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FUNCTIONAL ANALYSIS AND
THEORY OF SUMMABILITY

Matemaatika- ja mehaanika-alaseid
töid

TARTU  1993

TARTU ÜLIKOOLI TOIMETISED
ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS
ALUSTATUD 1893. a. VIHIK 960

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ON THE SOLUTIONS OF NONLINEAR SYSTEMS OF
ELLIPTIC EQUATIONS WITH GROUP SYMMETRIES

Z. Balanov and V. Ajevski

1. Introduction

Let us consider a linear boundary-value elliptic problem

$$Lu(x) \equiv \sum_{|\sigma| \leq 2m} a_\sigma(x) D^\sigma u(x) = 0, \quad x \in \Omega, \quad (1)$$

$$B_i u(x) \equiv \sum_{|\sigma| \leq m_i} b_{i\sigma}(x) D^\sigma u(x) = 0, \quad x \in \partial\Omega,$$

where $1 \leq i \leq m$, $m_i < 2m$, $\bar{\Omega}$ is the closure of the boundary region Ω in \mathbf{R}^n with C^∞ -smooth boundary. L is an elliptic operator with C^∞ -smooth coefficients $a_\sigma : \bar{\Omega} \rightarrow \text{Hom}(\mathbf{R}^q, \mathbf{R}^q)$, where the Euclidean space \mathbf{R}^q is the range of values of the function $u(\cdot)$. Let $\{B_i\}$ be a family of differential operators with C^∞ -smooth coefficients $b_{i\sigma} : \partial\Omega \rightarrow \text{Hom}(\mathbf{R}^q, \mathbf{R}^q)$ and let $|\sigma| = \sigma_1 + \dots + \sigma_n$ be the norm of the multi-index $\sigma = (\sigma_1, \dots, \sigma_n)$ (see [1,8-10]). Let us suppose that our boundary conditions $B = \{B_i\}$ are "well posed" (see [9], p.43); this means that the pair (L, B) defines a Fredholm operator in the corresponding Hölder space. We'll denote the index of (L, B) by ν and suppose that $\nu > 0$. Then the set of solutions of problem (1) is a linear subspace whose dimension is positive. In particular, this means that for any $r \geq 0$ there exists a solution u_0 of problem (1) such that $\|u_0\| = r$.

Now let us consider a nonlinear elliptic boundary-value problem which is associated with problem (1):

$$\begin{aligned} Lu(x) &= \varphi(x, u(x), Du(x), \dots, D^{2m-1}u(x)), & x \in \Omega, \\ B_i u(x) &= 0, & x \in \partial\Omega \end{aligned} \quad (2)$$

where φ is a smooth function.

The goal of our paper is to show that some symmetry conditions of $a_\sigma, b_{i\sigma}$ and φ guarantee the existence of solutions to problem (2) which have an arbitrarily large norm in the corresponding Hölder spaces.

It is necessary to note that in the case where φ is an odd function and $q = 1$, the relevant result was gotten by P. Rabinowitz [10]. Later W. Marzantowicz [8] generalized Rabinowitz's theorem to the situation where a compact Lie group G acts orthogonally on \mathbf{R}^q without G -fixed points (besides the origin) and $a_\sigma, b_{i\sigma}$ and φ are G -equivariant. In this paper the assumption of the absence of G -fixed points is weakened to a simple homotopic condition. We also construct an example which can be treated by our theorem but not by Rabinowitz-Marzantowicz Theorem (the corresponding action has nonzero G -fixed points). Of course we use a sharper version of the Borsuk-Ulam Theorem (see [2,3,6]) than W. Marzantowicz [8]. We follow P. Rabinowitz's scheme [10], as does W. Marzantowicz [8].

Our result was announced in [4,5].

2. Auxiliary Information and Formulation of Result

We'll gather some known facts and describe the situation discussed here.

If Ω is as above, then let us denote by $C^j(\bar{\Omega}, \mathbf{R}^q)$ the space of j -smooth \mathbf{R}^q -valued functions defined on $\bar{\Omega}$. For the given $0 < \mu < 1$ let us denote by $C^{j+\mu}(\bar{\Omega}, \mathbf{R}^q)$ the space of all $u \in C^j(\bar{\Omega}, \mathbf{R}^q)$ such that

$$\|u\|_{j+\mu} = \|u_j\| + \sum_{|\sigma|=j} \sup_{\substack{x \neq y \\ x, y \in \Omega}} \frac{\|D^\sigma u(x) - D^\sigma u(y)\|}{\|x - y\|^\mu} \quad (3)$$

is finite, where

$$\|u\|_j = \sum_{|\sigma| \leq j} \max_{x \in \bar{\Omega}} \|D^\sigma u(x)\|$$

(see, for example, [9], p. 44).

Formula (3) defines the well-known Hölder norm in $C^{j+\mu}(\bar{\Omega}, \mathbf{R}^q)$.

Let us suppose that an orthogonal action of a finite group G on \mathbf{R}^q is given and assume that the general common divisor of orbit lengths for this action on the unit sphere on $\mathbf{R}^q \ni (\mathbf{R}^q)^G$ is greater than one. Formula $(gu)(x) = g(u(x))$ for $g \in G$ defines the structure of a Banach G -space on $C^{j+\mu}(\bar{\Omega}, \mathbf{R}^q)$; it is clear that \mathbf{R}^q and $C^{j+\mu}(\bar{\Omega}, \mathbf{R}^q)$ have the same orbit types.

We assume that the matrices $a_\sigma(x)$ and $b_{i\sigma}(x')$ commute with the action of G on \mathbf{R}^q for all $x \in \Omega$ and $x' \in \partial\bar{\Omega}$ respectively. It is easy to see that this is a necessary and sufficient condition for operators L and B to be equivariant in the relevant functional spaces (see, for example, [8]).

We suppose that the Fredholm index of the linear operator $P = (L, B)$, where

$$P : \{u \in C^{2m+\mu}(\bar{\Omega}, \mathbf{R}^q); \quad Bu = 0 \quad \text{on} \quad \partial\Omega\} \rightarrow C^\mu(\bar{\Omega}, \mathbf{R}^q),$$

is positive and in addition that there exists the G -equivariant linear embedding of the co-kernel into the kernel (clearly the last condition holds automatically if the co-kernel is trivial).

Lemma 1 (see [1,8,9]).

(i) Let $f \in C^\mu(\bar{\Omega}, \mathbf{R}^q)$. If u is a solution of the problem

$$\begin{aligned} Lu(x) &= f(x), & x \in \Omega, \\ Bu(x) &= 0, & x \in \partial\Omega, \end{aligned}$$

then $u \in C^{2m+\mu}(\bar{\Omega}, \mathbf{R}^q)$.

(ii) For every function $u \in C^{2m+\mu}(\bar{\Omega}, \mathbf{R}^q) \ominus \ker P$ the following estimation

$$\|u\|_{2m+\mu} \leq C \cdot \|Pu\|_\mu$$

holds, where C depends on P, μ and does not depend on u .

For the sake of simplicity we suppose that the restriction

$$P : \{u \in C^{2m+\mu}(\bar{\Omega}, \mathbf{R}^q), \quad Bu = 0 \quad \text{on} \quad \partial\Omega\}^G \rightarrow (C^\mu(\bar{\Omega}, \mathbf{R}^q))^G$$

of the operator P is invertible.

Let us denote the space of k -linear operators from \mathbf{R}^n to \mathbf{R}^q by $L^k(\mathbf{R}^n, \mathbf{R}^q)$. Since G acts trivially on $\bar{\Omega}$, the diagonal action of G on

$$T = \bar{\Omega} \times \mathbf{R}^q \times L^1(\mathbf{R}^n, \mathbf{R}^q) \times \dots \times L^{2m-1}(\mathbf{R}^n, \mathbf{R}^q)$$

is defined correctly. Suppose that $\varphi : T \rightarrow \mathbf{R}^q$ is equivariant; then the formula

$$(\phi(u))(x) = \varphi(x, u(x), Du(x), \dots, D^{2m-1}u(x))$$

defines an equivariant mapping

$$\phi : C^{2m-1+\mu}(\bar{\Omega}, \mathbf{R}^q) \rightarrow C^\mu(\bar{\Omega}, \mathbf{R}^q).$$

Let us suppose finally that there exist numbers $D > 0$ and $0 < \gamma < 1$ such that for every $u \in (C^{2m-1+\mu}(\bar{\Omega}, \mathbf{R}^q))^G$ the following inequality is true:

$$\|\varphi(x, u(x), Du(x), \dots, D^{2m-1}u(x))\|_{\mathbf{R}^q} < D \left(1 + \sum_{|\sigma| \leq 2m-1} \|D^\sigma u(x)\|_{\mathbf{R}^q} \right)^\gamma. \quad (4)$$

We prove the following

Theorem. *Under the above assumptions with respect to L, B, φ and G there exists $r_0 \geq 0$ such that for every $r \geq r_0$ problem (2) has a solution $u^{(0)} \in C^\infty(\bar{\Omega}, \mathbf{R}^q)$ and $\|u^{(0)}\|_{2m-1+m} = r$. If G acts on \mathbf{R}^q without nonzero G -fixed points then $r_0 = 0$.*

For the proof of this theorem we need the following version of the Borsuk-Ulam Theorem.

Lemma 2 (see [2,3,6]). Let us suppose that a finite group G acts isometrically on the Banach space E and let $S \subset E$ be an arbitrary sphere whose center coincides with the origin 0. Suppose that $A \subset S$ is a closed

invariant subset. that $(H_1), \dots, (H_\ell)$ are all the orbit types in $S \setminus A$ and that $F : S \rightarrow E \setminus \{0\}$ is a compact operator which commutes with G such that the vector field $(I - F)|_A$ is equivariantly homotopic to $I|_A$, where I is the identity operator. Let us assume that $p = \text{G.C.D.}\{|G/H_i|\}_{k=1}^\ell > 1$. Then

$$0 \neq \deg(I - F, S, 0) \equiv 1 \pmod{p}.$$

Here $\deg(\dots)$ is the Leray-Schauder degree.

3. Proof of Theorem

Let us denote the kernel span $\{v_1, \dots, v_k\}$ of the operator P by V_k and its co-kernel span $\{w_1, \dots, w_d\}$ by W_d , where v_1, \dots, v_k and w_1, \dots, w_d are relevant linearly independent systems. Let us denote $C^{2m-1+\mu}(\bar{\Omega}, \mathbf{R}^q)$ by E . Since V_k is isomorphic to \mathbf{R}^k and W_d is isomorphic to \mathbf{R}^d , we can write

$$E = \mathbf{R}^k \times \bar{E} = \mathbf{R}^d \times \tilde{E}, \quad (5)$$

where \bar{E} and \tilde{E} are orthogonal complements of \mathbf{R}^k and \mathbf{R}^d correspondingly with respect to the norm of the space $L_2(\bar{\Omega}, \mathbf{R}^q)$; this means that $h \in \bar{E}$ if and only if $\langle h, v_i \rangle = 0$ with respect to $L_2(\bar{\Omega}, \mathbf{R}^q)$, where $i = 1, \dots, k$. Thus if $u \in E$ then

$$u = \sum_{i=1}^k \bar{a}_i v_i + \bar{u} = \sum_{s=1}^d \tilde{a}_s w_s + \tilde{u}$$

or $u = (\bar{a}, \bar{u}) = (\tilde{a}, \tilde{u})$.

Let the operator $\tilde{P} : E \rightarrow \tilde{E}$ be an orthogonal projection with respect to $L_2(\bar{\Omega}, \mathbf{R}^q)$ defined by the formula $\tilde{P}u = \tilde{u}$. Let us consider an additional nonhomogeneous problem

$$\begin{aligned} L\bar{U}(x) &= \tilde{P}f(x), & x \in \Omega, \\ B\bar{U}(x) &= 0, & x \in \partial\Omega, \end{aligned} \quad (6)$$

which is associated with problem (1), where $f \in E$. By virtue of Lemma 1(i) there exists a unique $\bar{U}_0 \in C^{2m+\mu}(\bar{\Omega}, \mathbf{R}^q) \cap \bar{E}$ satisfying (6).

Let us rewrite (2) as the operator equation in space E . Let $u = (\bar{a}, \bar{u}) \in E$ and $Fu = (\bar{A}, \bar{U}) = (A(u), T(u))$, where $\bar{A} = (\bar{A}_1, \dots, \bar{A}_k)$ is defined by formula:

$$\bar{A}_i = \begin{cases} \bar{a}_i - \langle \varphi(\cdot, u, Du, \dots, D^{2m-1}u), w_i \rangle, & 1 \leq i \leq d; \\ \bar{a}_i, & d+1 \leq i \leq k \end{cases} \quad (7)$$

(here $k > d$ by virtue of the assumption $\text{ind}(L, B) > 0$), operator T is defined with regard to (6) as the unique solution of the system

$$\begin{aligned} L\bar{U}(x) &= \tilde{P}\varphi(x, u(x), Du(x), \dots, D^{2m-1}u(x)), & x \in \Omega, \\ B\bar{U}(x) &= 0, & x \in \partial\Omega. \end{aligned} \quad (8)$$

By virtue of (5), (6) and (8) the operator F maps space E into itself. Since $a_\sigma(x), b_{i\sigma}(x)$ and φ are equivariant and there exists an equivariant embedding for the co-kernel of P into its kernel one can show that F is equivariant. In addition statement (ii) of Lemma 1 and the assumption of the smoothness give us (for neighboring points u and $u_0 \in E$) the following chain of the inequalities:

$$\begin{aligned} \|\bar{U} - \bar{U}_0\|_{2m-1+\mu} &\leq \|\bar{U} - \bar{U}_0\|_{2m+\mu} \leq \\ &\leq C\|L(\bar{U} - \bar{U}_0)\|_\mu = \\ &= C\|\tilde{P}\varphi(\cdot, u, Du, \dots, D^{2m-1}u) - \tilde{P}\varphi(\cdot, u_0, Du_0, \dots, D^{2m-1}u_0)\|_\mu, \end{aligned} \quad (9)$$

from which the continuity of F follows immediately. Now from [9], p. 47, and estimation (9) the compactness of F follows.

Let us convince ourselves that the operator equation $Fu = u$ is equivalent to (2). Suppose that $u^{(0)} = (\bar{a}^{(0)}, \bar{u}^{(0)})$ is a solution of (2) and $\bar{A}^{(0)} = A(u^{(0)})$, $\bar{U}^{(0)} = T(u^{(0)})$. Then

$$\varphi(\cdot, u^{(0)}, Du^{(0)}, \dots, D^{2m-1}u^{(0)}) = Lu^{(0)} \in \tilde{E},$$

from which (see (7))

$$\langle \varphi(\cdot, u^{(0)}, Du^{(0)}, \dots, D^{2m-1}u^{(0)}), w_i \rangle = 0,$$

i.e. $\bar{A}^{(0)} = \bar{a}_i^{(0)}$ for all $i = 1, \dots, k$. In addition

$$\begin{aligned} \varphi(\cdot, u^{(0)}, Du^{(0)}, \dots, D^{2m-1}u^{(0)}) &= Lu^{(0)} = L(\bar{a}^{(0)}, \bar{u}^{(0)}) = \\ &= L\bar{u}^{(0)} = L\bar{U}^{(0)}. \end{aligned}$$

Since L is an injective operator on the complement with respect to $\ker P$ we conclude that $\bar{u}^{(0)} = \bar{U}^{(0)}$. Thus

$$Fu^{(0)} = F(\bar{a}^{(0)}, \bar{u}^{(0)}) = (\bar{a}^{(0)}, \bar{u}^{(0)}) = u^{(0)}.$$

Conversely, if $u^{(0)}$ is a fixed point for F , then $\bar{A}_i^{(0)} = \bar{a}_i^{(0)}$ ($i = 1, \dots, k$), i.e. (see (5) and (7))

$$\begin{aligned} \varphi(\cdot, u^{(0)}, Du^{(0)}, \dots, D^{2m-1}u^{(0)}) &= \tilde{P}\varphi(\cdot, u^{(0)}, Du^{(0)}, \dots, D^{2m-1}u^{(0)}) = \\ &= L\bar{U}^{(0)} = L(\bar{a}^{(0)}, \bar{u}^{(0)}) = Lu^{(0)}. \end{aligned}$$

Now to complete the proof let us denote by S_r the sphere in E with radius equal to r and center coinciding with the origin. Estimation (4) guarantees the existence of $r_0 \geq 0$ (see [9], p. 49), such that for all $r \geq r_0$ the vector field $(I - F)|_{E^G \cap S_r}$ is non-degenerate and homotopic to I .

Let us now suppose that for some $r_1 \geq r_0$ problem (2) does not have a solution on the sphere S_{r_1} . This means that the vector field $I - F$ is non-degenerate on S_{r_1} and thus the Leray-Schauder degree $\deg(Id - F, S_{r_1}, 0)$ is defined correctly for it. Then on the one hand $\deg(I - F, S_{r_1}, 0) \neq 0$ according to Lemma 2; on the other hand an image of $I - F$ has a non-zero co-dimension (since $k > d$) and hence (see [9,10]) $\deg(I - F, S_{r_1}, 0) = 0$, which is a contradiction.

In order to prove that the obtained solution belongs to $C^\infty(\bar{\Omega}, \mathbf{R}^q)$ one can use standard arguments of [9], p.50. This completes the proof of our Theorem.

4. An Example

Here we construct an example, illustrating our result.

Let Ω be the open unit disk on the plane and $x = (x_1, x_2)$. Let us consider a system

$$\Delta u_1(x) = \exp\left(-\left(u_1(x) + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2 + u_2^2(x) + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2\right), \quad x \in \Omega$$

$$\Delta u_2(x) = u_2(x) \cdot \exp(u_1(x)),$$

$$u_1(x) = 0,$$

$$\frac{\partial u_2}{\partial x_1} = 0, \quad x \in \partial\Omega \quad (10)$$

and show that it satisfies the conditions of our Theorem (here Δ is the Laplacian).

Let us define an action of group \mathbf{Z}_2 on the space \mathbf{R}^2 -range of values of the vector function $u = (u_1, u_2)$ - by means of formula: $g(u_1, u_2) = (u_1, -u_2)$.

The equivariance of operator B which is defined by the boundary conditions (10) is obvious since $\partial(-u_2)/\partial x_1 = -\partial u_2/\partial x_1$ and g does not change the coordinate u_1 . It is clear that the linear operator L , which is defined with the help of the Laplacians, is also equivariant. Let us convince ourselves that φ is equivariant. We have:

$$\begin{aligned} \varphi(g(u_1, u_2)) &= \varphi(u_1, -u_2) = \\ &= \left(\exp\left(-\left(u_1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2 + (-u_2)^2 + \left(-\frac{\partial u_2}{\partial x_1}\right)^2 + \left(-\frac{\partial u_2}{\partial x_2}\right)^2, -u_2 \cdot \exp(u_1) \right) = \\ &= \left(\exp\left(-\left(u_1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2 + u_2^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2, -u_2 \cdot \exp(u_1) \right) = \\ &= g\varphi(u_1, u_2). \end{aligned}$$

Now let us calculate the index of the linear problem associated with (10). It is clear that the desired index is equal to the sum of the indices of the Dirichlet problem and the problem with the direct derivative. It is well-known that the first of them is equal to zero (see, for example, [7], §10.5); the second is equal to $2 - 2\delta$, where δ is the winding number of the vector field induced by the direct derivative (see [7], p. 266). In our situation condition $\partial u_2/\partial x_1 = 0$ implies $\delta = 0$. Thus the index we are interested is equal to $2 > 0$.

It is necessary to note that the co-kernel of the linear operator which is associated with (10) is trivial.

At last let us verify estimation (4). The set E^G consists of functions of the form of $(u_1(x), 0)$. So the relevant restriction of φ has the following form:

$$\varphi = \exp \left(- \left(u_1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right)^2 \right).$$

Since

$$\left| \exp \left(- \left(u_1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right)^2 \right) \right| < 1 \cdot \left(1 + \left| \frac{\partial u_1}{\partial x_1} \right| + \left| \frac{\partial u_1}{\partial x_2} \right| \right)^{1/2}$$

we can conclude that (4) is trivially true.

To finish the proof it is necessary to note that the invertibility of the operator $P|E^G$ is obvious because we are dealing with the Dirichlet problem.

5. Concluding Remarks

1) It is clear that if $u^{(0)}$ is a solution of (2), then $gu^{(0)}$ is also a solution for every $g \in G$. Thus on the relevant spheres we can guarantee at least r solutions of (2), where r is the minimal length of the non-trivial orbits.

2) We have established our result under the assumption that estimation (4) holds. But from the proof it follows that it is sufficient to require that $(I-F)|E^G \cap S_r$ is homotopic to the identity field for arbitrarily large r . One should note, however, that this condition is less observable than estimation (4).

3) Using a scheme from (10) it is easy to estimate the genus of the set of solutions of (2).

The authors would like to thank P.P. Zabrejko for useful remarks.

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Received November 10, 1992

Produktsätze für Potenzreihenverfahren und verallgemeinerte Nörlund-Mittel

Simson Baron und Hubert Tietz

1. Einleitung. Es sei $\{p_n\}$ eine Folge nichtnegativer reeller Zahlen mit $p_0 > 0$ und $P_n := p_0 + \dots + p_n \rightarrow \infty$, für welche die Potenzreihe

$$p(x) := \sum_{n=0}^{\infty} p_n x^n \quad (1.1)$$

den Konvergenzradius 1 hat. Die Folge $s = \{s_n\}$ komplexer Zahlen heißt J_p -limitierbar zum Wert σ ($J_p\text{-lim } s_n = \sigma$), wenn die Reihe

$$p_s(x) := \sum_{n=0}^{\infty} p_n s_n x^n \quad (1.2)$$

für $0 < x < 1$ konvergiert und

$$\lim_{x \rightarrow 1^-} p_s(x)/p(x) = \sigma$$

gilt. Neben diesem *Potenzreihenverfahren* J_p betrachten wir auch das Potenzreihenverfahren J_q , dessen zugehörige Folge $\{q_n\}$ die gleichen Eigenschaften haben soll wie die obige Folge $\{p_n\}$, und wir verwenden die entsprechenden Bezeichnungen wie für $\{p_n\}$. Die bekanntesten Potenzreihenverfahren sind das *verallgemeinerte Abel-Verfahren* A_λ mit $p_n =$

$\binom{n+\lambda-1}{n}$ für $\lambda > 0$ (A_1 ist das klassische *Abel-Verfahren*) und das *logarithmische Verfahren* L mit $p_n = 1/(n+1)$. Schließlich sei $\{r_n\}$ eine Folge nichtnegativer reeller Zahlen mit $R_n := r_0 + \dots + r_n$ und

$$(p * r)_n := p_0 r_n + \dots + p_n r_0 \neq 0,$$

für welche die Potenzreihe

$$r(x) := \sum_{n=0}^{\infty} r_n x^n$$

mindestens den Konvergenzradius 1 hat. Die Folge $\{s_n\}$ heißt dann N_{rp} -*limitierbar* zum Wert σ ($N_{rp}\text{-lim } s_n = \sigma$), wenn für die Folge $\{(N_{rp}s)_n\}$ mit

$$(N_{rp}s)_n := \frac{1}{(p * r)_n} \sum_{\nu=0}^n p_\nu r_{n-\nu} s_\nu, \quad (1.3)$$

$\lim (N_{rp}s)_n = \sigma$ gilt. Dieses *verallgemeinerte Nörlund-Verfahren* N_{rp} geht für $p_n = 1$ über in das *Nörlund-Verfahren* N_r (darunter für $r_n = \binom{n+\alpha-1}{n}$ mit $\alpha \geq 0$ das *Cesàro-Verfahren* C_α) und ergibt mit $r_n = 1$ das *Verfahren* M_p der *bewichteten Mittel* (darunter für $p_n = 1/(n+1)$ das *logarithmische Verfahren* l und für $p_n = 1$ das *Cesàro-Verfahren* C_1). Für $p_n = \binom{n+\beta}{n}$ mit $\beta > -1$ und $r_n = \binom{n+\alpha-1}{n}$ mit $\alpha \geq 0$ erhalten wir das *verallgemeinerte Cesàro-Verfahren* $C_{\alpha,\beta}$. Zu diesen Verfahren vergleiche man etwa [2] und [23].

In Nr. 3 dieser Arbeit geben wir hinreichende Bedingungen dafür an, daß aus $J_p\text{-lim } s_n = \sigma$ stets $J_q\text{-lim } (N_{rp}s)_n = \sigma$ folgt. Satz 3.1 enthält viele bekannte Resultate als Spezialfälle. In Nr. 4 bringen wir einige Anwendungen auf Tauber-Sätze. Ähnliche Ergebnisse lassen sich für entsprechende Verfahren zur Limitierung von Doppelfolgen gewinnen. Wir werden an anderer Stelle darauf zurückkommen.

2. Bezeichnungen und Definitionen. Der Folgenindex soll, wenn nichts Besonderes gesagt ist, von 0 an laufen, und Folgenglieder mit einem negativen Index sind gleich 0 zu setzen. Die Landau-Symbole O und O_L haben ihre übliche Bedeutung.

Die Folge $\{\mu_n\}$ heißt *totalmonoton*, wenn

$$\Delta^j \mu_n := \sum_{k=0}^j (-1)^k \binom{j}{k} \mu_{n+k} \geq 0 \quad \text{für } j, n = 0, 1, \dots \quad (2.1)$$

gilt. Bekanntlich ist (2.1) äquivalent dazu, daß es eine auf $[0,1]$ definierte, reellwertige nichtabnehmende Funktion α gibt mit

$$\mu_n = \int_0^1 x^n d\alpha(x). \quad (2.2)$$

Gilt (2.2) mit einer auf $[0,1]$ definierten, reellwertigen Funktion von beschränkter Variation, so heißt $\{\mu_n\}$ eine *Momentenfolge*. Man vergleiche hierzu etwa [5, Chapter XI].

3. Produktsätze. Der folgende Produktsatz enthält viele bekannte Resultate als Spezialfälle.

SATZ 3.1. *Ist $\{q_n/(p \cdot r)_n\}$ totalmonoton, so folgt aus $J_p\text{-lim } s_n = \sigma$ stets $J_q\text{-lim } (N_{r,p}s)_n = \sigma$.*

Beweis. Ist $J_p\text{-lim } s_n = \sigma$, so genügt es zu zeigen, daß für jedes $x \in (0,1)$ die Reihe

$$\sum_{n=0}^{\infty} q_n(N_{r,p}s)_n x^n \quad (3.1)$$

konvergiert und, mit einer auf $[0,1]$ nichtabnehmenden Funktion α , die Darstellung

$$\frac{1}{q(x)} \sum_{n=0}^{\infty} q_n(N_{r,p}s)_n x^n = \frac{1}{q(x)} \int_0^x \pi_s(u) p(u) r(u) d\alpha\left(\frac{u}{x}\right) \quad (3.2)$$

gilt, wobei hier $\pi_s(x) := p_s(x)/p(x)$ für $0 < x < 1$ gesetzt ist. Mit $\{s_n\} = \{1\}$ folgt aus (3.2) dann nämlich

$$1 = \frac{1}{q(x)} \int_0^x p(u) r(u) d\alpha\left(\frac{u}{x}\right) = \frac{1}{q(x)} \int_0^x p(u) r(u) |d\alpha\left(\frac{u}{x}\right)|, \quad (3.3)$$

und wegen $q(x) \rightarrow \infty$ für $x \rightarrow 1-$ weiter, bei festem $y \in (0,1)$,

$$\frac{1}{q(x)} \int_0^y p(u) r(u) d\alpha\left(\frac{u}{x}\right) \rightarrow 0 \text{ für } x \rightarrow 1-.$$

Die durch die rechte Seite von (3.2) gegebene Integraltransformation von $\pi_s(x)$ ist somit permanent, d.h. aus der Voraussetzung $\pi_s(x) \rightarrow \sigma$ für

$x \rightarrow 1-$ folgt $J_q\text{-lim } (N_{rps})_n = \sigma$ (vgl. [5, Beweis von Theorem 6]). Sei also $x \in (0, 1)$ fest. Da $\{q_n/(p * r)_n\}$ totalmonoton ist, gibt es eine auf $[0, 1]$ nichtabnehmende Funktion α mit

$$\frac{q_n}{(p * r)_n} x^n = x^n \int_0^1 t^n d\alpha(t) = \int_0^x u^n d\alpha\left(\frac{u}{x}\right)$$

für jedes $n = 0, 1, \dots$. Demnach gilt für jedes $\nu = 0, 1, \dots$ wegen der gleichmäßigen Konvergenz der Potenzreihe für $r(u)$

$$\sum_{n=\nu}^{\infty} \frac{q_n r_{n-\nu}}{(p * r)_n} x^n = \sum_{n=\nu}^{\infty} r_{n-\nu} \int_0^x u^n d\alpha\left(\frac{u}{x}\right) = \int_0^x u^\nu r(u) d\alpha\left(\frac{u}{x}\right), \quad (3.4)$$

insbesondere also

$$\sum_{n=\nu}^{\infty} \frac{q_n r_{n-\nu}}{(p * r)_n} x^n \leq x^\nu r(x) [\alpha(1) - \alpha(0)], \quad (3.5)$$

so daß die Doppelreihe

$$\sum_{\nu=0}^{\infty} p_\nu s_\nu \sum_{n=\nu}^{\infty} \frac{q_n r_{n-\nu}}{(p * r)_n} x^n \quad (3.6)$$

absolut konvergiert. Vertauschung der Summationsreihenfolge zeigt aber, daß der Wert der Reihe (3.6) gerade der Wert der Reihe (3.1) ist, insbesondere die Reihe (3.1) also konvergiert. Weiter ergibt sich aus (3.6) mit (3.4)

$$\sum_{n=0}^{\infty} q_n (N_{rps})_n x^n = \sum_{\nu=0}^{\infty} p_\nu s_\nu \int_0^x u^\nu r(u) d\alpha\left(\frac{u}{x}\right) = \int_0^x p_s(u) r(u) d\alpha\left(\frac{u}{x}\right),$$

wovon man (3.2) abliest. Die in der letzten Gleichung enthaltene Vertauschung von Summation und Integration ist wegen (3.5) und der gleichmäßigen Konvergenz der Potenzreihe für $p_s(u)$ auf $[0, x]$ erlaubt.

Für Satz 3.1 genügt es natürlich, wenn $\{q_n/(p * r)_n\}$ erst von einer Stelle an totalmonoton ist. Ferner wird diese Voraussetzung nur für die Bedingung

$$\frac{1}{q(x)} \int_0^x p(u) r(u) |d\alpha\left(\frac{u}{x}\right)| = O(1) \quad \text{für } x \rightarrow 1- \quad (3.7)$$

voll ausgenützt, die implizit in (3.3) enthalten ist. Man kann diese Voraussetzung deshalb durch

$$(3.7) \wedge \{q_n/(p * r)_n\} \text{ ist eine Momentenfolge}$$

ersetzen. Ein Resultat dieser Art (mit $p_n = 1$) findet sich bei Hoischen [6, Satz 1].

Wir betrachten einige Spezialfälle von Satz 3.1.

KOROLLAR 3.2. *Ist $\{q_n/P_n\}$ totalmonoton, so folgt aus J_p -lim $s_n = \sigma$ stets J_q -lim $(M_p s)_n = \sigma$.*

Dies ergibt sich aus Satz 3.1 mit $r_n = 1$. Mit $q_n = 1$ erhalten wir aus Korollar 3.2 folgende Verallgemeinerung eines Resultats von Mikhailin [17, Lemma 1]: *Ist $\{1/P_n\}$ totalmonoton, so folgt aus J_p -lim $s_n = \sigma$ stets A_1 -lim $(M_p s)_n = \sigma$. Ein Spezialfall hiervon stammt von Kokhanovskii [12, Lemma 2], nämlich: *Aus L -lim $s_n = \sigma$ folgt stets A_1 -lim $(I s)_n = \sigma$. Dies folgt aus dem voranstehenden Ergebnis, da, mit**

$$h_n := \sum_{\nu=0}^n \frac{1}{\nu+1}, \quad (3.8)$$

die Folge $\{1/h_n\}$ totalmonoton ist. Für jedes n ist nämlich

$$\Delta^1 \frac{1}{h_n} = \frac{1}{n+2} \frac{1}{h_n} \frac{1}{h_{n+1}}. \quad (3.9)$$

Ferner gilt: *Bei beliebigen Folgen $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ ist für $j, n = 0, 1, \dots$ stets*

$$\Delta^j (x_n y_n z_n) = \sum_{\nu=0}^j \binom{j}{\nu} \Delta^\nu x_n \cdot \sum_{\mu=0}^{j-\nu} \binom{j-\nu}{\mu} \Delta^\mu y_{n+\nu} \cdot \Delta^{j-\nu-\mu} z_{n+\nu+\mu}. \quad (3.10)$$

Da nun die Folge $\{1/(n+2)\}$ totalmonoton ist (vgl. [5, Seite 253]), ergibt sich aus (3.9) und (3.10) durch Induktion nach j , daß $\{1/h_n\}$ totalmonoton ist.

Aus Korollar 3.2 erhalten wir mit $q_n = p_n$ das

KOROLLAR 3.3. *Ist $\{p_n/P_n\}$ totalmonoton, so folgt aus J_p -lim $s_n = \sigma$ stets J_p -lim $(M_p s)_n = \sigma$.*

Da die Folge $\{1/h_n\}$, mit h_n aus (3.8), totalmonoton ist, steckt in Korollar 3.3 folgendes Ergebnis von Kwee [16, Lemma 3]: *Aus L -lim $s_n = \sigma$ folgt stets L -lim $(I s)_n = \sigma$.*

KOROLLAR 3.4. *Ist $\{1/R_n\}$ totalmonoton, so folgt aus A_1 -lim $s_n = \sigma$ stets A_1 -lim $(N_r s)_n = \sigma$.*

Dies ergibt sich auch aus einem Resultat von Hoischen [6, Satz 1]. Da die Folge $\{1/\binom{n+\alpha}{n}\}$ für $\alpha \geq 0$ totalmonoton ist (vgl. etwa Borwein [3, Seite 348]), enthält Korollar 3.4 insbesondere folgenden Satz von Amir (Jakimovski) [1, Theorem 2.1 mit $\alpha = 0$] und Szász [18]: *Ist $\alpha \geq 0$, so folgt aus A_1 -lim $s_n = \sigma$ stets A_1 -lim $(C_\alpha s)_n = \sigma$.* Der Fall $\alpha = 1$ findet sich schon bei Zygmund [24, Seite 189].

Für $\alpha \geq 0$ und $\beta > -1$ erhalten wir mit $p_n = \binom{n+\beta}{n}$ und $r_n = \binom{n+\alpha-1}{n}$ das zu Korollar 3.4 verwandte Resultat: *Ist $\{q_n/\binom{n+\alpha+\beta}{n}\}$ totalmonoton, so folgt aus $A_{\beta+1}$ -lim $s_n = \sigma$ stets J_q -lim $(C_{\alpha,\beta} s)_n = \sigma$.*

KOROLLAR 3.5. *Ist $\{q_n/p_n\}$ totalmonoton, so folgt aus J_p -lim $s_n = \sigma$ stets J_q -lim $s_n = \sigma$.*

Dies folgt aus Satz 3.1 mit $r_0 = 1$ und $r_n = 0$ für $n > 0$. Varianten und Verallgemeinerungen zu Korollar 3.5 finden sich in den Arbeiten von Borwein [3, Theorem A], Hoischen [7, Theorem 1] und Fischer [4, Satz 3.1.2]. Insbesondere gilt also: *Ist $\{1/p_n\}$ totalmonoton, so folgt aus J_p -lim $s_n = \sigma$ stets A_1 -lim $s_n = \sigma$, und: *Ist $\{q_n\}$ totalmonoton, so folgt aus A_1 -lim $s_n = \sigma$ stets J_q -lim $s_n = \sigma$.**

Wir beschließen diesen Abschnitt mit einem Resultat anderer Art. Dazu sei, bei gegebener Folge $\{s_n\}$, die Folge $\{\delta_n\}$ durch

$$\delta_n := s_n - (M_p s)_n = P_n^{-1} \sum_{\nu=1}^n P_{\nu-1} (s_\nu - s_{\nu-1}) \quad (3.11)$$

definiert. Ist nun $\{p_n/P_n\}$ totalmonoton und J_p -lim $s_n = \sigma$, so ist nach Korollar 3.3 auch J_p -lim $(M_p s)_n = \sigma$. Wegen der Linearität von J_p ergibt sich also das

KOROLLAR 3.6. *Ist $\{p_n/P_n\}$ totalmonoton, so folgt aus J_p -lim $s_n = \sigma$ stets J_p -lim $\delta_n = 0$.*

4. Tauber-Sätze. Die Ergebnisse aus Abschnitt 3 gestatten die Herleitung von Tauber-Sätzen für J_p aus bekannten Tauber-Sätzen für J_q . Wir zeigen dies nur an wenigen Beispielen.

SATZ 4.1. *Ist $\{q_n/P_n\}$ totalmonoton und gilt*

$$Q_n/Q_m \rightarrow 1 \text{ für } 1 < n/m \rightarrow 1 \quad (m \rightarrow \infty), \quad (4.1)$$

so folgt aus J_p -lim $s_n = \sigma$ und

$$\liminf \{(M_p s)_n - (M_p s)_m\} \geq 0 \text{ für } Q_n/Q_m \rightarrow 1 \quad (n > m \rightarrow \infty) \quad (4.2)$$

stets M_p -lim $s_n = \sigma$.

Beweis. Nach [20, Satz 3.9] folgt unter der Voraussetzung (4.1), aus J_q -lim $s_n = \sigma$ und

$$\liminf (s_n - s_m) \geq 0 \text{ für } Q_n/Q_m \rightarrow 1 \quad (n > m \rightarrow \infty)$$

stets $\lim s_n = \sigma$. Somit ergibt sich Satz 4.1 aus Korollar 3.2.

Ein Spezialfall von Satz 4.1 findet sich bei Kokhanovskii [12, Theorem 1]. In Satz 4.1 darf, mit δ_n aus (3.11), die Bedingung (4.2) durch

$$Q_n p_n \delta_n = O_L(P_{n-1} q_n) \quad (4.3)$$

ersetzt werden, da (4.2) aus (4.1) \wedge (4.3) folgt. Wegen $(M_p s)_\nu - (M_p s)_{\nu-1} = p_\nu P_{\nu-1}^{-1} \delta_\nu$ für $\nu > 0$ folgt mit (4.3) nämlich, daß es eine Konstante $K > 0$ gibt mit

$$\begin{aligned} (M_p s)_n - (M_p s)_m &= \sum_{\nu=m+1}^n [(M_p s)_\nu - (M_p s)_{\nu-1}] \\ &= \sum_{\nu=m+1}^n p_\nu P_{\nu-1}^{-1} \delta_\nu \geq -K \sum_{\nu=m+1}^n \frac{q_\nu}{Q_\nu} \geq -K \left(\frac{Q_n}{Q_m} - 1 \right). \end{aligned}$$

SATZ 4.2. Ist $\{p_n/P_n\}$ totalmonoton und gilt

$$P_n/P_m \rightarrow 1 \text{ für } 1 < n/m \rightarrow 1 \quad (m \rightarrow \infty), \quad (4.4)$$

so folgt, mit δ_n von (3.11), aus J_p -lim $s_n = \sigma$ und

$$\liminf (\delta_n - \delta_m) \geq 0 \text{ für } P_n/P_m \rightarrow 1 \quad (n > m \rightarrow \infty) \quad (4.5)$$

stets $\lim s_n = \sigma$.

Beweis. Es sei J_p -lim $s_n = \sigma$. Nach Korollar 3.6 erhalten wir dann J_p -lim $\delta_n = 0$. Daraus folgt mit (4.4) und (4.5) nach [20, Satz 3.9] sogar $\lim \delta_n = 0$. Dies aber ist nach [21, Satz 3.2] eine Tauber-Bedingung für das Verfahren J_p .

Der Spezialfall $J_p = A_1$ von Satz 4.2 stammt von Jakimovski [8, Theorem 1].

In [19–21] finden sich weitere Resultate, die zur Gewinnung von Tauber-Sätzen der voranstehenden Art geeignet sind. Auch die nachfolgende Auswahl neuerer Arbeiten könnte in dieser Weise ausgenützt werden: Tietz, Trautner [22], Kratz, Stadtmüller [13–15] sowie Kiesel [10] und Kiesel, Stadtmüller [11].

Als letztes Beispiel bringen wir eine kleine Anwendung auf Lückenumkehrsätze. Ist $\{k_n\}$ eine feste Indexfolge mit $0 \leq k_0 < k_1 < \dots$, so heißt

$$s_n - s_{n-1} = 0 \text{ für } n \neq k_0, k_1, \dots \quad (4.6)$$

eine *Lückenbedingung*. Lückenbedingungen für Potenzreihenverfahren J_q , für die $\{q_n\}$ totalmonoton ist, werden zum Beispiel von Jakimovski, Meyer-König, Zeller [9] untersucht.

SATZ 4.3. *Ist $\{q_n\}$ totalmonoton und gibt es eine Konstante $\lambda > 1$ mit*

$$Q_{k_{l+1}} \geq \lambda Q_{k_l} \text{ für } l = 0, 1, \dots, \quad (4.7)$$

so folgt aus $A_1\text{-lim } s_n = \sigma$ und (4.6) stets $\lim s_n = \sigma$.

Beweis. Da $\{q_n\}$ totalmonoton ist, folgt, wie nach Korollar 3.5 bemerkt wurde, aus $A_1\text{-lim } s_n = \sigma$ auch $J_q\text{-lim } s_n = \sigma$. Nach [22, Satz 6.1] ergibt sich wegen (4.6) und (4.7) dann $\lim s_n = \sigma$.

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Received September 22, 1992

Korrutisteoreemid astmeridade menetluste ja

Üldistatud Nörlundi keskmiste korral

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Resümee

Olgu J_p astmeridade menetlus, mis on defineeritud avaldiste (1.1), (1.2) ja

$$\lim_{x \rightarrow 1^-} p_x(x)/p(x)$$

kaudu, J_q teine samasugune menetlus ning N_{rp} üldistatud Nörlundi menetlus, mis on defineeritud keskmistega (1.3) kaudu. Näiteks kehtib järgmine

TEOREEM 3.1. *Olgu $\{q_n/(p * r)_n\}$ totaalmonotoonne jada. Siis*

$$J_q\text{-}\lim(N_{rp} s)_n = \sigma$$

alati, kui $J_p\text{-}\lim s_n = \sigma$.

Teoreemist 3.1 järelduvad erijuhuna paljud tuntud tulemused. Rakendustena on töö neljandas osas esitatud mõned Tauberi teoreemid.

Weak wedge spaces and theorems of Mazur-Orlicz type

Johann Boos and Toivo Leiger

A known theorem of Mazur-Orlicz type due to the authors (see [5, Theorem 4.5], [3, Theorem 1] and [4, Theorem 2.2]) says that the implication

$$(*) \quad M \cap W_E \subset F \implies M \cap W_E \subset W_F$$

holds for every L_φ -space F (in particular for each separable FK -space), for any FK -space E containing the set φ of all finite (real or complex valued) sequences, and for each sequence space M having suitable factor sequences, for example, $M := m$; thereby W_E denotes the set of all elements of E being weakly sectionally convergent. In his thesis [7] D. Seydel has shown that $M := bs$, where bs is the set of all sequences having bounded partial sums, does not satisfy the mentioned assumption to the space of all factor sequences of M . However, he has also proved that $(*)$ remains true if E is an FK -space containing c_0 , F is a separable FK -space and $M := Z(m)$, where Z is a matrix method being regular for null sequences; note, if Σ is the summation matrix then $Z := \Sigma^{-1}$ is regular for null sequences and $bs = \Sigma^{-1}(m)$.

In Seydel's proof the assumption that E contains c_0 is essential. The main result of this note (see Theorem 5) is that the above theorem of the authors remains true if $M := bs$ and E contains (in comparison to c_0) the smaller space cs . In the proof we use the notion of weak wedge spaces due to G. Bennett [1] and the mentioned theorem of the authors in case of $M := m$.

Let ω denote the linear space of all scalar (real or complex) sequences. By a sequence space E we mean any linear subspace of ω . A sequence space E carrying a locally convex topology τ_E is called a K -space if the inclusion map $i: E \rightarrow \omega$ is continuous where ω has the topology of coordinatewise convergence. A K -space E with a Fréchet topology is called an FK -space. If, in addition, the topology is normable then E is called a BK -space.

If E is any sequence space then the β -dual of E is given by

$$E^{\beta} := \left\{ x \in \omega \mid \sum_k x_k y_k \text{ converges for each } y \in E \right\}.$$

For any $x = (x_k) \in \omega$ and $n \in \mathbb{N}$ the n^{th} section of x is

$$x^{[n]} := \sum_{k=1}^n x_k e^k$$

where $e^k := (\delta_{ik})_{i \in \mathbb{N}}$ is the k^{th} coordinate vector.

If (E, τ_E) is a K -space containing φ , the space of finitely non-zero sequences, we set

$$E^f := \left\{ (f(e^k))_{k \in \mathbb{N}} \mid f \in E' \right\},$$

$$B_E := \left\{ x \in E \mid \{x^{[n]} \mid n \in \mathbb{N}\} \text{ is bounded} \right\},$$

$$F_E := \left\{ x \in E \mid (x^{[n]})_{n \in \mathbb{N}} \text{ is } \sigma(E, E')\text{-Cauchy} \right\},$$

$$W_E := \left\{ x \in E \mid x^{[n]} \rightarrow x \text{ } (\sigma(E, E')) \right\},$$

$$S_E := \left\{ x \in E \mid x^{[n]} \rightarrow x \text{ } (\tau_E) \right\}$$

where E' denotes the topological dual of (E, τ_E) and $\sigma(E, E')$ denotes the weak topology.

Let m, c, c_0, bv, bv_0, bs and cs denote, respectively, the BK -spaces of all bounded, convergent, ... and summable sequences with their natural topologies (see [9]). A K -space E containing c is called conservative. We say that E is *conull* if $e := (1, 1, \dots) \in W_E$. A K -space E containing φ is called a *weak wedge space* if $e^k \rightarrow 0$ ($\sigma(E, E')$).

If $B = (b_{nk})$ is an infinite matrix the (*convergence*) domain

$$c_B := \left\{ x \in \omega \mid Bx := \left(\sum_k b_{nk} x_k \right)_{n \in \mathbb{N}} \in c \right\}$$

of B is an FK -space (see [9]). Furthermore, if $\varphi \subset c_B$ we write W_B instead of W_{c_B} .

Let (E, τ) be a locally convex space with algebraic dual E^* . For any subspace S of E^* we use the notations

$$\overline{S} := \left\{ g \in E^* \mid \exists (g_n) \text{ in } S : g_n \rightarrow g \text{ } (\sigma(E^*, E)) \right\},$$

$$\underline{S} := \bigcap \left\{ V < E^* \mid S \subset V = \overline{V} \right\}.$$

If E is a K -space containing φ then it is called L_φ -space if $\underline{\varphi} \cap E' = E'$; for details see [5] and [2].

Throughout this note we consider K -spaces containing cs . If cs is continuously embedded in E then we have the inclusion $E^f \subset cs^f = bv$. In particular this is true for each FK -space E including cs . Moreover, by [8, Theorem 4] we get for each FK -space E the equivalence

$$E^f \subset bv \iff E \supset cs.$$

Assuming $E^f \cap (bv \setminus bv_0) \neq \emptyset$, we get a $z := (f(e^k)) \in bv \setminus c_0$ where $f \in E'$ and the series $\sum_k x_k z_k$ converges if and only if $x \in cs$. This statement is contained in the following lemma.

Lemma 1. For each $z \in bv \setminus bv_0$ we have $\{z\}^{\beta} = cs$.

Proof. Without loss of generality we may assume $z_k \neq 0$ ($k \in \mathbb{N}$). By Abels partial summation we have

$$(1) \quad \sum_{k=1}^n x_k z_k = \sum_{k=1}^{n-1} \Delta z_k \sum_{i=1}^k x_i + z_n \sum_{i=1}^n x_i.$$

Using the notations $y := \Sigma x := (\sum_{i=1}^k x_i)_k$ and

$$B = (b_{nk}) := \begin{pmatrix} z_1 & & & \\ \Delta z_1 & z_2 & & \\ \Delta z_1 & \Delta z_2 & z_3 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \\ \\ 0 \\ \end{matrix}$$

this identity means $\sum_{k=1}^n x_k z_k = (By)_n$ ($n \in \mathbb{N}$). The matrix B is a modification of Mazurs matrix (see [9, Example 1.8.12]), thus $c_B = c$ (see [6, Lemma]). Therefore, $\sum_k x_k z_k$ converges if and only if $y \in c$, that is $x \in cs$. \square

Proposition 2. For any FK -space E containing cs the following statements hold:

- (a) $bs \cap E \subset B_E$.
- (b) If $E^f \not\subset bv_0$ then $F_E = W_E = S_E = cs$.

Proof. Let $f \in E'$ and $z := (f(e^k))$. By (1) we get that for each $x \in bs \cap E$ the sequence of the partial sums of $\sum_k x_k z_k$ is bounded. Thus $x \in B_E$. The statement (b) comes from Lemma 1 and the monotonicity of FK -topologies. \square

Let $E = (E, \tau_E)$ be a K -space containing cs and let τ_E be generated by a family $\{p_\alpha\}_{\alpha \in A}$ of seminorms. Then $E_{\Sigma^{-1}} = \Sigma[E]$ endowed with the family of seminorms $\{p_\alpha \circ \Sigma^{-1}\}_{\alpha \in A}$ is a conservative K -space. Therefore the K -spaces $(E_{\Sigma^{-1}}, \tau_{E_{\Sigma^{-1}}})$ and (E, τ_E) are topologically and algebraically isomorphic with respect to the map $\Sigma^{-1} : E_{\Sigma^{-1}} \rightarrow E$. Obviously, we get

$$\Sigma e^k = e - \sum_{i=1}^{k-1} e^i \quad (k \in \mathbb{N})$$

implying

$$e^k \rightarrow 0 (\sigma(E, E')) \iff e = \sum_k e^k (\sigma(E_{\Sigma^{-1}}, (E_{\Sigma^{-1}})')) \iff e \in W_{E_{\Sigma^{-1}}}.$$

By that we get the

Proposition 3. *A K -space E containing φ is a weak wedge space if and only if the K -space $E_{\Sigma^{-1}}$ is conull.*

Proposition 4. *Let E be a K -space containing φ . Then E is a weak wedge space if and only if*

$$(2) \quad \Sigma[bs \cap W_E] = m \cap W_{E_{\Sigma^{-1}}}.$$

Proof. Let $e^k \rightarrow 0$ ($\sigma(E, E')$), $x \in bs \cap E$ and $y := \Sigma x \in m \cap E_{\Sigma^{-1}}$. Note $y^{[m]} \rightarrow y$ ($\sigma(E_{\Sigma^{-1}}, (E_{\Sigma^{-1}})')$) $\iff \Sigma^{-1}y^{[m]} \rightarrow \Sigma^{-1}y = x$ ($\sigma(E, E')$).

For each $m \in \mathbb{N}$ we get

$$\Sigma^{-1}y^{[m]} = (x_1, \dots, x_m, -\sum_{i=1}^m x_i, 0, 0, \dots) = \sum_{k=1}^m x_k e^k - \sum_{i=1}^m x_i e^{m+1},$$

whereby $\sum_{i=1}^m x_i e^{m+1} \xrightarrow{m \rightarrow \infty} 0$ ($\sigma(E, E')$). By that we obtain

$$\Sigma^{-1}y^{[m]} \rightarrow x$$
 ($\sigma(E, E')$) $\iff x^{[m]} \rightarrow x$ ($\sigma(E, E')$),

that is: $y \in W_{E_{\Sigma^{-1}}} \iff x \in W_E$. To prove the converse direction of the statement, we assume the validity of (2). Thus we obtain always $e = \Sigma e^1 \in m \cap W_{E_{\Sigma^{-1}}}$ as $e^1 \in bs \cap W_E$. On account of Proposition 3 E is a weak wedge space. \square

Now, we are going to formulate the main result of this note.

Theorem 5. *Let E be an FK -space containing cs . Then the implication*

$$bs \cap W_E \subset F \implies bs \cap W_E \subset W_F.$$

holds for any L_φ - K -space F .

In case of FK -spaces E containing c_0 and separable FK -spaces F this result is due to D. Seydel [7, 3.21]. He has also shown that his result –and therefore the statement in Theorem 5– is not contained in [5, Theorem 4.5].

Proof of Theorem 5. First of all, let E be a weak wedge space. Then $bs \cap W_E \subset F$ implies $m \cap W_{E_{\Sigma^{-1}}} \subset F_{\Sigma^{-1}}$ on account of Proposition 4. Since $F_{\Sigma^{-1}}$ is also an L_φ -space (see [2, Theorem 4.5]), we get $m \cap W_{E_{\Sigma^{-1}}} \subset W_{F_{\Sigma^{-1}}}$ by [5, Theorem 4.5] thus $bs \cap W_E \subset W_F$ by Proposition 4. In case of $e^k \not\rightarrow 0$ ($\sigma(E, E')$) we obtain $W_E = cs$ by Proposition 2. Because of [5, Theorem 4.2] this gives us

$$bs \cap W_E = cs \subset F \implies cs \subset W_F. \quad \square$$

The proof of Theorem 5 shows us that the theorem holds in case of weak wedge FK -spaces E containing φ . Therefore, the question arises whether it is true in general for FK -spaces E containing φ .

Proposition 6. *Let E be an FK -space containing cs . If E is a weak wedge space then the following statements hold:*

- (a) *An L_φ - K -space F containing $bs \cap W_E$ is a weak wedge space.*
- (b) *$bs \cap W_E \supseteq cs$.*
- (c) *$(bv_0, \sigma(bv_0, bs \cap W_E))$ is sequentially complete.*
- (d) *If E is an L_φ -space, then $bs \cap E = bs \cap W_E$ if and only if $(bv_0, \sigma(bv_0, bs \cap E))$ is sequentially complete.*

Proof. (a) Exactly as in the proof of Theorem 5 the inclusion $bs \cap W_E \subset F$ implies $m \cap W_{E_{T-1}} \subset W_{F_{T-1}}$, thus $e \in W_{F_{T-1}}$. Thus, F is a weak wedge space by Proposition 3.

(b) By Proposition 4 the identity $bs \cap W_E = cs$ is equivalent to $m \cap W_{E_{T-1}} = c$. However, that contradicts Theorem 4.5 of [5].

(c) This statement is an immediate corollary of Theorem 5 and [5, Theorem 4.4].

(d) In case of $bs \cap E = bs \cap W_E$ the sequential completeness of $(bv_0, \sigma(bv_0, bs \cap E))$ is immediately implied by (c). For a proof of the converse implication we remark that on account of [5, Theorem 4.2] the sequential completeness of $(bv_0, \sigma(bv_0, bs \cap E))$ implies the continuity of the embedding

$$i : (bs \cap E, \sigma(bs \cap E, bv_0)) \longrightarrow (E, \sigma(E, E')).$$

Because of $x^{[n]} \longrightarrow x(\sigma(bs \cap E, bv_0))$ we obtain $x^{[n]} \longrightarrow x(\sigma(E, E'))$, that is $x \in W_E$ for each $x \in bs \cap E$. \square

Remark 7. If an FK -space E with $E \supset cs$ is not a weak wedge space, then in general, in Proposition 6 the statements (a)–(d) do not remain valid. In that case we have $bs \cap W_E = cs$, where $(bv, \sigma(bv, cs))$ is sequentially complete, however, the subspace bv_0 is not sequentially closed (since $e^{[n]} \longrightarrow e(\sigma(bv, cs))$, $e^{[n]} \in bv_0$ and $e \in bv \setminus bv_0$) thus not sequentially complete. Indeed, for each matrix B we get the implication $cs \subset c_B \Rightarrow cs \subset W_B$ by the monotonicity of the FK -topologies; applying [5, Theorem 4.4] by that we obtain the $\sigma(bv, cs)$ -sequential completeness of $bv = cs^\beta$. To prove that bv_0 is not sequentially closed in $(bv, \sigma(bv, cs))$ we consider a normal matrix $A = (a_{nk})$ being a regular series to series transformation. For each $x \in cs$ we have $\langle a^{(n)}, x \rangle := \sum_{k=1}^n a_{nk} x_k \longrightarrow \sum_k x_k = \langle e, x \rangle$,

that is $a^{(n)} := (a_{nk})_k \xrightarrow{n \rightarrow \infty} e(\sigma(bv, cs))$. Consequently, there exists a sequence $a^{(n)}$ in bv_0 converging in $(bv, \sigma(bv, cs))$ to $e \in bv \setminus bv_0$.

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Received July 8, 1992

Nõrgad kiilruumid ja Mazur-Orliczi tüüpi teoreemid

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Artiklis [5] tõestatud Mazur-Orliczi tüüpi teoreemi kohaselt kehtib iga FK -ruumi E ja L_φ -ruumi F korral implikatsioon

$$(*) \quad M \cap W_E \subset F \Rightarrow M \cap W_E \subset W_F,$$

kus M on jadaruum, mille faktorjadad rahuldavad teatavaid tingimusi. D. Seydel [7] näitas, et juhul $M := bs$ (tõestatud osasummadega ridade ruum) ei ole mainitud tingimused täidetud, kuid väide (*) kehtib eeldusel $E \supset c_0$. Käesolevas artiklis tõestatakse väide (*) $M := bs$ korral, kui $E \supset cs$. Seejuures kasutatakse autorite eelpoolmainitud teoreemi juhul $M := m$ ning G. Bennetti [1] poolt defineeritud nõrkade kiilruumide omadusi.

ALMOST COMMUTATIVITY OF SPECTRALLY BOUNDED ALGEBRAS

Arne Kokk

Introduction. It is well-known that in any complex commutative unital Banach algebra A the spectrum of an element $a \in A$ is just the range of the corresponding Gelfand map \hat{a} , so that the spectral radius r_A of A is submultiplicative and, equivalently, subadditive on A . Besides, it is also well-known that each of these conditions separately implies a given complex unital Banach algebra A to be commutative modulo the Jacobson radical and, as a matter of fact (see, for example, [2,3,20]), the following conditions on a complex unital Banach algebra A are equivalent:

(1) the spectral radius r_A of A is submultiplicative, i.e. there exists $K > 0$ such that $r_A(ab) \leq Kr_A(a)r_A(b)$ for all a, b in A ;

(2) the spectral radius r_A of A is subadditive, i.e. there exists $K > 0$ such that $r_A(a+b) \leq K(r_A(a) + r_A(b))$ for all a, b in A ;

(3) algebra A is almost commutative, i.e. algebra A is commutative modulo the Jacobson radical.

At first sight it would seem that many of the proofs of the equivalences (1)-(3) strongly rely on the existence of Banach algebra norm on A because they use the E. Vesentini theorem for subharmonic functions [17] or an improvement for Banach algebras of the classical Jacobson density theorem obtained by

A. M. Sinclair [16, p. 36].

However, as we obtained in [10], nearly the same happens to be true under weaker hypotheses on A . More precisely, by means of the Hirschfeld-Zelazko theorem [8], we showed in [10], among other things, that if A is a spectrally bounded algebra with an identity such that

$$(1') \quad r_A(ab) \leq r_A(a)r_A(b) \quad (a, b \in A),$$

then A is almost commutative.

The main object of this paper is to proceed along the line and to prove even more. Namely, we show that the conditions (1)-(3) are still equivalent within the context of Gelfand-Mazur Q -algebras. (Precise definitions will follow below): It is perhaps worth pointing out that there exist Gelfand-Mazur Q -algebras which by no means can be made into Banach algebras. Take, for example, the complex algebra of all complex-valued infinitely differentiable functions on an interval [4, p. 95].

1. Preliminaries. Throughout the following all algebras are assumed to be associative and over the field \mathbb{C} .

If A is an algebra without identity, then A_1 stands for the algebra formed by adjoining an identity e to A ; and if A has an identity, then $A_1 = A$.

For any linear subspace B of an algebra A ; B^* is the algebraic dual of B equipped with the weak $*$ -topology, $\text{Hom}A$ is the subset of all non-zero multiplicative functionals of A^* and $c(A)$ is the set of all n -tuples $a = (a_1, a_2, \dots, a_n)$ of elements of $a_i \in A$ with arbitrary finite length n . If $\text{Hom}A$ is non-empty and $a = (a_1, a_2, \dots, a_n)$ is an n -tuple in $c(A)$, then the Gelfand transform \hat{a} of a is a function on the space

$\text{Hom} A$ with values in \mathbb{C}^n :

$$\hat{\alpha}(\Lambda) = (\Lambda(a_1), \Lambda(a_2), \dots, \Lambda(a_n)) \quad (\Lambda \in \text{Hom} A).$$

For each subset S of A , $L(S)$ (resp. $\langle S \rangle$) is the linear span of S (resp. the subalgebra of A generated by S), and we write in the sequel $L(a)$ in place of $L(\langle a \rangle)$ and $\langle a_1, a_2 \rangle$ in place of $\langle \langle a_1, a_2 \rangle \rangle$.

Further, the set of all quasi-invertible elements of A is denoted by $q\text{-Inv} A$, $\text{Rad} A$ is the Jacobson radical of A , and algebra A is said to be *almost commutative* if $[a, b] = ab - ba$ is contained in $\text{Rad} A$ for all a, b in A .

The spectrum $\sigma_A(a)$ of an element a of an algebra A is

$$\sigma_A(a) = \{\alpha \in \mathbb{C} : a - \alpha e \text{ is not invertible in } A\}$$

and the spectral radius of a is

$$r_A(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\},$$

where $r_A(a) = 0$ if $\sigma_A(a)$ is empty.

An algebra A is said to be *spectrally bounded* if the spectrum of every element in A is bounded or, equivalently, if the spectrum of every element in A is compact [14].

If B is a linear subspace of A we let $S(B, A)$ denote the set of all *spectral states* of B with respect to A , that is

$$S(B, A) = \{f \in B^* : f(b) \in \text{conv}(\sigma_A(b)) \text{ for all } b \in B\}$$

(Here $\text{conv}(\sigma_A(b))$ stands for the convex hull of $\sigma_A(b)$ in \mathbb{C}). $S(B, A)$ is a convex subset of B^* and the set of all its extreme points is denoted by $\text{ext} S(B, A)$. It is a routine matter to verify that if A is spectrally

bounded, $SB(A)$ is compact.

The left (resp. right) joint spectrum $\sigma_A^l(a)$ (resp. $\sigma_A^r(a)$) of an n -tuple $a = (a_1, a_2, \dots, a_n) \in c(A)$ with respect to A is defined to be the set of all those $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in \mathbb{C}^n for which the n -tuple $(a_1 - \alpha_1 e, a_2 - \alpha_2 e, \dots, a_n - \alpha_n e)$ generates a proper left (resp. right) ideal in A . The Harte joint spectrum $\sigma_A^H(a)$ of $a \in c(A)$ with respect to A is the set $\sigma_A^l(a) \cup \sigma_A^r(a)$.

In what follows $\tilde{\sigma}_A$ will denote one of the joint spectra $\sigma_A^l, \sigma_A^r, \sigma_A^H$; and $\tilde{\sigma}_A$ is said to have the projection property if

$$\pi_k^n(\tilde{\sigma}_A(a_1, a_2, \dots, a_n)) = \tilde{\sigma}_A(a_1, a_2, \dots, a_k)$$

for every $(a_1, a_2, \dots, a_n) \in c(A)$, where $1 \leq k \leq n$ and π_k^n is the projection of \mathbb{C}^n onto \mathbb{C}^k given by

$$\pi_k^n((\lambda_1, \dots, \lambda_n)) = (\lambda_1, \dots, \lambda_k) \quad ((\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n).$$

By a topological algebra we mean an algebra, which is also a Hausdorff topological vector space in such a way that the ring multiplication, considered as a bilinear map of $A \times A$ into A , is continuous. A topological algebra A is called a Q -algebra if the set $q\text{-Inv}A$ is open in the topology of A .

We recall that every Q -algebra is always spectrally bounded [11, p. 80].

Furthermore, a topological algebra A is said to be locally bounded if A has a bounded neighbourhood of the zero element, and if that is the case, the topology of A can be defined by a submultiplicative α -norm ($0 < \alpha \leq 1$) [11, p. 42]. Finally, a Gelfand-Mazur algebra is a topological algebra A such that for every proper closed two-sided maximal (maximal as a left or

as a right) modular ideal M of A the quotient algebra A/M is topologically isomorphic to \mathbb{C} ; and if τ is a topology on an algebra A such that (A, τ) is a Gelfand-Mazur Q -algebra, then we say that τ is a Gelfand-Mazur Q -topology on A . For different classes of Gelfand-Mazur algebras see, for example, [1, 11]. In particular, every locally bounded algebra is Gelfand-Mazur algebra [11, p. 308].

2. Main result. Now we are ready to prove the following theorem.

THEOREM 1. *Let A be a spectrally bounded algebra. The following assertions are equivalent:*

1) if $a \in A$ and $\alpha \in \sigma_{\Delta}(a)$ then there is $\Lambda \in A^*$ satisfying $\Lambda(a) = \alpha$ and $\Lambda(b) \in \sigma_{\Delta}(b)$ for any $b \in A$;

$$2) \sigma_{\Delta}(a) = \{\Lambda(a) : \Lambda \in \text{Hom}A_1\} \quad (a \in A);$$

$$3) \sigma_{\Delta}(ab) \subseteq \sigma_{\Delta}(a)\sigma_{\Delta}(b) \quad (a, b \in A);$$

$$4) \sigma_{\Delta}(a + b) \subseteq \sigma_{\Delta}(a) + \sigma_{\Delta}(b) \quad (a, b \in A);$$

$$5) r_{\Delta}(a) = \sup\{|\Lambda(a)| : \Lambda \in \text{Hom}A_1\} \quad (a \in A);$$

6) r_{Δ} is submultiplicative on A , i.e. there exists $K > 0$ such that $r_{\Delta}(ab) \leq Kr_{\Delta}(a)r_{\Delta}(b)$ for all a, b in A ;

7) r_{Δ} is subadditive on A , i.e. there exists $K > 0$ such that $r_{\Delta}(a + b) \leq K(r_{\Delta}(a) + r_{\Delta}(b))$ for all a, b in A ;

8) algebra A is almost commutative and $A/\text{Rad}A$ can be equipped with a Gelfand-Mazur Q -topology;

9) $\sigma_{\Delta}^l(a) \subseteq \sigma_{\Delta}^r(a)$ for any $a \in c(A)$ and $A/\text{Rad}A$ can be equipped with a Gelfand-Mazur Q -topology;

$$10) \tilde{\sigma}_{\Delta}(a) = \{\hat{\alpha}(\Lambda) : \Lambda \in \text{Hom}A_1\} \quad (a \in c(A));$$

11) $\tilde{\sigma}_{\Delta}$ admits the projection property;

12) if $S \subset A$ and $\lambda \in L(S)^*$ satisfies $\hat{S}(\lambda) \in \tilde{\sigma}_A(s)$ for all $s \in c(S)$, then λ has an extension $\bar{\lambda} \in \text{Hom}A_1$;

13) if $a \in A$ and $\lambda \in L(a)^*$ is such that $\lambda(a) \in \tilde{\sigma}_A(a)$ then λ has an extension $\bar{\lambda} \in \text{Hom}A_1$;

14) for any subalgebra $B \subset A_1$, sharing the identity of A_1 , each $f \in \text{ext}S(B, A_1)$ admits an extension $\bar{f} \in \text{Hom}A_1$;

15) for every $a \in A$ each $f \in \text{ext}S(\langle a, e \rangle, A_1)$ admits an extension $\bar{f} \in \text{Hom}A_1$.

Proof. Implications 2) \rightarrow 3), 2) \rightarrow 4), 2) \rightarrow 5), 3) \rightarrow 6), 4) \rightarrow 7), 5) \rightarrow 6), 10) \rightarrow 11), 12) \rightarrow 13), and 14) \rightarrow 15) are evident. Moreover, 11) \rightarrow 12) and 2) \rightarrow 14) are valid by Lemma 2 and Lemma 3 from [10] and by [10, pp. 126-127] respectively.

1) \Leftrightarrow 2). Put $S = \{\lambda \in A_1^* : \lambda(a) \in \sigma_A(a) \text{ for each } a \in A\}$. An easy calculation shows that we need only to establish $S \subset \text{Hom}A_1$. To this end let λ be an arbitrary functional in S . Then $\lambda(a + \alpha e)$ is contained in the spectrum of $a + \alpha e$ for any $a \in A$ and $\alpha \in \mathbb{C}$. Thus, S is precisely $\text{Hom}A_1$ [14] and this proves the equivalence 1) \Leftrightarrow 2).

6) \Leftrightarrow 7). Recall that in any Banach algebra A the spectral radius r_A is submultiplicative if and only if it is subadditive (see, for example [3]). As a matter of fact, the same argument remains true in any spectrally bounded algebra as well.

6) & 7) \rightarrow 8). We begin with the following preliminary result, well-known for normed algebras [15].

LEMMA 2. *Let A be a locally bounded algebra with an identity. Then $[a, b]$ cannot be the identity of A*

for any a, b in A .

Proof of Lemma 2. If there were a, b in A with $[a, b] = e$, then $[a^n, b] = na^{n-1}$ ($n = 2, 3, \dots$). By hypothesis, A is a locally bounded algebra, and so, as we mentioned earlier, there exists a submultiplicative α -norm $\| \cdot \|$ ($0 < \alpha \leq 1$) on A defining the topology of A . Hence

$$n^\alpha \|a^{n-1}\| \leq 2 \|a\| \|a^{n-1}\| \|b\|$$

for every $n \in \mathbb{N}$, which clearly is impossible because $a^{n-1} \neq 0$ ($n = 2, 3, \dots$). Lemma is proved.

To prove 7) \Rightarrow 8) we need only to consider the case $A \neq \text{Rad}A$.

Let $B = A/\text{Rad}A$ and denote by π the quotient homomorphism of A onto B . Clearly $r_B(\pi(a)) = r_A(a)$ for any a in A and, since r_A is submultiplicative on A , $\text{Rad}A = \{a \in A: r_A(a) = 0\}$. Now it is easy to check that the sets

$$U_\varepsilon = \{b \in B: r_B(b) < \varepsilon\} \quad (\varepsilon > 0)$$

define a local basis of a topology τ on B such that (B, τ) is a locally bounded Q -algebra [11, p. 59] and, consequently, τ is a Gelfand-Mazur Q -topology on B . Moreover, we can define a topology τ' with the same properties on B_1 as well [11, p. 35].

Next we shall show that $[a, b] \in \text{Rad}A$ for every a, b in A .

So, let $a, b \in A$, $c = \pi([a, b])$, and consider the subalgebra $\langle c \rangle$ in B generated by c . If $r_B(c) > 0$ then the spectral mapping theorem yields that an algebra norm $\| \cdot \|$ on $\langle c \rangle$ can be defined by $\|d\| = r_B(d)$ ($d \in \langle c \rangle$). Let D be the completion of $\langle c \rangle$ and consider $\langle c \rangle$ as a subalgebra of D . Then

$$r_D(c) = \lim \sqrt[n]{\|c^n\|} = r_B(c)$$

and, therefore, there is $\alpha \in \sigma_D(c)$ with $|\alpha| = r_B(c)$. Thus, setting $d = c/\alpha$, we have $1 \in \sigma_D(d)$ and $r_D(d) = 1$. By well-known properties of Banach algebras [4, p. 17] we can now find a sequence $\{x_n\}$ ($n \geq 1$) of elements of $\langle c \rangle$ with $r_B(x_n) \rightarrow 1$ ($n \rightarrow \infty$) and $r_B(dx_n - x_n) \rightarrow 0$ ($n \rightarrow \infty$). But this, in turn, implies that the element $d - e$ generates a proper two-sided ideal in B_1 . In fact, if there were $m \in \mathbb{N}$, $a_k, b_k, h, g \in B$ ($k = 1, 2, \dots, m$) and $\lambda \in \mathbb{C}$ satisfying

$$\sum_{k=1}^m a_k(d - e)b_k + h(d - e) + (d - e)g + \lambda(d - e) = e$$

then, using the fact that $r_B(xy) = r_B(yx)$ ($x, y \in B$) and that r_B is, by hypothesis, subadditive and submultiplicative on B , we would obtain

$$r_B\left(\left(\sum_{k=1}^m a_k(d - e)b_k + h(d - e) + (d - e)g + \lambda(d - e)\right)x_n\right) \rightarrow 0$$

($n \rightarrow \infty$).

This of course contradicts the fact $r_B(x_n) \rightarrow 1$ ($n \rightarrow \infty$).

Consequently, because (B_1, τ') is a Q-algebra, there exists a proper closed two-sided ideal I in B_1 containing the element $d - e$ [11, p. 67]. Now, as it is easy to be seen, the quotient algebra B_1/I is a locally bounded Q-algebra under the quotient topology, and the I -coset of $[\pi(a/\alpha), \pi(b)]$ is identity in B_1/I . But this is impossible by the above lemma. We conclude that $r_A([a, b]) = 0$ for every a, b in A . In other words, every commutator is in $\text{Rad}A$ which is what we wanted to prove.

8) \rightarrow 9) Let $a = (a_1, a_2, \dots, a_n) \in c(A)$ and take any n -tuple $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \sigma_A^H(a)$. If $A \neq A_1$ and $\alpha_k = 0$ for all $k = 1, 2, \dots, n$ then $(\alpha_1, \alpha_2, \dots, \alpha_n) = \hat{a}(\Lambda_0)$, where $\Lambda_0 \in \text{Hom}A_1$ is such that $\ker \Lambda_0 = A$. Otherwise there exists a maximal left or right ideal M in A_1 containing the elements $a_k - \alpha_k e$ ($k = 1, \dots, n$) such that $J = M \cap A$ is a two-sided maximal regular left or right ideal in A [12, § 7]. Moreover, as every commutator of elements of A is in the Jacobson radical of A , J is two-sided. If π is again the quotient homomorphism of A onto $B = A/\text{Rad}A$, then $\pi(J)$ is a maximal regular ideal in B , and so there is $\lambda \in \text{Hom}B$ with $\pi(J) = \ker \lambda$. Now $\Lambda = \lambda \circ \pi \in \text{Hom}A$ and $J = \ker \Lambda$. Define $\bar{\Lambda}(a + \alpha) = \Lambda(a) + \alpha$ for all a in A and α in \mathbb{C} . It is easy to be seen that $\bar{\Lambda} \in \text{Hom}A$ and that

$$\bar{\Lambda}(a_k - \alpha_k e)a = \Lambda(a_k a - \alpha_k a) = 0$$

for every $a \in A$ and $k = 1, 2, \dots, n$. Therefore $(\bar{\Lambda}(a_k) - \alpha_k)\Lambda(a) = 0$ ($a \in A$, $k = 1, 2, \dots, n$), and because $\ker \Lambda \neq A$, $(\alpha_1, \alpha_2, \dots, \alpha_n) = \hat{a}(\bar{\Lambda})$. Thus $\sigma_A^H(a) \subseteq \sigma_A^r(a)$.

9) \rightarrow 10) Note that for each a in $c(A_1)$ the left joint spectrum of a is contained in the right joint spectrum of a with respect to the algebra A_1 . Hence every maximal left ideal in A_1 is two-sided [6]. The rest of the proof is analogous to that of 8) \rightarrow 9).

13) \rightarrow 2) Let $a \in A$ and $\alpha \in \sigma_A(a)$ be chosen arbitrarily. The spectral mapping theorem yields that $\alpha = \lambda(a)$ for some $\lambda \in L(a)^*$. Now, by hypothesis, λ has an extension $\bar{\lambda} \in \text{Hom}A_1$ and thus 2) follows.

15) \rightarrow 2) Let $a \in A$. If $\alpha \in \sigma_A(a)$ then, again by the spectral mapping theorem, $\alpha = \lambda(a)$ for a suitable $\lambda \in \text{SC}(a, e, A_1)$ and so, according to the Krein-Milman

theorem,

$$r_A(a) = \sup\{|f(a)| : f \in \text{ext}SC\langle a, e \rangle, A_1\} = \sup\{|\Lambda(a)| : \Lambda \in \text{Hom}A_1\}.$$

The proof of the theorem is completed.

We conclude with some additional remarks.

REMARK 3. The equivalences given in Theorem 1 are known for Banach algebras (see [2,3,5,6,7,9,13,18,20]).

REMARK 4. There exist almost commutative spectrally bounded algebras, which cannot be made into Gelfand-Mazur Q -algebras under any topology. Consider, for example, any commutative division algebra which is not \mathbb{C} .

REMARK 5. In [19] it is proved that if A is a Banach algebra and $a \in A$, then $[a, b] \in \text{Rad}A$ for all $b \in A$ if and only if

$$s(a) = \sup\{r_A(a + b) - r_A(b) : b \in A\} < \infty.$$

Thus, a given Banach algebra A is almost commutative if and only if

$$(4) s(a) < \infty \text{ for any } a \text{ in } A.$$

It would be of interest to know whether this is still true within the context of spectrally bounded algebras.

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Received September 29, 1992

Spektraalselt tõkestatud algebrate peaaegu kommutatiivsus

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Resümee

Käesolevas töös antakse 15 tarvilikku ja piisavat tingimust spektraalselt tõkestatud algebra peaaegu kommutatiivsuseks.

ON STRONG BOUNDEDNESS AND SUMMABILITY WITH
RESPECT TO A SEQUENCE OF MODULI

Enno Kolk

1. Introduction. Let X be a Banach space over the field \mathbb{K} , where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. By $s(X)$, $m(X)$, $c(X)$ and $c_0(X)$ we denote the vector spaces of all X -valued sequences $x = (x_k) = (x_k)_{k \in \mathbb{N}}$ of all bounded sequences in X , of all convergent sequences in X and of all convergent to null sequences in X , respectively. In the case $X = \mathbb{K}$ we write s , m , c and c_0 instead of $s(X)$, $m(X)$, $c(X)$ and $c_0(X)$.

Let λ and μ be two subspaces of s and $A = (a_{nk})$ an infinite matrix with $a_{nk} \in \mathbb{K}$. If for each $u = (u_k) \in \lambda$ the series

$$A_n u = \sum_k a_{nk} u_k = \sum_{k=1}^{\infty} a_{nk} u_k \quad (n \in \mathbb{N})$$

converge and $(A_n u) \in \mu$ then we say that A maps λ into μ . By a theorem of Hahn (see [8]) A maps m into m if and only if

$$(H) \quad \sup_n \sum_k |a_{nk}| = M < \infty.$$

We denote by \mathcal{R}^+ the set of all such matrices A with $a_{nk} \geq 0$.

A matrix A is called regular if A maps c into c and $\lim_n A_n u = \lim_k u_k$ for all $u \in c$. The well-known Silvermann-Toeplitz theorem (see [8]) asserts that A is regular if and only if $\lim_n a_{nk} = 0$ ($k \in \mathbb{N}$),

$\lim_n \sum_k a_{nk} = 1$ and the condition (H) is satisfied. We

use the notation \mathcal{J}^+ for the set of all non-negative regular matrices. For example, the Cesaro matrix C_1 , defined by $a_{nk} = 1/n$ for $k \leq n$ and $a_{nk} = 0$ otherwise, belongs to \mathcal{J}^+ .

For $K \subset \mathbb{N}$ let $\chi(K)$ be the characteristic sequence of K . Let $A \in \mathcal{J}^+$. Following Freedman and Sember [3], a set $K \subset \mathbb{N}$ will be said to have A-density

$$\delta_A(K) = \lim_n A_n \chi(K)$$

when $(A_n \chi(K)) \in c$. In [5,6] the notion of statistical convergence, originally introduced in [2], was extended in the following way. For $A \in \mathcal{J}^+$ a sequence $x = (x_k) \in s(X)$ is said to be A-statistically convergent to $x_0 \in X$ if for every $\varepsilon > 0$,

$$\delta_A(\{k : \|x_k - x_0\| \geq \varepsilon\}) = 0.$$

Connor [1] defines this notion for number sequences in terms of strong A-summability. The sets of all A-statistically convergent to null and A-statistically convergent sequences in X are denoted by $st_0(A, X)$ and $st(A, X)$, respectively.

The class of sequences which are strongly summable with respect to a modulus was introduced by Maddox [7] and extended by Connor [1]. In [4,6] a further extension of these definitions was given by replacing one modulus with a sequence of moduli. We first recall the notion of modulus.

DEFINITION 1. A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus if

- (i) $f(t) = 0$ if and only if $t = 0$,
- (ii) $f(t + u) \leq f(t) + f(u)$ for all $t \geq 0, u \geq 0$,
- (iii) f is increasing,
- (iv) f is continuous from the right at 0.

The notion of strong summability with respect to a sequence of moduli was given in [4,6].

DEFINITION 2. Let $p > 0$ and $A \in \mathcal{J}^+$. For a sequence of moduli $F = (f_k)$ we define

$$w_0^p(A, F, X) = \{x \in s(X) : \lim_n \sum_k a_{nk} [f_k(|x_k|)]^p = 0\}$$

and

$$w^p(A, F, X) = \{x \in s(X) : \exists x_0 \in X, (x_k - x_0) \in w_0^p(A, F, X)\}.$$

If $(x_k - x_0) \in w_0^p(A, F, X)$, we say that x is strongly A -summable to x_0 with respect to the sequence of moduli F .

Similarly we define strong boundedness with respect to a sequence of moduli.

DEFINITION 3. Let $p > 0$ and $A \in \mathcal{K}^+$. For a sequence of moduli $F = (f_k)$ we define

$$w_\infty^p(A, F, X) = \{x \in s(X) : \sup_n \sum_k a_{nk} [f_k(|x_k|)]^p < \infty\}.$$

A sequence $x \in w_\infty^p(A, F, X)$ is called strongly A -bounded with respect to the sequence of moduli F .

We write f instead of F if $f_k = f$ ($k \in \mathbb{N}$) and we omit F in the special case $f_k(t) = t$ ($k \in \mathbb{N}$). In the case $X = \mathbb{K}$ we omit also X . Thus $w_0^p(A, X) = w_0^p(A, F, X)$ for $f_k(t) = t$ ($k \in \mathbb{N}$). $w_0^p(A) = w_0^p(A, \mathbb{K})$, $st(A) = st(A, \mathbb{K})$, and so on.

A sequence space $\lambda \subset s$ is said to be normal if $(v_k) \in \lambda$ whenever $|v_k| \leq |u_k|$ ($k \in \mathbb{N}$) for some $(u_k) \in \lambda$. For example, the sequence spaces m , c_0 , $w_\infty^p(A)$ and $w_0^p(A)$ are normal. For a normal sequence

space λ and for a sequence of moduli $F = (f_k)$ we can consider the sequence spaces

$$\lambda(F) = \{u \in s : (f_k(|u_k|)) \in \lambda\}$$

and

$$\lambda(F, X) = \{x \in s(X) : (f_k(\|x_k\|)) \in \lambda\}.$$

It is clear that above defined sequence spaces $w_0^p(A, F, X)$ and $w_\infty^p(A, F, X)$ are of type $\lambda(F, X)$, where $\lambda = w_0^p(A)$ and $\lambda = w_\infty^p(A)$, respectively.

The results of Connor [1] about the connections between $st(A)$ and $w^1(A, f)$ were generalized in [6]. In this note we investigate the relations between $\lambda(X)$ and $\lambda(F, X)$ where $\lambda \in \{m, c_0, w_\infty^p(A), w_0^p(A)\}$. We give a correction to [6] and extend some results of [1].

2. Strong boundedness with respect to a sequence of moduli. Let $F = (f_k)$ be a sequence of moduli. Our main results will be formulated by means of the conditions

$$(F1) \inf_k f_k(t) > 0 \quad (t > 0),$$

$$(F2) \sup_k f_k(t) < \infty \quad (t > 0),$$

$$(F3) \lim_{t \rightarrow 0^+} \sup_k f_k(t) = 0.$$

We start with two lemmas.

LEMMA 1. *The condition (F1) holds if and only if there exists a $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$.*

Proof. Let $\inf_k f_k(t_0) > 0$ where $t_0 > 0$. For all $t < t_0$ there is a $n \in \mathbb{N}$ such that $t \geq t_0/2^n$. By Definition 1 we have $f_k(t_0/2^n) \geq (1/2^n)f_k(t_0)$. Hence

$\inf_k f_k(t) > 0$ for $t < t_0$. Further, since f_k increases, $f_k(t) \geq f_k(t_0)$ for each $t > t_0$. This shows that $\inf_k f_k(t) > 0$ also for $t > t_0$.

LEMMA 2. *The condition (F2) holds if and only if there is a $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$.*

Proof. Let $\sup_k f_k(t_0) < \infty$. For all $t > 0$ there is a natural number n such that $t \leq nt_0$ and so by Definition 1 we have $f_k(t) \leq nf_k(t_0)$. Consequently $\sup_k f_k(t) < \infty$.

REMARK 1. By Lemma 2 it is clear that (F3) implies (F2).

We now prove some theorems about the inclusion $\lambda(X) \subset \lambda(F, X)$ for various sequence spaces λ .

THEOREM 1. *Let λ be a normal sequence space with $e \in \lambda$, where $e = (1, 1, \dots)$. Then $\lambda(X) \subset \lambda(F, X)$ whenever (F2) holds.*

Proof. If (F2) is true then $\sup_k f_k(1) = L < \infty$. Let $(x_k) \in \lambda(X)$. Using the inequality $\|x_k\| \leq [\|x_k\|] + 1$, where $[t]$ denotes the integer part of t , by Definition 1 we get

$$f_k(\|x_k\|) \leq ([\|x_k\|] + 1)f_k(1) \leq L(\|x_k\| + 1) \quad (k \in \mathbb{N}).$$

Thus, since $e \in \lambda$, by linearity and normality of λ we have $(f_k(\|x_k\|)) \in \lambda$ giving $x \in \lambda(F, X)$. The theorem is proved.

In the case $\lambda = m$ from the inclusion $m(X) \subset$

$\subset m(F, X)$ it follows that $(f_k(1)) \in m$ and so by Lemma 2, (F2) is true. Thus, for $\lambda = m$, Theorem 1 gives the following extension of Theorem 2 of [4].

COROLLARY 1. *The inclusion $m(X) \subset m(F, X)$ is true if and only if (F2) holds.*

Let $\lambda = w_{\infty}^p(A)$, where $p > 0$ and $A \in \mathcal{X}^+$. Clearly $w_{\infty}^p(A)$ is normal and by (H), $e \in w_{\infty}^p(A)$. Thus by Theorem 1, (F2) is sufficient for the inclusion

$$w_{\infty}^p(A, X) \subset w_{\infty}^p(A, F, X). \quad (1)$$

Conversely, let (1) holds for all $A \in \mathcal{M}$, where $\mathcal{M} \subset \mathcal{X}^+$. If \mathcal{M} contains the unit matrix E , then (1) is true for $A = E$ and, by $w_{\infty}^p(E, X) = m(X)$ and $w_{\infty}^p(E, F, X) = m(F, X)$, Corollary 1 shows that (F2) must hold. Consequently we have proved the following version of Theorem 4 [4].

THEOREM 2. *Let $p > 0$ and $\mathcal{M} \subset \mathcal{X}^+$ with $E \in \mathcal{M}$. The inclusion (1) holds for all $A \in \mathcal{M}$ if and only if (F2) is satisfied.*

Let f be a modulus. For every constant sequence F with $f_k = f$ ($k \in \mathbb{N}$), (F2) is automatically satisfied and we get

COROLLARY 2. *Let $p > 0$ and $A \in \mathcal{X}^+$. The inclusion $w_{\infty}^p(A, X) \subset w_{\infty}^p(A, f, X)$ is true for any modulus f .*

Maddox [7] proved Corollary 2 in the case $p = 1$, $A = C_1$ and $X = \mathbb{K}$.

3. Strong summability with respect to a sequence of moduli. For $p > 0$ and $A \in \mathcal{X}^+$ the space $w_0^p(A)$ is normal. So Theorem 1 is valid for $\lambda = w_0^p(A)$ if $e \in w_0^p(A)$, i.e.

$$\lim_n \sum_k a_{nk} = 0. \quad (2)$$

In the case $A = C_1$, the condition (2) is not valid, but Maddox [7] proved that $w_0^1(C_1) \subset w_0^1(C_1, f)$ for any modulus f . Connor [1] showed that this is true for any $A \in \mathcal{J}^+$. Here we prove the following extension of these results.

THEOREM 3. Let $p > 0$ and $\mathcal{M} \subset \mathcal{X}^+$ with $E \in \mathcal{M}$. Then the following statements are equivalent for all $A \in \mathcal{M}$:

- (i) $w_0^p(A, X) \subset w_0^p(A, F, X)$,
- (ii) $w_0^p(A, X) \cap m(X) \subset w_0^p(A, F, X) \cap m(X)$,
- (iii) (F3) is satisfied.

Proof. The implication (i) \rightarrow (ii) is obvious.

For $A = E$ we have $w_0^p(A, X) = c_0(X)$ and $w_0^p(A, F, X) = c_0(F, X)$, so (ii) gives $c_0(X) \subset c_0(F, X)$. If we now suppose that (F3) fails, then there exists a number $\varepsilon_0 > 0$, an infinite index sequence (k_i) and a positive sequence $(t_i) \in c_0$ such that

$$f_{k_i}(t_i) \geq \varepsilon_0 \quad (i \in \mathbb{N}). \quad (3)$$

For a fixed element $z \in X$ with $\|z\| = 1$ we define the sequence (x_k) by $x_{k_i} = t_i z$ and $x_k = 0$ for $k \neq k_i$ ($i \in \mathbb{N}$). Then $(x_k) \in c_0(X)$ which implies $(x_k) \in c_0(F, X)$. But in view of (3) we have

$$f_{k_i}(\|x_{k_i}\|) = f_{k_i}(t_i) \geq \varepsilon_0 \quad (i \in \mathbb{N}),$$

contrary to $(x_k) \in c_0(F, X)$. Thus (F3) must hold and so (ii) \rightarrow (iii).

Finally, if (F3) is true then for every $\varepsilon > 0$ there is a number δ such that $0 < \delta < 1$ and $f_k(t) < \varepsilon$ ($k \in \mathbb{N}$) for $t \leq \delta$. For a sequence $(x_k) \in w_0^P(A, X)$ let $\sigma_n = \sum_k a_{nk} \|x_k\|^P$, so that $\lim_n \sigma_n = 0$. We split the sum $\sum_k a_{nk} [f_k(\|x_k\|)]^P$ into two sums Σ_1 and Σ_2 over $\{k : \|x_k\| \leq \delta\}$ and $\{k : \|x_k\| > \delta\}$, respectively. Then by (H),

$$\Sigma_1 < \varepsilon M. \quad (4)$$

Further, since $\sup_k f_k(\delta) = G < \infty$ by (F3) and Lemma 2, for $\|x_k\| > \delta$ we have by Definition 1 that

$$\begin{aligned} f_k(\|x_k\|) &\leq f_k((1 + [\|x_k\|]/\delta)\delta) \leq (1 + \|x_k\|/\delta)f_k(\delta) \leq \\ &\leq (2\delta^{-1}G)\|x_k\| \quad (k \in \mathbb{N}). \end{aligned}$$

Hence $\Sigma_2 \leq (2\delta^{-1}G)^P \sigma_n$, which together with (4) yields $(x_k) \in w_0^P(A, F, X)$. Consequently (iii) \rightarrow (i) and the theorem is proved.

For $M = \{E\}$ from Theorem 3 we deduce the following extension of Theorem 3 of [4].

COROLLARY 3. *The inclusion $c_0(X) \subset c_0(F, X)$ holds if and only if (F3) is satisfied.*

From Theorem 3, using the definitions of $w^P(A, X)$ and $w^P(A, F, X)$, we immediately get also an extension of Proposition 3 of [1].

COROLLARY 4. *Let $p > 0$ and $E \in \mathcal{M} \subset \mathcal{K}^+$. The in-*

clusion $w^p(A, X) \subset w^p(A, F, X)$ holds for all $A \in \mathcal{M}$ if and only if (F3) is satisfied.

REMARK 2. Theorem 3.3 of [6] assert that the condition (F3) is necessary in order that $st_0(A, X) \subset w_0^p(A, F, X)$ ($p > 0$, $A \in \mathcal{U}^+$), but the proof in [6] is incorrect. In fact the necessity of (F3) can be obtain for the matrix class \mathcal{J}^+ , for example, from Corollary 3 in view of equalities $st_0(E, X) = c_0(X)$ and $w_0^p(E, F, X) = c_0(F, X)$. Thus Theorem 3.3 [6] holds for the matrix class \mathcal{J}^+ instead of \mathcal{U}^+ .

In [1] it is proved that the equality $w^1(A, f) \cap m = w^1(A) \cap m$ holds if $A \in \mathcal{J}^+$ and f is a modulus. By Corollaries 3.8 and 3.7 of [6] we extend this result in the following way.

COROLLARY 5. Let $p > 0$, let f be a modulus and let $F = (f_k)$ be a sequence of moduli. Then for all $A \in \mathcal{J}^+$

$$w^p(A, f, X) \cap m(X) = w^p(A, X) \cap m(X)$$

and

$$w^p(A, F, X) \cap m(X) = w^p(A, X) \cap m(X)$$

if and only if (F1) and (F3) hold.

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Received December 14, 1992

Tugev tškestatus ja summeeruvus moodulite jada suhtes
Enno Kolk
Resümee

Artiklis uuritakse Banachi ruumis tugevalt tškestatud ja tugevalt summeeruvate jadade hulkade ning moodulite jada suhtes tugevalt tškestatud ja tugevalt summeeruvate jadade hulkade sisalduvust. Siin tškestatud teoreemid sisaldavad erijuhuna Connori [1] vastavaid tulemusi arvjadade kohta.

ON CORES OF SUMMABILITY METHODS GENERATED
BY WEIGHTED MEANS

Leiki Loone and Epp Tohver

Let $A = (a_{mk})$ be a matrix with $a_{mk} \in \mathbb{R}$. A sequence $x = (\xi_k)$ is said to be A -summable to a if

$$\lim_m \sum_k a_{mk} \xi_k = a.$$

Let $A_m = (a_{mnk})$ ($m = 0, 1, \dots$) be a sequence of matrices with $a_{mnk} \in \mathbb{R}$.

A sequence $x = (\xi_k)$ is called α -summable to a if

$$\lim_m \sum_k a_{mnk} \xi_k = a \text{ uniformly in } n.$$

In the sequel we need the following notations:

$$\omega := \{x = (\xi_k) \mid \xi_k \in \mathbb{R}\}$$

(the set of all sequences);

$$m := \{x = (\xi_k) \in \omega \mid \sup_k |\xi_k| < \infty\}$$

(the set of all bounded sequences);

$$c := \{x = (\xi_k) \in \omega \mid \exists \lim_k \xi_k = a\}$$

(the set of all convergent sequences);

$$c_A := \{x = (\xi_k) \in \omega \mid \exists \lim_m \sum_k a_{mk} \xi_k\}$$

(the set of all A -summable sequences);

$c_\alpha := \{x = (\zeta_k) \in \omega \mid x \text{ is } \alpha\text{-summable}\};$

$m_A := \{x = (\zeta_k) \in \omega \mid \exists M > 0 : \left| \sum_k a_{mk} \zeta_k \right| < M\}$

(the set of all A-bounded sequences);

$c_0 := \{x = (\zeta_k) \in c \mid \lim \zeta_k = 0\}$

(the set of all null sequences);

$c_{0A} := \{x = (\zeta_k) \in \omega \mid (\sum_k a_{mk} \zeta_k) \in c_0\}$

(the set of all A-null sequences).

The set m is a Banach space with a norm

$$\|x\| = \sup_k |\zeta_k|.$$

Let K^0 be the set of all linear continuous functionals on m satisfying the following conditions:

1° $\langle e_k, f \rangle = 0 \quad \forall k = 0, 1, \dots,$

2° $\langle e, f \rangle = 1,$

3° $\|f\| = 1,$

where $e_k = (\underbrace{0, \dots, 0}_k, 1, 0, \dots)$ and $e = (1, 1, \dots, 1, \dots)$.

This set K^0 determines the Knopp's core on m . It means that for every $x = (\zeta_k) \in m$ the set

$$K^0(x) = \{\langle x, f \rangle : f \in K^0\}$$

is the Knopp's core of x , i.e.

$$K^0(x) = \left[\liminf_k \zeta_k, \overline{\lim}_k \zeta_k \right]$$

(see [4]).

Well-known necessary and sufficient conditions for the inclusion

$$K^0(Ax) \subset K^0(x) \quad \forall x \in m \quad (1)$$

are as follows

$$1. \lim_n a_{nk} = 0 \quad \forall k = 0, 1, \dots, \quad (2)$$

$$2. \lim_n \sum_k a_{nk} = 1, \quad (3)$$

$$3. \lim_n \sum_k |a_{nk}| = 1, \quad (4)$$

(see [1]). Let tA be the conjugate matrix to the matrix A . The set

$$K_A = \{{}^tAf : f \in K^0\}$$

determines the A -summability core on m (see [1]).

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and let Q be the set of all operators $q : \mathbb{N} \rightarrow \mathbb{N}$ and let $B_q = (a_{mq(m)k})$. The set

$$K_\alpha = \text{clco} \cup \{{}^tB_q(K^0) : q \in Q\} \quad (5)$$

defines the core which determines α -summability in m . Here "clco" denotes the close and convex hull of the set (see [2;6]).

The necessary and sufficient conditions for the inclusion

$$K_\alpha(x) \subset K^0(x) \quad \forall x \in m \quad (6)$$

are as follows

$$1. \lim_m \sup_n |a_{mnk}| = 0 \quad \forall k = 0, 1, \dots, \quad (7)$$

$$2. \lim_m \sum_k |a_{mnk}| = 1 \quad \text{uniformly in } n, \quad (8)$$

$$3. \lim_m \sup_n \sum_k |a_{mnk}| = 1, \quad (9)$$

(see [6]).

Let L be the set of all Banach functionals on m , i.e.

$$L = \{f \in K^0 \mid \langle Sx - x, f \rangle = 0 \quad \forall x \in m\},$$

where $S = (\delta_{kk+1})$ and δ_{nk} is Kronecker's symbol.

The set L determines almost convergency core on m .

The inclusion

$$K^0(Ax) \subset L(x) \quad \forall x \in m \quad (10)$$

holds iff (1) and

$$\lim_m \sum_k |a_{mk} - a_{mk+1}| = 0 \quad (11)$$

(see [5]).

The inclusion

$$K_\alpha(x) \subset L(x) \quad \forall x \in m$$

holds iff (3) and

$$\lim_m \sup_n \sum_k |a_{mnk} - a_{mnk+1}| = 0 \quad (12)$$

(see [6]).

Let $(p_k) \subset \omega$ and $p_0 = 1$, $p_k > 0 \quad \forall k = 1, 2, \dots$, with

$$P_m = \sum_{k=0}^m p_k \rightarrow \infty.$$

The weighted mean summability method (R, p_k) is defined by Riesz matrix $P = (a_{mk})$, where

$$a_{mk} = \begin{cases} \frac{p_k}{P_m}, & \text{if } k \leq m, \\ 0, & \text{if } k > m. \end{cases}$$

As $P_m \rightarrow \infty$, this method is regular (see [3]).

Let $P = (R, p_k)$ and $Q = (R, q_k)$ be two regular and positive Riesz matrices.

The inclusion

$$K^0(Px) \subset K^0(Qx) \quad \forall x \in m_Q \quad (13)$$

holds iff the inclusion (1) holds for the $A = PQ^{-1}$, where

$$a_{mk} = \begin{cases} \frac{Q_k}{P_m} \left(\frac{P_k}{q_k} - \frac{P_{k+1}}{q_{k+1}} \right) & \text{if } 0 \leq k < m - 1, \\ \frac{Q_m}{P_m} \cdot \frac{P_m}{q_m} & \text{if } k = m, \\ 0 & \text{if } k > m \end{cases}$$

(see [3]).

One can easily check that from the conditions (2) - (4) imply the following

LEMMA 1. *The inclusion (13) holds iff*

$$\lim_m \frac{1}{P_m} \left[\sum_{k=0}^{m-1} Q_k \left| \frac{P_k}{q_k} - \frac{P_{k+1}}{q_{k+1}} \right| + Q_m \frac{P_m}{q_m} \right] = 1. \quad (14)$$

It follows from the form of the condition (14) that for the inclusions of type (13) is important the ratio of P_{k+1} to P_k . This motivates the characterization of the Riesz-matrix (R, p_k) by the means of the sequence (u_k) , where

$$u_k := \frac{P_{k+1}}{P_k} - 1, \quad k = 0, 1, \dots$$

This sequence (u_k) is called ratio-sequence of (R, p_k) . It is obvious that if $p_0 = 1$ then between the sequences (p_k) and (u_k) is one-to-one relation and

$$p_{k+1} = \prod_{i=0}^k (1 + u_i), \quad k = 0, 1, 2, \dots,$$

where

$$\prod_{i=0}^k (1 + u_i) = (1 + u_0)(1 + u_1) \dots (1 + u_k).$$

Because of $p_k > 0$ we have $u_k > -1$ for every k .

COROLLARY 1.1. Let $u = (u_k)$ and $v = (v_k)$ be the ratio-sequences for the matrices P and Q respectively. If

$$u_k \leq v_k \quad \text{for every } k > k_0$$

then (13) holds.

Proof. Because of $u_k \leq v_k$ it holds that

$$\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \geq 0$$

and therefore (14) turns into the following

$$\begin{aligned} & \lim_m \frac{1}{p_m} \left[\sum_{k=0}^{m-1} Q_k \left(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right) + Q_m \frac{p_m}{q_m} \right] = \\ & = \lim_m \frac{1}{p_m} \left[\sum_{k=0}^{m-1} Q_k \frac{p_k}{q_k} - \sum_{k=1}^m Q_{k-1} \frac{p_k}{q_k} + Q_m \frac{p_m}{q_m} \right] = \\ & = \lim_m \frac{1}{p_m} \sum_{k=0}^m (Q_k - Q_{k-1}) \frac{p_k}{q_k} - Q_m \frac{p_m}{q_m} + Q_m \frac{p_m}{q_m} = 1. \end{aligned}$$

Example 1. Let $\lim_k u_k = \underline{a} > -1$ and let $v_k = \underline{a} - \varepsilon$ for every k . If $\varepsilon \in (0, \underline{a} + 1)$ then

$$K_{\underline{a}-\varepsilon} \subset K_P,$$

where $K_{\underline{a}-\varepsilon} := K_Q$ and $Q = (R, (1 + \underline{a} - \varepsilon)^k)$.

The weighted mean α -method (R_m, P_k) is defined by the sequence of matrices (A_m) where $A_m = (a_{mnk})$ and

$$a_{mnk} = \begin{cases} \frac{P_{k-n}}{P_m}, & \text{if } n \leq k \leq m+n, \\ 0, & \text{if } k < n \text{ or } k > m+n. \end{cases}$$

The core of x determined by this α -method is denoted by $K_{\alpha(P)}(x)$.

THEOREM 2. *The inclusion*

$$K_{\alpha(P)}(x) \subset K^0(Qx) \quad \forall x \in m_Q \quad (15)$$

holds iff

1° Q is Mercerian i.e. $c_Q = c$.

$$2^\circ \limsup_m \sup_n \sum_{k=n}^{n+m-1} \frac{Q_k}{P_m} \left| \frac{P_{k-n}}{q_k} - \frac{P_{k-n+1}}{q_{k+1}} \right| + \frac{Q_{n+m}}{P_m} \cdot \frac{P_m}{q_{n+m}} = 1 \quad (16)$$

Proof. Let $G_m = A_m Q^{-1}$. The inclusion (15) holds iff (6) is valid for α -method (G_m) . It is easy to check that

$$g_{mnk} = \begin{cases} \frac{Q_k}{P_m} \left(\frac{P_{k-n}}{q_k} - \frac{P_{k-n+1}}{q_{k+1}} \right) & \text{if } n \leq k \leq n+m-1, \\ \frac{Q_k}{P_m} \cdot \frac{P_{k-n}}{q_k} & \text{if } k = n+m, \\ -\frac{Q_k}{P_m} \cdot \frac{P_{k-n+1}}{q_{k+1}} & \text{if } k = n-1, \\ 0 & \text{if } k < n-1 \text{ or } k > n+m. \end{cases}$$

The condition (8) is valid. Indeed,

$$\limsup_m \sup_n |g_{mnk}| = \lim_m \frac{Q_k}{P_m} \max_{n < k} \left\{ \left| \frac{P_{k-n}}{q_k} - \frac{P_{k+1-n}}{q_{k+1}} \right|, \frac{P_0}{q_{k+1}} \right\} = 0,$$

because $P_m \rightarrow \infty$.

It is easy to check that for every m and n

$$\sum_k g_{mnk} = 1,$$

therefore (9) holds. As

$$|g_{mnn-1}| = \frac{Q_{n-1}}{q_n} \cdot \frac{P_0}{P_m}$$

then for

$$\limsup_m \sup_n \sum_{k=n-1}^{n+m} |g_{mnk}| = 1 \quad (17)$$

is necessary that

$$\sup_n \frac{Q_{n-1}}{q_n} < \infty,$$

i.e. Q is Mercerian (see [3]). If Q is Mercerian then (17) is equivalent to (16) as $P_m \rightarrow \infty$.

COROLLARY 2.1. *If Q is Mercerian and for every k*

$$\max_{0 \leq n \leq k} u_n \leq v_k$$

then (15) holds.

Proof. It follows from Theorem 2 in a manner analogous to Corollary 1.1 following from Lemma 1.

Example 2. If $\overline{\lim}_k u_k = \bar{a} > 0$. Then for every $\varepsilon > 0$

$$K_{\alpha(P)} < K_{\bar{a} + \varepsilon}^{-},$$

where $K_{\bar{a} + \varepsilon}^{-} := K_Q$ and $Q = (R, (1 + \bar{a} + \varepsilon)^k)$, i.e. $v_k = \bar{a} + \varepsilon \quad \forall k \in \mathbb{N}$.

COROLLARY 2.2. *The inclusion*

$$L(x) \subset K^0(Px) \quad \forall x \in m_p \quad (18)$$

holds iff

1^o P is Mercerian

$$2^o \lim_m \frac{1}{m+1} \sup_n \sum_{k=n}^{n+m-1} P_k \left| \frac{1}{P_k} - \frac{1}{P_{k+1}} \right| = 1 \quad (19)$$

Proof follows from Theorem 2 using the fact that $L = K_{\alpha(P_0)}$, where P_0 is the matrix of arithmetical means.

COROLLARY 2.3. *If $\lim_k |u_k| = 0$ then (18) is not valid.*

Proof follows immediately from the fact, that for the Mercerian P

$$\lim_m \frac{1}{m+1} \sup_n \sum_{k=n}^{m+n-1} \frac{P_k}{P_{k+1}} |u_k| = 0$$

if $\lim_k |u_k| = 0$.

THEOREM 3. *The inclusion*

$$K^0(Px) \subset L(x) \quad \forall x \in m \quad (20)$$

holds iff

$$1^\circ \lim_m \frac{P_m}{P_m} = 0, \quad (21)$$

$$2^\circ |u| \in c_{op}, \text{ where } |u| = (|u_k|). \quad (22)$$

Proof. As P is positive then the necessary and sufficient condition for (20) is (11) which in this case has the form

$$\lim_m \sum_{k=0}^{m-1} \left| \frac{P_k}{P_m} - \frac{P_{k+1}}{P_m} \right| + \frac{P_m}{P_m} = 0.$$

Therefore (20) holds iff (21) holds and

$$\lim_m \sum_{k=0}^{m-1} \left| \frac{P_k}{P_m} - \frac{P_{k+1}}{P_m} \right| = 0.$$

The last equality is equivalent to (22) as

$$\begin{aligned} \lim_m \sum_{k=0}^{m-1} \left| \frac{P_k}{P_m} - \frac{P_{k+1}}{P_m} \right| &= \lim_m \left(1 - \frac{P_m}{P_m} \right) \sum_{k=0}^{m-1} \frac{P_k}{P_{m-1}} |u_k| = \\ &= \lim_m \sum_{k=0}^{m-1} \frac{P_k}{P_{m-1}} |u_k|. \end{aligned}$$

COROLLARY 3.1. *The inclusion*

$$K_{\alpha P}(x) \subset L(x) \quad \forall x \in m \quad (23)$$

holds iff holds the inclusion (20).

Proof. As P is positive then (6) holds. The condition (12) turns into the condition (11) for every α -method $\alpha(P)$, as

$$\begin{aligned} \sup_n \sum_{k=n}^{n+m-1} \left| \frac{P_{k-n}}{P_m} - \frac{P_{k+1-n}}{P_m} \right| + \frac{P_m}{P_m} &= \\ = \sum_{k=0}^{m-1} \left| \frac{P_k}{P_m} - \frac{P_{k+1}}{P_m} \right| + \frac{P_m}{P_m} \end{aligned}$$

COROLLARY 3.2. 1. If $\lim_k |u_k| > 0$ then (20) is

not valid,

2. If $\lim_k |u_k| = 0$ then (20) holds iff holds (21),

3. If $\sum_k |u_k| < \infty$ then (20) holds.

Proof. 1. The matrix P is positive and regular. Since such matrices are core-regular,

$$K^0(P|u|) \subset K^0(|u|)$$

holds. If (20) is valid then (22) holds and therefore

$$\{0\} = K^0(P|u|) \subset K^0(|u|).$$

It means that

$$\lim_k |u_k| = 0.$$

2. As P is regular this statement is an immediate corollary from Theorem 3.

3. Let $\prod_k (1 + u_k) = (1 + u_0)(1 + u_1) \dots (1 + u_k) \dots$

It is well-known that

$$\text{if } \sum_k |u_k| < \infty \quad \text{then} \quad \prod_k (1 + u_k) < \infty.$$

As

$$\prod_{k=0}^n (1 + u_k) = P_{n+1},$$

we have the case when exists

$$\lim_n P_n = c < \infty.$$

Due to the assumption $P_m \rightarrow \infty$ follows (21). Therefore, because of $|u| \in c_0$, the inclusion (20) holds.

THEOREM 4. *The inclusion*

$$K_{\alpha(P)}(x) = K^0(Px) \quad \forall x \in m_p \quad (24)$$

holds iff there exists

$$\lim_k u_k = a > 0 \quad (a \neq \infty). \quad (25)$$

Proof. Since for every regular matrix P the inclusion

$$K_P \subset K_{\alpha(P)}$$

holds, it is sufficient to show that (15) holds for $P = Q$ iff (25) is valid.

1. If (15) then by Theorem 2 P is Mercerian and

$$\limsup_m \sup_n \left[\sum_{k=n}^{n+m-1} \frac{P_k}{P_{k+1}} \cdot \frac{P_{k-n}}{P_m} |u_{k-n+1} - u_{k+1}| + \frac{P_{n+m}}{P_m} \cdot \frac{P_m}{P_{m+1}} \right] = 1.$$

As

$$\sup_n \frac{P_{n+m}}{P_{n+m}} \cdot \frac{P_m}{P_m} \geq \frac{P_{0+m}}{P_{0+m}} \cdot \frac{P_m}{P_m} = 1$$

it is necessary that

$$\limsup_m \sup_n \frac{P_{m+n}}{P_{m+n}} \cdot \frac{P_m}{P_m} = 1. \quad (26)$$

Matrix P is Mercerian, therefore exists $c > 0$ such that

$$\lim_m \frac{P_m}{P_m} = \frac{1}{c}.$$

and therefore it follows directly from (26) that

$$\lim_m \frac{P_m}{P_m} = c, \text{ where } c \in (0, 1].$$

Now statement (16) has turned into statement

$$\limsup_m \sup_n \sum_{k=n}^{n+m-1} \frac{P_k}{P_{k+1}} \cdot \frac{P_{k-n}}{P_m} |u_{k-n+1} - u_{k+1}| = 0. \quad (27)$$

Because of (26), the statement (27) is valid iff

$$\limsup_m \sup_n |u_{m+n} - u_m| = 0, \quad (28)$$

i.e. there exists $a \in \mathbb{R}$ such that

$$\lim_m u_m = a.$$

Now it remains to show that $a > 0$. As

$$\begin{aligned} 1 &= \lim_m \frac{P_{m+1}}{P_{m+1}} \cdot \frac{P_m}{P_m} = \lim_m \frac{P_{m+1}}{P_m} \left[1 - \frac{P_{m+1}}{P_{m+1}} \right] = \\ &= \lim_m (1 + u_m)(1 - c), \end{aligned}$$

therefore

$$\lim_m u_m = \frac{c}{1-c}, \text{ where } c \in (0, 1],$$

i.e. $a > 0$ (As $a \in \mathbb{R}$ we get in addition that $c < 1$).

2. If (25) holds then by Example 1 the inclusion

$$K_{a-\varepsilon} \subset K_P$$

holds for $\varepsilon \in (0, a)$. As matrix $(R, (1 + a - \varepsilon)^k)$ is Mercerian therefore matrix P is Mercerian.

In addition

$$\frac{P_{m+1}}{P_{m+1}} - \frac{P_m}{P_m} \geq \frac{P_{m+1}}{P_{m+1}} - \frac{P_m}{P_{m+1}} = \frac{u_m P_m}{P_{m+1}} \geq 0.$$

It means that the sequence $\left(\frac{P_m}{P_m}\right)$ is monotonous and consequently has limit $c > 0$. Therefore (26) is valid and accordingly from (28) follows (27), i.e. the inclusion (15) holds.

COROLLARY 4.1. *If $\lim_k u_k = a > 0$ then*

$$K^0(Px) = K_a(x) \quad \forall x \in m,$$

where $K_a := K_Q$ and $Q = (R, (1 + a)^k)$.

Proof. If $\lim_k u_k = a > 0$ then we use from the proof of Theorem 4 that

$$\lim_m \frac{P_m}{P_m} = \frac{a}{a+1}$$

and obtain

$$\begin{aligned} \lim_m \frac{Q_m}{P_m} \cdot \frac{P_m}{Q_m} &= \lim_m \frac{[(a+1)^{m+1} - 1]}{a \cdot (1+a)^m} \cdot \frac{P_m}{P_m} = \\ &= \lim_m \left[\frac{a+1}{a} - \frac{1}{a(1+a)^m} \right] \cdot \frac{P_m}{P_m} = 1. \end{aligned}$$

As Q is Mercerian and (25) holds then

$$(\alpha_k) = \left[\frac{Q_k}{q_{k+1}} |u_k - a| \right] \in c_0.$$

Matrix P is Mercerian therefore $(\alpha_k) \in c_{0p}$ and by Lemma 1

$$K^0(Px) \subset K_A(x) \quad \forall x \in m.$$

The inverse inclusion follows from Lemma 1 using the facts that

$$\lim_m \frac{P_m}{Q_m} \cdot \frac{q_m}{p_m} = 1$$

and $(\beta_k) \in c_0 = c_{0Q}$, where

$$\beta_k = \frac{P_k}{p_{k+1}} |u_k - a|.$$

REMARK. It follows from the corollaries 1.1 and 2.2 that for any $\varepsilon \in (0,1)$

$$K_{-\varepsilon} \subset K_{p_0} \subseteq L \subset K_\varepsilon.$$

Therefore the core-inclusions in the case of $a = 0$ are dependent on the sequence (u_m) not only on the limit a (see also Corollary 3.2).

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Received January 11, 1993

Rieszi kaalutud keskmiste abil defineeritud summeeruvusmeetodite tuumasisalduvusest

Leiki Loone ja Epp Tohver

Resümee

Käesolevas töös vaadeldakse erinevate Rieszi menetluse baasil defineeritud tuumade vahelkorda. Sisalduvusi kirjeldatakse kasutades suhete-jada (u_k), kus

$$u_k := \frac{p_{k+1}}{p_k} - 1 \quad \forall n = 0, 1, \dots$$

ja kus (p_k) on Rieszi menetlust (R, p_k) määrav jada. Kasutades töödes [4-6] saadud üldisi tulemusi leitakse tarvilikud ja piisavad tingimused sisalduvusteks (15), (18), (23) ja (24). Osutub, et

$$K_{\alpha(P)} = K_p(x) \quad \forall x \in m_p$$

parajasti siis, kui

$$\lim_k u_k = a > 0,$$

ning sellisel juhul kehtib võrdus

$$K_p = K_a.$$

Siin hulgad K_p , $K_{\alpha(P)}$ ja K_a on vastavalt Rieszi menetlust P , α -menetlust $\alpha(P)$ ja Rieszi menetlust $(R, (1+a)^k)$ defineerivad tuumamäärajad.

A CLASS OF DUAL SEQUENCE SPACES

Ivor J. Maddox

Dedicated to the memory of Professor Brian Kuttner (1908-1992)

Introduction. In the theory of sequence spaces and infinite matrix transformations the properties of generalized Köthe-Toeplitz dual spaces are important.

A number of results on these dual spaces may be found in Maddox [4]. Let us recall the definitions of the α - and β -duals. Suppose that X and Y are Banach spaces over the complex field C , and that $(A_k) = (A_1, A_2, \dots)$ is a sequence of bounded linear operators A_k on X into Y . Denote by $s(X)$ the linear space of all sequences $x = (x_k)$ with x_k in X , and with the usual coordinatewise operations. If E is a non-empty subset of $s(X)$ then the usual α - and β -duals are defined by

$$E^\alpha = \{(A_k) : \sum \|A_k x_k\| < \infty \text{ for all } x \in E\}$$

$$E^\beta = \{(A_k) : \sum A_k x_k \text{ converges in } Y \text{ for all } x \in E\}.$$

Unless indicated otherwise, \sum without limits denotes a sum over $k \in N := \{1, 2, 3, \dots\}$, the natural numbers.

In case E is also a linear subspace of $s(X)$ then we say that E is a

sequence space.

Now suppose that F is a non-empty subset of $s(Y)$ and let us define

$$[E, F] = \{(A_k) : (A_k x_k) \in F \text{ for all } x \in E\}.$$

Then it is clear that the dual space $[E, F]$ is a natural generalization of E^α and E^β . For, if we define

$$\ell_1(Y) = \{y = (y_k) \in s(Y) : \sum \|y_k\| < \infty\} \quad (1)$$

and

$$\gamma(Y) = \{y = (y_k) \in s(Y) : \sum y_k \text{ converges in } Y\}$$

then

$$E^\alpha = [E, \ell_1(Y)] \quad \text{and} \quad E^\beta = [E, \gamma(Y)].$$

Suppose now that $p = (p_k) = (p_1, p_2, \dots)$ is a given sequence of strictly positive real numbers p_k , which is not in general assumed to be bounded. Numerous authors have studied, mainly in the scalar case, the sequence space

$$\ell(p, Y) := \{y \in s(Y) : \sum \|y_k\|^{p_k} < \infty\}.$$

In case $p_k = p_1$ for all $k \in N$ and $Y = C$, the complex field, then $\ell(p, Y) = \ell_{p_1}$, which is a classical Banach space when $p_1 \geq 1$, and a complete p_1 -normed space when $0 < p_1 < 1$.

In case $0 < p_k \leq 1$ for all $k \in N$, and $Y = C$, the space

$$\ell(p) := \ell(p, C) = \{y : \sum |y_k|^{p_k} < \infty\}$$

has been investigated in some detail by Simons [5], and others [1], [2], [3], have considered (p_k) without the restriction $0 < p_k \leq 1$.

Main results. Our main object is to characterize completely the dual space $[\ell_1(X), \ell(p, Y)]$, where $\ell_1(X)$ is defined by (1), being the case $p_k = 1$ for all $k \in N$. We suppose that $p = (p_k)$ is an arbitrary strictly positive sequence. Thus we shall determine necessary and sufficient conditions for (A_k) to be such that

$$\sum \|A_k x_k\|^{p_k} < \infty \quad \text{for all } x \in \ell_1(X).$$

As an interesting consequence of this result we deduce necessary and sufficient conditions on positive sequences $r = (r_k)$ and $s = (s_k)$ in order that

$$\ell(r, X) = \ell(s, X).$$

THEOREM 1. Let $p = (p_k)$ be an arbitrary strictly positive real sequence and write

$$S = \{k \in N : p_k \geq 1\} \quad \text{and} \quad T = \{k \in N : p_k < 1\}.$$

Then $(A_k) \in [\ell_1(X), \ell(p, Y)]$ if and only if

$$\sup_S \|A_k\| < \infty, \tag{2}$$

and

$$\text{there exists } M \in N \text{ such that } \sum_T \|A_k\|^{c_k} M^{-c_k} < \infty, \tag{3}$$

where $c_k := p_k / (1 - p_k)$ for $k \in T$.

Proof. For the sufficiency, let $x \in \ell_1(X)$. First, consider only those $k \in S$, so that $p_k \geq 1$. By (2) there is a positive constant H such that $\|A_k\| < H$ for all $k \in S$. Now choose $n \in N$ such that

$$\sum_{k=n}^{\infty} \|x_k\| < H^{-1}.$$

Then $H\|x_k\| < 1$ for all $k \geq n$, whence

$$(H\|x_k\|)^{p_k} \leq H\|x_k\|, \quad \text{for } k \geq n; k \in S.$$

Consequently,

$$\|A_k x_k\|^{p_k} \leq \|A_k\|^{p_k} \|x_k\|^{p_k} \leq H\|x_k\|$$

for $k \geq n$ and $k \in S$, whence

$$\sum_S \|A_k x_k\|^{p_k} < \infty. \quad (4)$$

Next, consider only those $k \in T$, so that $p_k < 1$. We shall use the inequality

$$by \leq b^r + y^r, \quad (5)$$

valid for $b \geq 0, y \geq 0$ and $r > 1$, with

$$\frac{1}{r} + \frac{1}{s} = 1.$$

Take the $M \in N$ which occurs in (3) and define

$$b = \|A_k\|^{p_k} M^{-p_k}, \quad y = M^{p_k} \|x_k\|^{p_k},$$

with $r = 1/p_k$. Then (5) yields

$$\|A_k x_k\|^{p_k} \leq by \leq (\|A_k\|/M)^{p_k} + M\|x_k\|, \quad (6)$$

and $sp_k = 1/(r-1) = p_k/(1-p_k) = c_k$. Hence (3), (4) and (6) imply that $(A_k) \in [\ell_1(X), \ell(p, Y)]$, which proves the sufficiency.

For the necessity, let $(A_k) \in [\ell_1(X), \ell(p, Y)]$. If S is empty we shall regard (2) as being satisfied automatically. Otherwise, if we suppose that (2) is false then there is a natural number sequence $(k(i))$ with $\|A_{k(i)}\| > 2i^2$ for all $i \in N$, whence there exists $z_{k(i)} \in X$ with $\|z_{k(i)}\| = 1$ such that $\|A_{k(i)} z_{k(i)}\| > i^2$. If we define $x_k = i^{-2} z_k$ for $k = k(i), i = 1, 2, \dots$, and $x_k = 0$ otherwise, we have $\sum \|x_k\| = \pi^2/6$ and for $k = k(i), \|A_k x_k\| = i^{-2} \|A_k z_k\| > 1$, whence $\|A_k x_k\|^{p_k} > 1$ for infinitely many k , contrary to the fact that $(A_k x_k) \in \ell(p, Y)$. Hence (2) is necessary.

If T is empty we regard (3) as satisfied. Otherwise we suppose (3) fails, and without loss of generality we assume that $T = N$. Thus we may determine integers $n(0), n(1), \dots$ with $0 = n(0) < n(1) < n(2) < \dots$ and

$$M(s) := \sum \|A\|^c (s+1)^{-c} > 1 \quad (7)$$

for $s = 0, 1, \dots$; the sum in (7) being over $1 + n(s-1) \leq k \leq n(s)$. For simplicity, we have written A instead of A_k and c instead of c_k .

Now take any k with $1 + n(s-1) \leq k \leq n(s)$. If $\|A_k\| = 0$ we define $x_k = 0$. If $\|A_k\| > 0$ we choose $z_k \in X, \|z_k\| = 1$, with $2\|A_k z_k\| > \|A_k\|$, and then define

$$x_k = \frac{z_k \|A_k\|^c}{M^{1/p} (s+1)^{1/p(1-p)}}. \quad (8)$$

In (8) we have written A instead of A_k , c instead of c_k , etc. with M for $M(s)$.

Using the facts that $M > 1$ and $p < 1$ we find that

$$\sum \|x_k\| \leq (s+1)^{-2},$$

the sum being over $1 + n(s-1) \leq k \leq n(s)$. Hence $x \in \ell_1(X)$. Also, continuing to omit the subscript k :

$$\|Ax\|^p \geq \frac{\|A\|^{p(1+c)}}{2^p M(s+1)^{1/(1-p)}}.$$

But $p(1+c) = c, 2^{-p} > 2^{-1}$ and $1/(1-p) = c+1$, whence we have

$$\sum_{k=1+n(s-1)}^{n(s)} \|A_k x_k\|^{p_k} \geq 2^{-1}(s+1)^{-1},$$

from which it follows that $\sum \|A_k x_k\|^{p_k}$ diverges. This proves the theorem.

Next, we employ the result of Theorem 1 to determine exactly when two spaces $\ell(r, X)$ and $\ell(s, X)$ are equal, for arbitrary positive sequences $r = (r_k)$ and $s = (s_k)$.

If p, q and t are any positive sequences then we define $pt := (p_k t_k)$, and likewise for other products. Also, we write $p^{-1} := (p_k^{-1})$. First we observe that

$$\ell(p, X) \subseteq \ell(q, X) \text{ if and only if } \ell(pt, X) \subseteq \ell(qt, X). \quad (9)$$

For, if $\ell(p, X) \subseteq \ell(q, X)$ and $x \in \ell(pt, X)$, let us define $y_k = 0$ if $\|x_k\| = 0$ and

$$y_k = x_k \|x_k\|^{t_k} / \|x_k\|, \text{ if } \|x_k\| > 0.$$

Then $y \in \ell(p, X) \subseteq \ell(q, X)$, whence

$$\sum \|x_k\|^{q_k t_k} = \sum \|y_k\|^{q_k} < \infty,$$

and so $x \in \ell(qt, X)$. Similarly for the converse.

THEOREM 2. Let $r = (r_k)$ and $s = (s_k)$ be arbitrary positive real sequences. Then

$$\ell(r, X) = \ell(s, X)$$

if and only if there exists $(M, H) \in N \times N$ such that

$$\sum_{r_k < s_k} M^{-c_k} < \infty \quad \text{and} \quad \sum_{r_k > s_k} H^{-d_k} < \infty, \quad (10)$$

where $c_k = r_k / (s_k - r_k)$ for $r_k < s_k$ and $d_k = s_k / (r_k - s_k)$ for $r_k > s_k$.

Proof. Let us define $p := rs^{-1}$. Then using (9) and the result of Theorem 1 we see that the statements (11), (12), (13), (14) below are equivalent:

$$\ell(r, X) = \ell(s, X) \quad (11)$$

$$\ell(p, X) = \ell_1(X) \quad (12)$$

$$\ell_1(X) \subseteq \ell(p, X) \quad \text{and} \quad \ell_1(X) \subseteq \ell(p^{-1}, X) \quad (13)$$

$$\text{there exists } (M, H) \in N \times N \text{ such that (10) holds.} \quad (14)$$

We remark that the statement $\ell_1(X) \subseteq \ell(p, X)$ is equivalent to the statement that $(A_k) \in [\ell_1(X), \ell(p, X)]$, where for all $k \in N$, $A_k = I$, the identity operator on X . Since $\|I\| = 1$, the requirement that $\sup_s \|I\| < \infty$ is satisfied automatically. This completes the proof.

In the special case in which $X = C$, the complex field, and with the

restriction that $0 < p_k \leq 1$ for all $k \in N$, Theorem 2 shows that $\ell(p) = \ell_1$ if and only if there exists $M \in N$ such that

$$\sum_{p_k < 1} M^{-c_k} < \infty,$$

where $c_k = p_k / (1 - p_k)$. This special case, which is due to J.H. Conway and H.T. Croft, appeared in the paper of Simons [5], though the proof there is quite different from ours.

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Received March 11, 1992

A NOTE ON M-IDEALS OF COMPACT OPERATORS

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1. Introduction. By now, many examples of those Banach spaces X are known for which $K(X)$ is an M -ideal in $L(X)$ (recently this list of examples was enlarged by [7]). Therefore it would be interesting to know how to obtain, departing from Banach spaces X such that $K(X)$ is an M -ideal in $L(X)$, new classes of M -ideals of compact operators.

In Section 4 of this note, we show that if $K(X)$ and $K(Y)$ are M -ideals in corresponding spaces of bounded operators, then $K(X, Y)$ is an M -ideal in $L(X, Y)$. More generally, we prove that in this case $K(E, F)$ is an M -ideal in $L(E, F)$ if E and F are closed subspaces of quotient spaces of X and Y , and E has the metric compact approximation property.

Our note is inspired by the recent paper [7] of Kalton. In [7] (cf. also [6]) a complete classification of those separable Banach spaces X with separable dual is given for which $K(X)$ is an M -ideal in $L(X)$. The above-mentioned results of Section 4 are based on an extension of Kalton's classification to the non-separable case, obtained in Section 3, and also on Corollary 2.3 from [10] giving sufficient conditions for a subspace E of a Banach space X to be an M -ideal in X . The results of Section 3 are prepared by those of Section 2 on strong variants of

Kalton's properties (M) and (M^*) from [7]; we shall call them properties (sM) and (sM^*) .

Let us fix some notation. For (real or complex) Banach spaces X and Y we denote by $L(X, Y)$ ($L(X)$ if $X = Y$) the Banach space of all bounded linear operators from X to Y , and by $K(X, Y)$ ($K(X)$ if $X = Y$) its subspace of compact operators. Let I_X or I denote the identity on X . For $A_\alpha \in L(X)$, where α is an index, we put $A^\alpha = I - A_\alpha$.

A closed subspace E of X is called an M -ideal in X if its annihilator E^\perp is complemented in the dual X^* by a subspace G such that for each $x^* \in X^*$

$$\|x^*\| = \|g\| + \|h\|$$

whenever $x^* = g + h$, $g \in G$, $h \in E^\perp$.

We denote by $B_X(a, r)$ the closed ball with center a and radius r in X , and by B_X the closed unit ball of X .

2. Properties (sM) and (sM^*) . We introduce strong variants of Kalton's properties (M) and (M^*) (cf. [7]) as follows.

We say that a Banach space X has property (sM) if

$$\limsup \|x + x_\lambda\| \leq \limsup \|y + x_\lambda\|$$

whenever $\|x\| \leq \|y\|$ and (x_λ) is a bounded weakly null net in X .

We say that X has property (sM^*) if

$$\limsup \|x^* + x_\lambda^*\| \leq \limsup \|y^* + x_\lambda^*\| \quad (1)$$

whenever $\|x^*\| \leq \|y^*\|$ and (x_λ^*) is a bounded weak*-null net in X^* .

Replacing nets by sequences in the above definitions gives us respectively the notions of properties (M) and (M^*) [7].

PROPOSITION 1. *If a Banach space X is separable, then properties (sM^*) and (M^*) are equivalent for X.*

If X^ is separable, then properties (sM) and (M) are equivalent for X.*

Proof. We consider only the case where X is separable. If X^* is separable, the argument is similar.

Evidently (sM^*) always implies (M^*) . To prove the converse, suppose X has property (M^*) . If X has no (sM^*) , then

$$\limsup \|x^* + x_\lambda\| > \limsup \|y^* + x_\lambda\|$$

for some $x^*, y^* \in X^*$ with $\|x^*\| \leq \|y^*\|$, and for a weak*-null net (x_λ^*) in some $B_{X^*}(0, r)$. By passing to a subnet, we may suppose that

$$a = \lim \|x^* + x_\lambda\| > \lim \|y^* + x_\lambda\| = b$$

(and both limits exist). Since $B_{X^*}(0, r)$ is metrizable in its weak*-topology, there is a neighbourhood base of zero (U_n) such that $U_1 \supset U_2 \supset \dots$. For $n = 1, 2, \dots$, pick $x_{\lambda(n)}^* \in U_n$ so that

$$|\|x^* + x_{\lambda(n)}^*\| - a| < 1/n$$

and

$$|\|y^* + x_{\lambda(n)}^*\| - b| < 1/n.$$

Then $x_{\lambda(n)}^* \rightarrow_n 0$ weak*, $\lim_n \|x^* + x_{\lambda(n)}^*\| = a$ and $\lim_n \|y^* + x_{\lambda(n)}^*\| = b$. Hence $a \leq b$ by property (M^*) . This contradiction proves that X has property (sM^*) .

Similarly to [7, Prop. 2.3], one can prove

PROPOSITION 2. *Let X be a Banach space with property (sM^*) . Then X has property (sM) and X is an M -ideal in X^{**} .*

Since for a reflexive Banach space X , property (sM) means property (sM^*) for X^* , and (sM) for X^* means (sM^*) for X , we have

COROLLARY 3. *For reflexive Banach spaces, properties (sM) and (sM^*) are equivalent.*

Similarly to Lemmas 2.1 and 2.2 in [7], one can prove the following lemma which extends Lemma 2.2 of [7] to the two spaces case.

LEMMA 4. *Let X and Y be Banach spaces and let $T \in L(X, Y)$ have $\|T\| \leq 1$.*

If both X and Y have property (sM) , (u_λ) and (v_λ) are relatively norm-compact nets in X and Y respectively with $\|v_\lambda\| \leq \|u_\lambda\|$ for each λ , and (x_λ) is a bounded weakly null net in X (all three nets defined on the same directed set), then

$$\limsup_{\lambda} \|v_{\lambda} + Tx_{\lambda}\| \leq \limsup_{\lambda} \|u_{\lambda} + x_{\lambda}\|.$$

If both X and Y have property (sM^*) , (u_{λ}^*) and (v_{λ}^*) are relatively norm-compact nets in X^* and Y^* respectively with $\|u_{\lambda}^*\| \leq \|v_{\lambda}^*\|$ for each λ , and (y_{λ}^*) is a bounded weak*-null net in Y^* (all three nets defined on the same directed set), then

$$\limsup_{\lambda} \|u_{\lambda}^* + T^*y_{\lambda}^*\| \leq \limsup_{\lambda} \|v_{\lambda}^* + y_{\lambda}^*\|.$$

3. Characterization of Banach spaces X for which $K(X)$ is an M -ideal in $L(X)$. The following Theorem 5 together with its Corollary 6 extend Theorems 2.4 and 2.6 of [7] to the non-separable case. Note that our proofs are somewhat easier than those of [7].

THEOREM 5. *The following statements about a Banach space X are equivalent.*

(a) $K(X)$ is an M -ideal in $\text{span } K(X) \cup \{I_X\}$.

(b) There is a net (K_{α}) in $K(X)$ such that $K_{\alpha}^*x^* \rightarrow x^*$ for all $x^* \in X^*$, $\limsup \|K_{\alpha}\| \leq 1$ (and consequently $\lim \|K_{\alpha}\| = 1$), and

$$\limsup_{\alpha} \|S + K^{\alpha}\| \leq 1 \quad \forall S \in B_{K(X)}.$$

(c) X has property (sM^*) and there is a net (K_{α}) in $B_{K(X)}$ such that $K_{\alpha}^*x^* \rightarrow x^*$ for all $x^* \in X^*$, and

$$\limsup_{\alpha} \|K_{\beta} + K^{\alpha}\| \leq 1 \quad \forall \beta.$$

(d) X has property (SM) and there is a net (K_α) in $B_{K(X)}$ such that $K_\alpha^* x^* \rightarrow x^*$ for all $x^* \in X^*$, and

$$\limsup_{\beta} \limsup_{\alpha} \|K_\beta + K^\alpha\| \leq 1. \quad (2)$$

(e) $K(X)$ is an M-ideal in $L(X)$.

Proof. (a) \Rightarrow (b) follows immediately from [15, Prop. 2.3] by using a convex combinations argument to obtain from a net of adjoint compact operators, converging to I_{X^*} in the weak operator topology, a net converging to I_{X^*} in the strong operator topology.

(b) \Rightarrow (c). To prove that X has property (SM^{*}), we first suppose $\|x^*\| < \|y^*\|$. Let (x_λ^*) be a bounded weak^{*}-null net in X^* . Then

$$\lim_{\lambda} \|K_\alpha^* x_\lambda^*\| = 0 \quad \forall \alpha.$$

Now choose $y \in X$ so that $y^*(y) = 1$ and $\|x^*\| < 1/\|y\|$. Denoting $S = x^* \otimes y$, we have $S \in B_{K(X)}$, $x^* = S^* y^*$ and $\lim_{\lambda} \|S^* x_\lambda^*\| = 0$. Hence, for all α ,

$$\begin{aligned} \limsup_{\lambda} \|x^* + x_\lambda^*\| &= \limsup_{\lambda} \|S^* y^* + K^{\alpha*} x_\lambda^*\| \leq \\ &\leq \limsup_{\lambda} \|S^*(y^* + x_\lambda^*) + K^{\alpha*}(y^* + x_\lambda^*)\| + \|K^{\alpha*} y^*\| \leq \\ &\leq \|S + K^\alpha\| \limsup_{\lambda} \|y^* + x_\lambda^*\| + \|K^{\alpha*} y^*\| \end{aligned}$$

and (1) follows by taking \limsup_{α} .

If $\|x^*\| \leq \|y^*\|$, then $\|x^*\| < \|ty^*\|$ for all $t > 1$.
 Consequently one has for all $t > 1$,

$$\limsup_{\lambda} \|x^* + x_{\lambda}^*\| \leq \limsup_{\lambda} \|y^* + x_{\lambda}^*\| + (t - 1)\|y^*\|.$$

Letting $t \rightarrow 1$ yields (1).

Finally, to get a net of compact operators satisfying (c), it is enough to norm the operators K_{α} in (b).

(c) \rightarrow (d) is clear since (sM^*) implies (sM) (see Proposition 2).

(d) \rightarrow (e). As easily follows from Corollary 2.3 of [10, p.38], it is enough to verify the condition

$$\limsup_{\alpha} \|S + TK^{\alpha}\| \leq 1 \quad \forall S \in B_{K(X)}, \quad \forall T \in B_{L(X)}.$$

But since

$$\limsup_{\alpha} \|S + TK^{\alpha}\| \leq \|S - SK_{\beta}\| + \limsup_{\alpha} \|SK_{\beta} + TK^{\alpha}\|$$

for all indices β , and $\|S - SK_{\beta}\| \rightarrow_{\beta} 0$, it is sufficient to prove that

$$\limsup_{\beta} \limsup_{\alpha} \|SK_{\beta} + TK^{\alpha}\| \leq 1. \quad (3)$$

For fixed β , let $x_{\alpha} \in B_X$ be such that

$$\limsup_{\alpha} \|SK_{\beta} + TK^{\alpha}\| = \limsup_{\alpha} \|SK_{\beta}x_{\alpha} + TK^{\alpha}x_{\alpha}\|.$$

As $(SK_{\beta}x_{\alpha})_{\alpha}$ and $(K_{\beta}x_{\alpha})_{\alpha}$ are relatively norm-compact

nets in X with $\|SK_{\beta}x_{\alpha}\| \leq \|K_{\beta}x_{\alpha}\|$, and $K^{\alpha}x_{\alpha} \rightarrow_{\alpha} 0$ weakly in X (this is obvious), we have by Lemma 4

$$\begin{aligned} \limsup_{\alpha} \|SK_{\beta}x_{\alpha} + TK^{\alpha}x_{\alpha}\| &\leq \limsup_{\alpha} \|K_{\beta}x_{\alpha} + K^{\alpha}x_{\alpha}\| \leq \\ &\leq \limsup_{\alpha} \|K_{\beta} + K^{\alpha}\|. \end{aligned}$$

And thus (3) follows by taking \limsup_{β} .

(e) \rightarrow (a) is obvious. This completes the proof of Theorem 5.

COROLLARY 6. *Let X be a Banach space. Then $K(X)$ is an M -ideal in $L(X)$ if and only if X has property (sM) and there is a net (K_{α}) in $K(X)$ such that $K_{\alpha}^{*}x^{*} \rightarrow x^{*}$ for all $x^{*} \in X^{*}$, and $\lim \|I - 2K_{\alpha}\| = 1$.*

Proof. If $K(X)$ is an M -ideal in $L(X)$, then X has property (sM) by Theorem 5, and the existence of (K_{α}) follows from [5, Th.V. 5.4]. Such a net of compact operators may be obtained also from the net (K_{α}) of Theorem 5, (b), proceeding similarly to [7, Th.2.4, (4) \rightarrow (5)] (note that here $X^{*} \otimes X^{**}$ is dense in $K(X)^{*}$ because X^{*} has the Radon-Nikodym property [8]).

To prove the converse, we shall appeal to (d) of Theorem 5. Clearly $\lim \|K_{\alpha}\| = 1$. Thus, by norming the K_{α} , we may suppose $K_{\alpha} \in B_{K(X)}$. Then condition (2) is also satisfied because, similarly to [7, Th. 2.4, (5) \rightarrow (3)], we have

$$\limsup_{\alpha} \|K_{\beta} + K^{\alpha}\| \leq \max \{1, \|I - 2K_{\beta}\|\}$$

for all β .

4. M-ideals of the form $K(X,Y)$. In this section we need the following result which is an immediate consequence of Corollary 2.3 from [10, p.38].

PROPOSITION 7. *Let X and Y be Banach spaces. Let (K_α) be a net in $B_{K(X)}$.*

If $K_\alpha^ x^* \rightarrow x^*$ for all $x^* \in X^*$, and*

$$\limsup_\alpha \|S + TK_\alpha\| \leq 1 \quad \forall S \in B_{K(X,Y)}, \quad \forall T \in B_{L(X,Y)}, \quad (4)$$

then $K(X,Y)$ is an M-ideal in $L(X,Y)$.

If $K_\alpha x \rightarrow x$ for all $x \in X$, and

$$\limsup_\alpha \|S + K_\alpha T\| \leq 1 \quad \forall S \in B_{K(Y,X)}, \quad \forall T \in B_{L(Y,X)}, \quad (5)$$

then $K(Y,X)$ is an M-ideal in $L(Y,X)$.

THEOREM 8. *Let X be a Banach space such that $K(X)$ is an M-ideal in $L(X)$. Then $K(X,Y)$ is an M-ideal in $L(X,Y)$ for all Banach spaces Y with property (sM) , and $K(Y,X)$ is an M-ideal in $L(Y,X)$ for all Banach spaces Y with property (sM^*) .*

Proof. By Theorem 5, X has property (sM^*) (and also (sM)). Let (K_α) be the net from condition (c) of Theorem 5.

To prove the first assertion, it suffices to verify condition (4). And this can be done extending

verbatim the proof of the implication (d) \rightarrow (e) of Theorem 5 to the two spaces case.

To prove the second statement, note that in the inequality of (c), K_β may be replaced by any convex combination of operators K_α . Therefore, as $K_\alpha \rightarrow I_X$ in the weak operator topology, we may suppose, using a convex combinations argument, that $K_\alpha \rightarrow I_X$ in the strong operator topology. Thus, it is enough to verify condition (5). But since

$$\limsup_{\alpha} \|S + K^{\alpha}T\| \leq \|S - K_{\beta}S\| + \limsup_{\alpha} \|S^*K_{\beta}^* + T^*K^{\alpha*}\|$$

for all β , and $\|S - K_{\beta}S\| \rightarrow_{\beta} 0$, it suffices to prove that

$$\limsup_{\alpha} \|S^*K_{\beta}^* + T^*K^{\alpha*}\| \leq 1$$

for all β . This can be done similarly to the proof of condition (3) appealing to the second assertion of Lemma 4.

REMARK. It is clear that in the case where X is separable, (sM) and (sM^{*}) in Theorem 8 may be replaced by (M) and (M^{*}) respectively.

An immediate consequence of Theorem 8 is

COROLLARY 9. *Let X and Y be Banach spaces such that $K(X)$ and $K(Y)$ are M -ideals in $L(X)$ and $L(Y)$. Then $K(X, Y)$ is an M -ideal in $L(X, Y)$.*

REMARK. Corollary 9 extends Corollaries 2.3 and 2.4 from [12].

As every modular sequence space $X = l_{(F_k)}$ (see [9] for the definition and properties) with separable dual can be equivalently normed so that $K(X)$ is an M-ideal in $L(X)$ [7, Prop. 4.1], we have

COROLLARY 10. *Let $l_{(F_k)}$ and $l_{(G_k)}$ be modular sequence spaces with separable duals. Then $l_{(F_k)}$ and $l_{(G_k)}$ can be equivalently normed so that $K(l_{(F_k)}, l_{(G_k)})$ is an M-ideal in $L(l_{(F_k)}, l_{(G_k)})$.*

REMARK. In particular, the same is true for Orlicz sequence spaces l_F and l_G with separable duals.

Now our aim is to extend Corollary 9 (and thus also Corollary 10) by replacing in its assertion X and Y by their quotients of closed subspaces. Note that the notion of a quotient of a closed subspace and that of a closed subspace of a quotient coincide (see e.g. [9], p.85). We need the following extension of Proposition from [11].

PROPOSITION 11. *Let X be a Banach space and $Z \subset Y \subset X$ its closed subspaces. Suppose there are bounded nets (K_α) in $K(X)$ and (T_β) in $K(Y/Z)$ such that $\lim K_\alpha^* x^* = x^*$ and $\lim T_\beta^* u^* = u^*$ for all $x^* \in X^*$ and $u^* \in (Y/Z)^*$. If $(X/Z)^*$ or Y^{**} has the Radon-Nikodym property, then there are nets (L_λ) and (S_λ) (defined on the same directed set) of convex combinations of (K_α) and (T_β) , respectively, such that*

$$\lim_{\lambda} \|QL_{\lambda}j - iS_{\lambda}q\| = 0,$$

where $Q : X \rightarrow X/Z$, $q : Y \rightarrow Y/Z$ are canonical surjections, and $j : Y \rightarrow X$, $i : Y/Z \rightarrow X/Z$ are canonical injections.

Proof. Let $A_{(\alpha,\beta)} = QK_{\alpha}j - iT_{\beta}q$, where $\{(\alpha,\beta)\}$ is directed by the product ordering. Then $A_{(\alpha,\beta)}$ is a bounded net in $K(Y,X/Z)$ such that $y^{**}(A_{(\alpha,\beta)}^*u^*) \rightarrow 0$ for all $y^{**} \in Y^{**}$ and $u^* \in (X/Z)^*$. Consequently (cf. [2], Cor. 1.2), $A_{(\alpha,\beta)} \rightarrow 0$ weakly in $K(Y,X/Z)$. Therefore we have a net $(B_{\lambda})_{\lambda} \in \wedge$ (with $\wedge = \{(\alpha,\beta)\} \times \mathbb{N}$) of convex combinations of $(A_{(\alpha,\beta)})$ which converges to zero in norm. The net (B_{λ}) clearly defines (L_{λ}) and (S_{λ}) as desired.

THEOREM 12. *Let X and Y be Banach spaces such that $K(X)$ and $K(Y)$ are M -ideals in $L(X)$ and $L(Y)$. Let E and F be closed subspaces of quotient spaces of X and Y . If E has the metric compact approximation property, then $K(E,F)$ is an M -ideal in $L(E,F)$.*

Proof. By Theorem 5, X and Y have (sM^*) . Since (sM^*) passes to quotients, (sM^*) implies (sM) , and (sM) passes to closed subspaces, both E and F have property (sM) . Now, if we can find a net (S_{λ}) in $B_{K(E)}$ satisfying $S_{\lambda}^*e^* \rightarrow e^*$ for all $e^* \in E^*$, and

$$\limsup_{\mu} \limsup_{\lambda} \|S_{\mu} + S_{\lambda}\| \leq 1, \quad (6)$$

then $K(E)$ will be an M -ideal in $L(E)$ by Theorem 5, (d), and therefore $K(E,F)$ will be an M -ideal in

$L(E, F)$ by Theorem 8.

To this end, suppose that (by a canonical identification) $E = W/Z$ for some closed subspaces $Z \subset W \subset X$. By [8], X is M -ideal in its bidual, and so do X/Z and E [4]. Therefore $(X/Z)^*$ has the Radon-Nikodym property [8], and there is a net (T_β) in $B_{K(E)}$ such that $T_\beta^* e^* \rightarrow e^*$ for all $e^* \in E^*$ [3, p.679]. Let (K_α) be the net from condition (c) of Theorem 5, and let (L_λ) and (S_λ) be the nets of convex combinations of (K_α) and (T_β) provided by Proposition 11. It is immediate that the properties of (K_α) are shared by (L_λ) .

Note that $S_\lambda \in B_{K(E)}$. For $e^* \in E^*$, pick $u^* \in (X/Z)^*$ such that $e^* = i^* u^*$. Then

$$\begin{aligned} \|S_\lambda^* e^*\| &= \|q^* S_\lambda^* i^* u^*\| \leq \|j^* L^* q^* u^*\| + \\ &+ \|QL_\lambda j - iS_\lambda q\| \|u^*\| \xrightarrow{\lambda} 0. \end{aligned}$$

Hence $S_\lambda^* e^* \rightarrow e^*$ for all $e^* \in E^*$. Since

$$\begin{aligned} \|S_\mu + S^\lambda\| &= \|i(S_\mu + S^\lambda)q\| \leq \|Q(L_\mu + L^\lambda)j\| + \\ &+ \|QL_\mu j - iS_\mu q\| + \|QL^\lambda j - iS^\lambda q\|, \end{aligned}$$

we have

$$\limsup_\lambda \|S_\mu + S^\lambda\| \leq 1 + \|QL_\mu j - iS_\mu q\|$$

for all μ , which implies (6) and completes the proof.

REMARK. Theorem 12 improves Theorem 2 of [11].

It is well-known that a Banach space E for which $K(E)$ is an M -ideal in $L(E)$ necessarily must enjoy the metric compact approximation property [4]. Therefore we immediately have

COROLLARY 13. *Let X be a Banach space such that $K(X)$ is an M -ideal in $L(X)$. Let E be a closed subspace of a quotient space of X . Then $K(E)$ is an M -ideal in $L(E)$ if and only if E has the metric compact approximation property.*

REMARK. Corollary 13 extends Theorem 2.5 of [7] to the non-separable case. It also provides an affirmative answer to the question (mentioned in [13]) whether the property of X that $K(X)$ is an M -ideal in $L(X)$ passes to quotients and closed subspaces of X having the metric compact approximation property.

Finally we apply the ideas of this section to obtain a characterization of (M_∞) -spaces. Recall that a Banach space Y is an (M_∞) -space if $K(Y \oplus_\infty Y)$ is an M -ideal in $L(Y \oplus_\infty Y)$. It is proved in [13] that each of the following two conditions is equivalent to the condition that Y is an (M_∞) -space.

(a) $K(X, Y)$ is an M -ideal for all Banach spaces X .

(b) There is a net (K_α) in $B_{K(Y)}$ such that $K_\alpha y \rightarrow y$ and $K_\alpha^* y^* \rightarrow y^*$ for all $y \in Y$ and $y^* \in Y^*$, and

$$\limsup_\alpha \sup \{ \|K_\alpha y_1 + K_\alpha^* y_2^*\| : y_1, y_2 \in B_Y \} \leq 1.$$

Inspired by Theorem 3.10 of [7], let us introduce the following property which is related to

the (M_∞) -space property but does not involve the compact approximation property like (M_∞) does.

We say that a Banach space Y has property (sK_∞) if

$$\limsup \|y + y_\lambda\| \leq 1$$

whenever $\|y\| = 1$ and (y_λ) is a bounded weakly null net in Y with $\limsup \|y_\lambda\| \leq 1$. If the above inequality holds only for sequences, then we say that Y has property (K_∞) .

Similarly to the proof of Proposition 2.5 in [12], one can prove that every (M_∞) -space has property (sK_∞) .

THEOREM 14. *Let Y be a Banach space with property (sK_∞) . Suppose there is a net (K_α) in $B_{K(Y)}$ such that $K_\alpha^* y^* \rightarrow y^*$ for all $y^* \in Y^*$. Then $K(X, Y)$ has property (U) in $L(X, Y)$ (i.e. every $g \in K(X, Y)^*$ has a unique norm preserving extension to an element of $L(X, Y)^*$) for all Banach spaces X . If moreover $\limsup \|K^\alpha\| \leq 1$, then Y is an (M_∞) -space.*

Proof. Since the case of finite dimensional Y is trivial, we assume that Y is infinite dimensional, and therefore $\limsup \|K^\alpha\| > 0$. By a convex combinations argument, we may also suppose that $K_\alpha y \rightarrow y$ for all $y \in Y$. Thus, as immediately follows from Corollary 2.3 in [10, p.38], to prove the first assertion, it suffices to verify the condition

$$\limsup_\alpha \|S + cK^\alpha T\| \leq 1 \quad \forall S \in B_{K(X, Y)}, \quad \forall T \in B_{L(X, Y)}$$

for some $c > 0$.

Let $c = \min \{1, 1/\limsup \|K^\alpha\|\}$. Choose $x_\alpha \in B_X$

such that

$$\limsup_{\alpha} \|S + cK^{\alpha}T\| = \limsup_{\alpha} \|Sx_{\alpha} + cK^{\alpha}Tx_{\alpha}\|.$$

Note that $y_{\alpha} = cK^{\alpha}Tx_{\alpha} \rightarrow_{\alpha} 0$ weakly and $\limsup_{\alpha} \|y_{\alpha}\| \leq 1$. Note also that (Sx_{α}) is relatively norm-compact. If

$$\limsup_{\alpha} \|Sx_{\alpha} + y_{\alpha}\| > 1,$$

then, by passing to a subnet, we should have

$$\limsup_{\alpha} \|z + y_{\alpha}\| > 1$$

for some $z \in B_Y$. Let $z = ty$ for some $t \in [0,1]$ and $y \in Y$ with $\|y\| = 1$. Then

$$\begin{aligned} \limsup_{\alpha} \|z + y_{\alpha}\| &\leq \\ &\leq t \limsup_{\alpha} \|y + y_{\alpha}\| + (1-t) \limsup_{\alpha} \|y_{\alpha}\| \leq 1. \end{aligned}$$

This is a contradiction, and we are done.

If $\limsup_{\alpha} \|K^{\alpha}\| \leq 1$, then $c = 1$ in the above inequality, and the second statement follows from Proposition 7.

Taking into account the remark before Theorem 14 and the definition (b) of (M_{∞}) -space, we have

COROLLARY 15. *A Banach space Y is an (M_{∞}) -space if and only if Y has property (sK_{∞}) and there is a net (K_{α}) in $B_{K(Y)}$ such that $K_{\alpha}^*y^* \rightarrow y^*$ for all $y^* \in Y^*$, and $\limsup_{\alpha} \|K^{\alpha}\| \leq 1$.*

It is clear that if in Theorem 14 (K_α) is a sequence, then the assumption of the (sK_∞) -property may be replaced by that of the (K_∞) -property. This remark implies e.g. to the case where Y^* is separable and has the approximation property, because then Y^* has the metric approximation property [1, p.247], and so there is a sequence of finite rank operators $K_n \in B_{K(Y)}$ such that $K_n^* y^* \rightarrow y^*$ for all $y^* \in Y^*$ [14, p.321]. Therefore we have

COROLLARY 16. *Let Y be a Banach space with property (K_∞) . If Y^* is separable and has the approximation property, then $K(X, Y)$ has property (U) in $L(X, Y)$ for all Banach spaces X .*

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Received October 07, 1992

INCLUSION THEOREMS FOR STRONG SUMMABILITY

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1. Introduction. Let $\alpha = (A_i)$ be a sequence of matrices $A_i = (a_{nik})$. A sequence $x = (x_k)$ is called α -summable if there exists

$$\lim_n \sum_k a_{nik} x_k = \alpha(x)$$

uniformly in i . Let $a_{nik} \geq 0$ and $p = (p_k)$ be a sequence of positive numbers. A sequence $x \doteq (x_k)$ is called strongly α -summable (to 1) with exponent p if

$$\lim_n \sum_k a_{nik} |x_k - 1|^{p_k} = 0$$

uniformly in i (notation $[c_\alpha]^p\text{-lim } x = 1$). The sets of α -summable, α -summable to zero, strongly α -summable, strongly α -summable to zero sequences are denoted, respectively, by c_α , $(c_\alpha)_0$, $[c_\alpha]^p$ and $[c_\alpha]^p_0$. In the case $\alpha = (A)$, $A = (a_{nk})$ we have $c_\alpha = c_A$ (the summability field of matrix method A) and $[c_\alpha]^p = [c_A]^p$ (the set of all strongly A -summable sequences).

The spaces of all convergent, all convergent to zero and all bounded sequences are, respectively, denoted by c , c_0 and m . Let $\beta = (B_i)$, $B_i = (b_{nik})$, $b_{nik} \geq 0$. The purpose of this paper is to study the inclusions $[c_\alpha]^p \subset [c_\beta]^q$, $[c_\alpha]^p \cap m \subset [c_\beta]^q$, $[c_\alpha] \subset c_\beta$, $[c_\alpha]^p \cap m \subset c_\beta$.

2. The inclusion $[c_\alpha]^p < [c_\beta]^q$, $p_k > q_k$. We write $[c_\alpha]^p < [c_\beta]^q$ (reg) if $[c_\alpha]^p < [c_\beta]^q$ and $[c_\alpha]^p\text{-lim } x = [c_\beta]^q\text{-lim } x$ for every $x \in [c_\alpha]^p$.

THEOREM 1. Suppose that $0 < q_k < p_k$, $r = \sup \frac{q_k}{p_k} < 1$, $\lambda = \inf \frac{q_k}{p_k} > 0$ and $b_{nik} \neq 0$ implies $a_{nik} \neq 0$. If the conditions

$$\sup_{n, i, k} \sum b_{nik} \frac{1}{1-r} a_{nik} \frac{r}{r-1} < \infty \quad (1)$$

and

$$\sup_{n, i, k} \sum b_{nik} \frac{1}{1-\lambda} a_{nik} \frac{\lambda}{\lambda-1} < \infty \quad (2)$$

are fulfilled then $[c_\alpha]^p < [c_\beta]^q$ (reg).

Proof. Let $x = (x_k) \in [c_\alpha]^p$ and $[c_\alpha]^p\text{-lim } x = l$.

Put $w_k = |x_k - l|^{p_k}$ and $\lambda_k = \frac{q_k}{p_k}$. Then

$$\lim_n \sum_k a_{nik} w_k = 0 \quad (3)$$

uniformly in i . Define

$$u_k = \begin{cases} w_k, & w_k \geq 1, \\ 0, & w_k < 1 \end{cases}$$

and

$$v_k = \begin{cases} w_k, & w_k \leq 1, \\ 0, & w_k > 1. \end{cases}$$

Then $w_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, $u_k \leq w_k$, $v_k \leq w_k$, $u_k^{\lambda_k} \leq u_k^r$ and $v_k^{\lambda_k} \leq v_k^\lambda$. By Hölder's inequality we obtain

$$\begin{aligned} \sum_k b_{nik} |x_k - 1|^{q_k} &= \sum_k b_{nik} w_k^{\lambda_k} = \\ &= \sum_k b_{nik} u_k^{\lambda_k} + \sum_k b_{nik} v_k^{\lambda_k} \leq \\ &\leq \sum_k (a_{nik} u_k)^r \frac{b_{nik}}{a_{nik}^r} + \sum_k (a_{nik} v_k)^\lambda \frac{b_{nik}}{a_{nik}^\lambda} \leq \\ &\leq \left(\sum_k a_{nik} w_k \right)^r \left(\sum_k b_{nik} \frac{1}{1-r} a_{nik}^{\frac{r}{r-1}} \right)^{1-r} + \\ &+ \left(\sum_k a_{nik} w_k \right)^\lambda \left(\sum_k b_{nik} \frac{1}{1-\lambda} a_{nik}^{\frac{\lambda}{\lambda-1}} \right)^{1-\lambda}. \end{aligned}$$

Now it follows by (1), (2) and (3) that $(x_k) \in [c_\beta]^q$ and $[c_\beta]^q$ - $\lim x = 1$. This completes the proof.

REMARKS. 1. By Hölder's inequality it is easy to show that in the case $\sup_{n, i, k} a_{nik} < \infty$ condition (2) follows from (1).

2. In the case $(c_\alpha)_0 \subset (c_\beta)_0$ the assumption $q_k < p_k$ may be replaced by $q_k \leq p_k$. Indeed, then we obtain (see the proof of theorem 1)

$$\sum_k b_{nik} u_k^{\lambda_k} \leq \sum_k b_{nik} u_k \leq \sum_k b_{nik} |x_k - 1|^{p_k} \rightarrow 0$$

($n \rightarrow \infty$, uniformly in i) for every $(x_k) \in [c_\alpha]^p$ as $(c_\alpha)_0 < (c_\beta)_0$. Then for $\alpha = \beta$ we have

THEOREM 2. Let $0 < q_k \leq p_k$, $\frac{p_k}{q_k} = 0(1)$ and $\sup_{n,i,k} \sum a_{nik} < \infty$. Then

$$[c_\alpha]^p \subset [c_\alpha]^q \text{ (reg.)}$$

For $\alpha = (A)$ this result is proved by Maddox in [3].

If $c \subset c_\alpha$ and $\alpha(x) = \lim x$ for every $x \in c$ we write $c \subset c_\alpha$ (reg).

THEOREM 3. Let $c \subset c_\alpha$ (reg) and $\lim_k \frac{p_k}{q_k} = \infty$. Then $[c_\alpha]^p \not\subset [c_\alpha]^q$.

Proof. It is known that if $c \subset c_\alpha$ (reg) then there is a bounded sequence $z = (z_k)$ consisting of 0's and 1's so that $z \in c_\alpha$. Let $x = (x_k)$, $x_k =$

$$= \left(\frac{z_n + 1}{z} \right)^{\frac{1}{q_k}} \text{ where } z > 2. \text{ Then } (x_k) \in [c_\alpha]^q \text{ but } (x_k) \notin [c_\alpha]^p \text{ as } \lim_k |x_k|^{p_k} = 0 \text{ and } c \subset c_\alpha \text{ (reg).}$$

3. The inclusion $[c_\alpha]^p \subset [c_\alpha]^q$, $p_k \leq q_k$. Let $\alpha = (A_i)$, $A_i = (a_{nik})$, $a_{nik} \geq 0$ and

$$\alpha_k = \sup_{n,i} a_{nik}$$

LEMMA 1. Let $p_k \geq r > 0$. If $(x_k) \in [c_\alpha]^p$ then

$$\alpha_k^{\frac{1}{p_k}} |x_k| = O(1).$$

This assertion is proved in [4]. Let $\beta = (B_i)$, $B_i = (b_{nik})$, $\beta_k = \sup_{n,i} b_{nik}$.

THEOREM 4. Suppose that $0 < r \leq p_k \leq q_k \leq M < \infty$,

$b_{nik} \neq 0$ implies $a_{nik} \neq 0$ and $\frac{b_{nik}}{a_{nik}} \leq \frac{\beta_k}{\alpha_k}$.

If

$$\frac{1}{\beta_k^{q_k}} \cdot \alpha_k^{-\frac{1}{p_k}} = O(1) \quad (4)$$

then $[c_\alpha]^p \subset [c_\beta]^q$.

Proof. Let $(x_k) \in [c_\alpha]^p$. Then by lemma 1 and condition (4) we have

$$\begin{aligned} & \sum_k b_{nik} |x_k - 1|^{q_k} = \\ & = \sum_k a_{nik} |x_k - 1|^{p_k} \frac{b_{nik}}{a_{nik}} |x_k - 1|^{q_k - p_k} \leq \\ & \leq \sum_k a_{nik} |x_k - 1|^{p_k} \frac{\beta_k}{\alpha_k} \left[\alpha_k^{-\frac{1}{p_k}} O(1) \right]^{q_k - p_k} = \\ & = O(1) \sum_k a_{nik} |x_k - 1|^{p_k} \left(\alpha_k^{\frac{1}{p_k}} \beta_k^{\frac{1}{q_k}} \right)^{q_k} = \\ & = O(1) \sum_k a_{nik} |x_k - 1|^{p_k} \rightarrow 0 \quad (n \rightarrow \infty, \text{uniformly in } i). \end{aligned}$$

Hence $[c_\alpha]^p < [c_\beta]^q$ (reg.).

In the case $\alpha = \beta$ we have

COROLLARY 4.1. Suppose that $0 < r \leq p_k \leq q_k \leq M < \omega$.

Then

$$\frac{p_k - q_k}{\alpha_k^{q_k} p_k} = O(1)$$

implies $[c_\alpha]^p < [c_\alpha]^q$ (reg.).

Let $\alpha = (A)$, $\beta = (B)$. It is known that in some restrictions for A and B condition (4) is necessary for $[c_A]^p < [c_B]^q$ (reg.) (see [1]). For example, if $A = B$ the following theorem is valid.

THEOREM 5. Suppose that A is regular, $\alpha_k = \sup_n a_{nk} > 0$ and $0 < r \leq p_k \leq q_k \leq M < \omega$. Then the inclusion $[c_A]^p < [c_A]^q$ (reg.) holds iff

$$\frac{p_k - q_k}{\alpha_k^{q_k} p_k} = O(1)$$

REMARK. Let $\alpha = (A)$, $\beta = (B)$. A and B be normal,

$a_{nk} \downarrow$, $\frac{b_{nk}}{a_{nk}} \downarrow$, as $n \uparrow$. Then $\frac{b_{nk}}{a_{nk}} \leq \frac{b_{kk}}{a_{kk}} \leq \frac{\beta_k}{\alpha_k}$.

4. The inclusion $[c_\alpha]^q \cap m < [c_\beta]^p$. The following theorem was proved in [9].

THEOREM 6. Let $\liminf \frac{p_k}{q_k} > 0$ then

$$[c_\alpha]^{q_k} \cap m \subset [c_\alpha]^{p_k}.$$

Then by theorem 2 and theorem 6 we have

COROLLARY 6.1. Let $q_k \leq p_k$, $\frac{p_k}{q_k} = O(1)$ and

$\sup_{n,i} \sum_k a_{nik} < \infty$. Then

$$[c_\alpha]^{q_k} \cap m = [c_\alpha]^{p_k} \cap m.$$

For $p = (1)$ we denote $[c_\alpha]^p = [c_\alpha]$.

COROLLARY 6.2. Let $0 < r \leq q_k \leq M < \infty$ and

$\sup_{n,i} \sum a_{nik} < \infty$. Then

$$[c_\alpha]^{q_k} \cap m = [c_\alpha] \cap m.$$

Proof. Let $\mu = (M)$. Then by corollary 6.1 we have

$$[c_\alpha]^{q_k} \cap m = [c_\alpha]^\mu \cap m$$

and by theorem 6

$$[c_\alpha]^\mu \cap m = [c_\alpha] \cap m$$

consequently

$$[c_\alpha]^{q_k} \cap m = [c_\alpha] \cap m.$$

COROLLARY 6.3. Let $q_k \leq p_k$, $\frac{p_k}{q_k} = O(1)$,

$\sup_{n,i} a_{nik} < \infty$ and $(c_\alpha)_0 < (c_\beta)_0$. Then

$$[c_\alpha]^q \cap m = [c_\beta]^p \cap m.$$

5. The inclusion $[c_\alpha]^p \subset c_\beta$. It is shown in [8] that if there exists $\lim \sum_n \sum_k a_{nik} = a \neq 0$ (uniformly in i) and $1 \leq p_k \leq M < \infty$ then $[c_\alpha]^p \subset c_\alpha$. For $0 < p_k < 1$ the inclusion might not hold. Let $\alpha = (A)$, $A = (C, 1)$. We denote $[c_A]^p = w(p)$, i.e.

$$w(p) = \{ x = (x_k) \mid \exists l, \lim_n \frac{1}{n+1} \sum_{k=0}^n |x_k - l|^{p_k} = 0 \}$$

The following theorem was announced in [6]

THEOREM 7. Let X be a FK-space and $0 < p_k = \tilde{p} < 1$, $k = 0, 1, \dots$. Then $w(p) \subset X$ implies $m \subset X$.

Let $X = c_A$, $A = (C, 1)$, then matrix method A is regular¹ and it is well known that for regular methods $m \subset c_A$. In this case by theorem 7 we have $[c_A]^p \subset c_A$.

Applying theorem 2 we may prove the following generalization of theorem 7

THEOREM 8. Let X be a FK-space and $q = (q_k)$ where $0 < r \leq q_k \leq \tilde{p} < 1$. Then $w(q) \subset X$ implies $m \subset X$.

¹The matrix method A is called regular if $c \subset c_A$ and $A\text{-lim } x = \lim x$ for every $x \in c$.

Proof. Let $p = (\tilde{p})$ then $\frac{\tilde{p}}{q_k} \leq \frac{\tilde{p}}{r}$ and by theorem 2 we obtain that $w(p) \subset w(q)$. Now $w(q) \subset X$ implies $w(p) \subset X$ and by theorem 7 we have $m \subset X$.

It is well known that if X is a coregular FK-space² then $m \subset X$ and hence we have

COROLLARY 7.1. *If X is a coregular FK-space then $w(p) \subset X$.*

6. The inclusion $[c_\alpha] \cap m \subset c_\beta$. Let \mathbb{N} be the set of natural numbers and $\varphi_Z = (\varphi_k^Z)$ be the characteristic sequence of the set $Z \subset \mathbb{N}$, i.e.

$$\varphi_k^Z = \begin{cases} 1, & k \in Z, \\ 0, & k \notin Z. \end{cases}$$

The next theorem characterizes bounded α -summable sequences.

THEOREM 9. [9] *The sequence $(x_k) \in [c_\alpha] \cap m$ iff there exists a set $Z \subset \mathbb{N}$ so that $\lim_{k \in \mathbb{N} \setminus Z} x_k = 1$ and $\varphi_Z \in (c_\alpha)_0$.*

Using theorem 9 we will investigate the inclusion $[c_\alpha] \cap m \subset c_\beta$. Let $c \subset c_\beta$ then there exist limits $\lim_n b_{nik} = b_k$ uniformly in i , $k = 0, 1, \dots$, (see [5], theorem 3).

²The FK-space X is called coregular if the sequence $\psi^n = e - \sum_{k=0}^n e_k$ is not convergent to zero in X .

THEOREM 10. Let $\alpha = (A_i)$, $a_{nik} \geq 0$, $\beta = (B_i)$ and $c \subset c_\beta$. The inclusion

$$[c_\alpha] \cap m \subset c_\beta$$

holds iff

$$\lim_n \sum_{k \in Z} |b_{nik} - b_k| = 0 \quad (5)$$

uniformly in i for every $Z \subset \mathbb{N}$ where $\varphi_Z \in (c_\alpha)_0$.

Proof. Sufficiency. Let $x = (x_k) \in [c_\alpha] \cap m$ and $[c_\alpha]^p\text{-lim } x = 1$. By theorem 9 there exists a set $Z \in \mathbb{N}$ such that the sequence $t = (t_k)$, where $t_k = \varphi_k^{\mathbb{N} \setminus Z} |x_k - 1|$ is convergent to zero and $\varphi_Z \in (c_\alpha)_0$. Let $\gamma = (C_i)$, $C_i = (c_{nik})$ where $c_{nik} = \varphi_k^Z |b_{nik} - b_k|$ and $\mathfrak{L} = (\bar{B}_i)$, $\bar{B}_i = (|b_{nik} - b_k|)$. Then $m \subset (c_\gamma)_0$ (see [5], theorem 4) and $c_0 \subset (c_{\mathfrak{L}})_0$ (see [5], theorem 3). Hence we have

$$\begin{aligned} & \lim_n \sum_k |b_{nik} - b_k| |x_k - 1| = \\ & = \lim_n \sum_k c_{nik} |x_k - 1| + \lim_n \sum_k |b_{nik} - b_k| t_k = 0 \end{aligned} \quad (6)$$

uniformly in i . As $c \subset c_\beta$ it follows from (6) that $(x_k) \in c_\beta$.

Necessity. Assume that there is a set $Z \subset \mathbb{N}$ where $(\varphi_Z) \in (c_\alpha)_0$ but (5) does not hold. Then there exists a bounded sequence (z_k) such that the limit $\lim_n \sum_{k \in Z} (b_{nik} - b_k) z_k$ does not exist uniformly in i (see [2]).

Let $x = (x_k)$, $x_k = \varphi_k^Z \cdot z_k$ then $(x_k) \in c_\beta$ but $(x_k) \notin [c_\alpha]_0 \cap m$. This completes the proof.

Let $b_{nik} \geq 0$ and $c \subset c_\beta$ (reg.) then $b_k = 0$ and we have proved that (5) implies $[c_\alpha] \cap m \subset [c_\beta]$. As

$[c_\beta] \subset c_\beta$ then (5) is necessary for $[c_\alpha] \cap m \subset [c_\beta]$.
Hence we have

COROLLARY 10.1. *Let $c \subset c_\beta$ (reg.) and $b_{nik} \geq 0$.
Then $[c_\alpha] \cap m \subset [c_\beta]$ iff*

$$\lim_n \sum_{k \in Z} b_{nik} = 0$$

for every $Z \in \mathbb{N}$ where $\varphi_Z \in (c_\alpha)_0$.

For core-regular α -methods this result is proved in [7].

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Received December 07, 1992

Sisalduvus tugeva summeeruvuse korral

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Resümee

Olgu $\alpha = (A_i)$ maatriksite $A_i = (a_{nik})$ jada. Jada $x = (x_k)$ nimetatakse α -summeeruvaks, kui $\lim_n \sum_k a_{nik} x_k = \alpha(x)$ ühtlaselt i suhtes. Olgu $a_{nik} \geq 0$ ja $p = (p_k)$ - positiivne arvjada. Jada (x_k) nimetatakse astmega p tugevalt α -summeeruvaks (arvuks 1), kui $(|x - 1|)$ on α -summeeruv nulliks. Tähistame sümboolitega c_α ja $[c_\alpha]^p$ vastavalt kõigi α -summeeruvate ja tugevalt α -summeeruvate jadade hulga, sümbooliga m kõigi tõkestatud jadade hulga.

Käesolevas artiklis on uuritud sisalduvusi $[c_\alpha]^p \subset [c_\beta]^q$, $[c_\alpha]^p \cap m \subset c_\beta$, $[c_\alpha]^p \cap m \subset [c_\beta]$.

EINE UNIVERSALE BEWEISMETHODE FÜR TAUBER-SÄTZE

Tamara Sõrmus

1. Es sei $A = (a_{nk})$ eine Dreiecksmatrix (d.h. $a_{nk} = 0$ für $k > n$) mit reellen Elementen. Eine Reihe

$$\sum_{k=0}^{\infty} u_k \quad (1)$$

komplexer Zahlen, wobei

$$U_n = \sum_{k=0}^n u_k$$

ist, heisst A-summierbar zum Wert U, wenn

$$U'_n = \sum_{k=0}^n a_{nk} U_k$$

und $\lim_n U'_n = U$ gilt. Die Folge (U_n) heisst dabei A-limitierbar zum Wert U und man schreibt $A\text{-}\lim_n U_n = U$. Definitionsmässig ist das Verfahren A permanent, wenn A summiert jede konvergente Reihe (1) (limitiert jede Folge (U_n) aus dem Raum c der konvergenten Folgen) zum Wert $\lim_n U_n$. Unter geeigneten zusätzlichen Bedingungen für die Reihe (1) (oder Folge (U_n)) kann man jedoch von $A\text{-}\lim_n U_n = U$ auf $\lim_n U_n = U$ zurückschliessen. Solche Bedingungen heissen Tauber-Bedingungen (TBn) vom A \rightarrow c-Typ, entsprechende Aussagen heissen Tauber-Sätze (kurz: T-Sätze). Für das Abel-Verfahren A_b (siehe [1] (Satz 85), [13]

(Satz 20.5)) gelten folgende wohlbekanntete Sätze von Tauber [7] und Littlewood [4].

SATZ A. Die Bedingung

$$ku_k = o(1) \quad (2)$$

ist eine TB vom $A_b \rightarrow c$ -Typ.

SATZ B. Die Bedingung

$$ku_k = O(1) \quad (3)$$

ist eine TB vom $A_b \rightarrow c$ -Typ.

Für das Cesaro-Verfahren (C,1) gilt der folgende T-Satz [1].

SATZ C. Die Bedingung (3) ist eine TB vom (C,1) $\rightarrow c$ -Typ.

Es gibt aber wichtige nichttriviale Summierungsverfahren, die keine divergenten Reihen summieren. Ein erstes Beispiel dazu gab Mercer [6].

SATZ D. Ist $\alpha > 0$ und konvergiert die Folge (U_n) mit

$$U'_n = \alpha U_n + (1 - \alpha) \left(\sum_{k=0}^n U_k \right) / (n + 1),$$

so konvergiert auch (U_n) .

Unter vielen Verallgemeinerungen des Satzes D stellen wir noch einen Satz von Love [5] vor.

SATZ E. Ist Verfahren $A = (a_{nk})$ permanent, $\alpha \in \mathbb{R}$ mit

$$\left| \frac{\alpha - 1}{\alpha} \right| < \left(\overline{\lim}_n \sum_{k=0}^n |a_{nk}| \right)^{-1}, \quad (4)$$

und konvergiert die Folge (U_n) mit

$$U_n = \alpha U_n + (1 - \alpha) \sum_{k=0}^n a_{nk} U_k, \quad (5)$$

so konvergiert auch (U_n) .

Aussagen dieser Art heissen Mercer-Sätze (kurz: M-Sätze). Offensichtlich kann man aus keinem T-Satz einen M-Satz ableiten, weil ein T-Satz gewisse T_{Bn} an Reihengliedern anlegt. Ist es aber möglich vermittels irgendeines passenden M-Satzes einen T-Satz zu beweisen?

In Teilen 2 und 3 der vorliegenden Arbeit wird eine, in gewissen Sinne, universale Methode zum Beweis verschiedenartigen T-Sätze beschrieben. Im wesentlichen müssen dabei gewisse M-Sätze angewandt worden.

2. Der Folgenindex soll, wenn nichts weiteres gesagt ist, von 0 an laufen. Es sei (p_k) eine Folge nichtnegativer Zahlen mit $p_0 > 0$ und

$$P_n = p_0 + \dots + p_n \rightarrow \infty. \quad (6)$$

Damit ist das Verfahren $A = (a_{nk})$ mit

$$a_{nk} = \begin{cases} p_k / P_n & (k = 0, \dots, n), \\ 0 & (k > n) \end{cases} \quad (7)$$

permanent. Wohlbekanntlich ist es ein Riesz-Verfahren (R, p_k) . Wir wollen für das (R, p_k) -Verfahren gleichzeitig zwei T-Sätze beweisen. Man vergleiche dazu auch die Resultate von Tietz [8,9] und Tietz und Trautner [10].

SATZ 1. Gilt (6) mit $p_0 > 0$ und ist (p_k) eine nichtnegative und nichtwachsende Folge, so ist

$$P_k u_k = o(p_k) \quad (8)$$

eine TB vom $(R, p_k) \rightarrow c$ -Typ.

SATZ 2. Gilt (6) mit $p_0 > 0$ und ist (p_k) eine wachsende Folge mit

$$\lim_n \frac{P_n}{p_n} = 0, \quad (9)$$

so ist (8) eine TB vom $(R, p_k) \rightarrow c$ -Typ.

Beweis. Zuerst stellen wir die (R, p_k) -Transformation der Reihe (1) folgendermassen vor:

$$\sum_{k=0}^n a_{nk} U_k = [\alpha U_n + (1 - \alpha) \sum_{k=0}^n b_{nk} U_k] - \sigma_n, \quad (10)$$

wobei

$$\sigma_n = \sum_{k=1}^n a_{nk}^* \frac{P_k}{P_k} \bar{\Delta} U_k \quad (11)$$

und $\bar{\Delta} U_k = u_k$ gilt. Zum Beweis unserer Behauptungen führen wir in (10) die Verfahren $B = (b_{nk})$ und $A^* = (a_{nk}^*)$ so ein, dass

$$a_{nk}^* = \begin{cases} P_k/P_n & (k = 1, \dots, n), \\ 0 & (k > n) \end{cases} \quad (12)$$

ist. Wegen (11) und (12) gilt

$$\sigma_n = \frac{1}{P_n} (P_n U_n - P_n U_{n-1} - \dots - P_2 U_1 - P_1 U_0). \quad (13)$$

Das Verfahren B ist jetzt durch (7) und (10)-(13) eindeutig bestimmt, so dass

$$b_{nk} = \begin{cases} \frac{1}{1-\alpha} \left(a_{n0} - \frac{p_1}{p_n} \right) & (k = 0), \\ \frac{1}{1-\alpha} \left(a_{nk} - \frac{p_{k+1}}{p_n} \right) & (k = 1, \dots, n-1), \\ \frac{1}{1-\alpha} (a_{nn} - \alpha + 1) & (k = n), \\ 0 & (k > n). \end{cases} \quad (14)$$

ist. Wegen unserer Bedingungen ist $A = (R, p_k)$ permanent. Daraus schliesst man unter (14) nach dem Satz von Toeplitz und Schur [11,12,13], dass auch B permanent ist. Wegen der Permanenz von (R, p_k) und (12) ist ebenfalls auch A^* permanent.

Es sei (U_n) eine A -limitierbare Folge. Wegen (8) gilt

$$A^* \text{-}\lim (P_n/p_n) \bar{\Delta} U_n = \lim_{n \rightarrow \infty} \sigma_n = 0.$$

Aus (10) folgt damit die Konvergenz der Folge (5) (mit B anstatt A). Um zu zeigen, dass (8) eine TB vom $(R, p_k) \rightarrow c$ -Typ ist, genügt es zum Schluss nach M-Satz E zu beweisen, dass das B-Verfahren der Bedingung (4) genügt. Unter (7) und (14) gilt entweder

$$\sum_{k=0}^n |b_{nk}| = \frac{1}{|1-\alpha|} \left(\left| 1 - \alpha + \frac{p_n}{p_n} \right| + \sum_{k=1}^{n-1} \frac{|p_k - p_{k+1}|}{p_n} + \frac{|-p_1|}{p_n} \right)$$

oder

$$\sum_{k=0}^n |b_{nk}| = \begin{cases} \frac{1}{|1-\alpha|} \left(\left| 1 - \alpha + \frac{p_n}{p_n} \right| + \frac{1}{p_n} (2p_1 - p_n) \right), \\ \frac{1}{|1-\alpha|} \left(\left| 1 - \alpha + \frac{p_n}{p_n} \right| + \frac{p_n}{p_n} \right), \end{cases} \quad (15)$$

wobei die erste Zeile für nichtwachsende (p_k), die zweite für wachsende (p_k) mit (9) gilt. Deshalb folgt aus (15)

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n |b_{nk}| = \frac{1}{|1-\alpha|} \cdot |1-\alpha| = 1. \quad (16)$$

Weil beliebiges $\alpha > 1/2$ die Bedingung (4) mit (16) erfüllt, so existiert nach M-Satz E auch $\lim_n U_n$, womit alles gezeigt ist.

BEMERKUNG 1.1. Der Satz 1 zeigt, dass mittels der gleichen Methode auch Satz C mit TB (2) bewiesen ist, weil die Voraussetzungen des Satzes 1 für $p_k = 1$ erfüllt sind. Wir erinnern dabei, dass $(C,1) = (R,1)$, $P_n = n + 1$ und $a_{nk} = (n + 1)^{-1}$.

BEMERKUNG 1.2. Die gleiche Methode erlaubt auch den T-Satz C mit der O-Bedingung (3) zu beweisen. Zu diesem Zweck genügt es im vorigen Beweis nur das Verfahren A^* so zu bestimmen, dass A^* jede beschränkte Folge limitiert. Zum Beispiel kann man

$$a_{nk}^* = \frac{1}{n^\rho} \quad (k = 1, \dots, n)$$

mit $\rho > 1$ nehmen.

BEMERKUNG 1.3. Ishiguro [2,3] hat gezeigt, dass

$$u_k = o(1/(k + 1)\ln(k + 1))$$

eine TB vom (1) + c-Typ ist, wobei (1) das permanente logarithmische Verfahren ist. Weil (1) = $(R, \frac{1}{k+1})$, wobei

$$a_{nk} = \frac{1/(k+1)}{\ln(n+2)} \quad \text{und} \quad \ln(n+2) \sim \sum_{k=0}^n \frac{1}{k+1},$$

so erfüllt die Folge $p_k = 1/(k+1)$ die Bedingungen des Satzes 1. Damit ist auch der T-Satz vom (1) \rightarrow c-Typ vermittels unserer gleichen Methode bewiesen.

Das sogenannte Zygmund-Verfahren (Z, ρ) (siehe z.B. [13]) ist wieder ein Sonderfall des (R, p_k) -Verfahrens mit

$$p_k = (k+1)^\rho - k^\rho, \quad P_n = (n+1)^\rho \quad (\rho > 0),$$

dabei ist (Z, ρ) permanent. Es ist klar, dass (Z, ρ) den Bedingungen des Satzes 1 genügt. Man soll noch hinzufügen, dass TB (8) mit $p_k = (k+1)^\rho - k^\rho$ die Form

$$(k+1)^\rho u_k = o((k+1)^\rho - k^\rho)$$

hat und äquivalent zur TB (2) ist. Wenden wir jetzt den Satz 1 auf (Z, ρ) -Verfahren an, so ergibt sich unmittelbar

KOROLLAR 1.1. Die Bedingung (2) ist eine TB vom $(Z, \rho) \rightarrow$ c-Typ ($\rho > 0$).

Das Verfahren (R, a^k) mit $a > 1$, also (R, p_k) mit $p_k = a^k$ und $P_n = \frac{a^{n+1} - 1}{a - 1}$, ist permanent. Alle Voraussetzungen des Satzes 2, ausser (9), sind erfüllt. Anstatt (9) gilt

$$\lim_n \frac{P_n}{p_n} = 1 - \frac{1}{a}. \quad (17)$$

Die TB (8) mit $p_k = a^k$ ist aber äquivalent zur Bedingung $u_k = o(1)$. Für das (R, a^k) -Verfahren kann

man doch unsere Beweismethode anwenden, weil die im Beweis vorkommende Bedingung (16) (für $\alpha > 2 - \frac{1}{a}$ mit (17)) gilt. Damit erhalten wir

SATZ 3. Gilt $a > 1$, so ist $u_k = o(1)$ eine TB vom $(R, a^k) \rightarrow c$ -Typ.

3. Es sei $q > 0$ und $A = (a_{nk})$, mit

$$a_{nk} = \begin{cases} \frac{\binom{n}{k} q^{n-k}}{(1+q)^n} & (k = 0, \dots, n), \\ 0 & (k > n). \end{cases} \quad (18)$$

Bekanntlich [12,13] ist damit ein permanentes Euler-Knopp-Verfahren (E, q) gegeben. Wir wollen zeigen, dass wir unsere Methode zum Beweis eines neuen T-Satzes wieder benutzen können. Dabei wird die TB vom $(E, q) \rightarrow c$ -Typ eine 0-Bedingung sein.

SATZ 4. Gilt $q > 0$, so ist

$$\sqrt{k} u_k = O(1) \quad (19)$$

eine TB vom $(E, q) \rightarrow c$ -Typ.

Beweis. Es sei (U_n) eine (E, q) -limitierbare Folge. Wie im Beweis des Satzes 1, wollen wir von (10) ausgehen, wobei

$$\sigma_n = \sum_{k=1}^n a_{nk}^* \sqrt{k} \bar{\Delta} U_k \quad (20)$$

und

$$a_{nk}^* = \begin{cases} \frac{\sqrt{k}}{n^2} & (k = 1, \dots, n), \\ 0 & (k > n). \end{cases} \quad (21)$$

Wegen der Beziehungen $a_{nk}^* > 0$, $\lim_n a_{nk}^* = 0$ ($k = 1, \dots, n$) und $\lim_n \sum_{k=0}^n |a_{nk}^*| = 0$ (siehe z.B. [13])

transformiert $A^* = (a_{nk}^*)$ alle beschränkte Folgen in o_n (den Raum der gegen Null konvergenten Folgen). Aus (20) und (21) ergibt sich

$$\sigma_n = \frac{1}{n} U_n - \frac{1}{n^2} \sum_{k=0}^{n-1} U_k. \quad (22)$$

Das Verfahren $B = (b_{nk})$ ist vermittleis (10), (18) und (20)-(22) eindeutig durch

$$b_{nk} = \begin{cases} \frac{1}{1-\alpha} \left(a_{nk} - \frac{1}{n^2} \right) & (k = 0, \dots, n-1), \\ \frac{1}{1-\alpha} \left(a_{nn} - \alpha + \frac{1}{n} \right) & (k = n), \\ 0 & (k > n). \end{cases} \quad (23)$$

bestimmt. Da $A = (E, q)$ ($q > 0$) permanent ist, ergibt sich aus (23) nach dem Satz von Toeplitz und Schur die Permanenz von B . Genauso, wie im Beweis des Satzes 1, gilt wegen (19)

$$\lim_n \sigma_n = A^* \text{-} \lim_n \sqrt{n} \bar{\Delta} U_n = 0.$$

Damit konvergiert auch die Folge (5) (mit B anstatt A). Nun bleibt es zu zeigen, dass B die Bedingung (4) erfüllt. Ist $\alpha > 2$, so gilt nach (23)

$$|b_{nn}| = \frac{1}{\alpha-1} \left(\alpha - \frac{1}{(1+q)^n} - \frac{1}{n} \right)$$

und

$$|b_{nk}| = \frac{1}{\alpha-1} \left(\frac{1}{n^2} - a_{nk} \right) \quad (k = 0, \dots, n-1), \quad (n > N),$$

weil es so ein $N > 0$ mit

$$a_{nk} \left(1 - \frac{1}{a_{nk} n^2}\right) < 0 \quad (n > N)$$

gibt. Deshalb haben wir

$$\begin{aligned} \overline{\lim}_n \sum_{k=0}^n |b_{nk}| &= \frac{1}{\alpha-1} \overline{\lim}_n \left(\alpha - \frac{1}{n} - a_{nn} + \sum_{k=0}^{n-1} \left(\frac{1}{n^2} - a_{nk} \right) \right) = \\ &= 1. \end{aligned}$$

Damit gilt (4) für jedes $\alpha > 2$ mit $\overline{\lim}_n \sum_{k=0}^n |b_{nk}| = 1$ und nach M-Satz E existiert $\lim_n U_n$, womit alles gezeigt ist.

BEMERKUNG 3.1. Bekanntlich enthält die Klasse von Hausdorff-Verfahren (H, μ_k) [12,13] das (E, q) -Verfahren. Weil der Satz 4 gilt, ist es nichts Unerwartetes, wenn wir behaupten, dass man mittels der gleichen Methode einen T-Satz auch für (H, μ_k) beweisen kann. Die oben beschriebene Methode zum Beweis von T-Sätzen hat sogar noch allgemeinere Anwendungen.

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Received October 31, 1992

**Universaalne tõestusmeetod Tauberi
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Resümee

Artiklis esitatakse teatavas mõttes universaalne tõestusmeetod Tauberi teoreemide (T-teoreemide) jaoks. Meetodi ideeks on jada (U_n) A-teisenduse esitamine kahe teatava uue teisenduse vahena, mis on alati teostatav. Üks neist kahest summeerib T-tingimusega antud jada, teine aga on jada (U_n) teatav Merceri teisendus. See võimaldab jada (U_n) koonduvust selgitada hoopis Merceri teoreemi abil. Meetodi universaalsus seisneb tema rakendatavuses paljude T-teoreemide tõestamiseks ja sealjuures ka erikujuliste T-tingimuste, näiteks o- ja O-tingimuste, puhul. Uue meetodi demonstreerimiseks on tõestatud neli teoreemi märkuste ja järeldustega. Kahte viimast teoreemi pole käesoleva töö autor kirjandusest leidnud. On põhjust märkida, et selle meetodiga saab tõestada T-teoreeme ka üldiste maatriksmenetluste jaoks.

CONVEXITY CONDITIONS FOR FAMILIES
OF SUMMABILITY METHODS

A. Tali

The present paper extends the authors research on convex families of summability methods started in [6-10]. In the papers mentioned there were proved the theorems that give the *sufficient* conditions for the convexity of the families of summability methods. Some convex families of summability methods were found. The main results of the present paper are theorems 1.5 - 1.7 in section 1 that give *necessary and sufficient* conditions for the convexity of a family of normal matrix methods. The proofs of these theorems base on the Quotient Theorem of H. Baumann (see [1], Theorem 1). In section 2 and 3 the applications of general convexity theorems to certain families are introduced. The convexity theorems for Nörlund methods (N, p_n^α, q_n) known from the papers [3, 5, 6, 7, 10] can be inferred from the proved results as immediate corollaries.

1. The necessary and sufficient conditions for the
convexity of a family of normal summability methods

1.1 Let us consider sequences $x = (\xi_n)$ with $\xi_n \in \mathbb{C}$ (or $\xi_n \in \mathbb{R}$) for $n = 0, 1, 2, \dots$. Let A_α be a family of summability methods given by sequence-to-sequence

transformations of¹ $x \in \omega A_\alpha$ into $A_\alpha x = (\eta_n^\alpha)$ where $\eta_n \in \mathbb{C}$ (or $\eta_n \in \mathbb{R}$) and α is a continuous parameter with values $\alpha > \alpha_0$. We denote, further, by cA_α the summability field of method A_α , by mA_α the set of all sequences x bounded by A_α and by c_0A_α the set of all sequences summed to zero by A_α .

The family of summability methods A_α is said to be convex if for every $\alpha < \beta$ and for every $\alpha < \gamma < \beta$ the conditions

$$mA_\alpha \subset mA_\beta, \quad cA_\alpha \subset cA_\beta \quad (1.1)$$

and

$$cA_\gamma \supset (mA_\alpha \cap cA_\beta) \quad (1.2)$$

hold (see [7]). The family A_α is said to be zero-convex (0-convex) if the conditions (1.1) and (1.2) hold with c_0 instead of c in them (see [7]).

We notice that if the family A_α is convex and all methods A_α are pairwise consistent, then the family A_α is 0-convex also.

The proofs of the convexity and 0-convexity theorems can be simplified by the next three trivial lemmas.

LEMMA 1.1. *If for every $\alpha > \alpha_0$ and $0 < \delta < 1$ the conditions*

$$mA_\alpha \subset mA_{\alpha+\delta}, \quad cA_\alpha \subset cA_{\alpha+\delta} \quad (1.3)$$

and

$$cA_{\alpha+\delta} \supset (mA_\alpha \cap cA_{\alpha+1}) \quad (1.4)$$

hold, then the family A_α is convex.

¹ We denote the transformation of $x \in \omega A_\alpha$ into $A_\alpha x = (\eta_n^\alpha)$ also by A_α . The notation ωA_α is used here for the set where the transformation A_α is applied. As a special case the transformations A_α can be matrix transformations $A_\alpha = (a_{nk}^\alpha)$.

LEMMA 1.2. If for every $\alpha > \alpha_0$ and $0 < \delta < 1$ the conditions

$$mA_\alpha \subset mA_{\alpha+\delta}, \quad c_0 A_\alpha \subset c_0 A_{\alpha+\delta}$$

and

$$c_0 A_{\alpha+\delta} \supset (mA_\alpha \cap c_0 A_{\alpha+1})$$

hold, then the family A_α is 0-convex.

LEMMA 1.3. Let A_α be linear transformations for every $\alpha > \alpha_0$, transforming the sequence $x' = (\xi'_n)$ with $\xi'_n = 1$ ($n = 0, 1, 2, \dots$) into sequences $A_\alpha x' = (\eta'_n)^\alpha$ where

$$\lim_n \eta'_n{}^\alpha = a_\alpha \neq 0. \quad (1.5)$$

If the family A_α is 0-convex then it is convex.

In particular, for matrix transformations $A_\alpha = (a_{nk}^\alpha)$ the condition (1.5) is:

$$\lim_n \sum_k a_{nk}^\alpha = a_\alpha \neq 0.$$

1.2. The idea of the techniques the author has used in her earlier research on convexity of families A_α follows immediately from Lemma 1.3 and the next theorem² (see [7], Theorem 1) that gives the sufficient conditions for 0-convexity of the family A_α .

THEOREM 1.1. Suppose the following conditions hold.

$$1) \quad mA_\alpha \subset mA_\beta, \quad c_0 A_\alpha \subset c_0 A_\beta \quad \text{for every } \alpha < \beta. \quad (1.8)$$

2) For each α , $0 < \delta < 1$ and³ $a < \kappa < b$ there exists a $c_0 \rightarrow c_0$ matrix $Q_{\alpha\delta\kappa} = (q_{nk}^{\alpha\delta\kappa})$, a matrix $R_{\alpha\delta\kappa} =$

² The author has used theorem 1.1 in her research on convex families of strong summability methods also (see [7]).

³ a and b are the real numbers fixed somehow.

$= (r_{nk}^{\alpha\delta})$ and a number $\rho^{\alpha\delta}(x) > 0$ satisfying

$$|\eta_n^{\alpha+\delta}| \leq \left| \sum_k q_{nk}^{\alpha\delta} \eta_k^{\alpha+1} \right| + \left| \sum_k r_{nk}^{\alpha\delta} \eta_k^\alpha \right| \quad (1.6)$$

for every $n = 0, 1, 2, \dots$ and every $x \in c_0 A_{\alpha+1} \cap mA_\alpha$, and

$$\sum_k |r_{nk}^{\alpha\delta}| \leq \rho^{\alpha\delta}(x), \quad \lim_{x \rightarrow b-} \rho^{\alpha\delta}(x) = 0.$$

Then the family A_α is 0-convex.

The main idea of present paper is to better the conditions of Theorem 1.1 in order to get necessary and sufficient conditions for the convexity of the family A_α . This will be done with the help of following theorem of Baumann (see [1], Theorem 1).

THEOREM 1.2. Let A and B be regular matrix methods. Then the following statements are equivalent.

1) $cB \supset (cA \cap m)$.

2) For every $\varepsilon > 0$ there exists a row-finite and column-finite regular matrix method Q_ε and a matrix $R_\varepsilon = (r_{nk}^\varepsilon)$ satisfying $B = Q_\varepsilon A + R_\varepsilon$ and $\limsup_n \sum_k |r_{nk}^\varepsilon| < \varepsilon$.

Suppose further that the methods $A_\alpha = (a_{nk}^\alpha)$ are normal matrix methods, i.e. $a_{nk}^\alpha = 0$ for all $k > n$ and $a_{nn}^\alpha \neq 0$ ($n = 0, 1, \dots$). Then there exists the inverse matrix A_α^{-1} for every A_α . Let us denote by $D_{\alpha\delta}$ the product $A_{\alpha+\delta} A_\alpha^{-1}$ of matrices $A_{\alpha+\delta}$ and A_α^{-1} where $\delta > 0$, i.e. $D_{\alpha\delta} = A_{\alpha+\delta} A_\alpha^{-1}$. Next we will prove the theorem that gives the necessary and sufficient conditions for convexity of a family of normal summability methods $A_\alpha = (a_{nk}^\alpha)$.

THEOREM 1.3. Let $A_\alpha = (a_{nk}^\alpha)$ be normal matrix methods for all $\alpha > \alpha_0$. Then the family A_α is convex and the methods A_α are pairwise consistent if and only if

the following conditions hold for every $\alpha > \alpha_0$ and $0 < \delta < 1$.

1) The matrix method $D_{\alpha\delta} = A_{\alpha+\delta} A_{\alpha}^{-1}$ is regular.

2) $cD_{\alpha\delta} > (cD_{\alpha\delta} \cap m)$ (with consistency).

Proof. It follows from Lemma 1.1 that the family A_{α} is convex and methods A_{α} are consistent if and only if the conditions (1.3) and (1.4) hold for every $\alpha > \alpha_0$ and $0 < \delta < 1$. We notice that the condition 1) and the condition (1.3) are equivalent here. Let us show that the condition 2) is equivalent to the relation (1.4). The relation (1.4) can be rewritten in the form of the next implication:

$$A_{\alpha} x \in m, A_{\alpha+1} x \in c \Rightarrow A_{\alpha+\delta} x \in c. \quad (1.7)$$

With the help of matrices $D_{\alpha\delta} = A_{\alpha+\delta} A_{\alpha}^{-1}$ we get that $A_{\alpha+1} x = D_{\alpha\delta} (A_{\alpha} x)$ and $A_{\alpha+\delta} x = D_{\alpha\delta} (A_{\alpha} x)$. Using the normality of methods A_{α} and the notation $y = A_{\alpha} x$ we can present the implication (1.7) in the equivalent form:

$$y \in m, D_{\alpha\delta} y \in c \Rightarrow D_{\alpha\delta} y \in c.$$

It means that the relation (1.4) is equivalent to the condition 2) for every $\alpha > \alpha_0$ and $0 < \delta < 1$.

Obviously, these considerations keep consistency.

REMARK 1.1. As it can be seen from the proof of Theorem 1.3, if we replace in the condition 1) of this theorem the regularity of the method $D_{\alpha\delta}$ by its conservativity and take the condition 2) without the text in brackets, then we get the necessary and sufficient conditions 1) and 2) for the convexity of a family of normal matrix methods A_{α} (without the additional condition of consistency).

Thus we can state that the convexity of a family of normal matrix methods A_{α} depends only on connection matrices $D_{\alpha\delta}$. Let us agree here that in the following part of our paper we are interested only in these

convex families where the methods A_α are consistent.

We see from the proof of Theorem 1.3 also that the restrictions on the methods A_α can be weakened so that the conditions 1) and 2) of Theorem 1.3 remain sufficient for the convexity of the family A_α .

THEOREM 1.4. Let the summability methods A_α and $A_{\alpha+\delta}$ for every $\alpha > \alpha_0$ and $0 < \delta < 1$ be connected by the row-finite matrix $D_{\alpha\delta}$ so that $A_{\alpha+\delta}x = D_{\alpha\delta}(A_\alpha x)$ for each $x \in \omega A_\alpha$.

Suppose the following conditions hold for every $\alpha > \alpha_0$ and $0 < \delta < 1$.

1) The matrix method $D_{\alpha\delta}$ is regular.

2) $cD_{\alpha\delta} > (cD_\alpha \cap m)$ (with consistency).

Then the family A_α is convex and the methods A_α are pairwise consistent.

Proof is analogous to the proof of sufficiency of the conditions 1) and 2) of Theorem 1.3 and will be therefore omitted.

The next result follows now from Theorems 1.3 and 1.4 as immediate corollary and gives us an idea for constructing new convex families with the help of any convex family of normal matrix methods A_α .

COROLLARY 1.1. Let $A_\alpha = (a_{nk}^\alpha)$ be normal matrix methods with the connection matrices $D_{\alpha\delta} = A_{\alpha+\delta} A_\alpha^{-1}$ for all $\alpha > \alpha_0$ and $0 < \delta < 1$. Let B_α be a further family of summability methods with connection matrices $D_{\alpha\delta}$, i.e. $B_{\alpha+\delta}x = D_{\alpha\delta}(B_\alpha x)$ for all $\alpha > \alpha_0$, $0 < \delta < 1$ and $x \in \omega B_\alpha$. Then the convexity (with consistency) of the family A_α implies the convexity (with consistency) of the family B_α .

We note that the different families with the same connection matrices $D_{\alpha\delta}$ can be easily constructed (see [6,7]). For example, if the family A_α ($\alpha > \alpha_0$) has the connection matrices $D_{\alpha\delta}$ and if A is any summability

method with $Ax \in \omega A_\alpha$ for every $x \in \omega A$ and $\alpha > \alpha_0$, then the family of methods B_α defined by the transformations $B_\alpha x = A_\alpha(Ax)$ has the same connection matrices $D_{\alpha\delta}$.

In order to get for a family of normal matrix methods A_α the exact convexity conditions which would be more constructive than the conditions of Theorem 1.3, we will use Baumann Theorem 1.2. By applying Theorem 1.2 to $A = D_{\alpha_1}$ and $B = D_{\alpha_2}$ we can infer the following result immediately from Theorem 1.3.

THEOREM 1.5. Let $A_\alpha = (a_{nk}^\alpha)$ be normal matrix methods for all $\alpha > \alpha_0$. The family A_α is convex and the methods A_α are pairwise consistent if and only if the following conditions hold for every $\alpha > \alpha_0$ and $0 < \delta < 1$.

1) The matrix method $D_{\alpha\delta} = A_{\alpha+\delta} A_\alpha^{-1}$ is regular.

2) For each $\varepsilon > 0$ there exists a regular row-finite and column-finite matrix method $Q_{\alpha\delta\varepsilon}$ and a matrix $R_{\alpha\delta\varepsilon} = (r_{nk}^{\alpha\delta\varepsilon})$ satisfying

$$D_{\alpha\delta} = Q_{\alpha\delta\varepsilon} D_{\alpha_1} + R_{\alpha\delta\varepsilon} \quad (1.8)$$

and

$$\lim_n \sup_k \sum |r_{nk}^{\alpha\delta\varepsilon}| < \varepsilon. \quad (1.9)$$

By weakening the condition 2) of theorem 1.5 we get the next theorem.

THEOREM 1.6. Let $A_\alpha = (a_{nk}^\alpha)$ be normal matrix methods for every $\alpha > \alpha_0$. The family A_α is convex and the methods A_α are pairwise consistent if and only if the following conditions hold for every $\alpha > \alpha_0$ and $0 < \delta < 1$.

1) The matrix method $D_{\alpha\delta} = A_{\alpha+\delta} A_\alpha^{-1}$ is regular.

2) For each $\varepsilon > 0$ there exists a row-finite $c_0 \rightarrow c_0$ matrix $Q_{\alpha\delta\varepsilon}$ and a matrix $R_{\alpha\delta\varepsilon} = (r_{nk}^{\alpha\delta\varepsilon})$ satisfying conditions (1.8) and (1.9).

Proof. Necessity of the conditions 1) and 2) follows immediately from Theorem 1.5 because the con-

dition 1) is the same in both theorems and the condition 2) in Theorem 1.5 is stronger than in the present theorem.

Sufficiency. Suppose the conditions 1) and 2) hold for every $\alpha > \alpha_0$ and $0 < \delta < 1$. The condition 2) implies the relation $c_{\alpha} D_{\alpha\delta} > (c_{\alpha} D_{\alpha\delta} \cap m)$. The last inclusion, in its turn, implies the relation $c D_{\alpha\delta} > (c D_{\alpha\delta} \cap m)$ (with consistency) because the methods $D_{\alpha\delta}$ and D_{α} are regular. Therefore the convexity of the family A_{α} (with consistency) follows from Theorem 1.3.

By weakening further the condition 2) of Theorem 1.6 we come to the following theorem.

THEOREM 1.7. Let $A_{\alpha} = (a_{nk}^{\alpha})$ be normal matrix methods for all $\alpha > \alpha_0$. The family A_{α} is convex and methods A_{α} are pairwise consistent if and only if the following conditions hold for all $\alpha > \alpha_0$ and $0 < \delta < 1$.

1) The matrix method $D_{\alpha\delta} = A_{\alpha+\delta} A_{\alpha}^{-1}$ is regular.

2) For each $\varepsilon > 0$ there exists a $c_{\alpha} \rightarrow c_0$ matrix $Q_{\alpha\delta\varepsilon} = (q_{nk}^{\alpha\delta\varepsilon})$ and a matrix $R_{\alpha\delta\varepsilon} = (r_{nk}^{\alpha\delta\varepsilon})$ satisfying the conditions (1.9) and

$$\left| \sum_k d_{nk}^{\alpha\delta} \xi_k \right| \leq \left| \sum_k q_{nk}^{\alpha\delta\varepsilon} \sum_{\nu} d_{k\nu}^{\alpha_1} \xi_k \right| + \left| \sum_k r_{nk}^{\alpha\delta\varepsilon} \xi_k \right| \quad (1.10)$$

for every $n = 0, 1, 2, \dots$ and $x \in c_{\alpha} D_{\alpha_1} \cap m$.

Proof. Necessity of the conditions 1) and 2) follows immediately from Theorem 1.6 because the condition 2) of the present theorem is weaker than condition 2) of Theorem 1.6.

Sufficiency of the conditions 1) and 2) can be shown in the same way as in the proof of Theorem 1.6.

REMARK 1.2. If we replace in the condition 2) of Theorem 1.7 the inequality (1.10) by the equivalent inequality

$$|\eta_n^{\alpha+\delta}| \leq \left| \sum_k q_{nk}^{\alpha\delta} \eta_k^{\alpha+1} \right| + \left| \sum_k r_{nk}^{\alpha\delta} \eta_k^\alpha \right|$$

(for every $n = 0, 1, 2, \dots$ and $x \in c_0 A_{\alpha+1} \cap mA_\alpha$), then we get the condition which is weaker than the condition 2) of Theorem 1.1.

REMARK 1.3. The condition 2) has three different constructive forms in Theorems 1.5-1.7. If we want to prove that a family A_α of normal matrix methods (satisfying the condition 1) of these theorems) is convex, it is useful to know Theorem 1.7, but if we want to show that A_α is not convex, then Theorem 1.5 would be more useful.

2. A method for constructing the quotient representations for certain connection matrices between summability methods

Let A_α (where $\alpha > \alpha_0$) be summability methods given by sequence-to-sequence transformations of $x \in \omega A_\alpha$ into $A_\alpha x = (\eta_n^\alpha)$, respectively. Suppose for every $\alpha > \alpha_0$, $\delta > 0$ and $x \in \omega A_\alpha$ the relation⁴

$$\eta_n^{\alpha+\delta} = \frac{1}{b_n^{\alpha+\delta}} \sum_{k=0}^n c_{nk}^{\alpha\delta} b_k^\alpha \eta_k^\alpha \quad (2.1)$$

is satisfied where $(c_{nk}^{\alpha\delta})$ is a complex (or real) matrix with $c_{nk}^{\alpha\delta} = 0$ for $k > n$ and $c_{nk}^{\alpha\delta} = c_n^\alpha \neq 0$ for $k \leq n$ ((c_n^α) is a sequence), and (b_n^α) is a complex (or real) sequence with $b_n^\alpha \neq 0$ ($n = 0, 1, 2, \dots$). So we have for every A_α and $A_{\alpha+\delta}$ the connection

$$A_{\alpha+\delta} x = D_{\alpha\delta} (A_\alpha x)$$

⁴ It will be seen in section 3 that the relation (2.1) is satisfied, for instance, for the families of generalized Nörlund summability methods.

where $x \in \omega A_\alpha$ and $D_{\alpha\delta} = (d_{nk}^{\alpha\delta})$ is matrix defined by the relation (2.1), i.e. $d_{nk}^{\alpha\delta} = b_k^\alpha c_{nk}^{\alpha\delta} / b_n^{\alpha+\delta}$.

Further we will construct for every connection matrix $D_{\alpha\delta}$ with $\alpha > \alpha_0$ and $0 < \delta < 1$ a quotient representation

$$D_{\alpha\delta} = Q_{\alpha\delta\kappa} D_{\alpha\alpha} + R_{\alpha\delta\kappa}, \quad (2.2)$$

where κ is any number from interval $]\frac{1}{2}, 1[$ and $Q_{\alpha\delta\kappa}$ and $R_{\alpha\delta\kappa}$ are certain matrices (depending on α, δ and κ). We need the quotient representation (2.2) to get the sufficient conditions for the convexity of family A_α with the help of the general convexity theorems given in section 1.

Let us fix any $\alpha > \alpha_0$, any $0 < \delta < 1$, any complex (or real) sequence $y = (\eta_n)$ and denote the sequence $D_{\alpha\delta} y$ by $(\mu_n^{\alpha\delta})$, i.e.

$$\mu_n^{\alpha\delta} = \frac{1}{b_n^{\alpha+\delta}} \sum_{k=0}^n c_{nk}^{\alpha\delta} b_k^\alpha \eta_k. \quad (2.3)$$

Let us fix also a $\frac{1}{2} < \kappa < 1$, denote by $N = [\kappa n]$ the integer part of the number κn and divide the sum (2.3) into two parts:

$$\mu_n^{\alpha\delta} = \frac{1}{b_n^{\alpha+\delta}} \sum_{k=0}^N c_{nk}^{\alpha\delta} b_k^\alpha \eta_k + \frac{1}{b_n^{\alpha+\delta}} \sum_{k=N+1}^n c_{nk}^{\alpha\delta} b_k^\alpha \eta_k.$$

Transforming the first of the two sums on the right by Abel transformation we obtain that⁵

$$\begin{aligned} \mu_n^{\alpha\delta} &= \frac{1}{b_n^{\alpha+\delta}} \sum_{k=0}^{N-1} \Delta_k c_{nk}^{\alpha\delta} \sum_{\nu=0}^k b_\nu^\alpha \eta_\nu + \frac{1}{b_n^{\alpha+\delta}} c_{nN}^{\alpha\delta} \sum_{\nu=0}^N b_\nu^\alpha \eta_\nu + \\ &+ \frac{1}{b_n^{\alpha+\delta}} \sum_{k=N+1}^n c_{nk}^{\alpha\delta} b_k^\alpha \eta_k. \end{aligned}$$

⁵ $\Delta_k c_{nk}^{\alpha\delta} = c_{nk}^{\alpha\delta} - c_{n, k+1}^{\alpha\delta}$.

Thus we have got for the sequence $D_{\alpha\delta}y = (\mu_n^{\alpha\delta})$ the representation

$$\mu_n^{\alpha\delta} = \sum_{k=0}^{N-1} \Delta_k c_{nk}^{\alpha\delta} b_k^{\alpha+1} \mu_k^{\alpha} / c_k^{\alpha} b_n^{\alpha+\delta} + c_{nN}^{\alpha\delta} b_N^{\alpha+1} \mu_N^{\alpha} / c_N^{\alpha} b_n^{\alpha+\delta} + \sum_{k=N+1}^n c_{nk}^{\alpha\delta} b_k^{\alpha} \eta_k / b_n^{\alpha+\delta} \quad (n = 0, 1, 2, \dots). \quad (2.4)$$

Denoting by $Q_{\alpha\delta\pi} = (q_{nk}^{\alpha\delta\pi})$ and $R_{\alpha\delta\pi} = (r_{nk}^{\alpha\delta\pi})$ the matrices defined by

$$q_{nk}^{\alpha\delta\pi} = \begin{cases} \Delta_k c_{nk}^{\alpha\delta} b_k^{\alpha+1} / c_k^{\alpha} b_n^{\alpha+\delta} & \text{if } k < N, \\ c_{nN}^{\alpha\delta} b_N^{\alpha+1} / c_N^{\alpha} b_n^{\alpha+\delta} & \text{if } k = N, \\ 0 & \text{if } k > N \end{cases} \quad (2.5)$$

and

$$r_{nk}^{\alpha\delta\pi} = \begin{cases} 0 & \text{if } k \leq N, \\ c_{nk}^{\alpha\delta} b_k^{\alpha} / b_n^{\alpha+\delta} & \text{if } k > N, \end{cases} \quad (2.6)$$

we can present the equality (2.4) in the following form:

$$D_{\alpha\delta}y = Q_{\alpha\delta\pi}(D_{\alpha\delta}y) + R_{\alpha\delta\pi}(y).$$

So we have constructed for the matrix $D_{\alpha\delta}$ the quotient representation^a (2.2) where the matrices $Q_{\alpha\delta\pi}$ and $R_{\alpha\delta\pi}$ are defined by (2.5) and (2.6), respectively. It follows immediately from the representation (2.2) that the condition

$$D_{\alpha\delta}(A_{\alpha}x) = Q_{\alpha\delta\pi}(D_{\alpha\delta}(A_{\alpha}x)) + R_{\alpha\delta\pi}(A_{\alpha}x)$$

holds for every $x \in \omega A_{\alpha}$. The last equality gives for

^a The idea of the method described above for constructing the quotient representations was used by the author first in the paper [6] and further also in the papers [7,8,10].

the method $A_{\alpha+\delta}$ the following representation:

$$A_{\alpha+\delta}x = Q_{\alpha\delta\kappa}(A_{\alpha+1}x) + R_{\alpha\delta\kappa}(A_{\alpha}x) \quad (x \in \omega A_{\alpha}).$$

Therefore the inequality (1.6) is satisfied for every $n = 0, 1, 2, \dots$ and $x \in \omega A_{\alpha}$, and the next result can be obtained as a direct application of Theorem 1.1 to the family considered here.

THEOREM 2.1. *Let the methods A_{α} and $A_{\alpha+\delta}$ be connected by the relation $A_{\alpha+\delta}x = D_{\alpha\delta}(A_{\alpha}x)$ for each $\alpha > \alpha_0$, $0 < \delta < 1$ and $x \in \omega A_{\alpha}$ where the matrix $D_{\alpha\delta}$ is defined by (2.1).*

Suppose the following conditions hold for every $\alpha > \alpha_0$, $0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$.

1) *The matrix $D_{\alpha\delta}$ is a $c_0 \rightarrow c_0$ matrix.*

2) *The matrix $Q_{\alpha\delta\kappa} = (q_{nk}^{\alpha\delta\kappa})$ defined by (2.5) is a $c_0 \rightarrow c_0$ matrix.*

3) *The matrix $R_{\alpha\delta\kappa} = (r_{nk}^{\alpha\delta\kappa})$ defined by (2.6) satisfies the condition*

$$\sum_{k=N+1}^n |r_{nk}^{\alpha\delta\kappa}| \leq \varphi_{\alpha\delta}(\kappa)$$

where $N = [n\kappa]$ and $\lim_{\kappa \rightarrow 1-} \varphi_{\alpha\delta}(\kappa) = 0$.

Then the family A_{α} is 0-convex.

We remark that the conditions 1)-3) of Theorem 2.1 put the restrictions on the matrices $D_{\alpha\delta}$ not on the methods A_{α} . The only restriction on methods A_{α} is the connection (2.1). Different families A_{α} that satisfy the same condition (2.1) can be easily constructed (see [6,7]).

The next result can be inferred from Theorem 2.1 with the help of Lemma 1.3 as an immediate corollary.

COROLLARY 2.1. *If the family A_{α} satisfy the presumptions and the conditions 1)-3) of Theorem 2.1 and the presumptions of Lemma 1.3 (with $\alpha_0 = 1$), then*

the family A_α is convex and the methods A_α are pairwise consistent.

By applying Theorem 1.4 we get the following result.

THEOREM 2.2. *Let the methods A_α and $A_{\alpha+\delta}$ ($\alpha > \alpha_0$, $0 < \delta < 1$) be connected with the matrix $D_{\alpha\delta}$ defined by (2.1). If the matrix method $D_{\alpha\delta}$ is regular and satisfies the conditions 2) and 3) of Theorem 2.1 for every $\alpha > \alpha_0$, $0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$, then the family A_α is convex and the methods A_α are pairwise consistent.*

Proof. The representation (2.2) together with conditions 2)-3) of Theorem 2.1 and regularity of the methods $D_{\alpha\delta}$ implies the validity of the conditions 1) and 2) of Theorem 1.4. Therefore the family A_α is convex (with consistency).

In a special case if the methods A_α ($\alpha > \alpha_0$) are normal matrix methods, Theorem 2.2 can be also taken for an immediate consequence of each of Theorems 1.3, 1.6 and 1.7.

REMARK 2.1. *The method for constructing the quotient representation for the connection matrix $D_{\alpha\delta}$ (and at the same time for the method $A_{\alpha+\delta}$) described above can be extended to the methods connected with the integral analogue of the transformation (2.1) (see [6], Theorems 5 and 6, and [7], Theorem 6'). This method can also be extended to certain methods of strong summability (see [8], Theorem 5 and Lemma 1).*

3. Convex families of generalized Nörlund summability methods

With the help of the convexity Theorem 2.2 (or Corollary 2.1) sufficient convexity conditions for certain families of generalized Nörlund methods can be

obtained.

We state that the sequence $(A_n^{\alpha\beta})$ is defined formally by the power series

$$f_{\alpha\beta}(x) = (1-x)^{-(\alpha+1)} \left(\log \frac{e}{1-x}\right)^\beta = \sum_{n=0}^{\infty} A_n^{\alpha\beta} x^n \quad (3.1)$$

where α, β, x are real numbers and e is the base of Napierian logarithm. Let us consider the generalized Nörlund summability methods $A_\alpha = (N, p_n^{\alpha\beta}, q_n)$ (see [10]) defined by the relation

$$\eta_n^\alpha = \frac{1}{R_n^{\alpha\beta}} \sum_{k=0}^n p_{n-k}^{\alpha\beta} q_k \xi_k \quad (3.2)$$

where

$$R_n^{\alpha\beta} = \sum_{k=0}^n p_{n-k}^{\alpha\beta} q_k, \quad (3.3)$$

$p_n^{\alpha\beta} = \sum_{k=0}^n A_{n-k}^{\alpha-1, \beta} p_k$, β_0 is the real number, p_n, q_n are the complex (or real) numbers and $R_n^{\alpha\beta} \neq 0$ ($n=0, 1, 2, \dots$).

REMARK 3.1. We notice that, in particular, if $\beta_0 = 0$ then the methods $(N, p_n^{\alpha\beta}, q_n)$ become the generalized Nörlund methods (N, p_n^α, q_n) (see [5] and [10]). If besides the condition $\beta_0 = 0$ there is $q_n = 1$ ($n = 0, 1, 2, \dots$) then we get the Nörlund methods (N, p_n^α) (see [3] and [10]). In particular, if $q_n = A_n^{\gamma_0 \sigma_0}$ (γ_0 and σ_0 are fixed real numbers) and $p_n = A_n^{-1, 0} = A_n^{-1}$ then the methods $(N, p_n^{\alpha\beta}, q_n)$ become the quasi-Cesàro methods $(C, \alpha, \beta_0, \gamma_0, \sigma_0)$ (see [4]). If $\beta_0 = \sigma_0 = 0$ then the methods $(C, \alpha, \beta_0, \gamma_0, \sigma_0)$ become the generalized Cesàro methods (C, α, γ_0) ; if we add to the previous conditions the presumption $\gamma_0 = 0$ then we get the Cesàro

⁷ In particular, if $\beta = 0$ then $A_n^{\alpha, 0} = A_n^\alpha$ are the Cesàro numbers.

methods (C, ω).

As it was indicated in [10] the methods $A_{\alpha+\delta} = (N, p_n^{\alpha+\delta, \beta_0}, q_n)$ and $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ for every $\alpha > \alpha_0$ and $\delta > 0$ are connected with the relation

$$\eta_n^{\alpha+\delta} = \frac{1}{R_n^{\alpha+\delta, \beta_0}} \sum_{k=0}^n A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} \eta_k^\alpha \quad (3.4)$$

and the condition

$$R_n^{\alpha+\delta, \beta_0} = \sum_{k=0}^n A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} \quad (3.5)$$

is satisfied also.

We notice that the relation (3.4) is a special case of the relation (2.1) where $c_{nk}^{\alpha\delta} = A_{n-k}^{\delta-1}$ are the Cesàro numbers for $k \leq n$ and $c_{nk}^{\alpha\delta} = 0$ for $k > n$ and $b_n^\alpha = R_n^{\alpha\beta_0}$. Thus the family $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ ($\alpha > \alpha_0$) has the connection matrices $D_{\alpha\delta} = (d_{nk}^{\alpha\delta})$ ($\alpha > \alpha_0$, $\delta > 0$) defined by (3.4), i.e. $d_{nk}^{\alpha\delta} = A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} / R_n^{\alpha+\delta, \beta_0}$ for $k \leq n$ and $d_{nk}^{\alpha\delta} = 0$ for $k > n$ ($n = 0, 1, 2, \dots$). By applying Theorem 2.2 (or Corollary 2.1) we will get the next result⁸.

THEOREM 3.1. *If for every $\alpha > \alpha_0$ and $0 < \delta < 1$ the conditions⁹*

$$1) |R_n^{\alpha\beta_0} / R_{n+k}^{\alpha\beta_0}| \leq N_\alpha \text{ for every } n, k = 0, 1, 2, \dots$$

⁸This theorem is formulated and proved in [10] (Theorem 2.1). Here we present a different proof taking for starting point Theorem 2.1.

⁹ N_α and $K_{\alpha\delta}$ are the constants depending only on α or only on α and δ , respectively. Further, in θ - and σ -conditions we also note the dependence of constants on some variables by indexes.

and

$$2) |R_n^{\alpha+\delta, \beta_0} / R_n^{\alpha\beta_0}| \geq K_{\alpha\delta} n^\delta \text{ for every } n = 0, 1, 2, \dots$$

hold then the family $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ is convex for $\alpha > \alpha_0$ and methods A_α are pairwise consistent.

Proof. We will show that the matrix method $D_{\alpha\delta}$ is regular and the conditions 2) and 3) of Theorem 2.1 are satisfied for all $\alpha > \alpha_0$, $0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$. Let us prove that the matrix method $D_{\alpha\delta} = (d_{nk}^{\alpha\delta})$ defined by the relation (3.4) is regular for every $\alpha > \alpha_0$ and $0 < \delta < 1$. Using the conditions 1) and 2) and the properties of the Cesàro numbers $A_{n-k}^{\delta-1}$ we get:

$$\begin{aligned} d_{nk}^{\alpha\delta} &= A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} / R_n^{\alpha+\delta, \beta_0} = o_\alpha(1) |A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} / R_n^{\alpha+\delta, \beta_0}| = \\ &= o_{\alpha\delta}(1) n^{-\delta} (n-k+1)^{\delta-1} = o_{\alpha\delta\kappa}(1) \quad (n \rightarrow \infty) \end{aligned}$$

for every $k = 0, 1, 2, \dots$ and

$$\begin{aligned} \sum_{k=0}^n |d_{nk}^{\alpha\delta}| &= \sum_{k=0}^n |A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} / R_n^{\alpha+\delta, \beta_0}| = \\ &= o_\alpha(1) |R_n^{\alpha\beta_0} / R_n^{\alpha+\delta, \beta_0}| \sum_{k=0}^n A_{n-k}^{\delta-1} = o_{\alpha\delta}(1) n^{-\delta} n^\delta = o_{\alpha\delta}(1). \end{aligned}$$

By the condition (3.5) we get

$$\sum_{k=0}^n d_{nk}^{\alpha\delta} = \sum_{k=0}^n A_{n-k}^{\delta-1} R_k^{\alpha\beta_0} / R_n^{\alpha\beta_0} = 1 \quad (n = 0, 1, 2, \dots).$$

Thus we have realized that the method $D_{\alpha\delta}$ is regular for every $\alpha > \alpha_0$ and $0 < \delta < 1$. It follows from the conditions (3.5) and 1) that

$$|R_n^{\alpha+\delta, \beta_0}| \leq N_\alpha |R_n^{\alpha\beta_0}| \sum_{k=0}^n A_{n-k}^{\delta-1} = o_{\alpha\delta}(1) |R_n^{\alpha\beta_0}| n^\delta$$

and thus the condition

$$R_n^{\alpha+\delta, \beta_0} / R_n^{\alpha\beta_0} = o_{\alpha\delta}(1) n^\delta \quad (3.6)$$

holds. Next we show that the matrices $Q_{\alpha\delta\kappa} = (q_{nk}^{\alpha\delta\kappa})$ and

$q_{\alpha\delta\kappa} = (r_{nk}^{\alpha\delta\kappa})$ defined by the relations (3.4) and (2.5) satisfy the conditions 2) and 3) of theorem 2.), respectively. Let us prove that the matrix $Q_{\alpha\delta\kappa} = (q_{nk}^{\alpha\delta\kappa})$ where

$$q_{nk}^{\alpha\delta\kappa} = \begin{cases} A_{n-k}^{\delta-2} R_k^{\alpha+1, \beta_0} / R_n^{\alpha+\delta} & \text{if } k < N = (n), \\ A_{n-N}^{\delta-1} R_N^{\alpha+1, \beta_0} / R_n^{\alpha+\delta} & \text{if } k = N, \\ 0 & \text{if } k > N \end{cases}$$

is a $c_0 \rightarrow c_0$ matrix for every $\alpha > \alpha_0$, $0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$. By the conditions 1) and (3.6) we obtain

$$\begin{aligned} q_{nk}^{\alpha\delta\kappa} &= A_{n-k}^{\delta-2} R_k^{\alpha+1, \beta_0} / R_n^{\alpha+\delta} = \theta_{\alpha\delta} (1) |R_n^{\alpha+1} / R_n^{\alpha+\delta}| (n-k)^{\delta-2} = \\ &= \theta_{\alpha\delta} (1) (n-N)^{\delta-2} n^{1-\delta} = \theta_{\alpha\delta\kappa} (1) \quad (n \rightarrow \infty) \end{aligned}$$

for each $k = 0, 1, 2, \dots$ and

$$\begin{aligned} \sum_{k=0}^N |q_{nk}^{\alpha\delta\kappa}| &= \sum_{k=0}^{N-1} |A_{n-k}^{\delta-2} R_k^{\alpha+1, \beta_0} / R_n^{\alpha+\delta, \beta_0}| + \\ &+ |A_{n-N}^{\delta-1} R_N^{\alpha+1, \beta_0} / R_n^{\alpha+\delta, \beta_0}| = \theta_{\alpha\delta} (1) n^{1-\delta} (n-N)^{\delta-2} N + \\ &+ \theta_{\alpha\delta} (1) (n-N)^{\delta-1} n^{1-\delta} = \theta_{\alpha\delta\kappa} (1) + \theta_{\alpha\delta\kappa} (1) = \theta_{\alpha\delta\kappa} (1). \end{aligned}$$

Thus we have seen that $Q_{\alpha\delta\kappa}$ is a $c_0 \rightarrow c_0$ matrix for every $\alpha > \alpha_0$, $0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$. It remains to prove that the matrix $R_{\alpha\delta\kappa} = (r_{nk}^{\alpha\delta\kappa})$ where

$$r_{nk}^{\alpha\delta\kappa} = \begin{cases} 0 & \text{if } k \leq N \text{ or if } k > n, \\ A_{n-k}^{\delta-1} R_k^{\alpha, \beta_0} / R_n^{\alpha+\delta, \beta_0} & \text{if } N < k \leq n \end{cases}$$

satisfies the condition 3) of theorem 2.1 for every $\alpha > \alpha_0$, $0 < \delta < 1$ and $\frac{1}{2} < \kappa < 1$. By the conditions 1) and 2) we get:

$$\sum_{k=N+1}^n |r_{nk}^{\alpha\delta\kappa}| = \sum_{k=N+1}^n |A_{n-k}^{\delta-1} R_k^{\alpha, \beta_0} / R_n^{\alpha+\delta, \beta_0}| =$$

$$\begin{aligned}
&= \theta_{\alpha} (1) |R_n^{\alpha\beta_0} / R_n^{\alpha+\delta, \beta_0}| \sum_{k=N+1}^n A_{n-k}^{\delta-1} = \theta_{\alpha\delta} (1) n^{-\delta} (n-N)^{\delta} = \\
&= \theta_{\alpha\delta} (1) n^{-\delta} n^{\delta} (1-\pi)^{\delta} = \theta_{\alpha\delta} (1) (1-\pi)^{\delta}.
\end{aligned}$$

Thus we have realized that $R_{\alpha\delta\pi}$ satisfies the condition 3) of theorem 2.1 with $\varphi_{\alpha\delta}(\pi) = L_{\alpha\delta} (1-\pi)^{\delta}$ where $L_{\alpha\delta}$ is a constant depending on α and δ . The convexity of the family A_{α} (with consistency) follows now from Theorem 2.2 because all its conditions are satisfied¹⁰.

We notice that the condition 1) of the proved theorem holds if all the sequences $(|R_n^{\alpha\beta_0}|)$ are non-decreasing.

The next corollary follows immediately from Theorem 3.1 (see also [10], Corollary 2.1, [7], Corollary 1 and Remark 3 and [6], Corollary 3).

COROLLARY 3.1. a) If $p_0 > 0$, $p_n \geq 0$, $q_n > 0$ for $n = 0, 1, 2, \dots$, and condition 2) of Theorem 3.1 holds for $\beta_0 = 0$, $\alpha > 1$ and $0 < \delta < 1$, then the family $A_{\alpha} = (CN, p_n^{\alpha\beta_0}, q_n) = (CN, p_n^{\alpha}, q_n)$ is convex for $\alpha > 1$.

b) If $p_0 > 0$, $p_n \geq 0$, $q_n = 1$ for $n = 0, 1, 2, \dots$ and condition 2) of Theorem 3.1 holds for $\beta_0 = 0$, $\alpha > 0$ and $0 < \delta < 1$, then the family $A_{\alpha} = (CN, p_n^{\alpha})$ is convex for $\alpha > 0$.

c) The family $A_{\alpha} = (C, \alpha, \beta_0, \gamma_0, \sigma_0)$ is convex for $\alpha > \gamma_0$. In particular, the family $A_{\alpha} = (C, \alpha, \gamma_0)$ is convex for $\alpha > \gamma_0$ and the family $A_{\alpha} = (C, \omega)$ is convex for $\alpha > 0$.

Proof. In the presumptions of statements a) and

¹⁰ All the conditions of Corollary 2.1 are satisfied too and therefore the convexity of the family A can be concluded from Corollary 2.1 also.^α

b) the conditions $R_n^{\alpha\beta_0} > 0$ and $R_n^{\alpha\beta_0} \uparrow$ hold. Therefore the condition 1) of Theorem 3.1 is satisfied for $\alpha > 1$ and $\alpha > 0$, respectively, and statements a) and b) follow from Theorem 3.1. For the families considered in statement c) the conditions 1) and 2) of Theorem 3.1 are satisfied for $\alpha > \gamma_0$ and $0 < \delta < 1$ (see [10]) and thus the statement c) is true.

As we realize from the proof of Theorem 3.1, the presumptions of Theorem 3.1 can be weakened so that the conditions of Theorem 2.2 stay satisfied. Thus we infer from theorem 2.2 the next result.

THEOREM 3.2. *Suppose the following conditions hold.*

1) $|R_n^\alpha| R_n^{\alpha\beta_0} / R_{n+k}^{\alpha\beta_0} \leq N_\alpha (n, k = 0, 1, 2, \dots) \dots$
for every $\alpha > \alpha_0 + 1$.

2) $K_{\alpha\delta} n^\delta \leq |R_n^{\alpha+\delta, \beta_0} / R_n^{\alpha\beta_0}| \leq L_{\alpha\delta} n^\delta$ ($n = 1, 2, \dots$)
for every $\alpha > \alpha_0$ and $0 < \delta < 1$.

3) The matrix method $D_{\alpha\delta}$ defined by (3.4) is regular for every $\alpha > \alpha_0$ and $0 < \delta < 1$.

Then the family $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ is convex for $\alpha > \alpha_0$ and the methods A_α are pairwise consistent.

Immediately from Theorem 3.2 we can infer the next results first two of which are known from the papers [5, 3].

COROLLARY 3.2. a) If $p_0 > 0$, $p_n \geq 0$, $q_n > 0$ for every $n = 0, 1, 2, \dots$ and the condition 2) of Theorem 3.2 holds for $\beta_0 = 0$ and $\alpha_0 = 0$ then the family $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n) = (N, p_n^\alpha, q_n)$ is convex for $\alpha > 0$ (see [5]).

b) If $p_0 > 0$, $p_n \geq 0$, $q_n = 1$ for every $n = 0, 1, 2, \dots$ and the condition 2) of Theorem 3.2 holds for $\beta_0 = 0$ and $\alpha_0 = -1$ then the family $A_\alpha = (N, p_n^\alpha)$ is convex for $\alpha > -1$ (see [3]).

c) The family $A_\alpha = (C, \alpha, \beta_0, \gamma_0, \sigma_0)$ is convex for

$\alpha > \gamma_0 - 1$ (see [3]). In particular, the family $A_\alpha = (C, \alpha, \gamma_0)$ is convex for $\alpha > \gamma_0 - 1$ and the family $A_\alpha = (C, \omega)$ is convex for $\alpha > -1$.

Proof. This corollary follows immediately from Theorem 3.2. In presumptions of every statement a)-c) all the conditions of Theorem 3.2 are satisfied as the matrix method $D_{\alpha\delta}$ defined by (3.4) is regular for every $\alpha > \alpha_0$ and $0 < \delta < 1$, and the sequences $(R_n^{\alpha\beta})$ satisfy the conditions 1) and 2) of Theorem 3.2.

As a result we can say that the convexity theorems for families $A_\alpha = (N, p_n^\alpha, q_n)$ and $A_\alpha = (N, p_n^\alpha)$ proved by R. Sinha ([5]) and F.P. Cass ([3]), respectively, follow from Theorem 3.1 or 3.2 as immediate corollaries (see statements a) and b) of Corollaries 3.1 and 3.2). It can be added that both of Theorems 3.1 and 3.2 state the convexity of these families in the weaker restrictions on methods A_α than the convexity theorems in the papers [5] and [3]. The convexity of the family $A_\alpha = (C, \alpha, \beta_0, \gamma_0, \sigma_0)$ follows from each of the Theorems 3.1 and 3.2 also (see statements c) of Corollaries 3.1 and 3.2). The convexity of the families $A_\alpha = (C, \alpha, \beta_0)$ and $A_\alpha = (C, \omega)$ can be inferred from the convexity of the family $A_\alpha = (C, \alpha, \beta_0, \gamma_0, \sigma_0)$ as special cases (see statements c) of Corollaries 3.1 and 3.2). The convexity of the $A_\alpha = (C, \omega)$ was first proved by G.H. Hardy and J.E. Littlewood in 1912. Afterwards the convexity of this family has been proved in different ways by several authors. The references on these proofs can be found in [2]. For example, J. Boos and R. Neuser ([2]) proved the convexity of the family $A_\alpha = (C, \omega)$ by using the quotient representations different from these that are used for summability methods in the present paper. Thus we conclude that

all convexity theorems¹¹ for special matrix methods already published by other authors (and which are known to the author) can be obtained from Theorems 3.1 and 3.2 as immediate corollaries.

The proofs of above mentioned theorems for special families A_α in the papers [2-4] base on matrices A_α but the proofs of Theorems 3.1, 3.2, 2.1 and 2.2 base on the connection matrices $D_{\alpha\delta}$ between methods A_α and $A_{\alpha+\delta}$ ($\delta > 0$) (see (2.1)). Taking as starting point the connection matrices $D_{\alpha\delta}$ we can prove the more general convexity theorems than can be proved by starting from the matrices A_α . We point out once more that the different families of summability methods connected by the same matrices $D_{\alpha\delta}$ can be easily constructed. In particular, let the family $A_\alpha = (N, p_n^{\alpha\beta}, q_n)$ connected by the matrices $D_{\alpha\delta}$ satisfy the conditions of Theorem 3.1 and be therefore convex. Then every other family B_α of summability methods connected by the same matrices $D_{\alpha\delta}$ is convex too (because the matrices $D_{\alpha\delta}$ satisfy all the conditions of Theorem 1.4). For example, if $A'_\alpha = (N, p_n^{\alpha\beta}, q'_n)$ is another family of generalized Nörlund methods with $A'_\alpha = (\eta'_n, \alpha)$, then the family B_α where $B_\alpha x = (\eta'_n, \alpha) R'_n / R_n$ is convex.

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¹¹ We mean under convexity theorems the theorems that state the convexity of a family A_α .

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Received November 15, 1992

ON THE INCLUSION OF THE POISSON-ABEL TYPE METHODS
FOR INTEGRALS

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Introduction. The classical inclusion theorem for Abelian summability due to G. Hardy ([2], theorem 28) has been generalized in various directions. J. Hudak [10] considered the inclusion between the Abel and Stieltjes methods. The variants for integrals were proved by K. Zhusipov [10]. The inclusion for absolute summability has been investigated for series by D. Rath [3] and for integrals by H. Diamond, B. Küttner, L. A. Raphael [1]. Some new Abelian type methods were introduced by the author, and various inclusion theorems for them in the case of ordinary summability [4-6] and absolute summability of integrals [7] were proved.

In Section 1 some inclusion theorems relating the summability of divergent integrals of the form $\int_0^{\infty} a(u)du$ under various Abelian type methods are proved. As in [4-6], the integral representations of the functions related to the summability methods play the decisive role in the proof of the theorems.

In Section 2 the results of [7] for absolute summability are analysed. The notion of absolute summability for integrals entails the relations that do not hold in case of series. The given examples help to delineate the boundaries of the summability methods.

1. Inclusion theorems for ordinary summability

Let $\lambda(u)$ be a positive and non-decreasing function in $[0, +\infty)$, $\lim_{u \rightarrow +\infty} \lambda(u) = +\infty$. For a given integral

$$\int_0^{+\infty} a(u) du \quad (1)$$

we consider for $\alpha > 0$ functions

$$\phi(\alpha) = \int_0^{+\infty} e^{-\alpha\lambda(u)} a(u) du, \quad (2)$$

$$\psi_{\square}(\alpha) = \int_0^{+\infty} (1 + \alpha\lambda(u))^{-\square} a(u) du, \quad (3)$$

$$\rho_{\square}(\alpha) = \int_0^{+\infty} (1 + \alpha\lambda(u))^{-\square} e^{-\alpha\lambda(u)} a(u) du, \quad (4)$$

$$h(\alpha) = \int_0^{+\infty} \lambda^{-\alpha}(u) a(u) du, \quad (5)$$

$$\chi(\alpha) = \int_0^{+\infty} \lambda^{-\alpha}(u) e^{-\alpha\lambda(u)} a(u) du. \quad (6)$$

DEFINITION 1. Integral (1) is summable by the Poisson-Abel method $PA(\lambda(u))$ (the Stieltjes method $T_{\square}(\lambda(u))$, $AT_{\square}(\lambda(u))$, the Mellin method $PA(\ln\lambda(u))$, $PA(\ln\lambda(u) + \lambda(u))$) to the sum z if the integrals (2) (or the corresponding ones (3)-(6)) converge for all $\alpha > 0$ and exists the limit

$$\lim_{\alpha \rightarrow +\infty} \phi(\alpha) = z \quad (\text{or } \lim_{\alpha \rightarrow +\infty} \psi_{\square}(\alpha) = z, \lim_{\alpha \rightarrow +\infty} \rho_{\square}(\alpha) = z, \lim_{\alpha \rightarrow +\infty} h(\alpha) = z, \lim_{\alpha \rightarrow +\infty} \chi(\alpha) = z).$$

Clearly, all the methods are regular.

In [4-6] some inclusion theorems related to the methods defined by (2)-(5) or their analogs for series were obtained. We will prove some results for the method defined by (6).

THEOREM 1. If integral (6) converges for all $\alpha > 0$ then

$$\chi(\omega) = \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{+\infty} (t-\alpha)^{\alpha-1} \phi(t) dt \quad (\alpha < 1). \quad (7)$$

Proof. Let be given β, τ such that $0 < \alpha < \beta \leq t \leq \tau < +\infty$.
Using (2), we get

$$\frac{1}{\Gamma(\alpha)} \int_{\beta}^{\tau} (t-\alpha)^{\alpha-1} \phi(t) dt = \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\tau} (t-\alpha)^{\alpha-1} \int_{\alpha}^{+\infty} e^{-t\lambda(u)} a(u) du dt. \quad (8)$$

Representing

$$\phi(t) = \int_{\alpha}^{+\infty} [e^{-\alpha\lambda(u)} a(u)] e^{(\alpha-t)\lambda(u)} du, \quad (9)$$

we see that integral $\int_{\alpha}^{+\infty} e^{-\alpha\lambda(u)} a(u) du$ converges and factor $e^{(\alpha-t)\lambda(u)}$ is non-increasing and bounded for u . Therefore, integral (2) is uniformly converging in t , and the change of the order of integration in (8) is allowed. Hence, we have

$$\frac{1}{\Gamma(\alpha)} \int_{\beta}^{\tau} (t-\alpha)^{\alpha-1} \phi(t) dt = \int_{\alpha}^{+\infty} a(u) du \frac{1}{\Gamma(\alpha)} \int_{\beta}^{\tau} (t-\alpha)^{\alpha-1} e^{-t\lambda(u)} dt. \quad (10)$$

Let $\beta \rightarrow \alpha$ and $\tau \rightarrow +\infty$. On the right we can calculate the limit under the integral sign if the integrals

$$A = \int_{\alpha}^{+\infty} a(u) du \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{\alpha+\varepsilon} (t-\alpha)^{\alpha-1} e^{-t\lambda(u)} dt \quad (\varepsilon > 0),$$

$$B = \int_{\alpha}^{+\infty} a(u) du \frac{1}{\Gamma(\alpha)} \int_{\tau}^{+\infty} (t-\alpha)^{\alpha-1} e^{-t\lambda(u)} dt$$

converge to zero as $\varepsilon \rightarrow 0$ and $\tau \rightarrow +\infty$.

Rewriting the first of them in the form

$$A = \int_{\alpha}^{+\infty} e^{-\alpha\lambda(u)} du \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{\alpha+\varepsilon} e^{(\alpha-t)\lambda(u)} (t-\alpha)^{\alpha-1} dt =$$

$$= \int_0^{+\infty} e^{-\alpha\lambda(u)} g(u, \alpha, \varepsilon) du, \quad (11)$$

we see that integral $\int_0^{+\infty} e^{-\alpha\lambda(u)} du$ converges and factor $g(u, \alpha, \varepsilon)$ is non-increasing and bounded for u . Thus, integral (11) converges uniformly in ε . Now it is clear that $A \rightarrow 0$ as $\varepsilon \rightarrow 0+$. Analogously, $B \rightarrow 0$ as $\tau \rightarrow +\infty$. Therefore, from (10) we get

$$\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (t-\omega)^{\alpha-1} \phi(t) dt = \int_0^{+\infty} a(u) du \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (t-\omega)^{\alpha-1} e^{-t\lambda(u)} dt,$$

whence, by the formula

$$\int_0^{+\infty} (t-\omega)^{\alpha-1} e^{-t\nu} dt = \Gamma(\alpha) \nu^{-\alpha} e^{-\alpha\nu} \quad (12)$$

([8], formula (9.132)), the representation (7) follows immediately.

Now we are able to prove the inclusion theorem for the methods defined by (2) and (6).

THEOREM 2. *If integral (1) is summable by the method $PA(\lambda(u))$ to the sum z then it is summable to the same z by the method $PA(\ln\lambda(u) + \lambda(u))$.*

Proof. Let

$$\lim_{t \rightarrow 0+} \phi(t) = z. \quad (13)$$

Then for $\delta > \alpha$

$$\begin{aligned} |\chi(\alpha) - z| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (t-\omega)^{\alpha-1} \phi(t) dt - z \right| \leq \\ &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{\delta} (t-\omega)^{\alpha-1} \phi(t) dt - \frac{1}{\Gamma(\alpha)} \int_0^{\delta} (t-\omega)^{\alpha-1} z dt \right| + \end{aligned}$$

$$+ \left| \left[\frac{1}{\Gamma(\alpha)} \int_{\alpha}^{\delta} (t-\alpha)^{\alpha-1} \phi(t) dt - 1 \right] z \right| + \left| \frac{1}{\Gamma(\alpha)} \int_{\delta}^{+\infty} (t-\alpha)^{\alpha-1} \phi(t) dt \right| =$$

$$= A_1 + A_2 + A_3.$$

Since

$$A_1 \leq \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{\delta} (t-\alpha)^{\alpha-1} |\phi(t) - z| dt,$$

we can, due to (13), choose δ sufficiently small to get $A_1 < \varepsilon/3$. Next,

$$A_2 = \left| \frac{(\delta-\alpha)^{\alpha}}{\Gamma(\alpha+1)} - 1 \right| |z| < \varepsilon/3$$

by the choice of α sufficiently close to 0. As

$$e^{\lambda(\alpha)t} \phi(t) = \int_{\alpha}^{+\infty} e^{-\delta(\lambda(u)-\lambda(\alpha))} a(u) \cdot e^{(\delta-t)(\lambda(u)-\lambda(\alpha))} du$$

and the integral of the first factor converges and the second factor is non-increasing and bounded for all $t \geq \delta$, the integral is uniformly converging in $[\delta, +\infty)$. Therefore, the regarded function is bounded, i.e.

$$|e^{\lambda(\alpha)t} \phi(t)| \leq M$$

for all $t \geq \delta$. Now, due to $\lim_{\alpha \rightarrow 0+} \Gamma(\alpha) = \infty$, for sufficiently small α

$$A_3 \leq \frac{M}{\Gamma(\alpha)} \int_{\delta}^{+\infty} (t-\alpha)^{\alpha-1} e^{-\lambda(\alpha)t} dt \leq \frac{M}{\Gamma(\alpha)} (\delta-\alpha)^{\alpha-1} \int_{\delta}^{+\infty} e^{-\lambda(\alpha)t} dt =$$

$$= \frac{M(\delta-\alpha)^{\alpha-1}}{\lambda(\alpha)\Gamma(\alpha)} e^{-\lambda(\alpha)\delta} < \varepsilon/3.$$

Finally, for sufficiently small α we have

$$A_1 + A_2 + A_3 < \varepsilon,$$

i.e. $\lim_{\alpha \rightarrow 0+} \chi(\alpha) = z$. This completes the proof.

THEOREM 3. *If for $\alpha > 0$ integral*

$$\alpha(\omega) = \int_0^{+\infty} [\lambda(u) \ln \lambda(u)]^{-\alpha} a(u) du \quad (14)$$

converges, then

$$\alpha(\omega) = \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{+\infty} (t-\alpha)^{\alpha-1} h(t) dt \quad (0 < \alpha < 1). \quad (15)$$

Proof. It is obvious that the convergence of integral (14) entails the convergence of integral (5). Applying the procedure analogous to the one in the proof of theorem 1 we get the equality

$$\frac{1}{\Gamma(\alpha)} \int_{\beta}^{\tau} (t-\alpha)^{\alpha-1} h(t) dt = \frac{1}{\Gamma(\alpha)} \int_0^{\tau} a(u) du \int_{\beta}^{\tau} (t-\alpha)^{\alpha-1} \lambda^{-t}(u) dt.$$

As $\beta \rightarrow \alpha$ and $\tau \rightarrow +\infty$, it follows from this, due to (12), that

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{+\infty} (t-\alpha)^{\alpha-1} h(t) dt &= \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} a(u) du \int_{\alpha}^{+\infty} (t-\alpha)^{\alpha-1} \lambda^{-t}(u) dt = \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} a(u) du \int_{\alpha}^{+\infty} (t-\alpha)^{\alpha-1} e^{-t \ln \lambda(u)} dt = \alpha(\omega). \end{aligned}$$

THEOREM 4. *If integral (1) is summable by the method $PA(\ln \lambda(u))$ to the sum z then it is summable to the same z by the method $PA(\ln(\lambda(u) \ln \lambda(u)))$.*

Proof. It is analogous to the proof of theorem 2 with (14) in place of formula (7).

THEOREM 5. *If integral (14) converges and integral (1) is summable by the method $T_s(\lambda(u))$ or*

$AT_{\bullet}(\lambda(u))$ to z then integral (1) is summable by the method $PA(\ln(\lambda(u))\ln\lambda(u))$ to the same z .

Proof. It follows immediately from theorem 4 and the integral analogs of theorems 4 and 6 in [4].

2. Boundaries of absolute summability for the Poisson-Abel type methods

We will analyse our results in [7] for absolute summability. We write $f(\alpha) \in BV(0, c)$ if $f(\alpha)$ has bounded variation on $(0, c)$.

DEFINITION 2 ([7], definitions 3 and 4). *Integral (1) is absolutely summable by method $PA(\lambda(u))$ (or $T_{\bullet}(\lambda(u))$, $AT_{\bullet}(\lambda(u))$, $PA(\ln\lambda(u))$) if it is summable by the respective method and $\phi(\alpha) \in BV(0, +\infty)$ (or $\psi_{\bullet}(\alpha), \rho(\alpha) \in BV(0, +\infty)$, $h(\alpha) \in BV(0, c)$ (for some $c > 0$)).*

The following proposition is from [7] (cf. Theorems 1-4).

PROPOSITION. *It holds*

- 1) $|PA(\lambda(u))| < |AT_{\bullet}(\lambda(u))|,$
- 2) $|PA(\lambda(u))| < |T_{\bullet}(\lambda(u))|,$
- 3) $|AT_{\bullet}(\lambda(u))| < |PA(\ln\lambda(u))|,$
- 4) $|T_{\bullet}(\lambda(u))| < |PA(\ln\lambda(u))|.$

Our aim is to demonstrate on examples the relations between summability and absolute summability for the considered methods.

Let us take $\lambda(u) = u$.

THEOREM 6. *There exist integrals (1) that are*

- 1) $|PA(u)|$ -summable but not $T_{\bullet}(u)$ -summable

- and $PA\langle \ln u \rangle$ -summable;
- 2) $|PA\langle u \rangle|$ -summable and $|T_s\langle u \rangle|$ -summable but not $PA\langle \ln u \rangle$ -summable;
- 3) $|PA\langle \ln u \rangle|$ -summable but not $PA\langle u \rangle$ -summable and $T_s\langle u \rangle$ -summable.

Proof. 1) Let

$$I_1 = \int_0^{+\infty} u^s \cos u \, du \quad (s \geq 1).$$

This integral diverges, since, due to

$$\int_{2n\pi-1/n}^{2n\pi} u^s \cos u \, du > (2n\pi-1/n) \sin(1/n),$$

the Cauchy condition does not hold. Integral I_1 is $PA\langle u \rangle$ -summable, because, by the formula 7.47 in [8],

$$I_2 = \int_0^{+\infty} e^{-\alpha u} u^s \cos u \, du = \Gamma(s+1) (\alpha^2+1)^{-(s+1)/2} \cos[(s+1) \arctan \alpha^{-1}]$$

$$\text{and } \lim_{\alpha \rightarrow 0+} I_2 = -\Gamma(s+1) \sin(\pi s/2).$$

We must show that $I_2 \in BV(0, +\infty)$. Function $(\alpha^2+1)^{-(s+1)/2}$ is decreasing, and so it has bounded variation. As $0 < \alpha < +\infty$, then $(s+1)\pi/2 > (s+1)\arctan \alpha^{-1} > 0$. Since the interval $(0, (s+1)\pi/2)$ may be divided into a finite number of subintervals on which the function $\cos[(s+1)\arctan \alpha^{-1}]$ is monotonous, the latter function has bounded variation on $(0, +\infty)$. Obviously, the integral $I_2 \in BV(0, +\infty)$.

For $T_s\langle u \rangle$ - and $PA\langle \ln u \rangle$ -summability the integrals

$$\int_0^{+\infty} (1+\alpha u)^{-s} u^s \cos u \, du \quad \text{and} \quad \int_0^{+\infty} u^{s-\alpha} \cos u \, du$$

must converge for $\alpha > 0$. In a similar manner as for integral I_1 , one can show that this does not hold for

all $\alpha > 0$.

It completes the proof.

2) Let

$$I_3 = \int_0^{+\infty} u^{s-\varepsilon} \cos u \, du \quad (\operatorname{Re} s, s-\varepsilon > 1, \varepsilon > 0).$$

According to 1) integral I_3 is $|PA(u)|$ -summable. First of all, we show that

$$I_4 = \int_0^{+\infty} (1+\alpha u)^{-s} u^{s-\varepsilon} \cos u \, du$$

converges for $\alpha > 0$. Rewriting

$$I_4 = \int_0^{\pi/2} (1+\alpha u)^{-s} u^{s-\varepsilon} \cos u \, du + \\ + \sum_{n=0}^{+\infty} \int_{(2n+1)\pi/2}^{(2n+3)\pi/2} (1+\alpha u)^{-s} u^{s-\varepsilon} \cos u \, du$$

and calculating the integrals in the second term by substitution $u = n\pi + t$, we get

$$I_4 = \int_0^{\pi/2} (1+\alpha u)^{-s} u^{s-\varepsilon} \cos u \, du + \sum_{n=0}^{+\infty} (-1)^{n+1} v_n, \quad (16)$$

where

$$v_n = - \int_{\pi/2}^{3\pi/2} (1+\alpha(n\pi+t))^{-s} (n\pi+t)^{s-\varepsilon} \cos t \, dt \leq \\ \leq \alpha^{-s} \int_{\pi/2}^{3\pi/2} (n\pi+t)^{-\varepsilon} dt < \alpha^{-s} n^{-\varepsilon} \pi^{1-\varepsilon}.$$

From the latter follows $\lim v_n = 0$. As for $z > (s-\varepsilon)/\alpha$

the function $z^{\alpha-\varepsilon}(1+\alpha z)^{-\alpha}$ decreases, there exists n_0 such that v_n will decrease for $n > n_0$. Thus the series in (16) converges, and this implies the convergence of I_4 . Now, by the assertion 2) of Proposition, I_3 is $|T_\alpha\langle u \rangle|$ -summable. Integral I_3 is not $PA\langle \ln u \rangle$ -summable, since

$$\int_0^{+\infty} u^{\alpha-\varepsilon} \cos u \, du$$

diverges for sufficiently small α .

3) Let

$$I_5 = \int_0^{+\infty} u^i (1+u)^{-i} \, du \quad (i = \sqrt{-1}).$$

By elementary calculus, one can show that this integral diverges. For $PA\langle \ln u \rangle$ -summability, by formula 10.11 in [8], we get

$$I_5 = \int_0^{+\infty} u^{i-\alpha} (1+u)^{-i} \, du = \pi \operatorname{cosec}[\pi(\alpha-i)] \quad (0 < \alpha < 1),$$

from which

$$\lim_{\alpha \rightarrow 0+} I_5 = -\pi / \sin \pi i.$$

Analogously to part 1), one proves that $I_5 \in BV(0, c)$ with a sufficiently small $c > 0$. Thus, I_5 is $|PA\langle \ln u \rangle|$ -summable. For the $T_\alpha\langle u \rangle$ -transform of I_5 we get (formula 10.14 in [8])

$$I_7 = \int_0^{+\infty} u^i (1+u)^{-i} (1+\alpha u)^{-s} \, du = \\ = \Gamma(i+1) \Gamma(s-i) \Gamma^{-1}(s+1) F(s, i+1; s+1, i-\alpha),$$

where F is hypergeometric function. Taking into account the properties of F , it follows that for the given values of parameters the limit of I_7 , as $\alpha \rightarrow 0$, does not exist. So I_5 is not $T_\alpha\langle u \rangle$ -summable, and, by theorem 3

[6], also not $PA(u)$ -summable.

The theorem is proved.

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Received September 24, 1992

Poisson-Abeli tüüpi menetluste sisaldavusest
integraalide korral
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Resümee

Artiklites [4-7] on uuritud Poisson-Abeli tüüpi menetluste sisaldavust hariliku ja absoluutse summeeruvuse korral. Käesoleva töö esimeses osas vaadeldakse integraalide summeeruvust erinevate Poisson-Abeli tüüpi menetlustega. On tõestatud viis teoreemi nende menetluste sisaldavuse kohta.

Olgu $\lambda(u) > 0$ vahemikus $[0, +\infty)$, $\int_0^{+\infty} a(u) du < +\infty$. Integraali $\int_0^{+\infty} a(u) du$ nimetatakse summeeruvaks menetlusega $PA(\lambda(u))$ arvuks z , kui

$$\lim_{\alpha \rightarrow 0^+} \int_0^{+\infty} e^{-\alpha \lambda(u)} a(u) du = z.$$

TEOREEM 2. Kui integraal $\int_0^{+\infty} a(u) du$ on summeeruv menetlusega $PA(\lambda(u))$ arvuks z , siis on ta summeeruv menetlusega $PA(\ln \lambda(u) + \lambda(u))$ samaks arvuks.

Töö teises osas vaadeldakse integraalide absoluutset summeeruvust ja täpsustatakse artikli [7] tulemusi. Kolme näitega selgitatakse tõestatud teoreemide väidete ulatust hariliku ja absoluutse summeeruvuse korral.

SUMMABILITY FACTORS AND TAUBERIAN THEOREMS FOR DOUBLE SERIES

Shlomo Yanetz

The first paper on connections between summability factors and Tauberian theorems for simple series was by Kangro [4], who proved three theorems where conditions were formulated in terms of summability factors. The theorems of Kangro [4] are generalizations of the theorems of Meyer-König and Tietz [6],[7],[8] and Leviatan [5].

In order to find connections between summability factors of double series and Tauberian theorems we must generalize some important theorems about summability factors in simple series.

§1. Basic definitions and notations.

The following notation and definitions will be used in this section.

Let

$$\sum_{m,n=0}^{\infty} u_{mn} \quad (1.1)$$

be an infinite double series with real or complex numbers, with partial sums

$$U_{mn} = \sum_{k,\ell=0}^{m,n} u_{k\ell}. \quad (1.2)$$

Let $A = (a_{mnkl})$ be an infinite normal double matrix of real or complex numbers.

Let

$$U'_{mn} = \sum_{k,\ell=0}^{m,n} a_{mnkl} U_{k\ell}. \quad (1.3)$$

DEFINITION 1.1. We say that (1.1) is summable by A , or in short (1.1) is A -summable, to the sum U' if

$$\exists \lim_{m,n} U'_{mn} = U'. \quad (1.4)$$

In this case we call the matrix A a method of summability.

DEFINITION 1.2. We call a double series (1.1) boundedly A -summable, or in short A_b -summable to the sum U' , if (1.3) satisfies the conditions (1.4) and

$$U'_{mn} = O(1).$$

In this case we also denote the method of summability A by A_b . Analogously B_b -summability will be defined with the aid of the double matrix $B = (b_{m n k l})$.

Let A'_b be the set of all A_b -summable double series. If (1.1) is A_b -summable to the sum U' , we write

$$A_b \left\{ \sum_{m,n} u_{mn} \right\} = U'. \quad (1.5)$$

The number (1.5) is called the A -sum of the double series (1.1). The set A'_b is called the *field of bounded A -summability* or in short the summability *field of A_b* .

DEFINITION 1.3. If whenever (U_{mn}) is a bounded convergent sequence, (U'_{mn}) boundedly converges to the same value, then method A is said to be *regular*.

In the sequel we will consider only regular methods.

DEFINITION 1.4. We say that two methods A_b and B_b are *consistent* if the equality

$$B_b \left\{ \sum_{m,n} u_{mn} \right\} = A_b \left\{ \sum_{m,n} u_{mn} \right\} \quad (1.6)$$

holds for every double series (1.1) from $A'_b \cap B'_b$.

Let ρ be an arbitrary set of double series (1.1). We denote by r the set of double sequences $x = (U_{k\ell})$ associated with ρ , i.e.

$$r = \left\{ x : \sum_{k,\ell} \bar{\Delta}_{k\ell} U_{k\ell} \in \rho \right\}. \quad (1.7)$$

For example, if ρ is the set of bounded convergent double series, i.e. $\rho = b\gamma$, then $r = bc$ is the set of bounded convergent double sequences; if ρ is the set of bounded double series converging to zero, i.e. $\rho = b\gamma_0$, then $r = bc\eta$ is the set of bounded double sequences converging to zero; if $\rho = \mu$ is the set of double series with bounded partial sums, then r is the set m of bounded double sequences; if ρ is the set ℓ of absolutely convergent double series, then r is the set a of absolutely converging double sequences, that is

$$a = \{x : U_{k\ell} = \Omega(1)\}.$$

Here

$$x_{mn} = \Omega(y_{mn})$$

denotes that $\sum_{m,n} |\bar{\Delta}_{mn}(x_{mn}/y_{mn})| < \infty$.

If ρ is the set ℓ_0 of double series absolutely converging to zero, then $r = a_0$ is the set of double sequences x , for which $U_{k\ell} = \omega(1)$.

Here

$$x_{mn} = \omega(y_{mn})$$

denotes that $x_{mn} = \Omega(y_{mn})$ and $x_{mn} = o(y_{mn})$.

From now on we use the following notation:

- 1) $r\text{-}\lim_{m,n} s_{mn} = 0$ means $\lim_{m,n} s_{mn} = \lim_m s_{mn} = \lim_n s_{mn} = 0$,
- 2) $b\text{-}\lim_{m,n} s_{mn} = 0$ means $\lim_{m,n} s_{mn} = 0$ and $s_{mi} = O(1)$,
- 3) $s_{mn} = o_b(1)$ means $b\text{-}\lim_{m,n} s_{mn} = 0$.

DEFINITION 1.5. The numbers (ε_{mn}) are called *summability factors of type* (ρ, B_b) , if for every double series $\sum_{m,n} u_{mn} \in \rho$, it always follows that

$$\sum_{m,n} \varepsilon_{mn} u_{mn} \in B'_b. \quad (1.8)$$

Let A and B be linear methods of summability of double series (1.1), with field of bounded summability A'_b and B'_b respectively. Let T and T_0 be certain sets of double series (1.1). We denote three double series (1.1) by x, y and z .

DEFINITION 1.6. The condition $x \in T$ will be called *B_b -Tauberian for A_b* if

$$A'_b \cap T \subset B'_b. \quad (1.9)$$

In particular, when B is the convergence method $E = C^{0,0}$ then the E_b -Tauberian condition will simply be called a *Tauberian condition*.

§2. Main Lemma.

Now we formulate the following main lemma.

MAIN LEMMA 2.1. Let $A_b \supset B_b$ and A_b be consistent with B_b . If the condition $x \in T_0$ is B_b -Tauberian for A_b , then the condition $x \in T$ is also B_b -Tauberian for A_b , if every element $x \in T$ can be represented by

$$x = y + z, \quad (2.1)$$

where $y \in T_0$ and $z \in B'_b$.

PROOF. In order to prove the lemma, we will take an arbitrary element $x \in A'_b \cap T$ and assume that there exists the representation (2.1) for x . If we can prove that $y \in B'_b$, then it follows that $z \in B'_b$ and therefore $x \in B'_b$, which completes the proof of inclusion (1.9).

The inclusion $B'_b \subset A'_b$ yields that $z \in A'_b$. Since $A'_b \cap T \subset A'_b$, and if we consider (2.1), we have $y = x - z \in A'_b$ (since A'_b is an additive group),

and because $y \in T_0$ we obtain $y \in A'_b \cap T_0$. However, since the condition $y \in T_0$ is B_b -Tauberian for A_b it follows that $y \in B'_b$.

This completes the proof of the main lemma 2.1.

In the sequel we use the following notations:

$$\begin{aligned}\bar{\Delta}_m x_{mn} &= x_{mn} - x_{m-1,n}, \\ \bar{\Delta}_n x_{mn} &= x_{mn} - x_{m,n-1}, \\ \bar{\Delta}_{mn} x_{mn} &= \bar{\Delta}_m(\bar{\Delta}_n x_{mn}) = \bar{\Delta}_n(\bar{\Delta}_m x_{mn}) = \\ &= x_{mn} - x_{m-1,n} - x_{m,n-1} + x_{m-1,n-1}.\end{aligned}$$

Let the numbers $\lambda_{mn} \neq 0$, $\mu_{mn} \neq 0$, where $\bar{\Delta}_{mn}\mu_{mn} \neq 0$ and $\mu_{-1,n} = \mu_{m,-1} = 0$.

Let us assume that μ_{mn} and λ_{mn} are factorizable, that is,

$$\mu_{mn} = \mu'_m \cdot \mu''_n, \quad \lambda_{mn} = \lambda'_m \cdot \lambda''_n,$$

then

$$\begin{aligned}\frac{\bar{\Delta}_n \mu_{m-1,n}}{\lambda_{mn}} &= \frac{\mu'_{m-1}}{\lambda'_m} \cdot \frac{\bar{\Delta}_n \mu''_n}{\lambda''_n}, \\ \frac{\bar{\Delta}_m \mu_{m,n-1}}{\lambda_{mn}} &= \frac{\bar{\Delta}_m \mu'_m}{\lambda'_m} \cdot \frac{\mu''_{n-1}}{\lambda''_n}.\end{aligned}\tag{2.2}$$

For every double series (1.1) a double sequence (V_{mn}) will be constructed by the formula

$$V_{mn} = \frac{1}{\mu_{mn}} \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell}.\tag{2.3}$$

Let us denote

$$\begin{aligned}\alpha_{mn} &= \mu_{m-1,n-1}/\lambda_{mn}, & \gamma_{mn} &= (\bar{\Delta}_{mn}\mu_{mn})/\lambda_{mn}, \\ \beta'_{mn} &= (\bar{\Delta}_n \mu_{m-1,n})/\lambda_{mn}, & \beta''_{mn} &= (\bar{\Delta}_m \mu_{m,n-1})/\lambda_{mn}.\end{aligned}\tag{2.4}$$

One can now state

THEOREM 2.2. Let $A_b \supset B_b$, where A_b is consistent with B_b . If the following conditions are fulfilled:

- 1) the numbers α_{mn} are summability factors of type (ρ, B_b) ,
- 2) the double series

$$\omega' = \sum_{m,n} \beta'_{mn} \bar{\Delta}_m V_{mn}$$

and

$$\omega'' = \sum_{m,n} \beta''_{mn} \bar{\Delta}_n V_{mn}$$

are boundedly B -summable,

- 3) condition $(u_{mn}/\gamma_{mn}) \in r$ is B_b -Tauberian for A_b ,

then condition

$$(V_{mn}) \in r \tag{2.5}$$

is also B_b -Tauberian for A_b .

PROOF. From (2.5) we obtain

$$u_{mn} = \frac{1}{\lambda_{mn}} \bar{\Delta}_{mn} (\mu_{mn} V_{mn}).$$

We use the formula (see formula (15.31) in [1]) for the difference of products of double sequences and we obtain

$$u_{mn} = \alpha_{mn} \bar{\Delta}_{mn} V_{mn} + \beta'_{mn} \bar{\Delta}_m V_{mn} + \beta''_{mn} \bar{\Delta}_n V_{mn} + \gamma_{mn} V_{mn}. \tag{2.6}$$

With the aid of the formal series

$$x = \sum_{m,n} u_{mn}, \quad y = \sum_{m,n} \gamma_{mn} V_{mn}, \quad z = \omega + \omega' + \omega'',$$

where

$$\omega = \sum_{m,n} \alpha_{mn} \bar{\Delta}_{mn} V_{mn}, \quad \omega' = \sum_{m,n} \beta'_{mn} \bar{\Delta}_m V_{mn}, \quad \omega'' = \sum_{m,n} \beta''_{mn} \bar{\Delta}_n V_{mn} \tag{2.7}$$

from (2.6) we get expansion (2.1).

If condition (2.5) is satisfied then the double series $\sum_{m,n} \bar{\Delta}_{mn} V_{mn} \in \rho$, and from conditions 1) and 2) it follows that $z \in B'_b$.

On the other hand, we denote

$$T_0 = \left\{ \sum_{m,n} u_{mn} : (u_{mn}/\gamma_{mn}) \in r \right\}. \quad (2.8)$$

From (2.5) we conclude that $y \in T_0$, if in T_0 we take $u_{mn} = \gamma_{mn} V_{mn}$. According to 3), condition $x \in T_0$ is B_b -Tauberian for A_b . Thus the statement of theorem 2.2 follows from the main lemma 2.1 if one sets

$$T = \left\{ \sum_{m,n} u_{mn} : (V_{mn}) \in r \right\}, \quad (2.9)$$

because every element x has the representation (2.1).

§3. Summability factors of type $\varepsilon_{mn} \in (b\gamma_0, E_b)$.

In theorem 2.2 we shall take the method $B = E$ and $\rho = b\gamma_0$. Then we have the case where the summability factor is of type $(b\gamma_0, E_b)$. Now we will find these conditions for this type using the following lemma.

LEMMA 3.1. In order that $\varepsilon_{mn} \in (b\gamma_0, E_b)$, conditions

$$\sum_{m,n} |\Delta_{mn} \varepsilon_{mn}| < \infty \quad (3.1)$$

and

$$\lim_n \Delta_m \varepsilon_{mn} = 0, \quad \lim_m \Delta_n \varepsilon_{mn} = 0 \quad (3.2)$$

are necessary and sufficient.

PROOF. By definition (1.9) we have to find conditions for the bounded convergence of double series (1.8), if the bounded double series (1.1) converges to zero. We denote partial sums of double series (1.1) by U'_{mn} and U''_{mn} . Then using the transformations of Abel-Hardy we get

$$U''_{mn} = \sum_{k,\ell=0}^{m,n} a_{mnk\ell} U'_{k\ell},$$

where

$$a_{m n k \ell} = \begin{cases} \Delta_{k \ell} \varepsilon_{k \ell}, & k, \ell < m, n, \\ \Delta_k \varepsilon_{k n}, & k < m, \ell = n, \\ \Delta_\ell \varepsilon_{m \ell}, & k = m, \ell < n, \\ b_{m n}, & k = m, \ell = n. \end{cases}$$

Therefore, $\varepsilon_{m n} \in (b\gamma_0, E_b)$, iff the matrix $A = (a_{m n k \ell})$ satisfies the conditions for transformations $A : bcn \rightarrow bc$. By Hamilton's theorem (see [3], p.49, theorem 19), the necessary and sufficient conditions are

$$\sum_{k, \ell=0}^{m, n} |a_{m n k \ell}| = O(1), \quad (c_1)$$

and

$$\lim_{m, n} \sum_{k=0}^m |a_{m n k \ell} - a_{k \ell}| = 0, \quad (d_4)$$

$$\lim_{m, n} \sum_{\ell=0}^n |a_{m n k \ell} - a_{k \ell}| = 0,$$

where

$$a_{k \ell} = \lim_{m, n} a_{m n k \ell}.$$

The meaning of (c_1) is

$$\sum_{k, \ell=0}^{m, n} |a_{m n k \ell}| = \sum_{k, \ell=0}^{m-1, n-1} |\Delta_{k \ell} \varepsilon_{k \ell}| + \sum_{k=0}^{m-1} |\Delta_k \varepsilon_{k n}| + \sum_{\ell=0}^{n-1} |\Delta_\ell \varepsilon_{m \ell}| + |\varepsilon_{m n}| = O(1).$$

In conclusion, condition (c_1) transforms into the following conditions:

$$\sum_{k, \ell} |\Delta_{k \ell} \varepsilon_{k \ell}| < \infty, \quad (3.3)$$

$$\sum_k |\Delta_k \varepsilon_{k n}| = O(1), \quad \sum_\ell |\Delta_\ell \varepsilon_{m \ell}| = O(1), \quad (3.4)$$

$$\varepsilon_{m n} = O(1). \quad (3.5)$$

For (d_4) we have

$$\begin{aligned} \lim_{m, n} \sum_{k=0}^m |a_{m n k \ell} - a_{k \ell}| &= \lim_{m, n} \left[\sum_{k=0}^{m-1} |a_{m n k \ell} - a_{k \ell}| + |a_{m n m \ell} - a_{m \ell}| \right] \\ &= \lim_{m, n} |a_{m n m \ell} - a_{m \ell}| = \lim_{m, n} |\Delta_\ell \varepsilon_{m \ell} - \Delta_{m \ell} \varepsilon_{m \ell}| \\ &= \lim_{m, n} |\Delta_\ell (\varepsilon_{m \ell} - \Delta_{m \ell} \varepsilon_{m \ell})| = \lim_{m, n} |\Delta_\ell (\varepsilon_{m \ell} - \varepsilon_{m \ell} + \varepsilon_{m+1, \ell})| \\ &= \lim_{m, n} |\Delta_\ell \varepsilon_{m+1, \ell}| = 0. \end{aligned}$$

Therefore, condition (d_4) is replaced by condition (3.6).

The second part of condition (d_4) can be checked analogously.

$$\lim_m \Delta_\ell \varepsilon_{m+1, \ell} = 0, \quad \lim_n \Delta_k \varepsilon_{k, n+1} = 0. \quad (3.6)$$

This proves lemma 3.1. Q.E.D.

LEMMA 3.2. Necessary and sufficient conditions for $\varepsilon_{mn} \in (b\gamma, E_b)$ are (3.1) and (3.2).

PROOF. See Hamilton ([2], p.283).

From lemmas 3.1 and 3.2 it follows that

$$\varepsilon_{mn} \in (b\gamma_0, E_b) \Leftrightarrow \varepsilon_{mn} \in (b\gamma, E_b).$$

COROLLARY 3.3. If ε_{mn} is factorizable,

$$\varepsilon_{mn} = \varepsilon'_m \cdot \varepsilon''_{mn}$$

then $\varepsilon_{mn} \in (b\gamma_0, E_b)$ and $\varepsilon_{mn} \in (b\gamma, E_b)$ if and only if (3.1) and

$$r\text{-}\lim_{m,n} \varepsilon_{mn} = 0 \quad (3.7)$$

are fulfilled.

PROOF. Since ε_{mn} is factorizable, then from (3.2) we obtain

$$\lim_n \Delta_m \varepsilon_{mn} = \Delta_m \varepsilon'_m \lim_n \varepsilon''_n = 0$$

and therefore $\lim_n \varepsilon''_n = 0$. Therefore in this case condition (3.2) is equivalent to (3.7).

§4. Connections between summability factors of double series and Tauberian theorems.

Denote, for $m, n \geq 0$

$$h'_m = \overline{\Delta}_m \mu'_m / \lambda'_m, \quad h''_n = \overline{\Delta}_n \mu''_n / \lambda''_n. \quad (4.1)$$

Now we apply corollary 3.5, and from theorem 2.2 we get

COROLLARY 4.1. Let A be a regular method of summability of double series. If the following conditions are fulfilled:

- 1) $\sum_{m,n} |\bar{\Delta}_{mn} \alpha_{mn}| < \infty$,
- 2) $r\text{-}\lim \alpha_{mn} = 0$,
- 3) $\sum_{m,n} |h_{mn}| < \infty$,
- 4) $\sum_m \left| \frac{\mu'_{m-1}}{\lambda'_m} \bar{\Delta}_m V_{mn} \right| = O(1)$,
 $\sum_n \left| \frac{\mu''_{n-1}}{\lambda''_n} \bar{\Delta}_n V_{mn} \right| = O(1)$,
- 5) condition

$$b\text{-}\lim_{m,n} (u_{mn}/\gamma_{mn}) = 0, \quad (4.2)$$

is Tauberian for A_b ,

then

$$V_{mn} = o_b(1)$$

is also Tauberian for A_b .

PROOF. We must check all the conditions of theorem 2.2 with $B = E$ and $p = b\gamma_0$. Since A is regular, then $A_b \supset E_b$ and A_b is consistent with E_b . From conditions 1) and 2), by Corollary 3.3 it follows that α_{mn} are summability factors of type $(b\gamma_0, E_b)$. For ω' , we substitute (2.2) in (2.4) and consider conditions 3) and 4). Then we get

$$\begin{aligned} |\omega'| &\leq \sum_{m,n} \left| \frac{\mu'_{m-1}}{\lambda'_m} h''_n \bar{\Delta}_m V_{mn} \right| = \sum_n |h''_n| \sum_m \left| \frac{\mu'_{m-1}}{\lambda'_m} V_{mn} \right| = \\ &= O(1) \sum_n |h''_n| = O(1), \end{aligned}$$

and consequently ω' is b -convergent. Analogously for ω'' . Condition 3) of theorem 2.2 is the condition 5) of our corollary 4.1. Then by Theorem 2.2 condition $V_{mn} = o_b(1)$ is Tauberian for A_b . Q.E.D.

If we apply lemmas 3.1 and 3.2, and from theorem 2.2 we get

COROLLARY 4.2. Let A be a regular method of summability of double series. If the following conditions are fulfilled:

- 1) $\sum_{m,n} |\Delta_{mn} \alpha_{mn}| < \infty$,
- 2) $\lim_n \Delta_m \alpha_{mn} = 0$, $\lim_m \Delta_n \alpha_{mn} = 0$
- 3) $\sum_{m,n} |h_{mn}| < \infty$,
- 4) $\sum_m \left| \frac{\mu'_{m-1}}{\lambda'_m} \bar{\Delta}_m V_{mn} \right| = O(1)$,
 $\sum_n \left| \frac{\mu''_{n-1}}{\lambda''_n} \bar{\Delta}_n V_{mn} \right| = O(1)$,
- 5) condition (4.2) is Tauberian for A_b ,

then

$$(V_{mn}) \in bc$$

is also Tauberian for A_b .

PROOF: From conditions 1) and 2) by lemma 3.2 it follows that α_{mn} are summability factors of type $(b\gamma, E_b)$. For ω' we consider conditions 3) and 4), and we obtain that ω' is even absolute convergent.

Condition 3) of Theorem 2.2 is condition 5) of corollary 4.2.

Analogously for ω'' , because condition 5) is Tauberian for A_b . Then by theorem 2.2, condition $V_{mn} \in bc$ is also Tauberian for A_b . Denote, for $m, n \geq 0$

$$\tau_{mn} = \frac{1}{\mu'_n} \sum_l^n \lambda_l u_{ml}, \quad \sigma_{mn} = \frac{1}{\mu'_m} \sum_k^m \lambda_k u_{kn}. \quad (4.3)$$

Now we shall prove an analogous theorem to 2.2 for another representation of (2.6).

THEOREM 4.3. Let A be a regular method of summability of double series.

If the following conditions are fulfilled:

- 1) $\sum_{m,n} |\Delta_{mn} \alpha_{mn}| < \infty$,
- 2) $r\text{-}\lim \alpha_{mn} = 0$
- 3) $\sum_{m,n} |h_{mn}| < \infty$,
- 4) $\lim_n \sum_m \tau_{mn} = 0$, $\lim_m \sum_n \sigma_{mn} = 0$,

5) condition

$$(u_{mn}) \in bc$$

is Tauberian for A_b ,

then

$$(V_{mn}) \in bc$$

is also Tauberian for A_b .

PROOF. We examine

$$\beta'_{mn} \bar{\Delta}_m V_{mn} + \beta''_{mn} \bar{\Delta}_n V_{mn} + \gamma_{mn} V_{mn}, \quad (4.4)$$

part of the sum (2.6).

If we substitute (2.4) and consider (2.2) in (4.3) we get

$$\frac{\bar{\Delta}_n \mu_{m-1,n}}{\lambda_{mn}} \cdot \bar{\Delta}_m V_{mn} + \frac{\bar{\Delta}_m \mu_{m,n-1}}{\lambda_{mn}} \bar{\Delta}_n V_{mn} + \frac{\bar{\Delta}_{mn} \mu_{mn}}{\lambda_{mn}} V_{mn}.$$

After performing elementary operations on the last expansion and substituting (2.3), we obtain

$$\begin{aligned} & \frac{\mu_{mn}}{\lambda_{mn}} V_{mn} - \frac{1}{\lambda_{mn}} \left[\sum_{k,\ell=0}^{m-1,n} \lambda_{k\ell} u_{k\ell} + \sum_{k,\ell=0}^{m,n-1} \lambda_{k\ell} u_{k\ell} \right] - \\ & - \alpha_{mn} \left[\frac{1}{\mu_{mn}} \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} - \frac{1}{\mu_{m-1,m}} \sum_{\ell=0}^{m-1,n} \lambda_{k\ell} u_{k\ell} - \frac{1}{\mu_{m,n-1}} \sum_{k,\ell=0}^{m,n-1} \lambda_{k\ell} u_{k\ell} \right]. \end{aligned}$$

Then if we consider that

$$\begin{aligned} \sum_{k,\ell=0}^{m,n-1} \lambda_{k\ell} u_{k\ell} &= \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} - \sum_{k=0}^m \lambda_{kn} u_{kn}, \\ \sum_{k,\ell=0}^{m-1,n} \lambda_{k\ell} u_{k\ell} &= \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} - \sum_{\ell=0}^n \lambda_{m\ell} u_{m\ell}, \end{aligned} \quad (4.5)$$

we obtain

$$\begin{aligned} & \frac{1}{\lambda_{mn}} \sum_{\ell=0}^n \lambda_{k\ell} u_{k\ell} - \frac{1}{\lambda_{mn}} \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} + \frac{1}{\lambda_{mn}} \sum_{k=0}^m \lambda_{k\ell} u_{kn} - \\ & - \alpha_{mn} \cdot \frac{1}{\mu_{mn}} \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} + \alpha_{mn} \frac{1}{\mu_{m-1,n}} \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} - \\ & - \alpha_{mn} \cdot \frac{1}{\mu_{m-1,n}} \sum_{\ell=0}^n \lambda_{m\ell} u_{m\ell} + \alpha_{mn} \frac{1}{\mu_{m,n-1}} \sum_{k=0}^m \lambda_{kn} u_{kn} + \\ & + \alpha_{mn} \cdot \frac{1}{\mu_{m,n-1}} \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell}. \end{aligned}$$

Combining terms,

$$\begin{aligned} & \left[\frac{1}{\lambda_{mn}} - \frac{\alpha_{mn}}{\mu_{m-1,n}} \right] \sum_{\ell=0}^m \lambda_{m\ell} u_{m\ell} + \left[\frac{1}{\lambda_{mn}} - \frac{\alpha_{mn}}{\mu_{m,n-1}} \right] \sum_{k=0}^m \lambda_{kn} u_{kn} + \\ & + \left[\frac{\alpha_{mn}}{\mu_{m-1,n}} + \frac{\alpha_{mn}}{\mu_{m,n-1}} - \frac{1}{\lambda_{mn}} - \frac{\alpha_{mn}}{\mu_{mn}} \right] \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell}. \end{aligned}$$

In the last expression we substitute (2.3), and we recall the fact that all the numbers are factorizable. Then we get

$$\begin{aligned} & \left[\frac{1}{\lambda_{mn}} - \frac{\mu'_{m-1} \mu''_{n-1}}{\lambda_{mn} \mu'_{m-1} \mu''_{n-1}} \right] \sum_{\ell=0}^n \lambda_{m\ell} u_{m\ell} + \left[\frac{1}{\lambda_{mn}} - \frac{\mu'_{m-1} \mu''_{n-1}}{\lambda_{mn} \mu'_{m-1} \mu''_{n-1}} \right] \sum_{k=0}^m \lambda_{k\ell} u_{kn} + \\ & + \left[\frac{\mu'_{m-1} \mu''_{n-1}}{\mu'_{m-1} \mu''_{n-1} \lambda_{mn}} + \frac{\mu'_{m-1} \mu''_{n-1}}{\mu'_{m-1} \mu''_{n-1} \lambda_{mn}} - \frac{1}{\lambda_{mn}} - \frac{\mu'_{m-1} \mu''_{n-1}}{\lambda_{mn} \mu'_{m-1} \mu''_{n-1}} \right] \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} = \\ & = \left[\frac{1}{\lambda'_m \lambda''_n} - \frac{\mu''_{n-1}}{\lambda'_m \lambda''_n \mu''_{n-1}} \right] \lambda'_m \sum_{\ell=0}^n \lambda''_{\ell} u_{m\ell} + \left[\frac{\lambda''_n}{\lambda'_m \lambda''_n} - \frac{\mu'_{m-1} \lambda''_n}{\lambda'_m \lambda''_n \mu'_m} \right] \sum_{k=0}^m \lambda'_k u_{kn} + \\ & + \frac{1}{\lambda_{mn}} \left[\frac{\mu''_{n-1}}{\mu''_n} + \frac{\mu'_{m-1}}{\mu'_m} - 1 - \frac{\mu'_{m-1} \mu''_{n-1}}{\mu'_m \mu''_n} \right] \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} = \\ & = \frac{\bar{\Delta} \mu''_n}{\lambda''_n} \cdot \frac{1}{\mu''_{n-1}} \sum_{\ell=0}^n \lambda''_{\ell} u_{m\ell} + \frac{\bar{\Delta} \mu'_m}{\lambda'_m} \cdot \frac{1}{\mu'_m} \sum_{k=0}^m \lambda'_k u_{kn} + \\ & + \frac{1}{\lambda_{mn}} \left[\frac{\mu'_{n-1} \mu'_m + \mu'_{m-1} \mu''_n - \mu'_m \mu''_n - \mu'_{m-1} \mu''_{n-1}}{\mu'_m \mu''_n} \right] \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell}. \end{aligned}$$

Now we substitute (2.3), (4.1) and (4.2); then we get

$$\begin{aligned} h_n'' \tau_{mn} + h_m' \sigma_{mn} + \frac{1}{\lambda_{mn}} \left[\frac{\mu_{m-1}' \bar{\Delta} \mu_n'' - \mu_m' (\mu_n'' - \mu_{n-1}'')}{\mu_m' \mu_n''} \right] \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell} = \\ = h_n'' \tau_{mn} + h_m' \sigma_{mn} - \left[\frac{\bar{\Delta} \mu_n'' \bar{\Delta} \mu_m'}{\lambda_{mn} \mu_m' \mu_n''} \right] \sum_{k,\ell=0}^{m,n} \lambda_{k\ell} u_{k\ell}. \end{aligned}$$

If we consider (4.1) and (2.3), we obtain

$$h_n'' \tau_{mn} + h_m' \sigma_{mn} - h_m' h_n'' V_{mn}, \quad (4.6)$$

and, therefore, every $x \in T$ can be represented by

$$x = z + y,$$

where

$$x = \sum_{m,n} u_{mn}, \quad y = \sum_{m,n} h_m' h_n'' V_{mn}, \quad z = \omega + \omega' + \omega'',$$

and

$$\omega = \sum_{m,n} \alpha_{mn} \bar{\Delta}_{mn} V_{mn}, \quad \omega' = \sum_{m,n} h_n'' \tau_{mn}, \quad \omega'' = \sum_{m,n} h_m' \sigma_{mn}. \quad (4.7)$$

Finally, we have

$$u_{mn} = \alpha_{mn} \bar{\Delta}_{mn} V_{mn} + h_n'' \tau_{mn} + h_m' \sigma_{mn} - h_m' h_n'' V_{mn}. \quad (4.8)$$

The proof of theorem 4.3 follows from (4.8). Indeed, from conditions 1), 2) by corollary 3.3 it follows that $\sum_{m,n} \alpha_{mn} \bar{\Delta}_{mn} V_{mn} \in b\gamma$, because (2.5) and α_{mn} are summability factors of type $(b\gamma, E_b)$. For ω' we consider conditions 3) and 4). From 4) it follows that $\sum_m \tau_{mn} = o_b(1)$ and $\sum_n \sigma_{mn} = o_b(1)$, i.e., for ω' we have

$$|\omega'| = \left| \sum_{m,n} h_n'' \tau_{mn} \right| \leq \sum_n |h_n''| \left| \sum_m \tau_{mn} \right| = O(1),$$

and analogously for ω'' . Condition 5) is Tauberian for A_b ; therefore $(V_{mn}) \in bc$ is also Tauberian for A_b .

Analogous theorems for simple series were proved by W. Meyer-König and H. Tietz (see [8], Theorems 2.1 and 2.13, p.180), G. Kangro (see [4], pp.24-26) and D. Leviatan (see [5], Theorem 3, i-iii, p.13).

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Received December 9, 1992

TÜ toimetised. Vihik 960. FUNCTIONAL ANALYSIS AND THEORY OF
SUMMABILITY. Matemaatika- ja mehaanika-alaseid teid. Tartu
Ülikool, EE2400 Tartu, Ülikooli 18. Vastutav toimetaja E. Kolk.
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