\cdot : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ and*

$\{, \} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ is called a **Poisson algebra** if (\mathcal{P}, \cdot) is a commutative algebra, $(\mathcal{P}, \{, \})$ is a Lie algebra, and the compatibility condition

$$\{z, x \cdot y\} = \{z, x\} \cdot y + x \cdot \{z, y\} \quad (2.1)$$

holds for any $x, y, z \in \mathcal{P}$.

In this case, the bilinear map $\{, \} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ is called the *Poisson bracket*. The compatibility condition (2.1) has the structure of the Leibniz rule and shows that the Poisson bracket is a derivation of the algebra (\mathcal{P}, \cdot) . In what follows, we will denote a Poisson algebra by $(\mathcal{P}, \cdot, \{, \})$.

Definition 7. A *derivation of a Poisson algebra* is a linear map $D : \mathcal{P} \rightarrow \mathcal{P}$ that is simultaneously a derivation of both the commutative algebra (\mathcal{P}, \cdot) and the Lie algebra $(\mathcal{P}, \{, \})$, that is,

1. $D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$,
2. $D(\{x, y\}) = \{D(x), y\} + \{x, D(y)\}$,

for any $x, y \in \mathcal{P}$.

Let (\mathcal{A}, \cdot) be a commutative algebra. Define the Poisson bracket by the formula $\{x, y\} = 0$, where $x, y \in \mathcal{A}$. Then $(\mathcal{A}, \cdot, \{, \})$ is a Poisson algebra. Similarly, for any Lie algebra $(\mathcal{L}, [,])$ one can define multiplication by $x \cdot y = 0$ for all $x, y \in \mathcal{L}$. The resulting structure $(\mathcal{L}, \cdot, [,])$ is also a Poisson algebra. Such Poisson algebras will be called *trivial*.

Poisson algebra plays an important role in classical mechanics. Before presenting the Poisson bracket itself, it is necessary to introduce additional notation and briefly discuss the required physical terminology. In Hamiltonian mechanics, energy is considered as a function of the coordinates of a particle and the components of its momentum. For example, if we consider the motion of a point particle in

Euclidean space \mathbb{R}^n , then one example of a *Hamiltonian function* or *Hamiltonian* is the function

$$H = \frac{1}{2m} \sum_{k=1}^n p_k^2 + V(x_1, \dots, x_n),$$

where x_1, \dots, x_n are the *coordinates* of the particle, p_1, \dots, p_n are the components of its *momentum*, and m is the mass of the particle under consideration. The function V , depending on the coordinates, represents the potential energy. It is known [12] that Newton's laws can be written in terms of the Hamiltonian as

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x_i} \end{aligned} \tag{2.2}$$

In classical mechanics, equations (2.2) are called the *Hamilton equations*.

Next, we present one of the most well-known examples of a Poisson algebra based on Hamiltonian mechanics. Consider the space \mathbb{R}^{2n} . Elements of this space will be written in the form $(x_1, \dots, x_n, p_1, \dots, p_n)$. In classical mechanics, the space \mathbb{R}^{2n} is called the *phase space*, x_1, \dots, x_n are the *generalized coordinates*, and p_1, \dots, p_n are the *generalized momenta*. Also consider the algebra of infinitely differentiable functions on phase space $C^\infty(\mathbb{R}^{2n})$. The Poisson bracket of two functions $f, g \in C^\infty(\mathbb{R}^{2n})$ is defined by

$$\{f, g\} = \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right). \tag{2.3}$$

It can be shown that the Poisson bracket (2.3) is bilinear and skew-symmetric. Moreover, for any smooth functions $f, g, h \in C^\infty(\mathbb{R}^{2n})$, the following properties hold:

- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,
- $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

The proof of these properties can be found in [12]. Thus, the commutative algebra $C^\infty(\mathbb{R}^{2n})$ (with pointwise multiplication of functions), equipped with the Poisson bracket (2.3), is a Poisson algebra.

A more general example of a Poisson algebra can be constructed on an arbitrary commutative algebra (\mathcal{A}, \cdot) using derivations. Let D_1, D_2 be two commuting derivations of the algebra \mathcal{A} , that is, $D_1 \circ D_2 = D_2 \circ D_1$. Define the bracket by the formula

$$\{x, y\} = D_1(x)D_2(y) - D_2(x)D_1(y).$$

It can be shown that $(\mathcal{A}, \{, \})$ is a Lie algebra and $(\mathcal{A}, \cdot, \{, \})$ is a Poisson algebra.

2.2 Transposed Poisson Algebras

Previously, we introduced the notion of a Poisson algebra. A Poisson algebra is an algebra with two operations: a commutative multiplication and a Lie bracket. In a Poisson algebra, the Lie bracket is a derivation with respect to the commutative multiplication. In [4], a structure was introduced and studied in which the roles of these operations are reversed, that is, the commutative multiplication is a derivation of the Lie bracket. A transposed Poisson algebra is defined as follows:

Definition 8. *A vector space \mathcal{T} with two bilinear maps $\cdot : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ and $[,] : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ is called a **transposed Poisson algebra** if (\mathcal{T}, \cdot) is a commutative algebra, $(\mathcal{T}, [,])$ is a Lie algebra, and the compatibility condition*

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y] \tag{2.4}$$

holds for any $x, y, z \in \mathcal{T}$.

In this case, we will call the bilinear map $[,]$ the *transposed Poisson bracket*. The compatibility condition (2.4) is called the *transposed Leibniz rule*, since the roles of the two operations are reversed in it. A *derivation* of a transposed Poisson algebra is

defined analogously to a derivation of a Poisson algebra (Definition 7). By analogy with Poisson algebras, one can define *trivial* transposed Poisson algebras.

The following theorem provides a way to construct a broad and important class of transposed Poisson algebras.

Theorem 1. *Let (\mathcal{A}, \cdot) be a commutative algebra and $D : \mathcal{A} \rightarrow \mathcal{A}$ its derivation. Define the bracket by the formula*

$$[x, y] = D(x) \cdot y - x \cdot D(y). \quad (2.5)$$

Then $(\mathcal{A}, \cdot, [,]) is a transposed Poisson algebra.$

Proof. In Proposition 3, it was shown that the bracket (2.5) is a Lie bracket. To show that $(\mathcal{A}, \cdot, [,]) is a transposed Poisson algebra, it is necessary to prove that the compatibility condition (2.4) holds. Using the Leibniz rule for the derivation D and taking into account that the algebra (\mathcal{A}, \cdot) is commutative, we obtain$

$$\begin{aligned} [z \cdot x, y] + [x, z \cdot y] &= D(z \cdot x) \cdot y - z \cdot x \cdot D(y) + D(x) \cdot z \cdot y - x \cdot D(z \cdot y) \\ &= D(z) \cdot x \cdot y + z \cdot D(x) \cdot y - z \cdot x \cdot D(y) + D(x) \cdot z \cdot y \\ &\quad - x \cdot D(z) \cdot y - x \cdot z \cdot D(y) \\ &= 2z \cdot D(x) \cdot y - 2z \cdot x \cdot D(y) \\ &= 2z \cdot (D(x) \cdot y - x \cdot D(y)) \\ &= 2z \cdot [x, y]. \end{aligned}$$

□

In particular, this theorem demonstrates the necessity of the coefficient 2 in the transposed Leibniz rule (2.4).

Since Poisson algebras and transposed Poisson algebras have many similarities, it is natural to ask under what conditions these structures coincide. The answer to this question can be found in [4]. We formulate this result in the following proposition

Proposition 4. *Let $(\mathcal{L}, \cdot, [,])$ be such a structure that (\mathcal{L}, \cdot) is a commutative algebra and $(\mathcal{L}, [,])$ is a Lie algebra. The algebra $(\mathcal{L}, \cdot, [,])$ is simultaneously a Poisson algebra and a transposed Poisson algebra if and only if*

$$x \cdot [y, z] = [x \cdot y, z] = 0$$

for any $x, y, z \in \mathcal{L}$.

In the paper [4], in addition to the transposed Poisson algebra, another algebraic structure of interest for this work was defined.

Definition 9. *Let (\mathcal{L}, \cdot) be a commutative algebra and $(\mathcal{L}, [, ,])$ a 3-Lie algebra. The algebra $(\mathcal{L}, \cdot, [, ,])$ is called a **transposed Poisson 3-Lie algebra** if the compatibility condition*

$$3u \cdot [x, y, z] = [u \cdot x, y, z] + [x, u \cdot y, z] + [x, y, u \cdot z], \quad (2.6)$$

holds for any $x, y, z, u \in \mathcal{L}$.

In [4], a theorem is proved that provides a way to construct a broad class of transposed Poisson 3-Lie algebras.

Theorem 2. *Let $(\mathcal{L}, \cdot, [,])$ be a transposed Poisson algebra and $D : \mathcal{L} \rightarrow \mathcal{L}$ its derivation. Define a ternary bracket by the formula*

$$[x, y, z] = D(x)[y, z] + D(y)[z, x] + D(z)[x, y]. \quad (2.7)$$

Then $(\mathcal{L}, \cdot, [, ,])$ is a transposed Poisson 3-Lie algebra.

The main result of this master thesis is a generalization of this theorem to the case of superalgebras (Theorem 6).

3 Superalgebras

This chapter is devoted to superalgebras and Lie superalgebras, which arose in theoretical physics in connection with the development of supersymmetry theory. Superalgebras are a particular case of graded algebras. More precisely, superalgebras are \mathbb{Z}_2 -graded algebras, that is, algebras in which elements are assigned a *parity*, or a grading from the ring \mathbb{Z}_2 . Since in this work we consider exclusively \mathbb{Z}_2 -graded algebras, we do not describe the more general theory of graded algebras and use the term superalgebra, as is commonly done in the literature. In the first section of this chapter, we give the definition of a superalgebra and all important notions related to the concept of a superalgebra. One of the most important notions for this thesis is that of a derivation of a superalgebra. The concept of a superalgebra makes it possible to generalize many algebraic structures. In this work, we are primarily interested in Lie algebras and their generalizations. Therefore, in this section we provide a description of Lie superalgebras and 3-Lie superalgebras.

In the final section of this chapter, we present the definition of a transposed Poisson superalgebra, based on the notions of a superalgebra and a Lie superalgebra. This structure was first defined and studied in the paper [1]. Since the goal of this work is to construct a transposed 3-Lie Poisson superalgebra based on a transposed Poisson superalgebra and its even derivation, in this section we present in detail all results related to transposed Poisson superalgebras.

3.1 Superalgebras

This section is devoted to the notion of a superalgebra. In order to define a superalgebra, it is necessary to introduce some notation. The elements of the ring \mathbb{Z}_2 will be denoted by the symbols $\bar{0}$ and $\bar{1}$. In what follows, we will use $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. Let V be a vector space represented as a direct sum of two of its non-trivial subspaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If an element x belongs either to the space $V_{\bar{0}}$

or to the space $V_{\bar{1}}$, it is called *homogeneous*. For each homogeneous element x , its *parity* is denoted by \bar{x} and is defined as follows:

$$\bar{x} = \begin{cases} \bar{0}, & \text{if } x \in V_{\bar{0}}, \\ \bar{1}, & \text{if } x \in V_{\bar{1}}. \end{cases} \quad (3.1)$$

Elements of the subspace $V_{\bar{0}}$ are called *even*, and elements of the subspace $V_{\bar{1}}$ are called *odd*. The zero vector is considered to be both even and odd. Thus, any element of the vector space V can be represented as a sum of an even and an odd element. For brevity, we will call the vector space V , represented as a direct sum of two of its subspaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with parity (3.1), a *vector superspace*. In what follows, when we speak of the parity of an element, we assume that the element is homogeneous. Now we can give the definition of a superalgebra.

Definition 10. *An algebra (\mathcal{A}, \cdot) is called a **superalgebra** if $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ is a vector superspace and the following condition holds:*

$$\overline{x \cdot y} = \bar{x} + \bar{y}. \quad (3.2)$$

Condition (3.2) is the standard requirement for the algebra operation to be compatible with the parity. Thus, when we say in this work that (\mathcal{A}, \cdot) is a superalgebra, we mean that the multiplication is associative and is compatible with parities of elements. From condition (3.2) it follows that in a superalgebra the following properties hold:

$$\mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{0}}, \quad \mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}} \cdot \mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{0}}.$$

These properties show that the subspace $\mathcal{A}_{\bar{0}}$ is a subalgebra of the algebra \mathcal{A} , and the subspace $\mathcal{A}_{\bar{1}}$ is a bimodule over the algebra $\mathcal{A}_{\bar{0}}$.

In this work, we are particularly interested in commutative superalgebras. One of the important tools widely used in computations in superalgebras is the *Koszul*

sign rule. This rule can be formulated as follows: if in an algebraic expression two neighboring elements with parities a and b are interchanged, then this interchange is accompanied by multiplication of the expression by the factor $(-1)^{ab}$. Thus, a commutative superalgebra is defined as follows:

Definition 11. A superalgebra (\mathcal{A}, \cdot) is called a **commutative superalgebra** if the following condition holds:

$$x \cdot y = (-1)^{\bar{x}\bar{y}} y \cdot x. \quad (3.3)$$

In the literature on superalgebras, condition (3.3) is also called graded commutativity or supercommutativity.

Superalgebras arise naturally in geometry and theoretical physics. The Grassmann algebra is a commutative superalgebra [14]. Grassmann algebras underlie the theory of supermanifolds. Differential forms defined on a smooth manifold also form a commutative superalgebra. The Clifford algebra is an example of a noncommutative superalgebra [16].

For the purposes of this work, the notion of a derivation of a superalgebra is of particular importance. A linear map $D : \mathcal{A} \rightarrow \mathcal{A}$ is called

- **even**, denoted $\bar{D} = \bar{0}$, if $\overline{D(x)} = \bar{x}$,
- **odd**, denoted $\bar{D} = \bar{1}$, if $\overline{D(x)} = \bar{x} + \bar{1}$.

Definition 12. A linear map $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a **derivation** of the superalgebra (\mathcal{A}, \cdot) if for all elements $x, y \in \mathcal{A}$ the following condition holds:

$$D(x \cdot y) = D(x) \cdot y + (-1)^{\bar{x}\bar{D}} x \cdot D(y). \quad (3.4)$$

Property (3.4) is called the *Leibniz rule*.

Next, we give the definition of a Lie superalgebra.

Definition 13. A vector superspace $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ with a bilinear map $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ is called a **Lie superalgebra** if for all $x, y, z \in \mathcal{L}$ the following conditions hold:

$$\overline{[x, y]} = \bar{x} + \bar{y}, \quad (3.5)$$

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x], \quad (3.6)$$

$$(-1)^{\bar{x}\bar{z}}[[x, y], z] + (-1)^{\bar{y}\bar{x}}[[y, z], x] + (-1)^{\bar{z}\bar{y}}[[z, x], y] = 0. \quad (3.7)$$

Property (3.6) will be called skew-symmetry, and identity (3.7) the Jacobi identity, as in the case of Lie algebras. It is worth noting that the construction of the commutator from Example 2 naturally extends to superalgebras.

Example 3. Let (\mathcal{A}, \cdot) be a superalgebra. Define the bracket $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ by the formula

$$[x, y] = x \cdot y - (-1)^{\bar{x}\bar{y}}y \cdot x. \quad (3.8)$$

It can be shown that $(\mathcal{A}, [\cdot, \cdot])$ is a Lie superalgebra. The bracket (3.8) is sometimes called a *graded commutator*.

A large number of important examples of Lie superalgebras can be found in [14].

Finally, we give the definition of a 3-Lie superalgebra.

Definition 14. A vector superspace $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ with a trilinear bracket $[\cdot, \cdot, \cdot] : \mathcal{L}^3 \rightarrow \mathcal{L}$ is called a **3-Lie superalgebra** if for any elements $x, y, z, u, v \in \mathcal{L}$ the following conditions hold:

$$\overline{[x, y, z]} = \bar{x} + \bar{y} + \bar{z}, \quad (3.9)$$

$$-(-1)^{\bar{x}\bar{y}}[y, x, z] = [x, y, z] = -(-1)^{\bar{y}\bar{z}}[x, z, y], \quad (3.10)$$

$$\begin{aligned} [[x, y, z], u, v] = & (-1)^{(\bar{y}+\bar{z})(\bar{u}+\bar{v})} [[x, u, v], y, z] \\ & + (-1)^{\bar{x}(\bar{y}+\bar{z})+(\bar{x}+\bar{z})(\bar{u}+\bar{v})} [[y, u, v], z, x] \\ & + (-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{u}+\bar{v})} [[z, u, v], x, y]. \end{aligned} \quad (3.11)$$

Condition (3.10) will be called skew-symmetry. Condition (3.11) will be called the Filippov-Jacobi identity.

3.2 Transposed Poisson Superalgebras

This section is devoted to the one of the main structures for this thesis, namely the transposed Poisson superalgebra. The transposed Poisson superalgebra is a generalization of the transposed Poisson algebra. Such a generalization was first introduced and studied in the paper [1]. In this section we give the definition of a transposed Poisson superalgebra, present a wide class of transposed Poisson superalgebras constructed on a superalgebra by means of its even derivation, and recall other important results on transposed Poisson superalgebras proved in [1]. Among them, the most important for this work are the identities satisfied in a transposed Poisson superalgebra. These non-trivial identities are essential for proving the main results of this work.

Let $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1$ be a vector superspace with two bilinear operations $\cdot : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ and $[\cdot, \cdot] : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ such that (\mathcal{T}, \cdot) is a commutative superalgebra and $(\mathcal{T}, [\cdot, \cdot])$ is a Lie superalgebra.

Definition 15. *A triple $(\mathcal{T}, \cdot, [\cdot, \cdot])$ is called a **transposed Poisson superalgebra** if the following compatibility condition holds*

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{\bar{x}\bar{z}}[x, z \cdot y], \quad (3.12)$$

for any $x, y, z \in \mathcal{T}$.

Theorem 2 provides a wide class of transposed Poisson algebras constructed on a commutative algebra by means of its derivation. This theorem can be extended to transposed Poisson superalgebras. This was done in [1], where the following theorem was proved:

Theorem 3. *Let (\mathcal{A}, \cdot) be a commutative superalgebra and $D : \mathcal{A} \rightarrow \mathcal{A}$ its even derivation. Define a bracket by*

$$[x, y] = x \cdot D(y) - (-1)^{\bar{x}\bar{y}} y \cdot D(x). \quad (3.13)$$

Then $(\mathcal{A}, \cdot, [,])$ is a transposed Poisson superalgebra.

In [1], geometric realizations of this construction are also provided. Let M^n be an n -dimensional smooth manifold and let $\Omega(M^n) = \bigoplus_p \Omega^p(M^n)$ be the algebra of differential forms on M^n , where the algebraic operation is the exterior product \wedge . This algebra is graded if the degree p of a form $\omega \in \Omega^p(M^n)$ is used as grading. If instead we take $p \pmod{2}$ as grading, we obtain a vector superspace $\Omega(M^n) = \Omega_0(M^n) \oplus \Omega_1(M^n)$. This vector superspace is a commutative superalgebra with respect to the exterior product \wedge , since

$$\omega \wedge \theta = (-1)^{pq} \theta \wedge \omega, \quad \omega \in \Omega^p(M^n), \theta \in \Omega^q(M^n).$$

Let X be a smooth vector field on M^n . For any integer p , the interior product operator (contraction) $i_X : \Omega^p(M^n) \rightarrow \Omega^{p-1}(M^n)$ and the exterior derivative $d : \Omega^p(M^n) \rightarrow \Omega^{p+1}(M^n)$ are odd derivations of the commutative superalgebra of differential forms. The Lie derivative $\mathcal{L}_X = d \circ i_X + i_X \circ d$ is an even derivation of this superalgebra, i.e. $\mathcal{L}_X : \Omega^p(M^n) \rightarrow \Omega^p(M^n)$ and satisfies the Leibniz rule

$$\mathcal{L}_X(\omega \wedge \theta) = \mathcal{L}_X(\omega) \wedge \theta + \omega \wedge \mathcal{L}_X(\theta).$$

Using formula (3.13), for any vector field X we define a bracket of two differential forms $\omega \in \Omega^p(M^n)$ and $\theta \in \Omega^q(M^n)$ by

$$[\omega, \theta]_X = \omega \wedge \mathcal{L}_X(\theta) - (-1)^{pq} \theta \wedge \mathcal{L}_X(\omega), \quad (3.14)$$

and extend this bracket by linearity to the whole superalgebra $\Omega(M^n)$. The com-

mutative superalgebra of differential forms $(\Omega(M^n), \wedge)$ equipped with the bracket (3.14) is a transposed Poisson superalgebra.

In [1], a classification of transposed Poisson superalgebras in low dimensions is also given. Let $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1$ be a vector superspace of type $(1, 1)$. Let $e_0 \in \mathcal{T}_0$ and $e_1 \in \mathcal{T}_1$ be basis vectors. It is shown in [1] that there exist four transposed Poisson superalgebras on \mathcal{T} :

1. $e_0 \cdot e_0 = e_0, e_0 \cdot e_1 = 0, e_1 \cdot e_1 = 0,$
 $[e_0, e_0] = 0, [e_0, e_1] = 0, [e_1, e_1] = 0.$
2. $e_0 \cdot e_0 = e_0, e_0 \cdot e_1 = e_1, e_1 \cdot e_1 = 0,$
 $[e_0, e_0] = 0, [e_0, e_1] = 0, [e_1, e_1] = 0.$
3. $e_0 \cdot e_0 = e_0, e_0 \cdot e_1 = 0, e_1 \cdot e_1 = 0,$
 $[e_0, e_0] = 0, [e_0, e_1] = 0, [e_1, e_1] = e_0.$
4. $e_0 \cdot e_0 = e_0, e_0 \cdot e_1 = e_1, e_1 \cdot e_1 = 0,$
 $[e_0, e_0] = 0, [e_0, e_1] = e_1, [e_1, e_1] = 0.$

In [1], the following identities were proved, which play a crucial role in obtaining all results related to transposed Poisson superalgebras. Since this result is important for this thesis, we state it in full as a theorem.

Theorem 4. *Let $(P, \cdot, [,]) be a transposed Poisson superalgebra. Then for any elements $h, x, y, z, u, v \in P$ the following identities hold:$*

$$(-1)^{\bar{x}\bar{z}}x[y, z] + (-1)^{\bar{x}\bar{y}}y[z, x] + (-1)^{\bar{y}\bar{z}}z[x, y] = 0, \quad (3.15)$$

$$(-1)^{\bar{x}\bar{z}}[h \cdot [x, y], z] + (-1)^{\bar{x}\bar{y}}[h \cdot [y, z], x] + (-1)^{\bar{y}\bar{z}}[h \cdot [z, x], y] = 0, \quad (3.16)$$

$$(-1)^{\bar{x}\bar{z}}[h \cdot x, [y, z]] + (-1)^{\bar{x}\bar{y}}[h \cdot y, [z, x]] + (-1)^{\bar{y}\bar{z}}[h \cdot z, [x, y]] = 0, \quad (3.17)$$

$$(-1)^{\bar{x}\bar{z}}[h, x][y, z] + (-1)^{\bar{x}\bar{y}}[h, y][z, x] + (-1)^{\bar{y}\bar{z}}[h, z][x, y] = 0, \quad (3.18)$$

$$2u \cdot v \cdot [x, y] = (-1)^{\bar{x}\bar{v}}[u \cdot x, v \cdot y] + (-1)^{\bar{u}(\bar{x}+\bar{v})}[v \cdot x, u \cdot y] = 0, \quad (3.19)$$

$$(-1)^{\bar{u}(\bar{y}+\bar{v})}x \cdot [u, y \cdot v] + (-1)^{(\bar{x}+\bar{y})\bar{v}}v \cdot [x \cdot y, u] + (-1)^{\bar{x}(\bar{y}+\bar{v})}y \cdot [v, x] \cdot u = 0. \quad (3.20)$$

4 Transposed Poisson 3-Lie Superalgebras

This chapter is the key part of the present work, since it contains the main original result. In this chapter we introduce a new notion of a transposed Poisson 3-Lie superalgebra. The transposed Poisson 3-Lie superalgebra is a generalization of the transposed Poisson 3-Lie algebra, whose definition was first given in the paper [4]. The results presented in this chapter continue the investigations carried out in the thesis [20] and in the paper [2]. In these works, a ternary Lie bracket on a transposed Poisson superalgebra was constructed and studied. In the present chapter we prove that this bracket satisfies an identity which can be regarded as a ternary analogue of the compatibility condition in a transposed Poisson superalgebra (3.12). This identity is the key compatibility condition in the definition of a transposed Poisson 3-Lie superalgebra. Finally, using the proved theorem, we construct a wide class of transposed Poisson 3-Lie superalgebras.

Definition 16. *A triple $(\mathcal{T}, \cdot, [\ , \ , \])$ is called a **transposed Poisson 3-Lie superalgebra** if (\mathcal{T}, \cdot) is a commutative superalgebra, $(\mathcal{T}, [\ , \ , \])$ is a 3-Lie superalgebra, and for any elements $x, y, z, u \in \mathcal{T}$ the following compatibility condition holds:*

$$3u[x, y, z] = [ux, y, z] + (-1)^{\bar{x}\bar{u}}[x, uy, z] + (-1)^{\bar{u}(\bar{x}+\bar{y})}[x, y, uz]. \quad (4.1)$$

In the paper [2], a graded ternary bracket was introduced:

$$[x, y, z] := D(x) \cdot [y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}D(y) \cdot [z, x] + (-1)^{(\bar{x}+\bar{y})\bar{z}}D(z) \cdot [x, y], \quad (4.2)$$

and it was shown that this ternary bracket defines a 3-Lie superalgebra structure. Thus, the following theorem was proved:

Theorem 5. *Let $(\mathcal{T}, \cdot, [\ , \ , \])$ be a transposed Poisson superalgebra and let D be its even derivation. Define the ternary bracket by formula (4.2). Then $(\mathcal{T}, [\ , \ , \])$ is a*

3-Lie superalgebra.

Our goal in this chapter is to show that the bracket (4.2) also satisfies the compatibility condition (4.1), that is, we obtain a transposed Poisson 3-Lie superalgebra.

Theorem 6. *Let $(\mathcal{T}, \cdot, [, ,])$ be a transposed Poisson superalgebra and let $D : \mathcal{T} \rightarrow \mathcal{T}$ be its even derivation. Then the ternary bracket (4.2) satisfies the compatibility condition (4.1), and $(\mathcal{T}, \cdot, [, ,])$ is a transposed Poisson 3-Lie superalgebra.*

Proof. From Theorem 5 it follows that the ternary bracket (4.2) defines a 3-Lie superalgebra on \mathcal{T} . Now we show that this construction also satisfies the compatibility condition (4.1). For this reason, let us compute both sides of the equality (4.1). The left-hand side is equal to

$$\begin{aligned} \text{LHS} &:= 3u[x, y, z] \\ &= 3uD(x)[y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}3uD(y)[z, x] + (-1)^{(\bar{x}+\bar{y})\bar{z}}3uD(z)[x, y]. \end{aligned} \quad (4.3)$$

The terms in the right-hand side can be computed using Leibniz rule for derivation D as follows:

$$\begin{aligned} \text{I} &:= [ux, y, z] \\ &= D(ux)[y, z] + (-1)^{(\bar{u}+\bar{x})(\bar{y}+\bar{z})}D(y)[z, ux] + (-1)^{(\bar{u}+\bar{x}+\bar{y})\bar{z}}D(z)[ux, y] \\ &= D(u)x[y, z] + uD(x)[y, z] + (-1)^{(\bar{u}+\bar{x})(\bar{y}+\bar{z})}D(y)[z, ux] + (-1)^{(\bar{u}+\bar{x}+\bar{y})\bar{z}}D(z)[ux, y], \end{aligned} \quad (4.4)$$

$$\begin{aligned} \text{II} &:= (-1)^{\bar{x}\bar{u}}[x, uy, z] \\ &= (-1)^{\bar{x}\bar{u}}\left(D(x)[uy, z] + (-1)^{\bar{x}(\bar{u}+\bar{y}+\bar{z})}D(uy)[z, x] + (-1)^{(\bar{x}+\bar{u}+\bar{y})\bar{z}}D(z)[x, uy]\right) \\ &= (-1)^{\bar{x}\bar{u}}D(x)[uy, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}D(u)y[z, x] + (-1)^{\bar{x}(\bar{y}+\bar{z})}uD(y)[z, x] \\ &\quad + (-1)^{\bar{x}\bar{u}+(\bar{x}+\bar{u}+\bar{y})\bar{z}}D(z)[x, uy], \end{aligned} \quad (4.5)$$

$$\begin{aligned}
\text{III} &:= (-1)^{\bar{u}(\bar{x}+\bar{y})}[x, y, uz] \\
&= (-1)^{\bar{u}(\bar{x}+\bar{y})} \left(D(x)[y, uz] + (-1)^{\bar{x}(\bar{y}+\bar{u}+\bar{z})} D(y)[uz, x] + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{z})} D(uz)[x, y] \right) \\
&= (-1)^{\bar{u}(\bar{x}+\bar{y})} D(x)[y, uz] + (-1)^{\bar{u}\bar{y}+\bar{x}(\bar{y}+\bar{z})} D(y)[uz, x] + (-1)^{(\bar{x}+\bar{y})\bar{z}} D(u)z[x, y] \\
&\quad + (-1)^{(\bar{x}+\bar{y})\bar{z}} uD(z)[x, y].
\end{aligned} \tag{4.6}$$

In the left-hand side there is no term with the factor $D(u)$. So consider the terms with factor $D(u)$ in the right-hand side. Their sum is

$$D(u)(x[y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}y[z, x] + (-1)^{(\bar{x}+\bar{y})\bar{z}}z[x, y]).$$

This sum equals to zero due to identity (3.15). Next, consider the terms with the factor $D(x)$

$$uD(x)[y, z] + (-1)^{\bar{x}\bar{u}}D(x)[uy, z] + (-1)^{\bar{u}(\bar{x}+\bar{y})}D(x)[y, uz]. \tag{4.7}$$

The last two terms can be transformed using the compatibility condition (3.12) of transposed Poisson superalgebra as follows:

$$\begin{aligned}
(-1)^{\bar{x}\bar{u}}D(x)[uy, z] + (-1)^{\bar{u}(\bar{x}+\bar{y})}D(x)[y, uz] &= (-1)^{\bar{x}\bar{u}}D(x)([uy, z] + (-1)^{\bar{u}\bar{y}}[y, uz]) \\
&= (-1)^{\bar{x}\bar{u}}D(x) \cdot 2u[x, y] \\
&= 2uD(x)[y, z].
\end{aligned}$$

Thus the terms with $D(x)$ (4.7) together equal to $3uD(x)[y, z]$. Analogously the sum of terms with $D(y)$ in the right-hand side is equal to $(-1)^{\bar{x}(\bar{y}+\bar{z})}3uD(y)[z, x]$ and the sum of terms with $D(z)$ is equal to $(-1)^{(\bar{x}+\bar{y})\bar{z}}3uD(z)[x, y]$. Thus the left-hand side is equal to the right-hand side and the equality (4.1) holds.

□

Theorem 6 shows that there exists a wide class of transposed 3-Lie Poisson superalgebras. Moreover, combining the results of the papers [1], [2] with the proved Theorem 6, we obtain the following construction of a transposed 3-Lie Poisson superalgebra based on a commutative superalgebra and its two even commuting derivations:

Corollary 6.1. *Let (\mathcal{A}, \cdot) be a commutative superalgebra and D_1, D_2 its even commuting derivations. Define a ternary operation as follows:*

$$\begin{aligned} [x, y, z] := & D_2(x)(yD_1(z) - (-1)^{\bar{y}\bar{z}}zD_1(y)) \\ & + (-1)^{\bar{x}(\bar{y}+\bar{z})}D_2(y)(zD_1(x) - (-1)^{\bar{x}\bar{z}}xD_1(z)) \\ & + (-1)^{(\bar{x}+\bar{y})\bar{z}}D_2(z)(xD_1(y) - (-1)^{\bar{x}\bar{y}}yD_1(x)). \end{aligned} \quad (4.8)$$

Then $(\mathcal{A}, \cdot, [, ,])$ is transposed Poisson 3-Lie superalgebra.

Proof. Remark that on commutative superalgebra (\mathcal{A}, \cdot) with even derivation D_1 we can define Lie superalgebra with the bracket

$$[x, y] := xD_1(y) - (-1)^{xy}yD_1(x).$$

Due to Theorem 3, $(\mathcal{A}, \cdot, [, ,])$ is a transposed Poisson superalgebra. Thereby to use the previous Theorem 6 we have to show that D_2 is even derivation of Lie superalgebra $(\mathcal{A}, [, ,])$, that is, we have to prove the following:

$$D_2([x, y]) = [D_2(x), y] + [x, D_2(y)].$$

The proof is a direct computation, given that D_2 is even derivation of (\mathcal{A}, \cdot) and

that two derivations D_1, D_2 commute, $D_2 \circ D_1 = D_1 \circ D_2$.

$$\begin{aligned}
D_2([x, y]) &= D_2(xD_1(y) - (-1)^{xy}yD_1(x)) \\
&= D_2(xD_1(y)) - (-1)^{\bar{x}\bar{y}}D_2(yD_1(x)) \\
&= D_2(x)D_1(y) + xD_2(D_1(y)) - (-1)^{\bar{x}\bar{y}}(D_2(y)D_1(x) + yD_2(D_1(x))) \\
&= \left(D_2(x)D_1(y) - (-1)^{\bar{x}\bar{y}}yD_1(D_2(x)) \right) + \left(xD_1(D_2(y)) - (-1)^{\bar{x}\bar{y}}D_2(y)D_1(x) \right) \\
&= [D_2(x), y] + [x, D_2(y)].
\end{aligned}$$

Thus by Theorem 6 we get that $(\mathcal{A}, \cdot, [, ,])$ is transposed Poisson 3-Lie superalgebra. □

Thus, in this chapter, a new notion of a transposed Poisson 3-Lie superalgebra has been introduced, and a wide class of such algebras has been constructed. This structure has the potential to be applicable in various areas of modern geometry and theoretical physics.

Conclusion

This thesis continues the study of the structure of 3-Lie superalgebras presented in the paper [2]. Assuming only a basic knowledge of abstract algebra from the reader, we introduce all necessary notions and constructions required to define the new structure of a transposed Poisson 3-Lie superalgebra. We provide a generalization of the structure first introduced in [4]. The main result of the thesis is a theorem stating that the construction from [2] defines a transposed Poisson 3-Lie superalgebra structure. Using a transposed Poisson superalgebra and its even derivation, we construct a transposed Poisson 3-Lie superalgebra. Moreover, using a commutative superalgebra and its two even commuting derivations, we construct a wide class of transposed Poisson 3-Lie superalgebras.

It should be noted that in all constructions presented in this thesis, the condition for a derivations to be even is essential. The next direction of research in this area is to extend the obtained results to the case of odd derivations. For this purpose, it may be necessary to incorporate additional algebraic structures such as Jordan algebras [2].

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