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Inest Ferdinandi Mindingii disquisitio de formae, in quam geometra britannicus Hamilton integralia mechanices analyticae redegit, origine genuina.

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quinque lustra

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Universitatis Literarum Dorpatensis

Rector et Senatus.

De formae, in quam geometra britannicus Hamilton integralia mechanices analyticae redegit, origine genuina.

Ex quo celeberrimum ac certe sagacissimum Hamiltonii inventum geometris innotuit, statim Jacobi ejus excolendi atque augendi curam suscepit, quem paulo post alii insignes geometrae secuti sunt, inter quos hoc loco inprimis juvat in memoriam revocare Petropolitanum Ostrogradsky, quem nuper praematura morte abreptum dolemus. Hi omnes inventum Hamiltonianum latius extendere aliisque quaestionibus, quae primo saltem adspectu a proposita longe remotae esse videbantur, adaptare tanto successu conati sunt, ut methodi quae nunc ad tractanda problemata analytico-mechanica nec non in aliis analyseos sublimioris partibus huic generi affinibus usu veniunt, generalitate ac elegantia cum aetate prioribus comparari nequeant.

Sed quo latius patet campus investigationum ab inventis Hamiltonianis proficiscentium, tanto magis necessarium est principia quibus inventa illa nituntur, omni cura perscrutari, quae in eis latent inter se diversa, probe distinguere et quae in disquisitionum continuatione permixta occurrunt, in stabiliendis fundamentis segregare. Quoniam igitur praesens quaestio cum e calculo sublimiori tum e legibus mechanices pendeat, operae pretium erit, quid priori, quid alteri parti debeatur, dignoscere. Qua cogitatione motus cum recenti tempore omni studio in id incubuissem, ut primaria theoremata Hamiltoniana ad propositiones algebraicas quantum fieri posset simplicissimas revocarem, non dubitavi praesentem occasionem capessere fontemque monstrare quem illis derivandis aperuisse mihi visus sum. Attamen urgente jam tempore primaria tantum disquisitionis meae momenta succincte exposuisse hoc loco satis habebo.

Ponamus ex argumentis variabilibus $p_1\ p_2 \ldots p_n$ eorumque differentialibus conflatum esse aggregatum hujusmodi, quod cum quantitatem repraesentet infinite parvam secundi ordinis, adscito novo argumento t, symbolo $\Omega\ dt^2$ designare convenit, puta:

$$\begin{aligned}\Omega dt^2 = E_{1.1}\, dp_1^2 + 2E_{1.2}\, dp_1\, dp_2 + E_{2.2}\, dp_2^2 + 2E_{1.3}\, dp_1\, dp_3 + 2E_{2.3}\, dp_2\, dp_3 \\ + E_{3.3}\, dp_3^2 + 2E_{1.4}\, dp_1\, dp_4 + \ldots\ldots\ldots\ldots \qquad \text{I.} \\ \ldots\ldots + E_{n.n}\, dp_n^2,\end{aligned}$$

in quo factores litera E denotati dato modo ex argumentis p formati concipiantur. Aggregatum Ω essentialiter positivum esse suppono, sive ita comparatum, ut neque negativum evadere neque in nihilum abire possit, quicunque valores reales quantitatibus $\frac{dp_1}{dt}\ \frac{dp_2}{dt} \ldots \frac{dp_n}{dt}$ pro lubitu attribuantur, dum argumenta p vel omnino, vel saltem intra certos limites, prout fert natura problematum, arbitrarios valores induant.

His praemissis statim propero ad enuntiandum theorema in praesenti quaestione fundamentale hoc:

Aggregatum signo Ωdt^2 denotatum revocari potest ad summam n quadratorum, $n-1$ quantitates constantes arbitrarias involventium, et quidem ita ut unum e dictis quadratis radicem habeat immediate integrabilem.

Theorema propositum formula sequente exhibetur:

$$\begin{aligned}\Omega dt^2 = dV^2 + (C_{1.1}\, dp_1 + C_{1.2}\, dp_2 + C_{1.3}\, dp_3 + \ldots\ldots + C_{1.n}\, dp_n)^2 \\ + (C_{2.2}\, dp_2 + C_{2.3}\, dp_3 + \ldots\ldots\ldots\ldots + C_{2.n}\, dp_n)^2 \\ + (C_{3.3}\, dp_3 + C_{3.4}\, dp_4 + \ldots\ldots + C_{3.n}\, dp_n)^2 \\ + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{II.} \\ + \ldots\ldots\ldots\ldots\ldots\ldots \\ + (C_{n-1,n-1}\, dp_{n-1} + C_{n-1,n}\, dp_n)^2,\end{aligned}$$

in qua literae C diversis indicibus auctae nec non V functiones argumentorum p repraesentant, in quibus determinandis quaestio tota versatur.

Sit brevitatis gratia $\frac{dV}{dp_1} = v_1, \ldots$ generaliter $\frac{dV}{dp_\mu} = v_\mu$, ideoque

$$dV = v_1\, dp_1 + v_2\, dp_2 + \ldots + v_n\, dp_n.$$

Jam patet comparata forma data I. cum proposita II. prodire conditiones sequentes

$$\begin{aligned}&E_{1.1} - v_1^2 = C_{1.1}^2,\ E_{1.2} - v_1\, v_2 = C_{1.1}\, C_{1.2},\ \ldots\ldots\ E_{1.n} - v_1\, v_n = C_{1.1}\, C_{1.n} \\ &E_{2.2} - v_2^2 = C_{1.2}^2 + C_{2.2}^2,\ E_{2.3} - v_2\, v_3 = C_{1.2}\, C_{1.3} + C_{2.2}\, C_{2.3},\ \ldots\ldots \\ &E_{3.3} - v_3^2 = C_{1.3}^2 + C_{2.3}^2 + C_{3.3}^2,\ E_{3.4} - v_{3.4} = C_{1.3}\, C_{1.4} + C_{2.3}\, C_{2.4} + C_{3.3}\, C_{3.4},\end{aligned}$$

quibus continuatis tandem pervenitur ad has:

$$\begin{aligned}E_{n-1,n} - v_{n-1}\, v_n = C_{1.n-1}\, C_{1.n} + C_{2.n-1}\, C_{2.n} + \ldots + C_{n-1,n-1}\, C_{n,n-1}, \\ E_{n.n} - v_n^2 = C_{1.n}^2 + C_{2.n}^2 + \ldots\ldots + C_{n-1.n}^2.\end{aligned}$$

Generaliter habemus:

$$E_{\mu,\nu} - v_\mu\, v_\nu = C_{1.\mu}\, C_{1.\nu} + C_{2.\mu}\, C_{2.\nu} + \ldots\ldots + C_{\mu.\mu}\, C_{\mu.\nu},$$ siquidem $\mu < \nu$ vel $\mu = \nu$ esse concipitur.

Ex aequationibus propositis, quarum numerus est $\frac{n\,\overline{n+1)}}{2}$, facile eliminantur incognitae C, numero $\frac{n(n+1)}{2} - 1$; quo facto pervenitur ad aequationem differentialem partialem primi ordinis determinando V inservientem.

Quem in finem ex elementis E primo loco formetur complexus signo $\Delta\ (E\mu{,}\nu)$ denotandus, quem determinantem hodierni mathematici appellare solent. Statuamus igitur:

$$\Delta\ (E\mu{,}\nu) = \begin{vmatrix} E_{1.1} & E_{1.2} & E_{1.3} & \ldots\ldots & E_{1.n} \\ E_{1.2} & E_{2.2} & E_{2.3} & \ldots\ldots & E_{2.n} \\ E_{1.3} & E_{2.3} & E_{3.3} & \ldots\ldots & E_{3.n} \\ . & . & . & \ldots\ldots & . \\ . & . & . & \ldots\ldots & . \\ . & . & . & \ldots\ldots & . \\ E_{1.n} & E_{2.n} & E_{3.n} & \ldots\ldots & E_{n.n} \end{vmatrix} \qquad \text{III.}$$

qui complexus, manifesto symmetricus quoniam $E_{\mu,\nu} = E_{\nu,\mu}$, substituto $E_{\mu,\nu} - v_\mu\, v_\nu$ in locum $E_{\mu,\nu}$, abit in sequentem:

$$\Delta\ (E_{\mu,\nu} - v_\mu\, v_\nu) = \Delta', \qquad \text{IV.}$$

quem introductis valoribus argumentornm $E_{\mu,\nu} - v_\mu\, v_\nu$ ex aequationibus II. prodeuntibus facile intelligitur in nihilum abire. Quod, cum brevitati consulendum sit, simplici exemplo proposito sufficienter probabitur. Igitur si $n = 4$, complexus Δ' sequenti tabula repraesentatur, in qua pro $C_{\mu,\nu}$ compendii causa simpliciter $(\mu.\nu)$ scripsimus:

$$\begin{array}{llll}
(1.1)(1.1) & (1.1)(1.2)\;.\quad\bullet & (1.1)(1.3)\;.\quad\bullet\quad.\quad\bullet & (1.1)(1.4)\;.\quad\bullet\quad.\quad\bullet \\
(1.1)(1.2) & (1.2)(1.2)+(2.2)(2.2) & (1.2)(1.3)+(2.2)(2.3)\;.\quad\bullet & (1.2)(1.4)+(2.2)(2.4)\;.\quad\bullet \\
(1.1)(1.3) & (1.2)(1.3)+(2.2)(2.3) & (1.3)(1.3)+(2.3)(2.3)+(3.3)(3.3) & (1.3)(1.4)+(2.3)(2.4)+(3.3)(3.4) \\
(1.1)(1.4) & (1.2)(1.4)+(2.2)(2.4) & (1.3)(1.4)+(2.3)(2.4)+(3.3)(3.4) & (1.4)(1.4)+(2.4)(2.4)+(3.4)(3.4)
\end{array}$$

quem complexum ad nihilum redire lector peritus facile intelliget candemque conclusionem generaliter valere perspiciet.

Ita eliminatis omnibus C delati sumus ad aequationem $\Delta' = o$, sive

$$\Delta' = \begin{vmatrix}
E_{1.1} - v_1^2 & E_{1.2} - v_1 v_2 & \cdots & E_{1.n} - v_1 v_n \\
E_{1.2} - v_1 v_2 & E_{2.2} - v_2^2 & \cdots & E_{2.n} - v_2 v_n \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
E_{1.n} - v_1 v_n & E_{2.n} - v_2 v_n & \cdots & E_{n.n} - v_n^2
\end{vmatrix} = 0, \quad V.$$

quae relationem inter differentialia partialia functionis V atque argumenta p intercedentem sistit, qua ad determinandum V utendum erit.

Ponamus solutionem completam hujus aequationis innotuisse, quae igitur n quantitates constantes arbitrarias secum feret, quarum tamen una non nisi additione ad valorem V accedit, ita ut dV $n-1$ quantitates constantes arbitrarias $\alpha_1\ \alpha_2 \ldots\ldots \alpha_{n-1}$ involvat.

His concessis reliquae incognitae C alia post aliam statim determinantur; fit enim ex aequationibus III.

$$C_{1.1}^2 = E_{1.1} - v_1^2, \quad C_{1.2} = \frac{E_{1.2} - v_1 v_2}{C_{1.1}}, \ldots\ldots$$

$$C_{2.2}^2 = E_{2.2} - v_2^2 - C_{1.1}^2, \ldots\ldots\ldots\ldots$$

nec superfluum est monere, etiam signorum in formula II. nullam occurrere ambiguitatem, quippe quam adjunctis deinceps in singulis terminis et in factorum et in divisorum loco quantitatibus $C_{1.1}^2,\ C_{2.2}^2,\ C_{3.3}^2 \ldots\ldots$ ita scribere licet

$$\Omega dt^2 = dV^2 + \frac{(C_{1.1}^2 dp_1 + C_{1.1} C_{1.2} dp_2 + \cdots + C_{1.1} C_{1.n} dp_n)^2}{C_{1.1}^2}$$

$$+ \frac{(C_{2.2}^2 dp_2 + \cdots + C_{2.2} C_{2.n} dp_n)^2}{C_{2.2}^2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ \frac{(C_{n-1,n-1}^2 dp_{n-1} + C_{n-1,n-1} C_{n-1,n} dp_n)^2}{C_{n-1,n-1}^2},$$

unde dato V ad determinandas reliquas incognitas nulla amplius radicum extractione opus esse perspicitur.

Nunc stabilito theoremate fundamentali consideremus complexum Δ' in aequatione V. ob oculos positum. Quem si secundum potestates et producta argumentorum $v_1\ v_2 \ldots v_n$ evolvimus, primum patet terminos imparis ordinis omnino abesse, deinde facile intelligitur omnes terminos ordinis quarti aut quarto altioris in nihilum abire, ita ut praeter primum terminum Δ_0 (quem hucusque symbolo $\Delta(E\mu,\nu)$ denotatum formula III. exhibet) aggregatum non nisi secundi ordinis restet, quod abhinc signo $-\Delta_2$ repraesentabimus. Quo pacto aequatio V. fit

$$\Delta' = \Delta_0 - \Delta_2 = 0. \qquad \text{VI.}$$

Ut evolutio aggregati Δ_2 commode instituatur, designetur symbolo $\Delta[-\mu, -\nu]$ complexus determinans is qui oritur, si e complexu Δ_0 terminos μ^{tae} lineae horizontalis una cum terminis ν^{tae} lineae verticalis elidimus. Manifesto propter symmetricam conformationem complexus Δ_0 directiones horizontales et verticales inter se commutare licet, unde habetur $\Delta[-\mu, -\nu] = \Delta[-\nu, -\mu]$. Ita posito $n = 3$, erit

$$\Delta_0 = \begin{vmatrix} E_{1.1} & E_{1.2} & E_{1.3} \\ E_{1.2} & E_{2.2} & E_{2.3} \\ E_{1.3} & E_{2.3} & E_{3.3} \end{vmatrix}$$

$$\Delta[-2, -3] = \Delta[-3, -2] = \begin{vmatrix} E_{1.1} & E_{1.2} \\ E_{1.3} & E_{2.3} \end{vmatrix} = \begin{vmatrix} E_{1.1} & E_{1.3} \\ E_{1.2} & E_{2.3} \end{vmatrix}$$

Quo scribendi compendio utentes facile nanciscimur evolutionem quaesitam hanc:

$$\Delta_2 = \Sigma(-1)^{\mu+\nu} v_\mu v_\nu\, \Delta[-\mu, -\nu], \qquad \text{VII.}$$

in qua summatio indicata omnes combinationes numerorum μ et ν, e serie $1\ 2\ 3 \ldots n$ depromendorum, etiam mutati ordinis ratione habita, complectitur. Qua de causa si aggregatum Δ_2 bifariam ita distribuimus, ut in priori parte colligantur omnes termini, in quibus $\mu = \nu$, in altera ii quorum indices inter se differunt, termini posterioris partes omnes factore 2 affecti prodibunt, itaque erit:

$$\Delta_2 = v_1^2\, \Delta[-1, -1] + v_2^2 \Delta[-2, -2] + \ldots\ldots + v_n^2 \Delta[-n, -n]$$

$$
\begin{aligned}
& -2v_1v_2\Delta[-1,-2] + 2v_1v_3\Delta[-1,-3] \ldots\ldots - (-1)^n 2v_1v_n\Delta[-1,-n] \\
& -2v_1v_2\Delta[-2,-3] \;.\;.\;.\;.\;.\;.\;.\;.\;.\;. + (-1)^n 2v_2v_n\Delta[-2,-n] \\
& .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\
& .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\
& \qquad\qquad\qquad\qquad -2v_{n-1}v_n\Delta[-(n-1),-n].
\end{aligned}
$$

Hinc porro deducimus

$$
\tfrac{1}{2}\frac{d\Delta_2}{dv_\mu} =
$$

$$
(-1)^\mu \Big\{ -v_1\Delta[-1,-\mu] + v_2\Delta[-2,-\mu] \ldots + (-1)^\mu v_\mu\Delta[-\mu,-\mu] \ldots + (-1)^n v_n\Delta[-\mu,-n] \Big\}
$$

sive quod idem est;

$$
\tfrac{1}{2}\frac{d\Delta_2}{dv_\mu} = \begin{vmatrix} E_{1.1} & E_{1.2} & \ldots & E_{1.\mu-1} & v_1 & E_{1.\mu+1} & \ldots & E_{1.n} \\ E_{1.1} & E_{2.2} & \ldots & E_{2.\mu-1} & v_2 & E_{1.\mu+1} & \ldots & E_{2.n} \\ E_{1.3} & E_{2.3} & \ldots & E_{3.\mu-1} & v_3 & E_{3.\mu+1} & \ldots & E_{3.n} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ E_{1.n} & E_{2.n} & \ldots & E_{\mu-1,n} & v_n & E_{\mu+1,n} & \ldots & E_{n.n} \end{vmatrix} \qquad \text{VIII.}
$$

Ponamus in aequatione I. omnibus E adjungi factorem communem F, ideoque Ω transire in $F.\Omega$; tunc manifesto $\triangle_0$ mutatur in $F^n.\triangle_0$, nec non $\triangle_2$ in $F^{n-1}.\triangle_0$; unde aequatio VI. abit in hanc:

$$
F^n.\triangle_0 - F^{n-1}.\triangle_2 = 0,
$$

sive rejecto factore F^{n-1},

$$
F.\triangle_0 - \triangle_2 = 0. \qquad \text{IX.}
$$

Jam ad conclusiones quasdam e praecedentibus colligendas convertamur.

Aequatio II. quamquam prorsus idem quod I. aggregatum repraesentans, nihilominus a dextra parte quantitates constantes arbitrarias $\alpha_1\ \alpha_2 \ldots \alpha_{n-1}$ involvit, quae sinistrae parti sive formae primitivae aggregati Ω omnino desunt. Unde patet partem dextram mutatis his constantibus nullum incrumentum capere posse, sive ejus differentiale e mutatione uniuscujusque dictarum constantium (quam litera α simpliciter denotamus) necessario evanescere. Ita differentiando secundum α obtinemus:

$$
0 = dV.\,d\frac{dV}{d\alpha} + (C_{1.1}\,dp_1 + \ldots + C_{1.n}\,dp_n)\Big(\frac{dC_{1.1}}{d\alpha}\,dp_1 + \ldots + \frac{dC_{1.n}}{d\alpha}\,dp_n\Big)
$$
$$
+ \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.
$$

Verumtamen ut scriptionem formulae satis prolixae contrahamus, commodioribus notationibus utemur. Ponamus

$$
\begin{aligned}
C_{1.1}\,dp_1 + C_{1.2}\,dp_2 + \ldots + C_{1.n}\,dp_n &= L_1\,dt \\
C_{2.2}\,dp_2 + \ldots + C_{2.n}\,dp_n &= L_2\,dt \qquad \text{X.} \\
.\;.\;.\;.\;.\;.\;. &
\end{aligned}
$$

vel generaliter:

$$
C_{\mu,\mu}\,dp_\mu + C_{\mu,\mu+1}\,dp_{\mu+1} + \ldots + C_{\mu,n}\,dp_n = L_\mu\,dt,
$$

quo pacto aggregatum II. hanc speciem induit:

$$
\Omega\,dt^2 = dV^2 + (L_1{}^2 + L_2{}^2 + \ldots + L_{n-1}{}^2)\,dt^2
$$

ejusque differentiale e mutato α oriundum fit:

$$
o = dV d\frac{dV}{d\alpha} + \Big(L_1\frac{dL_1}{d\alpha} + L_2\frac{dL_2}{d\alpha} + \ldots + L_{n-1}\frac{dL_{n-1}}{d\alpha}\Big)\,dt^2. \qquad \text{XI.}
$$

Nihil impedit quominus supponamus, inter argumenta $p_1\ p_2 \ldots p_n$ hucusque prorsus inter se independentia subsistere aequationes differentiales sequentes:

$$
L_1 = o,\ L_2 = o, \ldots\ldots L_{n-1} = o; \qquad \text{XII.}
$$

qua ex suppositione sequitur

$$
\Omega\,dt^2 = dV^2 \text{ nec non } dV.d\frac{dV}{d\alpha} = o.
$$

Aequatio prior docet dV non evanescere, cum ex hypothesi nostra Ω vi aequationum XII. in nihilum abire nequeat; unde ex altera aequatione colligitur:

$$
d\frac{dV}{d\alpha} = o.
$$

Quae conclusio cum ad omnes constantes α aeque pertineat, docet e systemate XII. $n-1$ aequationum differentialium necessario consequi alterum systema totidem aequationum differentialium, puta

$$
d\frac{dV}{d\alpha_1} = o,\ d\frac{dV}{d\alpha_2} = o, \ldots\ d\frac{dV}{d\alpha_{n-1}} = o \qquad \text{XIII.}
$$

quod igitur easdem quas systema XII. relationes inter differentialia $dp_1\ dp_2 \ldots dp_n$ sistet, ita ut alterum systema alteri plane aequipolleat. Unde colliguntur utriusque systematis integralia haec, in quibus novas constantes litera β denotamus:

$$\frac{dV}{d\alpha_1} = \beta_1, \frac{dV}{d\alpha_2} = \beta_2, \ldots \frac{dV}{d\alpha_{n-1}} = \beta_{n-1}.$$

Statuamus datum aggregatum Ω continere constantem quandam quantitatem indeterminatam h, quae igitur ut in sinistra, ita etiam in dextra parte aequationis II. aderit. Quae si secundum h differentiatur, prodit:

$$\tfrac{1}{2}\frac{d\Omega}{dh}. dt^2 = dV.\, d\frac{dV}{dh} + \left(L_1 \frac{dL}{dh} + \ldots \right) dt^2,$$

sive iterum admissis conditionibus XII.

$$\tfrac{1}{2}\frac{d\Omega}{dh}\, dt^2 = dV.d\frac{dV}{dh},$$

unde accedente aequatione

$$dV = \sqrt{\Omega}.dt,$$

colligitur:

$$\frac{1}{2\sqrt{\Omega}}\frac{d\Omega}{dh}\, dt = d\,\frac{dV}{dh},$$

sive

$$\frac{d\sqrt{\Omega}}{dh}\, dt = d\,\frac{dV}{dh}. \qquad \text{XV.}$$

Jam redeamus ad casum supra memoratum, quo aggregati Ω elementa E omnia factore communi F aucta esse supponebamus. Quo in casu cum Ω transeat in $F.\Omega$, praecedens aequatio fit:

$$\frac{d\sqrt{F.\Omega}}{dh}\, dt = d\,\frac{dV}{dh} \qquad \text{XVI.}$$

ubi probe notandum, V non amplius ex aequatione VI., sed ex aequatione IX., quae praesenti casui convenit, derivandum esse.

Denique si constans h non in Ω, sed in factore F solo occurrit, ex aequatione XVI. obtinemus:

$$\sqrt{\Omega}.\frac{d\sqrt{F}}{dh}\, dt = d\,\frac{dV}{dh}$$

sive

$$\frac{\sqrt{\Omega}}{2\sqrt{F}}.\frac{dF}{dh}\, dt = d\,\frac{dV}{dh}. \qquad \text{XVII.}$$

His omnibus quam arcte connexa sit doctrina Hamiltoniana, lectorem fugere nequit. Nihilominus quaestionem attentione dignissimam accuratius examinasse juvabit.

Proposito problemate mechanico, ejus naturae ut vires corpora accelerantes functionem potentialem admittant, concipiamus coordinatas argumentis $p_1\, p_2 \ldots p_n$ ita expressas esse, ut conditiones systematis, si quae adsunt, nulla inter argumenta p relatione interposita intactae serventur. Designet more consueto t tempus, $U + h$ functionem potentialem constante h auctam, T vim vivam systematis, ponatur insuper compendii causa

$$\frac{dp_1}{dt} = q_1, \frac{dp_2}{dt} = q_2, \ldots \frac{dp_n}{dt} = q_n$$

ideoque $\quad T = \frac{1}{2}(E_{1.1}\, q_1^2 + 2\, E_{1.2}\, q_1\, q_2 + E_{2.2}\, q_2^2 + \ldots\ldots + E_{n.n}\, q_n^2)$

sive $\quad T = \frac{1}{2}\,\Omega.$

His statutis obtinemus, accito factore

$$F = 2\,(U + h)$$

aggregatum

$$F\Omega = 4\,(U + h)\, T = 2\,(U + h)\,(E_{1.1}\, dp_1^2 + \ldots + E_{n.n}\, dp_n^2)$$

quod transformationibus supra explicatis subjectum, ad aequationes Hamiltonianas perducit.

Data enim solutione completa V aequationis differentialis partialis

$$\Delta = 2\,(U + h)\,\Delta_0 - \Delta_2 = o,$$

secundum praecedentia datur etiam producti $4\,(U + h)\, T$ transformatio plane identica

$$4\,(U + h)\, T = \left(\frac{dV}{dt}\right)^2 + L_1^2 + L_2^2 + \ldots + L_{n-1}^2,$$

qua aequationum differentialium $L_1 = o$, $L_2 = o, \ldots L_{n-1} = o$ integralia exhibentur in forma $\frac{dV}{d\alpha_1} = \beta_1, \ldots \frac{dV}{d\alpha_{n-1}} = \beta_{n-1}$.

Ex iisdem aequationibus differentialibus etiam sequitur:

$$\frac{dV}{dt} = 2\sqrt{(U + h)\, T}.$$

Insuper e formula XVII, quae posito $F = 2\,(U + h)$ nec non $\Omega = 2\, T$ praesenti casui convenit, elicitur:

$$\frac{\sqrt{2T}}{2\sqrt{2\,U + 2\,h}}.\frac{d\,(2\,U + 2\,h)}{dh} = \frac{d\left(\frac{dV}{dh}\right)}{dt}$$

sive

$$\sqrt{\frac{T}{U + h}} = \frac{d\left(\frac{dV}{dh}\right)}{dt}. \qquad \text{XVIII.}$$

Aequationes hucusque inventae sunt merae transformationes algebraicae, quas lex virium vivarum nullo modo attingit, cujus adsciscendi nunc demum occasio praebetur. Statuamus igitur dicta e lege mechanica

$$T = U + h,$$

unde videmus prodire aequationem $d\dfrac{\left(\frac{dV}{dh}\right)}{dt} = 1$

integrale notum

$$\frac{dV}{dh} = t + \tau \qquad \text{XIX.}$$

quod reliquum erat, suppeditantem.

His adjungenda est relatio

$$\frac{dV}{dt} = 2\sqrt{(U+h)\,T},$$

quae ex hypothesi $T = U + h$ abit in

$$\frac{dV}{dt} = 2\,T = 2\,(U + h) \qquad \text{XX.}$$

sive $$V = 2\int T dt = 2\int (U + h)\,dt.$$

Ita delati sumus ad definitionem primitivam functionis characteristicae V, cui Hamilton disquisitiones suas superstruxit et qua vim vivam systematis accumulatam (vel potius secundum usum hodiernum et re vera commodiorem duplicem vim vivam acc.) repraesentari monuit. Huc igitur progressi finem nostro scripto imponere lectoremque ad opera laudati auctoris ejusque successorum delegare possemus. Sed ne quid desideretur quod ad principia doctrinae de qua agitur, attinet, restat ut monstremus quomodo e praecedentibus, eliminatis constantibus $\alpha_1\ \alpha_2\ \ldots\ \alpha_{n-1}$, prodeant aequationes differentiales secundi ordinis, a quibus solutio problematum mechanicorum proficiscitur. Nam in praesenti quaestione, in qua functio V ex aequatione differentiali partiali eruta supponitur, aequationes differentiales secundi ordinis quasi inverso decursu ultimo tandem loco post integralia prima et secunda obviam fiunt.

Cum sit $dV = v_1 q_1 + v_2 q_2 + \ldots\ldots + v_n q_n$

$$2T = \frac{dT}{dq_1}q_1 + \frac{dT}{dq_2}q_2 + \ldots\ldots + \frac{dT}{dq_n}q_n,$$

formulam XX. $\dfrac{dV}{dt} = 2T$

ita scribimus:

$$v_1 q_1 + v_2 q_2 + \ldots + v_n q_n = \frac{dT}{dq_1}q_1 + \frac{dT}{dq_2}q_2 + \ldots + \frac{dT}{dq_n}q_n.$$

Aequatio $$\Delta_2 = 2(U + h)\,\Delta_0$$

secundum α (sive unam constantiam $\alpha_1\ \alpha_2\ \ldots\ \alpha_{n-1}$) differentiata, cum a dextra parte nulla harum constantium occurrat, suppeditat systema $n-1$ aequationum $\dfrac{d\Delta_2}{d\alpha} = 0$, i. e.

$$\frac{d\Delta_2}{dv_1}\cdot\frac{dv_1}{d\alpha} + \frac{d\Delta_2}{dv_2}\cdot\frac{dv_2}{d\alpha} + \ldots + \frac{d\Delta_2}{dv_n}\cdot\frac{dv_n}{d\alpha} = o, \qquad \text{XXII.}$$

quibus si comparantur totidem aequationes formae $d\dfrac{dV}{d\alpha} = o$

sive $$\frac{dv_1}{d\alpha}q_1 + \frac{dv_2}{d\alpha}q_2 + \ldots\ldots + \frac{dv_n}{d\alpha}q_n = o, \qquad \text{XXIII.}$$

manifesto obtinemus:

$$q_1 : q_2 \ldots\ldots : q_n = \frac{d\Delta_2}{dv_1} : \frac{d\Delta_2}{dv_2} \ldots\ldots : \frac{d\Delta_2}{dv_n}$$

sive $$\frac{1}{q_1}\frac{d\Delta_2}{dv_1} = \frac{1}{q_2}\frac{d\Delta_2}{dv_2} = \ldots\ldots = \frac{1}{q_n}\frac{d\Delta_2}{dv_n}. \qquad \text{XXIV.}$$

Porro consideremus complexum eum qui oritur, si in complexu Δ_0 (III) terminis seriei verticalis $E_{1,\mu}\ E_{2,\mu}\ \ldots\ E_{\mu,\mu}\ \ldots\ E_{\mu,n}$

ex ordine substituantur $\dfrac{dT}{dq_1}\ \dfrac{dT}{dq_2}\ \ldots\ \dfrac{dT}{dq_\mu}\ \ldots\ \dfrac{dT}{dq_n}.$

Quem complexum litera Θ_μ designantes habemus

$$\Theta_\mu = \begin{vmatrix} E_{1.1} & E_{1.2} & \ldots & E_{1.\mu-1} & \frac{dT}{dq_1} & E_{1.\mu+1} & \ldots & E_{1.n} \\ E_{1.2} & E_{2.2} & \ldots & E_{2,\mu-1} & \frac{dT}{dq_2} & E_{2,\mu+2} & \ldots & E_{2.n} \\ \cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \ldots & \cdot \\ E_{1.n} & E_{1.n} & \ldots & E_{\mu-1,n} & \frac{dT}{dq_n} & E_{\mu+1,n} & \ldots & E_{n.n} \end{vmatrix} \qquad \text{XXV.}$$

Jam vero cum sit

$$\frac{dT}{dq_\mu} = E_{1.\mu}\,q_1 + E_{2.\mu}\,q_2 + \ldots\ldots + E_{\mu.n}\,q_n$$

liquet terminos complexus propositi factore q_ν affectos evanescere omnes, nisi sit $\nu = \mu$, ideoque simpliciter prodire

$$\Theta_\mu = \triangle_0 \, q_\mu,$$

unde colligitur:

$$\frac{\Theta_1}{q_1} = \frac{\Theta_2}{q_2} = \ldots\ldots = \frac{\Theta_n}{q_n}, \qquad \text{XXVI.}$$

quarum quantitatum valor communis est $\triangle_0$.

Comparanti aequationes XXVI. cum aequationibus XXIV. simulque ad formas complexuum $\frac{d\Delta_2}{dv_\mu}$ et Θ_μ (VIII. et XXV.) respicienti obvium erit, rationes $v_1 : v_2 : \ldots : v_n$ nec non rationes $\frac{dT}{dq_1} : \frac{dT}{dq_2} \ldots : \frac{dT}{dq_n}$ utrasque eidem systemati $n-1$ aequationum linearium satisfacere ideoque aequalibus valoribus gaudere. Est igitur

$$\frac{1}{v_1}\frac{dT}{dq_1} = \frac{1}{v_2}\frac{dT}{dq_2} = \ldots = \frac{1}{v_n}\frac{dT}{dq_n};$$

hinc autem ope formulae XXI. colligitur esse:

$$v_1 = \frac{dT}{dq_1},\ v_2 = \frac{dT}{dq_2}, \ldots v_n = \frac{dT}{dq_n} \qquad \text{XXVII.}$$

sive

$$\frac{dV}{dp_1} = \frac{dT}{dq_1}, \ldots\ldots \frac{dV}{dp_n} = \frac{dT}{dq_n},$$

quae est nota forma integralium praesentis problematis intermediorum, constantes arbitrarias $\alpha_1\ \alpha_2 \ldots \alpha_{n-1}$ involventium.

Denique ut ad aequationes differentiales secundi ordinis, eliminatis omnibus α, descendamus, proficiscamur a formulis:

$$T = U + h,\ \frac{dV}{dh} = 2\,T,$$

quarum combinatio suggerit:

$$\frac{dV}{dt} - T - U - h = o,$$

sive

$$v_1\,q_1 + v_2\,q_2 + \ldots + v_n\,q_n - T - U - h = o, \qquad \text{XXVIII.}$$

qua in aequatione valores quantitatum $q_1\ q_2 \ldots q_n$ in $\frac{dV}{dt}$ et in T occurrentium ope formularum XXVII. per argumenta $p_1\ p_2 \ldots p_n$ et constantes $\alpha_1\ \alpha_2 \ldots \alpha_{n-1}$ expressos concipi oportet, quibus substitutis aggregatum propositum identice evanescet. His perpensis differentiando secundum argumentum p, obtinemus:

$$\frac{dv_1}{dp_1}q_1 + \frac{dv_2}{dp_1}q_2 + \ldots + \frac{dv_n}{dp_1}q_n - \frac{dT}{dp_1} - \frac{dU}{dp_1}$$
$$+ v, \frac{dq_1}{dp_1} + v_2\frac{dq_2}{dp_1} + \ldots + v_n\frac{dq_n}{dp_1} - \frac{dT}{dq_1}\frac{dq_1}{dp_1} - \frac{dT}{dp_2}\frac{dq_2}{dp_1} \ldots - \frac{dT}{dq_n}\frac{dq_n}{dp_1} = o.$$

Cum autem ex aequationibus XXVII. bini lineae inferioris termini se mutuo destruant, proposita in hanc contrahitur:

$$\frac{dv_1}{dp_1}q_1 + \frac{dv_2}{dp_1}q_2 + \ldots + \frac{dv_n}{dp_1}q_n = \frac{dT}{dp_1} + \frac{dU}{dp_1}.$$

Manifesto aggregatum ad laevam nihil aliud est quam

$$\frac{d^2V}{dp_1^{\,2}}\frac{dp_1}{dt} + \ldots + \frac{d^2V}{dp_1\,dp_n}\frac{dp_n}{dt} = \frac{d\left(\frac{dV}{dp_1}\right)}{dt} = \frac{dv_1}{dt}.$$

Unde accedente valore (XXVII.)

$$v, = \frac{dT}{dq_1}$$

constantes α omnes simul eliminantur atque eruitur

$$\frac{d\left(\frac{dT}{dq_1}\right)}{dt} = \frac{dT}{dp_1} + \frac{dU}{dp_1},$$

quae est quaesita aequatio differentialis secundi ordinis ad argumentum p, pertinens; reliquae eodem modo inveniuntur.

In praecedentibus non attigi proprietates minimum valorem integralis

$$\int\sqrt{\Omega}\,dt \text{ sive } \int\sqrt{E_{1.1}\,dp_1^{\,2} + 2\,E_{1.2}\,dp_1\,dp_2 + \ldots + E_{n.n}\,dp_n^{\,2}}$$

spectantes, quibus eruendis calculus variationum in usum vocari solet. Sed verum fundamentum hujus doctrinae, ad quam theoria linearum brevissimarum, nec non, quod multo latius patet, principium minimae actionis pertinet, manifesto positum est in discerptione aggregati Ωdt^2 in plura quadrata, quarum unum radicem habet per se integrabilem. Quae discerptio cum sit aggregati Ω transformatio plane identica, nihilominus tamen numerum sufficientem constantium arbitrariarum involvens, quarum ope iis conditionibus satisfieri potest quas limites integrationis postulant, non solum aequationes differentiales minimi, quae nostris in signis erunt $L_1 = 0$, $L_2 = 0, \ldots L_{n-1} = 0$, ipsumque minimum $\int dV$, verum etiam terminos, ultra quos integrationem extendere, nisi cassante minimo, non licet, non indicare nequit. Attamen hanc rem uberiore explicatione egentem, temporis angustiis pressus, in praesens dimittam et ad observationes quasdam particulares me convertam, quae mihi hujusmodi quaestionibus insistenti obviam fuerunt.

Prima observatio spectat theoriam linearum brevissimarum, quae plerumque a calculo variationum proficiscitur. Igitur operae pretium videtur esse, quomodo e nostris principiis, nullo e calculo variationum adjumento petito, aequationes differentiales necessariae facillime demanent, ostendere.

Tota enim quaestio de lineis brevissimis eo redit, ut aggregatum $Edp^2 + 2Fdpdq + Gdq^2$ in formam $dr^2 + m^2\, d\varphi^2$ redigatur, siquidem hoc loco, quod commodissimum erit, notationes Disquisitionum circa superficies curvas adoptamus. Igitur ut aequatio

$$Edp^2 + 2\, Fdpdq + Gdq^2 = dr^2 + m^2\, d\varphi^2$$

fiat identica, incognitae r, φ, m e relationibus eruendae sunt quae sequuntur:

$$\left(\frac{dr}{dp}\right)^2 + m^2 \left(\frac{dr}{d\varphi}\right)^2 = E$$

$$\frac{dr}{dp} \cdot \frac{dr}{dq} + m^2 \frac{d\varphi}{dp} \cdot \frac{d\varphi}{dq} = F$$

$$\left(\frac{dr}{dq}\right)^2 + m^2 \left(\frac{d\varphi}{dq}\right)^2 = G,$$

unde demanant conditiones eaedem quas disquisitimes landatae sistunt, puta:

$$E \left(\frac{dr}{dq}\right)^2 - 2\, F\frac{dr}{dp} \cdot \frac{dr}{dq} + G \left(\frac{dr}{dp}\right)^2 = EG - F^2$$

$$\left(E\frac{dr}{dq} - F\frac{dr}{dp}\right) \frac{d\varphi}{dp} + \left(G\, \frac{dr}{dp} - F\frac{dr}{dq}\right) \frac{d\varphi}{dq} = 0$$

$$EG - F^2 = \left(\frac{dr}{dp} \cdot \frac{d\varphi}{dq} - \frac{dr}{dq} \cdot \frac{d\varphi}{dp}\right) m^2$$

sive
$$E\left(\frac{d\varphi}{dq}\right)^2 - 2\, F\frac{d\varphi}{dp} \frac{d\varphi}{dq} + G \left(\frac{d\varphi}{dp}\right)^2 = \frac{EG - F^2}{m^2},$$

quarum secunda etiam e prima, ope differentiationis secundum constantem α in r contentam institutae, deducitur ponendo $\frac{dr}{d\alpha} = \varphi$.

Deinde angulus θ, ab elemento dr lineae brevissimae et elemento $\sqrt{E}\,.\,dp$ comprehensus (cf. disqu. art. 18) ex eodem principio, calculo variationum non adhibito, ita determinatur.

Aequationem $Edp^2 + 2\, Idpdq + Gdq^2 = dr^2 + m^2\, d\varphi^2$

sive posito $EG - F^2 = \Delta^2$

$$\frac{(Edp + Fdq)^2 + \Delta^2\, dq^2)}{E} = dr^2 + m^2\, d\varphi^2$$

introducto novo argumento variabili indeterminato θ ita dispertire licet ut sit

$$\frac{(Edp + Fdq) \cos\theta + \Delta \sin\theta\, dq}{\sqrt{E}} = dr$$

$$-\frac{(Edp + Fdq) \sin\theta + \Delta \cos\theta\, dq}{\sqrt{E}} = m d\varphi.$$

Jam ut re vera dr sit elementum lineae brevissimae, cujus aequatio erit $d\varphi = 0$, postulatur ut simul expressio proposita elementi dr sit differentiale completum, ideoque

sive
$$\frac{d\,(\sqrt{E} \cos\theta)}{dq} = \frac{d\left(\frac{F \cos\theta + \Delta \sin\theta}{\sqrt{E}}\right)}{dp},$$

$$\left(\frac{d\sqrt{E}}{dq} - \frac{d\left(\frac{F}{\sqrt{E}}\right)}{dp}\right) \cos\theta - \frac{d\left(\frac{\Delta}{\sqrt{E}}\right)}{dp} \sin\theta = \sqrt{E} \sin\theta \frac{d\theta}{dq} + \frac{\Delta \cos\theta - F \sin\theta}{\sqrt{E}} \cdot \frac{d\theta}{dp}.$$

Hinc fit accedente conditione $d\varphi = 0$, qua datur

$$\cot g\, \theta = \frac{Edp + Fdq}{\Delta\, dq},$$

$$dr\,.\,\cos\theta = \frac{Edp + Fdq}{\sqrt{E}},\ dr \sin\theta = \frac{\Delta}{\sqrt{E}}\, dq:$$

$$\frac{\Delta \cos\theta - F \sin\theta}{\sqrt{E}}\, dr = \Delta dp,\ \sqrt{E} \sin\theta\, dr = \Delta dq,$$

ideoque

$$\left\{\sqrt{E} \sin\theta\, \frac{d\theta}{dq} + \frac{\Delta \cos\theta - I \sin\theta}{\sqrt{E}} \frac{d\theta}{dp}\right\} dr = \Delta \left(\frac{d\theta}{dq}\, dq + \frac{d\theta}{dp}\, dp\right) = \Delta\, d\theta;$$

unde conditio integrabilitatis hanc formam nanciscitur:

$$\left\{\frac{d\sqrt{E}}{dq} - \frac{d\left(\frac{F}{\sqrt{E}}\right)}{dp}\right\} \frac{Edp + Fdq}{\sqrt{E}} - \frac{d\left(\frac{\Delta}{\sqrt{E}}\right)}{dp} \cdot \frac{\Delta}{\sqrt{E}}\, dq = \Delta d\Theta,$$

quae brevi calculo subducto in sequentem contrahitur:

$$\Delta d\Theta = \tfrac{1}{2}\, \frac{F}{E} dE + \tfrac{1}{2}\, \frac{dE}{dq}\, dp - \frac{dF}{dp}\, dp - \tfrac{1}{2}\, \frac{dG}{dp}\, dq,$$

eandem quam disquisitiones l. l. exhibent.

Altera quam ajungere placet observatio versatur in substitutione analytica maxime memorabili, qua variabiles $x_1\ x_2 \dots x_{n+1}$, relatione

$$\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\dots+\frac{x_{n+1}^2}{a_{n+1}}=1$$

inter se connexae, exprimuntur ope radicum p aequationis

$$\frac{x_1^2}{a_1(a_1-p)}+\frac{x_2^2}{a_2(a_2-p)}+\dots+\frac{x_{n+1}^2}{a_{n+1}(a_{n+1}-p)}=0,$$

quas radices symbolis $p_1\ p_2 \dots p_n$ designare convenit.

Hanc substitutionem egregie tractavit in diarii Crelliani volumine 22. geometra Hammensis Haedenkamp, ibique inter alia plura quaestionem de valore minimo integralis

$$\int ds=\int\sqrt{dx_1^2+dx_2^2+\dots+dx_{n+1}^2}$$

elegantissime absolvit. Jam quomodo hoc in casu discerptio aggregati ds^2 in plura quadrata, qualem postulamus, commodissime instituatur, paucis ostendere juvat.

Statuatur, ut est praesenti casui aptum,

$$\Omega dt^2 = E_1\, dp_1^2 + E_2\, dp_2^2+\dots+E_n\, dp_n^2$$
$$\varphi p = p-p_1 . p-p_2 \dots p-p_n$$
$$\Pi = p^{n-1}+\alpha_1 p^{n-2}+\alpha_2 p^{n-3}+\dots+\alpha_{n-1};$$

sint $\alpha_1\ \alpha_2 \dots \alpha_{n-1}$ constantes arbitrariae, denique mutato p in p_μ, transeat Π in Π_μ; tunc erit ex principiis notissimis

$$\Sigma \frac{\Pi_\mu}{\varphi' p_\mu}=1$$

nec non

$$\Sigma \frac{p_\mu^{n-\nu-1}}{\varphi' p_\mu}=0,$$

siquidem ν unum e numeris $1\ 2\ 3 \dots n-1$ repraesentat, dum summationes indictae valores $\mu = 1\ 2\ 3\dots n-1$ amplectuntur.

Quaesita discerptio statim obtinetur aggregatum Ωdt^2 cum aggregato $\Sigma\frac{\Pi_\mu}{\varphi' p_\mu}$ multiplicando productumque secundum schema sequens disponendo:

$$(a_1^2+a_2^2+a_3^2+\dots+a_n^2)(b_1^2+b_2^2+b_3^2+\dots+b_n^2)$$
$$=(a_1 b_1+a_2 b_2+\dots+a_n b_n)^2+\Sigma\Sigma(a_\mu b_\nu - a_\nu b_\mu)^2,$$

in quo duplex summatio omnes valores indicum μ et ν inde ab 1 usque ad n (inclusis extremis) amplectitur, unde aggregatum $\frac{n.n-1}{2}$ quadratorum formabitur. Protinus statuendo

$$a_\mu = \sqrt{E_\mu}\,.\,dp_\mu,\ b_\mu=\sqrt{\frac{\Pi_\mu}{\varphi' p_\mu}}$$

ideoque

$$a_1^2+a_2^2+\dots+a_n^2=\Omega dt^2$$
$$b_1^2+b_2^2+\dots+b_n^2=\Sigma\frac{\Pi_\mu}{\varphi' p_\mu}=1,$$

colligitur:

$$\Omega\, dt^2 = \left\{\sqrt{\frac{\Pi_1E_1}{\varphi' p_1}}\,dp_1+\sqrt{\frac{\Pi_2E_2}{\varphi' p_2}}\,dp_2+\dots+\sqrt{\frac{\Pi_nE_n}{\varphi' p_n}}\,dp_n\right\}^2$$
$$+\Sigma\Sigma\left\{\sqrt{\frac{\Pi_\nu E_\mu}{\varphi' p_\nu}}\,dp_\mu - \sqrt{\frac{\Pi_\mu E_\nu}{\varphi' p_\mu}}\,dp_\nu\right\}^2$$

Haec tamen transformatio nullius esset usus, nisi in casu praesenti singulari feliciter contingeret, ut $\frac{E_\mu}{\varphi' p_\mu}$ ideoque etiam $\frac{\Pi_\mu E_\mu}{\varphi' p_\mu}$ a solo argumento p_μ penderent, cujus proprietatis beneficio fit ut in proposita aequatione pars prima ad dextram occurrens sit quadratum differentialis exacti. Reliquae partes ad dextram positae in aggregatum $n-1$ quadratorum facile quidem contrahuntur, qua tamen reductione nihil proficimus. Patet enim totum aggregatum propositum in nihilum abire statuendo

$$\frac{\sqrt{E_1}\,dp_1}{\sqrt{\frac{\Pi_1}{\varphi' p_1}}}=\frac{\sqrt{E_2}\,dp_2}{\sqrt{\frac{\Pi_2}{\varphi' p_2}}}=\dots=\frac{\sqrt{E_n}\,dp_n}{\sqrt{\frac{\Pi_n}{\varphi' p_n}}},$$

quibus igitur aequationibus conditiones minimi exprimuntur; ipsum autem minimum fit aggregatum partium singillatim integrabilium, puta

$$V=\int\sqrt{\frac{\Pi_1E_1}{\varphi'p_1}}\,dp_1+\int\sqrt{\frac{\Pi_2E_2}{\varphi'p_2}}\,dp_2+\dots+\int\sqrt{\frac{\Pi_nE_n}{\varphi'p_n}}\,dp_n,$$

unde demanant integralia

$$\frac{dV}{d\alpha_1}=\beta_1,\ \frac{dV}{d\alpha_2}=\beta_2,\dots\frac{dV}{d\alpha_{n-1}}=\beta_{n-1},$$

quae necessario conditionibus minimi supra exhibitis satisfaciunt. Quod quomodo fiat, ita explicatur. Primum erit

$$d\frac{dV}{d\alpha_\nu} = \frac{1}{2}\,\Sigma\,\frac{\sqrt{E_\mu}\,.\,dp_\mu}{\sqrt{\Pi_\mu\,.\,\varphi^1 p_\mu}}\,.\,\frac{d\Pi_\mu}{d\alpha_\nu},\ \text{nec non}\ \frac{d\Pi_\mu}{d\alpha_\nu} = p^{n-\nu-1};$$

deinde introducta in locum elementi $\sqrt{E_\mu}\,dp_\mu$ quantitate huic elemento proportionali $\sqrt{\frac{\Pi_\mu}{\varphi^1 p_\mu}}$, obtinemus summam

$$\Sigma\,\frac{p_\mu^{n-\nu-1}}{\varphi^1 p_\mu}$$

quae, ut supra monuimus, in nihilum redit; unde protinus sequitur:

$$d\frac{dV}{d\alpha_\nu} = 0;\ \text{quod erat probandum.}$$

Sed jam appropinquante die festo, cui has pagellas dicare constitutum est, quae sunt reliqua in posterum dimittimus, satis in praesens habentes, principium mere algebraicum explicuisse, e quo primaria saltem theoriäe Hamiltonianae momenta tamquam corollaria demanant.

Datum Dorpati Livonorum, die 27. m. Julii a. 1864.