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Esimese järgu kvantkromodünaamilised kiirgusparandid polariseeritud W^+-bosoni hadronlagunemisel

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Sisukord

1 Sissejuhatus

Uue füüsika nähtuste leidmine vajab Standardmudeli täpseid teoreetilisi arvutusi. Seda saab saavutada ainult siis, kui arvutusmudelitesse lisatakse kõrgema järgu kiirgusparandeid. Kõrgema järgu liikmed ilmuvad kui reaalsed parandid, sidudes endas footonite ja gluuonite kiirgust kui virtuaalseid osakesi, mis esinevad Feynmani diagrammides suletud silmuste näol. Täpseid mõõtmisi, mis on tundlikud kiirgusparandite suhtes, võib vaadelda, kui elektronõrga teooria testimist silmuselise parandi tasemel. Veelgi enam, kvantväljateooria (KVT) peab püsima ennustatavana ka siis, kui kõrgema järgu liikmeid lisatakse (nt peenstruktuurikonstant α). Samas kui avastatakse anomaalia, siis võib see tähendada mainitud uue füüsika ilminguid (nt supersümmeetria või mitme Higgs-bosoni mudelid).

Uut osakestekiirendit *Large Hadron Collider*-it (LHC) CERNis (masskeskme energiaga 14 TeV) võib vaadelda kui suurt top-kvargi tehast ja seetõttu võimaldab see tulevikus rohkesti täpsustavaid mõõtmisi top-kvargi füüsikas - nii tekke- kui ka lagunemisprotsessides. Top-kvargid lagunevad rohkem kui 99.9%-lise tõenäosusega protsessi $t \to b + W^+$ [1,2,3] kohaselt, kus W-bosoni polarisatsioon antud lagunemises on jaotunud u. 70% pikisuunaliseks ja u 30% põiki(miinus)suunaliseks. Seda saab võrrelda Drell-Yan protsessis W-bosoni tekitamisega, kus W on 100% põikisuunaliselt polariseeritud. Seega pakub top-kvargi lagunemisel saadav W unikaalse võimaluse testida pikisuunaliselt polariseeritud W-bosoni lagunemist, mis omakorda võimaldab uurida elektronõrga sümmeetria rikkumise mehhanismi. W-bosoni polarisatsiooni saab kasutada ka selleks, et tema tekkimisel paarsust rikkuvaid interaktsioone analüüsida. W-boson on ennastanalüüsiv, mis tähendab seda, et saame mõõta W-bosoni polarisatsiooni, mõõtes tema lagunemisproduktide nurkjaotust.

Et hinnata kiirgusparandite mõju W-bosoni tekkimisel protsessis $t \to b + W^+$ on arvutatud ühesilmuselised kvantkromodünaamilised (KKD) ja elektronõrgad parandid protsessile $t \to b + W^+(\uparrow)$ (ka protsessile $b \to c + W^+(\uparrow)$)[4,5,6,7,8]. Samuti on arvutatud ühesilmuselised elektronõrgad parandid mittepolariseeritud W leptonlagunemisele $W^+ \to l^+ + \nu_l$ [9]. Massitud (st. kõrge energia korral) ühesilmuselised KKD parandid hadronlagunemisele $W^+ \to Q + \bar{q}$ (kus Q on up-tüüpi kvark ja $\bar{q}$ on down-tüüpi antikvark) saab arvutada [9] järgmise juhtiva järguni (NLO) tulemuste kaudu [10,11,12,13]. Kuid siiski ei ole senini arvutatud protsessi nii, et arvesse oleks võetud erinevaid massiparameetreid.

Käesoleva magistritöö eesmärk on analüütiliselt esitada esimese järgu kvantkromodünaamilised kiirgusparandid polariseeritud W^+-bosoni hadronlagunemisel nii, et arvesse on võetud ka lagunemisproduktide massiparameetrid. Selleks lahendame esmalt protsessi, kus polariseeritud W^+-boson laguneb kaheks kvargiks, seejärel lahendame verteksparandi protsessi ning lõpetuseks uurime protsessi, kus peale kahe kvargi kiirgub ka pehme gluuon (nn *bremsstrahlung*).

Töö esimene osa kujutab sissejuhatust teemasse. Seal anname ülevaate uuritava teema olulisusest. Samas tutvustame ka kasutatavat matemaatilist aparatuuri. Töö teises osas lahendame analüütiliselt polariseeritud W^+-bosoni lagunemise kaheks kvargiks ja leiame verteksparandid. Verteksparandite peatükis lahendame kujutegurite renormeerimise juures ka ultravioletthajumiste probleemi. Kolmandas peatükis uurime protsessi, kus polariseeritud W^+-bosoni lagunemisel peale kvargi ja antikvargi kiirgub ka pehme gluuon. Neljandas peatükis lahendame infrapunaste hajuvuste probleemi ja esitame uuritava protsessi täistulemused, mis on töö põhitulemusteks. Lisaks esitame täistulemuste piirjuhud väikeste masside ja läve piirkonnas. Viiendas osas esitame tulemused graafiliselt ning võrdleme neid piirjuhtude vastustega. Magistritöö lõpetavad kokkuvõte, kasutatud kirjanduse loetelu ning lisad A, B, C ja D.

Antud töös kasutame üldjuhul sama tähistust nagu Peskin ja Schroeder [14], lisaks eelmainitule on uurimuses läbivalt veel kasutatud raamatuid [15,16,17]. Ühikud on valitud nii, et $\hbar = c = 1$. Diraci võrrandis ja adjungeeritud võrrandis

$$(\not{p} - m)\psi(p,s) = 0, \qquad \bar{\psi}(p,s)(\not{p} - m) = 0, \qquad \bar{\psi}(p,s) = \psi^\dagger(p,s)\gamma^0 \tag{1}$$

(kus on kasutatud tähistust $\not{p} = p_\mu\gamma^\mu$) esinevad Diraci maatriksid γ^μ, mis on neljadimensionaalsed algebralised elemendid, mille jaoks kehtivad erinevad esitused. Nende hulgas on osakeste–antiosakeste esitus ja spiraalesitus. Antud töös kasutame osakeste–antiosakeste esitust

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{2}$$

kus $\mathbb{1}_2$ on kahedimensionaalne ühikmaatriks ja σ_i $(i = 1,2,3)$ on kahedimensionaalsed Pauli maatriksid

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Algebra on defineeritud kui

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{4}$$

Veel esinevad elemendid

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] := \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right), \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5. \tag{5}$$

γ_5 jaoks kehtib lisareegel

$$\{\gamma_5, \gamma^\mu\} = \gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 0. \tag{6}$$

Diraci maatriksite ja maatriksi γ_5 jälg on null, $\mathrm{tr}(\gamma^\mu) = \mathrm{tr}(\gamma_5) = 0$, neljadimensionaalse ühikmaatriksi jälg on $\mathrm{tr}(\mathbb{1}) = 4$. Lisaks on veel reeglid

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu}, \qquad \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) \;=\; 0, \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right), \qquad \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;=\; -4i\varepsilon^{\mu\nu\rho\sigma}, \\
\mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0, \qquad \mathrm{tr}(\gamma_5\gamma^{\mu_1}\cdots\gamma^{\mu_n}) \;=\; 0, \qquad \text{kui } n \text{ on paaritu} \\
\mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 4\Big(g^{\mu_1\mu_2}\mathrm{tr}(\gamma^{\mu_3}\cdots\gamma^{\mu_n}) - g^{\mu_1\mu_3}\mathrm{tr}(\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_n}) + \\
&\quad + g^{\mu_1\mu_4}\mathrm{tr}(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_5}\cdots\gamma^{\mu_n}) \mp \ldots + g^{\mu_1\mu_n}\mathrm{tr}(\gamma^{\mu_2}\cdots\gamma^{\mu_{n-1}})\Big).
\end{aligned} \tag{7}$$

2 Kahe osakese protsess: Borni diagramm ja silmusdiagramm.

Fermi kuldreegli järgi saab W^+-bosoni lagunemisäära kirja panna kui

$$\Gamma_i(W^+ \to X_i) = \frac{1}{2m_W}\int dPS_i\, |\mathcal{M}|^2\,, \tag{8}$$

kus $i = 2, 3$ kirjeldab tekkivate osakeste arvu, $X = Q + \bar{q}$ puhul $i = 2$, $X = Q + \bar{q} + G$ puhul $i = 3$. Me alustame selles peatükis kaheosakeselise (pea)protsessiga, kolmeosakeseliste protsessi käsitleme peatükis 3. dPS_i märgib faasiruumi osa ning $\mathcal{M}$ on nn Feynmani amplituud, mis kirjeldab lagunemisprotsessi.

2.1 Kahe osakese kinemaatika ja faasiruum

Tähistame q-ga W^+-bosoni impulsi, p_1-ga kvargi Q impulsi ja p_2-ga antikvargi $\bar{q}$ impulsi. m_W tähistab algosakese ehk W-bosoni massi. Arvutame kahe osakese kinemaatika paigalsüsteemis. Nelivektorites $p_1 = (E_1; \vec{p}_1)$ ja $p_2 = (E_2; \vec{p}_2)$ on meil neli tundmatut suurust, aga samuti ka neli järgnevat seosevõrrandit

$$E_1^2 = \vec{p}_1^{\,2} + m_1^2, \qquad E_2^2 = \vec{p}_2^{\,2} + m_2^2, \qquad E_1 + E_2 = \sqrt{q^2}, \qquad \vec{p}_1 + \vec{p}_2 = \vec{0}. \tag{9}$$

Saame

$$\vec{p}_1^2 + m_1^2 = \vec{p}_2^2 + m_2^2 = E_2^2 = (\sqrt{q^2} - E_1)^2 = q^2 + E_1^2 - 2E_1\sqrt{q^2} = q^2 + m_1^2 + \vec{p}_1^2 - 2E_1\sqrt{q^2} \quad (10)$$

ehk

$$2E_1\sqrt{q^2} = q^2 + m_1^2 - m_2^2 = (1 + \mu_1 - \mu_2)q^2 \quad \Leftrightarrow \quad E_1 = \frac{1}{2}(1 + \mu_1 - \mu_2)\sqrt{q^2} \quad (11)$$

ja vastavalt $E_2 = \sqrt{q^2} - E_1 = \frac{1}{2}(1 - \mu_1 + \mu_2)\sqrt{q^2}$ ning

$$\vec{p}_1^2 = E_1^2 - m_1^2 = \frac{1}{4}(1 + \mu_1 - \mu_2)^2 q^2 - \mu_1 q^2 = \frac{1}{4}\lambda(1, \mu_1, \mu_2)q^2, \quad (12)$$

kus

$$\lambda(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \quad (13)$$

on Källéni funktsioon. Kui valime z-telje kvargi liikumise suunas, on meil

$$\begin{aligned} p_1 &= \frac{1}{2}\left(1 + \mu_1 - \mu_2; 0, 0, \sqrt{\lambda(1, \mu_1, \mu_2)}\right), \\ p_2 &= \frac{1}{2}\left(1 - \mu_1 + \mu_2; 0, 0, -\sqrt{\lambda(1, \mu_1, \mu_2)}\right). \end{aligned} \quad (14)$$

Kahe osakese faasiruum on nüüd esitatav valemiga

$$\begin{aligned} dPS_2 &= (2\pi)^4\delta^{(4)}(p_1 + p_2 - q)\prod_{i=1}^{2}\frac{d^4p_i}{(2\pi)^4}(2\pi)\delta(p_i^2 - m_i^2)\theta(p_i^0) = \\ &= \frac{d^4p_1}{(2\pi)^4}(2\pi)\delta(p_1^2 - m_1^2)\theta(E_1)\delta\left((q - p_1)^2 - m_2^2\right)\theta(q^0 - E_1), \end{aligned} \quad (15)$$

milles asenduse $p_2 = q - p_1$ tegemiseks on kasutatud neljadimensionaalset deltafunktsiooni. Kasutades võrdust

$$\frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2)\theta(E) = \frac{d^3p}{(2\pi)^3 2E}, \qquad E = \sqrt{\vec{p}^2 + m^2}, \quad (16)$$

saame

$$dPS_2 = \frac{d^3p_1}{(2\pi)^3 2E_1}\delta\left((q - p_1)^2 - m_2^2\right)\theta(q^0 - E_1)\Big|_{E_1 = \sqrt{\vec{p}_1^2 + m_1^2}}. \quad (17)$$

Samas on teada, et

$$(q - p_1)^2 - m_2^2 = q^2 - 2E_1\sqrt{q^2} + m_1^2 - m_2^2 = -2\sqrt{q^2}\left(E_1 - \frac{1}{2\sqrt{q^2}}(q^2 + m_1^2 - m_2^2)\right) \quad (18)$$

ja sellega saame leida

$$\delta\left((q - p_1)^2 - m_2^2\right) = \frac{1}{2\sqrt{q^2}}\delta\left(E_1 - \frac{1}{2\sqrt{q^2}}(q^2 + m_1^2 - m_2^2)\right). \quad (19)$$

Lõpuks võime kolmedimensionaalse integraalimõõdu esitada polaarkoordinaatides ning ära kasutada $E_1^2 = \vec{p}_1^2 + m_1^2 \Rightarrow 2E_1 dE_1 = 2|\vec{p}_1| d|\vec{p}_1|$,

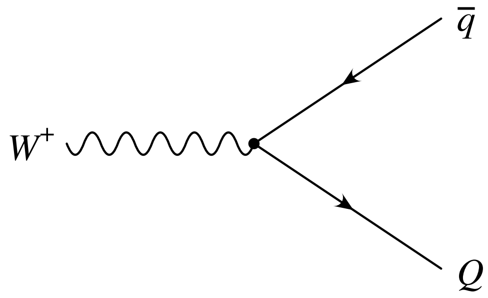
$$d^3 p_1 = |\vec{p}_1|^2 d|\vec{p}_1| d\varphi d(\cos\theta) = E_1 \sqrt{E_1^2 - m_1^2} dE_1 d\varphi d(\cos\theta). \tag{20}$$

Kordaja E_1 taandub. Deltafunktsiooni ära kasutades kaotame integreerimise üle E_1. Kuna ei ole oodata, et tulemused sõltuksid asimuutnurgast φ-st, siis üle selle integreerimine annab $\int d\varphi = 2\pi$. Kokkuvõtteks saame kirjutada, et

$$\begin{aligned} dPS_2 &= \frac{1}{(2\pi)^2 2\sqrt{q^2}} \delta\left(E_1 - \frac{1}{2\sqrt{q^2}}(q^2 + m_1^2 - m_2^2)\right) \frac{E_1}{2E_1} \sqrt{E_1^2 - m_1^2} dE_1 d\varphi d(\cos\theta) = \\ &= \frac{2\pi}{4(2\pi)^2 \sqrt{q^2}} \sqrt{\frac{1}{4q^2}\left(q^2 + m_1^2 - m_2^2\right)^2 - m_1^2}\; d(\cos\theta) = \\ &= \frac{1}{16\pi q^2} \sqrt{\lambda(q^2, m_1^2, m_2^2)}\; d(\cos\theta). \end{aligned} \tag{21}$$

2.2 Borni diagrammi maatrikselementide arvutamine

Vaatleme jooniselt 1 nn Borni taseme Feynmani diagrammi. Feynmani reeglite järgi vastab



Joonis 1: Borni taseme Feynmani diagramm protsessis $W^+(\uparrow) \to Q + \bar{q}$

sellele diagrammile avaldis

$$\mathcal{M} = \bar{u}(p_1, s_1) \left(-i\frac{g_W}{\sqrt{2}} V_{iI} \gamma^\alpha \frac{\mathbb{1} - \gamma_5}{2}\right) v(p_2, s_2) \varepsilon_\alpha(q, \lambda), \tag{22}$$

kus $u(p_1, s_1)$ on kvargi Q spiinor, $v(p_2, s_2)$ antikvargi $\bar{q}$ spiinor ja $\varepsilon_\alpha(q, \lambda)$ on W^+ bosoni lainefunktsioon. Nende vahel asub verteksfunktsioon

$$-i\frac{g_W}{\sqrt{2}} V_{iI} \gamma^\alpha \frac{\mathbb{1} - \gamma_5}{2}, \tag{23}$$

kus g_W on elektronõrk laeng, V_{iI} on Kobayashi–Maskawa segumaatriksi element, ja γ^α ja γ_5 on tuntud Diraci maatriksid. Et arvutada maatriksi absoluutväärtuse ruutu, läheb meil vaja ka suurust $\mathcal{M}^*$. Diraci maatrikside algebrat meeles pidades ning seda, et $\bar{u}(p,s) = u^\dagger(p,s)\gamma^0$, saame

$$\mathcal{M}^* = \bar{v}(p_2,s_2)\left(+i\frac{g_W}{\sqrt{2}}V_{iI}^*\gamma^\alpha\frac{\mathbb{1}-\gamma_5}{2}\right)u(p_1,s_1)\varepsilon_\alpha^*(q,\lambda). \tag{24}$$

Lõpuks tuleb veel summeerida üle mittevaadeldavate lõpposakeste spinnolekute. Selle jaoks kasutame võrrandeid

$$\sum_s u(p,s)\bar{u}(p,s) = (\not{p}+m), \qquad \sum_s v(p,s)\bar{v}(p,s) = (\not{p}-m). \tag{25}$$

Tulemuseks saame

$$|\mathcal{M}|^2 = \sum_{s_1,s_2}\mathcal{M}^*\mathcal{M} = \frac{g_W^2}{8}|V_{iI}|^2\mathrm{tr}\left((\not{p}_2-m_2)\gamma^\alpha(\mathbb{1}-\gamma_5)(\not{p}_1+m_1)\gamma^\beta(\mathbb{1}-\gamma_5)\right)\varepsilon_\alpha^*(q,\lambda)\varepsilon_\beta(q,\lambda). \tag{26}$$

Võrrandi (26) jäljeosaks saame

$$\begin{aligned}\mathrm{tr}_0^{\mu\nu} &= \mathrm{Tr}\left((\not{p}_1+m_1)\gamma^\mu(1-\gamma_5)(\not{p}_2-m_2)\gamma^\nu(1-\gamma_5)\right) = \\ &= 8\left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1p_2)g^{\mu\nu} + i\epsilon^{\mu\nu\rho\sigma}p_{1\rho}p_{2\sigma}\right),\end{aligned} \tag{27}$$

millest on näha, et massid taanduvad välja. Võrrandis (25) oleme summeerinud üle kvarkide spinnolekute, kuna neid protsessis ei vaadelda. Samas saame summeerida ka üle W-bosoni polarisatsiooniolekute, kui neid ei vaadelda. Summeerimine annab

$$\sum_\lambda \varepsilon_\alpha^*(q,\lambda)\varepsilon_\beta(q,\lambda) = -g_{\alpha\beta}. \tag{28}$$

Sellega saame

$$\mathrm{tr}_{0g} = -\mathrm{tr}_0^{\alpha\beta}g_{\alpha\beta} = 16(p_1p_2). \tag{29}$$

2.3 Borni taseme tulemused

Nagu sissejuhatuses mainitud, on W-boson iseanalüüsiv. See tähendab, et W-bosoni polarisatsioon avaldub tema laguproduktide liikumissuundade anisotroopiana. Sellist anisotroopiat saab kirjeldada nurksõltuvusena W-bosoni polarisatsiooni ja kvarkide liikumissuuna vahel. Sellepärast ei summeerita enam üle W-bosoni polarisatsiooni λ, vaid analüüsitakse erinevaid polarisatsiooni panuseid. Kinemaatiliselt lähtume taustsüsteemist, milles eelmise lagunemisprotsessi $t \to W + b$ top-kvark on paigal ja W-boson liigub z-telje suunas. Kui vahetame taustsüsteemi nii, et see liigub sama kiirusega kui W-boson z-telje suunas, siis on z-telg kui W-bosoni liikumissuund „kinni külmutatud”, ilma et W-boson antud taustsüsteemis ise liiguks. Selle uue z-telje suuna suhtes peame nüüd kvarkide liikumissuunda pöörama. Siiani olid

$$p_1 = \frac{1}{2}\sqrt{q^2}(1+\mu_1-\mu_2; 0, 0, \sqrt{\lambda}), \qquad p_2 = \frac{1}{2}\sqrt{q^2}(1-\mu_1+\mu_2; 0, 0, -\sqrt{\lambda}). \tag{30}$$

Need võrdused kehtisid eeldusel, et kvark liigub z-telje positiivses suunas. Kui me aga kasutame nn „kinni külmutatud” taustsüsteemi, kus z-teljel on W^+-bosoni endine liikumise suund, siis peame omakorda kvargi suunda nurga θ võrra pöörama. Leppides kokku, et see pööre toimub ümber positiivse y-telje, saame impulsid kirjutada kujul

$$\begin{aligned} p_1 &= \frac{1}{2}\sqrt{q^2}(1+\mu_1-\mu_2; \sqrt{\lambda}\sin\theta, 0, \sqrt{\lambda}\cos\theta), \\ p_2 &= \frac{1}{2}\sqrt{q^2}(1+\mu_1-\mu_2; -\sqrt{\lambda}\sin\theta, 0, -\sqrt{\lambda}\cos\theta). \end{aligned} \tag{31}$$

Kui võtame kvargid massituks, st. energiad on suured, siis jäljemaatriks on

$$\mathrm{tr}_0^{\mu\nu} = 8(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1 p_2) g^{\mu\nu} + i\epsilon^{\mu\nu\rho\sigma} p_{1\rho} p_{2\sigma} \tag{32}$$

ja

$$p_1 = \frac{1}{2}\sqrt{q^2}(1; \sin\theta, 0, \cos\theta), \qquad p_2 = \frac{1}{2}\sqrt{q^2}(1; -\sin\theta, 0, -\cos\theta). \tag{33}$$

Kasutades võrdust $p_1 p_2 = q^2(1+\sin^2\theta+\cos^2\theta)/4 = q^2/2$, saame tulemuseks

$$(\mathrm{tr}_0^{\mu\nu}) = 4q^4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta & i\sin\theta & -\sin\theta\cos\theta \\ 0 & -i\cos\theta & 1 & i\sin\theta \\ 0 & -\sin\theta\cos\theta & -i\sin\theta & \sin^2\theta \end{pmatrix}. \tag{34}$$

Polarisatsiooni nelivektoritega[1]

$$\varepsilon(0) = (1; 0, 0, 0), \qquad \varepsilon(\pm) = \frac{1}{\sqrt{2}}(0; \mp 1, -i, 0), \qquad \varepsilon(3) = (0; 0, 0, 1) \tag{35}$$

võrrandit (34) ahendades saame nüüd vastavad jaotused:

$$\begin{aligned} \mathrm{tr}_0^{00} &= \varepsilon_\alpha(0)\mathrm{tr}_0^{\alpha\beta}\varepsilon_\beta^*(0) = 0, \\ \mathrm{tr}_0^{++} &= \varepsilon_\alpha(+)\mathrm{tr}_0^{\alpha\beta}\varepsilon_\beta^*(+) = 2q^2(-1, -i)\begin{pmatrix} \cos^2\theta & i\cos\theta \\ -i\cos\theta & 1 \end{pmatrix}\begin{pmatrix} -1 \\ i \end{pmatrix} = 2q^2(1+\cos\theta)^2, \end{aligned}$$

[1]Tihti kasutatakse puhast z-suuna vektorit $\lambda = 0$ ja esimene vektor jäetakse välja. Antud töös arvutame aga kõik (ka ajalised) polarisatsioonid.

$$
\begin{aligned}
\mathrm{tr}_0^{--} &= \varepsilon_\alpha(-)\mathrm{tr}_0^{\alpha\beta}\varepsilon^*_\beta(-) = 2q^2(1,-i)\begin{pmatrix}\cos^2\theta & i\cos\theta\\ -i\cos\theta & 1\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 2q^2(1-\cos\theta)^2,\\
\mathrm{tr}_0^{33} &= \varepsilon_\alpha(3)\mathrm{tr}_0^{\alpha\beta}\varepsilon^*_\beta(3) = 4q^2\sin^2\theta.
\end{aligned} \tag{36}
$$

See vastab kirjanduses avaldatud tulemusele [5], et kui $t \to X_b + W^+$ lagunemisele järgneb polariseeritud W^+-bosoni lagunemine massituteks osakesteks, siis relatiivne lagunemismäär on

$$
\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} = \frac{3}{8}(1+\cos)^2\mathcal{F}_+ + \frac{3}{8}(1-\cos\theta)^2\mathcal{F}_- + \frac{3}{4}\sin^2\theta\mathcal{F}_3. \tag{37}
$$

Pöördudes nüüd tagasi mittekaduvate masside juurde, saame jäljemaatriksi $(\mathrm{tr}_0^{\mu\nu})/4q^2 =$

$$
= \begin{pmatrix}
\mu_1+\mu_2-(\mu_1-\mu_2)^2 & (\mu_1-\mu_2)\sqrt{\lambda}\sin\theta & 0 & (\mu_1-\mu_2)\sqrt{\lambda}\cos\theta\\
(\mu_1-\mu_2)\sqrt{\lambda}\sin\theta & 1-\mu_1-\mu_2-\lambda\sin^2\theta & i\sqrt{\lambda}\cos\theta & -\lambda\sin\theta\cos\theta\\
0 & -i\sqrt{\lambda}\cos\theta & 1-\mu_1-\mu_2 & i\sqrt{\lambda}\sin\theta\\
(\mu_1-\mu_2)\sqrt{\lambda}\cos\theta & -\lambda\sin\theta\cos\theta & -i\sqrt{\lambda}\sin\theta & 1-\mu_1-\mu_2-\lambda\cos^2\theta
\end{pmatrix} \tag{38}
$$

ja vastavad projektsioonid

$$
\begin{aligned}
\mathrm{tr}_0^{00} &= \varepsilon_\alpha(0)\mathrm{tr}_0^{\alpha\beta}\varepsilon^*_\beta(0) = 4q^2\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right),\\
\mathrm{tr}_0^{++} &= \varepsilon_\alpha(+)\mathrm{tr}_0^{\alpha\beta}\varepsilon^*_\beta(+) = 2q^2\left(-(\mu_1-\mu_2)^2+(1+\sqrt{\lambda}\cos\theta)^2\right),\\
\mathrm{tr}_0^{--} &= \varepsilon_\alpha(-)\mathrm{tr}_0^{\alpha\beta}\varepsilon^*_\beta(-) = 2q^2\left(-(\mu_1-\mu_2)^2+(1-\sqrt{\lambda}\cos\theta)^2\right),\\
\mathrm{tr}_0^{33} &= \varepsilon_\alpha(3)\mathrm{tr}_0^{\alpha\beta}\varepsilon^*_\beta(3) = 4q^2\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2+\lambda\sin^2\theta\right).
\end{aligned} \tag{39}
$$

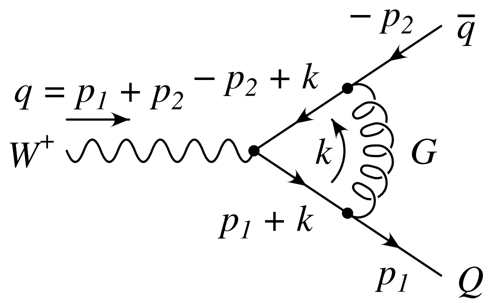
Pannes tähele, et $\mathrm{tr}_0^{++}+\mathrm{tr}_0^{--} = \mathrm{tr}_0^{11}+\mathrm{tr}_0^{22}$, saame näidata, et

$$
\begin{aligned}
&-\mathrm{tr}_0^{00}+\mathrm{tr}_0^{++}+\mathrm{tr}_0^{--}+\mathrm{tr}_0^{33} =\\
&= -4q^2\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)+2q^2\left(-(\mu_1-\mu_2)^2+(1+\sqrt{\lambda}\cos\theta)^2\right)+\\
&\quad +2q^2\left(-(\mu_1-\mu_2)^2+(1-\sqrt{\lambda}\cos\theta)^2\right)+4q^2\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2+\lambda\sin^2\theta\right) =\\
&= 4q^2\lambda\sin^2\theta-4q^2(\mu_1-\mu_2)^2+4q^2(1+\lambda\cos^2\theta) = 8q^2(1-\mu_1-\mu_2) = 16(p_1p_2),
\end{aligned} \tag{40}
$$

mis on kooskõlas valemiga (29).

2.4 Verteksparand

Kahe osakese protsessi täistulemuseks on meil vaja arvutada ka verteksparand. Joonisest 2



Joonis 2: Verteksparandi Feynmani diagramm $W^+(\uparrow) \to Q + \bar{q}$ protsessile

lähtudes saame Feynmani reegleid kasutades antud protsessi maatrikselemendiks järgneva avaldise

$$\begin{aligned}
\mathcal{M}^\mu &= \bar{u}(p_1,s_1)\int \frac{d^Dk}{(2\pi)^D}(-ig_s\gamma^\alpha T_a)\frac{i}{\not p_1+\not k-m_1}\times \\
&\times\left(-i\frac{g_W}{\sqrt{2}}V_{iI}\gamma^\mu\frac{1-\gamma_5}{2}\right)\frac{i}{-\not p_2+\not k-m_2}(-ig_s\gamma^\beta T_b)\frac{-ig_{\alpha\beta}\delta_{ab}}{k^2}v(p_2,s_2) = \\
&= -\frac{g_W g_s^2}{2\sqrt{2}}V_{iI}T_aT_a\bar{u}(p_1,s_1)\int\frac{d^Dk}{(2\pi)^D}\frac{\gamma^\alpha(\not p_1+\not k+m_1)\gamma^\mu(1-\gamma_5)(-\not p_2+\not k+m_2)\gamma_\alpha}{((p_1+k)^2-m_1^2)((p_2-k)^2-m_2)k^2}v(p_2,s_2) = \\
&=: -i\frac{g_W}{\sqrt{2}}V_{iI}\bar{u}(p_1,s_1)\Delta\Gamma^\mu_L v(p_2,s_2). \qquad (41)
\end{aligned}$$

Kui arvutaksime integraali neljadimensionaalses aegruumis, oleks integraal hajuv, mida ei ole võimalik arvutada. Selleks, et integraali arvutada saaks, peame seda regulariseerima. Regulariseerimine tähendab lihtsustatult seda, et muudame integraalis mõnda parameetrit nii, et integraal saab lõplikuks. Endine hajuvus väljendub siis selle muudatuse kaudu ja on sellepärast käsitletav. Järgnevalt räägime hajuvusest alati selle muudatuse mõttes. On olemas erinevad regulariseerimismeetodid. Puudiagrammide peatükis kasutame näiteks seda, et valime gluuoni massi mittekaduvaks, st massiregulariseerimist. Siin aga on see meetod sobimatu. Selle asemel kasutame dimensionaalset regulariseerimist. See tähendab, et me asendame integraalis neljadimensionaalse integreerimise D-dimensionaalse integreerimisega, kus $D = 4 - 2\varepsilon$ on neljast ainult natuke erinev. Nagu hiljem näeme, väljendub hajuvus siis panustega $1/\varepsilon$.

Aga jätkame integraaliga. Kuna integraali lugeja seisab kahe spinori vahel, siis saame lihtsustamiseks kasutada Diraci võrrandeid

$$\bar{u}(p_1,s_1)(\not p_1 - m_1) = 0 = (\not p_2 + m_2)v(p_2,s_2). \qquad (42)$$

Me saame

$$\bar{u}(p_1,s_1)\gamma^\alpha(\not p_1+m_1) = 2p_1^\alpha\bar{u}(p_1,s_1)+\bar{u}(p_1,s_1)(-\not p_1+m_1)\gamma^\alpha = 2p_1^\alpha\bar{u}(p_1,s_1),$$

$$(-\not{p}_2+m_2)\gamma_\alpha v(p_2,s_2) \;=\; -2p_{2\alpha}v(p_2,s_2)+\gamma_\alpha(\not{p}_2+m_2)v(p_2,s_2) \;=\; -2p_{2\alpha}v(p_2,s_2) \tag{43}$$

ja sellega lugeja L jaoks

$$\begin{aligned} L &= \bar{u}(p_1,s_1)\gamma^\alpha(\not{p}_1+\not{k}+m_1)\gamma^\mu(1-\gamma_5)(-\not{p}_2+\not{k}+m_2)\gamma_\alpha v(p_2,s_2) \;= \\ &= \bar{u}(p_1,s_1)(\gamma^\alpha\not{k}+2p_1^\alpha)\gamma^\mu(1-\gamma_5)(\not{k}\gamma_\alpha-2p_{2\alpha})v(p_2,s_2) \;= \\ &= \bar{u}(p_1,s_1)\gamma^\alpha\not{k}\gamma^\mu(1-\gamma_5)\not{k}\gamma_\alpha v(p_2,s_2)-2\bar{u}(p_1,s_1)\not{p}_2\not{k}\gamma^\mu(1-\gamma_5)v(p_2,s_2)+ \\ &\quad +2\bar{u}(p_1,s_1)\gamma^\mu(1-\gamma_5)\not{k}\not{p}_1 v(p_2,s_2)-4p_1p_2\bar{u}(p_1,s_1)\gamma^\mu(1-\gamma_5)v(p_2,s_2). \end{aligned} \tag{44}$$

Vaatame võrrandi (44) panuseid eraldi (neist viimane liige jääb samaks)

$$\begin{aligned} \bar{u}\gamma^\alpha\not{k}\gamma^\mu(1-\gamma_5)\not{k}\gamma_\alpha v &= 2k^2\bar{u}\gamma^\mu(1-\gamma_5)v-\bar{u}\gamma^\alpha\not{k}\gamma^\mu(1-\gamma_5)\gamma_\alpha\not{k}v \;= \\ &= 2k^2\bar{u}\gamma^\mu(1-\gamma_5)v-\bar{u}\gamma^\alpha\not{k}\gamma^\mu\gamma_\alpha(1+\gamma_5)\not{k}v \;= \\ &= 2k^2\bar{u}\gamma^\mu(1-\gamma_5)v-2\bar{u}\gamma^\mu\not{k}(1+\gamma_5)\not{k}v+\bar{u}\gamma^\alpha\not{k}\gamma_\alpha\gamma^\mu(1+\gamma_5)\not{k}v \;= \\ &= 2k^2\bar{u}\gamma^\mu(1-\gamma_5)v-2k^2\bar{u}\gamma^\mu(1-\gamma_5)v+(2-D)\bar{u}\not{k}\gamma^\mu\not{k}(1-\gamma_5)v \;= \\ &= (D-2)(k^2g^{\mu\nu}-2k^\mu k^\nu)\bar{u}\gamma_\nu(1-\gamma_5)v \end{aligned} \tag{45}$$

ning

$$\begin{aligned} -2\bar{u}\not{p}_2\not{k}\gamma^\mu(1-\gamma_5)v &= -4(p_2k)\bar{u}\gamma^\mu(1-\gamma_5)v+2\bar{u}\not{k}\not{p}_2\gamma^\mu(1-\gamma_5)v \;= \\ &= -4(p_2k)\bar{u}\gamma^\mu(1-\gamma_5)v+4p_2^\mu\bar{u}\not{k}(1-\gamma_5)v-2\bar{u}\not{k}\gamma^\mu\not{p}_2(1-\gamma_5)v \;= \\ &= -4(p_2k)\bar{u}\gamma^\mu(1-\gamma_5)v+4p_2^\mu\bar{u}\not{k}(1-\gamma_5)v-2\bar{u}\not{k}\gamma^\mu(1+\gamma_5)\not{p}_2v \;= \\ &= -4(p_2k)\bar{u}\gamma^\mu(1-\gamma_5)v+4p_2^\mu\bar{u}\not{k}(1-\gamma_5)v+2m_2\bar{u}\not{k}(1-\gamma_5)\gamma^\mu v \end{aligned} \tag{46}$$

ja

$$\begin{aligned} 2\bar{u}\gamma^\mu(1-\gamma_5)\not{k}\not{p}_1v &= 4(p_1k)\bar{u}\gamma^\mu(1-\gamma_5)v-2\bar{u}\gamma^\mu(1-\gamma_5)\not{p}_1\not{k}v \;= \\ &= 4(p_1k)\bar{u}\gamma^\mu(1-\gamma_5)v-2\bar{u}\gamma^\mu\not{p}_1(1+\gamma_5)\not{k}v \;= \\ &= 4(p_1k)\bar{u}\gamma^\mu(1-\gamma_5)v-4p_1^\mu\bar{u}(1-\gamma_5)\not{k}v+2\bar{u}\not{p}_1\gamma^\mu(1+\gamma_5)\not{k}v \;= \\ &= 4(p_1k)\bar{u}\gamma^\mu(1-\gamma_5)v-4p_1^\mu\bar{u}\not{k}(1-\gamma_5)v+2m_1\bar{u}\gamma^\mu\not{k}(1-\gamma_5)v, \end{aligned} \tag{47}$$

Arvesse võttes eelnevaid lihtsustusi saame, et

$$
\begin{aligned}
L &= \Big((D-2)(k^2 g^{\mu\nu} - 2k^\mu k^\nu) + 4((p_1-p_2)k - p_1p_2)g^{\mu\nu} - 4(p_1-p_2)^\mu k^\nu\Big) \times \\
&\qquad \times \bar{u}(p_1,s_1)\gamma_\nu(1-\gamma_5)v(p_2,s_2) + \\
&\quad + 2m_1 k^\nu \bar{u}(p_1,s_1)\gamma^\mu\gamma_\nu(1-\gamma_5)v(p_2,s_2) + 2m_2 k^\nu \bar{u}(p_1,s_1)\gamma_\nu(1-\gamma_5)\gamma^\mu v(p_2,s_2).
\end{aligned} \tag{48}
$$

2.5 Verteksparandi arvutamine Feynmani parameetritega

Verteksparandi integraali arvutame kasutades Feynmani parameetreid. Kehtib valem

$$
\frac{1}{A_1^{\alpha_1}\cdots A_m^{\alpha_m}} = \frac{\Gamma(\alpha_1+\ldots+\alpha_m)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}\int_0^1 \frac{x_1^{\alpha_1-1}dx_1\cdots x_m^{\alpha_m-1}dx_m}{(x_1A_1+\ldots+x_mA_m)^{\alpha_1+\ldots+\alpha_m}}\delta(x_1+\ldots+x_m-1), \tag{49}
$$

kus $\Gamma(z)$ on Euleri gammafunktsioon ehk faktoriaali üldistus omadusega

$$
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \Gamma(z+1) = \Gamma(z), \qquad \Gamma(1) = 1. \tag{50}
$$

Meil läheb vaja kolme parameetrit, mis on seotud võrrandi $x_1 + x_2 + x_3 = 1$ kaudu. Võrrandis saab x_3 asendada avaldisega $1 - x_1 - x_2$, ja rajad on $0 \le x_1 \le 1$ ja $0 \le x_2 \le 1 - x_1$. Ilma lugejata on meil

$$
\begin{aligned}
&\frac{1}{((p_1+k)^2-m_1^2)((p_2-k)^2-m_2^2)k^2} = \\
&= \frac{\Gamma(3)}{\Gamma(1)^3}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{(k^2+2x_1p_1k+x_1(p_1^2-m_1^2)-2x_2p_2k+x_2(p_2^2-m_2^2))^3}.
\end{aligned} \tag{51}
$$

Kuna tegu on ainult esimese järgu protsessiga, saame oletada, et mõlemad tekkivad osakesed on massi pinnal, st $p_1^2 = m_1^2$ ja $p_2^2 = m_2^2$. See lihtsustab arvutamist. Jätkame nüüd nimetajaga

$$
N = (k^2 + 2x_1p_1k - 2x_2p_2k)^3 = \Big((k + x_1p_1 - x_2p_2)^2 - (x_1p_1 - x_2p_2)^2\Big)^3. \tag{52}
$$

Lõpuks saame üle k integreerimisel teha asenduse $k \to k - x_1p_1 + x_2p_2$, sest integreerimisrajad on siin $\pm\infty$. Sellega lihtsustub nimetaja $N = (k^2 - (x_1p_1 - x_2p_2)^2)^3$, aga lugeja läheb keerulisemaks. Lugeja rühmitame erinevate k astmete järgi. Lugeja $L_0 = 1$ jaoks ei muutu midagi. Lugeja $L_1^\mu = k^\mu$ on asendatud $L_1^\mu = k^\mu - xp_1^\mu + x_2p_2^\mu$ ja lugeja $L_2^{\mu\nu} = k^\mu k^\nu$ avaldisega $L_2^{\mu\nu} = (k^\mu - x_1p_1^\mu + x_2p_2^\mu)(k^\nu - x_1p_1^\nu + x_2p_2^\nu)$. Lõpuks kaovad kõik lineaarsed k-sõltuvused, sest integraal on k suhtes sümmeetriline. $k^\mu k^\nu$ ei kao ainult siis, kui $\mu = \nu$. Sellega on vastav integraal meetrikaga

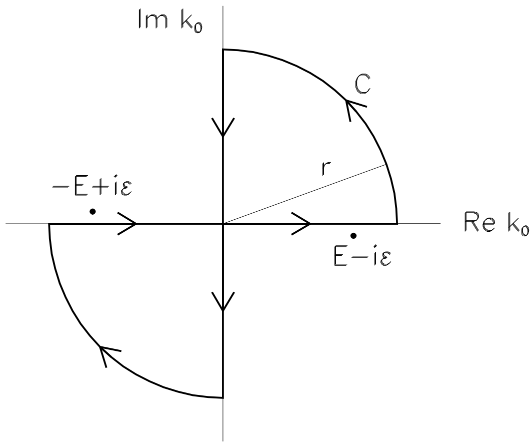
$g^{\mu\nu}$ proportsionaalne, ja saame kontraktsiooni teel skalaarse panuse arvutada. Seega tuleb nüüd arvutada kolm järgnevat integraali

$$
\begin{aligned}
I_0 &= -i\int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+2p_1k)(k^2-2p_2k)k^2} = \\
&= -i\Gamma(3)\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-(x_1p_1-x_2p_2)^2)^3},\\
I_1^\mu &= -i\int \frac{d^Dk}{(2\pi)^D}\frac{k^\mu}{(k^2+2p_1k)(k^2-2p_2k)k^2} = \\
&= -i\Gamma(3)\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dk}{(2\pi)^D}\frac{-(x_1p_1-x_2p_2)^\mu}{(k^2-(x_1p_1-x_2p_2)^2)^3},\\
I_2^{\mu\nu} &= -i\int \frac{d^Dk}{(2\pi)^D}\frac{k^\mu k^\nu}{(k^2+2p_1k)(k^2-2p_2k)k^2} = \\
&= -i\Gamma(3)\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dk}{(2\pi)^D}\frac{(k-x_1p_1+x_2p_2)^\mu(k-x_1p_1+x_2p_2)^\nu}{(k^2-(x_1p_1-x_2p_2)^2)^3} = \\
&= -\frac{i}{4}g^{\mu\nu}\Gamma(3)\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-(x_1p_1-x_2p_2)^2)^2} + \\
&\quad - i\Gamma(3)\int_0^1 dx_1\int_0^{1-x_1}dx_2\int\frac{d^Dk}{(2\pi)^D}\frac{(x_1p_1-x_2p_2)^\mu(x_1p_1-x_2p_2)^\nu}{(k^2-(x_1p_1-x_2p_2)^2)^3}. \qquad (53)
\end{aligned}
$$

2.6 Verteksparandi põhiintegraal

Tuleb välja, et üle k integreerimiseks on vaja arvutada põhiintegraal

$$
I_\alpha(D) = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{(-k^2+m^2+i\epsilon)^\alpha}. \qquad (54)
$$



Arvutamiseks eraldame esiteks aja ja $(D-1)$-dimensionaalse ruumi integreerimise. Lisaks kasutame asjaolu, et integrandi poolused Feynmani propagaatori jaoks ei ole reaalteljel, vaid natuke imaginaarsuunas nihutatud. Nimetaja nullkohad k_0 suhtes on

$$
k_0 = \pm\left(\sqrt{\vec{k}^2+m^2} - i\epsilon\right) = \pm(E-i\epsilon). \qquad (55)
$$

Kui me asendame integreerimistee $[-\infty,+\infty]$ komplekstasandis oleva tee $[i\infty,-i\infty]$ kaudu ja ühendame mõlemad teed veerandkaarte kaudu, siis ei sisalda suletud tee enam pooluseid. Kaared võib ära kaotada, sest integrand kaob, kui k_0^2 on piisavalt suur, nii et asendame Cauchy teoreemi järgi ühe integraalitee teisega. Lõpuks kasutame ik_0 asemel uut suurust k_0 ja oleme sellega Eukleidilises piirkonnas [2]. Sellega saame põhiintegraali esitada kujul

$$I_\alpha(D) = i\int \frac{d^Dk_E}{(2\pi)^D}\frac{1}{(k_E^2+m^2)^\alpha}, \tag{56}$$

kus väikest imaginaarosa enam ei ole, sest integreerimistee on nüüd poolustest kaugel. Nüüd saame D-dimensionaalse integreerimise läbi viia, sest kõik on radiaalsümmeetriline. D-dimensionaalne ühiselement on $2\pi^{D/2}/\Gamma(D/2)$. Me saame ($\kappa^2 = k^2$)

$$I_\alpha(D) = \frac{2i}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^\infty \frac{\kappa^{D-1}d\kappa}{(\kappa^2+m^2)^\alpha}. \tag{57}$$

Asendusega $y = m^2/(\kappa^2+m^2)$ on meil

$$\begin{aligned} I_\alpha(D) &= \frac{i(m^2)^{D/2-\alpha}}{(4\pi)^{D/2}\Gamma(D/2)}\int_0^1 y^{D/2-1}(1-y)^{\alpha-D/2-1}dy = \\ &= \frac{i(m^2)^{D/2-\alpha}}{(4\pi)^{D/2}\Gamma(D/2)}\frac{\Gamma(D/2)\Gamma(\alpha-D/2)}{\Gamma(\alpha)} = \frac{i(m^2)^{D/2-\alpha}}{(4\pi)^{D/2}}\frac{\Gamma(\alpha-D/2)}{\Gamma(\alpha)}, \end{aligned} \tag{58}$$

kus me kasutasime Euleri betafunktsiooni

$$B(m,n) := \int_0^1 y^{m-1}(1-y)^{n-1}dy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{59}$$

Seda põhiintegraali kasutades on meil nüüd

$$\begin{aligned} I_0 &= \frac{-\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^1 dx_1\int_0^{1-x_1}dx_2\left((x_1p_1-x_2p_2)^2\right)^{D/2-3}, \\ I_1^\mu &= \frac{\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^1 dx_1\int_0^{1-x_1}dx_2(x_1p_1-x_2p_2)^\mu\left((x_1p_1-x_2p_2)^2\right)^{D/2-3}, \\ I_2^{\mu\nu} &= \frac{\Gamma(2-D/2)}{2(4\pi)^{D/2}}g^{\mu\nu}\int_0^1 dx_1\int_0^{1-x_1}dx_2\left((x_1p_1-x_2p_2)^2\right)^{D/2-2} + \\ &\quad - \frac{\Gamma(3-D/2)}{(4\pi)^{D/2}}\int_0^1 dx_1\int_0^{1-x_1}dx_2(x_1p_1-x_2p_2)^\mu(x_1p_1-x_2p_2)^\nu\left((x_1p_1-x_2p_2)^2\right)^{D/2-3}, \\ I_2 &= I_2^{\mu\nu}g_{\mu\nu} = \left(\frac{D}{2}-\left(2-\frac{D}{2}\right)\right)I_0' = (D-2)I_0', \qquad \text{kus} \\ I_0' &= \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}\int_0^1 dx_1\int_0^{1-x_1}dx_2\left((x_1p_1-x_2p_2)^2\right)^{D/2-2}. \end{aligned} \tag{60}$$

Nende integraalide kaudu väljendades saame

$$\begin{aligned} \Delta\Gamma_L^\mu &= g_s^2C_F\Big((D-2)(I_2g^{\mu\nu}-2I_2^{\mu\nu}) + 4\left((p_1-p_2)I_1g^{\mu\nu}-(p_1-p_2)^\mu I_1^\nu\right) + \\ &\quad - 4(p_1p_2)I_0g^{\mu\nu}\Big)\gamma_\nu\frac{1-\gamma_5}{2} + 2C_FI_1^\nu\left(m_1\gamma^\mu\gamma_\nu\frac{1-\gamma_5}{2}+\gamma_\nu\frac{1-\gamma_5}{2}\gamma^\mu m_2\right). \end{aligned} \tag{61}$$

Tulemus sisaldab nii alguspärast struktuuri $\gamma_\nu(1-\gamma_5)/2$ kui ka massist sõltuvaid eristruktuure (ei ilmu Borni tasemel), mis on oodatav, kuna massiline arvutamine viib meid segarenormeerimiseni. Kuid esmalt peame siiski arvutama integraalid I_0, I_1^μ ja $I_2^{\mu\nu}$.

[2] Antud arvutuskäik võetakse kokku terminiga *Wicki pööre*

2.7 Verteksfunktsioon

Alustame üldintegraaliga (56). Võttes abiks samamassilised arvutamised [10,11,12] esitame integranditekitaja nüüd veidi teistmoodi

$$\mu_2 x_2^2 + \mu_1 x_1^2 - (1-\mu_1-\mu_2)x_1x_2 = A_+(\sqrt{\mu_2}x_2 + \sqrt{\mu_1}x_1)^2 + A_-(\sqrt{\mu_2}x_2 - \sqrt{\mu_1}x_1)^2, \qquad (62)$$

kus [3]

$$A_+ = -\frac{1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2}{4\sqrt{\mu_1\mu_2}}, \qquad A_- = \frac{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2}{4\sqrt{\mu_1\mu_2}}. \qquad (63)$$

Kui me nüüd uued muutujad

$$u := \sqrt{\mu_2}x_2 + \sqrt{\mu_1}x_1, \qquad w := \frac{\sqrt{\mu_2}x_2 - \sqrt{\mu_1}x_1}{\sqrt{\mu_2}x_2 + \sqrt{\mu_1}x_1} \qquad (64)$$

ning kiiruse v masskeskme-süsteemi suhtes

$$v^2 := \frac{1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2}{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2} \qquad (65)$$

kasutusse võtame, siis lihtsustub integranditekitaja

$$(x_1p_1 - x_2p_2)^2 = u^2\left(\frac{w^2-v^2}{1-v^2}\right)q^2. \qquad (66)$$

Integraalimõõt muutub nagu $2\sqrt{\mu_1\mu_2}dx_1dx_2 = u\,du\,dw$, rajade jaoks saame omakorda

$$-1 \le w \le +1, \qquad 0 \le u \le \frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w} =: u_0(w). \qquad (67)$$

Sellega saame integreerimise läbi viia

$$\begin{aligned}
\mathcal{I}_\alpha(q^2) &= \frac{\Gamma(\alpha)}{(4\pi)^{2-\varepsilon}}\int_0^1 dx_1\int_0^{1-x_1}dx_2\left((x_1p_1-x_2p_2)^2\right)^{-\alpha} = \\
&= \frac{\Gamma(\alpha)(q^2)^{-\alpha}}{2(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\alpha}dw\int_0^{u_0(w)}u^{1-2\alpha}du = \\
&= \frac{\Gamma(\alpha)(q^2)^{-\alpha}}{4(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}(1-\alpha)}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\alpha}u_0(w)^{2-2\alpha}dw = \\
&= \frac{-\Gamma(\alpha-1)(q^2)^{-\alpha}}{4(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\alpha}\left(\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right)^{2-2\alpha}dw. \qquad (68)
\end{aligned}$$

Leiame nüüd need integraalid samas järjekorras, nagu kirja panime.

[3] Kui me võtta arvesse, et μ_1 ja μ_2 saavad ka negatiivsed olla (mittefüüsikalises piirkonnas $v > 1$), on täpsem kirjutada avaldise $\sqrt{\mu_1\mu_2}$ asemel $\sqrt{\mu_1}\sqrt{\mu_2}$.

Integraal $I_0'(q^2)$

Integraali $I_0'(q^2)$ puhul on $\alpha = 2 - D/2 = \varepsilon$, nii et

$$\begin{aligned}
I_0'(q^2) &= \frac{-\Gamma(\varepsilon-1)(q^2)^{-\varepsilon}}{4(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\varepsilon}\left(\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right)^{2-2\varepsilon}dw\ = \\
&= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\Bigg[\frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon(1-\varepsilon)}\int_{-1}^{+1}\frac{\sqrt{\mu_1\mu_2}dw}{((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w)^2}+ \\
&\quad+\int_{-1}^{+1}\frac{2\sqrt{\mu_1\mu_2}dw}{((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w)^2}\ln((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w)+ \\
&\quad-\int_{-1}^{+1}\frac{\sqrt{\mu_1\mu_2}dw}{((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w)^2}\ln\left(\frac{w^2-v^2}{1-v^2}\right)\Bigg] = \\
&=: I_{00}'(q^2)+I_{01}'(q^2)+I_{02}'(q^2),
\end{aligned} \tag{69}$$

kus kasutasime valemit $\Gamma(1+\varepsilon) = \varepsilon(\varepsilon-1)\Gamma(\varepsilon-1)$ ja $z^\varepsilon = 1+\varepsilon\ln z$. Singulaarne osa on

$$\begin{aligned}
I_{00}'(q^2) &= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon(1-\varepsilon)}\int_{-1}^{+1}\frac{\sqrt{\mu_1\mu_2}dw}{((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w)^2} = \\
&= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon(1-\varepsilon)}\frac{\sqrt{\mu_1\mu_2}}{\sqrt{\mu_1}-\sqrt{\mu_2}}\int_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}}\frac{dw'}{w'^2} = \\
&= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon(1-\varepsilon)}\frac{\sqrt{\mu_1\mu_2}}{\sqrt{\mu_1}-\sqrt{\mu_2}}\left(\frac{1}{2\sqrt{\mu_2}}-\frac{1}{2\sqrt{\mu_1}}\right) = \\
&= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon(1-\varepsilon)}\frac{\sqrt{\mu_1\mu_2}(\sqrt{\mu_1}-\sqrt{\mu_2})}{2\sqrt{\mu_1\mu_2}(\sqrt{\mu_1}-\sqrt{\mu_2})} = \\
&= \frac{\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon(1-\varepsilon)} = \frac{\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\left(\frac{1}{\varepsilon}+1-\ln(4\mu_1\mu_2)\right),
\end{aligned} \tag{70}$$

kus me kasutasime võrdust $w' = (\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w$ ja $\overline{\text{MS}}$-skeemi. Küsitavana võib paista piirjuht $\mu_1 = \mu_2$, sest me taandasime kordajad $(\sqrt{\mu_1}-\sqrt{\mu_2})$. Sel juhul on integrand aga w-st sõltumatu ja integraal on

$$\int_{-1}^{+1}\frac{\mu_1 dw}{(2\sqrt{\mu_1})^2} = \frac{1}{4}\int_{-1}^{+1}dw = \frac{1}{2}, \tag{71}$$

nii et saame sama tulemuse. Iseloomustav on, et singulaarne osa ei sõltu μ_1-st ega μ_2-st. Järgmise osa jaoks saame võtta $\varepsilon = 0$ ja kasutada sama asendust

$$\begin{aligned}
I_{01}'(q^2) &= \frac{2\sqrt{\mu_1\mu_2}}{(4\pi)^2}\int_{-1}^{+1}\frac{\ln\big((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\big)}{((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w)^2}dw = \\
&= \frac{2\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\int_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}}\frac{dw'}{w'^2}\ln w' = \\
&= \frac{2\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\left[-\frac{1}{w'}(1+\ln w')\right]_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\left(\frac{1}{2\sqrt{\mu_2}}(1+\ln(2\sqrt{\mu_2}))-\frac{1}{2\sqrt{\mu_1}}(1+\ln(2\sqrt{\mu_1}))\right) = \\
&= \frac{1}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\left(\sqrt{\mu_1}(1+\ln(2\sqrt{\mu_2}))-\sqrt{\mu_2}(1+\ln(2\sqrt{\mu_1}))\right) = \\
&= \frac{1}{(4\pi)^2}\left(1+\frac{1}{2}\ln(4\sqrt{\mu_1\mu_2})\right)+\frac{1}{2(4\pi)^2}\left(\frac{\sqrt{\mu_1}+\sqrt{\mu_2}}{\sqrt{\mu_1}-\sqrt{\mu_2}}\right)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right). \qquad (72)
\end{aligned}
$$

Viimane vastus on piirjuhul $\mu_1 = \mu_2$ ohutu, sest

$$
\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) = \ln\left(1-\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) \approx \frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}} \qquad (73)
$$

ning poolus taandub jälle. Ka viimase osa jaoks saame eelnevat asendust kasutada. Kuna $w = (w'-\sqrt{\mu_1}-\sqrt{\mu_2})/(\sqrt{\mu_1}-\sqrt{\mu_2})$ on logaritmi argument

$$
\frac{w^2-v^2}{1-v^2} = \frac{(w'-\sqrt{\mu_1}-\sqrt{\mu_2})^2-(\sqrt{\mu_1}-\sqrt{\mu_2})^2v^2}{(\sqrt{\mu_1}-\sqrt{\mu_2})^2(1-v^2)} = \frac{(w'-w'_+)(w'-w'_-)}{(\sqrt{\mu_1}-\sqrt{\mu_2})^2(1-v^2)}, \qquad (74)
$$

kus $w'_\pm = (\sqrt{\mu_1}+\sqrt{\mu_2}) \pm (\sqrt{\mu_1}-\sqrt{\mu_2})v$, siis kasutame nüüd avaldist

$$
\int \frac{dw'}{w'^2}\ln(w'-w'_\pm) = \left(\frac{1}{w'_\pm}-\frac{1}{w'}\right)\ln(w'-w'_\pm) - \frac{1}{w'_\pm}\ln w', \qquad (75)
$$

et arvutada

$$
\begin{aligned}
I'_{02}(q^2) &= \frac{-1}{(4\pi)^2}\int_{-1}^{+1}\frac{\sqrt{\mu_1\mu_2}\,dw}{\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right)^2}\ln\left(\frac{w^2-v^2}{1-v^2}\right) = \\
&= \frac{-\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\int_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}}\frac{dw'}{w'^2}\ln\left(\frac{(w'-w'_+)(w'-w'_-)}{(\sqrt{\mu_1}-\sqrt{\mu_2})^2(1-v^2)}\right) = \\
&= \frac{-\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\Bigg[\left(\frac{1}{w'_+}-\frac{1}{w'}\right)\ln(w'-w'_+)+\left(\frac{1}{w'_-}-\frac{1}{w'}\right)\ln(w'-w'_-)+ \\
&\qquad -\frac{1}{w'_+}\ln w'-\frac{1}{w'_-}\ln w'+\frac{1}{w'}\ln\left((\sqrt{\mu_1}-\sqrt{\mu_2})^2(1-v^2)\right)\Bigg]_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}} = \\
&= \frac{-\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\Bigg[-\frac{1}{w'}\ln\left(\frac{(w'-w'_+)(w'-w'_-)}{(\sqrt{\mu_1}-\sqrt{\mu_2})^2(1-v^2)}\right)+ \\
&\qquad +\frac{1}{w'_+}\ln(w'-w'_+)+\frac{1}{w'_-}\ln(w'-w'_-)-\frac{1}{w'_+}\ln w'-\frac{1}{w'_-}\ln w'\Bigg]_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}}. \qquad (76)
\end{aligned}
$$

Nüüd on meil

$$
2\sqrt{\mu_2}-w'_\pm = -(\sqrt{\mu_1}-\sqrt{\mu_2})(1\pm v), \qquad 2\sqrt{\mu_1}-w'_\pm = (\sqrt{\mu_1}-\sqrt{\mu_2})(1\mp v), \qquad (77)
$$

nii et esimene logaritm antud radades kaob ja teise/kolmanda logaritmi asemel saame

$$
\ln(w'-w'_\pm)\Big|_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}} = \ln\left(-\frac{1\mp v}{1\pm v}\right) = \pm\ln\left(\frac{v-1}{v+1}\right). \qquad (78)
$$

See on koht, kus jätkame $v > 1$ jaoks, kuigi vaid $v < 1$ on füüsikaline piirkond. Saame

$$\begin{aligned} I'_{02}(q^2) &= \frac{-\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\left[\frac{1}{w'_+}\ln(w'-w'_+)+\frac{1}{w'_-}\ln(w'-w'_-)-\left(\frac{1}{w'_+}+\frac{1}{w'_-}\right)\ln w'\right]_{2\sqrt{\mu_2}}^{2\sqrt{\mu_1}} = \\ &= \frac{\sqrt{\mu_1\mu_2}}{(4\pi)^2(\sqrt{\mu_1}-\sqrt{\mu_2})}\left(\left(\frac{1}{w'_-}-\frac{1}{w'_+}\right)\ln\left(\frac{v-1}{v+1}\right)+\left(\frac{1}{w'_-}+\frac{1}{w'_+}\right)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right). \end{aligned} \quad (79)$$

Arvutame nüüd

$$\begin{aligned} w'_+w'_- &= (\sqrt{\mu_1}+\sqrt{\mu_2})^2-(\sqrt{\mu_1}-\sqrt{\mu_2})^2v^2 = \\ &= \frac{(\sqrt{\mu_1}+\sqrt{\mu_2})^2\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\left(1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2\right)}{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2} = \\ &= \frac{(\sqrt{\mu_1}+\sqrt{\mu_2})^2-(\sqrt{\mu_1}-\sqrt{\mu_2})^2}{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2} = \frac{4\sqrt{\mu_1\mu_2}}{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2} = 1-v^2, \end{aligned} \quad (80)$$

nii et

$$\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}-\sqrt{\mu_2})}\left(\frac{1}{w'_-}-\frac{1}{w'_+}\right) = \sqrt{(1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2)\,(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2)} = \sqrt{\lambda}, \quad (81)$$

$$\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}-\sqrt{\mu_2})}\left(\frac{1}{w'_-}+\frac{1}{w'_+}\right) = \left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\frac{\sqrt{\mu_1}+\sqrt{\mu_2}}{\sqrt{\mu_1}-\sqrt{\mu_2}} = \frac{\sqrt{\mu_1}+\sqrt{\mu_2}}{\sqrt{\mu_1}-\sqrt{\mu_2}}-(\mu_1-\mu_2).$$

Sellega saame lõpuks viimase osa vastuseks

$$I'_{02}(q^2) = \frac{1}{2(4\pi)^2}\left(\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)+\left(\left(\frac{\sqrt{\mu_1}+\sqrt{\mu_2}}{\sqrt{\mu_1}-\sqrt{\mu_2}}\right)-(\mu_1-\mu_2)\right)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right). \quad (82)$$

Kokku võttes on

$$I'_0(q^2) = \frac{\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\left(\frac{1}{\varepsilon}+3-\ln(\sqrt{\mu_1\mu_2})+\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right). \quad (83)$$

Integraal $I_0(q^2)$

Integraal $I_0(q^2)$ on jälle teiste (vektor- ja tensor-)integraalide allikas. Kasutame $\mathcal{I}_\alpha(q^2)$ juhul $\alpha = 3-D/2 = 1+\varepsilon$ ja leiame, et integraal on siiski divergentne, kusjuures integrandi kordajad vahetavad oma ülesanded

$$\begin{aligned} I_0(q^2) &= \frac{\Gamma(\varepsilon)(q^2)^{-1-\varepsilon}}{4(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-1-\varepsilon}\left(\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right)^{-2\varepsilon}dw = \\ &= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-1-\varepsilon}\frac{1-v^2}{4\sqrt{\mu_1\mu_2}}\left[\frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon}\int_{-1}^{+1}\frac{dw}{w^2-v^2}+\right. \\ &\left.-\int_{-1}^{+1}\frac{dw}{w^2-v^2}\ln\left(\frac{w^2-v^2}{1-v^2}\right)+\int_{-1}^{+1}\frac{2dw}{w^2-v^2}\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right)\right] = \\ &=: I_{00}(q^2)+I_{01}(q^2)+I_{02}(q^2). \end{aligned} \quad (84)$$

Taas mittefüüsikalises piirkonnas jätkates arvutame

$$
\begin{aligned}
I_{00}(q^2) &= -\frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-1-\varepsilon}\frac{1-v^2}{4\sqrt{\mu_1\mu_2}}\ \frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon}\int_{-1}^{+1}\frac{dw}{v^2-w^2} = \\
&= -\frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-1-\varepsilon}\frac{1-v^2}{8v\sqrt{\mu_1\mu_2}}\ \frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon}\int_{-1}^{+1}\left(\frac{dw}{v+w}+\frac{dw}{v-w}\right) = \\
&= -\frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-1-\varepsilon}\frac{1-v^2}{8v\sqrt{\mu_1\mu_2}}\ \frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon}\Big[\ln(v+w)-\ln(v-w)\Big]_{-1}^{+1} = \\
&= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-1-\varepsilon}\frac{1-v^2}{4v\sqrt{\mu_1\mu_2}}\ \frac{(4\mu_1\mu_2)^{-\varepsilon}}{\varepsilon}\ln\left(\frac{v-1}{v+1}\right) = \\
&= \frac{\Gamma(1+\varepsilon)(1-v^2)}{4(4\pi)^{2-\varepsilon}q^2v\sqrt{\mu_1\mu_2}}(q^2)^{-\varepsilon}\left(\frac{1}{\varepsilon}-\ln(4\mu_1\mu_2)\right)\ln\left(\frac{v-1}{v+1}\right). \qquad (85)
\end{aligned}
$$

Järgmine integraal $I_{01}(q^2)$ annab dilogaritmid ja logaritmide ruudud, sest

$$
\int_{-1}^{+1}\frac{dw}{v^2-w^2}\ln\left(\frac{v^2-w^2}{v^2-1}\right) = \frac{1}{2v}\int_{-1}^{+1}\left(\frac{dw}{v+w}+\frac{dw}{v-w}\right)\left(\ln\left(\frac{v+w}{v+1}\right)+\ln\left(\frac{v-w}{v-1}\right)\right). \qquad (86)
$$

Lihtsamad osad on

$$
\begin{aligned}
\int_{-1}^{+1}\frac{dw}{v+w}\ln\left(\frac{v+w}{v+1}\right) &= \int_{(v-1)/(v+1)}^{1}\frac{dw'}{w'}\ln w' = \left[\frac{1}{2}\ln^2 w'\right]_{(v-1)/(v+1)}^{1} = -\frac{1}{2}\ln^2\left(\frac{v-1}{v+1}\right) \quad \text{ja} \\
\int_{-1}^{+1}\frac{dw}{v-w}\ln\left(\frac{v-w}{v-1}\right) &= -\int_{(v+1)/(v-1)}^{1}\frac{dw'}{w'}\ln w' = -\left[\frac{1}{2}\ln^2 w'\right]_{(v+1)/(v-1)}^{1} = \frac{1}{2}\ln^2\left(\frac{v+1}{v-1}\right), \qquad (87)
\end{aligned}
$$

kus me tegime asenduse $w' = (v\pm w)/(v\pm 1)$, keerulisemad aga

$$
\begin{aligned}
\int_{-1}^{+1}\frac{dw}{v+w}\ln\left(\frac{v-w}{v-1}\right) &= \int_{(v-1)/(v+1)}^{1}\frac{dw'}{w'}\ln\left(\frac{2v-(v+1)w'}{v-1}\right) = \\
&= \left[-\operatorname{li}_2\left(\frac{v+1}{2v}w'\right)+\ln\left(\frac{2v}{v-1}\right)\ln w'\right]_{(v-1)/(v+1)}^{1} = \\
&= \operatorname{Li}_2\left(\frac{v-1}{2v}\right)-\operatorname{Li}_2\left(\frac{v+1}{2v}\right)-\ln\left(\frac{2v}{v-1}\right)\ln\left(\frac{v-1}{v+1}\right) \quad \text{ja} \\
\int_{-1}^{+1}\frac{dw}{v-w}\ln\left(\frac{v+w}{v+1}\right) &= -\int_{(v+1)/(v-1)}^{1}\frac{dw'}{w'}\ln\left(\frac{2v-(v-1)w'}{v+1}\right) = \\
&= -\left[-\operatorname{li}_2\left(\frac{v-1}{2v}w'\right)+\ln\left(\frac{2v}{v+1}\right)\ln w'\right]_{(v+1)/(v-1)}^{1} = \\
&= \operatorname{Li}_2\left(\frac{v-1}{2v}\right)-\operatorname{Li}_2\left(\frac{v+1}{2v}\right)+\ln\left(\frac{2v}{v+1}\right)\ln\left(\frac{v+1}{v-1}\right). \qquad (88)
\end{aligned}
$$

Kokku võttes saame

$$
2v\int_{-1}^{+1}\frac{dw}{v^2-w^2}\ln\left(\frac{v^2-w^2}{v^2-1}\right) = 2\operatorname{Li}_2\left(\frac{v-1}{2v}\right)-2\operatorname{Li}_2\left(\frac{v+1}{2v}\right)-\ln\left(\frac{4v^2}{v^2-1}\right)\ln\left(\frac{v-1}{v+1}\right) =: 2L(v) \qquad (89)
$$

ja sellega

$$I_{01}(q^2) = \frac{(1-v^2)L(v)}{4(4\pi)^2 q^2 v\sqrt{\mu_1\mu_2}}. \tag{90}$$

Viimane osa $I_{02}(q^2)$ jaguneb jälle kaheks osaks, sest

$$\begin{aligned}
&\int_{-1}^{+1} \frac{2dw}{w^2-v^2} \ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right) = \\
&= \frac{1}{v}\int_{-1}^{+1}\left(\frac{dw}{w+v}-\frac{dw}{w-v}\right)\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right). \qquad (91)
\end{aligned}$$

Me arvutame

$$\begin{aligned}
&\int_{-1}^{+1}\frac{dw}{w\pm v}\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right) = \qquad w' = w \pm v \\
&= \int_{-1\pm v}^{1\pm v}\frac{dw'}{w'}\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})\mp(\sqrt{\mu_1}-\sqrt{\mu_2})v+(\sqrt{\mu_1}-\sqrt{\mu_2})w'\right) = \\
&= \left[-\mathrm{li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})w'}{(\sqrt{\mu_2}+\sqrt{\mu_1})\mp(\sqrt{\mu_1}-\sqrt{\mu_2})v}\right)+\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})\mp(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)\ln w'\right]_{-1\pm v}^{1\pm v} = \\
&= -\mathrm{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(1\pm v)}{(\sqrt{\mu_1}+\sqrt{\mu_2})\mp(\sqrt{\mu_1}-\sqrt{\mu_2})v}\right)+\mathrm{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(-1\pm v)}{(\sqrt{\mu_1}+\sqrt{\mu_2})\mp(\sqrt{\mu_1}-\sqrt{\mu_2})v}\right)+ \\
&\quad +\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})\mp(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)\ln\left(\frac{1\pm v}{-1\pm v}\right). \qquad (92)
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^{+1}\frac{dw}{w+v}\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right) = \\
&= \mathrm{Li}_2\left(\frac{(\sqrt{\mu_1}-\sqrt{\mu_2})(v+1)}{2\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{(\sqrt{\mu_1}-\sqrt{\mu_2})(v-1)}{2\sqrt{\mu_2}}\right)+ \qquad (93) \\
&\quad +\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)\ln\left(\frac{\sqrt{\mu_2}(v+1)}{\sqrt{\mu_1}(v-1)}\right)+\frac{1}{2}\ln^2(2\sqrt{\mu_1})-\frac{1}{2}\ln^2(2\sqrt{\mu_2}), \\
&\int_{-1}^{+1}\frac{dw}{w-v}\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right) = \\
&= \mathrm{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(v-1)}{2\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(v+1)}{2\sqrt{\mu_2}}\right)+ \qquad (94) \\
&\quad +\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)\ln\left(\frac{\sqrt{\mu_2}(v-1)}{\sqrt{\mu_1}(v+1)}\right)+\frac{1}{2}\ln^2(2\sqrt{\mu_1})-\frac{1}{2}\ln^2(2\sqrt{\mu_2})
\end{aligned}$$

ja sellepärast

$$\begin{aligned}
&\int_{-1}^{+1}\left(\frac{dw}{w+v}-\frac{dw}{w-v}\right)\ln\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right) = \\
&= \mathrm{Li}_2\left(\frac{(\sqrt{\mu_1}-\sqrt{\mu_2})(v+1)}{2\sqrt{\mu_1}}\right)-\mathrm{Li}_2\left(\frac{(\sqrt{\mu_1}-\sqrt{\mu_2})(v-1)}{2\sqrt{\mu_2}}\right)+ \\
&\quad -\mathrm{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(v-1)}{2\sqrt{\mu_1}}\right)+\mathrm{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(v+1)}{2\sqrt{\mu_2}}\right)+
\end{aligned}$$

$$+\ln\left(\frac{(\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v}\right)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)-\ln(1-v^2)\ln\left(\frac{v-1}{v+1}\right) =$$

$$=:\ L(v)-L'(\mu_1,\mu_2)-\ln(1-v^2)\ln\left(\frac{v-1}{v+1}\right) \qquad (95)$$

ning

$$I_{02}(q^2)=\frac{(1-v^2)L'(\mu_1,\mu_2)}{4(4\pi)^2v\sqrt{\mu_1\mu_2}}. \qquad (96)$$

Kokku võttes saame ($(1-v^2)/4v\sqrt{\mu_1\mu_2}=1/\sqrt{\lambda}$ kasutades)

$$I_0(q^2)=\frac{\Gamma(1+\varepsilon)(q^2)^{-\varepsilon}}{(4\pi)^{2-\varepsilon}q^2\sqrt{\lambda}}\left[\left(\frac{1}{\varepsilon}-\ln\left(\sqrt{\mu_1\mu_2}\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\right)\ln\left(\frac{v-1}{v+1}\right)+L'\right]. \qquad (97)$$

Integraal $I_1^\mu(q^2)$

Kui lisame $-I_0(q^2)$ integrandile kordaja $(x_1p_1-x_2p_2)^\mu$, saame $I_1^\mu(q^2)$. Kuna aga

$$(x_1p_1-x_2p_2)^\mu=\frac{1}{2}u\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\mu-\frac{1+w}{\sqrt{\mu_2}}p_2^\mu\right), \qquad (98)$$

siis saame lisakordaja u, mis muudab integraali mittesingulaarseks. Võrreldes võrrandiga (68) on meil nüüd

$$\mathcal{I}_\alpha^\mu(q^2) = \frac{\Gamma(\alpha)}{(4\pi)^{2-\varepsilon}}\int_0^1 dx_1\int_0^{1-x_1}dx_2\left((x_1p_1-x_2p_2)^2\right)^{-\alpha}(x_1p_1^\mu-x_2p_2^\mu) =$$

$$= \frac{\Gamma(\alpha)(q^2)^{-\alpha}}{4(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\alpha}\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\mu-\frac{1+w}{\sqrt{\mu_2}}p_2^\mu\right)dw\int_0^{u_0(w)}u^{2-2\alpha}du =$$

$$= \frac{\Gamma(\alpha)(q^2)^{-\alpha}}{4(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}(3-2\alpha)}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\alpha}\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\mu-\frac{1+w}{\sqrt{\mu_2}}p_2^\mu\right)u_0(w)^{3-2\alpha}dw =$$

$$= \frac{\Gamma(\alpha)(q^2)^{-\alpha}}{4(4\pi)^{2-\varepsilon}\sqrt{\mu_1\mu_2}(3-2\alpha)}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\alpha}\times$$

$$\times\left(\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right)^{3-2\alpha}\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\mu-\frac{1+w}{\sqrt{\mu_2}}p_2^\mu\right)dw, \qquad (99)$$

mida saab kohe $\alpha=1$ jaoks arvutada. Saame

$$I_1^\mu(q^2) = \mathcal{I}_1^\mu(q^2) = \frac{1}{4(4\pi)^2q^2\sqrt{\mu_1\mu_2}}\int_{-1}^{+1}\frac{1-v^2}{w^2-v^2}\times \qquad (100)$$

$$\times\left(\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right)\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\mu-\frac{1+w}{\sqrt{\mu_2}}p_2^\mu\right)dw \ =:\ I_1^1(q^2)p_1^\mu-I_1^2(q^2)p_2^\mu.$$

Osade jaoks arvutame

$$2(4\pi)^2q^2\sqrt{\mu_1}I_1^1(q^2) = \int_{-1}^{+1}\frac{(1-v^2)(1-w)dw}{(w^2-v^2)\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right)} =$$

$$
\begin{aligned}
&= -\int_{-1}^{+1}\left(\frac{(1-v^2)(1-v)dw}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v)(v-w)}+\right.\\
&\quad\left.+\frac{(1-v^2)(1+v)dw}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v)(v+w)}-\frac{2(\sqrt{\mu_1}-\sqrt{\mu_2})\sqrt{\mu_1}dw}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right) =\\
&= \frac{(1-v^2)(1-v)}{2v\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)}\ln\left(\frac{v-1}{v+1}\right)+\\
&\quad+\frac{(1-v^2)(1+v)}{2v\left((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)}\ln\left(\frac{v-1}{v+1}\right)+2\sqrt{\mu_1}\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) =\\
&= \frac{(1-\mu_1+\mu_2)(1-v^2)}{2v\sqrt{\mu_2}}\ln\left(\frac{v-1}{v+1}\right)+2\sqrt{\mu_1}\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) =\\
&= 2(1-\mu_1+\mu_2)\frac{\sqrt{\mu_1}}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+2\sqrt{\mu_1}\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right),
\end{aligned}
\tag{101}
$$

kus me kasutasime

$$
\frac{1-v^2}{2v}=\frac{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2-1+(\sqrt{\mu_2}+\sqrt{\mu_1})^2}{2\sqrt{(1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2)(1-(\sqrt{\mu_2}-\sqrt{\mu_1})^2)}}=2\frac{\sqrt{\mu_1\mu_2}}{\sqrt{\lambda}}. \tag{102}
$$

Samuti arvutame eraldi

$$
\begin{aligned}
2(4\pi)^2q^2\sqrt{\mu_2}I_2^1(q^2) &= \int_{-1}^{+1}\frac{(1-v^2)(1+w)dw}{(w^2-v^2)\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\right)} =\\
&= -\int_{-1}^{+1}\left(\frac{(1-v^2)(1-v)dw}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v)(v+w)}+\right.\\
&\quad\left.+\frac{(1-v^2)(1+v)dw}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v)(v-w)}+\frac{2(\sqrt{\mu_1}-\sqrt{\mu_2})\sqrt{\mu_2}dw}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right) =\\
&= \frac{(1-v^2)(1-v)}{2v\left((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)}\ln\left(\frac{v-1}{v+1}\right)+\\
&\quad+\frac{(1-v^2)(1+v)}{2v\left((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v\right)}\ln\left(\frac{v-1}{v+1}\right)+2\sqrt{\mu_2}\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) =\\
&= 2(1+\mu_1-\mu_2)\frac{\sqrt{\mu_2}}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+2\sqrt{\mu_2}\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)
\end{aligned}
\tag{103}
$$

ja saame vastuseks

$$
\begin{aligned}
I_1^\mu(q^2) &= \frac{1}{(4\pi)^2q^2\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)(p_1-p_2)^\mu+\\
&\quad+\frac{1}{(4\pi)^2q^2}\left(\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)-\frac{\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)\right)(p_1+p_2)^\mu.
\end{aligned}
\tag{104}
$$

Integraal $I_2^{\mu\nu}(q^2)$

Integraali $I_2^{\mu\nu}(q^2)$ jaoks kasutame

$$
\mathcal{I}_\alpha^{\mu\nu}(q^2) = \frac{\Gamma(\alpha)}{(4\pi)^{2-\varepsilon}}\int_0^1 dx_1\int_0^{1-x_1}dx_2\left((x_1p_1-x_2p_2)^2\right)^{-\alpha}(x_1p_1^\mu-x_2p_2^\mu)(x_1p_1^\nu-x_2p_2^\nu) =
$$

$$= \frac{\Gamma(\alpha)(q^2)^{-\alpha}}{(4\pi)^{2-\varepsilon}8\sqrt{\mu_1\mu_2}(4-2\alpha)}\int_{-1}^{+1}\left(\frac{w^2-v^2}{1-v^2}\right)^{-\alpha}\left(\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right)^{4-2\alpha}\times$$
$$\times\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\mu-\frac{1+w}{\sqrt{\mu_2}}p_2^\mu\right)\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\nu-\frac{1+w}{\sqrt{\mu_2}}p_2^\nu\right)dw. \qquad (105)$$

Selle saab jälle otsekohe $\alpha = 1$ jaoks arvutada, et saada

$$\mathcal{I}_1^{\mu\nu}(q^2) = \frac{1}{16(4\pi)^2q^2\sqrt{\mu_1\mu_2}}\int_{-1}^{+1}\frac{1-v^2}{w-v^2}\left(\frac{2\sqrt{\mu_1\mu_2}}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}\right)^2\times$$
$$\times\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\mu-\frac{1+w}{\sqrt{\mu_2}}p_2^\mu\right)\left(\frac{1-w}{\sqrt{\mu_1}}p_1^\nu-\frac{1+w}{\sqrt{\mu_2}}p_2^\nu\right)dw =$$
$$= I_2^{11}p_1^\mu p_1^\nu - I_2^{12}(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) + I_2^{22}p_2^\mu p_2^\nu. \qquad (106)$$

Me saame

$$4(4\pi)^2q^2\sqrt{\mu_1}I_2^{11}(q^2)/\sqrt{\mu_2} = \int_{-1}^{+1}\frac{(1-v^2)(1-w)^2dw}{(w^2-v^2)\Big((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\Big)^2} =$$
$$= \int_{-1}^{+1}\Bigg(-\frac{(1-v)^2(1-v^2)dw}{2v\Big((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v\Big)^2(v-w)}+$$
$$-\frac{(1+v)^2(1-v^2)dw}{2v\Big((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v\Big)^2(v+w)}+$$
$$+\frac{2\sqrt{\mu_1}(\sqrt{\mu_1}-\sqrt{\mu_2})(1-\mu_1+\mu_2)dw}{\sqrt{\mu_2}\Big((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\Big)}+\frac{4\mu_1dw}{\Big((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\Big)^2}\Bigg) =$$
$$= \frac{(1-v)^2(1-v^2)}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v)^2}\ln\left(\frac{v-1}{v+1}\right)+$$
$$+\frac{(1+v)^2(1-v^2)}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v)^2}\ln\left(\frac{v-1}{v+1}\right)+$$
$$+2\sqrt{\mu_1}\frac{1-\mu_1+\mu_2)}{\sqrt{\mu_2}}\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)-\frac{4\mu_1}{\sqrt{\mu_1}-\sqrt{\mu_2}}\left(\frac{1}{2\sqrt{\mu_1}}-\frac{1}{2\sqrt{\mu_2}}\right) =$$
$$= \frac{1-v^2}{v}\left(\frac{\lambda}{2\mu_2}+1\right)\ln\left(\frac{v-1}{v+1}\right)+2\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+2\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}} =$$
$$= 2\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\left(\frac{1-2\mu_1+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right). \qquad (107)$$

Tulemuse $I_2^{22}(q^2)$ jaoks saame vastavalt

$$4(4\pi)^2q^2\sqrt{\mu_2}I_2^{22}(q^2)/\sqrt{\mu_1} = \int_{-1}^{+1}\frac{(1-v^2)(1+w)^2dw}{(w^2-v^2)\Big((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w\Big)^2} =$$
$$= 2\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\left(\frac{1-2\mu_2+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right), \qquad (108)$$

mis on sama nagu enne, ainult μ_1 ja μ_2 on ära vahetatud. Lõpuks

$$\begin{aligned}
4(4\pi)^2q^2I_2^{12}(q^2) &= \int_{-1}^{+1}\frac{(1-v^2)(1-w^2)dw}{(w^2-v^2)\Big((\sqrt{\mu_1}+\sqrt{\mu_2})=(\sqrt{\mu_1}-\sqrt{\mu_2})w\Big)^2} = \\
&= \int_{-1}^{+1}\Bigg(-\frac{(1-v^2)^2dw}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v)^2(v-w)}+ \\
&\quad -\frac{(1-v^2)^2dw}{2v((\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v)^2(v+w)}+ \\
&\quad +\frac{2(\sqrt{\mu_1}-\sqrt{\mu_2})(\mu_1-\mu_2)dw}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w}-\frac{4\sqrt{\mu_1\mu_2}dw}{((\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})w)^2}\Bigg) = \\
&= 2\left(\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)-1\right). \qquad (109)
\end{aligned}$$

Kokku võttes saame $I_2^{\mu\nu}(q^2)=I_2^g(q^2)g^{\mu\nu}-\mathcal{I}_1^{\mu\nu}(q^2)$ või

$$\begin{aligned}
I_2^{\mu\nu}(q^2) &= \frac{1}{2}I_0'(q^2)g^{\mu\nu}-\frac{1}{2(4\pi)^2q^2}\times \\
&\Bigg[\left(\frac{1-2\mu_1+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu p_1^\nu+ \\
&-\left(\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)-1\right)(p_1^\mu p_2^\nu+p_2^\mu p_1^\nu)+ \\
&+\left(\frac{1-2\mu_2+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu p_2^\nu\Bigg] = \\
&= \frac{\Gamma(1+\varepsilon)}{2(4\pi)^{2-\varepsilon}q^2}(q^2)^{-\varepsilon}\Bigg[\frac{1}{2}\Bigg(\frac{1}{\varepsilon}+3-\ln(\sqrt{\mu_1\mu_2})+ \\
&\qquad +\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\Bigg)q^2g^{\mu\nu}+ \\
&-\left(\frac{1-2\mu_1+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu p_1^\nu+ \\
&+\left(\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)-1\right)(p_1^\mu p_2^\nu+p_2^\mu p_1^\nu)+ \\
&-\left(\frac{1-2\mu_2+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu p_2^\nu\Bigg] \qquad (110)
\end{aligned}$$

ja

$$\begin{aligned}
I_2(q^2) &= 2(1-\varepsilon)I_0'(q^2) = \\
&= \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(q^2)^{-\varepsilon}\left(\frac{1}{\varepsilon}+2-\ln(\sqrt{\mu_1\mu_2})+\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right). \qquad (111)
\end{aligned}$$

Γ_L arvutamise poole edasi minnes leiame nüüd

$$(D-2)\Big(I_2(q^2)g^{\mu\nu}-2I_2^{\mu\nu}(q^2)\Big) = \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}q^2}(q^2)^{-\varepsilon}\times$$

$$\left[\left(\frac{1}{\varepsilon}-\ln\left(\sqrt{\mu_1\mu_2}\right)+\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right)q^2g^{\mu\nu}+\right.$$
$$+2\left(\frac{(\mu_1-\mu_2)^2-2\mu_1+1}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu p_1^\nu+$$
$$+2\left(\frac{(\mu_1-\mu_2)^2-\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)(p_1^\mu p_2^\nu+p_2^\mu p_1^\nu)+$$
$$\left.+2\left(\frac{(\mu_1-\mu_2)^2-2\mu_2+1}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu p_2^\nu\right]. \tag{112}$$

Arvutame veel

$$4(p_1-p_2)I_1g^{\mu\nu}=\frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}q^2}(q^2)^{-\varepsilon}\left(4(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)-4\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)\right)q^2g^{\mu\nu}, \tag{113}$$

$$\begin{aligned}
-4(p_1-p_2)^\mu I_1^\nu \;&=\; \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}q^2}(q^2)^{-\varepsilon}\times\\
&\times\left[2\left(-\frac{2}{\sqrt{\lambda}}(1-\mu_1+\mu_2)\ln\left(\frac{v-1}{v+1}\right)-2\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right)p_1^\mu p_1^\nu+\right.\\
&+2\left(\frac{2}{\sqrt{\lambda}}(1+\mu_1-\mu_2)\ln\left(\frac{v-1}{v+1}\right)-2\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right)p_1^\mu p_2^\nu+\\
&+2\left(\frac{2}{\sqrt{\lambda}}(1-\mu_1+\mu_2)\ln\left(\frac{v-1}{v+1}\right)-2\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)\right)p_2^\mu p_1^\nu+\\
&\left.+2\left(-\frac{2}{\sqrt{\lambda}}(1+\mu_1-\mu_2)\ln\left(\frac{v-1}{v+1}\right)-2\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)\right)p_2^\mu p_2^\nu\right]
\end{aligned} \tag{114}$$

ja

$$\begin{aligned}
-4(p_1p_2)I_0g^{\mu\nu} \;&=\; -\frac{2\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}q^2}(q^2)^{-\varepsilon}\frac{1-\mu_1-\mu_2}{\sqrt{\lambda}}q^2g^{\mu\nu}\times\\
&\times\left[\frac{1}{\varepsilon}-\ln\left(\sqrt{\mu_1\mu_2}\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ln\left(\frac{v-1}{v+1}\right)+L'\right].
\end{aligned} \tag{115}$$

Kui kõik liikmed kokku võtta, saame

$$\begin{aligned}
&(D-2)\left(I_2(q^2)g^{\mu\nu}-2I_2^{\mu\nu}(q^2)\right)+4\left((p_1-p_2)I_1g^{\mu\nu}-(p_1-p_2)^\mu I_1^\nu-(p_1p_2)I_0g^{\mu\nu}\right)\;=\\
&=\;\frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}q^2}(\sqrt{\mu_1\mu_2}q^2)^{-\varepsilon}\left[\left(\frac{1}{\varepsilon}-3\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)+3(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+\right.\right.\\
&\quad-\frac{2}{\sqrt{\lambda}}(1-\mu_1-\mu_2)\left(\left(\frac{1}{\varepsilon}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ln\left(\frac{v-1}{v+1}\right)+L'\right)\Bigg)q^2g^{\mu\nu}+\\
&\quad+2\left(\frac{(1+\mu_1-\mu_2)^2-2\mu_1-2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu p_1^\nu+\\
&\quad+2\left(\frac{(1+\mu_1-\mu_2)^2+1-\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(2+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu p_2^\nu+
\end{aligned}$$

$$+2\left(\frac{(1-\mu_1+\mu_2)^2+1-\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(2-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu p_1^\nu+$$
$$+2\left(\frac{(1-\mu_1+\mu_2)^2-2\mu_2-2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu p_2^\nu\Bigg]. \quad (116)$$

Järgmine samm on ahendamine avaldisega $\bar{u}(p_1)\gamma_\nu\frac{1}{2}(1-\gamma_5)v(p_2)$

$$\begin{aligned}
g^{\mu\nu}\bar{u}(p_1)\gamma_\nu\frac{1-\gamma_5}{2}v(p_2) &= \bar{u}(p_1)\gamma^\mu\frac{1-\gamma_5}{2}v(p_2),\\
p_1^\nu\bar{u}(p_1)\gamma_\nu\frac{1-\gamma_5}{2}v(p_2) &= \bar{u}(p_1)\not{p}_1\frac{1-\gamma_5}{2}v(p_2) = m_1\bar{u}(p_1)\frac{1-\gamma_5}{2}v(p_2),\\
p_2^\nu\bar{u}(p_1)\gamma_\nu\frac{1-\gamma_5}{2}v(p_2) &= \bar{u}(p_1)\frac{1+\gamma_5}{2}\not{p}_2 v(p_2) = -m_2\bar{u}(p_1)\frac{1+\gamma_5}{2}v(p_2).
\end{aligned} \quad (117)$$

Sellega saame

$$\begin{aligned}
\Delta\Gamma^\mu_{L1} = {} & \frac{g_s^2 C_F\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(\sqrt{\mu_1\mu_2}q^2)^{-\varepsilon}\Bigg[\Bigg(\frac{1}{\varepsilon}-3\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)+3(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+\\
& -\frac{2}{\sqrt{\lambda}}(1-\mu_1-\mu_2)\left(\left(\frac{1}{\varepsilon}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ln\left(\frac{v-1}{v+1}\right)+L'\right)\Bigg)\gamma^\mu\frac{1-\gamma_5}{2}+\\
& +\frac{2m_1}{q^2}\left(\frac{(1+\mu_1-\mu_2)^2-2\mu_1-2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu\frac{1-\gamma_5}{2}+\\
& -\frac{2m_2}{q^2}\left(\frac{(1+\mu_1-\mu_2)^2+1-\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(2+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu\frac{1+\gamma_5}{2}+\\
& +\frac{2m_1}{q^2}\left(\frac{(1-\mu_1+\mu_2)^2+1-\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(2-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu\frac{1-\gamma_5}{2}+\\
& -\frac{2m_2}{q^2}\left(\frac{(1-\mu_1+\mu_2)^2-2\mu_2-2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu\frac{1+\gamma_5}{2}\Bigg].
\end{aligned} \quad (118)$$

Teine osa on

$$\begin{aligned}
\Delta\Gamma^\mu_{L2} &= 2g_s^2C_F I_1^\nu\left(m_1\gamma^\mu\gamma_\nu\frac{1-\gamma_5}{2}+m_2\gamma_\nu\gamma^\mu\frac{1+\gamma_5}{2}\right) = \\
&= 2g_s^2C_F I_1^1\left(2m_1p_1^\mu\frac{1-\gamma_5}{2}-m_1^2\gamma^\mu\frac{1-\gamma_5}{2}+m_1m_2\gamma^\mu\frac{1+\gamma_5}{2}\right)+\\
&\quad +2g_s^2C_F I_1^2\left(2m_2p_2^\mu\frac{1+\gamma_5}{2}-m_1m_2\gamma^\mu\frac{1+\gamma_5}{2}+m_2^2\gamma^\mu\frac{1-\gamma_5}{2}\right) = \\
&= \frac{-2g_s^2C_F}{(4\pi)^2}\left(\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right)\gamma^\mu\frac{1-\gamma_5}{2}+\\
&\quad +\frac{4g_s^2C_F\sqrt{\mu_1\mu_2}}{(4\pi)^2\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)\gamma^\mu\frac{1+\gamma_5}{2}+\\
&\quad +\frac{4g_s^2C_Fm_1}{(4\pi)^2q^2}\left(\frac{1-\mu_1+\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right)p_1^\mu\frac{1-\gamma_5}{2}+\\
&\quad -\frac{4g_s^2C_Fm_2}{(4\pi)^2q^2}\left(\frac{1+\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)\right)p_2^\mu\frac{1+\gamma_5}{2}.
\end{aligned} \quad (119)$$

Kokku on

$$\begin{aligned}
\Delta\Gamma^\mu_L &= g_s^2 C_F \frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}(\sqrt{\mu_1\mu_2}q^2)^{-\varepsilon}\left[4\frac{\sqrt{\mu_1\mu_2}}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)\gamma^\mu\frac{1+\gamma_5}{2}+\right.\\
&+\left(\frac{1}{\varepsilon}-2\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-3\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)+(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+\right.\\
&\qquad\left.-\frac{2}{\sqrt{\lambda}}(1-\mu_1-\mu_2)\left(\left(\frac{1}{\varepsilon}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ln\left(\frac{v-1}{v+1}\right)+L'\right)\right)\gamma^\mu\frac{1-\gamma_5}{2}+\\
&+\frac{2m_1}{q^2}\left(\frac{1-2\mu_1+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu\frac{1-\gamma_5}{2}+\\
&-\frac{2m_2}{q^2}\left(\frac{1-\mu_1-\mu_2+(1+\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(2+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right)p_1^\mu\frac{1+\gamma_5}{2}+\\
&+\frac{2m_1}{q^2}\left(\frac{1-\mu_1-\mu_2+(1-\mu_1+\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-(2-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu\frac{1-\gamma_5}{2}+\\
&-\left.\frac{2m_2}{q^2}\left(\frac{1-2\mu_2+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)+(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right)p_2^\mu\frac{1+\gamma_5}{2}\right]. \qquad (120)
\end{aligned}$$

2.8 Kujutegurid ja renormeerimine

Täistulemuse parandatud verteksi $\Gamma^\mu_L = \gamma^\mu(1-\gamma_5)/2+\Delta\Gamma^\mu_L$ jaoks saab kirjutada nn kujutegurite kaudu,

$$\Gamma^\mu_L = (1+A_L)\gamma^\mu\frac{1-\gamma_5}{2}+A_R\gamma^\mu\frac{1+\gamma_5}{2}+B^1_L p_1^\mu\frac{1-\gamma_5}{2}+B^1_R p_1^\mu\frac{1+\gamma_5}{2}+B^2_L p_2^\mu\frac{1-\gamma_5}{2}+B^2_R p_2^\mu\frac{1+\gamma_5}{2}. \qquad (121)$$

Esimene kujutegur

$$\begin{aligned}
A^b_L &= \frac{\alpha_s}{4\pi}C_F\Gamma(1+\varepsilon)\left(\frac{4\pi\mu^2}{\sqrt{\mu_1\mu_2}q^2}\right)^\varepsilon\times\\
&\times\left[\frac{1}{\varepsilon}-2\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{v-1}{v+1}\right)-3\sqrt{\lambda}\ln\left(\frac{v-1}{v+1}\right)+(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+\right.\\
&\qquad\left.-\frac{2}{\sqrt{\lambda}}(1-\mu_1-\mu_2)\left(\left(\frac{1}{\varepsilon}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ln\left(\frac{v-1}{v+1}\right)+L'\right)\right] \qquad (122)
\end{aligned}$$

on ainus hajuv tegur. Renormeerimine on ilmnevate hajuvuste käsitlemine. Renormeeritavate teooriate (nagu Standardmudeli omad) hea omadus on see, et singulaarsusi saab lõpuks kasutada parameetrite, nagu massid ja laengud, ümber normeerimiseks (sellest tuleb nimi „renormeerimine“). Renormeerimiseks on vaja hajuvuste raamatupidamist. Selleks saab hajuvused koguda nn renormeerimisteguritesse, mis osaliselt omavahel taanduvad, nii et lõpuks jäävad nad ainult parameetrite külge. Seosed, mis näitavad, et renormeerimistegurid taanduvad, on Wardi identsused ja nende sugulased, nt Slavnov–Taylori identsused. Need saab ka pragmaatiliselt esitada.

Selleks, et teooria oleks renormeeritav, peavad renormeerimistegurid verteksites kokku langema laengu renormeerimisteguriga Z_g^{-1}. Kui $Z_1(m_1, m_2)$ on parandatud kahe massiga osakese ja W-bosoni vahelise verteksi renormeerimistegur ja $Z_2(m)$ parandatud fermionpropagaatori renormeerimistegur massiga m, siis kehtib nende jaoks seos

$$Z_1(m_1, m_2) = \sqrt{Z_2(m_1) Z_2(m_2)}. \tag{123}$$

Renormeerimistegurit $Z_2(m)$ ennast me siin ei arvuta. Tulemus nn massipinna (OS) skeemis on

$$\begin{aligned} Z_2^{\rm OS}(m) &= 1 - \frac{\alpha_s}{4\pi} C_F \Bigg(\left(\frac{1}{\varepsilon_{\rm UV}} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right) + 2\left(\frac{1}{\varepsilon_{\rm IR}} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right) + 4\Bigg) = \\ &= 1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon} \left(\frac{1}{\varepsilon_{\rm UV}} + \frac{2}{\varepsilon_{\rm IR}} + 4\right), \end{aligned} \tag{124}$$

millest tuleb

$$Z_1^{\rm OS}(m_1, m_2) = 1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi\mu^2}{m_1 m_2}\right)^{\varepsilon} \left(\frac{1}{\varepsilon_{\rm UV}} + \frac{2}{\varepsilon_{\rm IR}} + 4\right) =: 1 + \Delta Z_1^{\rm OS}(m_1, m_2). \tag{125}$$

Valemi $A_L = A_L^b + \Delta Z_1$ arvutamisel taandub ultraviolett(UV)-hajuvus. Sellega on UV-renormeeritud kujutegur

$$\begin{aligned} A_L &= -\frac{\alpha_s}{4\pi} C_F \Gamma(1+\varepsilon) \left(\frac{4\pi\mu^2}{\sqrt{\mu_1\mu_2} q^2}\right)^{\varepsilon} \times \\ &\times \Bigg[\frac{2}{\varepsilon} + 2\frac{\mu_1 + \mu_2 - (\mu_1 - \mu_2)^2}{\sqrt{\lambda}} \ln\left(\frac{v-1}{v+1}\right) + 3\sqrt{\lambda} \ln\left(\frac{v-1}{v+1}\right) - (\mu_1 - \mu_2) \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + \\ &\quad + \frac{2}{\sqrt{\lambda}} (1 - \mu_1 - \mu_2) \left(\left(\frac{1}{\varepsilon} - \ln\left(1 - (\sqrt{\mu_1} - \sqrt{\mu_2})^2\right)\right) \ln\left(\frac{v-1}{v+1}\right) + L' \right) + 4 \Bigg], \end{aligned} \tag{126}$$

mis sisaldab ainult infrapunaseid(IR) hajuvusi. Teised kujutegurid on ($\varepsilon = 0$ puhul)

$$\begin{aligned} A_R &= \frac{\alpha_s}{4\pi} C_F \left[4\frac{\sqrt{\mu_1\mu_2}}{\sqrt{\lambda}} \ln\left(\frac{v-1}{v+1}\right)\right], \\ B_L^1 &= \frac{\alpha_s}{4\pi} C_F \frac{2m_1}{q^2} \left[\frac{1 - 2\mu_1 + (\mu_1 - \mu_2)^2}{\sqrt{\lambda}} \ln\left(\frac{v-1}{v+1}\right) + (1 - \mu_1 + \mu_2) \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + 1\right], \\ B_R^1 &= -\frac{\alpha_s}{4\pi} C_F \frac{2m_2}{q^2} \left[\frac{1 - \mu_1 - \mu_2 + (1 + \mu_1 - \mu_2)^2}{\sqrt{\lambda}} \ln\left(\frac{v-1}{v+1}\right) - (2 + \mu_1 - \mu_2) \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + 1\right], \\ B_L^2 &= \frac{\alpha_s}{4\pi} C_F \frac{2m_1}{q^2} \left[\frac{1 - \mu_1 - \mu_2 + (1 - \mu_1 + \mu_2)^2}{\sqrt{\lambda}} \ln\left(\frac{v-1}{v+1}\right) - (2 - \mu_1 + \mu_2) \ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) + 1\right], \\ B_R^2 &= -\frac{\alpha_s}{4\pi} C_F \frac{2m_2}{q^2} \left[\frac{1 - 2\mu_2 + (\mu_1 - \mu_2)^2}{\sqrt{\lambda}} \ln\left(\frac{v-1}{v+1}\right) + (1 + \mu_1 - \mu_2) \ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) + 1\right]. \end{aligned} \tag{127}$$

Siiani arvutasime mitte-füüsikalises piirkonnas $v > 1$. Jätkamine annab

$$\ln\left(\frac{v-1}{v+1}\right) = \ln\left(\frac{1-v}{1+v}\right) + i\pi. \tag{128}$$

Mis vajab veel tähelepanu on L'. Saab näidata, et (vt võrrandit (95))

$$\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{2\sqrt{\mu_1}}(1\pm v)\leq 1, \qquad \frac{\sqrt{\mu_2}-\sqrt{\mu_1}}{2\sqrt{\mu_2}}(1\pm v)\leq 1 \tag{129}$$

ning

$$\frac{(\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v}>0, \tag{130}$$

nii et $L'-L$ on reaalne. Seepärast kehtib $\operatorname{Re}L'=(L'-L)+\operatorname{Re}L$ ja $\operatorname{Im}L'=\operatorname{Im}L$, kus

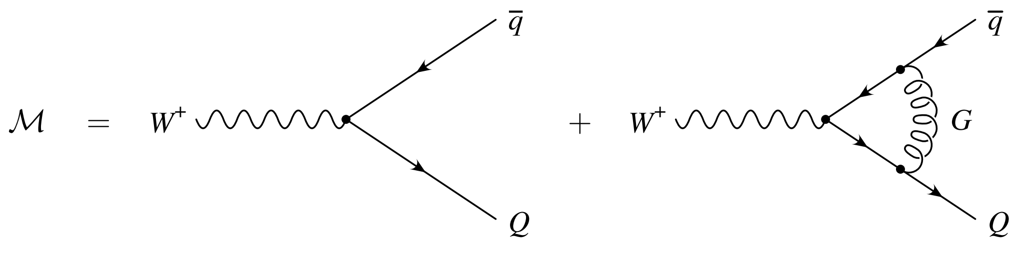
$$L(v)=\operatorname{Li}_2\left(\frac{2v}{1+v}\right)-\operatorname{Li}_2\left(\frac{-2v}{1-v}\right)+i\pi\ln\left(\frac{1-v^2}{4v^2}\right)-\pi^2. \tag{131}$$

Kujutegurite reaalväärtused on seepärast

$$\begin{aligned}
\operatorname{Re}A_L &= -\frac{\alpha_s}{4\pi}C_F\Gamma(1+\varepsilon)\left(\frac{4\pi\mu^2}{\sqrt{\mu_1\mu_2}q^2}\right)^{\varepsilon}\times \\
&\times\left[\frac{2}{\varepsilon}+2\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)+3\sqrt{\lambda}\ln\left(\frac{1-v}{1+v}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+\right.\\
&\left.+\frac{2}{\sqrt{\lambda}}(1-\mu_1-\mu_2)\left(\left(\frac{1}{\varepsilon}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ln\left(\frac{1-v}{1+v}\right)+\operatorname{Re}L'\right)+4\right],\\
\operatorname{Re}A_R &= \frac{\alpha_s}{4\pi}C_F\left[4\frac{\sqrt{\mu_1\mu_2}}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)\right],\\
\operatorname{Re}B_L^1 &= \frac{\alpha_s}{4\pi}C_F\frac{2m_1}{q^2}\left[\frac{1-2\mu_1+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)+(1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right],\\
\operatorname{Re}B_R^1 &= -\frac{\alpha_s}{4\pi}C_F\frac{2m_2}{q^2}\left[\frac{1-\mu_1-\mu_2+(1+\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)-(2+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+1\right],\\
\operatorname{Re}B_L^2 &= \frac{\alpha_s}{4\pi}C_F\frac{2m_1}{q^2}\left[\frac{1-\mu_1-\mu_2+(1-\mu_1+\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)-(2-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right],\\
\operatorname{Re}B_R^2 &= -\frac{\alpha_s}{4\pi}C_F\frac{2m_2}{q^2}\left[\frac{1-2\mu_2+(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)+(1+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)+1\right].
\end{aligned} \tag{132}$$

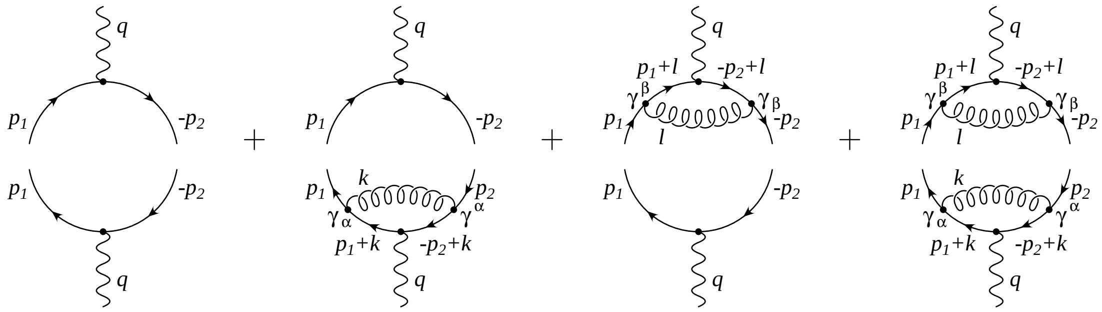
2.9 Esimese järgu silmuseliste parandite tulemused

Esimese järgu parandi määramiseks arvutame vastavate maatrikselementide summa



(133)

absoluutväärtuse ruudu, mille summeerime üle kõikide mittevaadeldavate spinnolekute. Saame neli panust, mis on esitatud alloleval skeemil.



(134)

Neist esimene on tavaline Borni taseme panus. Teised kaks on olulised panused, mida loetakse esimese järgu silmuselisteks paranditeks. Viimase neist võime arvestamata jätta, sest tegu on teise järgu panusega. Kasutades parandatud verteksi kaaskompleksväärtust

$$\begin{aligned}\bar{\Gamma}_L^{\mu} &= \gamma^0 \Gamma_L^{\mu\dagger} \gamma^0 = (1 + A_L^*)\gamma^{\mu}\frac{1-\gamma_5}{2} + A_R^*\gamma^{\mu}\frac{1+\gamma_5}{2} + \\ &+ B_L^{1*} p_1^{\mu}\frac{1+\gamma_5}{2} + B_R^{1*} p_1^{\mu}\frac{1-\gamma_5}{2} + B_L^{2*} p_2^{\mu}\frac{1+\gamma_5}{2} + B_R^{2*} p_2^{\mu}\frac{1-\gamma_5}{2}\end{aligned} \tag{135}$$

saame arvutada

$$h^{\mu\nu} = \mathrm{tr}\left(\Gamma_L^{\mu}(\not{p}_1 + m_1)\bar{\Gamma}_L^{\nu}(\not{p}_2 - m_2)\right) \tag{136}$$

ja ahendada kõikide projektoritega $g_{\mu\nu}$, $\varepsilon_{\mu}^{0}\varepsilon_{\nu}^{0*}$, $\varepsilon_{\mu}^{3}\varepsilon_{\nu}^{3*}$ ja $\varepsilon_{\mu}^{\pm}\varepsilon_{\nu}^{\pm*}$. Tulemused on järgnevalt esitatud. Tuleb tähele panna, mis juhtub Diraci struktuuridega, kui arvutatakse kaaskompleksi

$$\gamma^0\left(\gamma^{\mu}\frac{1\pm\gamma_5}{2}\right)^{\dagger}\gamma^0 = \gamma^0\frac{1\pm\gamma_5^{\dagger}}{2}\gamma^{\mu\dagger}\gamma^0 = \gamma^0\frac{1\pm\gamma_5}{2}\gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 = \frac{1\mp\gamma_5}{2}\gamma^{\mu} = \gamma^{\mu}\frac{1\pm\gamma_5}{2}, \tag{137}$$

nii et A-struktuurid säiluvad, aga

$$\gamma^0\left(p_i^{\mu}\frac{1\pm\gamma_5}{2}\right)^{\dagger}\gamma^0 = p_i^{\mu}\gamma^0\frac{1\pm\gamma_5^{\dagger}}{2}\gamma^0 = p_i^{\mu}\gamma^0\frac{1\pm\gamma_5}{2}\gamma^0 = p_i^{\mu}\frac{1\mp\gamma_5}{2}, \tag{138}$$

ehk B-struktuurid muutuvad oma käelisuse. Sellega aga tuleb järgnevates tulemustes välja, et vaja läheb ainult kujutegurite reaalväärtusi.

Mittepolariseeritud panused

Kui me ahendame $h_{\mu\nu}$ võrrandist (136) meetrilise tensori komponentidega $-g^{\mu\nu}$, siis saame täissumma panused $h = -g^{\mu\nu}h_{\mu\nu}$,

$$\begin{aligned}h(Born) &= 2(q^2 - m_1^2 - m_2^2), \\ h(loop) &= 2(q^2 - m_1^2 - m_2^2)\Big(2\,\mathrm{Re}\,A_L - m_1\,\mathrm{Re}\,B_L^1 + m_2\,\mathrm{Re}\,B_R^2\Big) + \\ &+ 4m_1m_2\Big(4\,\mathrm{Re}\,A_R + m_1\,\mathrm{Re}\,B_R^1 - m_2\,\mathrm{Re}\,B_L^2\Big),\end{aligned} \tag{139}$$

kus kasutasime kinemaatikat $p_1^2 = m_1^2$, $p_2^2 = m_2^2$ ja $p_1 p_2 = (q^2 - m_1^2 - m_2^2)/2$.

Panused ajapolarisatsioonile

Lisatingimustega $\varepsilon(0)p_1 = (q^2+m_1^2-m_2^2)/2\sqrt{q}$ ja $\varepsilon(0)p_2 = (q^2-m_1^2+m_2^2)/2\sqrt{q}$ saame ahendades $\varepsilon^\mu(0)\varepsilon^{\nu *}(0)$-ga ajapolarisatsiooni puhul

$$\begin{aligned} h_{00}(Born) &= q^2 - m_1^2 - m_2^2 - q^2\lambda, \qquad (\lambda = \lambda(1,\mu_1,\mu_2)) \\ h_{00}(loop) &= 2\left(q^2 - m_1^2 - m_2^2 - q^2\lambda\right)\mathrm{Re}\,A_L - 4m_1m_2\,\mathrm{Re}\,A_R + \\ &+ \frac{1}{q^2}\left(q^4 - (m_1^2 - m_2^2)^2\right)\left(m_1\,\mathrm{Re}\,B_L^1 - m_2\,\mathrm{Re}\,B_R^2\right) + \\ &- \frac{m_2}{q^2}(q^2 + m_1^2 - m_2^2)^2\,\mathrm{Re}\,B_R^1 + \frac{m_1}{q^2}(q^2 - m_1^2 + m_2^2)^2\,\mathrm{Re}\,B_L^2. \end{aligned} \tag{140}$$

Panused pikipolarisatsioonile

Kasutades $\varepsilon(3)p_1 = \sqrt{q^2}\sqrt{\lambda}\cos\theta/2$ ja $\varepsilon(3)p_2 = -\sqrt{q^2}\sqrt{\lambda}\cos\theta/2$ ning ahendades $\varepsilon^\mu(3)\varepsilon^{\nu *}(3)$-ga saame

$$\begin{aligned} h_{33}(Born) &= (q^2 - m_1^2 - m_2^2) - q^2\lambda\cos^2\theta, \\ h_{33}(loop) &= 2(q^2 - m_1^2 - m_2^2)\,\mathrm{Re}\,A_L + 4m_1m_2\,\mathrm{Re}\,A_R + \\ &- q^2\lambda\left(2\,\mathrm{Re}\,A_L + m_1(\mathrm{Re}\,B_L^1 - \mathrm{Re}\,B_L^2) + m_2(\mathrm{Re}\,B_R^1 - \mathrm{Re}\,B_R^2)\right)\cos^2\theta. \end{aligned} \tag{141}$$

Panused põikpolarisatsioonile

Lõpuks kasutades avaldisi $\varepsilon(\pm)^{(*)}p_1 = \mp\sqrt{q^2}\sqrt{\lambda}\sin\theta/2\sqrt{2}$ ja $\varepsilon(\pm)^{(*)}p_2 = \pm\sqrt{q^2}\sqrt{\lambda}\sin\theta/2\sqrt{2}$ ning reeglit $\epsilon(\varepsilon(\pm),\varepsilon(\pm)^*,p_1,p_2) = \pm i\sqrt{\lambda}q^2/2$ ja ahendades $\varepsilon^\mu(\pm)\varepsilon^{\nu *}(\pm)$-ga saame

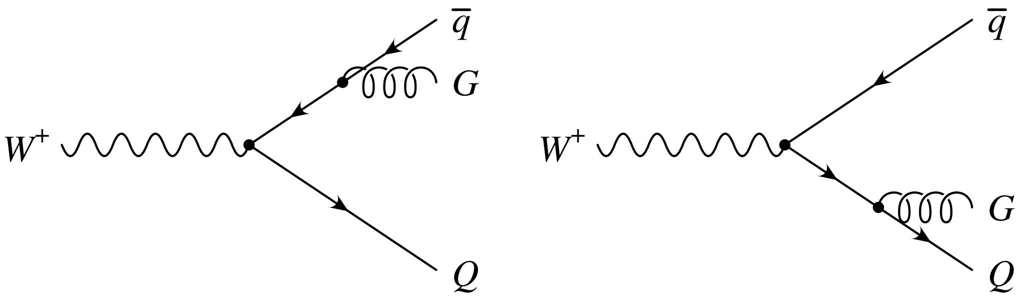
$$\begin{aligned} h_{\pm\pm}(Born) &= (q^2 - m_1^2 - m_2^2) - \frac{1}{2}q^2\lambda(1-\cos^2\theta) \mp q^2\sqrt{\lambda}\cos\theta, \\ h_{\pm\pm}(loop) &= 2(q^2 - m_1^2 - m_2^2)\,\mathrm{Re}\,A_L - q^2\lambda\,\mathrm{Re}\,A_L(1-\cos^2\theta) + \\ &+ 4m_1m_2\,\mathrm{Re}\,A_R \mp 2q^2\sqrt{\lambda}\,\mathrm{Re}\,A_L\cos\theta + \\ &- \frac{1}{2}q^2\lambda\left(m_1(\mathrm{Re}\,B_L^1 - \mathrm{Re}\,B_L^2) + m_2(\mathrm{Re}\,B_R^1 - \mathrm{Re}\,B_R^2)\right)(1-\cos^2\theta). \end{aligned} \tag{142}$$

3 Puudiagrammid

Borni taseme diagramm koos silmuselise panusega moodustab kaheosakeselise protsessi. Võib arvata, et sellega on kõik vajalik arvutatud. Selgub aga, et silmuselisel parandil on IR-hajuvus. Seda on uurinud juba Bloch ja Nordsieck [18] ning hiljem Lee ja Nauenberg [19]. Osutub, et ilmnevaid IR-hajuvusi saab taandada, kui võtta arvesse ka pehme gluuoniga kolmeosakeselist protsessi $W \to Q + \bar{q} + G$. Niisugust protsessi ei saa eksperimendis kaheosakeselisest protsessist eristada, kui gluuoni energia on piisavalt väike. Kuna aga tehniliselt on keeruline eraldada pehmeid ja kalke gluuoneid, siis võetakse IR-hajuvuste taandamiseks terve kolmeosakeselise protsessi panus arvesse. Sellega vaadeldakse sellest punktist alatest mitte puhast kaheosakeselist protsessi, vaid kahe- ja kolmeosakeseliste protsesside segu.

3.1 Kolme osakese protsess

Kolme osakese protsessi Feynmani graafikud on esitatud joonisel 3. Kuna pehme gluuon võib kiirguda nii kvargilt kui ka antikvagilt, koosneb protsess kahest liikmest. Vastavalt Feynmani



Joonis 3: Kolme osakese protsesi Feynmani graafikud

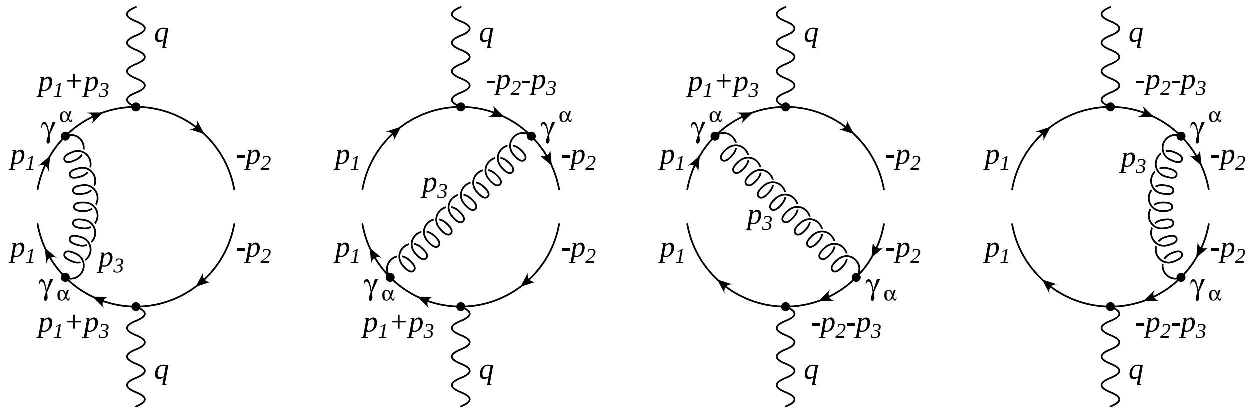
reeglitele saame amplituudi kirja panna järgmiselt:

$$\begin{aligned}\mathcal{M}^\mu &= \bar{u}(p_1,s_1)(-i\gamma^\alpha T_a)\frac{i}{\not{p}_1+\not{p}_2+m_1}\left(-i\frac{g_W}{\sqrt{2}}V_{iI}\gamma^\mu\frac{1-\gamma_5}{2}\right)v(p_2,s_2)\varepsilon^*_\alpha(q,\lambda)+\\ &+\bar{u}(p_1,s_1)\left(-i\frac{g_W}{\sqrt{2}}V_{iI}\gamma^\mu\frac{1-\gamma_5}{2}\right)\frac{i}{-\not{p}_2+\not{p}_3+m_2}(-i\gamma^\alpha T_a)v(p_2,s_2)\varepsilon^*_\alpha(q,\lambda). \qquad (143)\end{aligned}$$

Amplituudi absoluutväärtuse ruudu arvutamiseks läheb meil vaja ka suurust $\mathcal{M}^*$

$$\begin{aligned}\mathcal{M}^{*\mu} &= \bar{v}(p_2,s_2)(i\gamma^\alpha T_a)\frac{i}{\not{p}_1+\not{p}_2+m_1}\left(i\frac{g_W}{\sqrt{2}}V^*_{iI}\gamma^\mu\frac{1-\gamma_5}{2}\right)u(p_1,s_1)\varepsilon_\alpha(q,\lambda)+\\ &+\bar{v}(p_2,s_2)\left(i\frac{g_W}{\sqrt{2}}V^*_{iI}\gamma^\mu\frac{1-\gamma_5}{2}\right)\frac{i}{-\not{p}_2+\not{p}_3+m_2}(-i\gamma^\alpha T_a)u(p_1,s_1)\varepsilon_\alpha(q,\lambda). \qquad (144)\end{aligned}$$

Kolme osakese lagunemisprotsessi amplituudi ruudu arvutamisel on meil neli diagrammi, mida



Joonis 4: Kolme osakese protsesi matrikselemendi ruudud

tähistame jooniselt 4 vasakult paremale (vastavalt gluuoni alg- ja lõpp-punktile) „11“, „12“, „21“ ja „22“. Panuseks „11“ saame

$$\begin{aligned}
\mathrm{tr}_{11}^{\mu\nu} &= \mathrm{Tr}\Big((\not p_1 + m_1)(-i\gamma^\alpha T_a)\frac{i}{\not p_1 + \not p_3 - m_1}\left(-i\frac{g_W}{\sqrt{2}}V_{iI}\gamma^\mu\frac{1-\gamma_5}{2}\right) \times \\
&\qquad \times (\not p_2 - m_2)\left(i\frac{g_W}{\sqrt{2}}V_{iI}^*\gamma^\nu\frac{1-\gamma_5}{2}\right)\frac{i}{\not p_1 + \not p_3 - m_1}(-i\gamma_\alpha T_a)\Big) = \\
&= \frac{g_W^2|V_{iI}|^2 T_a T_a}{8((p_1+p_3)^2 - m_1^2)^2} \times \\
&\quad \times \mathrm{Tr}\left((\not p_1 + m_1)\gamma^\alpha(\not p_1 + \not p_3 + m_1)\gamma^\mu(1-\gamma_5)(\not p_2 - m_2)\gamma^\nu(1-\gamma_5)(\not p_1 + \not p_3 + m_1)\gamma_\alpha\right).
\end{aligned} \tag{145}$$

Tulemuse saamiseks kasutame teisendusi[4]

$$\frac{i}{\not p_1 + \not p_3 - m_1} = i\frac{\not p_1 + \not p_3 + m_1}{(p_1+p_3)^2 - m_1^2}, \qquad \frac{i}{-\not p_2 - \not p_3 - m_2} = i\frac{-\not p_2 - \not p_3 + m_2}{(p_2+p_3)^2 - m_2^2} \tag{146}$$

Järgnevate elementide jaoks saame tulemuseks

$$\begin{aligned}
\mathrm{tr}_{12}^{\mu\nu} &= \frac{g_W^2|V_{iI}|^2 C_F}{8((p_1+p_3)^2 - m_1^2)((p_2+p_3)^2 - m_2^2)} \times \\
&\times \mathrm{Tr}\left((\not p_1 + m_1)\gamma^\alpha(\not p_1 + \not p_3 + m_1)\gamma^\mu(1-\gamma_5)(\not p_2 - m_2)\gamma_\alpha(-\not p_2 - \not p_3 + m_2)\gamma^\nu(1-\gamma_5)\right),
\end{aligned} \tag{147}$$

$$\begin{aligned}
\mathrm{tr}_{21}^{\mu\nu} &= \frac{g_W^2|V_{iI}|^2 C_F}{8((p_2+p_3)^2 - m_2^2)((p_1+p_3)^2 - m_1^2)} \times \\
&\times \mathrm{Tr}\left((\not p_1 + m_1)\gamma^\mu(1-\gamma_5)(-\not p_2 - \not p_3 + m_2)\gamma^\alpha(\not p_2 - m_2)\gamma^\nu(1-\gamma_5)(\not p_1 + \not p_3 + m_1)\gamma_\alpha\right),
\end{aligned} \tag{148}$$

$$\begin{aligned}
\mathrm{tr}_{22}^{\mu\nu} &= \frac{g_W^2|V_{iI}|^2 C_F}{8((p_2+p_3)^2 - m_2^2)^2} \times \\
&\times \mathrm{Tr}\left((\not p_1 + m_1)\gamma^\mu(1-\gamma_5)(-\not p_2 - \not p_3 + m_2)\gamma^\alpha(\not p_2 - m_2)\gamma_\alpha(-\not p_2 - \not p_3 + m_2)\gamma^\nu(1-\gamma_5)\right)
\end{aligned} \tag{149}$$

[4] $T_a T_a = C_F = (N_c^2 - 1)/2N_c$, kus N_c on värvide arv. $N_c = 3$ puhul on $C_F = 4/3$.

Tuleb tähele panna, et

$$\gamma^0\left(\frac{i}{\not{p}-m}\right)^\dagger\gamma^0=\gamma^0\frac{(i(\not{p}+m))^\dagger}{p^2-m^2}\gamma^0=-i\frac{\gamma^0\not{p}^\dagger\gamma^0+m}{p^2-m^2}=-i\frac{\not{p}+m}{p^2-m^2}=\frac{-i}{\not{p}-m}, \tag{150}$$

aga samal ajal ka $\gamma^0(-i\gamma^\alpha)^\dagger\gamma^0=i\gamma^0\gamma^\alpha\gamma^0=i\gamma^\alpha$, nii et need kaks märki kompenseerivad teineteist.

3.2 Sobivad parameetrid

Faasiruumi integreerimiseks oleks sobiv, kui nimetajakordajad võtaksid kõige lihtsama kuju. Sellepärast defineerime dimensioonitud suurused y_1 ja y_2

$$\begin{aligned} y_1q^2 &:= (p_1+p_3)^2-m_1^2 = p_1^2+2p_1p_3+p_3^2-m_1^2 = 2p_1p_3+p_3^2,\\ y_2q^2 &:= (p_2+p_3)^2-m_2^2 = p_2^2+2p_2p_3+p_3^2-m_2^2 = 2p_2p_3+p_3^2, \end{aligned} \tag{151}$$

kus kvargid on massi pinnal, nii et $p_i^2=m_i^2$ ja kirjutame need vastavalt

$$p_1^2=m_1^2=\mu_1q^2,\qquad p_2^2=m_2^2=\mu_2q^2,\qquad p_3^2=m_3^2=\Lambda q^2, \tag{152}$$

kus oleme kasutanud IR-regulariseerimiseks väikest gluuoni massi $\sqrt{\Lambda q^2}$. Puuduva skalaarkorrutise p_1p_2 saame impulsi q kaudu

$$q^2=(p_1+p_2+p_3)^2=p_1^2+2p_1p_2+2p_1p_3+p_2^2+2p_2p_3+p_3^2. \tag{153}$$

Kokku võttes saame kirjutada välja kõik vajalikud skalaarkorrutised

$$\begin{aligned} p_1p_1 &= \mu_1q^2,\\ p_1p_2 &= \frac{1}{2}\left(1-(\mu_1+y_1)-(\mu_2+y_2)+\Lambda\right)q^2,\\ p_1p_3 &= \frac{1}{2}(y_1-\Lambda)q^2,\\ p_2p_2 &= \mu_2q^2,\\ p_2p_3 &= \frac{1}{2}(y_2-\Lambda)q^2,\\ p_3p_3 &= \Lambda q^2. \end{aligned} \tag{154}$$

3.3 Kolme osakese protsessi kinemaatika paigalsüsteemis

Kui me kasutame süsteemi, milles lagunev W^+-boson on paigal, siis $q=(\sqrt{q^2};\vec{0})$. Sellisel juhul on meil lihtne osakeste energiad arvutada. Energia on nelivektori $p_i=(E_i;\vec{p}_i)$ nullkomponent ja saame selle esitada skalaarkorrutise kaudu järgnevalt

$$\begin{aligned}
E_1 &= p_1 q/\sqrt{q^2} = (p_1^2 + p_1 p_2 + p_1 p_3)/\sqrt{q^2} = \frac{1}{2}\left(1+\mu_1-(\mu_2+y_2)\right)\sqrt{q^2},\\
E_2 &= p_2 q/\sqrt{q^2} = (p_1 p_2 + p_2^2 + p_2 p_3)/\sqrt{q^2} = \frac{1}{2}\left(1+\mu_2-(\mu_1+y_1)\right)\sqrt{q^2},\\
E_3 &= p_3 q/\sqrt{q^2} = (p_1 p_3 + p_2 p_3 + p_3^2)/\sqrt{q^2} = \frac{1}{2}\left(y_1+y_2\right)\sqrt{q^2}. \qquad (155)
\end{aligned}$$

Lõpuks saame arvutada ka kolmvektorite absoluutväärtused. Kuna

$$\vec{p}_1^{\,2} = E_1^2 - m_1^2 = \frac{1}{4}\left(\left(1+\mu_1-(\mu_2+y_2)\right)-4\mu_1\right)q^2 = \frac{1}{4}\lambda(1,\mu_1,\mu_2+y_2)q^2, \qquad (156)$$

saame

$$\begin{aligned}
|\vec{p}_1| &= \frac{1}{2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\sqrt{q^2},\\
|\vec{p}_2| &= \frac{1}{2}\sqrt{\lambda(1,\mu_1+y_1,\mu_2)}\sqrt{q^2},\\
|\vec{p}_3| &= \frac{1}{2}\sqrt{(y_1+y_2)^2-4\Lambda}\sqrt{q^2}. \qquad (157)
\end{aligned}$$

3.4 Kolme osakese faasiruum

Kolme osakese faasiruumi arvutamisel lähtume valemist (8), kuid tuleb tähele panna, et olukord on veidi teine, kui see oli kahe osakese protsessi puhul. Meil on nüüd

$$dPS_3 = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3-q)\prod_{i=1}^{3}\frac{d^4p_i}{(2\pi)^4}(2\pi)\delta(p_i^2-m_i^2)\theta(p_i^0). \qquad (158)$$

Antud juhul eemaldame gluuoni impulsi p_3 kasutades ära neljadimensionaalse deltafunktsiooni omadust

$$\begin{aligned}
dPS_3 &= \frac{d^4p_1}{(2\pi)^4}(2\pi)\delta(p_1^2-m_1^2)\theta(E_1)\frac{d^4p_2}{(2\pi)^4}(2\pi)\delta(p_2^2-m_2^2)\theta(E_2)(2\pi)\delta(p_3^2-m_3^2)\theta(E_3)\Big|_{p_3=q-p_1-p_2} = \\
&= \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}(2\pi)\delta\left((q-p_1-p_2)^2-\Lambda q^2\right)\theta(q^0-E_1-E_2)\Big|_{E_i=\sqrt{\vec{p}_i^{\,2}-m_i^2}}. \qquad (159)
\end{aligned}$$

Kolmedimensionaalsed integraalmõõdud kirjutame jälle polaarkoordinaatides

$$d^3p_i = |\vec{p}_i|^2 d|\vec{p}_i| d\varphi_i d(\cos\theta_i) = E_i\sqrt{E_i^2-m_i^2}\,dE_i d\varphi_i d\cos(\theta_i). \qquad (160)$$

Nüüd ei kao integraal üle E_1 ega E_2, nagu kahe osakese faasiruumi korral. Energiate asemel saame aga eelnevalt tutvustatud suurusi y_i kasutada

$$\begin{aligned}
E_1 &= \frac{1}{2}\left(1+\mu_1-(\mu_2+y_2)\right)\sqrt{q^2} \quad\Rightarrow\quad dE_1 = -\frac{1}{2}\sqrt{q^2}dy_2\\
E_2 &= \frac{1}{2}\left(1-(\mu_1+y_1)+\mu_2\right)\sqrt{q^2} \quad\Rightarrow\quad dE_2 = -\frac{1}{2}\sqrt{q^2}dy_1 \qquad (161)
\end{aligned}$$

ja saame

$$dPS_3 = \frac{1}{(2\pi)^5}\delta\left((q-p_1p_2)^2-\Lambda q^2\right)\theta(q^0-E_1-E_2)\Big|_{E_i=\sqrt{\vec{p}_i^2+m_i^2}} \times$$
$$\times \frac{1}{8}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}q^2dy_2d\varphi_1d(\cos\theta_1) \times$$
$$\times \frac{1}{8}\sqrt{\lambda(1,\mu_1+y_1,\mu_2)}q^2dy_1d\varphi_2d(\cos\theta_2). \quad (162)$$

Lõpuks säilitame nurgaintegraalide hulgast ainult integreerimise üle $\theta := \theta_1$ ja $\varphi := \varphi_1$. Nurk θ_2 on siis relatiivne nurk $\vec{p}_1$ ja $\vec{p}_2$ vahel ja seda kasutame faasiruumi piiride määramiseks. Ülejäänud deltafunktsioon kirjeldab faasiruumi piire. Argument

$$\begin{aligned}(q-p_1-p_2)^2-\Lambda q^2 &= q^2-2qp_1-2qp_2+p_1^2+2p_1p_2+p_2^2-\Lambda q^2 = \\ &= q^2-2E_1\sqrt{q^2}-2E_2\sqrt{q^2}+m_1^2+m_2^2-\Lambda q^2+2p_1p_2 = \\ &= \left(1-(1+\mu_1-(\mu_2+y_2))-(1-(\mu_1+y_1)+\mu_2)+\mu_1+\mu_2-\Lambda\right)q^2+2p_1p_2 = \\ &= 2p_1p_2-\left(1-(\mu_1+y_1)-(\mu_2+y_2)+\Lambda\right)q^2 \end{aligned} \quad (163)$$

paistab triviaalselt kaduvat, sest skalaarkorrutis parameetritena võrdub

$$\frac{1}{2}\left(1-(\mu_1+y_1)-(\mu_2+y_2)+\Lambda\right)q^2. \quad (164)$$

Oletades aga, et me ei tea seda ning kirjutame teise lahutatava panuse asemel $2\mu_{12}q^2$, siis saame

$$\begin{aligned}\delta\left((q-p_1-p_2)^2-\Lambda q^2\right) &= \delta(2p_1p_2-2\mu_{12}q^2) = \delta\left(2E_1E_2-2|\vec{p}_1||\vec{p}_2|\cos\theta_2-2\mu_{12}q^2\right) = \\ &= \frac{1}{2|\vec{p}_1||\vec{p}_2|}\delta\left(\frac{E_1E_2-\mu_{12}q^2}{|\vec{p}_1||\vec{p}_2|}-\cos\theta_2\right), \end{aligned} \quad (165)$$

ja seda võime üle $d(\cos\theta_2)$ integreerida (ehk selle vastu taandada). Lisaks uuele tingimusele taanduvad kordajad $|\vec{p}_1||\vec{p}_2|$. Võttes arvesse, et ülejäänud asimuutnurga φ_2 integraal annab kordaja 2π, saame lõpptulemuseks

$$dPS_3 = \frac{q^2}{32(2\pi)^4}dy_1dy_2d\varphi d(\cos\theta) = \frac{q^2}{32(2\pi)^3}dy_1dy_2d(\cos\theta). \quad (166)$$

3.5 Faasiruumi piirid

Tingimus

$$-1 \le \cos\theta_2 = \frac{E_1E_2-\mu_{12}q^2}{|\vec{p}_1||\vec{p}_2|} \le +1 \qquad \Leftrightarrow \qquad E_1E_2-\mu_{12}q^2 \pm |\vec{p}_1||\vec{p}_2| \quad (167)$$

kirjeldab faasiruumi piire. Kasutades parameetrilist esitust avaldame y_1 teiste muutujate funktsioonina

$$y_{1\pm}(y_2) = \frac{1}{2(\mu_2+y_2)}\Big(y_2\,(1-\mu_1-(\mu_2+y_2))+\Lambda\,(1-\mu_1+(\mu_2+y_2))+ \\ \pm\sqrt{(y_2-\Lambda)^2-4\Lambda\mu_2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\Big) =: \frac{A(y_2)\pm B(y_2)}{C(y_2)}, \quad (168)$$

kus

$$A(y_2) := y_2\,(1-\mu_1-(\mu_2+y_2))+\Lambda\,(1-\mu_1+(\mu_2+y_2))$$
$$B(y_2) := \sqrt{(y_2-\Lambda)^2-4\Lambda\mu_2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}$$
$$C(y_2) := \frac{1}{2(\mu_2+y_2)} \quad (169)$$

Tingimus, et ruutjuured on mittenegatiivsed, määrab meile y_2 määramispiirkonna. Selleks saame

$$y_2 \geq y_{2-} = \Lambda + 2\sqrt{\Lambda\mu_2}, \qquad y_2 \leq y_{2+} = (1-\sqrt{\mu_1})^2 - \mu_2 \leq 1. \quad (170)$$

Ülemine piir ei kattu nii kaua alumise piiriga (st faasiruum kaob), kui $\sqrt{\mu_1}+\sqrt{\mu_2}+\sqrt{\Lambda} \leq 1$. See aga tähendab füüsikalistes suurustes, et $m_1+m_2+m_3 \leq \sqrt{q^2}$, nii et W^+-bosoni lagunemisest tekkivad kolm reaalset (mittevirtuaalset) osakest.

3.6 Nurgasõltuvused kolme osakese faasiruumis

Kui tekivad kvark, antikvark ja gluuon, on meil veel üks nurk, nimelt ϕ, mis vastab antikvark–gluuoni tasandi pööramisele ümber kvargi liikumise suuna. Kui me vaatame ainult kvargi nurgajaotisi, siis see nurgasõltuvus langeb välja, aga peame seda ikka vahepeal arvesse võtma. Enne pööramise arvutamist tuleb aga esitada kvargisüsteemis olev kinemaatika. Meil on juba teada, et

$$E_1 = \frac{1}{2}\sqrt{q^2}\,(1+\mu_1-(\mu_2+y_2)) \qquad |\vec{p}_1| = \frac{1}{2}\sqrt{q^2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)},$$
$$E_2 = \frac{1}{2}\sqrt{q^2}\,(1-(\mu_1+y_1)+\mu_2) \qquad |\vec{p}_2| = \frac{1}{2}\sqrt{q^2}\sqrt{\lambda(1,\mu_1+y_1,\mu_2)},$$
$$E_3 = \frac{1}{2}\sqrt{q^2}(y_1+y_2) \qquad |\vec{p}_3| = \frac{1}{2}\sqrt{q^2}\sqrt{(y_1+y_2)^2-4\Lambda} \quad (171)$$

(panuse Λ võib järgnevates arvutamistes alati ära jätta). Lahtine on veel küsimus, milline on nurk θ_{12} on $\vec{p}_1$ ja $\vec{p}_2$ vahel. Lahenduseks kasutame

$$p_1p_2 = E_1E_2 - \frac{1}{4}q^2\sqrt{\lambda(1,\mu_1,\mu_2+y_2)\lambda(1,\mu_1+y_1,\mu_2)}\cos\theta_{12} =$$

$$= \frac{1}{2}q^2\left(1 - (\mu_1 + y_1) - (\mu_2 + y_2) + \Lambda\right), \tag{172}$$

millest tulevad

$$\cos\theta_{12} = -\frac{\lambda - y_1(1-\mu_1+\mu_2) - y_2(1+\mu_1-\mu_2) - y_1y_2}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)\lambda(1,\mu_1+y_1,\mu_2)}} \quad \text{ja} \tag{173}$$

$$\sin\theta_{12} = 2\frac{\sqrt{y_1y_2(1-y_1-y_2) - (\mu_1y_2+\mu_2y_1)(y_1+y_2)}}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)\lambda(1,\mu_1+y_1,\mu_2)}} \tag{174}$$

(paneme tähele, et $\pi/2 < \theta_{12} < \pi$, sest on kokku lepitud, et antikvark kiiratakse x-telje suunas). Kui me korrutame absoluutväärtusega $|\vec{p}_2|$, saame nelivektori p_2 z- ja x-komponendid kvarksüsteemis

$$p_{2z} = -\frac{1}{2}\sqrt{q^2}\frac{\lambda(1,\mu_1,\mu_2) - y_1(1-\mu_1+\mu_2) - y_2(1+\mu_1-\mu_2) - y_1y_2}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}},$$

$$p_{2x} = \sqrt{q^2}\frac{\sqrt{y_1y_2(1-y_1-y_2) - (\mu_1y_2+\mu_2y_1)(y_1+y_2)}}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}}. \tag{175}$$

Vastavalt saame

$$\cos\theta_{13} = \frac{E_1E_3 - p_1p_3}{|\vec{p}_1||\vec{p}_3|} = -\frac{y_1(1-\mu_1+\mu_2) - y_2(1+\mu_1-\mu_2) + y_1y_2 + y_2^2}{\sqrt{((y_1+y_2)^2 - 4\Lambda)\lambda(1,\mu_1+y_1,\mu_2)}} \quad \text{ja} \tag{176}$$

$$\sin\theta_{13} = -2\frac{\sqrt{y_1y_2(1-y_1-y_2) - (\mu_1y_2+\mu_2y_1)(y_1+y_2)}}{\sqrt{((y_1+y_2)^2 - 4\Lambda)\lambda(1,\mu_1+y_1,\mu_2)}}, \tag{177}$$

($\pi < \theta_{13} < 3\pi/2$), nii et

$$p_{3z} = -\frac{1}{2}\sqrt{q^2}\frac{y_1(1-\mu_1+\mu_2) - y_2(1+\mu_1-\mu_2) + y_1y_2 + y_2^2}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}},$$

$$p_{2x} = -\sqrt{q^2}\frac{\sqrt{y_1y_2(1-y_1-y_2) - (\mu_1y_2+\mu_2y_1)(y_1+y_2)}}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}}. \tag{178}$$

3.7 Kvarksüsteemist polarisatsioonisüsteemi

Loomulikult saame sellest tulemusest $p_{2x} + p_{3x} = 0$ ja

$$p_{2z} + p_{3z} = -\frac{1}{2}\sqrt{q^2}\frac{\lambda - 2y_2(1+\mu_1-\mu_2) + y_2^2}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}} = -\frac{1}{2}\sqrt{q^2}\frac{\lambda(1,\mu_1,\mu_2+y_2)}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}} =$$

$$= -\frac{1}{2}\sqrt{q^2}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)} = -p_{1z} \quad \Rightarrow \quad p_{1z} + p_{2z} + p_{3z} = 0. \tag{179}$$

Nagu juba mainitud, on meil nüüd vaja kaks pööramist läbi viia. Esimene on nurga ϕ kaudu ümber z-telje,

$$
\begin{aligned}
p_1 = (E_1; 0, 0, p_{1z}) &\rightarrow (E_1, 0, 0, p_{1z}) \\
p_2 = (E_2; p_{2x}, 0, p_{2z}) &\rightarrow (E_2; p_{2x}\cos\phi, p_{2x}\sin\phi, p_{2z}) \\
p_3 = (E_3; p_{3x}, 0, p_{3z}) &\rightarrow (E_3; p_{3x}\cos\phi, p_{3x}\sin\phi, p_{3z}), \qquad (180)
\end{aligned}
$$

teine aga nurga θ kaudu ümber y-telje, mille jaoks kehtib

$$
p''_x = p'_x \cos\theta + p'_z \sin\theta, \qquad p''_z = p'_z \cos\theta - p'_x \sin\theta \qquad (181)
$$

kus $p'_x = p_{2x}\cos\phi$, $p'_z = p_{2z}$ ja sellega

$$
\begin{aligned}
p_1 &= (E_1; p_{1z}\sin\theta, 0, p_{1z}\cos\theta) \\
p_2 &= (E_2; p_{2x}\cos\phi\cos\theta + p_{2z}\sin\theta, p_{2x}\sin\phi, p_{2z}\cos\theta - p_{2x}\cos\phi\sin\theta) \\
p_3 &= (E_3; p_{3x}\cos\phi\cos\theta + p_{3z}\sin\theta, p_{3x}\sin\phi, p_{3z}\cos\theta - p_{3x}\cos\phi\sin\theta). \qquad (182)
\end{aligned}
$$

Kuna impulsside skalaarkorrutised on pöörete suhtes invariantsed, saame lõpuks arvutada ka skalaarkorrutised impulsside ja polarisatsioonivektorite vahel. Tulemused $\varepsilon(0)$ ja $\varepsilon(3)$ jaoks on ilmsed. $\varepsilon(\pm)$ puhul saame

$$
\begin{aligned}
\varepsilon(\pm)p_1 &= \mp\frac{1}{\sqrt{2}}p_{1z}\sin\theta, \\
\varepsilon(\pm)p_2 &= -\frac{1}{\sqrt{2}}(\pm(p_{2z}\sin\theta + p_{2x}\cos\phi\cos\theta) + ip_{2x}\sin\phi), \\
\varepsilon(\pm)p_3 &= -\frac{1}{\sqrt{2}}(\pm(p_{3z}\sin\theta + p_{3x}\cos\phi\cos\theta) + ip_{3x}\sin\phi). \qquad (183)
\end{aligned}
$$

Sellisel juhul peame ka need korrutised arvutama, kus ilmub tensor $\epsilon^{\mu\nu\rho\sigma}$. Me saame[5]

$$
\epsilon(\varepsilon(\pm), p_1, p_2, p_3) = \frac{1}{\sqrt{2}}\sqrt{q^2}p_{1z}p_{2x}(\mp\sin\phi\cos\theta + i\cos\phi) \qquad (184)
$$

ja

$$
\begin{aligned}
\epsilon(\varepsilon(\pm), \varepsilon^*(\pm), p_1, p_2) &= \pm i\left((p_{1z}E_2 - E_1p_{2z})\cos\theta + E_1p_{2x}\cos\phi\sin\theta\right), \\
\epsilon(\varepsilon(\pm), \varepsilon^*(\pm), p_1, p_3) &= \pm i\left((p_{1z}E_3 - E_1p_{3z})\cos\theta + E_1p_{3x}\cos\phi\sin\theta\right), \\
\epsilon(\varepsilon(\pm), \varepsilon^*(\pm), p_2, p_3) &= \pm i\left((p_{2z}E_3 - E_2p_{3z})\cos\theta + (E_2p_{3x} - p_{2x}E_3)\cos\phi\sin\theta\right). \qquad (185)
\end{aligned}
$$

[5] $\epsilon(\varepsilon, p_1, p_2, p_3) := \epsilon_{\mu\mu\rho\sigma}\varepsilon^\mu p_1^\nu p_2^\rho p_3^\sigma$

3.8 Integreerimine

Pöördume nüüd peatüki algusesse tagasi ja kirjutame maatrikselemendite absoluutväärtuste ruudud y_1 ja y_2 kaudu. Siis saame kogusumma jaoks järgmise avaldise

$$\begin{aligned}\sum_{Spin}|\mathcal{M}|^2 &= -\frac{8\mu_1}{y_1^2}(1-\mu_1-\mu_2)+\frac{8}{y_1}(1-\mu_1-\mu_2)-\frac{8\mu_2}{y_2^2}(1-\mu_1-\mu_2)-\frac{8}{y_2}(1-\mu_1-\mu_2)+\\ &+\frac{8}{y_1y_2}(1-\mu_1-\mu_2)^2+\frac{4y_1}{y_2}+\frac{4y_2}{y_1},\end{aligned}\tag{186}$$

kusjuures saime lugejates $\Lambda = 0$ valida. Põhiintegrandid on seega

$$1,\quad \frac{1}{y_2},\ \frac{1}{y_2^2},\quad ,\frac{1}{y_1},\ \frac{1}{y_1^2},\quad \frac{y_1}{y_2},\ \frac{y_1}{y_2^2},\quad \frac{y_2}{y_1},\ \frac{y_2}{y_1^2},\quad \frac{1}{y_2y_1}.\tag{187}$$

Esmalt integreerime üle y_1 ja siis üle y_2. Integreerimine üle y_1 on formaalne ja esitatav järgnevalt

$$\int_{y_{1-}}^{y_{1+}} y_1 dy_1 = \frac{1}{2}\left(y_{1+}^2-y_{1-}^2\right) = \frac{1}{2C^2}\left((A+B)^2-(A-B)^2\right) = 2\frac{AB}{C^2},\tag{188}$$

$$\int_{y_{1-}}^{y_{1+}} dy_1 = y_{1+}-y_{1-} = \frac{1}{C}\left((A+B)-(A-B)\right) = 2\frac{B}{C},\tag{189}$$

$$\int_{y_{1-}}^{y_{1+}} \frac{dy_1}{y_1} = \ln(y_{1+})-\ln(y_{1-}) = \ln\left(\frac{A+B}{C}\right)-\ln\left(\frac{A-B}{C}\right) = \ln\left(\frac{A+B}{A-B}\right),\tag{190}$$

$$\int_{y_{1-}}^{y_{1+}} \frac{dy_1}{y_1^2} = -\frac{1}{y_{1+}}+\frac{1}{y_{1-}} = -\frac{C}{A+B}+\frac{C}{A-B} = \frac{2CB}{A^2-B^2}.\tag{191}$$

Integreerimine üle y_2 kasutab (piirjuhul $\Lambda \to 0$) alati asendusi

$$\begin{aligned}&y_2 = 1+\mu_1-\mu_2-\sqrt{\mu_1}\left(\frac{1}{z}+z\right) \qquad z(y_2) = \frac{1}{2\sqrt{\mu_1}}\left(1+\mu_1-\mu_2-y_2-\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}\right)\\ &dy_2 = \sqrt{\mu_1}\left(\frac{1}{z^2}-1\right)dz \qquad \sqrt{\lambda(1,\mu_1,\mu_2+y_2)} = \sqrt{\mu_1}\left(\frac{1}{z}-z\right).\end{aligned}\tag{192}$$

Sellega on

$$\begin{aligned}y_2(z) &= \frac{\sqrt{\mu_1}}{z}(z_+-z)(z-z_-), \qquad z_\pm := \frac{1}{2\sqrt{\mu_1}}\left(1+\mu_1-\mu_2\pm\sqrt{\lambda(1,\mu_1,\mu_2)}\right),\\ A(y_2(z)) &= \left(\sqrt{\mu_1}\left(\frac{1}{z}+z\right)-2\mu_1\right)y_2(z),\\ B(y_2(z)) &= \sqrt{\mu_1}\left(\frac{1}{z}-z\right)y_2(z),\\ C(y_2(z)) &= 2\left(z-\sqrt{\mu_1}\right)\left(\frac{1}{z}-\sqrt{\mu_1}\right),\end{aligned}\tag{193}$$

Oluline on ka seos $A(y_2(z))^2 + B(y_2(z))^2 = 2\mu_1 y_2(z)^2 C(y_2(z))$. Rajad on

$$z(0) = z_- = z_+^{-1}, \qquad z\left((1-\sqrt{\mu_1})^2-\mu_2\right) = 1.\tag{194}$$

Ainus erand on IR-hajuvad integraalid, aga nende jaoks on divergentsed osad samuti universaalsed. Ilmnevad integraalid saab klassifitseerida y_1 ja y_2 astmete järgi. Lisaks ilmuvad polariseeritud panuste puhul erinevad $\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}$ astmed vastavalt dünaamikale (vt. pt. 3.6). Sellega defineerime neli erinevat integraalitüüpi

$$\begin{aligned} I(n_1,n_2) &= \int \sqrt{\lambda(1,\mu_1,\mu_2+y_2)}y_1^{n_1}y_2^{n_2}dy_1dy_2, \\ S(n_1,n_2) &= \int y_1^{n_1}y_2^{n_2}dy_1dy_2, \\ J(n_1,n_2) &= \int \frac{y_1^{n_1}y_2^{n_2}dy_1dy_2}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}}, \\ T(n_1,n_2) &= \int \frac{y_1^{n_1}y_2^{n_2}dy_1dy_2}{\lambda(1,\mu_1,\mu_2+y_2)} \end{aligned} \tag{195}$$

3.9 Integreerimispõhimõtted

Enne, kui alustada üldise integreerimisega peame eraldama divergentsed osad. Divergentsed integraalid on üldiselt need, kus valemites (195) $(n_1,n_2)=(-2,0)$, $(-1,-1)$ ja $(0,-2)$. Tuleb välja, et hajuvuse korral saame kasutada asendust

$$\sqrt{\lambda(1,\mu_1,\mu_2+y_2)} \to \sqrt{\lambda}, \quad \frac{1}{y_1^2} \to S_D(-2,0), \quad \frac{1}{y_1y_2} \to S_D(-1,-1), \quad \frac{1}{y_2^2} \to S_D(0,-2), \tag{196}$$

kus $\sqrt{\lambda} := \sqrt{\lambda(1,\mu_1,\mu_2)}$. Esimese asenduse järgi saab integraali kordaja $\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}$ integraalides (195) kordajana $\sqrt{\lambda}$ ette võtta. Sellega jäävad kõik sama (n_1,n_2)-ga integraalid samaks, mis väljendab mainitud universaalsust ja mis viib teistele asendustele võrrandis (196). Arvutamise käigus, mis ei ole siin lahti seletatud, ilmub hajuva panuse liikmena kordaja

$$\ln\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2}{\sqrt{\Lambda\mu_2}}\right). \tag{197}$$

Kasulikuks osutuvad divergentsed avaldised, mis on $\mu_1 \leftrightarrow \mu_2$ suhtes sümmeetrilised. $S(-2,0)$ ja $S(0,-2)$ korral saab kasutada kuju

$$\ln\left(\frac{\lambda}{\sqrt{\Lambda\mu_1\mu_2}}\right) = \ln\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2}{\sqrt{\Lambda\mu_2}}\right) + \ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}}\right), \tag{198}$$

kus me kasutasime $((1-\sqrt{\mu_1})^2-\mu_2)((1+\sqrt{\mu_1})^2-\mu_2)=\lambda$. Selleks puuduv teine panus tuleb nüüd integraalist

$$\int_{z_-}^1 \left(\frac{-2dz}{z-z_+} + \frac{dz}{z}\right) = \Big[-2\ln(z_+-z) + \ln z\Big]_{z_-}^1 =$$

$$= \quad 2\ln\left(\frac{z_+ - z_-}{z_+ - 1}\right) - \ln z_- \;=\; 2\ln(1+z_-) - \ln z_- =$$
$$= \quad \ln\left(\frac{(1+\sqrt{\mu_1})^2 - \mu_2}{\sqrt{\mu_1}}\right). \qquad (199)$$

Kui lisame selle integraali divergentsetele osadele (ja vastavalt konvergentsetest osadest taandame), siis lihtsustuvad divergentsed integraalid veelgi ja saame teha asenduse(196)

$$\begin{aligned}
& \sqrt{\lambda(1,\mu_1,\mu_2+y_2)} \;\to\; \sqrt{\lambda(1,\mu_1,\mu_2)} \;=:\; \sqrt{\lambda}, \\
\frac{1}{y_1^2} \;\to\; & S'_D(-2,0) \;=\; \frac{\sqrt{\lambda}}{\mu_1}\left(\ln\left(\frac{\lambda}{\sqrt{\Lambda\mu_1\mu_2}}\right) - 1\right), \\
\frac{1}{y_1 y_2} \;\to\; & S'_D(-1,-1) \;=\; \ln\left(\frac{\lambda}{\sqrt{\Lambda\mu_1\mu_2}}\right)\ln\alpha_+ + \frac{1}{2}\mathrm{Li}_2(1-\alpha_+) - \frac{1}{2}\mathrm{Li}_2(1-\alpha_-), \\
\frac{1}{y_2^2} \;\to\; & S'_D(0,-2) \;=\; \frac{\sqrt{\lambda}}{\mu_2}\left(\ln\left(\frac{\lambda}{\sqrt{\Lambda\mu_1\mu_2}}\right) - 1\right), \qquad (200)
\end{aligned}$$

kus

$$\alpha_+ = \frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}} = \alpha_-^{-1}. \qquad (201)$$

3.10 IR-hajuvate integraalide konvergentsed osad

Enne eelmainitud parandust olid divergentsed integraalid piirjuhul $\Lambda \to 0$

$$\begin{aligned}
S_D(-2,0) \;&\to\; \frac{\sqrt{\lambda}}{\mu_1}\int_{z_-}^{1}\left(\frac{dz}{z-z_-} + \frac{dz}{z-z_+} - \frac{dz}{z}\right), \\
S_D(-1,-1) \;&\to\; \ln\alpha_+\int_{z_-}^{1}\left(\frac{dz}{z-z_-} + \frac{dz}{z-z_+} - \frac{dz}{z}\right), \\
S_D(0,-2) \;&\to\; \frac{\sqrt{\lambda}}{\mu_2}\int_{z_-}^{1}\left(\frac{dz}{z-z_-} + \frac{dz}{z-z_+} - \frac{dz}{z}\right). \qquad (202)
\end{aligned}$$

Lisades veel panuse (199) koos vastava kordajaga saame aga

$$\begin{aligned}
S'_D(-2,0) \;&\to\; \frac{\sqrt{\lambda}}{\mu_1}\int_{z_-}^{1}\left(\frac{dz}{z-z_-} - \frac{dz}{z-z_+}\right), \\
S'_D(-1,-1) \;&\to\; \ln\alpha_+\int_{z_-}^{1}\left(\frac{dz}{z-z_-} - \frac{dz}{z-z_+}\right), \\
S'_D(0,-2) \;&\to\; \frac{\sqrt{\lambda}}{\mu_2}\int_{z_-}^{1}\left(\frac{dz}{z-z_-} - \frac{dz}{z-z_+}\right). \qquad (203)
\end{aligned}$$

Integraalid $S(-2,0)$, $S(-1,-1)$ ja $S(0,-2)$ ja teiste tüüpide (I,J,T) vastavad integraalid saab nüüd arvutada, kuna vastupanused taandavad $S(-2,0)$ ja $S(0,-2)$ hajuva panuse $(z-z_-)^{-1}$

ning $S(-1,-1)$ korral pehmendavad panust nii, et hajuvust enam ei teki. Töö edasises käigus teeme asendused

$$\begin{aligned}\sqrt{\lambda(1,\mu_1,\mu_2+y_2)} &\to \sqrt{\lambda},\\ \frac{1}{y_1^2} &\to -S_D'(-2,0),\\ \frac{1}{y_1y_2} &\to -S_D'(-1,-1),\\ \frac{1}{y_2^2} &\to -S_D'(0,-2)\end{aligned}\tag{204}$$

ja säilitame tulemuse vastupanuseks.

3.11 Logaritmilised singulaarpanused

Logaritmilised panused tulevad astmest y_1^{-1}, sest esimene integraal annab

$$\int_{y_{1-}(y_2)}^{y_{1+}(y_2)}\frac{dy_1}{y_1}=\ln\left(\frac{A+B}{A-B}\right)=\ln\left(\frac{1-\sqrt{\mu_1}z}{z(z-\sqrt{\mu_1})}\right):=L(z).\tag{205}$$

Ülemise raja $z=1$ juures on $L(1)=0$, alumise raja $z=z_-$ juures aga

$$L(z_-)=\ln\left(\frac{1-\sqrt{\mu_1}z_-}{z_-(z_--\sqrt{\mu_1})}\right)=\ln\left(\frac{z_+-\sqrt{\mu_1}}{z_--\sqrt{\mu_1}}\right)=\ln\left(\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right)=\ln\alpha_+.\tag{206}$$

See on aga just vastupanuse kordaja. Defineerime sellepärast integraalid

$$\begin{aligned}I_z^\ell(0) &= \int_{z_-}^1 (L(z)-L(z_-))\left(\frac{dz}{z-z_-}-\frac{dz}{z-z_+}\right) \qquad \text{ja}\\ S_z^\ell(0) &= \int_{z_-}^1\left(L(z)\left(\frac{dz}{z-z_-}+\frac{dz}{z-z_+}-\frac{dz}{z}\right)-L(z_-)\left(\frac{dz}{z-z_-}-\frac{dz}{z-z_+}\right)\right).\end{aligned}\tag{207}$$

Kordaja $(L(z)-L(z_-))$ pehmendab hajuvust ja võimaldab sellega arvutada integraali. Integraal $I_z^\ell(0)$ erineb integraalist $\sqrt{\lambda}J_C(-1,-1)$ kordaja $L(z_-)\times(199)$ poolest nii, et tulemus lihtsustub võrreldes $\sqrt{\lambda}J_C(-1,-1)$-ga,

$$I_z^\ell(0)=\mathrm{Li}_2(-z_+)-\mathrm{Li}_2(-z_-)+\mathrm{Li}_2\left(\frac{z_+-\sqrt{\mu_1}}{\sqrt{\mu_1}z_+-1}\right)-\mathrm{Li}_2\left(\frac{\sqrt{\mu_1}z_+-1}{z_+-\sqrt{\mu_1}}\right).\tag{208}$$

Teine integraal $S_z^\ell(0)$ osutub keerulisemaks,

$$S_z^\ell(0) = S_C(-1,-1)-2\ln\alpha_+\ln\left(\frac{z_+-z_-}{z_+-1}\right)+\ln\alpha_+\ln z_- =$$

$$
\begin{aligned}
&= \mathrm{Li}_2(\alpha_-) + \mathrm{Li}_2(\alpha_- z_+^2) - \mathrm{Li}_2(1-z_-^2) - \frac{\pi^2}{3} - \frac{1}{2}\ln^2\alpha_- + \ln\sqrt{\mu_1}\ln(\alpha_- z_+^2) + \\
&\quad - 2\ln\alpha_+\ln z_+ - 2\ln\alpha_+\ln\left(\frac{z_+-z_-}{z_+-\sqrt{\mu_1}}\right) + 2\ln z_+\ln\left(\frac{z_+-z_-}{z_--\sqrt{\mu_1}}\right) = \\
&= \mathrm{Li}_2(\alpha_-) + \mathrm{Li}_2(\alpha_- z_+^2) - \mathrm{Li}_2(1-z_-^2) - \frac{\pi^2}{3} - \frac{1}{2}\ln^2\alpha_- + \ln\sqrt{\mu_1}\ln(\alpha_- z_+^2) + \\
&\quad - 2\ln\alpha_+\ln\left(\frac{z_+-z_-}{z_+-\sqrt{\mu_1}}\right) + 2\ln z_+\ln\left(\frac{z_+-z_-}{z_+-\sqrt{\mu_1}}\right) = \\
&= \mathrm{Li}_2(\alpha_-) + \mathrm{Li}_2(\alpha_- z_+^2) + \mathrm{Li}_2(z_-^2) - \frac{\pi^2}{2} - \frac{1}{2}\ln^2\alpha_- + \ln\sqrt{\mu_1}\ln(\alpha_- z_+^2) + \\
&\quad - 2\ln\alpha_+\ln\left(\frac{z_+-z_-}{z_+-\sqrt{\mu_1}}\right) + 2\ln z_+\ln\left(\frac{z_+-z_-}{z_+-\sqrt{\mu_1}}\right) + 2\ln z_-\ln(1-z_-^2) = \\
&= \mathrm{Li}_2(\alpha_-) + \mathrm{Li}_2(\alpha_- z_+^2) + \mathrm{Li}_2(z_-^2) - \frac{\pi^2}{2} - \frac{1}{2}\ln^2\alpha_- + \ln\sqrt{\mu_1}\ln(\alpha_- z_+^2) + \\
&\quad - 2\ln\alpha_+\ln\left(\frac{z_+-z_-}{z_+-\sqrt{\mu_1}}\right) + 2\ln z_+\ln\left(\frac{z_+}{z_+-\sqrt{\mu_1}}\right), \qquad (209)
\end{aligned}
$$

mille leidmiseks oleme kasutanud teisendusi

$$
\begin{aligned}
(1+\mu_1-\mu_2-\sqrt{\lambda})(1+\mu_1-\mu_2+\sqrt{\lambda}) &= 4\mu_1, \\
(1-\mu_1-\mu_2-\sqrt{\lambda})(1-\mu_1-\mu_2+\sqrt{\lambda}) &= 4\mu_1\mu_2. \qquad (210)
\end{aligned}
$$

Kasutades veel

$$
\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) = \ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right), \qquad (211)
$$

saame uuritavale avaldisele anda kuju

$$
\begin{aligned}
S_z^\ell(0) &= \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \\
&\quad - \frac{\pi^2}{2} - \frac{1}{2}\ln^2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \ln(\sqrt{\mu_1})\ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \\
&\quad + 2\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)\ln\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + 2\ln z_+\ln\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \\
&\quad - \frac{\pi^2}{2} - \frac{1}{2}\ln^2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \ln(\sqrt{\mu_1})\ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \\
&\quad + \ln\left(\frac{\lambda}{\mu_1\mu_2}\right)\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \ln^2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) - \ln\mu_2\ln z_+ + \\
&\quad + \frac{1}{2}\ln^2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) - \frac{1}{2}\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) =
\end{aligned}
$$

$$
\begin{aligned}
= \; & \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& -\frac{\pi^2}{2}+\frac{1}{2}\ln^2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\frac{1}{2}\ln^2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& -\frac{1}{2}\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\ln\left(\frac{\lambda}{\mu_1\mu_2}\right)\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& +\ln(\sqrt{\mu_1})\ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\ln(\sqrt{\mu_2})\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \; =
\end{aligned}
$$

$$
\begin{aligned}
= \; & \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& -\frac{\pi^2}{2}+\frac{1}{2}\ln^2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\frac{1}{2}\ln^2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& +\frac{1}{2}\ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\ln\left(\frac{\lambda}{2\mu_1\mu_2}\right)\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& +\ln(2\sqrt{\mu_1})\ln\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\ln(2\sqrt{\mu_2})\ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \; =
\end{aligned}
$$

$$
\begin{aligned}
= \; & \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& -\frac{\pi^2}{2}+\frac{1}{2}\ln^2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\ln\left(\frac{\lambda}{2\mu_1\mu_2}\right)\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+ \\
& -\frac{1}{2}\ln(1-\mu_1+\mu_2-\sqrt{\lambda})\ln(1+\mu_1-\mu_2-\sqrt{\lambda})+ \\
& \quad +\frac{1}{2}\ln(1-\mu_1+\mu_2-\sqrt{\lambda})\ln(1+\mu_1-\mu_2+\sqrt{\lambda})+ \\
& \qquad +\frac{1}{2}\ln(1-\mu_1+\mu_2+\sqrt{\lambda})\ln(1+\mu_1-\mu_2-\sqrt{\lambda})+ \\
& \qquad\quad -\frac{1}{2}\ln(1-\mu_1+\mu_2+\sqrt{\lambda})\ln(1+\mu_1-\mu_2+\sqrt{\lambda})+ \\
& +\frac{1}{2}\ln(1-\mu_1+\mu_2-\sqrt{\lambda})\ln(1+\mu_1-\mu_2-\sqrt{\lambda})+ \\
& \quad +\frac{1}{2}\ln(1-\mu_1+\mu_2-\sqrt{\lambda})\ln(1+\mu_1-\mu_2+\sqrt{\lambda})+ \\
& \qquad -\frac{1}{2}\ln(1-\mu_1+\mu_2+\sqrt{\lambda})\ln(1+\mu_1-\mu_2-\sqrt{\lambda})+ \\
& \qquad\quad -\frac{1}{2}\ln(1-\mu_1+\mu_2+\sqrt{\lambda})\ln(1+\mu_1-\mu_2+\sqrt{\lambda})+ \\
& +\frac{1}{2}\ln(1-\mu_1+\mu_2-\sqrt{\lambda})\ln(1+\mu_1-\mu_2-\sqrt{\lambda})+ \\
& \quad -\frac{1}{2}\ln(1-\mu_1+\mu_2-\sqrt{\lambda})\ln(1+\mu_1-\mu_2+\sqrt{\lambda})+ \\
& \qquad +\frac{1}{2}\ln(1-\mu_1+\mu_2+\sqrt{\lambda})\ln(1+\mu_1-\mu_2-\sqrt{\lambda})+
\end{aligned}
$$

$$-\frac{1}{2}\ln(1-\mu_1+\mu_2+\sqrt{\lambda})\ln(1+\mu_1-\mu_2+\sqrt{\lambda}) \;=$$

$$\begin{aligned}
=\;& \mathrm{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right)+\mathrm{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right)+\\
&-\frac{\pi^2}{2}+\frac{1}{2}\ln^2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\ln\left(\frac{\lambda}{2\mu_1\mu_2}\right)\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right)+\\
&+2\ln(2\sqrt{\mu_1})\ln(2\sqrt{\mu_2})-2\ln(1-\mu_1+\mu_2+\sqrt{\lambda})\ln(1+\mu_1-\mu_2+\sqrt{\lambda}). \qquad (212)
\end{aligned}$$

3.12 Teised logaritmilised panused

Kui singulaarsused on logaritmilistest panustest eraldatud, siis jäävad veel mittehajuvad osad. Kõige lihtsam panus on

$$I^\ell(0)=\int_{z_-}^1\frac{dz}{z}L=\mathrm{Li}_2(\sqrt{\mu_1}z_+)+\mathrm{Li}_2(\sqrt{\mu_1}z_-)-2\mathrm{Li}_2(\sqrt{\mu_1})+\ln^2 z_-=S^\ell(0). \qquad (213)$$

Sellele järgnevad teised „taasilmuvad integraalid“

$$\begin{aligned}
I^\ell(1) &= \sqrt{\mu_1}\int_{z_-}^1\left(\frac{1}{z^2}+1\right)\ln\left(\frac{1-\sqrt{\mu_1}z}{z(z-\sqrt{\mu_1})}\right)dz \;=\\
&= \sqrt{\lambda}\ln\left(\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right)-(1-\mu_1)\ln\left(\frac{(1-\sqrt{\mu_1})^2}{\mu_2}\right)-\left((1-\sqrt{\mu_1})^2-\mu_2\right),\\
I^\ell(2) &= 2\mu_1\int_{z_-}^1\left(\frac{1}{z^3}+z\right)\ln\left(\frac{1-\sqrt{\mu_1}z}{z(z-\sqrt{\mu_1})}\right)dz \;=\\
&= (1+\mu_1-\mu_2)\sqrt{\lambda}\ln\left(\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right)-(1-\mu_1^2)\ln\left(\frac{(1-\sqrt{\mu_1})^2}{\mu_2}\right)+\\
&\quad -\frac{1}{2}\lambda+(1-\mu_1)\left((1-\sqrt{\mu_1})^2-\mu_2\right),\\
I^\ell(3) &= 3\mu_1\sqrt{\mu_1}\int_{z_-}^1\left(\frac{1}{z^4}+z^2\right)\ln\left(\frac{1-\sqrt{\mu_1}z}{z(z-\sqrt{\mu_1})}\right)dz \;=\\
&= (\lambda+3\mu_1)\sqrt{\lambda}\ln\left(\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right)-(1-\mu_1^3)\ln\left(\frac{(1-\sqrt{\mu_1})^2}{\mu_2}\right)+\\
&\quad -(1+\mu_1-\mu_2)\frac{\lambda}{3}-\left((1-\sqrt{\mu_1})^2-\mu_2\right)\frac{\mu_1}{3}+(1-\mu_1)\frac{\lambda}{2}+(1-\mu_1^2)\left((1-\sqrt{\mu_1})^2-\mu_2\right)
\end{aligned} \qquad (214)$$

ja

$$S^\ell(1) \;=\; \sqrt{\mu_1}\int_{z_-}^1\left(\frac{1}{z^2}-1\right)\ln\left(\frac{1-\sqrt{\mu_1}z}{z(z-\sqrt{\mu_1})}\right)dz \;=$$

$$
\begin{aligned}
&= \mu_2 \ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) - \ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) - \sqrt{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
S^\ell(2) &= 2\mu_1 \int_{z_-}^1 \left(\frac{1}{z^3} - z\right) \ln\left(\frac{1-\sqrt{\mu_1}z}{z(z-\sqrt{\mu_1})}\right) dz \ = \\
&= \mu_2(2-\mu_2+2\mu_1) \ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \\
&\qquad - \ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \frac{1}{2}(1+\mu_2-3\mu_1)\sqrt{\lambda},
\end{aligned}
$$

$$
\begin{aligned}
S^\ell(3) &= 3\mu_1\sqrt{\mu_1} \int_{z_-}^1 \left(\frac{1}{z^4} - z^2\right) \ln\left(\frac{1-\sqrt{\mu_1}z}{z(z-\sqrt{\mu_1})}\right) dz \ = \\
&= \mu_2(3+\mu_2^2+3\mu_1^2-3\mu_2+3\mu_1-3\mu_2\mu_1) \ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \\
&\qquad - \ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \frac{1}{6}(7-2\mu_2^2-11\mu_1^2+\mu_2-2\mu_1+7\mu_2\mu_1)\sqrt{\lambda}.
\end{aligned}
\tag{215}
$$

Lõpuks on järele jäänud veel kaks logaritmilist integraali

$$
\begin{aligned}
S_1^\ell(0) &= \int_{z_-}^1 L\left(\frac{dz}{z-1} - \frac{dz}{z+1}\right) \ = \ -2\sqrt{\mu_1}T(-1,0) \ = \\
&= \mathrm{Li}_2(z_-) - \mathrm{Li}_2(-z_-) - \frac{\pi^2}{4} + \ln z_- \ln\left(\frac{1-z_-}{1+z_-}\right) + \\
&\quad - \mathrm{Li}_2\left(\frac{(1+\sqrt{\mu_1})(1-z_-)}{(1-\sqrt{\mu_1})(1+z_-)}\right) + \mathrm{Li}_2\left(-\frac{(1+\sqrt{\mu_1})(1-z_-)}{(1-\sqrt{\mu_1})(1+z_-)}\right) \qquad \text{ja}
\end{aligned}
\tag{216}
$$

$$
\begin{aligned}
I_1^\ell(0) &= \int_{z_-}^1 L\left(\frac{dz}{z-1} + \frac{dz}{z+1} - \frac{dz}{z}\right) \ = \ T(-1,-1) - (1+\mu_1-\mu_2)T(-1,0) \ = \\
&= \mathrm{Li}_2(\mu_1) - \mathrm{Li}_2(\sqrt{\mu_1}z_+) - \mathrm{Li}_2(\sqrt{\mu_1}z_-) - \frac{\pi^2}{6} + \\
&\quad + \frac{1}{2}\mathrm{Li}_2\left(\frac{(z_- - \sqrt{\mu_1})^2}{(1-\sqrt{\mu_1}z_-)^2}\right) + \frac{1}{2}\mathrm{Li}_2(z_-^2) - 2\mathrm{Li}_2\left(\frac{\sqrt{\mu_1}(\sqrt{\mu_1}-z_-)}{1-\sqrt{\mu_1}z_-}\right) + \\
&\quad + \ln\left(\frac{1-z_-^2}{1-\mu_1}\right) \ln\left(\frac{z_- - \sqrt{\mu_1}}{1-\sqrt{\mu_1}z_-}\right) + \ln z_- \ln(z_+ - z_-).
\end{aligned}
\tag{217}
$$

Tuleb välja, et võimalik singulaarsus raja $z = 1$ lähedal ei häiri tulemust, vaid taandub välja. Seega on võimalik kõik logaritmilised panused asendada eelnevalt leitud integraalidega $S_z^\ell(0)$, $S_0^\ell(0)$, $S^\ell(n)$, $I_z^\ell(0)$, $I_0^\ell(0)$ ja $I^\ell(n)$ $(n = 0, 1, 2, 3)$.

3.13 Mittelogaritmilised singulaarpanused

Ainus mittelogaritmiline singulaarpanus on

$$I_z(0) = \int_{z_-}^{1} \left(\frac{2dz}{z - z_+} - \frac{dz}{z} \right) = -\ln \left(\frac{(1 + \sqrt{\mu_1})^2 - \mu_2}{\sqrt{\mu_1}} \right), \qquad S_z(0) = 0. \tag{218}$$

Selleks, et seda näha, vaatame kõiki mittelogaritmilisi singulaarseid integraale

$$\begin{aligned}
I(-2,0) &= \frac{\lambda}{\mu_1} \left(\frac{1}{z - z_-} + \frac{1}{z - z_+} - \frac{1}{z} \right) + \ldots & I(0,-2) &= \frac{\lambda}{\mu_2} \left(\frac{1}{z - z_-} + \frac{1}{z - z_+} \right) + \ldots \\
S(-2,0) &= \frac{\sqrt{\lambda}}{\mu_1} \left(\frac{1}{z - z_-} - \frac{1}{z - z_+} \right) + \ldots & S(0,-2) &= \frac{\sqrt{\lambda}}{\mu_2} \left(\frac{1}{z - z_-} - \frac{1}{z - z_+} \right) + \ldots \\
J(-2,0) &= \frac{1}{\mu_1} \left(\frac{1}{z - z_-} + \frac{1}{z - z_+} - \frac{1}{z} \right) & J(0,-2) &= \frac{1}{\mu_2} \left(\frac{1}{z - z_-} + \frac{1}{z - z_+} \right) \\
T(-2,0) &= \frac{1}{\mu_1\sqrt{\lambda}} \left(\frac{1}{z - z_-} - \frac{1}{z - z_+} \right) + \ldots & T(0,-2) &= \frac{1}{\mu_2\sqrt{\lambda}} \left(\frac{1}{z - z_-} - \frac{1}{z - z_+} \right) + \ldots
\end{aligned} \tag{219}$$

kus punktid tähistavad teisi, $z_\pm$-st mittesõltuvaid panuseid. Kui lahutame vastupanused (203) koos vastavate kordajatega, saame integraalide $I(-2,0)$ ja $J(-2,0)$ puhul just $I_z(0)$, integraalide $S(-2,0)$, $S(0,-2)$, $T(-2,0)$ ja $T(0,-2)$ puhul aga $S_z(0)$. Integraalide $I(0,-2)$ ja $J(0,-2)$ puhul ei vii taandumine täpselt panusele $I_z(0)$, aga vahet $1/z$ suhtes saab järgnevalt mainitud panustega kokku võtta.

3.14 Teised mittelogaritmilised panused

Logaritme mitte sisaldavad teised panused on

$$\begin{aligned}
S_+(0) &= \int_{z_-}^{1} \frac{dz}{z} = -\ln z_- = \frac{1}{2} \ln \left(\frac{1 + \mu_1 - \mu_2 + \sqrt{\lambda}}{1 + \mu_1 - \mu_2 - \sqrt{\lambda}} \right), \\
S_+(1) &= \sqrt{\mu_1} \int_{z_-}^{1} \left[\frac{1}{z^2} + 1 \right] dz = \sqrt{\mu_1} \left[z - \frac{1}{z} \right]_{z_-}^{1} = \sqrt{\mu_1}(z_+ - z_-) = \sqrt{\lambda}, \\
S_+(2) &= 2\mu_1 \int_{z_-}^{1} \left[\frac{1}{z^3} + z \right] dz = \mu_1 \left[z^2 - \frac{1}{z^2} \right]_{z_-}^{1} = \mu_1(z_+^2 - z_-^2) = (1 + \mu_1 - \mu_2)\sqrt{\lambda}, \\
S_+(3) &= 3\mu_1\sqrt{\mu_1} \int_{z_-}^{1} \left[\frac{1}{z^4} + z^2 \right] dz = \mu_1\sqrt{\mu_1} \left[z^3 - \frac{1}{z^3} \right]_{z_-}^{1} = \\
&= \mu_1\sqrt{\mu_1}(z_+^3 - z_-^3) = (\lambda + 3\mu_1)\sqrt{\lambda}, \\
S_-(0) &= \int_{z_-}^{1} \left[\frac{1}{z - \sqrt{\mu_1}} + \frac{\sqrt{\mu_1}}{1 - \sqrt{\mu_1}z} - \frac{1}{z} \right] dz = \left[\ln \left(\frac{z - \sqrt{\mu_1}}{1 - \sqrt{\mu_1}z} \right) - \ln z \right]_{z_-}^{1} = \\
&= \ln \left(\frac{1 - \sqrt{\mu_1}z^{-1}}{1 - \sqrt{\mu_1}z} \right) \Bigg|_{z_-}^{1} = \ln \left(\frac{1 - \sqrt{\mu_1}z_-}{1 - \sqrt{\mu_1}z_+} \right) = \ln \left(\frac{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}} \right),
\end{aligned}$$

$$
\begin{aligned}
S_-(1) &= \mu_2 \int_{z_-}^1 \left[\frac{\sqrt{\mu_1}}{(z-\sqrt{\mu_1})^2} + \frac{\sqrt{\mu_1}}{(1-\sqrt{\mu_1}z)^2} \right] dz \;=\; -\mu_2 \left[\frac{\sqrt{\mu_1}}{z-\sqrt{\mu_1}} - \frac{1}{1-\sqrt{\mu_1}z} \right]_{z_-}^1 \;= \\
&= \mu_2 \left[\frac{\sqrt{\mu_1}}{z_- - \sqrt{\mu_1}} - \frac{1}{1-\sqrt{\mu_1}z_-} + 1 \right] \;=\; \mu_2 \left[\frac{\sqrt{\mu_1}}{z_- - \sqrt{\mu_1}} - \frac{\sqrt{\mu_1}}{z_+ - \sqrt{\mu_1}} \right] \;=\; \sqrt{\lambda}, \\
S_-(2) &= 2\mu_2^2 \int_{z_-}^1 \left[\frac{\mu_1}{(z-\sqrt{\mu_1})^3} + \frac{\sqrt{\mu_1}}{(1-\sqrt{\mu_1}z)^3} - \frac{\sqrt{\mu_1}}{(1-\sqrt{\mu_1}z)^2} \right] dz \;= \\
&= -\mu_2^2 \left[\frac{\mu_1}{(z-\sqrt{\mu_1})^2} - \frac{1}{(1-\sqrt{\mu_1}z)^2} - \frac{2}{1-\sqrt{\mu_1}z} \right]_{z_-}^1 \;= \\
&= \mu_2^2 \left[\frac{\mu_1}{(z_- - \sqrt{\mu_1})^2} - \frac{1}{(1-\sqrt{\mu_1}z_-)^2} - \frac{2}{1-\sqrt{\mu_1}z_-} + 1 \right] \;= \\
&= \mu_2^2 \left[\frac{\mu_1}{(z_- - \sqrt{\mu_1})^2} - \frac{\mu_1}{(z_+ - \sqrt{\mu_1})^2} \right] \;=\; (1-\mu_1-\mu_2)\sqrt{\lambda} \qquad (220)
\end{aligned}
$$

ja

$$
\begin{aligned}
I_+(0) &= \int_{z_-}^1 \frac{dz}{z} \;=\; -\ln z_- \;=\; \frac{1}{2} \ln\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{1+\mu_1-\mu_2-\sqrt{\lambda}} \right), \\
I_+(1) &= \sqrt{\mu_1} \int_{z_-}^1 \left(\frac{1}{z^2} - 1 \right) dz \;=\; -\sqrt{\mu_1} \left[z + \frac{1}{z} \right]_{z_-}^1 \;=\; \sqrt{\mu_1}\,(z_+ + z_- - 2)\,, \\
I_+(2) &= 2\mu_1 \int_{z_-}^1 \left(\frac{1}{z^3} - z \right) dz \;=\; -\mu_1 \left[z^2 + \frac{1}{z^2} \right]_{z_-}^1 \;=\; \mu_1 \left(z_+^2 + z_-^2 - 2 \right), \\
I_+(3) &= 3\mu_1\sqrt{\mu_1} \int_{z_-}^1 \left(\frac{1}{z^4} - z^2 \right) dz \;=\; \mu_1\sqrt{\mu_1} \left(z_+^3 + z_-^3 - 2 \right), \\
I_+(4) &= 4\mu_1^2 \int_{z_-}^1 \left(\frac{1}{z^5} - z^3 \right) dz \;=\; \mu_1^2 \left(z_+^4 + z_-^4 - 2 \right), \\
I_+(5) &= 5\mu_1^2\sqrt{\mu_1} \int_{z_-}^1 \left(\frac{1}{z^6} - z^4 \right) dz \;=\; \mu_1^2\sqrt{\mu_1} \left(z_+^5 + z_-^5 - 2 \right), \\
I_-(0) &= \int_{z_-}^1 \left(\frac{1}{z-\sqrt{\mu_1}} - \frac{\sqrt{\mu_1}}{1-\sqrt{\mu_1}z} - \frac{1}{z} \right) dz \;=\; \left[\ln\left((z-\sqrt{\mu_1})(1-\sqrt{\mu_1}z) \right) - \ln z \right]_{z_-}^1 \;= \\
&= \ln\left((1-\sqrt{\mu_1})(1-\sqrt{\mu_1}) \right) - \ln\left((z_- - \sqrt{\mu_1})(1-\sqrt{\mu_1}z_-) \right) + \ln z_- \;= \\
&= 2\ln(1-\sqrt{\mu_1}) - \ln(\mu_2 z_-) + \ln z_- \;=\; \ln\left(\frac{(1-\sqrt{\mu_1})^2}{\mu_2} \right), \\
I_-(1) &= \int_{z_-}^1 \left(\frac{\sqrt{\mu_1}}{(z-\sqrt{\mu_1})^2} - \frac{\sqrt{\mu_1}}{(1-\sqrt{\mu_1}z)^2} \right) dz \;=\; -\left[\frac{\sqrt{\mu_1}}{z-\sqrt{\mu_1}} + \frac{1}{1-\sqrt{\mu_1}z} \right]_{z_-}^1 \;= \\
&= \left[\frac{-(1-\mu_1)z}{(z-\sqrt{\mu_1})(1-\sqrt{\mu_1}z)} \right]_{z_-}^1 \;=\; \frac{1-\mu_1}{(z_- - \sqrt{\mu_1})(z_+ - \sqrt{\mu_1})} - \frac{1-\mu_1}{(1-\sqrt{\mu_1})^2} \;= \\
&= \frac{1-\mu_1}{\mu_2} - \frac{1-\mu_1}{(1-\sqrt{\mu_1})^2} \;=\; \frac{\lambda}{(1-\mu_1)\mu_2} - \frac{1}{1-\mu_1} \left((1-\sqrt{\mu_1})^2 - \mu_2 \right). \qquad (221)
\end{aligned}
$$

3.15 Puudiagrammide tulemuste kokkuvõte

Lisades A–D kirjutame integraalide $I(n_1,n_2)$, $S(n_1,n_2)$, $J(n_1,n_2)$ ja $T(n_1,n_2)$ ilmnevad kujud põhiintegraalide suhtes lahti. Tulemused sõltuvad siis panustest

$$D := \ln\left(\frac{\lambda}{\sqrt{\Lambda\mu_1\mu_2}}\right) - 1, \qquad D^\ell := \ln\left(\frac{\lambda}{\sqrt{\Lambda\mu_1\mu_2}}\right)\ln\alpha_+ + \frac{1}{2}\mathrm{Li}_2(1-\alpha_+) - \frac{1}{2}\mathrm{Li}_2(1-\alpha_-), \quad (222)$$

dilogaritmilistest panustest $S_z(0)$, $I_z(0)$, $S^\ell(0)$, $I^\ell(0)$, $S_1^\ell(0)$ ja $I_1^\ell(0)$, logaritmilistest panustest

$$\ell_1 = \ln\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{1+\mu_1-\mu_2-\sqrt{\lambda}}\right), \qquad \ell_2 = \ln\left(\frac{1-\mu_1+\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right),$$

$$\ell_0 = \ln\left(\frac{(1-\sqrt{\mu_1})^2}{\mu_2}\right), \qquad \ell_4 = \ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}}\right) \quad (223)$$

ja $\ell_3 = \ln\alpha_+ = \ell_1+\ell_2$ ning $\sqrt{\lambda}$-st. Lõpptulemuses defineerime kõige olulisema osana divergentsestest panustest sõltuvad suurused

$$D_S := (1-\mu_1-\mu_2)\left(D^\ell + S_z(0)\right) - 2\sqrt{\lambda}D + \frac{3}{4}\left((1+\mu_1-\mu_2)\ell_1 + (1-\mu_1+\mu_2)\ell_2 + \sqrt{\lambda}\right),$$

$$D_I := (1-\mu_1-\mu_2)\left(D^\ell + I_z(0)\right) - 2\sqrt{\lambda}D + \frac{3}{4}\left((1+\mu_1-\mu_2)\ell_1 + (1-\mu_1+\mu_2)\ell_2 + \sqrt{\lambda}\right). \quad (224)$$

Tulemused on ($N = \alpha_s C_F q^2/4\pi\sqrt{\lambda}$)

$$h(tree) = N\Big[8(1-\mu_1-\mu_2)D_S + 4\mu_1(1+\mu_1)\ell_1 + 4\mu_2(1+\mu_2)\ell_2 - 8(\mu_1+\mu_2)\sqrt{\lambda}\Big], \quad (225)$$

$$h_{00}(tree) = N\Big[-4\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)D_S - 2\mu_1(-\mu_1-3\mu_2+\mu_1^2+\mu_1\mu_2+4\mu_2^2)\ell_1 +$$
$$-2\mu_2(-3\mu_1-\mu_2+4\mu_1^2+\mu_1\mu_2+\mu_2^2)\ell_2 + 2(-\mu_1-\mu_2+\mu_1^2-8\mu_1\mu_2+\mu_2^2)\sqrt{\lambda}\Big], \quad (226)$$

$$h'_{33}(tree) = N\Big[4(1-\mu_1-\mu_2)D_S - 4\mu_1(1+7\mu_1-\mu_2)I_1^\ell(0) +$$
$$-2\sqrt{\mu_1}(1-12\mu_1-2\mu_2-5\mu_1^2+4\mu_1\mu_2+\mu_2^2)S_1^\ell(0) +$$
$$-2\mu_1(6+4\mu_1-7\mu_2)\ell_1 + 2\mu_2(2+3\mu_1)\ell_2 - 2(1-11\mu_1+\mu_2)\sqrt{\lambda} +$$
$$-\cos^2\theta\Big\{4\lambda D_S - 12\mu_1(1+7\mu_1-\mu_2)I_1^\ell(0) +$$
$$-6\sqrt{\mu_1}(1-12\mu_1-2\mu_2-5\mu_1^2+4\mu_1\mu_2+\mu_2^2)S_1^\ell(0) +$$
$$-2\mu_1(20+13\mu_1-24\mu_2+\mu_1^2+\mu_1\mu_2+4\mu_2^2)\ell_1 +$$

$$+ 2\mu_2(4 + 12\mu_1 - \mu_2 - 4\mu_1^2 - \mu_1\mu_2 - \mu_2^2)\ell_2 +$$

$$- 2(3 - 36\mu_1 - \mu_1^2 + 8\mu_1\mu_2 - \mu_2^2)\sqrt{\lambda}\Big\}\Big], \tag{227}$$

$$h'_{\pm\pm}(tree) \; = \; N\Big[2(1-\mu_1+\mu_2)(1+\mu_1-\mu_2)D_S + 2\mu_1(1+7\mu_1-\mu_2)I_1^\ell(0) +$$

$$+ \sqrt{\mu_1}(1 - 12\mu_1 - 2\mu_2 - 5\mu_1^2 + 4\mu_1\mu_2 + \mu_2^2)S_1^\ell(0) +$$

$$+ \mu_1(8 + 5\mu_1 - 10\mu_2 + \mu_1^2 + \mu_1\mu_2 + 4\mu_2^2)\ell_1 +$$

$$+ \mu_2(-6\mu_1 + \mu_2 + 4\mu_1^2 + \mu_1\mu_2 + \mu_2^2)\ell_2 +$$

$$+ (1 - 14\mu_1 - 2\mu_2 - \mu_1^2 + 8\mu_1\mu_2 - \mu_2^2)\sqrt{\lambda} +$$

$$- \cos^2\theta\Big\{ - 2\lambda D_S + 6\mu_1(1 + 7\mu_1 - \mu_2)I_1^\ell(0) +$$

$$+ 3\sqrt{\mu_1}(1 - 12\mu_1 - 2\mu_2 - 5\mu_1^2 + 4\mu_1\mu_2 + \mu_2^2)S_1^\ell(0) +$$

$$+ \mu_1(20 + 13\mu_1 - 24\mu_2 + \mu_1^2 + \mu_1\mu_2 + 4\mu_2^2)\ell_1 +$$

$$- \mu_2(4 + 12\mu_1 - \mu_2 - 4\mu_1^2 - \mu_1\mu_2 - \mu_2^2)\ell_2 +$$

$$+ (3 - 36\mu_1 - \mu_1^2 + 8\mu_1\mu_2 - \mu_2^2)\sqrt{\lambda}\Big\} +$$

$$\mp \cos\theta\Big\{4\sqrt{\lambda}D_I - 4(1 - 3\mu_1 - 2\mu_2 - \mu_1^2 + \mu_2^2)I^\ell(0) +$$

$$+ 2(2 - \mu_1 + \mu_2 - \mu_1^2 + \mu_1\mu_2)\ell_0 + 8\lambda\ell_4 +$$

$$- 4\sqrt{\lambda}(1 + 2\mu_1 - \mu_2)\ell_1 - 2\sqrt{\lambda}(2 + \mu_1 + \mu_2)\ell_2 +$$

$$- (3 + 14\sqrt{\mu_1} - 3\mu_1 + 3\mu_2)\left((1 - \sqrt{\mu_1})^2 - \mu_2\right)\Big\}\Big]. \tag{228}$$

Osa tulemustest on antud primmiga, kuna nad on leitud algosakeste süsteemis, kus kombineeruvad protsessid $t(\uparrow) \to W^+(\uparrow) + b$ kujuteguritega H_{++}, H_{--}, H_{33} [5,6] ja $W^+(\uparrow) \to Q + \bar{q}$ kujuteguritega h'_{++}, h'_{--}, h'_{33} lagunemistõenäosuseks $W(\theta)$ summana

$$W(\theta) \sim H_{++}h'_{++} + H_{--}h'_{--} + H_{33}h'_{33}. \tag{229}$$

Standardne esitus on aga see, kus igat käelisust mõõdetakse lõpposakeste süsteemis. Seal on lagunemistõenäosus

$$\begin{aligned}
W(\theta) &\sim \sum_{m,m'=\pm,3} H_{mm} d^1_{mm'}(\theta) d^1_{mm'}(\theta) h_{m'm'} = \\
&= \frac{3}{8}(1+\cos\theta)^2(H_{++}h_{++}+H_{--}h_{--}) + \frac{3}{8}(1-\cos\theta)^2(H_{++}h_{--}+H_{--}h_{++}) + \\
&\quad + \frac{3}{4}\sin^2\theta(H_{++}h_{33}+H_{33}h_{++}+H_{--}h_{33}+H_{33}h_{--}) + \frac{3}{2}\cos^2\theta H_{33}h_{33}. \qquad (230)
\end{aligned}$$

Võrreldes võrrandit (230) võrrandiga (229) ja võttes arvesse sobivat normeerimist, saame (kus relatiivne normeerimine on valitud nii, et $h_{++}+h_{--}+h_{33} = h'_{++}+h'_{--}+h'_{33}$)

$$\begin{aligned}
\frac{3}{2}h'_{++} &= \frac{3}{8}(1+\cos\theta)^2 h_{++} + \frac{3}{8}(1-\cos\theta)^2 h_{--} + \frac{3}{4}\sin^2\theta h_{33}, \\
\frac{3}{2}h'_{--} &= \frac{3}{8}(1+\cos\theta)^2 h_{--} + \frac{3}{8}(1-\cos\theta)^2 h_{++} + \frac{3}{4}\sin^2\theta h_{33}, \\
\frac{3}{2}h'_{33} &= \frac{3}{4}\sin^2\theta(h_{++}+h_{--}) + \frac{3}{2}\cos^2\theta h_{33}. \qquad (231)
\end{aligned}$$

Kirjutades lahti erinevad nurgasõltuvused saame

$$\begin{aligned}
\frac{3}{2}h'_{++} &= \frac{3}{8}(h_{++}+h_{--}+2h_{33}) + \frac{3}{4}\cos\theta(h_{++}-h_{--}) + \frac{3}{8}\cos^2\theta(h_{++}+h_{--}-2h_{33}), \\
\frac{3}{2}h'_{--} &= \frac{3}{8}(h_{++}+h_{--}+2h_{33}) - \frac{3}{4}\cos\theta(h_{++}-h_{--}) + \frac{3}{8}\cos^2\theta(h_{++}+h_{--}-2h_{33}), \\
\frac{3}{2}h'_{33} &= \frac{3}{4}(h_{++}+h_{--}) - \frac{3}{4}\cos^2\theta(h_{++}+h_{--}-2h_{33}) \qquad (232)
\end{aligned}$$

(paneme tähele, et $h_{++}+h_{--}-2h_{33}$ ilmuvad kahes erinevas kohas, mis on meie tulemustega kooskõlas). Võrdlemine annab meile

$$\begin{aligned}
\frac{1}{2}(h_{++}+h_{--}) &= N\Big[4(1-\mu_1-\mu_2)D_S - 4\mu_1(1+7\mu_1-\mu_2)I_1^\ell(0) + \\
&\quad - 2\sqrt{\mu_1}(1-12\mu_1-2\mu_2-5\mu_1^2+4\mu_1\mu_2+\mu_2^2)S_1^\ell(0) + \\
&\quad - 2\mu_1(6+4\mu_1-7\mu_2)\ell_1 + 2\mu_2(2+3\mu_1)\ell_2 - 2(1-11\mu_1+\mu_2)\sqrt{\lambda}\Big], \qquad (233)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}(h_{++}-h_{--}) &= N\Big[-4\sqrt{\lambda}D_I + 4(1-3\mu_1-2\mu_2-\mu_1^2+\mu_2^2)I^\ell(0) + \\
&\quad - 2(2-\mu_1+\mu_2-\mu_1^2+\mu_1\mu_2)\ell_0 - 8\lambda\ell_4 + \\
&\quad + 4\sqrt{\lambda}(1+2\mu_1-\mu_2)\ell_1 + 2\sqrt{\lambda}(2+\mu_1+\mu_2)\ell_2 + \\
&\quad + (3+14\sqrt{\mu_1}-3\mu_1+3\mu_2)\Big((1-\sqrt{\mu_1})^2-\mu_2\Big)\Big], \qquad (234)
\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(h_{++}+h_{--}-2h_{33}) &= N\Big[4\lambda D_S - 12\mu_1(1+7\mu_1-\mu_2)I_1^\ell(0) + \\ &- 6\sqrt{\mu_1}(1-12\mu_1-2\mu_2-5\mu_1^2+4\mu_1\mu_2+\mu_2^2)S_1^\ell(0) + \\ &- 2\mu_1(20+13\mu_1-24\mu_2+\mu_1^2+\mu_1\mu_2+4\mu_2^2)\ell_1 + \\ &+ 2\mu_2(4+12\mu_1-\mu_2-4\mu_1^2-\mu_1\mu_2-\mu_2^2)\ell_2 + \\ &- 2(3-36\mu_1-\mu_1^2+8\mu_1\mu_2-\mu_2^2)\sqrt{\lambda}\Big] \end{aligned} \quad (235)$$

ja $h_{++}+h_{--}+2h_{33}$. Veel lihtsam tulemus (peale $h_{++}\pm h_{--}$) on

$$\begin{aligned} h_{++}+h_{--}+h_{33} &= N\Big[4\left(2-\mu_1-\mu_2-(\mu_1-\mu_2)^2\right)D_S + \\ &+ 2\mu_1(2+\mu_1-3\mu_2+\mu_1^2+\mu_1\mu_2+4\mu_2^2)\ell_1 + \\ &+ 2\mu_2(2-3\mu_1+\mu_2+4\mu_1^2+\mu_1\mu_2+\mu_2^2)\ell_2 + \\ &- 2(3\mu_1+3\mu_2+\mu_1^2-8\mu_1\mu_2+\mu_2^2)\sqrt{\lambda}\Big]. \end{aligned} \quad (236)$$

Eraldi ei ole vaja h_{++}, h_{--} ja h_{33} välja tuua, sest panused on oluliselt erinevad. Ülemises süsteemis piisab kolmest suvalist panusest. Põhimõtteliselt oleme sellega lõpposakeste süsteemis arvutanud W-bosoni kiraalsuse komponendid, mis ei sõltu nurkadest.

4 Silmuselised panused ja täistulemus

Selles peatükis esitame tulemuste loetelu ja liidame vastavad panused. Selle käigus taanduvad allesjäänud IR-hajuvused silmuseliste ja puupanuste vahel. Selleks on aga vaja asendada massiregulariseerimise dimensionaalse regulariseerimisega vastava seosvalemi abil

$$\ln\left(\frac{\Lambda}{\mu}\right) \leftrightarrow \frac{1}{\epsilon} - \gamma_E + \ln(4\pi), \quad (237)$$

kus μ on renormeerimisskaala $\overline{\text{MS}}$-skeemis. Alguses aga taasesitame Borni taseme ja silmuselised panused ning leiame vajalikud kujutegurid.

4.1 Infrapunaste hajumiste taandumine

Peale Borni taseme panuste

$$h(Born) = 2q^2(1-\mu_1-\mu_2),$$

$$
\begin{aligned}
h_{00}(Born) &= -q^2(1-\mu_1-\mu_2-\lambda), \\
h'_{33}(Born) &= q^2(1-\mu_1-\mu_2) - q^2\lambda\cos^2\theta, \\
h'_{\pm\pm}(Born) &= q^2(1-\mu_1-\mu_2) - \frac{1}{2}q^2\lambda(1-\cos^2\theta) \mp q^2\sqrt{\lambda}\cos\theta
\end{aligned} \tag{238}
$$

on meil ka silmuselised panused

$$
\begin{aligned}
h(loop) &= 2q^2(1-\mu_1-\mu_2)(2\,\mathrm{Re}\,A_L - m_1\,\mathrm{Re}\,B^1_L + m_2\,\mathrm{Re}\,B^2_R) + \\
&\quad + 4q^2\sqrt{\mu_1\mu_2}(4\,\mathrm{Re}\,A_R + m_1\,\mathrm{Re}\,B^1_R - m_2\,\mathrm{Re}\,B^2_L), \\
h_{00}(loop) &= -2q^2(1-\mu_1-\mu_2-\lambda)\,\mathrm{Re}\,A_L + 4q^2\sqrt{\mu_1\mu_2}\,\mathrm{Re}\,A_R + \\
&\quad - q^2\left(1-(\mu_1-\mu_2)^2\right)(m_1\,\mathrm{Re}\,B^1_L - m_2\,\mathrm{Re}\,B^2_R) + \\
&\quad + q^2(1+\mu_1-\mu_2)^2 m_2\,\mathrm{Re}\,B^1_R - q^2(1-\mu_1+\mu_2)^2 m_1\,\mathrm{Re}\,B^2_L, \\
h'_{33}(loop) &= 2q^2(1-\mu_1-\mu_2)\,\mathrm{Re}\,A_L + 4q^2\sqrt{\mu_1\mu_2}\,\mathrm{Re}\,A_R + \\
&\quad - q^2\lambda\left(2\,\mathrm{Re}\,A_L + m_1(\mathrm{Re}\,B^1_L - \mathrm{Re}\,B^2_L) + m_2(\mathrm{Re}\,B^1_R - \mathrm{Re}\,B^2_R)\right)\cos^2\theta, \\
h'_{\pm\pm}(loop) &= 2q^2(1-\mu_1-\mu_2)\,\mathrm{Re}\,A_L - q^2\lambda\,\mathrm{Re}\,A_L(1-\cos^2\theta) + \\
&\quad + 4q^2\sqrt{\mu_1\mu_2}\,\mathrm{Re}\,A_R \mp 2q^2\sqrt{\lambda}\,\mathrm{Re}\,A_L\cos\theta + \\
&\quad - \frac{1}{2}q^2\lambda\left(m_1(\mathrm{Re}\,B^1_L - \mathrm{Re}\,B^2_L) + m_2(\mathrm{Re}\,B^1_R - \mathrm{Re}\,B^2_R)\right)(1-\cos^2\theta).
\end{aligned} \tag{239}
$$

Need panused saab samuti nurgast sõltumatule kujule viia

$$
\begin{aligned}
\frac{1}{2}(h_{++}+h_{--}+2h_{33}) &= 4q^2(1-\mu_1-\mu_2)\,\mathrm{Re}\,A_L - 2q^2\lambda\,\mathrm{Re}\,A_L + 8q^2\sqrt{\mu_1\mu_2}\,\mathrm{Re}\,A_R + \\
&\quad - q^2\lambda\left(m_1(\mathrm{Re}\,B^1_L - \mathrm{Re}\,B^2_L) + m_2(\mathrm{Re}\,B^1_R - \mathrm{Re}\,B^2_R)\right), \\
\frac{1}{2}(h_{++}+h_{--}-2h_{33}) &= 2q^2\lambda\,\mathrm{Re}\,A_L + q^2\lambda\left(m_1(\mathrm{Re}\,B^1_L - \mathrm{Re}\,B^2_L) + m_2(\mathrm{Re}\,B^1_R - \mathrm{Re}\,B^2_R)\right), \\
\frac{1}{2}(h_{++}+h_{--}) &= 2q^2(1-\mu_1-\mu_2)\,\mathrm{Re}\,A_L + 4q^2\sqrt{\mu_1\mu_2}\,\mathrm{Re}\,A_R, \\
\frac{1}{2}(h_{++}-h_{--}) &= -2q^2\sqrt{\lambda}\,\mathrm{Re}\,A_L.
\end{aligned} \tag{240}
$$

Enne kui me kõik puupanused liidame, lihtsustame ilmnevaid liikmeid. Eelnevalt on leitud tulemus

$$\mathrm{Re}\,A_R = \frac{\alpha_s}{4\pi}C_F\left[4\frac{\sqrt{\mu_1\mu_2}}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)\right] = -2\frac{\alpha_s C_F}{4\pi\sqrt{\lambda}}\sqrt{\mu_1\mu_2}\ell_3, \tag{241}$$

kus oleme kasutanud

$$\begin{aligned}
-2\ln\left(\frac{1-v}{1+v}\right) &= 2\ln\left(\frac{1+v}{1-v}\right) = \ln\left(\frac{(1+v)^2}{(1-v)^2}\right) = \ln\left(\frac{1+v^2+2v}{1+v^2-2v}\right) = \\
&= \ln\left(\frac{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2+1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2+2\sqrt{1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2}\sqrt{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2}}{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2+1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2-2\sqrt{1-(\sqrt{\mu_1}+\sqrt{\mu_2})^2}\sqrt{1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2}}\right) = \\
&= \ln\left(\frac{2-2\mu_1-2\mu_2+2\sqrt{1-2\mu_1-2\mu_2+(\mu_1-\mu_2)^2}}{2-2\mu_1-2\mu_2-2\sqrt{1-2\mu_1-2\mu_2+(\mu_1-\mu_2)^2}}\right) = \\
&= \ln\left(\frac{1-\mu_1-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) = \ell_3
\end{aligned} \tag{242}$$

Kujutegurid B_L^1, B_L^2, B_R^1 ja B_R^2 ilmuvad ainult ühel kindlal kujul

$$\begin{aligned}
m_1(\mathrm{Re}\,B_L^1-\mathrm{Re}\,B_L^2)+m_2(\mathrm{Re}\,B_R^1-\mathrm{Re}\,B_R^2) &= \\
= 2\frac{\alpha_s C_F}{4\pi}\mu_1\Bigg(&\frac{1-2\mu_1+(\mu_1-\mu_2)^2-1+\mu_1+\mu_2-(1-\mu_1+\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)+ \\
&+(1-\mu_1+\mu_2-2+\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\Bigg)+ \\
-2\frac{\alpha_s C_F}{4\pi}\mu_2\Bigg(&\frac{1-\mu_1-\mu_2+(1+\mu_1-\mu_2)^2-1+2\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)+ \\
&-(2+\mu_1-\mu_2-1-\mu_1+\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\Bigg) = \\
= 2\frac{\alpha_s C_F}{4\pi}\mu_1&\left(\frac{-1+\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)-\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right)+ \\
-2\frac{\alpha_s C_F}{4\pi}\mu_2&\left(\frac{1+\mu_1-\mu_2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)-\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right) = \\
= 2\frac{\alpha_s C_F}{4\pi}&\left(\frac{\mu_1(1-\mu_1+\mu_2)+\mu_2(1+\mu_1-\mu_2)}{\sqrt{\lambda}}\ln\left(\frac{1+v}{1-v}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\right) = \\
= \frac{\alpha_s C_F}{4\pi\sqrt{\lambda}}&\left(\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)\ell_3-(\mu_1-\mu_2)\sqrt{\lambda}\ell_B\right)
\end{aligned} \tag{243}$$

($\ell_B := \ln(\mu_1/\mu_2)$). Viimane samm on IR hajuvuste taandumine tulemustes

$$2ND_S+q^2\,\mathrm{Re}\,A_L =: 2NA_S, \qquad 2ND_I+q^2\,\mathrm{Re}\,A_L =: 2NA_I. \tag{244}$$

Selleks vaatame

$$\mathrm{Re}\,A_L = -\frac{\alpha_s}{4\pi}C_F\Gamma(1+\varepsilon)\left(\frac{4\pi\mu^2}{\sqrt{\mu_1\mu_2}q^2}\right)^{\varepsilon}\times$$

$$\times\left[\frac{2}{\varepsilon}+2\frac{\mu_1+\mu_2-(\mu_1-\mu_2)^2}{\sqrt{\lambda}}\ln\left(\frac{1-v}{1+v}\right)+3\sqrt{\lambda}\ln\left(\frac{1-v}{1+v}\right)-(\mu_1-\mu_2)\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+\right.$$
$$\left.+\frac{2}{\sqrt{\lambda}}(1-\mu_1-\mu_2)\left(\left(\frac{1}{\varepsilon}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ln\left(\frac{1-v}{1+v}\right)+\operatorname{Re}L'\right)+4\right]=$$
$$=-\frac{\alpha_s C_F}{4\pi\sqrt{\lambda}}\left[2\sqrt{\lambda}\left(\frac{1}{\varepsilon}-\gamma_E+\ln\left(\frac{4\pi\mu^2}{q^2}\right)-\ln\sqrt{\mu_1\mu_2}\right)+\right.$$
$$-\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)\ell_3-\frac{3}{2}\lambda\ell_3-(\mu_1-\mu_2)\sqrt{\lambda}\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+$$
$$-(1-\mu_1-\mu_2)\left(\frac{1}{\varepsilon}-\gamma_E+\ln\left(\frac{4\pi\mu^2}{q^2}\right)-\ln\sqrt{\mu_1\mu_2}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ell_3+$$
$$\left.+2(1-\mu_1-\mu_2)\operatorname{Re}L'(\mu_1,\mu_2)+4\sqrt{\lambda}\right],\tag{245}$$

kus

$$L(v)-L'(\mu_1,\mu_2)=\operatorname{Li}_2\left(\frac{(\sqrt{\mu_1}-\sqrt{\mu_2})(v+1)}{2\sqrt{\mu_1}}\right)-\operatorname{Li}_2\left(\frac{(\sqrt{\mu_1}-\sqrt{\mu_2})(v-1)}{2\sqrt{\mu_2}}\right)+$$
$$-\operatorname{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(v-1)}{2\sqrt{\mu_1}}\right)+\operatorname{Li}_2\left(\frac{-(\sqrt{\mu_1}-\sqrt{\mu_2})(v+1)}{2\sqrt{\mu_2}}\right)+$$
$$+\ln\left(\frac{(\sqrt{\mu_1}+\sqrt{\mu_2})-(\sqrt{\mu_1}-\sqrt{\mu_2})v}{(\sqrt{\mu_1}+\sqrt{\mu_2})+(\sqrt{\mu_1}-\sqrt{\mu_2})v}\right)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)=L_2(\mu_1,\mu_2)\tag{246}$$

ja

$$L(v)=\operatorname{Li}_2\left(\frac{2v}{1+v}\right)-\operatorname{Li}_2\left(\frac{-2v}{1-v}\right)+i\pi\ln\left(\frac{1-v^2}{4v^2}\right)-\pi^2\tag{247}$$

ning saame

$$2A_S=2D_S+\frac{q^2}{N}\operatorname{Re}A_L=$$
$$=(1-\mu_1-\mu_2)\left(-\left(\frac{1}{\varepsilon}-\gamma_E+\ln\left(\frac{4\pi\mu_1\mu_2}{\lambda^2}\right)\right)\ell_3+\operatorname{Li}_2(1-\alpha_+)-\operatorname{Li}_2(1-\alpha_-)+2S_z(0)\right)+$$
$$+2\sqrt{\lambda}\left(\frac{1}{\varepsilon}-\gamma_E+\ln\left(\frac{4\pi\mu_1\mu_2}{\lambda^2}\right)+2\right)-2\sqrt{\lambda}\left(\frac{1}{\varepsilon}-\gamma_E+\ln\left(\frac{4\pi\mu^2}{q^2}\right)-\ln\sqrt{\mu_1\mu_2}\right)+$$
$$+\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)\ell_3+\frac{3}{2}\lambda\ell_3+(\mu_1-\mu_2)\sqrt{\lambda}\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)+$$
$$+(1-\mu_1-\mu_2)\left(\frac{1}{\varepsilon}-\gamma_E+\ln\left(\frac{4\pi\mu^2}{q^2}\right)-\ln\sqrt{\mu_1\mu_2}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ell_3+$$
$$-2(1-\mu_1-\mu_2)\operatorname{Re}L'(\mu_1,\mu_2)-4\sqrt{\lambda}+\frac{3}{2}\left((1+\mu_1-\mu_2)\ell_1+(1-\mu_1+\mu_2)\ell_2+\sqrt{\lambda}\right)=$$
$$=(1-\mu_1-\mu_2)\left(\left(\ln\left(\frac{\lambda^2\mu^2}{q^2}\right)-3\ln\sqrt{\mu_1\mu_2}-\ln\left(1-(\sqrt{\mu_1}-\sqrt{\mu_2})^2\right)\right)\ell_3+\right.$$
$$\left.+\operatorname{Li}_2(1-\alpha_+)-\operatorname{Li}_2(1-\alpha_-)-2\operatorname{Re}L'(\mu_1,\mu_2)+2S_z(0)\right)+$$

$$
\begin{aligned}
&- 2\sqrt{\lambda}\left(\ln\left(\frac{\lambda^2\mu^2}{q^2}\right) - 3\ln\sqrt{\mu_1\mu_2}\right) + \left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)\ell_3 + \frac{3}{2}\lambda\ell_3 + \\
&+ (\mu_1-\mu_2)\sqrt{\lambda}\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + \frac{3}{2}\left((1+\mu_1-\mu_2)\ell_1 + (1-\mu_1+\mu_2)\ell_2 + \sqrt{\lambda}\right) \; = \\
= \; &(1-\mu_1-\mu_2)(t_A + 2S_z(0)) - 2\sqrt{\lambda}\ell_A + \left(1-\mu_1-\mu_2+\frac{1}{2}\lambda\right)\ell_3 + \\
&+ \frac{1}{2}(\mu_1-\mu_2)\sqrt{\lambda}\ell_B + \frac{3}{2}\left((1+\mu_1-\mu_2)\ell_1 + (1-\mu_1+\mu_2)\ell_2 + \sqrt{\lambda}\right) \qquad (248)
\end{aligned}
$$

ja $2A_I$ on seesama, kus $S_z(0)$ on vaid asendatud $I_z(0)$-ga. Hajuvused seega taanduvad.

4.2 Täistulemus

Me saame liitmise abil avaldada $h_4 := (h_{++} + h_{--} + 2h_{33})/2 = h_1 - h_3/2$,

$$
\begin{aligned}
h_3 \; := \; \frac{1}{2}(h_{++} + h_{--} - 2h_{33}) \; = \; & N\Big[4\lambda A_S - 12\mu_1(1+7\mu_1-\mu_2)I_1^\ell(0) + \\
& - 6\sqrt{\mu_1}(1 - 12\mu_1 - 5\mu_1^2 - 2\mu_2 + 4\mu_1\mu_2 + \mu_2^2)S_1^\ell(0) + \\
& - 2\mu_1(20 + 13\mu_1 + \mu_1^2 - 24\mu_2 + \mu_1\mu_2 + 4\mu_2^2)\ell_1 + \\
& + 2\mu_2(4 + 12\mu_1 - 4\mu_1^2 - \mu_2 - \mu_1\mu_2 - \mu_2^2)\ell_2 + \\
& + \lambda\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)\ell_3 - (\mu_1-\mu_2)\lambda\sqrt{\lambda}\ell_B + \\
& - 2(3 - 36\mu_1 - \mu_1^2 + 8\mu_1\mu_2 - \mu_2^2)\sqrt{\lambda}\Big], \qquad (249)
\end{aligned}
$$

$$
\begin{aligned}
h_2 \; := \; \frac{1}{2}(h_{++} - h_{--}) \; = \; & N\Big[-4\sqrt{\lambda}A_I + 4(1 - 3\mu_1 - \mu_1^2 - 2\mu_2 + \mu_2^2)I^\ell(0) + \\
& - 2(2 - \mu_1 - \mu_1^2 + \mu_2 + \mu_1\mu_2)\ell_0 - 8\lambda\ell_4 + \\
& + 4\sqrt{\lambda}(1 + 2\mu_1 - \mu_2)\ell_1 + 2\sqrt{\lambda}(2 + \mu_1 + \mu_2)\ell_2 + \\
& + (3 + 14\sqrt{\mu_1} - 3\mu_1 + 3\mu_2)\left((1-\sqrt{\mu_1})^2 - \mu_2\right)\Big], \qquad (250)
\end{aligned}
$$

$$
\begin{aligned}
h_1 \; := \; \frac{1}{2}(h_{++} + h_{--}) \; = \; & N\Big[4(1-\mu_1-\mu_2)A_S - 4\mu_1(1+7\mu_1-\mu_2)I_1^\ell(0) + \\
& - 2\sqrt{\mu_1}(1 - 12\mu_1 - 5\mu_1^2 - 2\mu_2 + 4\mu_1\mu_2 + \mu_2^2)S_1^\ell(0) + \\
& - 2\mu_1(6 + 4\mu_1 - 7\mu_2)\ell_1 + 2\mu_2(2 + 3\mu_1)\ell_2 + \\
& - 8\mu_1\mu_2\ell_3 - 2(1 - 11\mu_1 + \mu_2)\sqrt{\lambda}\Big] \qquad (251)
\end{aligned}
$$

ja lõpuks ka

$$
\begin{aligned}
h_0 \;:=\; h_{00} \;&=\; N\Big[-4\left(\mu_1+\mu_2-(\mu_1-\mu_2)^2\right)A_S + \\
&-\left(\left(5\mu_1-3\mu_2+3(\mu_1-\mu_2)^2\right)\lambda-8\mu_1+14\mu_1^2+2\mu_1\mu_2+18\mu_1^2\mu_2-6\mu_1^3\right)\ell_1 + \\
&-\left(\left(5\mu_2-3\mu_1+3(\mu_1-\mu_2)^2\right)\lambda-8\mu_2+14\mu_2^2+2\mu_1\mu_2+18\mu_1\mu_2^2-6\mu_2^3\right)\ell_2 + \\
&-6(\mu_1+\mu_2-\mu_1^2+4\mu_1\mu_2-\mu_2^2)\sqrt{\lambda}-3(\mu_1-\mu_2)\lambda\ell_B. \qquad (252)
\end{aligned}
$$

Tulemused h_0 kuni h_4 on antud töö põhitulemused. Need moodustavad parandid, mis parandavad Borni taseme tulemusi. Analüüsis nimetame neid järgmise juhtiva järgu (NLO) panusteks ja nendega parandatud tulemused nimetame $O(\alpha_s)$ tulemusteks.

4.3 Väikeste masside piirjuht

Arvutame siin piirjuhu, kus μ_1 ja μ_2 lähenevad sõltumatult nullile. Selleks vaatame esiteks läbi kõik need panused, mis on kokku pandud. Alustades arendamisest

$$
\sqrt{\lambda}=\sqrt{1+\mu_1^2+\mu_2^2-2\mu_1-2\mu_2-2\mu_1\mu_2} \;=\; 1-\mu_1-\mu_2-\mu_1\mu_2+O(\mu_i^3) \qquad (253)
$$

saame

$$
\begin{aligned}
\ell_0 \;&\to\; -\ln\mu_2, \\
\ell_1 \;&\to\; \ln\left(\frac{2}{1+\mu_1-\mu_2-1+\mu_1+\mu_2}\right) \;=\; -\ln\mu_1, \\
\ell_2 \;&\to\; \ln\left(\frac{2}{1-\mu_1+\mu_2-1+\mu_1+\mu_2}\right) \;=\; -\ln\mu_2, \\
\ell_3 \;&=\; \ell_1+\ell_2 \;\to\; -\ln\mu_1-\ln\mu_2, \\
\ell_4 \;&\to\; \ln\frac{1}{\sqrt{\mu_1}} \;=\; -\frac{1}{2}\ln\mu_1. \qquad (254)
\end{aligned}
$$

Kasutades

$$
\begin{aligned}
z_+ \;&\to\; \frac{1+\mu_1-\mu_2+1-\mu_1-\mu_2}{2\sqrt{\mu_1}} \;=\; \frac{1}{\sqrt{\mu_1}}, \\
z_- \;&\to\; \frac{1+\mu_1-\mu_1-1+\mu_1+\mu_2}{2\sqrt{\mu_1}} \;=\; \sqrt{\mu_1} \qquad (255)
\end{aligned}
$$

saame

$$
S^\ell(0) \;\to\; \mathrm{Li}_2\left(\sqrt{\mu_1}/\sqrt{\mu_1}\right)+\mathrm{Li}_2\left(\sqrt{\mu_1}\sqrt{\mu_1}\right)-2\mathrm{Li}_2\left(\sqrt{\mu_1}\right)+\ln^2\left(\sqrt{\mu_1}\right) \;=
$$

$$
\begin{aligned}
&= \mathrm{Li}_2(1) + \mathrm{Li}_2(\mu_1) - 2\mathrm{Li}_2\left(\sqrt{\mu_1}\right) + \frac{1}{4}\ln^2\mu_1 \;\rightarrow\; \frac{\pi^2}{6} + \frac{1}{2}\ln^2\mu_1 \;\leftarrow\; I^\ell(0), \\
S_z(0) &\rightarrow \mathrm{Li}_2(0) - \mathrm{Li}_2(1) + \mathrm{Li}_2(0) - \mathrm{Li}_2(1) + \mathrm{Li}_2(0) - \mathrm{Li}_2(1) + \frac{1}{2}\ln^2(\mu_1\mu_2) + \\
&\quad + \ln\left(\frac{1}{2\mu_1\mu_2}\right)\ln(\mu_1\mu_2) + 2\ln\left(2\sqrt{\mu_1}\right)\ln\left(2\sqrt{\mu_2}\right) - 2\ln^2 2 \;= \\
&= -\frac{\pi^2}{2} + \frac{1}{2}(\ln\mu_1 + \ln\mu_2)^2 - (\ln 2 + \ln\mu_1 + \ln\mu_2)(\ln\mu_1 + \ln\mu_2) + \\
&\quad + 2\left(\ln 2 + \frac{1}{2}\ln\mu_1\right)\left(\ln 2 + \frac{1}{2}\ln\mu_2\right) - 2\ln^2 2 \;= \\
&= -\frac{\pi^2}{2} + \frac{1}{2}\ln^2\mu_1 + \ln\mu_1\ln\mu_2 + \frac{1}{2}\ln^2\mu_2 - \ln 2(\ln\mu_1 + \ln\mu_2) - \ln^2\mu_1 + \\
&\quad - 2\ln\mu_1\ln\mu_2 - \ln^2\mu_2 + 2\ln^2 2 + \ln 2(\ln\mu_1 + \ln\mu_2) + \frac{1}{2}\ln\mu_1\ln\mu_2 - 2\ln^2 2 \;= \\
&= -\frac{\pi^2}{2} - \frac{1}{2}\ln^2\mu_1 - \frac{1}{2}\ln\mu_1\ln\mu_2 - \frac{1}{2}\ln^2\mu_2. \qquad (256)
\end{aligned}
$$

Integraali $I_z(0)$ arendamiseks on vaja ka z_+ täpsemalt arendada. Saame avaldada $\sqrt{\mu_1}z_+ \approx 1 - \mu_2$. Lisaks kasutades

$$
\begin{aligned}
\mathrm{Li}_2\left(\frac{-1}{\sqrt{\mu_1}}\right) &= -\mathrm{Li}_2\left(-\sqrt{\mu_1}\right) - \frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(\sqrt{\mu_1}\right) \;\approx\; -\frac{\pi^2}{6} - \frac{1}{8}\ln^2\mu_1, \\
\mathrm{Li}_2\left(\frac{z_+ - \sqrt{\mu_1}}{\sqrt{\mu_1}z_+ - 1}\right) &\rightarrow \mathrm{Li}_2\left(\frac{(\sqrt{\mu_1})^{-1} - \sqrt{\mu_1}}{1 - \mu_2 - 1}\right) = \mathrm{Li}_2\left(\frac{1 - \mu_1}{-\sqrt{\mu_1}\mu_2}\right) = \\
&\approx \mathrm{Li}_2\left(\frac{-1}{\sqrt{\mu_1}\mu_2}\right) \approx -\frac{\pi^2}{6} - \frac{1}{2}\ln^2\left(\sqrt{\mu_1}\mu_2\right) = \\
&= -\frac{\pi^2}{6} - \frac{1}{8}\ln^2\mu_1 - \frac{1}{2}\ln\mu_1\ln\mu_2 - \frac{1}{2}\ln^2\mu_2 \qquad (257)
\end{aligned}
$$

ning sellega kokkuvõttes on meil

$$
\begin{aligned}
I_z(0) &\rightarrow \mathrm{Li}_2\left(\frac{-1}{\sqrt{\mu_1}}\right) - \mathrm{Li}_2\left(-\sqrt{\mu_1}\right) + \mathrm{Li}_2\left(\frac{(1/\sqrt{\mu_1}) - \sqrt{\mu_1}}{\sqrt{\mu_1}z_+ - 1}\right) - \mathrm{Li}_2\left(\frac{(\sqrt{\mu_1}/\sqrt{\mu_1}) - 1}{(1/\sqrt{\mu_1}) - \sqrt{\mu_1}}\right) = \\
&\approx \mathrm{Li}_2\left(\frac{-1}{\sqrt{\mu_1}}\right) + \mathrm{Li}_2\left(\frac{(1/\sqrt{\mu_1}) - \sqrt{\mu_1}}{\sqrt{\mu_1}z_+ - 1}\right) \rightarrow -\frac{\pi^2}{3} - \frac{1}{4}\ln^2\mu_1 - \frac{1}{2}\ln\mu_1\ln\mu_2 - \frac{1}{2}\ln^2\mu_2. \\
&\qquad (258)
\end{aligned}
$$

Arendusi jätkates saame veel

$$
\begin{aligned}
S_1^\ell(0) &\rightarrow \mathrm{Li}_2\left(\sqrt{\mu_1}\right) - \mathrm{Li}_2\left(-\sqrt{\mu_1}\right) - \frac{\pi^2}{4} - \mathrm{Li}_2\left(\frac{(1+\sqrt{\mu_1})(1-\sqrt{\mu_1})}{(1-\sqrt{\mu_1})(1+\sqrt{\mu_1})}\right) + \\
&\quad + \mathrm{Li}_2\left(-\frac{(1+\sqrt{\mu_1})(1-\sqrt{\mu_1})}{(1-\sqrt{\mu_1})(1+\sqrt{\mu_1})}\right) + \ln\left(\sqrt{\mu_1}\right)\ln\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}} = \\
&\rightarrow -\frac{\pi^2}{4} - \frac{\pi^2}{6} - \frac{\pi^2}{12} = -\frac{\pi^2}{2},
\end{aligned}
$$

$$
\begin{aligned}
I_1^\ell(0) &\to \mathrm{Li}_2(\mu_1) - \mathrm{Li}_2\left(\sqrt{\mu_1}/\sqrt{\mu_1}\right) - \mathrm{Li}_2\left(\sqrt{\mu_1}\sqrt{\mu_1}\right) - \frac{\pi^2}{6} + \\
&+ \frac{1}{2}\mathrm{Li}_2\left(\frac{(\sqrt{\mu_1}-\sqrt{\mu_2})^2}{(1-\mu_1)^2}\right) + \frac{1}{2}\mathrm{Li}_2(\mu_1) - 2\mathrm{Li}_2\left(\frac{\sqrt{\mu_1}(\sqrt{\mu_1}-\sqrt{\mu_1})}{1-\sqrt{\mu_1}\sqrt{\mu_1}}\right) + \\
&+ \ln\left(\frac{1-\mu_1}{1-\mu_1}\right)\ln\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_1}}{1-\sqrt{\mu_1}\sqrt{\mu_1}}\right) + \ln\left(\sqrt{\mu_1}\right)\ln\left(\frac{1}{\sqrt{\mu_1}} - \sqrt{\mu_1}\right) = \\
&\to -\mathrm{Li}_2(1) - \frac{\pi^2}{6} + \ln\left(\sqrt{\mu_1}\right)\left(\ln(1-\mu_1) - \ln\left(\sqrt{\mu_1}\right)\right) = -\frac{\pi^2}{3} - \frac{1}{4}\ln^2\mu_1. \qquad (259)
\end{aligned}
$$

Kiiruse v arendamist $v(\mu_1,\mu_2) = 1 - 2\sqrt{\mu_1\mu_2} + O(\mu_i^2)$ kasutades saame

$$
\begin{aligned}
L_2(\mu_1,\mu_2) &\to \mathrm{Li}_2\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) - \mathrm{Li}_2\left(-\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_2}}\sqrt{\mu_1\mu_2}\right) + \\
&- \mathrm{Li}_2\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}}\sqrt{\mu_1\mu_2}\right) + \mathrm{Li}_2\left(-\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_2}}\right) + \ln\left(\frac{2\sqrt{\mu_2}}{2\sqrt{\mu_1}}\right)\ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) = \\
&\to \mathrm{Li}_2\left(1-\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) + \mathrm{Li}_2\left(1-\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + \ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) = \\
&= \frac{\pi^2}{3} - \mathrm{Li}_2\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) - \ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)\ln\left(1-\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) + \\
&- \mathrm{Li}_2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) - \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\ln\left(1-\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + \ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right). \qquad (260)
\end{aligned}
$$

Juhul kui $\sqrt{\mu_1} > \sqrt{\mu_2}$ on

$$
\begin{aligned}
L_2(\mu_1,\mu_2) &\to \frac{\pi^2}{2} + \frac{1}{2}\ln^2\left(-\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) - \ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)\ln\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) + \\
&- \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\ln\left(-\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_2}}\right) + \ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) = \\
&= \frac{\pi^2}{2} + \frac{1}{2}\ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + i\pi\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) - \frac{\pi^2}{2} - \ln\left(\frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\right)\ln\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) + \\
&- \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\ln\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_2}}\right) - i\pi\ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + \ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) = \\
&= \frac{1}{2}\ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) + \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\ln\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}}\right) + \\
&- \ln\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right)\ln\left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_2}}\right) + \ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) = \\
&= \frac{1}{2}\ln^2\left(\frac{\sqrt{\mu_1}}{\sqrt{\mu_2}}\right) = \frac{1}{8}\left(\ln\mu_1 - \ln\mu_2\right)^2. \qquad (261)
\end{aligned}
$$

Nüüd on

$$
\begin{aligned}
L\left(v(\mu_1,\mu_2)\right) &= \mathrm{Li}_2\left(\frac{2v(\mu_1,\mu_2)}{1+v(\mu_1,\mu_2)}\right) - \mathrm{Li}_2\left(\frac{-2v(\mu_1,\mu_2)}{1-v(\mu_1,\mu_2)}\right) - \pi^2 = \\
&\to \mathrm{Li}_2(1) - \mathrm{Li}_2\left(\frac{-1}{\sqrt{\mu_1\mu_2}}\right) - \pi^2 =
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\pi^2}{6} + \mathrm{Li}_2\left(-\sqrt{\mu_1\mu_2}\right) + \frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\sqrt{\mu_1\mu_2}\right) - \pi^2 \;= \\
&= -\frac{2\pi^2}{3} + \frac{1}{8}\left(\ln\mu_1 + \ln\mu_2\right)^2 \qquad (262)
\end{aligned}
$$

ja sellega

$$
\begin{aligned}
L'(\mu_1,\mu_2) &= L\left(v(\mu_1,\mu_2)\right) - L_2(\mu_1,\mu_2) \;= \\
&\to -\frac{2\pi^2}{3} + \frac{1}{8}\left(\ln\mu_1 + \ln\mu_2\right)^2 - \frac{1}{8}\left(\ln\mu_1 - \ln\mu_2\right)^2 \;= \\
&= -\frac{2\pi^2}{3} + \frac{1}{2}\ln\mu_1\ln\mu_2. \qquad (263)
\end{aligned}
$$

Viimaks kasutame

$$
\begin{aligned}
\mathrm{Li}_2(1-\alpha_+) &\to \mathrm{Li}_2\left(1-\frac{1}{\mu_1\mu_2}\right) \;=\; -\mathrm{Li}_2\left(\frac{1}{\mu_1\mu_2}\right) + \frac{\pi^2}{6} - \ln\left(1-\frac{1}{\mu_1\mu_2}\right)\ln\left(\frac{1}{\mu_1\mu_2}\right) \;= \\
&= \mathrm{Li}_2(\mu_1\mu_2) + \frac{\pi^2}{3} + \frac{1}{2}\ln^2\left(\frac{-1}{\mu_1\mu_2}\right) - \ln\left(1-\frac{1}{\mu_1\mu_2}\right)\ln\left(\frac{1}{\mu_1\mu_2}\right) \;= \\
&= \mathrm{Li}_2(\mu_1\mu_2) + \frac{\pi^2}{3} + \frac{1}{2}\left(\frac{1}{\mu_1\mu_2}\right) + i\pi\ln\left(\frac{1}{\mu_1\mu_2}\right) - \frac{\pi^2}{2} + \\
&\quad - i\pi\ln\left(\frac{1}{\mu_1\mu_2}\right) - \ln\left(\frac{1-\mu_1\mu_2}{\mu_1\mu_2}\right)\ln\left(\frac{1}{\mu_1\mu_2}\right) \;= \\
&= \mathrm{Li}_2(\mu_1\mu_2) - \frac{\pi^2}{6} + \frac{1}{2}\ln^2(\mu_1\mu_2) - \ln^2(\mu_1\mu_2) + \ln(1-\mu_1\mu_2)\ln(\mu_1\mu_2) \;= \\
&\to -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(\mu_1\mu_2) \qquad \text{ja} \\
\mathrm{Li}_2(1-\alpha_-) &\to \mathrm{Li}_2(1-\mu_1\mu_2) \;\to\; \frac{\pi^2}{6} \qquad (264)
\end{aligned}
$$

panuse t_A arvutamiseks.

$$
\begin{aligned}
t_A &\to \ell_A\ell_3 + \mathrm{Li}_2\left(1-\frac{1}{\mu_1\mu_2}\right) - \mathrm{Li}_2(1-\mu_1\mu_2) + L'(\mu_1\mu_2) \;= \\
&= \ell_A\ell_3 - \frac{\pi^2}{3} - \frac{1}{2}\left(\ln\mu_1 + \ln\mu_2\right)^2 - 2L'(\mu_1,\mu_2) \;= \\
&= \frac{3}{2}\left(\ln\mu_1 + \ln\mu_2\right)^2 - \frac{\pi^2}{3} - \frac{1}{2}\left(\ln\mu_1 + \ln\mu_2\right)^2 + \frac{4\pi^2}{3} - \ln\mu_1\ln\mu_2 \;= \\
&= \left(\ln\mu_1 + \ln\mu_2\right)^2 + \pi^2 - \ln\mu_1\ln\mu_2 \;= \\
&= \pi^2 + \ln^2\mu_1 + \ln\mu_1\ln\mu_2 + \ln^2\mu_2. \qquad (265)
\end{aligned}
$$

Selleks on vaja veel

$$
\begin{aligned}
\ell_A &= \ln\left(\frac{\lambda^2\mu^2}{q^2}\right) - 3\ln\left(\sqrt{\mu_1\mu_2}\right) \;\to \ln\left(\frac{\mu^2}{q^2}\right) - \frac{3}{2}\left(\ln\mu_1 + \ln\mu_2\right), \\
\ell_B &= \ln\left(\frac{\mu_1}{\mu_2}\right) \;=\; \ln\mu_1 - \ln\mu_2. \qquad (266)
\end{aligned}
$$

Panuseid kasutades saame, et piirjuhul, kui μ_1 ja μ_2 lähevad sõltumatult nullile on $h_1 \to 1+N$, $h_2 \to -1$ ja $h_3 \to 1-3N$ ning sellega

$$h_{++} \to 0+N, \qquad h_{--} \to 2q^2+N \quad \text{ja} \quad h_{33} \to 0+4N. \tag{267}$$

See on kooskõlas artikli [5] tulemustega, kuna piirjuhul tõepoolest $h_{--} \sim 1+\alpha_s/6\pi$, $h_{++} \sim \alpha_s/6\pi$ ja $h_{33} \sim 2\alpha_s/3\pi$. Borni tasemel normeerimiseks võtame arvesse, et üldiselt kehtib

$$h_{++} = h_1 + h_2, \qquad h_{--} = h_1 - h_2, \qquad h_{33} = h_1 - h_3, \tag{268}$$

millest tuleb

$$h_{\pm\pm} = q^2(1-\mu_1-\mu_2 \mp \sqrt{\lambda}) \to q^2(1\mp 1), \qquad h_{33} = q^2(1-\mu_1-\mu_2-\lambda) \to 0. \tag{269}$$

4.4 Tulemus lävel $\sqrt{\mu_1}+\sqrt{\mu_2}=1$

Teine piirjuht on tekkelävel, kui W^+-boson oma paigalsüsteemis laguneb kaheks kvargiks. See lävi on määratud tingimusega $\sqrt{\mu_1}+\sqrt{\mu_2}=1$. Tingimust kasutades selgub, et paljud panused lihtsustuvad otsekohe, ilma et neid peaks arvutama. Saame

$$\sqrt{\lambda(1,\mu_1,(1-\sqrt{\mu_1})^2)} = \sqrt{1+\mu_1^2+(1-\sqrt{\mu_1})^4-2\mu_1-2(1-\sqrt{\mu_1})^2-2\mu_1(1-\sqrt{\mu_1})^2} = 0 \tag{270}$$

ja sellepärast $\ell_0 \to 0$, $\ell_1 \to 0$, $\ell_2 \to 0$ ja $\ell_3 \to 0$. ℓ_4 jaoks saame

$$\ell_4 = \ln\left(\frac{(1+\sqrt{\mu_1})^2-(1-\sqrt{\mu_1})^2}{\sqrt{\mu_1}}\right) = \ln 4. \tag{271}$$

Edasi saame $z_\pm = 1$ ja sellega $S^\ell(0) = 0 = I^\ell(0)$. $S_z(0)$-i tuleb natuke detailsemalt uurida

$$\begin{aligned} S_z(0) &= \mathrm{Li}_2(1)+\mathrm{Li}_2(1)+\mathrm{Li}_2(1)-\frac{\pi^2}{2}+\frac{1}{2}\ln^2 1+\ln\left(\frac{\lambda}{2\mu_1\mu_2}\right)\ell_3 + \\ &\quad + 2\ln(2\sqrt{\mu_1})\ln(2-2\sqrt{\mu_1}) - 2\ln\left(1-\mu_1+(1-\sqrt{\mu_1})^2\right)\ln\left(1+\mu_1-(1-\sqrt{\mu_1})^2\right). \end{aligned} \tag{272}$$

Panuses taanduvad nii esimesed polülogaritmid $-3\mathrm{Li}_2(1) = -\pi^2/2$ vastu kui ka viimase rea kaksiklogaritmid. Ainus kriitiline osa on esimese rea viimane panus. Selleks on vaja arendada $\sqrt{\lambda}$ läve lähedal. Saame

$$\sqrt{\lambda(1,\mu_1,(1-\sqrt{\mu_1})^2-\varepsilon^2)} = 2\sqrt[4]{\mu_1}\varepsilon + O(\varepsilon^3) \tag{273}$$

ja

$$\begin{aligned} \ell_3 &= \ln\left(\frac{1-\mu_2-(1-\sqrt{\mu_1})^2+2\sqrt[4]{\mu_1}\varepsilon}{1-\mu_1-(1-\sqrt{\mu_1})^2-2\sqrt[4]{\mu_1}\varepsilon}\right) = \ln\left(\frac{\sqrt{\mu_1}-\mu_1+\sqrt[4]{\mu_1}\varepsilon}{\sqrt{\mu_1}-\mu_1-\sqrt[4]{\mu_1}\varepsilon}\right) = \\ &= \ln\left(1+\frac{\sqrt[4]{\mu_1}\varepsilon}{\sqrt{\mu_1}-\mu_1}\right)-\ln\left(1-\frac{\sqrt[4]{\mu_1}\varepsilon}{\sqrt{\mu_1}-\mu_1}\right) = \frac{2\sqrt[4]{\mu_1}\varepsilon}{\sqrt{\mu_1}-\mu_1}. \end{aligned} \tag{274}$$

Piirjuhul $\varepsilon \to 0$ saame

$$\ln(\lambda)\ell_3 \approx \ln(4\sqrt{\mu_1}\varepsilon^2)\frac{2\sqrt[4]{\mu_1}\varepsilon}{\sqrt{\mu_1}-\mu_1} \to 0. \tag{275}$$

Kokku saame siis nii $S_z(0) = 0$ kui ka $I_z(0) = 0$. Järgmiste panuste arvutamiseks kasutame $\alpha_\pm \to 1$. Sellega leiame

$$\begin{aligned} S_1^\ell(0) &= \mathrm{Li}_2(1)-\mathrm{Li}_2(-1)-\frac{\pi^2}{4}+\ln 1\ln\left(\frac{1-1}{1+1}\right)-\mathrm{Li}_2(0)+\mathrm{Li}_2(0) = \\ &= \frac{\pi^2}{6}+\frac{\pi^2}{12}-\frac{\pi^2}{4} = 0, \end{aligned} \tag{276}$$

$$\begin{aligned} I_1^\ell(0) &= \mathrm{Li}_2(\mu_1)-2\mathrm{Li}_2(\sqrt{\mu_1})-\frac{\pi^2}{6}+\frac{1}{2}\mathrm{Li}_2(1)+\frac{1}{2}\mathrm{Li}_2(1)-2\mathrm{Li}_2(-\sqrt{\mu_1})+ \\ &\quad +\ln\left(\frac{1-1}{1-\mu_1}\right)\ln 1+\ln 1\ln(1-1) = \\ &= \mathrm{Li}_2(\mu_1)-2\mathrm{Li}_2(\sqrt{\mu_1})-2\mathrm{Li}_2(-\sqrt{\mu_1}) = \mathrm{Li}_2(\mu_1)-\mathrm{Li}_2(\mu_1) = 0, \end{aligned} \tag{277}$$

kus kasutasime $\mathrm{Li}_2(z^2) = 2\mathrm{Li}_2(z)+2\mathrm{Li}_2(-z)$ tagurpidi. Järgmiste arvutamiste jaoks kasutme $v \to 0$ ja saame sellega

$$\mathrm{Re}\,L(v=0) = \mathrm{Li}_2(0)-\mathrm{Li}_2(0)-\pi^2 = -\pi^2, \tag{278}$$

mis koos tulemusega $L_2(\mu_1,(1-\sqrt{\mu_1})^2) = 0$ annab

$$L'\left(\mu_1,(1-\sqrt{\mu_1})^2\right) = L(0)-L_2\left(\mu_1,(1-\sqrt{\mu_1})^2\right) = -\pi^2. \tag{279}$$

See panus on tähtis t_A arvutamiseks. t_A sisaldab ka (ainukesena) ℓ_A-d, mis on hajuv tegur. Aga sellega ei ole probleemi, sest see esineb alati koos ℓ_3-ga ja kaob samal põhjusel, miks enne $\ln(\lambda)\ell_3$ kadus. Seega saame

$$\begin{aligned} t_A &= (\ell_A-\ln\left(1-(\sqrt{\mu_1}-(1-\sqrt{\mu_1}))^2\right)\ell_3+ \qquad \left(\ell_A = \ln\left(\frac{\lambda^2\mu^2}{q^2}\right)-3\ln(\sqrt{\mu_1\mu_2})\right) \\ &\quad +\mathrm{Li}_2(1-\alpha_+)-\mathrm{Li}_2(1-\alpha_-)-2L'\left(\mu_1,(1-\sqrt{\mu_1})^2\right) = 2\pi^2. \end{aligned} \tag{280}$$

Teine silmuliselisest parandist tulev logaritmiline panus on

$$\ell_B = \ln\left(\frac{\mu_1}{(1-\sqrt{\mu_1})^2}\right). \tag{281}$$

Seega

$$
\begin{aligned}
A_S &= \frac{1}{2}\Big((1-\mu_1-\mu_2)(t_A+2S_z(0))-2\sqrt{\lambda}\ell_A+\left(1-\mu_1-\mu_2+\frac{1}{2}\lambda\right)\ell_3+ \\
&\quad+\frac{1}{2}(\mu_1-\mu_2)\sqrt{\lambda}\ell_B+\frac{3}{2}\left((1+\mu_1-\mu_2)\ell_1+(1-\mu_1+\mu_2)\ell_2+\sqrt{\lambda}\right)\Big) = \\
&= 2\pi^2\left(1-\mu_1-(1-\sqrt{\mu_1})^2\right) = 2\pi^2\sqrt{\mu_1}(1-\sqrt{\mu_1}) = A_I. \qquad (282)
\end{aligned}
$$

Arvesse võttes kõiki vajaminevaid piirjuhte saame parandite jaoks uuritaval lävel tulemuseks

$$
h_0=-16\pi^2\mu_1\mu_2N, \qquad h_1=+16\pi^2\mu_1\mu_2N, \qquad h_2=0=h_3 \qquad (283)
$$

ja koos Borni taseme panusega

$$
\begin{aligned}
h_{00} &= -2\sqrt{\mu_1\mu_2}q^2-16\pi^2\mu_1\mu_2N, \\
h_{33} &= 2\sqrt{\mu_1\mu_2}q^2+16\pi^2\mu_1\mu_2N = h_{++} = h_{--}, \qquad (284)
\end{aligned}
$$

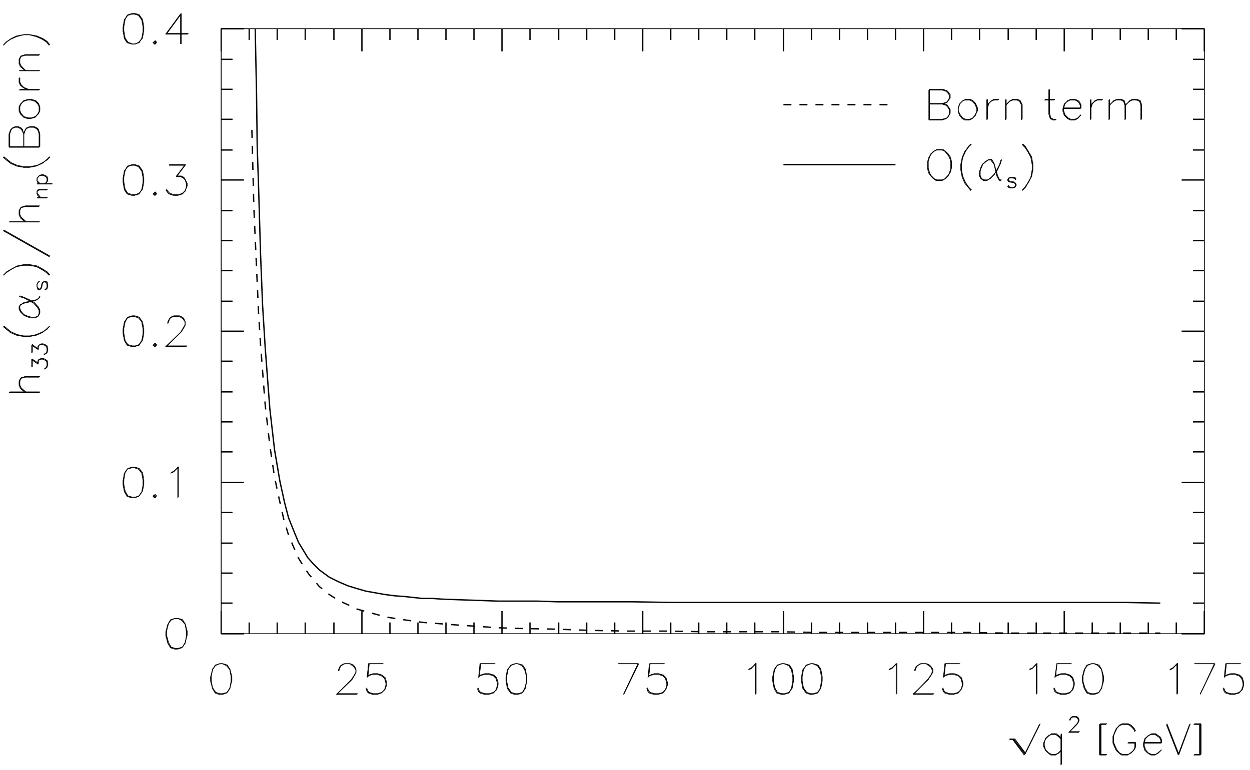
mis on kooskõlas kirjanduses avaldatud tulemustega, et lävel $\sqrt{\mu_1}+\sqrt{\mu_2}=1$ kehtib $h_{33}=h_{++}=h_{--}$.

5 Joonised

Antud peatükis esitame oma tulemused graafiliselt. Selleks kasutame masside jaoks väärtusi, mis on antud *Particle Data Group*'i poolt [20]. Lisaks peame arvesse võtma, et seosekonstant α_s renomeerimise teooria raames ei ole enam konstant, vaid sõltub masskeskmeenergiast [21].

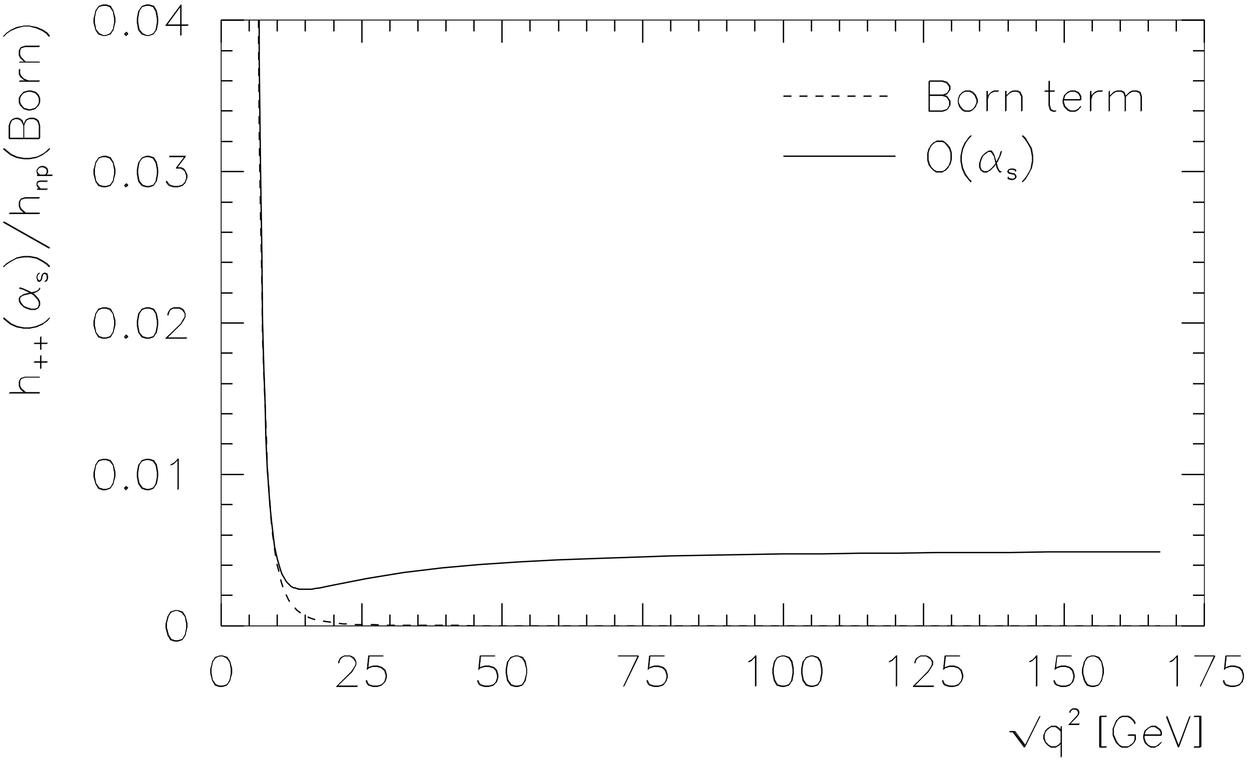
5.1 Polarisatsiooni tulemuste energiasõltuvused

Oletades, et W-boson võib vaheprotsessides mitte olla massipinnal, võime vaadelda protsessi energiasõltuvust. Sobilik energiavahemik on selleks piiratud altpoolt tekkeenergiaga $E_1 = m_b + m_c$ ja ülevalt poolt maksimaalse energiaga $E_2 = m_t - m_b$. Järgnevalt kasutame $m_t = 171.3 \pm 1.2$GeV, $m_b = 4.20 \pm 0.17$GeV, $m_c = 1.27 \pm 0.11$GeV ja $m_W = 80.398 \pm 0.025$GeV.
Jooniselt 5, kus on esitatud $h_{33}(\alpha_s)/h_{np}(Born)$ tulemus näeme, et madala energia piirjuhul on Borni taseme tulemus 1/3, samal ajal 1. järgu parandatud tulemus läheb lõpmatusse, mis on seletatav sellega, et häiritusarvutused madalate energiate puhul ei kehti (nn. Columbi singulaarsus). Kõrgete energiate piirjuhul läheb Borni tulemus nulliks, samas näeme, et parandatud tulemus jääb 0.02 juurde. See on kooskõlas valemiga (267) ($2\alpha_s/3\pi \to$0.02, kus $\alpha_s \to$0.1).



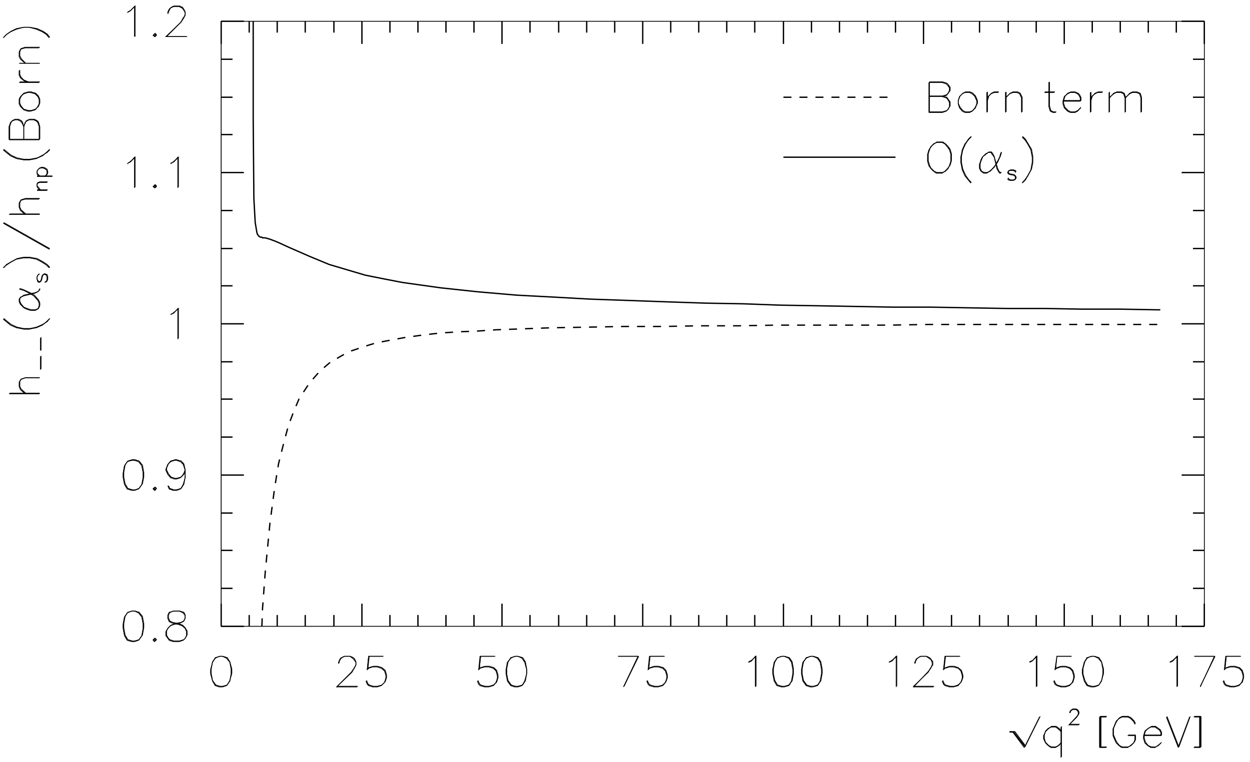
Joonis 5: $h_{33}(\alpha_s)$ panus

Joonisel 6 on kujutatud $h_{++}(\alpha_s)/h_{np}(Born)$ energia sõltuvus. Antud juhul näeme, et Borni taseme tulemus läheneb kõrge energia piirjuhul nullile. Parandatud tulemus $O(\alpha_s)$ on 0.005 ja see on valmeiga (267) kooskõlas ($\alpha_s/6\pi \rightarrow 0.005$, kus $\alpha_s \rightarrow 0.1$).



Joonis 6: $h_{++}(\alpha_s)$ panus

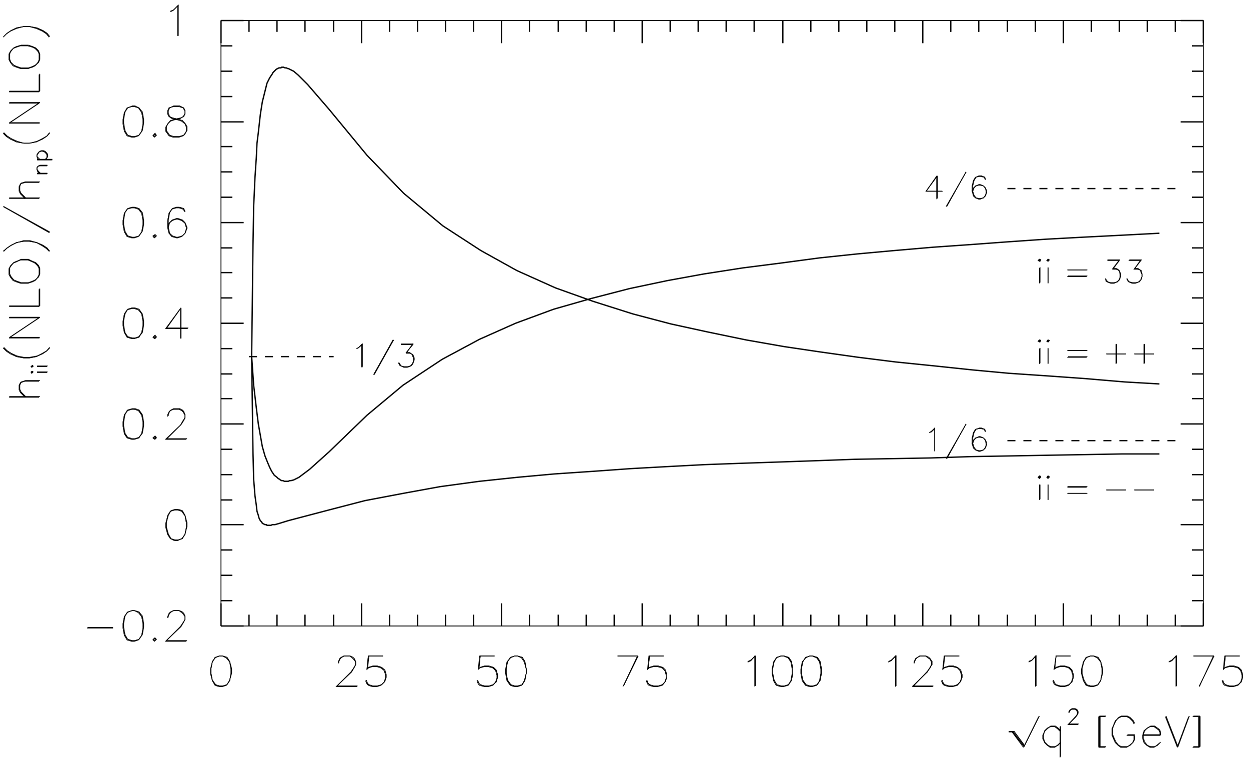
Jooniselt 7 näeme, et kõrgete energiate korral läheneb Borni taseme tulemus ühele. Esimese järgu parandi tulemus peaks olema 1.005, mis antud jooniselt pole veel nähtav. Seetõttu me näitame eraldi ka parandite energiasõltuvusi.



Joonis 7: $h_{--}(\alpha_s)$ panus

5.2 Parandite energiasõltuvus

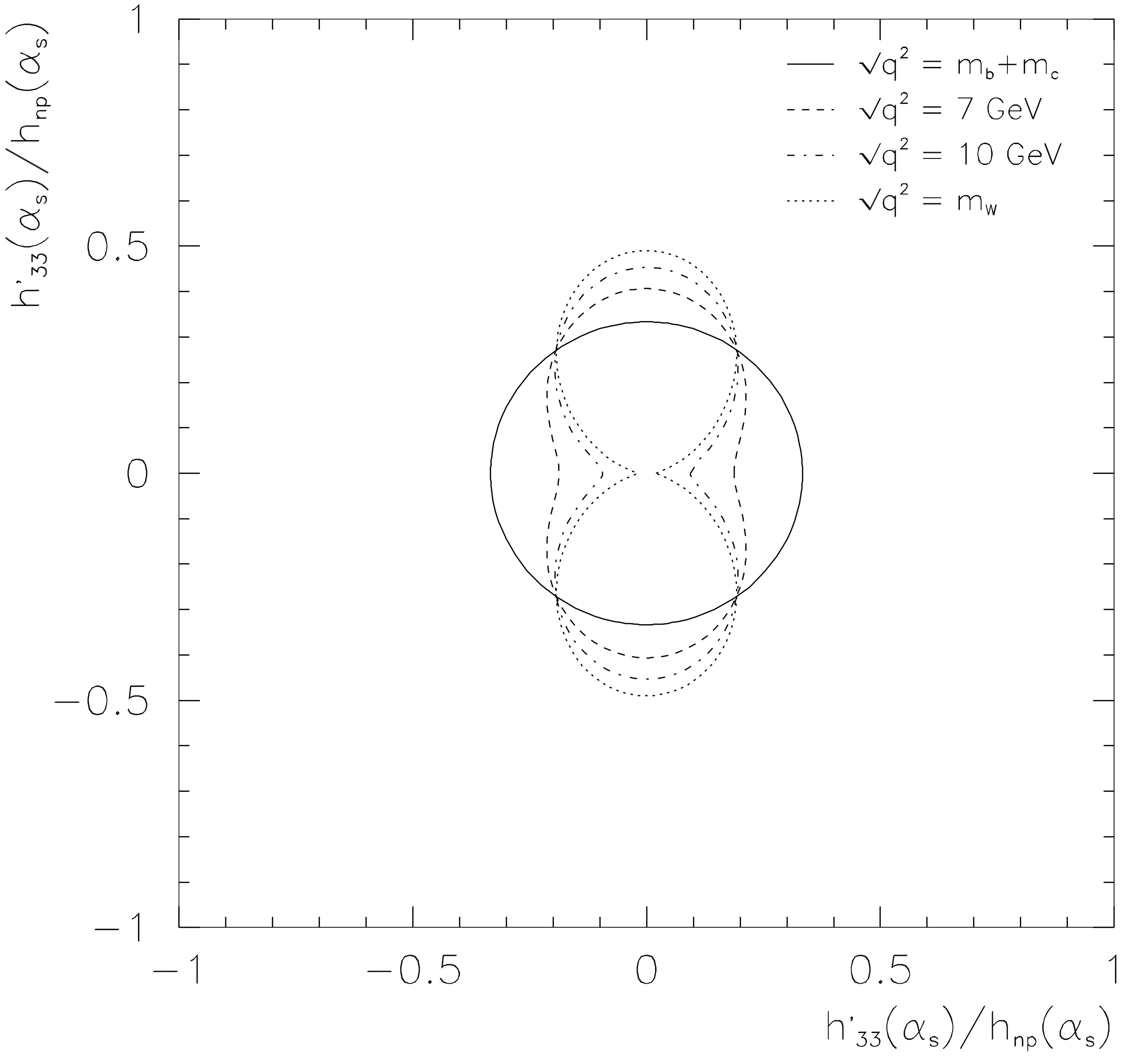
Joonisel 8 on esitatud parandite energiasõltuvused, kusjuures tulemused on normeeritud mittepolariseeritud parandi $h_{np}(NLO) = h_{33}(NLO) + h_{++}(NLO) + h_{--}(NLO)$ peale. Tänu normeerimisele singulaarsused taanduvad ja näeme, et lävel on tulemuseks 1/3, mis on heas kooskõlas valemiga (284). Kõrge energia piirjuhul $h_{np}(NLO) \to 6N$ ja vastavalt $h_{33}(NLO) \to 4/6$, $h_{++}(NLO) \to 1/6$ ja $h_{--}(NLO) \to 1/6$, mis jällegi on vastavuses valemiga (284).



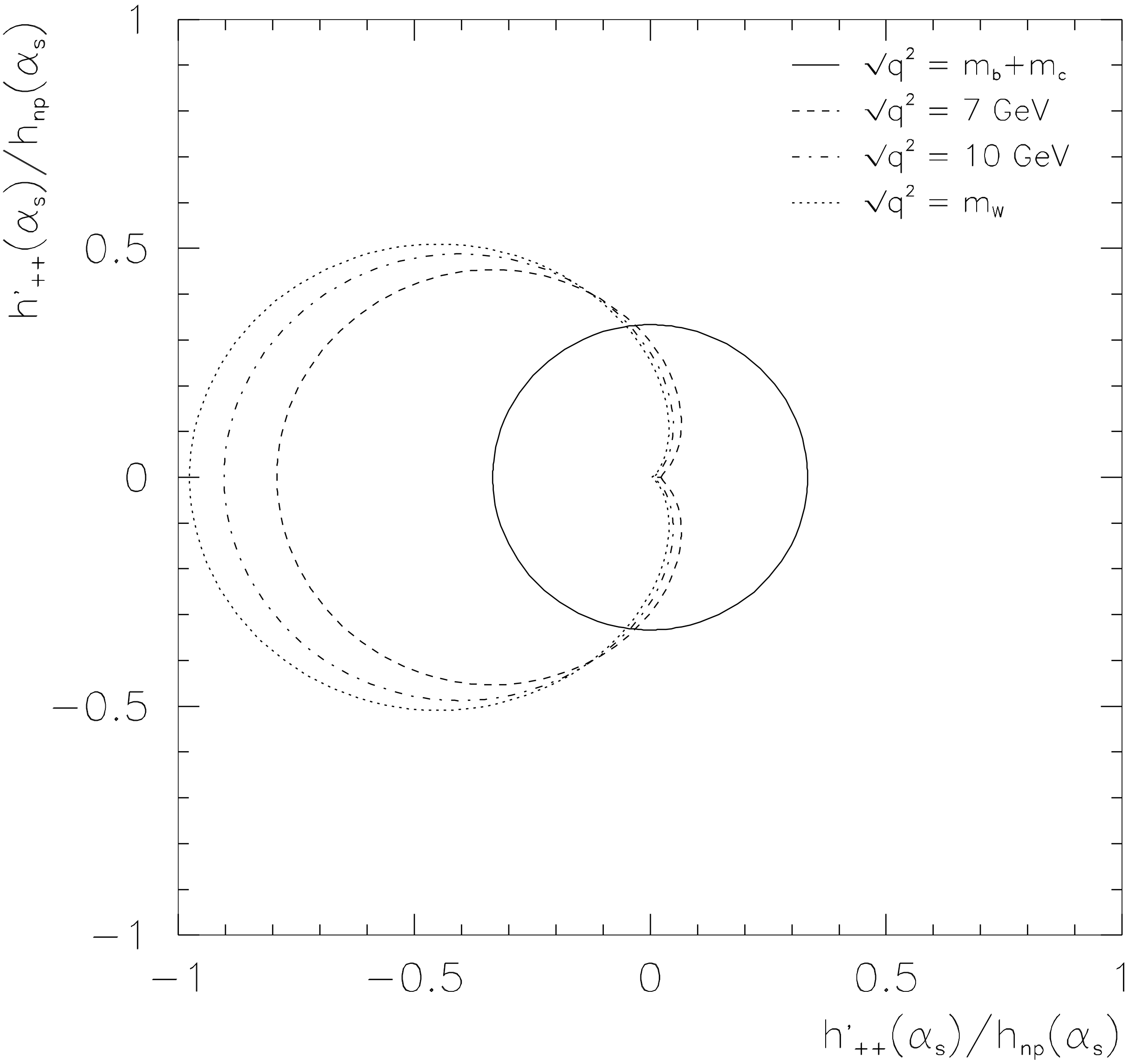
Joonis 8: Parandite energiasõltuvus

5.3 Polarisatsiooni tulemuste nurgasõltuvused

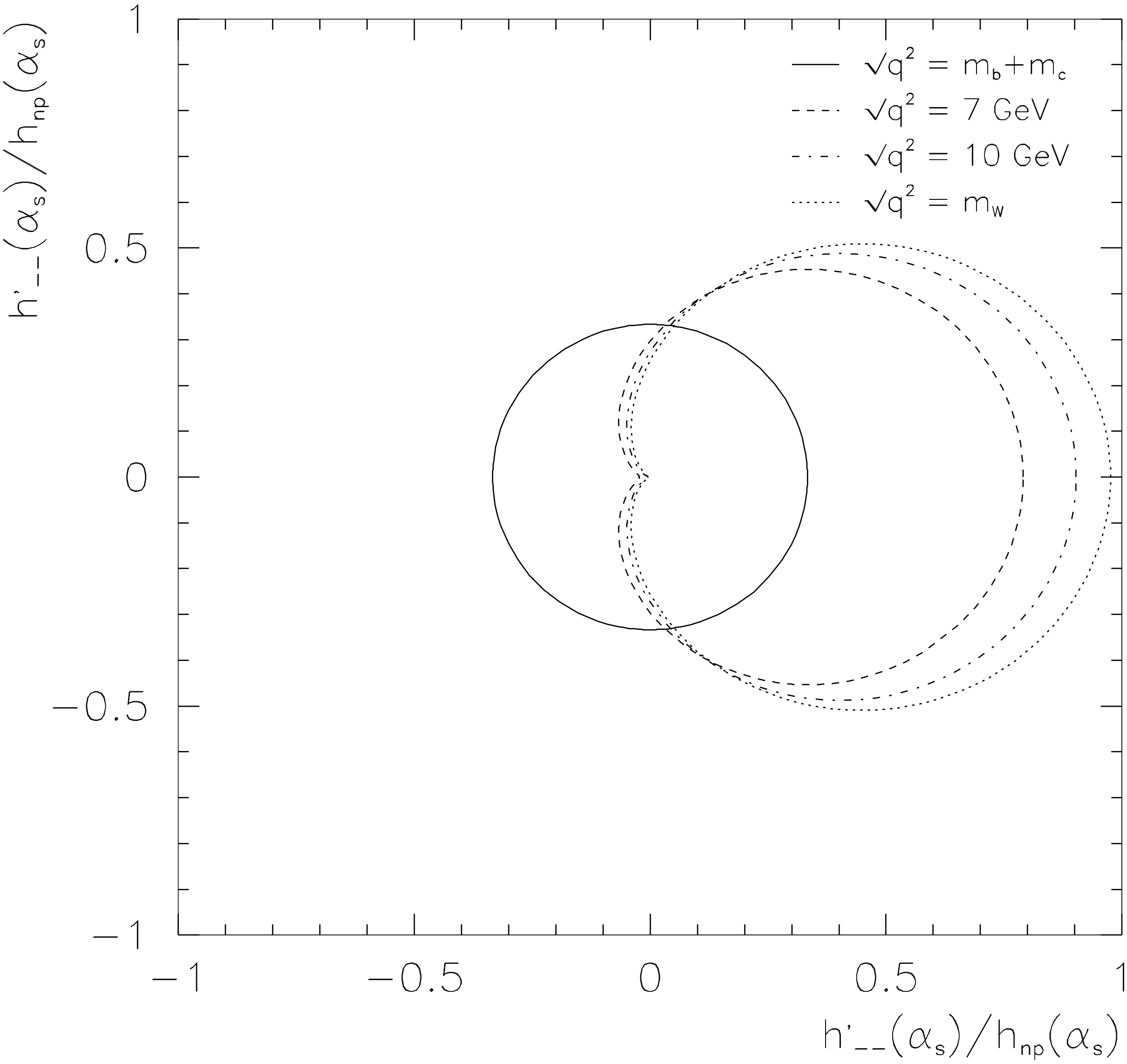
Joonistel 9, 10 ja 11 on esitatud polarisatsiooni nurgasõltuvuste tulemused. Taustsüsteem on valitud nii, et see liigub sama kiirusega kui W-boson z-telje suunas, siis on z-telg kui W-bosoni liikumissuund „kinni külmutatud", ilma et W-boson antud taustsüsteemis ise liiguks (vaadeldav z-telg algab 0-punktist ja liigub sirgelt paremale). Näeme, et $h_{33}(\alpha_s)$ korral on panus põikisuunas domineeriv. $h_{++}(\alpha_s)$ ja $h_{--}(\alpha_s)$ puhul on panus esimesel juhul „maha jäänud" ja teisel juhul „ette läinud".



Joonis 9: $h'_{33}(\alpha_s)$ nurgasõltuvus radiaalesituses

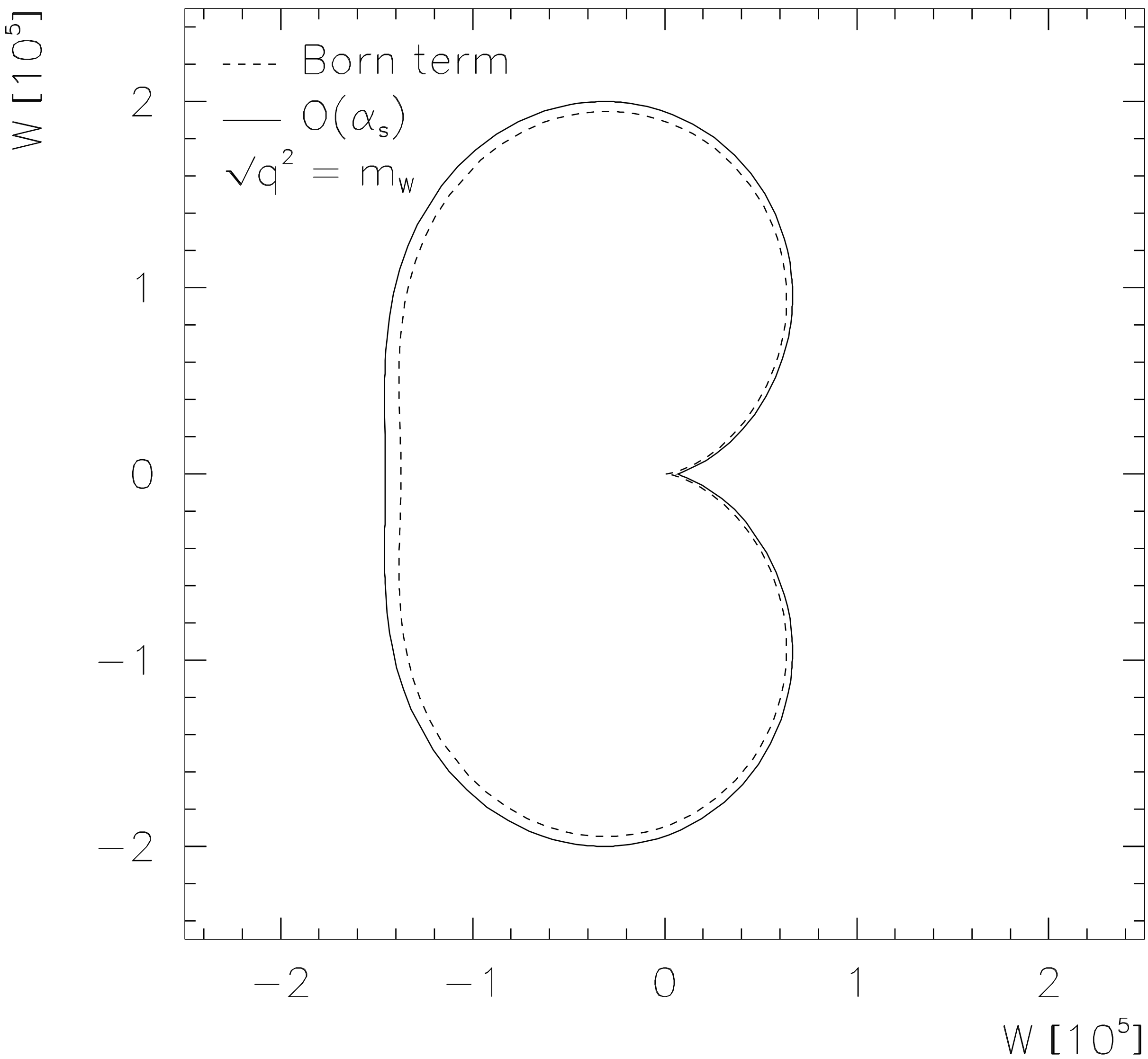


Joonis 10: $h_{++}(\alpha_s)$ nurgasõltuvus radiaalesituses



Joonis 11: $h_{--}(\alpha_s)$ nurgasõltuvus radiaalesituses

Joonisel 12 vaatleme t-kvargi lagunemisprotsessi tulemusena tekkinud W-bosoni lagunemist kvarkideks ja analüüsime selle koondprotsessi nurgasõltuvust. Selleks on kasutatud tulemusi viidetest [5,6] ja töö originaaltulemusi (229), et esitada $W(\theta)$ (kasutades radiaalesitust). Tõstame esile, et näiteks $\theta = \pi$ juures on parandus 6% Borni tulemusest, mis on eksperimendis juba mõõdetatav suurus.



Joonis 12: Täisprotsessi $t \to b + W^+(\to c + \bar{b})$ lagunemismäär $W(\theta)$ radiaalesituses

6 Kokkuvõte

Käesolevas magistritöös on uuritud esimese järgu kvantkromodünaamilisi kiirgusparandeid polariseeritud W^+-bosoni hadronlagunemisel. Sissejuhatuses on tutvustatud uuritava teema olulisust. Lisaks on antud lühike ülevaade kasutatud matemaatilisest aparatuurist. Töö teises osas leidsime polariseeritud W^+-bosoni Borni taseme tulemused ja samuti kaheosakese protsessi alla kuuluva verteksparandi. Verteksparandi osas on kujutegurite renormeerimise juures lahendatud ka ultravioletsete hajumiste probleem. Magistritöö kolmandas osas on uuritud protsessi, kus peale kahe kvargi kiirgub ka pehme gluuon (nn *bremsstrahlung*). Neljandas peatükis on lahendatud infrapunaste hajuvuste probleem ja esitatud uuritava protsessi täistulemused. Need moodustavad antud magistritöö põhitulemused, mis on plaanis ka publitseerida. Lisaks on esitatud täistulemuste piirjuhud väikeste masside ja protsessi läve piirkonnas. Lõpetuseks on töö viiendas peatükis esitatud tulemuste graafilised lahendused. Lisaks on neid võrreldud piirjuhtude vastustega.

Töö autor tänab südamest juhendajat Stefan Grootet, kes on alati pühendunult ja abivalmilt toeks olnud käesoleva uurimuse valmimisel.

7 Summary

First-order quantum chromodynamical radiative correction to the polarized W^+-boson hadronic decay

In this work we study first-order quantum chromodynamical radiative corrections to the polarized W-boson hadronic decay $W^+(\uparrow) \rightarrow Q + \bar{q}$ where Q is an up-type quark and $\bar{q}$ is a down-type anti-quark. The introduction describes the relevance of the thesis' subject. In the second chapter as the beginning of the original work we consider the Born level process where the $W^+(\uparrow)$-boson decays into a pair of quark and anti-quark. In the same chapter we calculate the vertex correction and eliminate the ultra-violet divergences. In the third chapter we examine the three particle process, where in addition to the quark and anti-quark pair a soft gluon is emitted. In the fourth chapter infra-red divergences are canceled and the total results for the process studied in this work is presented. In the fifth chapter we present our results graphically and compare these results with the results in the border limits.

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A Arvutatud I-integraalid

$$I(n_1,n_2)=\int\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}y_1^{n_1}y_2^{n_2}dy_1dy_2 \tag{A1}$$

$$\begin{aligned}
I(-2,0) &= \frac{\lambda}{\mu_1}(D-\ell_4)-\sqrt{\lambda}\frac{1-\mu_1-\mu_2}{2\mu_1}\ell_3+\\
&-\frac{1+\mu_1-\mu_2}{\mu_1}\Big((1-\sqrt{\mu_1})^2-\mu_2\Big)+\frac{\lambda}{2\mu_1},
\end{aligned} \tag{A2}$$

$$I(-2,1) = -3I_+(1)+\frac{1}{3\mu_1}I_+(3), \tag{A3}$$

$$I(-1,-1) = \sqrt{\lambda}(D^\ell+I_z(0))-(1+\mu_1-\mu_2)I^\ell(0)-I^\ell(1), \tag{A4}$$

$$I(-1,0) = -2\mu_1 I^\ell(0)+\frac{1}{2}I^\ell(2), \tag{A5}$$

$$I(0,-2) = \frac{\lambda}{\mu_2}(D-\ell_4)+\Big((1-\sqrt{\mu_1})^2-\mu_2\Big)-\frac{(1-\mu_1)^2}{\mu_2}\ell_0, \tag{A6}$$

$$I(0,-1) = (1-\mu_1)^2I_-(0)-(1+\mu_1)I_+(1)-\frac{1}{2}I_+(2), \tag{A7}$$

$$I(0,0) = -\mu_2(1-\mu_1)^2I_-(0)-(3\mu_1-\mu_2-\mu_1\mu_2)I_+(1)+\frac{1}{2}\mu_2I_+(2)+\frac{1}{3}I_+(3), \tag{A8}$$

$$\begin{aligned}
I(1,-1) &= \frac{1}{2}\Big\{-\mu_2(1-\mu_1)^2I_-(1)+(1-\mu_1)(1-2\mu_1+\mu_1^2+3\mu_2+\mu_1\mu_2)I_-(0)+\\
&-(1-3\mu_1-\mu_1^2+2\mu_2)I_+(1)-\frac{1}{2}(1-\mu_1+\mu_2)I_+(2)-\frac{1}{3}I_+(3)\Big\},
\end{aligned} \tag{A9}$$

$$\begin{aligned}
I(1,1) &= \frac{1}{2}\Big\{-\mu_2^3(1-\mu_1)^2I_-(1)+\mu_2^2(1-\mu_1)(3-6\mu_1+3\mu_1^2+3\mu_2+\mu_1\mu_2)I_-(0)+\\
&+(8\mu_1^2+6\mu_1^3-18\mu_1^2\mu_2-3\mu_2^2+9\mu_1\mu_2^2+3\mu_1^2\mu_2^2-2\mu_2^3)I_+(1)+\\
&-\frac{1}{2}(2\mu_1+6\mu_1^2-6\mu_1\mu_2+3\mu_2^2-3\mu_1\mu_2^2+\mu_2^3)I_+(2)+\\
&-\frac{1}{3}(\mu_1+2\mu_1^2-6\mu_1\mu_2+3\mu_2^2)I_+(3)+\frac{1}{4}(1+3\mu_1-3\mu_2)I_+(4)-\frac{1}{5}I_+(5)\Big\}.
\end{aligned} \tag{A10}$$

B Arvutatud S-integraalid

$$S(n_1,n_2) = \int y_1^{n_1} y_2^{n_2} dy_1 dy_2 \tag{B1}$$

$$S(-2,0) = \frac{\sqrt{\lambda}}{\mu_1}D - \frac{1+\mu_1-\mu_2}{2\mu_1}\ell_1 - \frac{1-\mu_1-\mu_2}{2\mu_1}\ell_3, \tag{B2}$$

$$S(-2,1) = \frac{1+\mu_1-\mu_2}{2\mu_1}\sqrt{\lambda} - \ell_1, \tag{B3}$$

$$S(-1,-1) = D^\ell + S_z(0), \tag{B4}$$

$$S(-1,0) = -\sqrt{\lambda} + \ell_1 - \mu_2\ell_3, \tag{B5}$$

$$S(-1,1) = -\frac{1}{4}(5+\mu_1-3\mu_2)\sqrt{\lambda} + \frac{1}{2}(1+2\mu_1-2\mu_2)\ell_1 + \frac{1}{2}\mu_2^2\ell_3, \tag{B6}$$

$$\begin{aligned} S(-1,2) = {} & \frac{1}{3}\Big\{ -\frac{1}{6}(20+38\mu_1-37\mu_2+2\mu_1^2-7\mu_1\mu_2+11\mu_2^2)\sqrt{\lambda} + \\ & + (1+6\mu_1-3\mu_2+3\mu_1^2-6\mu_1\mu_2+3\mu_2^2)\ell_1 - \mu_2^3\ell_3\Big\}, \end{aligned} \tag{B7}$$

$$S(0,-2) = \frac{\sqrt{\lambda}}{\mu_2}D - \frac{1-\mu_1+\mu_2}{2\mu_2}\ell_2 - \frac{1-\mu_1-\mu_2}{2\mu_2}\ell_3, \tag{B8}$$

$$S(0,-1) = -\sqrt{\lambda} + \ell_2 - \mu_1\ell_3, \tag{B9}$$

$$S(0,0) = \frac{1}{2}(1+\mu_1+\mu_2)\sqrt{\lambda} - (\mu_1-\mu_2)\ell_1 - \mu_2(1-\mu_1)\ell_3, \tag{B10}$$

$$\begin{aligned} S(0,1) = {} & \frac{1}{6}(1+10\mu_1-5\mu_2+\mu_1^2-5\mu_1\mu_2-2\mu_2^2)\sqrt{\lambda} + \\ & - \mu_1\left((1-\mu_2)^2+\mu_1\right)\ell_1 + \mu_2^2(1-\mu_1)\ell_2, \end{aligned} \tag{B11}$$

$$S(1,-2) = \frac{1-\mu_1+\mu_2}{2\mu_2}\sqrt{\lambda} - \ell_2, \tag{B12}$$

$$S(1,-1) = -\frac{1}{4}(5-3\mu_1+\mu_2)\sqrt{\lambda} + \frac{1}{2}(1-2\mu_1+2\mu_2)\ell_2 + \frac{1}{2}\mu_1^2\ell_3, \tag{B13}$$

$$\begin{aligned} S(1,0) = {} & \frac{1}{6}(1-5\mu_1+10\mu_2-2\mu_1^2-5\mu_1\mu_2+\mu_2^2)\sqrt{\lambda} + \\ & + \mu_1^2(1-\mu_2)\ell_1 - \mu_2\left((1-\mu_1)^2+\mu_2\right)\ell_2, \end{aligned} \tag{B14}$$

$$S(2,-1) = \frac{1}{3}\Big\{ -\frac{1}{6}(20-37\mu_1+38\mu_2+11\mu_1^2-7\mu_1\mu_2+2\mu_2^2)\sqrt{\lambda} + $$
$$+ (1-3\mu_1+6\mu_2+3\mu_1^2-6\mu_1\mu_2+3\mu_2^2)\ell_2 - \mu_1^3\ell_3\Big\}. \tag{B15}$$

C Arvutatud J-integraalid

$$J(n_1,n_2) = \int \frac{y_1^{n_1}y_2^{n_2}}{\sqrt{\lambda(1,\mu_1,\mu_2+y_2)}}dy_1dy_2 \tag{C1}$$

$$J(-2,0) = \frac{1}{\mu_1}\Big\{D-\ell_4-\frac{1-\mu_1-\mu_2}{2\sqrt{\lambda}}\ell_3\Big\}, \tag{C2}$$

$$J(-2,1) = \frac{1}{\mu_1}I_+(1), \tag{C3}$$

$$J(-2,2) = \frac{1}{\mu_1}\Big\{(1+\mu_1-\mu_2)I_+(1)-\frac{1}{2}I_+(2)\Big\}, \tag{C4}$$

$$J(-2,3) = \frac{1}{\mu_1}\Big\{(1+3\mu_1+\mu_1^2-2\mu_2-2\mu_1\mu_2+\mu_2^2)I_+(1) + $$
$$-(1+\mu_1-\mu_2)I_+(2)+\frac{1}{3}I_+(3)\Big\}, \tag{C5}$$

$$J(-1,-1) = \frac{1}{\sqrt{\lambda}}\Big\{D^\ell + I_z(0)\Big\}, \tag{C6}$$

$$J(-1,0) = I^\ell(0), \tag{C7}$$

$$J(-1,1) = (1+\mu_1-\mu_2)I^\ell(0)-I^\ell(1), \tag{C8}$$

$$J(-1,2) = (1+4\mu_1+\mu_1^2-2\mu_2-2\mu_1\mu_2+\mu_2^2)I^\ell(0) + $$
$$-2(1+\mu_1-\mu_2)I^\ell(1)+\frac{1}{2}I^\ell(2), \tag{C9}$$

$$J(0,-2) = \frac{1}{\mu_2}\Big\{D-\ell_4-I_-(0)\Big\}, \tag{C10}$$

$$J(0,-1) = I_-(0), \tag{C11}$$

$$J(0,0) = -\mu_2 I_-(0)+I_+(1), \tag{C12}$$

$$J(0,1) = \mu_2^2 I_-(0) + (1+\mu_1-2\mu_2)I_+(1) - \frac{1}{2}I_+(2), \tag{C13}$$

$$\begin{aligned} J(0,2) &= -\mu_2^3 I_-(0) + (1+3\mu_1+\mu_1^2-3\mu_2-3\mu_1\mu_2+3\mu_2^2)I_+(1) + \\ &\quad -\frac{1}{2}(2+2\mu_1-3\mu_2)I_+(2) + \frac{1}{3}I_+(3), \end{aligned} \tag{C14}$$

$$J(1,-2) = \frac{1}{2}\Big\{ I_-(1) - I_-(0) \Big\}, \tag{C15}$$

$$J(1,-1) = \frac{1}{2}\Big\{ -\mu_2 I_-(1) + (1-\mu_1+\mu_2)I_-(0) - I_+(1) \Big\}, \tag{C16}$$

$$J(1,0) = \frac{1}{2}\Big\{ \mu_2^2 I_-(1) - \mu_2(2-2\mu_1+\mu_2)I_-(0) - 2(\mu_1-\mu_2)I_+(1) + \frac{1}{2}I_+(2) \Big\}, \tag{C17}$$

$$\begin{aligned} J(1,1) &= \frac{1}{2}\Big\{ -\mu_2^3 I_-(1) + \mu_2^2(3-3\mu_1+\mu_2)I_-(0) + \\ &\quad -(3\mu_1+2\mu_1^2-6\mu_1\mu_2+3\mu_2^2)I_+(1) + \frac{1}{2}(1+3\mu_1-3\mu_2)I_+(2) - \frac{1}{3}I_+(3) \Big\}. \end{aligned} \tag{C18}$$

D Arvutatud T-integraalid

$$T(n_1,n_2) = \int \frac{y_1^{n_1} y_2^{n_2}}{\lambda(1,\mu_1,\mu_2+y_2)} dy_1 dy_2 \tag{D1}$$

$$T(-2,0) = \frac{1}{\mu_1\sqrt{\lambda}}\Big\{ D + 1 - \frac{1-\mu_1-\mu_2}{2\sqrt{\lambda}}\ell_3 \Big\}, \tag{D2}$$

$$T(-2,1) = \frac{1}{\mu_1}S_+(0), \tag{D3}$$

$$T(-2,2) = \frac{1}{\mu_1}\Big\{ (1+\mu_1-\mu_2)S_+(0) - S_+(1) \Big\}, \tag{D4}$$

$$\begin{aligned} T(-2,3) &= \frac{1}{\mu_1}\Big\{ (1+4\mu_1+\mu_1^2-2\mu_2-2\mu_1\mu_2+\mu_2^2)S_+(0) + \\ &\quad -2(1+\mu_1-\mu_2)S_+(1) + \frac{1}{2}S_+(2) \Big\}, \end{aligned} \tag{D5}$$

$$T(-1,-1) = \frac{1}{\lambda}\Big\{ D^\ell + S_z(0) - I_1^\ell(0) - \frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}}S_1^\ell(0) \Big\}, \tag{D6}$$

$$T(-1,0) = -\frac{1}{2\sqrt{\mu_1}} S_1^\ell(0), \tag{D7}$$

$$T(-1,1) = I_1^\ell(0) - \frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}} S_1^\ell(0), \tag{D8}$$

$$T(-1,2) = 2(1+\mu_1-\mu_2) I_1^\ell(0) - \frac{(1+\mu_1-\mu_2)^2+4\mu_1}{2\sqrt{\mu_1}} S_1^\ell(0) + S^\ell(1), \tag{D9}$$

$$\begin{aligned} T(-1,3) &= (3+10\mu_1+3\mu_1^2-6\mu_2-6\mu_1\mu_2+3\mu_2^2) I_1^\ell(0) + \\ &- \frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}} \left((1+\mu_1-\mu_2)^2+12\mu_1\right) S_1^\ell(0) + 3(1+\mu_1-\mu_2) S^\ell(1) - \frac{1}{2} S^\ell(2), \end{aligned} \tag{D10}$$

$$T(0,-2) = \frac{1}{\mu_2\sqrt{\lambda}} D - \frac{1}{2\mu_2(1-\mu_1)}(\ell_1+2\ell_2), \tag{D11}$$

$$T(0,-1) = \frac{1}{2(1-\mu_1)}(\ell_1+2\ell_2), \tag{D12}$$

$$T(0,0) = \frac{1}{2(1-\mu_1)} \left\{(1-\mu_1-\mu_2)\ell_1 - 2\mu_2\ell_2\right\}, \tag{D13}$$

$$T(0,1) = -\sqrt{\lambda} + \frac{1}{2(1-\mu_1)} \left\{ \left((1-\mu_2)^2-\mu_1^2+2\mu_1\mu_2\right)\ell_1 + 2\mu_2^2\ell_2\right\}, \tag{D14}$$

$$\begin{aligned} T(0,2) &= (1+4\mu_1+\mu_1^2-3\mu_2-3\mu_1\mu_2+3\mu_2^2) S_+(0) + \\ &- \frac{\mu_2^3}{1-\mu_1}\left(S_+(0)+S_-(0)\right) - (2+2\mu_1-3\mu_2) S_+(1) + \frac{1}{2} S_+(2), \end{aligned} \tag{D15}$$

$$T(1,-1) = -\frac{\sqrt{\lambda}}{2(1-\mu_1)} - \frac{1}{2(1-\mu_1)^2} \left\{\mu_1\mu_2\ell_1 - \left((1-\mu_1)^2-2\mu_1\mu_2\right)\ell_2\right\}, \tag{D16}$$

$$\begin{aligned} T(1,0) &= (1-\mu_1+\mu_2)\frac{\sqrt{\lambda}}{2(1-\mu_1)} + \\ &- \frac{1}{2(1-\mu_1)^2}\left\{\mu_1\left((1-\mu_1)^2-\mu_2^2\right)\ell_1 + 2\mu_2\left((1-\mu_1)^2-\mu_1\mu_2\right)\ell_2\right\}, \end{aligned} \tag{D17}$$

$$\begin{aligned} T(1,1) &= \frac{1}{2}\left(\frac{1}{2}(1+5\mu_1-5\mu_2) - \frac{\mu_2^2}{1-\mu_1}\right)\sqrt{\lambda} + \\ &- \frac{\mu_1}{2}\left(2+\mu_1-3\mu_2+\frac{\mu_2^3}{(1-\mu_1)^2}\right)\ell_1 + \frac{\mu_2^2}{2}\left(3-\frac{2\mu_1\mu_2}{(1-\mu_1)^2}\right)\ell_2, \end{aligned} \tag{D18}$$

$$
\begin{aligned}
T(1,2) &= \frac{\mu_2^3}{2(1-\mu_1)}S_-(1) - \frac{\mu_2^3}{(1-\mu_1)^2}\left(2-4\mu_1+2\mu_1^2-\mu_1\mu_2\right)(S_-(0)+S_+(0)) + \\
&- \left(3\mu_1+6\mu_1^2+\mu_1^3-8\mu_1\mu_2-4\mu_1^2\mu_2+6\mu_1\mu_2^2-2\mu_2^3\right)S_+(0) + \\
&+ \frac{1}{2}\left(1+9\mu_1+5\mu_1^2-4\mu_2-12\mu_1\mu_2+6\mu_2^2\right)S_+(1) + \\
&- \frac{1}{2}(1+2\mu_1-2\mu_2)S_+(2) + \frac{1}{6}S_+(3), \qquad \text{(D19)}
\end{aligned}
$$

$$
\begin{aligned}
T(2,-1) &= -\frac{5((1-\mu_1)^2-\mu_1\mu_2)+\mu_2}{6(1-\mu_1)^2}\sqrt{\lambda} - \frac{1}{6}(1-2\mu_1+2\mu_2)\ell_1 + \\
&+ \frac{1}{6}\left(1-\mu_1+4\mu_2-\frac{2\mu_2}{1-\mu_1}+\mu_1\frac{(1+3\mu_1)\mu_2^2}{(1-\mu_1)^3}\right)(\ell_1+2\ell_2)\Bigg\}, \qquad \text{(D20)}
\end{aligned}
$$

$$
\begin{aligned}
T(2,0) &= \frac{1}{6}\left(1-5\mu_1+10\mu_2+(1-5\mu_1)\frac{\mu_2^2}{(1-\mu_1)^2}\right)\sqrt{\lambda} + \\
&+ (1-\mu_1-\mu_2)\frac{\mu_1}{6(1-\mu_1)}\left(1+3\mu_1-6\mu_2+\frac{4\mu_2}{1-\mu_1}+\frac{(1+3\mu_1)\mu_2^2}{(1-\mu_1)^2}\right)\ell_1 + \\
&- \mu_2\left(1-\mu_1+2\mu_2-\frac{\mu_2}{1-\mu_1}+\mu_1\frac{(1+3\mu_1)\mu_2^2}{3(1-\mu_1)^3}\right)\ell_2, \qquad \text{(D21)}
\end{aligned}
$$

$$
\begin{aligned}
T(2,1) &= \frac{1}{18}\left(1-17\mu_1-11\mu_2-26\mu_1^2+52\mu_1\mu_2-47\mu_2^2-3(1-5\mu_1)\frac{\mu_2^3}{(1-\mu_1)^2}\right)\sqrt{\lambda} + \\
&+ \frac{\mu_1}{6}\left(1+10\mu_1-4\mu_2+3\mu_1^2-12\mu_1\mu_2+6\mu_2^2-\frac{4\mu_2^3}{1-\mu_1}+\frac{(1+3\mu_1)\mu_2^4}{(1-\mu_1)^3}\right)\ell_1 + \\
&+ \frac{2\mu_2^2}{3}\left(3-3\mu_1+4\mu_2-\frac{2\mu_2}{1-\mu_1}+\mu_1\frac{(1+3\mu_1)\mu_2^2}{2(1-\mu_1)^3}\right)\ell_2. \qquad \text{(D22)}
\end{aligned}
$$