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**Generation of Smooth Motion Trajectories for
a Rigid Body**

Mathematics
Bachelor's thesis (9 ECTS)

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TARTU 2026

**TAHKE KEHA SILEDATE LIIKUMISTRAJEKTOORIDE
GENEREERIMINE**

Bakalaureusetöö

Aleksandr Beditski

Lühikokkuvõte

Käesolev töö on pühendatud tahke keha siledate liikumistrajektooride genereerimise kirjeldamisele, arvestades selle algset ja lõplikku asendit ning orientatsiooni ruumis. Selleks tuuakse sisse sileda muutkonna, Riemanni geometria, Lie rühmade ning nendega seotud Lie algebrate mõisted. Trajektooride genereerimise ülesanne formuleeritakse seejärel geodeetika ülesandena rühmal $SE(3)$, mis on varustatud vasakpoolset invariantse Riemanni meetrikaga, tuletatud tahke keha kineetilisest energiast.

CERCS teaduseriala: P120 Arvuteooria, väljateooria, algebraline geometria, algebra, rühmateooria. P130 Funktsioonid, diferentsiaalvõrrandid. P150 Geomeetria, algebraline topoloogia.

Märksõnad: Eukleedilised rühmad, Lie rühmad, Lie algebra, muutkonnade teooria, trajektooride genereerimine, variatsioonarvutus.

**GENERATION OF SMOOTH MOTION TRAJECTORIES FOR A
RIGID BODY**

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Aleksandr Beditski

Abstract

The following thesis addresses the problem of generating smooth trajectories

for a rigid body, considering its initial and final position, as well as its orientation in space. For this purpose, concepts of smooth manifolds, Riemannian geometry, Lie groups and their associated Lie algebras are introduced. The problem of trajectory generation is then formulated as a geodesic problem on $SE(3)$ equipped with a left-invariant Riemannian metric, derived from the kinetic energy of the rigid body.

CERCS research specialization: P120 Number theory, field theory, algebraic geometry, algebra, group theory. P130 Functions, differential equations. P150 Geometry, algebraic topology.

Key Words: Euclidean group, Lie groups, Lie algebra, theory of manifolds, trajectory generation, variational calculus.

Contents

Introduction	5
1 Smooth Manifolds	6
1.1 Elementary notions of Point Set Topology	6
1.2 Differentiable Manifolds	7
1.3 Tangent Vectors and Differentials	10
1.4 Computations in Coordinates	15
1.5 Change of Basis	18
2 Vector Fields and Riemannian geometry	20
2.1 Tangent Bundle, Vector Fields and Frames	20
2.2 Riemannian Metrics	22
2.3 Covariant derivative, Connections and Christoffel symbols	22
2.4 Parallel transport and Levi-Civita Connections	26
2.5 Geodesic	29
3 Lie group of SE(3) and its Lie algebra	31
3.1 Lie group – Lie algebra correspondence	31
3.2 Matrix Lie group and Exponential map	34
3.3 The Lie group SE(3) and corresponding Lie algebra	37
3.4 The Choice of Metric	40
3.5 Connection and the Geodesic	43

Conclusion	48
References	49

Introduction

The problem of generating smooth trajectories for a rigid body arises naturally in the field of robotics. It is not only required to generate an arbitrary trajectory, but to also find a trajectory that is optimized in a physically meaningful way. This problem is naturally related to the field of differential geometry, as the configuration space of a rigid body is described as a smooth manifold known as the special Euclidean group $SE(3)$.

This work does not contain original results in the field of robotics or differential geometry, but rather aims to present the mathematical framework necessary to rigorously formulate and analyze the problem of generating and optimizing smooth motion trajectories for a rigid body.

The thesis is structured as follows. Section 1 introduces the foundational concepts of smooth manifolds. Section 2 develops the theory of vector fields and Riemannian geometry on smooth manifolds, including the notions of the tangent bundle, covariant derivatives, and geodesics. Section 3 extends these concepts to the Lie group $SE(3)$, establishes the correspondence between Lie groups and their Lie algebras, and formulates the trajectory generation problem as a geodesic problem on $SE(3)$ equipped with a left-invariant Riemannian metric derived from the kinetic energy of the rigid body. Necessary conditions for smooth optimal trajectories are then derived using the calculus of variations.

1 Smooth Manifolds

1.1 Elementary notions of Point Set Topology

Before beginning our discussion of smooth manifolds, we need to introduce some elementary notions from general topology. All of the definitions and related comments in this subsection are sourced from [6].

Definition 1.1.1. A *topology* on a set X is a collection \mathcal{T} of subsets of X having the following properties:

1. \emptyset and X are in \mathcal{T} .
2. The union of the elements of any subcollection of \mathcal{T} is in \mathcal{T} .
3. The intersection of any two elements of \mathcal{T} is in \mathcal{T} .

A set X for which a topology \mathcal{T} has been specified is called a *topological space*.

To be more precise, a topological space is an ordered pair (X, \mathcal{T}) consisting of a set X and a topology \mathcal{T} on X , but this specific mention of \mathcal{T} is usually omitted if no confusion will arise.

If X is a topological space with topology \mathcal{T} , we say that a subset U of X is an *open set* of X if U belongs to the collection \mathcal{T} . A subset $A \subset X$ is called a *closed set* if its complement A^c is open.

Definition 1.1.2. Let X be a set endowed with a topology \mathcal{T} . A collection \mathcal{B} of subsets of X is called a *basis* for the topology on X if an arbitrary $U \in \mathcal{T}$ can be represented as some union of sets from the collection \mathcal{B} .

Definition 1.1.3. A topological space is said to be *second-countable* if it has a countable basis.

Definition 1.1.4. Let X be a set endowed with a topology \mathcal{T} and let $x \in X$. An open set $U \in \mathcal{T}$ that contains x is said to be a *neighborhood* of x .

Definition 1.1.5. A topological space X is called a *Hausdorff space* if for each pair x, y of distinct points of X , there exist neighborhoods U_x and U_y of x and y , respectively, that are disjoint.

Definition 1.1.6. Let X, Y be topological spaces endowed with topologies $\mathcal{T}_X, \mathcal{T}_Y$, respectively, and let $f : X \rightarrow Y$. The map f is called *continuous* if for every open set $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V)$ is open in X , i.e., $f^{-1}(V) \in \mathcal{T}_X$.

Definition 1.1.7. Let X, Y be topological spaces endowed with topologies $\mathcal{T}_X, \mathcal{T}_Y$ respectively, and let $f : X \rightarrow Y$. The map f is called a *homeomorphism* if

1. f is bijective
2. f is a continuous map
3. f^{-1} is also a continuous map

Definition 1.1.8. A topological space X is said to be *connected* if it cannot be represented as a disjoint union of two non-empty open sets.

1.2 Differentiable Manifolds

Unless stated otherwise, definitions and results provided in the remaining subsections of this section are sourced either from [4] or from [11] (albeit we may adhere to a different notational style in some aspects).

Definition 1.2.1. An *n -dimensional topological manifold* M^n is a topological space that satisfies the following three conditions:

1. M^n is a Hausdorff space,

2. M^n is a second-countable topological space,
3. M^n is locally homeomorphic to an open subset of \mathbb{R}^n , i.e., for any point $p \in M^n$ there exists an open neighborhood which is homeomorphic to some open subset of \mathbb{R}^n .

Definition 1.2.2. A pair (U, ϕ) , where U is an open subset of some n -dimensional manifold M^n and ϕ is a homeomorphism between U and some open subset of \mathbb{R}^n , is called a *(local) coordinate chart* (or simply a *chart*).

Definition 1.2.3. Let M^n be an n -dimensional manifold and \mathcal{A} be an index set. A collection of charts $\{(U_\alpha, \phi_\alpha)_{\alpha \in \mathcal{A}}\}$ is called an *atlas* of M^n if

$$M^n = \bigcup_{\alpha \in \mathcal{A}} U_\alpha.$$

Definition 1.2.4. Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \rightarrow \mathbb{R}$. We say that f is of class C^k in U (or simply that $f \in C^k(U)$) if f is k -times differentiable in U . If $f : U \rightarrow \mathbb{R}^n$, then we say that $f \in C^k(U)$ if each component function $f_i = r_i \circ f$ is of class $C^k(U)$.

Definition 1.2.5. The vector space A over the field K is called an *algebra over K* or simply an *algebra* if A is equipped with a bilinear operation.

Let U be an open subset of \mathbb{R}^n and let $f, g \in C^\infty(U)$ and $\lambda \in \mathbb{R}$. Let p be in U . Then $C^\infty(U)$ endowed with addition and scalar multiplication defined as

- $(f + g)(p) = f(p) + g(p)$
- $(\lambda \cdot f)(p) = \lambda f(p)$

becomes a vector space. If we define the multiplication of f, g as

- $(f \cdot g)(p) = f(p) \cdot g(p)$

then $C^\infty(U)$ becomes an algebra, which will be called an *algebra of smooth functions* over U .

Definition 1.2.6. An atlas $\{(U_\alpha, \phi_\alpha)_{\alpha \in \mathcal{A}}\}$ of the manifold M^n is called *smooth* if for any $\alpha, \beta \in \mathcal{A}$, the map $\phi_\alpha \circ \phi_\beta^{-1}$ is smooth.

Definition 1.2.7. A smooth atlas $\mathcal{F} = \{(U_\alpha, \phi_\alpha)_{\alpha \in \mathcal{A}}\}$ of the manifold M^n is called a *smooth structure* if for each coordinate system (U, ϕ) such that $\phi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are smooth maps for any $\alpha \in \mathcal{A}$, then (U, ϕ) is also an element of \mathcal{F} .

Definition 1.2.8. An n -dimensional manifold M^n , equipped with a smooth structure \mathcal{F} is called a *smooth manifold*.

Analogously to the Euclidean case, one can define a smooth function on manifolds.

Definition 1.2.9. Let M^n be a manifold endowed with a smooth structure $\mathcal{F} = \{(U_\alpha, \phi_\alpha)_{\alpha \in \mathcal{A}}\}$. Also, let $U \subset M^n$ be U an open set. We say that $f : U \rightarrow \mathbb{R}$ is of class C^∞ or simply is a *smooth function*, if for any $\alpha \in \mathcal{A}$ the map $f \circ \phi_\alpha^{-1}$ will be a smooth function.

Let U be an open subset of M^n and let $f, g \in C^\infty(U)$ and $\lambda \in \mathbb{R}$. Let $p \in U$. Then $C^\infty(U)$ endowed with addition, scalar multiplication and multiplication defined as

- $(f + g)(p) = f(p) + g(p)$
- $(\lambda \cdot f)(p) = \lambda f(p)$
- $(f \cdot g)(p) = f(p) \cdot g(p)$

becomes an algebra.

Next, we introduce the notion of *diffeomorphism*.

Definition 1.2.10. Let M^n, N^m be smooth manifolds endowed with smooth structures $\{(U_\alpha, \phi_\alpha)_{\alpha \in \mathcal{A}}\}$ and $\{(V_\beta, \sigma_\beta)_{\beta \in \mathcal{B}}\}$, respectively. We say that $f : M^n \rightarrow N^m$ is of class $C^\infty(M^n, N^m)$ or simply is a *smooth map* if for any $\alpha \in \mathcal{A}$ and $\forall \beta \in \mathcal{B}$ the map $\sigma_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth. If f is bijective and its inverse is smooth as well then it is called a *diffeomorphism*.

1.3 Tangent Vectors and Differentials

The core idea of mathematical analysis revolves around the concept of *linear approximation*. This idea arises repeatedly in the study of the calculus of Euclidean spaces. For example, a parametrized curve can be approximated by its velocity vector or a surface by its tangent plane. In order to make sense of the calculus on manifolds, we first discuss the notion of a tangent vector in the Euclidean space and then extend it to smooth manifolds.

We begin by introducing the concept of a *geometric tangent vector*.

Definition 1.3.1. A *geometric tangent vector in the Euclidean space \mathbb{R}^n* (or equivalently *vector at a point p*) is a pair $(p; \vec{v})$, where $\vec{v} \in \mathbb{R}^n$. A vector at a point $(p; \vec{v})$ will be denoted as \mathbf{v}_p .

In the definition provided above, we treat \vec{v} as a vector at the point p . The vector space structure of \mathbb{R}^n induces the vector space structure in the set of geometric tangent vectors, which will be denoted as \mathbb{R}_p^n : for two geometric tangent vectors $\mathbf{v}_p = (p; \vec{v}), \mathbf{w}_p = (p; \vec{w})$ we define

$$\mathbf{v}_p + \mathbf{w}_p = (p; \vec{v} + \vec{w}), \quad a\mathbf{v}_p = (p; a\vec{v}), \quad a \in \mathbb{R}.$$

For the purpose of differential geometry, it is also useful to consider geometric tangent vectors from a slightly different angle. Any geometric tangent vector

\mathbf{v}_p induces the map $\mathbf{v}_p \triangleright : C^\infty(U) \rightarrow \mathbb{R}$, which is defined as a directional derivative in direction \vec{v} at the point p

$$\mathbf{v}_p \triangleright f := \left. \frac{d}{dt} (f(p + t\vec{v})) \right|_{t=0},$$

where U is an open subset of the vector space \mathbb{R}^n and $f \in C^\infty(U)$, $p \in U$.

It is easy to show that the directional derivative possesses two important properties: it is linear, and it satisfies the Leibniz rule:

- $\mathbf{v}_p \triangleright (f + \lambda g) = \mathbf{v}_p \triangleright f + \lambda \mathbf{v}_p \triangleright g$
- $\mathbf{v}_p \triangleright (fg) = f \mathbf{v}_p \triangleright (g) + g \mathbf{v}_p \triangleright (f)$

With this construction in mind, we make the following definition. If p is a point in the open set $U \subset \mathbb{R}^n$, a map $w : C^\infty(U) \rightarrow \mathbb{R}$ is called a *derivation at p* if it is linear over \mathbb{R} and satisfies the Leibniz rule:

$$w(fg) = f(p)w(g) + g(p)w(f),$$

where $f, g \in C^\infty(U)$. Now, let $T_p\mathbb{R}^n$ denote the set of all the derivations of $C^\infty(U)$ at point p . It is easy (almost trivial) to show that $T_p\mathbb{R}^n$ is a vector space under point-wise addition and scalar multiplication

$$(w_1 + w_2)(f) = w_1(f) + w_2(f), \quad (\lambda w_1)f = \lambda(w_1f),$$

where $w_1, w_2 \in T_p\mathbb{R}^n$ and $f \in C^\infty(U)$.

The most important (and perhaps somewhat surprising) fact about the vector space $T_p\mathbb{R}^n$ is that it is finite-dimensional and, in fact, isomorphic to the geometric tangent space \mathbb{R}_p^n constructed above.

The natural isomorphism between vector spaces $T_p\mathbb{R}^n$ and \mathbb{R}_p^n is also the reason why we define derivations over the set of smooth functions $C^\infty(U)$ but not over the set of functions that are differentiable k times $C^k(U)$, since for the latter there is no isomorphism between derivations over $C^k(U)$ and the tangent space \mathbb{R}_p^n [9].

The provided definition of a geometric tangent vector is given in the coordinate-free form. Suppose that $\{e_i \mid i \in \mathbb{N}, 1 \leq i \leq n\}$ is a standard (and orthonormal) basis of the Euclidean vector space \mathbb{R}^n . In this case, we can use the chain rule and obtain the following result:

$$\begin{aligned} \mathbf{v}_p \triangleright f &= \left. \frac{d}{dt} (f(p + t\vec{v})) \right|_{t=0} \\ &= \sum_{i=1}^n \left. \frac{d}{dt} (p^i + tv^i) \right|_{t=0} \left. \frac{\partial f}{\partial x^i} \right|_p \\ &= \sum_{i=1}^n v^i \left. \frac{\partial f}{\partial x^i} \right|_p, \end{aligned}$$

where v^i and p^i are i -th coordinates of the vectors \vec{v}, p respectively. From this it follows that an arbitrary vector of the space $T_p\mathbb{R}^n$ in coordinate form will look like

$$\sum_{i=1}^n a^i \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{defined by} \quad \left(\sum_{i=1}^n a^i \left. \frac{\partial}{\partial x^i} \right|_p \right) (f) = \sum_{i=1}^n a^i \left. \frac{\partial f}{\partial x^i} \right|_p$$

where $a^i \in \mathbb{R}$. Moreover, the set of derivations over the space of smooth functions $C^\infty(U)$ over the open subset $U \subset \mathbb{R}^n$:

$$\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$$

forms a basis of the vector space $T_p\mathbb{R}^n$, and now we are in a position to define

tangent vectors on manifolds.

Definition 1.3.2. Let M^n be a smooth manifold, and let p be a point of M^n . A linear map $v : C^\infty(M^n) \rightarrow \mathbb{R}$ is called a *derivation at p* if it satisfies

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M^n).$$

The set of all derivations of $C^\infty(M^n)$ at p , denoted by T_pM^n , is a vector space called the *tangent space to M^n at p* . An element of T_pM^n is called a *tangent vector at p* .

This definition may seem abstract, but in fact it is not. Indeed, pick an arbitrary point $p \in M^n$. Then there will be an (open) neighborhood U of p that is locally diffeomorphic to \mathbb{R}^n . For this reason, any smooth function $f \in C^\infty(U)$ can be viewed as a real smooth n -valued function. Indeed, let us choose the chart (U, φ) . Then $f \circ \varphi^{-1}$ will be a smooth function. Similarly, the derivations at the point p can be viewed as "locally" Euclidean derivations. Analogously to the Euclidean case, the following result holds:

Proposition 1.3.1. *If M^n is an n -dimensional smooth manifold, then for each $p \in M^n$, the tangent space T_pM^n is an n -dimensional vector space.*

To relate the abstract tangent space defined above on manifolds to geometric tangent spaces in the Euclidean space \mathbb{R}^n , we need to explore how smooth maps could affect tangent vectors. If we consider a smooth map between Euclidean spaces, then the total derivative of this map at a point, represented by its Jacobian matrix, would be the "best linear approximation" of the map in an infinitesimally small neighborhood. Unfortunately, the structure of smooth manifolds does not allow us to talk about linear transformations between them. Instead, it will be a linear map between tangent spaces.

Definition 1.3.3. If M^n, N^m are smooth manifolds of dimensions n, m respectively and $F : M^n \rightarrow N^m$ be a smooth map, then a map

$$dF_p : T_p M^n \rightarrow T_{F(p)} N^m$$

defined so that for every $v \in T_p M^n$ and $f \in C^\infty(N^m)$

$$dF_p(v)(f) = v \triangleright (f \circ F)$$

is called the *differential of F at p* .

Proposition 1.3.2. Let M^n, N^m, P^s be smooth manifolds of dimensions n, m, s , respectively, let $F : M^n \rightarrow N^m, G : N^m \rightarrow P^s$ be smooth maps, and let $p \in M^n$. Then

1. $dF_p : T_p M^n \rightarrow T_{F(p)} N^m$ is linear
2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p M^n \rightarrow T_{G \circ F(p)} P^s$
3. $d(Id_{M^n})_p = Id_{T_p M^n} : T_p M^n \rightarrow T_p M^n$
4. if F is a diffeomorphism, then $dF_p : T_p M^n \rightarrow T_{F(p)} N^m$ is an isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

The definition of a differential map may seem peculiar and abstract, but in fact it is quite useful and finds several important applications. For example, the proof of the Proposition 1.3.1 about the dimension of tangent space for a smooth manifold utilizes Proposition 1.3.2(4).

1.4 Computations in Coordinates

So far we have treated tangent vectors to a manifold in an abstract, coordinate free manner. Next, we will show how coordinate charts are used to relate the tangent space to a point on a manifold with the Euclidean tangent space. But before that, we need to address an important technical issue: the tangent space is defined in terms of smooth functions on the whole manifold while coordinate charts are in general defined on open subsets. The key idea, expressed in the following proposition, is that since operation of taking derivatives depends only on local properties of functions, properties in arbitrarily small neighborhoods of the point at which the derivative is being taken, then tangent vectors act locally.

Proposition 1.4.1. *Let M^n be a smooth manifold with the dimension $n \in \mathbb{N}$, $p \in M^n$, and $v \in T_p M^n$. If $f, g \in C^\infty(M^n)$ agree on some neighborhood of p , then $vf = vg$.*

The next proposition follows from Proposition 1.4.1 and shows that we can identify the tangent space to an open submanifold with the tangent space to the whole manifold.

Proposition 1.4.2. *Let M^n be a smooth manifold of dimension n , let $U \subseteq M^n$ be an open subset and let $\iota : U \rightarrow M^n$ be an inclusion map. For every $p \in U$, the differential $d\iota_p : T_p U \rightarrow T_p M^n$ is an isomorphism.*

Given a smooth manifold M^n of dimension n , smooth coordinate chart (U, φ) of M^n and a point $p \in U$, we get that φ is, in particular, a diffeomorphism from U to an open subset $\tilde{U} \subset \mathbb{R}^n$. If we combine Proposition 1.3.2 (4) and Proposition 1.4.2, we obtain that the map $d\varphi_p : T_p M^n \rightarrow T_{\varphi(p)} \mathbb{R}^n$ is an isomorphism.

As discussed above, the set of derivations $\frac{\partial}{\partial x^1}\Big|_{\varphi(p)}, \dots, \frac{\partial}{\partial x^n}\Big|_{\varphi(p)}$ forms a basis for the vector space $T_{\varphi(p)}\mathbb{R}^n$. Hence, the preimage of these vectors under the isomorphism $d\varphi_p$ forms a basis for T_pM^n . Next, we define the notation $\frac{\partial}{\partial x^i}\Big|_p$ as

$$\frac{\partial}{\partial x^i}\Big|_p = (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right) = d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right).$$

Passing $f \in C^\infty(U)$ to $\frac{\partial}{\partial x^i}\Big|_p$ we observe that $\frac{\partial}{\partial x^i}\Big|_p$ acts on f by

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}(f \circ \varphi^{-1}) = \frac{\partial \tilde{f}}{\partial x^i}(\tilde{p})$$

where $\tilde{f} = f \circ \varphi^{-1}$ is the coordinate representation of f , and $\tilde{p} = (p^1, \dots, p^n) = \varphi(p)$ is the coordinate representation of p . In other words, $\frac{\partial}{\partial x^i}\Big|_p$ is just the derivation that takes the i -th partial derivative of (the coordinate representation of) f at (the coordinate representation of) p .

Definition 1.4.1. Vectors $\frac{\partial}{\partial x^i}\Big|_p$ are called *coordinate vectors at p* associated with the given coordinate system (U, φ) .

Summarizing the above so far we get the following proposition

Proposition 1.4.3. *Let M^n be a smooth n -manifold, and let $p \in M^n$. Then T_pM^n is an n -dimensional vector space, and for any smooth chart (U, φ) containing p , the coordinate vectors $\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p$, where $x^i = r^i \circ \varphi$, form a basis for T_pM^n .*

From this we conclude that an arbitrary tangent vector $v \in T_pM^n$ can be written uniquely as a linear combination

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p.$$

Definition 1.4.2. The ordered basis $\left(\frac{\partial}{\partial x^i}\Big|_p\right)$ is called a *coordinate basis* for $T_p M^n$ and the numbers (v^1, \dots, v^n) are called the *components of v* with respect to the coordinate basis.

Given a vector v and coordinate chart (U, φ) , then components of v can be easily found from its action on the coordinate functions. Indeed, the components of v are given by $v^i = v(x^i)$, where $x^i = r^i \circ \varphi$, since

$$v(x^i) = \left(\sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\Big|_p\right)(x^i) = \sum_{j=1}^n v^j \frac{\partial x^i}{\partial x^j} = \sum_{j=1}^n v^j \delta_j^i = v^i.$$

In the case of differential map, we begin by considering the special case of a smooth map $F : U \rightarrow V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets of Euclidean spaces. For any $p \in U$, we will determine the matrix of $dF_p : T_p \mathbb{R}^n \rightarrow T_{F(p)} \mathbb{R}^m$ in terms of regular coordinate bases. Denoting the coordinates of a domain as (x^1, \dots, x^n) and the coordinates of a codomain as (y^1, \dots, y^m) , we use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$\begin{aligned} dF_p \left(\frac{\partial}{\partial x^i}\Big|_p \right) f &= \frac{\partial}{\partial x^i}\Big|_p (f \circ F) = \sum_{j=1}^m \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial F^j}{\partial x^i}(p) \\ &= \sum_{j=1}^m \left(\frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\Big|_{F(p)} \right) f. \end{aligned}$$

Hence

$$dF_p \left(\frac{\partial}{\partial x^i}\Big|_p \right) = \sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

If we consider a smooth map $F : M^n \rightarrow N^m$ between smooth manifolds and choose coordinate charts (U, φ) for M^n containing p and (V, ψ) for N^m containing $F(p)$, we obtain the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1} :$

$\psi(U \cap F^{-1}(V)) \rightarrow \psi(V)$. Let $\hat{p} = \varphi(p)$ denote the coordinate representation of p . Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \hat{F}$ and the computation above, we obtain

$$\begin{aligned}
dF_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) &= dF_p \left(d(\varphi^{-1})_{\hat{p}} \left(\frac{\partial}{\partial x^i} \Big|_{\hat{p}} \right) \right) \\
&= d(\psi^{-1})_{\hat{F}(\hat{p})} \left(d\hat{F}_{\hat{p}} \left(\frac{\partial}{\partial x^i} \Big|_{\hat{p}} \right) \right) \\
&= \sum_{j=1}^m d(\psi^{-1})_{\hat{F}(\hat{p})} \left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j} \Big|_{\hat{F}(\hat{p})} \right) \\
&= \sum_{j=1}^m \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j} \Big|_{\hat{F}(\hat{p})}.
\end{aligned}$$

1.5 Change of Basis

If we assume that (U, φ) and (V, ψ) are two smooth charts on a smooth n -dimensional manifold M^n and $x^i = r^i \circ \varphi, \tilde{x}^i = r^i \circ \psi$ are their respective coordinate functions, then the following question arises naturally: given the point $p \in U \cap V$, how are the two bases $\left(\frac{\partial}{\partial x^i} \Big|_p \right)$ and $\left(\frac{\partial}{\partial \tilde{x}^i} \Big|_p \right)$ related?

To answer this question, we will first introduce a shorthand notation for the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$ in the following form:

$$\psi \circ \varphi^{-1}(x) = (\tilde{x}^1(x), \dots, \tilde{x}^n(x)),$$

where $x = (x^1, \dots, x^n)$. In the expression $\tilde{x}^i(x)$ we consider \tilde{x}^i as a coordinate function and at the same time we think of x as a representation of a point, in this case, in $\varphi(U \cap V)$. From Proposition 1.3.2 it follows that the differential

$d(\psi \circ \varphi^{-1})_{\varphi(p)}$ can be written as

$$d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) = \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} (\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_{\psi(p)}.$$

From the definition of coordinate vectors, we get

$$\begin{aligned} \frac{\partial}{\partial x^i} \Big|_p &= d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) \\ &= d(\psi^{-1})_{\psi(p)} \circ d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) \\ &= d(\psi^{-1})_{\psi(p)} \left(\sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} (\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_{\psi(p)} \right) \\ &= \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} (\hat{p}) \frac{\partial}{\partial \tilde{x}^j} \Big|_p. \end{aligned}$$

Applying this result to the components of a vector $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p = \sum_{j=1}^n \tilde{v}^j \frac{\partial}{\partial \tilde{x}^j} \Big|_p$, we get that the components transform by the rule

$$\tilde{v}^j = \sum_{i=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} (\hat{p}) v^i.$$

2 Vector Fields and Riemannian geometry

2.1 Tangent Bundle, Vector Fields and Frames

In this and the following section, M^n will represent an n -dimensional smooth manifold. Also, going forward, when talking about manifolds, we will tacitly assume that they are smooth unless stated otherwise. The material of this and the following subsections of this section is based on [2], [4] and [11].

Definition 2.1.1. The set TM^n , defined as

$$TM^n := \bigcup_{p \in M^n} T_p M^n$$

is called a *tangent bundle* of M^n .

Definition 2.1.2. Given the tangent bundle TM^n , a *projection map* $\pi : TM^n \rightarrow M^n$ is a map that sends a vector from $T_p M^n$ to the point p , i.e., $\pi(p, v) = p$.

Proposition 2.1.1. *For any smooth n -dimensional manifold M^n , the tangent bundle TM^n has a natural topology and smooth structure that make it into a $2n$ -dimensional smooth manifold. With respect to this structure, the projection $TM^n \rightarrow M^n$ is smooth.*

Proposition 2.1.2. *If M^n can be covered by a single smooth chart, then TM^n is diffeomorphic to $M^n \times \mathbb{R}^n$.*

Definition 2.1.3. If M^n has a tangent bundle TM^n , then a *vector field* X is a (continuous) map $X : M^n \rightarrow TM^n$, with the property that

$$\pi \circ X = \text{Id}_{M^n}.$$

If this map is smooth, then X is called a *smooth* vector field. Going forward, when talking about smooth vector fields, we will omit the smooth part, and simply assume that the vector field is smooth, unless it is stated otherwise.

If we assume that $X : M^n \rightarrow TM^n$ is a vector field and (U, φ) is a smooth coordinate chart on M^n , we may write the value of the vector field X at any point $p \in U$ in terms of coordinate basis vectors ($x^i = r^i \circ \varphi$):

$$X_p := X(p) = X^i(p) \frac{\partial}{\partial x^i} \Big|_p,$$

i.e., vector fields define n functions $X^i : U \rightarrow \mathbb{R}$, which are called the *component functions* of X in the given chart. This definition leads to an important property of smooth vector fields:

Proposition 2.1.3. (*Smoothness Criterion for Vector Fields*). *Let $X : M^n \rightarrow TM^n$ be a vector field (not necessarily continuous). If (U, φ) is a smooth coordinate chart on M^n , then the restriction of X to U is smooth if and only if its component functions are smooth, with respect to this chart.*

Definition 2.1.4. An ordered k -tuple (X_1, \dots, X_k) of vector fields defined on some subset $A \subseteq M^n$ is said to be *linearly independent* if $(X_1|_p, \dots, X_k|_p)$ is a linearly independent k -tuple in T_pM^n for each $p \in A$. If this tuple spans T_pM^n at each $p \in A$, then it is said that this tuple spans the tangent bundle.

Definition 2.1.5. A *local frame* for M^n is an ordered n -tuple of vector fields (E_1, \dots, E_n) defined on an open subset $U \subseteq M^n$ that forms a basis for T_pM^n at each $p \in U$. If each of the vector fields in the tuple is smooth, then this tuple is called a *smooth frame*.

2.2 Riemannian Metrics

Definition 2.2.1. Given a smooth n -dimensional manifold M^n , a *Riemannian metric on M^n (or TM^n)* is a family $(\langle -, - \rangle_p)_{p \in M^n}$ of inner products on each tangent space $T_p M^n$, such that $\langle -, - \rangle_p$ depends smoothly on p , i.e., for every smooth chart (U, φ) , for every frame $(X_1, \dots, X^n$ on U), maps

$$p \mapsto \langle X_i(p), X_j(p) \rangle_p, \quad p \in U, \quad 1 \leq i, j \leq n,$$

are smooth. A smooth manifold with a Riemannian metric is called a *Riemannian manifold*.

Theorem 2.2.1. [3] *Every smooth manifold admits a Riemannian metric.*

2.3 Covariant derivative, Connections and Christoffel symbols

We will first introduce the notion of the covariant derivative for the Euclidean space \mathbb{R}^3 and then extend it to smooth manifolds. We will start by defining the *directional derivative*.

Definition 2.3.1. Given any two vector fields X and Y defined on some open subset $U \subseteq \mathbb{R}^3$, for every $p \in U$, the *directional derivative $D_X Y(p)$* of Y with respect to X is defined by

$$D_X Y(p) = \lim_{t \rightarrow 0} \frac{Y(p + tX(p)) - Y(p)}{t}.$$

It is easy to see that the directional derivative itself is a vector field. The next proposition provides some of the important properties of the directional derivative.

Proposition 2.3.1. *If X and Y are smooth vector fields from U to \mathbb{R}^3 on some open subset U of \mathbb{R}^3 , then their directional derivatives satisfy the following properties:*

$$\begin{aligned} D_{X_1+X_2}Y(p) &= D_{X_1}Y(p) + D_{X_2}Y(p), \\ D_{fX}Y(p) &= f(p)D_XY(p), \\ D_X(Y_1 + Y_2)(p) &= D_XY_1(p) + D_XY_2(p), \\ D_X(fY)(p) &= X[f](p)Y(p) + f(p)D_XY(p), \end{aligned}$$

for all $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(U)$ and all $f \in C^\infty(U)$.

In the Euclidean space, the tangent spaces at different points are canonically identified with \mathbb{R}^3 , so differentiating vector fields poses no difficulty. For a surface $M \subset \mathbb{R}^3$, however, tangent spaces T_pM and T_qM of two different points p and q generally speaking, are associated with two different planes, and the directional derivative $D_XY(p)$ may not lie in the same tangent space T_pM .

With this motivation in mind, we introduce the notion of a *covariant derivative*.

Definition 2.3.2. Let M be a (smooth) surface in \mathbb{R}^3 and let X, Y be two smooth vector fields defined on some open subset $U \subseteq \mathbb{R}^3$ such that $W \subseteq M$ for an open in M set W and $X(p), Y(p) \in T_pM$ for $p \in W$. Then for every $p \in W$ a *covariant derivative* $\nabla_XY(p)$ is defined as an orthogonal projection of a directional derivative $D_XY(p)$ onto the tangent (plane) space T_pM .

It can also be shown that this definition of a covariant derivative satisfies the properties of a directional derivative, that were provided in Proposition 2.3.1. Next we introduce the notion of a parallel vector along the vector field.

Definition 2.3.3. Let $c : [0, 1] \rightarrow M$ be a smooth curve on a surface M . A *vector field along c* is a map

$$Y : [0, 1] \rightarrow TM$$

such that $Y(t) \in T_{c(t)}M$ for all $t \in [0, 1]$. A vector field Y along c is said to be *parallel along c* if for all $t \in [0, 1]$

$$\nabla_{c'(t)}Y = 0.$$

With these definitions in mind, we are now able to describe the process of a *parallel transport* on a smooth surface $M \subset \mathbb{R}^3$. For the sake of simplicity, let's assume that M is also a connected set. Let's take two arbitrary points $p, q \in M$. Since M is connected, there exists a smooth curve $c : [0, 1] \rightarrow M$, such that $p = c(0), q = c(1)$. Then, for every tangent vector $Y_0 \in T_pM$, there exists one and only one parallel vector field Y along the curve c , with $Y(0) = Y_0$. By setting $Y_1 = Y(1)$, we obtain a linear map $\eta : T_pM \rightarrow T_qM$ defined as $Y_0 \mapsto Y_1$.

With the previous discussion we have managed to define parallel transform, which is a natural way of relating tangent spaces of a smooth surface in \mathbb{R}^3 . With this motivation in mind, we now define the notion of a connection for an arbitrary smooth manifold M^n .

Definition 2.3.4. [2] Let $\mathfrak{X}(M^n)$ be a module of vector fields on M^n over the module of smooth functions $C^\infty(M^n)$. A *connection* on M^n is an \mathbb{R} -bilinear map

$$\nabla : \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \rightarrow \mathfrak{X}(M^n),$$

which satisfies the following two conditions:

$$\nabla_{fX}Y = f\nabla_XY$$

$$\nabla_X(fY) = X[f]Y + f\nabla_XY,$$

for all $X, Y \in \mathfrak{X}(M^n)$ and all $f \in C^\infty(M^n)$, where $\nabla_XY := \nabla(X, Y)$. The vector field ∇_XY is called the *covariant derivative of Y with respect to X* .

It can easily be shown that Definition 2.3.2 satisfies the definition given above. The following propositions give some properties of a connection.

Proposition 2.3.2. *Every smooth manifold M^n possesses a connection.*

Proposition 2.3.3. [5] *Let M^n be a smooth manifold and let ∇ be a connection on M^n . For every open subset $U \subseteq M^n$, for every vector field $Y \in \mathfrak{X}(M^n)$, if $Y \equiv 0$ on U , then $\nabla_XY \equiv 0$ on U for all $X \in \mathfrak{X}(M^n)$.*

Proposition 2.3.4. [10] *For any two vector fields $X, Y \in \mathfrak{X}(M^n)$, if $X(p) = Y(p)$ for some $p \in M^n$, then*

$$(\nabla_XZ)(p) = (\nabla_YZ)(p) \quad \forall Z \in \mathfrak{X}(M^n).$$

Let's now consider some point $p \in U \subset M^n$, such that (U, x^1, \dots, x^n) is a smooth coordinate chart. The basis vectors of the tangent space T_pM^n in the local coordinate chart are: $\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p$. Let's define vector fields $\frac{\partial}{\partial x^i}$ on U as $\frac{\partial}{\partial x^i}(p) = \frac{\partial}{\partial x^i} \Big|_p$. Then the n -tuple of vector fields $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ is a local frame. We define *Christoffel symbols* in the following manner:

Definition 2.3.5. Let ∇ be a connection on a smooth manifold M^n . The

unique smooth functions Γ_{ij}^k on U , determined by

$$\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} \right) = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

are called the *Christoffel symbols* of the connection ∇ with respect to the coordinate chart (U, x^1, \dots, x^n) .

2.4 Parallel transport and Levi-Civita Connections

Definition 2.4.1. Given any smooth manifold M^n , a *smooth parametric curve* (for short, *curve*) on M^n is a smooth map $\gamma : I \rightarrow M^n$, where I is some open interval of \mathbb{R} . For a closed interval $[a, b] \subseteq \mathbb{R}$, a map $\gamma : [a, b] \rightarrow M^n$ is a *smooth curve from $p = \gamma(a)$ to $q = \gamma(b)$* if γ can be extended to a smooth curve $\tilde{\gamma} : (a - \varepsilon, b + \varepsilon) \rightarrow M^n$, for some $\varepsilon > 0$. Given any two points $p, q \in M^n$, a continuous map $\gamma : [a, b] \rightarrow M^n$ is a *piecewise smooth curve from p to q* if

1. There is a sequence

$$a = t_0 < t_1 < \dots < t_{k-1} < t_k = b$$

of numbers $t_i \in \mathbb{R}$, so that each map

$$\gamma_i = \gamma|_{[t_i, t_{i+1}]},$$

called a *curve segment*, is a smooth curve for $i = 0, \dots, k - 1$.

2. $\gamma(a) = p$ and $\gamma(b) = q$.

The set of all piecewise smooth curves from p to q is denoted by $\Omega(M^n; p, q)$, or briefly by $\Omega(p, q)$ (or even by Ω , when p and q are understood).

Definition 2.4.2. A *smooth vector field along the curve* γ is a smooth map $X : [a, b] \rightarrow TM^n$, such that

$$\pi(X(t)) = \gamma(t),$$

for all $t \in [a, b]$ ($X(t) \in T_{\gamma(t)}M$).

Proposition 2.4.1. Let ∇ be a connection on M^n and $\gamma : [a, b] \rightarrow M^n$ be a smooth curve in M^n . Then there exists a unique \mathbb{R} -linear map $\frac{D}{dt}$, defined on the vector space of smooth vector fields X along γ , which satisfies the following conditions.

1. For any smooth function $f : [a, b] \rightarrow \mathbb{R}$,

$$\frac{D(fX)}{dt} = \frac{df}{dt}X + f\frac{DX}{dt}.$$

2. If X is induced by a vector field $Z \in \mathfrak{X}(M)$, that is $X(t_0) = Z(\gamma(t_0))$ for all $t_0 \in [a, b]$, then

$$\frac{DX}{dt}(t_0) = (\nabla_{\gamma'(t_0)}Z)_{\gamma(t_0)}.$$

Definition 2.4.3. The operator $\frac{D}{dt}$ obtained from Proposition 2.4.1 is called the *covariant derivative along* γ and is also denoted by $\nabla_{\gamma'(t)}$.

Definition 2.4.4. Let ∇ be a connection on M^n . For every curve $\gamma : [a, b] \rightarrow M^n$ in M^n , a vector field X along γ is *parallel (along γ)* if

$$\frac{DX}{dt}(s) = 0 \quad \forall s \in [a, b].$$

Proposition 2.4.2. Let ∇ be a connection on M^n . For every C^1 curve $\gamma :$

$[a, b] \rightarrow M^n$ in M^n , for every $t \in [a, b]$ and every $v \in T_{\gamma(t)}M^n$, there is a unique parallel vector field X along γ such that $X(t) = v$.

Definition 2.4.5. Let ∇ be a connection on M^n . For every curve $\gamma : [a, b] \rightarrow M^n$ in M^n , for every $t \in [a, b]$, the *parallel transport from $\gamma(a)$ to $\gamma(t)$ along γ* is the linear map from $T_{\gamma(a)}M^n$ to $T_{\gamma(t)}M^n$ which associates to any $v \in T_{\gamma(a)}M^n$ the vector $X_v(t) \in T_{\gamma(t)}M^n$, where X_v is the unique parallel vector field along γ with $X_v(a) = v$.

Proposition 2.4.3. Let ∇ be a connection on M^n . For every C^1 curve $\gamma : [a, b] \rightarrow M^n$ in M^n , the parallel transport along γ defines for every $t \in [a, b]$ a linear isomorphism

$$P_\gamma : T_{\gamma(a)}M \rightarrow T_{\gamma(t)}M,$$

between the tangent spaces $T_{\gamma(a)}M^n$ and $T_{\gamma(t)}M^n$.

Given any two vector fields $Y, Z \in \mathfrak{X}(M)$, the smooth function $\langle Y, Z \rangle$ is defined as

$$\langle Y, Z \rangle(p) = \langle Y_p, Z_p \rangle_p,$$

for all $p \in M^n$.

Definition 2.4.6. Given any metric $\langle -, - \rangle$ on M^n , a connection ∇ on M^n is *compatible with the metric*, for short, a *metric connection*, if

$$X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M^n)$.

Definition 2.4.7. Given a metric $\langle -, - \rangle$ on M^n , a connection ∇ on M^n is said to be *torsion-free* if

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

for the vector fields $X, Y \in \mathfrak{X}(M^n)$. The vector field $[X, Y]$ is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \forall f \in C^\infty(M^n).$$

and is called a *Lie bracket* or a *commutator* of vector fields X and Y and is also a smooth vector field. We will discuss the properties of the Lie bracket more thoroughly in the next section.

Proposition 2.4.4. *Let M^n be any Riemannian manifold. Then there is a unique, metric, torsion-free connection ∇ on M^n .*

Definition 2.4.8. The connection obtained in Proposition 2.4.4 is called a Levi-Civita connection.

Proposition 2.4.5. *Let M^n be a Riemannian manifold and let ∇ be the Levi-Civita connection on M^n . Then ∇ is defined by the Koszul formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= X(\langle Y, Z \rangle) + Y(\langle X, Z \rangle) - Z(\langle X, Y \rangle) \\ &\quad - \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle + \langle Z, [Y, X] \rangle. \end{aligned}$$

Proposition 2.4.6. *Given any Riemannian manifold M^n and a metric connection ∇ on M^n , for every curve $\gamma : [a, b] \rightarrow M^n$ on M^n , if X and Y are two vector fields along γ , then*

$$\frac{d}{dt} \langle X(\gamma(t)), Y(\gamma(t)) \rangle = \left\langle \frac{DX}{dt}, Y(\gamma(t)) \right\rangle + \left\langle X(\gamma(t)), \frac{DY}{dt} \right\rangle.$$

2.5 Geodesic

Definition 2.5.1. Let (M^n, g) be a Riemannian manifold. Given any $p \in M^n$, for every $v \in T_p M^n$, the (*Riemannian*) *norm* of v , denoted $\|v\|$, is

defined by

$$\|v\| = \sqrt{g_p(v, v)}.$$

The Riemannian inner product $g_p(u, v)$ of two tangent vectors $u, v \in T_p M^n$ will also be denoted by $\langle u, v \rangle_p$, or simply $\langle u, v \rangle$.

Definition 2.5.2. Let (M^n, g) be a Riemannian manifold. Given any curve $\gamma \in \Omega(M^n; p, q)$, the *length* $L(\gamma)$ of γ is defined by

$$L(\gamma) = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|\gamma'(t)\| dt = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

Definition 2.5.3. Let M^n be any smooth manifold equipped with a connection ∇ . A curve

$$\gamma : I \rightarrow M$$

(where $I \subseteq \mathbb{R}$ is any interval) is a *geodesic* in and only if $\gamma'(t)$ is parallel along γ ; that is, iff

$$\frac{D\gamma'}{dt} = \nabla_{\gamma'} \gamma' = 0.$$

The notion of a geodesic only requires a connection on a manifold, generally geodesics can be defined on manifolds that are *not endowed with a Riemannian metric*. However, most useful properties of geodesics involve metric notions, and their proofs use the fact that the connection on the manifold is compatible with the metric and torsion-free. Therefore, from now on, we assume unless otherwise specified that our Riemannian manifold (M^n, g) is equipped with the Levi-Civita connection.

Proposition 2.5.1. *Let (M^n, g) be a Riemannian manifold. For every $p \in M^n$ and every $v \in T_p M^n$, there is some interval $(-\eta, \eta)$ and a unique geodesic $\gamma_v : (-\eta, \eta) \rightarrow M^n$, satisfying the conditions $\gamma_v(0) = p$, $\gamma'_v(0) = v$.*

3 Lie group of SE(3) and its Lie algebra

3.1 Lie group – Lie algebra correspondence

Unless stated otherwise, the definitions and results in this subsection are sourced from [4].

In the previous section we have introduced the notion of a *Lie bracket* as a vector field, obtained from the combination of two other vector fields on a manifold M^n . It can be easily shown that the previously defined Lie bracket is bilinear, antisymmetric and satisfies so-called *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

For all $X, Y, Z \in \mathfrak{X}(M^n)$. With this in mind, we introduce a more general structure that is called a *Lie Algebra*.

Definition 3.1.1. A *Lie algebra* (over \mathbb{R}) is a real vector space \mathfrak{g} endowed with a map called the *Lie bracket* (or the *commutator*)

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g},$$

usually denoted by $(X, Y) \mapsto [X, Y]$, that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

(i) **BILINEARITY:** For $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(ii) ANTISYMMETRY:

$$[X, Y] = -[Y, X].$$

(iii) JACOBI IDENTITY:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Definition 3.1.2. If \mathfrak{g} is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a *Lie subalgebra* of \mathfrak{g} if it is closed under the Lie bracket. In this case \mathfrak{h} is itself a Lie algebra with the restriction of the same bracket.

The Lie bracket of two elements of a Lie algebra is also an element of the Lie algebra and can be expressed as a linear combination of the basis vectors. Thus, the structure of a Lie algebra is fully determined by the behavior of a Lie bracket on basis vectors. This naturally induces the notion of *structure constants* of a Lie algebra.

Definition 3.1.3. [12] Let \mathfrak{g} be a Lie algebra and let $\{L_1, L_2, \dots\}$ be its basis. Then coefficients C_{ij}^k obtained from acting Lie bracket onto the basis vectors

$$[L_i, L_j] = \sum_k C_{ij}^k L_k.$$

are called *structure constants* of the Lie algebra \mathfrak{g} .

Next, we introduce seemingly distinct, yet tightly related to Lie algebra structure – a *Lie group*.

Definition 3.1.4. A *Lie group* is a smooth manifold M^n that is also a group, with the property that the multiplication map $m : M^n \times M^n \rightarrow M^n$

and inversion map $i : M^n \rightarrow M^n$, given by

$$m(g, h) = gh, \quad i(g) = g^{-1},$$

are both smooth. A Lie group is, in particular, a *topological group* (a topological space with a group structure such that the multiplication and inversion maps are continuous).

At first glance, Lie groups and Lie algebras are completely different and unrelated structures. But in reality, they are not unrelated. For every Lie group, there exists an associated Lie algebra. In order to explain how a Lie group induces a Lie algebra, we first introduce the notion of a *left translation*.

Definition 3.1.5. If G is a Lie group, any element $g \in G$ defines a map

$$L_g : G \rightarrow G,$$

called *left translation* defined as

$$L_g(h) = gh$$

It can be easily shown that the left translation is a diffeomorphism [4]. Then we introduce the notion of a *left-invariant vector field*.

Definition 3.1.6. Let G be a Lie group. A vector field X on G is said to be *left-invariant* if it is invariant under all left translations, in the sense that it is L_g -related to itself $\forall g \in G$. More explicitly, this means

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}, \quad \forall g, g' \in G.$$

For the left-invariant fields, the following proposition holds.

Proposition 3.1.1. *Let G be a Lie group, and suppose that X, Y are left-invariant vector fields on G . Then $[X, Y]$ is also a left-invariant vector field.*

In fact, if G is a Lie group, the set of all smooth left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$ and therefore is also a Lie algebra and is denoted as $\text{Lie}(G)$.

An important property of $\text{Lie}(G)$ is that its dimension equals the dimension of a manifold G and that the tangent space $T_e G$ at the identity element uniquely defines the Lie algebra of a Lie group. This result is formulated in the following theorem.

Theorem 3.1.1. *Let G be a Lie group. The map $\epsilon : \text{Lie}(G) \rightarrow T_e G$, given by $\epsilon(X) = X_e$, is a vector space isomorphism. Thus, $\text{Lie}(G)$ is a finite-dimensional vector space, with dimension equal to the dimension of G .*

The following result was proved by Cartan in 1930 and states that a closed subgroup of a Lie group is also a Lie group.

Theorem 3.1.2. *(Closed Subgroup Theorem) [1] Suppose that G is a Lie group and $H \subset G$ is a subgroup that is also a (topologically) closed subset of G . Then H is an embedded Lie subgroup.*

3.2 Matrix Lie group and Exponential map

The notion of the exponential map and related results are drawn from [2]. The remaining material in this subsection is mostly based on [12] (unless cited otherwise).

An important subclass of Lie groups is the so-called matrix Lie groups, the class of all closed subgroups of the general linear group of matrices. At first glance, it may not be clear how an arbitrary closed matrix group is a Lie group, or even how it induces a smooth structure. This property of matrix subgroups of $\text{GL}(n, \mathbb{R})$ follows from the Closed Subgroup Theorem (3.1.2) and from the fact that the general linear group is also a Lie group (see Example 1.27 [4]).

Before explaining how an arbitrary closed matrix subgroup of $\text{GL}(n, \mathbb{R})$ defines its Lie algebra, we need to introduce the notion of an *exponential map*, which provides a key link between a matrix Lie group and its Lie algebra.

Definition 3.2.1. Given an $n \times n$ (real or complex) matrix $A = (a_{ij})$, the exponential map $A \mapsto e^A$ is defined as an infinite series

$$e^A := I + \sum_{p=1}^{\infty} \frac{A^p}{p!} = \sum_{p=0}^{\infty} \frac{A^p}{p!}.$$

It can be shown that the exponential map is well-defined (see Proposition 2.1 [2]). The following proposition provides the properties of an exponential map.

Proposition 3.2.1. *Let $X, Y \in \text{Mat}_n(\mathbb{R})$, let $A \in \text{GL}_n(\mathbb{R})$ and let I be the identity matrix of $\text{GL}_n(\mathbb{R})$. Then the exponential map has the following properties:*

- | | |
|---|---|
| 1. $e^X e^{-X} = I$ | 4. $e^{AXA^{-1}} = Ae^X A^{-1}$ |
| 2. $e^{X^T} = (e^X)^T$ | 5. $\frac{de^{tX}}{dt} = Xe^{tX}, t \in \mathbb{R}$ |
| 3. $XY = YX \implies e^X e^Y = e^{X+Y}$ | 6. $ e^X = e^{\text{tr}(X)}$ |

where $\text{tr}(X)$ is the sum of diagonal elements of the matrix X .

The following crucial result relates an arbitrary matrix Lie group to its Lie algebra and was originally derived by von Neumann in 1929.

Theorem 3.2.1. [8] *A (topologically) closed subgroup G of $\text{GL}(n, \mathbb{R})$ is a matrix Lie group. Furthermore, the set \mathfrak{g} defined as*

$$\mathfrak{g} = \{X \in \text{Mat}(n, \mathbb{R}) \mid e^{tX} \in G \quad \forall t \in \mathbb{R}\}$$

is a vector space that is isomorphic to the tangent space $T_I G$ at the identity matrix I , and \mathfrak{g} forms a Lie algebra of G under the commutator $[-, -]$ defined as

$$[A, B] = AB - BA \quad \forall A, B \in \text{Mat}(n, \mathbb{R}).$$

On a matrix Lie group, a left-invariant vector field X is obtained by setting

$$X(A) = \hat{T}(A) = AT$$

where A is an element of a Lie group, T belongs to a corresponding Lie algebra and the notation \hat{T} indicates that the vector field is obtained by left-translating the Lie algebra element T . Note that by construction such left-invariant vector fields is isomorphic to a Lie algebra of a matrix group.

Let us now consider a Lie algebra \mathfrak{g} of a matrix Lie group G . If L_1, L_2, \dots, L_n is a basis for \mathfrak{g} , the set of the left-invariant vector fields $\{\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n\}$ is a basis of the space of the left-invariant vector fields. In addition, we have

$$[\hat{L}_i, \hat{L}_j] = \widehat{[L_i, L_j]} = \sum_{k=1}^n C_{ij}^k \hat{L}_k.$$

Furthermore, because at any point $A \in G$ the vectors $\hat{L}_1(A), \dots, \hat{L}_n(A)$ form a basis of the tangent space at that point, any vector field X can be expressed as

$$X = \sum_{i=1}^n X^i \hat{L}_i.$$

where the coefficients X^i vary over the manifold – if they are constant then X is left-invariant.

3.3 The Lie group $SE(3)$ and corresponding Lie algebra

In the remaining sections of this work we will provide an example of a metric suitable for trajectory optimization and describe the geometry of Special Euclidean group endowed with this metric. The definitions, related results and remarks in the remaining sections follow [12].

Consider a rigid body moving in free space, which is represented as a Euclidean vector space \mathbb{R}^3 with a fixed orthonormal reference frame $F = \{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$. At each moment of time t , the position and orientation of the rigid body are represented as an orthonormal frame $M(t) = \{O'(t), \vec{v}_1(t), \vec{v}_2(t), \vec{v}_3(t)\}$, as depicted in Figure (1). Let us choose the frame F such that $F = M(0)$. Then the configuration of a rigid body can be represented as a combination of two transformations: rotation of the rigid body, which is described by the rotation matrix $R \in SO(3)$, and displacement from point O to point O' described by a vector $d \in \mathbb{R}^3$.

Proposition 3.3.1. [7] *The set of matrices*

$$SE(n) = \left\{ \left(\begin{array}{c|c} R & d \\ \hline 0 & 1 \end{array} \right) \middle| R \in SO(n), d \in \mathbb{R}^n \right\}$$

is a group under the standard matrix multiplication.

Definition 3.3.1. The group $\text{SE}(n)$ endowed with standard matrix multiplication is called a *Special Euclidean group of dimension n* or simply a *Special Euclidean group*, if its dimension is clear from the context.

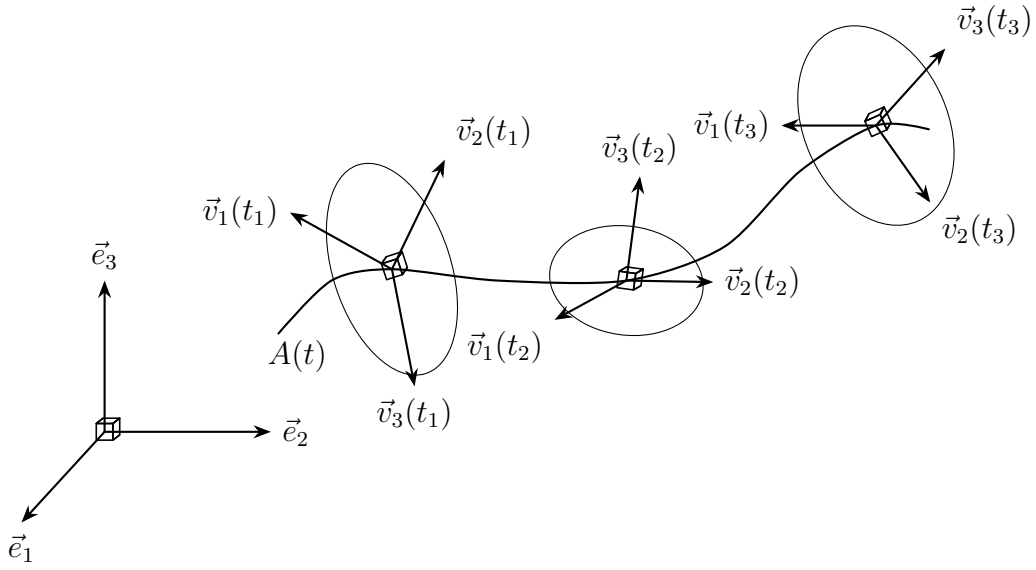


Figure 1: The inertial frame, and the moving frame attached to a rigid body.

It can be easily shown that $\text{SE}(n)$ is closed and hence, by the Closed Subgroup Theorem 3.1.2 it is a Lie group. The associated Lie algebra of $\text{SE}(n)$ is denoted by $\mathfrak{se}(n)$. For our discussion, we are interested in the specific case of the special Euclidean group of dimension three – $\text{SE}(3)$. The next proposition provides the structure of the Lie algebra $\mathfrak{se}(3)$ associated with $\text{SE}(3)$.

Proposition 3.3.2. [7] *The associated Lie algebra $\mathfrak{se}(3)$ of a Lie group $\text{SE}(3)$ is given by the set*

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix} \mid \Omega \in \text{Mat}(3, \mathbb{R}), v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}.$$

Furthermore, the set of matrices $\{L_1, L_2, L_3, L_4, L_5, L_6\}$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$L_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

forms a basis of $\mathfrak{se}(3)$.

Any 3×3 skew-symmetric matrix Ω is uniquely identified with a vector $\omega \in \mathbb{R}^3$ so that for any vector $x \in \mathbb{R}^3$, $\Omega x = \omega \times x$, where \times is the vector cross product. Hence each element $S \in \mathfrak{se}(3)$ is identified with a vector pair $\{\omega, v\}$.

Given a curve $A(t) : [-a, a] \rightarrow \text{SE}(3)$, an element $S(t)$ of the Lie algebra $\mathfrak{se}(3)$ can be associated to the tangent vector $\dot{A}(t)$ at an arbitrary time t by

$$S(t) = A^{-1}(t)\dot{A}(t) = \begin{bmatrix} R^T \dot{R} & R^T \dot{d} \\ 0 & 0 \end{bmatrix}.$$

A curve on $\text{SE}(3)$ physically represents a motion of the rigid body. If $\{\omega(t), v(t)\}$ is the vector pair corresponding to $S(t)$, then ω physically corresponds to the angular velocity of the rigid body while v is the linear velocity of the origin O' of the frame $\{M\}$, both expressed in the frame $\{M\}$.

The matrices L_1, L_2 , and L_3 from the Proposition 3.3.2 represent instantaneous rotations around, and L_4, L_5 , and L_6 instantaneous translations along, the Cartesian axes x, y , and z , respectively. The components of an element $S \in \mathfrak{se}(3)$ in this basis are given precisely by the velocity vector pair, $\{\omega, v\}$.

The Lie bracket of two elements $S_1, S_2 \in \mathfrak{se}(3)$ is defined by

$$[S_1, S_2] = S_1 S_2 - S_2 S_1.$$

It can be easily verified that if $\{\omega_1, v_1\}$ and $\{\omega_2, v_2\}$ are vector pairs corresponding to S_1 and S_2 , the vector pair $\{\omega, v\}$ corresponding to their Lie bracket $[S_1, S_2]$ is given by

$$\{\omega, v\} = \{\omega_1 \times \omega_2; \omega_1 \times v_2 + v_1 \times \omega_2\}.$$

3.4 The Choice of Metric

By using the notion of left translation, an inner product on the tangent space at the identity element of an arbitrary Lie group can be extended to a Riemannian metric. Indeed, let us define the inner product of $T_1, T_2 \in \mathfrak{se}(3)$ as

$$\langle T_1, T_2 \rangle_I = t_1^T W t_2, \tag{1}$$

where t_1 and t_2 are the 6×1 vectors of components of T_1 and T_2 with respect to some fixed basis, and W is a positive definite matrix. Given tangent vectors

V_1 and V_2 at an arbitrary point $A \in \text{SE}(3)$, the inner product can be extended to the tangent space $T_A \text{SE}(3)$ by

$$\langle V_1, V_2 \rangle_A = \langle A^{-1}V_1, A^{-1}V_2 \rangle_I.$$

A metric constructed in such a way is said to be *left-invariant*. In a similar fashion, a *right-invariant* metric can be defined.

To understand the physical meaning of left invariance, consider two motions $A_1(t)$ and $A_2(t)$ of a rigid body, both passing through a point A at time $t = t_0$, with velocity fields $V_1 = (dA_1/dt)$ and $V_2 = (dA_2/dt)$. If the inertial reference frame is displaced by $C \in \text{SE}(3)$, the motions in the new frame become $\tilde{A}_1(t) = CA_1(t)$ and $\tilde{A}_2(t) = CA_2(t)$, with velocity vector fields $\tilde{V}_1 = CV_1$ and $\tilde{V}_2 = CV_2$. This yields

$$\langle \tilde{V}_1, \tilde{V}_2 \rangle_{CA} = \langle A^{-1}C^{-1}\tilde{V}_1, A^{-1}C^{-1}\tilde{V}_2 \rangle_I = \langle V_1, V_2 \rangle_A,$$

showing that the metric value is unaffected by the change of inertial frame.

A desirable property of a planning method for generating smooth trajectories is that the resulting trajectories remain invariant under the change of the reference frame. It is also preferable that the planning method minimizes a physically meaningful function. A good candidate would be a metric that is both left- and right-invariant and therefore independent of the choice of a reference frame. The only problem is that $\text{SE}(3)$ does not admit such metrics. Because of this, we will focus on the left-invariant metric that is independent of the choice of the inertial reference frame, but is not independent of the choice of the body-fixed reference frame.

One such metric can be obtained by considering the dynamic properties of

the rigid body. The kinetic energy of a rigid body is a scalar quantity that does not depend on the choice of the inertial reference frame, and therefore induces a left-invariant metric on $SE(3)$. For this metric, the matrix W from (1) represents the inertia matrix, and $\frac{1}{2}\langle V, V \rangle$ corresponds to the kinetic energy of the rigid body moving with a velocity V . When the body-fixed frame is placed at the centroid and aligned with the principal axes, then

$$W = \begin{bmatrix} H & 0 \\ 0 & mI \end{bmatrix}$$

where m is the mass of the rigid body and H is

$$H = \begin{bmatrix} H_{xx} & 0 & 0 \\ 0 & H_{yy} & 0 \\ 0 & 0 & H_{zz} \end{bmatrix}$$

where H_{xx} , H_{yy} , and H_{zz} denote the moments of inertia about the x , y , and z axes respectively. The vector pair $\{\omega, v\}$, associated with the vector V , represents the instantaneous twist associated with the motion, expressed in the body-fixed reference frame. The norm of the vector V therefore can be expressed in the following form

$$\langle V, V \rangle = \omega^T H \omega + m v^T v.$$

If the body-fixed frame $\{M\}$ is displaced by the matrix

$$C = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

to a new frame $\{M\}_C$, then the kinetic energy does not change. From this it

follows that the matrix W_C defining the energy metric for the new description of the motion of the rigid body will take the following form

$$W_C = \begin{bmatrix} R^T H R - m R^T D^2 R & -m R^T D R \\ m R^T D R & m I \end{bmatrix}$$

where D is the skew-symmetric matrix corresponding to the vector d . This is therefore the most general form of the inertia matrix.

3.5 Connection and the Geodesic

In this final section we will derive the Riemannian connection that corresponds to the metric (1), and provide necessary and sufficient conditions for a curve to be a geodesic on $SE(3)$. Finding a geodesic yields the optimal trajectory for a respective metric, which takes into consideration starting and final positions of a rigid body, as well as its orientation in space. This is particularly useful in the field of robotics, since this type of minimization reduces energy consumption and lowers mechanical wear, which prolongs the lifespan of a robotic system.

We start by noting that by using the Koszul formula from Proposition 2.4.5 the following result follows immediately

Proposition 3.5.1. *If ∇ is the Riemannian connection compatible with a left-invariant metric described by a matrix $W = [w_{ij}]$ from (1), then the Christoffel symbols for the basis \hat{L}_i are given by*

$$\Gamma_{ji}^k = \frac{1}{2} \sum_m w_{km}^{-1} (C_{ij}^s w_{sm} + C_{mj}^s w_{si} + C_{mi}^s w_{sj}), \quad (2)$$

where C_{ij}^k are the structure constants of the Lie algebra and $w_{km}^{-1} = (W^{-1})_{km}$.

For the sake of simplicity, going forward we will use the Einstein summation notation (see [4]). According to (3.2), any vector field on SE(3) can be expressed as a linear combination of left-invariant vector fields. If $X = \sum_{i=1}^6 X^i \hat{L}_i$ and $Y = \sum_{i=1}^6 Y^i \hat{L}_i$ are any two vector fields, then

$$\begin{aligned}\nabla_X Y &= \nabla_{X^j \hat{L}_j} Y^i \hat{L}_i \\ &= \frac{dY^i}{dt} \hat{L}_i + X^j Y^j \nabla_{\hat{L}_i} \hat{L}_j \\ &= \frac{dY^i}{dt} \hat{L}_i + X^j Y^j \Gamma_{ji}^k \hat{L}_k.\end{aligned}$$

where d/dt is the derivative along the integral curve of X and Γ_{ji}^k are obtained from (2). In order to derive expressions for the Riemannian connection directly from Koszul formula, instead of computing Γ_{ji}^k we first prove the following result for the Lie group SE(3).

Proposition 3.5.2. *Let $X = X^i \hat{L}_i$, $Y = Y^i \hat{L}_i$, and $Z = Z^i \hat{L}_i$ be three arbitrary vector fields and let the corresponding vector pairs be $\{\omega_x, v_x\}$, $\{\omega_y, v_y\}$, and $\{\omega_z, v_z\}$, respectively. If ∇ is the Riemannian connection corresponding to a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$, then*

$$\begin{aligned}\langle Z, \nabla_X Y \rangle &= \langle Z, X(Y^i) \hat{L}_i \rangle \\ &+ \frac{1}{2} \left[\langle \{(\omega_z \times \omega_y), (\omega_z \times v_y + v_z \times \omega_y)\}, \{\omega_x, v_x\} \rangle \right. \\ &+ \langle \{(\omega_z \times \omega_x), (\omega_z \times v_x + v_z \times \omega_x)\}, \{\omega_y, v_y\} \rangle \\ &\left. + \langle \{(\omega_x \times \omega_y), (\omega_x \times v_y + v_x \times \omega_y)\}, \{\omega_z, v_z\} \rangle \right].\end{aligned}$$

Proof. The result follows directly from Koszul formula 2.4.5. The Lie bracket of any two vector fields is

$$[X, Y] = X^i Y^j [\hat{L}_i, \hat{L}_j] + X(Y^i) \hat{L}_i - Y(X^i) \hat{L}_i.$$

Rewritten in terms of the pairs $\{\omega_x, v_x\}$ and $\{\omega_y, v_y\}$, the first term becomes

$$X^i Y^j [\hat{L}_i, \hat{L}_j] = \{\omega_x \times \omega_y, \omega_x \times v_y + v_x \times \omega_y\}.$$

Thus

$$\begin{aligned} \langle Z, [X, Y] \rangle &= \langle Z, X(Y^i \hat{L}_i) \rangle - \langle Z, Y(X^i \hat{L}_i) \rangle \\ &+ \langle \{\omega_z, v_z\}, \{(\omega_x \times \omega_y), (\omega_x \times v_y + v_x \times \omega_y)\} \rangle. \end{aligned}$$

Furthermore, if W_{ij} forms the matrix W from (1), then

$$\begin{aligned} X \langle Y, Z \rangle &= X(Y^i W_{ij} Z^j) \\ &= X(Y^i) W_{ij} Z^j + Y^i W_{ij} X(Z^j) \\ &= \langle X(Y^i) \hat{L}_i, Z \rangle + \langle Y, X(Z^i) \hat{L}_i \rangle. \end{aligned}$$

Expanding all terms in the Koszul formula in a similar manner yields the desired result. \square

The next result provides the structure of a Riemannian connection on $SE(3)$ for the metric (1).

Theorem 3.5.1. *Let $X = X^i \hat{L}_i$ and $Y = Y^i \hat{L}_i$ be two arbitrary vector fields in $SE(3)$. If ∇ is the Riemannian connection corresponding to the Riemannian metric (1), then*

$$\begin{aligned} \nabla_X Y &= \left\{ \frac{d\omega_y}{dt} + \frac{1}{2} \left[(\omega_x \times \omega_y) + H^{-1}(\omega_x \times (H\omega_y)) \right. \right. \\ &\quad \left. \left. + H^{-1}(\omega_y \times (H\omega_x)) \right], \frac{dv_y}{dt} + \omega_x \times v_y \right\}. \end{aligned} \quad (3)$$

where d/dt is the derivative along the integral curve of X . The translational component of the expression $\nabla_X Y$ is independent of the choice of matrix H

and thus independent of the choice of the metric on $SO(3)$.

Proof. By using Proposition 3.5.2 we compute the right-hand side of Koszul formula 2.4.5 with the metric from (1):

$$\begin{aligned}
\langle Z, \nabla_X Y \rangle &= \langle Z, X(Y^i) \hat{L}_i \rangle \\
&+ \frac{1}{2} \left[(\omega_z \times \omega_y) \cdot (H\omega_x) + m(\omega_z \times v_y + v_z \times \omega_y) \cdot v_x \right. \\
&+ (\omega_z \times \omega_x) \cdot (H\omega_y) + m(\omega_z \times v_x + v_z \times \omega_x) \cdot v_y \\
&+ \left. (\omega_x \times \omega_y) \cdot (H\omega_z) + m(\omega_x \times v_y + v_x \times \omega_y) \cdot v_z \right] \\
&= \langle Z, X(Y^i) \hat{L}_i \rangle \\
&+ \frac{1}{2} \left[2m(\omega_x \times v_y) \cdot v_z - ((H\omega_x) \times \omega_y) \cdot \omega_z \right. \\
&+ \left. (\omega_x \times (H\omega_y)) \cdot \omega_z + (\omega_x \times \omega_y) \cdot (H\omega_z) \right] \\
&= \langle Z, X(Y^i) \hat{L}_i \rangle \\
&+ \frac{1}{2} \left[2m(\omega_x \times v_y) \cdot v_z - (H^{-1}((H\omega_x) \times \omega_y)) \cdot (H\omega_z) \right. \\
&+ \left. (H^{-1}(\omega_x \times (H\omega_y))) \cdot (H\omega_z) + (\omega_x \times \omega_y) \cdot (H\omega_z) \right] \\
&= \langle Z, X(Y^i) \hat{L}_i \rangle \\
&+ \left\langle Z, \left\{ \frac{1}{2} \left[(\omega_x \times \omega_y) + H^{-1}(\omega_x \times (H\omega_y)) \right. \right. \right. \\
&+ \left. \left. \left. H^{-1}(\omega_y \times (H\omega_x)) \right] \right\}, \omega_x \times v_y \right\rangle.
\end{aligned}$$

Since the above is true for an arbitrary Z , this proves the theorem. \square

Now we are in the position to provide necessary and sufficient conditions for a curve to be a geodesic with regard to the metric (1).

Proposition 3.5.3. *A curve $A(t)$ on $SE(3)$ equipped with the metric (1) is*

a geodesic if and only if the vector pair $\{\omega, v\}$ corresponding to the velocity vector field $V = \frac{dA}{dt}$ satisfies the equations

$$\begin{aligned}\frac{d\omega}{dt} &= -H^{-1}(\omega \times (H\omega)), \\ \frac{dv}{dt} &= -\omega \times v.\end{aligned}\tag{4}$$

The second equation in (4) can be simplified to

$$\ddot{d} = 0.$$

Proof. A curve $A(t)$ is a geodesic if and only if $\nabla_V V = 0$ is satisfied. Substituting for $\nabla_V V$ from Proposition 3.5.2, and letting

$$\{\omega_x, v_x\} = \{\omega_y, v_y\} = \{\omega, v\},$$

we get (4). The second equation in (4) can be written as

$$\dot{v} + \omega \times v = 0.$$

By writing $\Omega = R^T \dot{R}$ and $v = R^T \dot{d}$ and using the identity

$$\dot{R}^T = -R^T \dot{R} R^T,$$

we obtain

$$\begin{aligned}\dot{v} + \omega \times v &= \dot{v} + \Omega v \\ &= \left(\dot{R}^T \dot{d} + R^T \ddot{d} \right) + R^T \dot{R} R^T \dot{d} \\ &= R^T \ddot{d} = 0,\end{aligned}$$

which proves $\ddot{d} = 0$. □

Conclusion

This thesis presented a mathematical framework required to formulate the problem of smooth motion trajectory generation in three-dimensional space. For this purpose the concepts of point-set topology, smooth manifolds and Riemannian geometry were introduced, and then used to formulate the trajectory generation problem on the Lie group $SE(3)$, equipped with a left-invariant Riemannian metric derived from the kinetic energy of the rigid body.

As a result, we have described the geometry of $SE(3)$ equipped with a left-invariant Riemannian metric, that was mentioned above. The formula for computing corresponding Christoffel symbols for this specific metric was provided, as was the corresponding Riemannian connection described. In addition, the necessary and sufficient conditions for a curve to be a geodesic with regard to the mentioned metric was provided.

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