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ON SEMIGROUP AMALGAMS

Bachelor thesis

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# Introduction

Consider two groups  $G_1$  and  $G_2$  intersecting in a common subgroup  $U$ . Can we find a group  $W$  such that  $G_1$  and  $G_2$  are contained in it and their intersection is still  $U$ ? More formally, one asks if the group amalgam  $[U; G_1, G_2]$  is embeddable. This question (about the embeddability of group amalgams) was first posed by Otto Schreier. In 1927 he proved that all group amalgams are embeddable. It was then natural to ask if this result could be expanded to the class of semigroups. In 1957 N. Kimura constructed a counter example in his doctoral thesis, showing that this is not the case for semigroups. J. M. Howie proved in 1962 that a semigroup amalgam  $[U; S_1, S_2]$ , in which  $S_1$  and  $S_2$  are groups, is embeddable if and only if  $U$  is also a group. Howie's result was particularly interesting as it provided an infinite class of non-embeddable semigroup amalgams. In 1975 T. E. Hall generalized Schreier's result to the class of inverse semigroups (Journal of Algebra 34, 375–385 (1975)). The aim of this thesis is to consider some situations where semigroup amalgams fail to embed. In the first chapter frequently used terminology and some needed lemmas are introduced. In the second chapter we discuss the existence of pushouts, a categorical notion that is linked with amalgamation. The last chapter, which concentrates on semigroup amalgams, provides some results regarding non-embeddable semigroup amalgams and establishes a link between pushouts and amalgamation.

# 1. Preliminaries

In this chapter we define some important concepts that are frequently used in the sequel. We also give some preliminary results without proofs. The interested readers are however provided with the references to their origins.

A nonempty set  $S$  with an associative binary operation

$$* : S \times S \rightarrow S$$

is called a *semigroup*. We shall usually denote  $a * b$  (the image of  $(a, b)$  under  $*$ ) by  $ab$ .

Let  $X$  be a nonempty set. We call a subset  $R$  of  $X \times X$  a *relation* on  $X$ . To show that  $(x, y)$  belongs to  $R$  we also write  $xRy$ . The product (equally composition)  $A \circ B$  of relations  $A$  and  $B$  is defined as follows:

$$A \circ B = \{(x, y) : (x, z) \in A \text{ and } (z, y) \in B \text{ for some } z \in X\}.$$

The relation  $\{(x, x) : x \in X\}$  is called the *identity relation* and is denoted by  $1_X$ . The relation  $\{(x, y) : (y, x) \in R\}$  is called the *converse* of  $R$  and is denoted by  $R^{-1}$ . We call a relation  $R$  *reflexive* if  $1_X \subseteq R$  and *symmetric* if  $R^{-1} \subseteq R$  (which in fact leads to  $R = R^{-1}$ ). A relation  $R$  is called *transitive* if  $R \circ R \subseteq R$ . We call a relation  $R$  an *equivalence relation* if it is reflexive, symmetric and transitive. We say that a relation  $R$  on a semigroup  $S$  is *compatible* (with the binary operation on  $S$ ) if for any  $a$  belonging to  $S$  and  $(x, y) \in R$  we have  $(ax, ay) \in R$  and  $(xa, ya) \in R$  (that is  $R$  is right and left compatible). We call an equivalence relation  $R$  on a semigroup  $S$  a *congruence* if it is compatible. A mapping  $\phi$  from a semigroup  $S$  to a semigroup  $T$  is called a (semigroup) homomorphism if

$$\phi(s_1 s_2) = \phi(s_1) \phi(s_2) \quad \forall s_1, s_2 \in S.$$

Let  $S_1$  and  $S_2$  be semigroups and let  $\phi : S_1 \rightarrow S_2$  be a homomorphism then the kernel of  $\phi$  is defined as

$$\text{Ker}\phi = \{(a, b) \in S_1 \times S_1 : \phi(a) = \phi(b)\}.$$

It is easy to verify that  $\text{Ker}\phi$  is a congruence on  $S_1$ .

The following lemma shows that given a relation  $R$  on a nonempty set  $X$  one can find the smallest (with respect to inclusion) transitive relation  $R^\infty$  containing  $R$ .

**Lemma 1.1** (see [4, Lemma 1.4.8]).

$$R^\infty = \cup_{i=1}^{\infty} R^i,$$

where  $R^n = \underbrace{R \circ R \circ \dots \circ R}_{n\text{-times}}$  is defined inductively.

For every relation  $R$  on a nonempty set  $X$  we can also find (the smallest) equivalence relation  $R^e$  so that  $R \subseteq R^e$  and  $R^e \subseteq T$  for any equivalence relation  $T$  containing  $R$ . Note that the relation  $R^e$  always exists because (as one can easily observe) the intersection of equivalences is an equivalence and that at least the all-relation  $X \times X$  is an equivalence relation containing  $R$ . We call  $R^e$  the *equivalence relation generated by  $R$* . The following lemma gives an explicit method to find  $R^e$ .

**Lemma 1.2** (see [4, Proposition 1.4.9]). *The equivalence relation  $R^e$  generated by  $R$  is given by*

$$R^e = (R \cup R^{-1} \cup 1_S)^\infty.$$

It is also possible to speak, in a similar fashion, of the congruence relation  $R^\sharp$  generated by  $R$ . In order to give a recipe for calculating  $R^\sharp$  we need the smallest compatible relation  $R^c$  containing  $R$ . For a semigroup  $S$  we define

$$S^1 = \begin{cases} S & \text{if } S \text{ contains an identity} \\ S \cup \{1\} & \text{otherwise} \end{cases},$$

with  $s1 = 1s = s$  for all  $s \in S$  and  $11 = 1$ . One can easily see that the (extended) binary operation on  $S^1$  is associative. So  $S^1$  is a semigroup with identity, i.e a *monoid*.

**Lemma 1.3** (see [4, Lemma 1.5.5]). *Let  $R$  be a relation on a semigroup  $S$ . Then*

$$R^c = \{(axb, ayb) : a, b \in S^1, (x, y) \in R\}$$

*is the smallest (in the sense described above) compatible relation containing  $R$ .*

**Lemma 1.4** (see [4, Proposition 1.5.8]). *Given a relation  $R$  on a semigroup  $S$ ,*

$$R^\sharp = (R^c)^e.$$

There is yet an alternate description of  $R^\sharp$  that proves more practical in many situations. Let  $S$  be a semigroup and let  $a, b \in S$ . We say that  $a$  is connected to  $b$  by an elementary  $R$ -transition if

$$a = xcy,$$

$$b = xdy,$$

for some  $x$  and  $y$  in  $S^1$ , where either  $(a, b)$  or  $(b, a)$  belongs to  $R$ . Then we have the following lemma.

**Lemma 1.5** (see [4, Proposition 1.5.9]). *Let  $R$  be a relation on a semigroup  $S$  and let  $a, b \in S$ . Then  $(a, b) \in R^\sharp$  if and only if either  $a = b$  or for some  $n$  in  $\mathbb{N}$  there exists a sequence*

$$a = z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_n = b$$

*of elementary  $R$ -transitions connecting  $a$  to  $b$ .*

An element  $a$  of a semigroup  $S$  is called *idempotent* if  $a^2 = a$ . An element  $x$  of a semigroup  $S$  is called *regular* if there exists an element  $y \in S$  such that

$$x = xyx.$$

We call a semigroup *regular* if all of its elements are regular. Let  $x$  and  $y$  be elements of a regular semigroup with  $x = xyx$ . Then one can easily observe that  $x' = yxy$  satisfies the identities

$$xx'x = x,$$

$$x'xx' = x'.$$

We call  $x'$  (which may not be unique) *inverse* of  $x$ . Clearly if  $x'$  is inverse of  $x$  then  $x$  is inverse of  $x'$ . We call  $S$  an *inverse semigroup* if every element  $x$  in  $S$  has a unique inverse. Of course an inverse semigroup is always regular. A semigroup  $S$  is called *completely regular* if there exists a unary operation  $a \mapsto a^{-1}$  on  $S$  with the properties

$$(a^{-1})^{-1} = a, aa^{-1}a = a, aa^{-1} = a^{-1}a.$$

Completely regular semigroups are in fact unions of groups (see [4, Proposition 4.1.1]). A semigroup  $S$  is called a *Clifford semigroup* if it is completely regular and for all  $x, y \in S$

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

Let  $X$  be a nonempty set. Consider the set of all ‘words’ in the ‘alphabet’  $X$ ,

$$A = \{a_1a_2 \dots a_n : a_i \in X, n \in \mathbb{N}\}.$$

Define multiplication on  $A$  as follows. For  $a = a_1a_2 \dots a_n, b = b_1b_2 \dots b_m \in A, n, m \in \mathbb{N}$

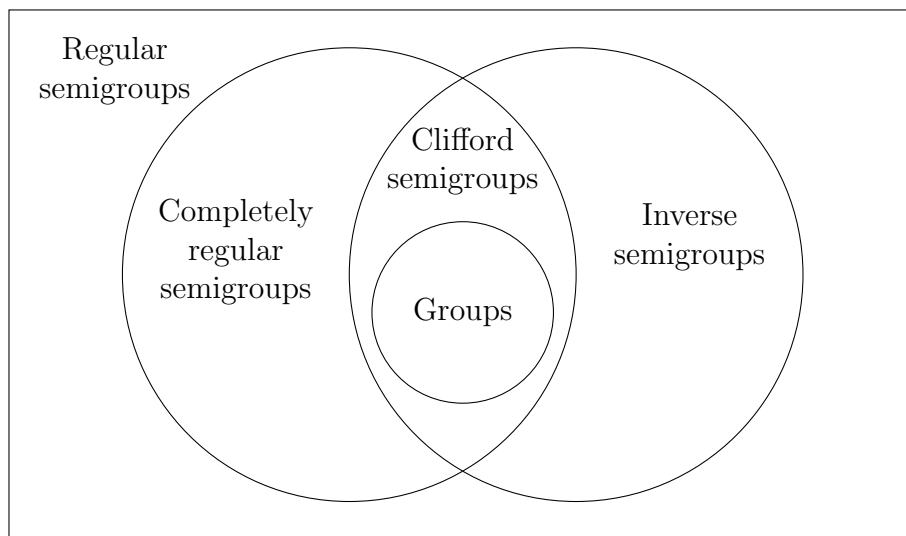
$$a \cdot b = a_1a_2 \dots a_nb_1b_2 \dots b_m.$$

The set  $A$  with this multiplication (which can easily be verified to be associative) is called the *free semigroup* on  $X$ .

**Theorem 1.6** (cf. [4, Theorem 5.1.1] ). *Let  $S$  be a semigroup. Then the following statements are equivalent:*

1.  $S$  is an inverse semigroup;
2.  $S$  is regular, and its idempotents commute.

The following diagram illustrates (for details see introduction of Chapter 4 and §5.1 of [4]) an interplay amongst different classes of semigroups, that we discussed above:



## 2. Some categorical notions

In this chapter we define categories and pushouts. The latter notion, in the category of semigroups, is related to the amalgamation, as we shall see in the next chapter.

**Definition 2.1.** A category consists of the following:

1. a class  $\mathcal{C}$  of objects;
2. for every pair  $(A, B)$  of objects, a set  $Mor(A, B)$ , whose elements are called morphisms;
3. for every three objects  $A, B, C$ , a composition map  $\circ$ ,

$\circ : Mor(A, B) \times Mor(B, C) \rightarrow Mor(A, C)$  given by

$$(f, g) \mapsto g \circ f \text{ and}$$

4. for every object  $A \in \mathcal{C}$ , a morphism  $id_A \in Mor(A, A)$ , called the identity morphism on  $A$ .

The following conditions need also to be satisfied.

- i.  $(A, B) \neq (A', B') \Rightarrow Mor(A, B) \cap Mor(A', B') = \emptyset$ .
- ii. (Associativity) For every  $A, B, C \in \mathcal{C}$  and for morphisms  $f \in Mor(A, B)$ ,  $g \in Mor(B, C)$ ,  $h \in Mor(C, D)$  we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- iii. (Unitality) For every  $A, B, C \in \mathcal{C}$  and for morphisms  $f \in Mor(A, B)$ ,  $g \in Mor(B, C)$  one has

$$id_B \circ f = f \text{ and } g \circ id_B = g.$$

An element  $f$  from  $Mor(A, B)$  will also be denoted by  $f : A \rightarrow B$ . Let  $X_1$  and  $X_2$  be objects of some category  $C$ . Then a morphism  $f : X_1 \rightarrow X_2$  is called a *monomorphism* if it is *left cancellative*, that is

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2,$$

for all morphisms  $g_1, g_2 : X \rightarrow X_1$ . In the category of semigroups a homomorphism is a monomorphism if and only if it is one-one. By a *diagram* in a category we shall mean a set of objects some of which are connected via arrows (which represent morphisms). We say that a diagram *commutes* if all morphisms and their compositions between any two objects are equal.

**Definition 2.2.** In a category  $C$ , an object  $P$  is called a *pushout* of the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \\ S_2 & & \end{array}, \quad (2.1)$$

if there exist  $\psi_1 \in Mor(S_1, P)$ ,  $\psi_2 \in Mor(S_2, P)$  such that the following conditions are satisfied.

1. The diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \psi_1 \\ S_2 & \xrightarrow{\psi_2} & P \end{array},$$

commutes.

2. If  $W$  is an object of  $C$  such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \lambda_1 \\ S_2 & \xrightarrow{\lambda_2} & W \end{array}$$

commutes for some morphisms  $\lambda_1$  and  $\lambda_2$ , then there exists a unique morphism  $\lambda : P \rightarrow W$  such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \psi_1 \\ S_2 & \xrightarrow{\psi_2} & P \end{array} \begin{array}{c} \searrow \lambda_1 \\ \searrow \lambda \\ \searrow \lambda_2 \end{array} \begin{array}{c} \\ \\ W \end{array}.$$

we now show that pushouts exist in the category of semigroups. For this a pushout is constructed. We begin by introducing the coproduct of semigroups.

**Definition 2.3.** Let  $C$  be a category and let  $X_1$  and  $X_2$  be objects of  $C$ . We call an object  $Y$  the coproduct of  $X_1$  and  $X_2$  if there exist morphisms  $\theta_i : X_i \rightarrow Y$ ,  $i \in \{1, 2\}$ , such that for any other object  $Q$  and morphisms  $\beta_i : X_i \rightarrow Q$  there exists a unique morphism  $\lambda : Y \rightarrow Q$  commuting the following diagram:

$$\begin{array}{ccc}
 X_2 & & \\
 \beta_2 \downarrow & \searrow \theta_2 & \\
 & & Y \\
 Q \longleftarrow & \lambda & \longrightarrow \\
 \beta_1 \uparrow & \nearrow \theta_1 & \\
 X_1 & & 
 \end{array} .$$

In the category of semigroups the coproduct  $S_1 * S_2$  of two semigroups  $S_1$  and  $S_2$  can be constructed as follows. Take

$$\begin{aligned}
 S_1 * S_2 = \{ & a_1 a_2 \dots a_n : a_1, a_2, \dots, a_n \in S_1 \dot{\cup} S_2 : \{a_j, a_{j+1}\} \not\subseteq S_i, \\
 & i \in \{1, 2\}, 1 \leq j \leq n-1 \}
 \end{aligned}$$

(note that we can always take an isomorphic copy of  $S_2$  disjoint from  $S_1$ ). For the sake of brevity we shall denote  $S_1 * S_2$  by  $F$ . Define multiplication on  $F$  as follows. Let  $a = a_1 a_2 \dots a_n$ ,  $b = b_1 b_2 \dots b_m \in F$  then

$$ab = \begin{cases} a_1 a_2 \dots a_n b_1 b_2 \dots b_m, & \text{if } a_n \text{ and } b_1 \text{ are from different semigroups} \\
 a_1 a_2 \dots a_{n-1} c b_2 \dots b_m, & \text{where } c = a_n b_1, \text{ otherwise.} \end{cases}$$

It is easy to see that this operation is associative. We now define morphisms  $\theta_i : S_i \rightarrow F$  by

$$\theta_i(a) = a,$$

where  $a$  on the right hand side represents the one-letter word in  $F$ . We leave it to the reader to verify that  $F$  satisfies the conditions of Definition 2.3.

Referring to Diagram 2.1, next define a relation

$$R = \{(\theta_1 \phi_1(u), \theta_2 \phi_2(u)) : u \in U\}$$

on  $F$  ( $= S_1 * S_2$ ). Let  $\rho$  be the smallest congruence on  $F$  containing  $R$ . Let us write

$$P = S_1 *_U S_2 = F/\rho.$$

Then we have a natural homomorphism:

$$\rho^{\natural} : F \rightarrow P,$$

defined by

$$w \mapsto \bar{w},$$

where  $\bar{w}$  denotes the equivalence class containing the word  $w$ . Define  $\mu_i : S_i \rightarrow P$ ,  $i \in \{1, 2\}$  by

$$\mu_i = \rho^{\natural} \circ \theta_i, \quad i \in \{1, 2\}.$$

This gives the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \theta_1 \\ S_2 & \xrightarrow{\theta_2} & F \\ & \searrow \mu_2 & \downarrow \rho^{\natural} \\ & & P \end{array} \quad \begin{array}{c} \nearrow \mu_1 \\ \cdot \end{array}$$

We show that the object  $P$  is a pushout of the diagram 2.1. Firstly observe that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \theta_1 \\ S_2 & \xrightarrow{\theta_2} & F \\ & \searrow \mu_2 & \downarrow \rho^{\natural} \\ & & P \end{array} \quad \begin{array}{c} \nearrow \mu_1 \\ \cdot \end{array}$$

commutes, because

$$\begin{aligned} \mu_1 \phi_1(u) &= \rho^{\natural} \theta_1 \phi_1(u) \\ &= \overline{\theta_1 \phi_1(u)} \quad \text{by definition of } R \\ &= \overline{\theta_2 \phi_2(u)} \\ &= \rho^{\natural} \theta_2 \phi_2(u) \\ &= \mu_2 \phi_2(u). \end{aligned}$$

Now let  $T$  be another semigroup such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \lambda_1 \\ S_2 & \xrightarrow{\lambda_2} & T \end{array} \quad (2.2)$$

Then since  $F$  is the coproduct of  $S_1$  and  $S_2$ , there exists a unique morphism  $\lambda : F \rightarrow T$ , such that in the diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\phi_1} & S_1 \\
 \phi_2 \downarrow & & \downarrow \theta_1 \\
 S_2 & \xrightarrow{\theta_2} & F \\
 & \searrow \lambda_2 & \searrow \lambda \\
 & & T
 \end{array}$$

we have  $\lambda \circ \theta_1 = \lambda$  and  $\lambda \circ \theta_2 = \lambda_2$ . Now because the above diagram commutes, we have

$$\begin{aligned}
 \lambda \theta_1 \phi_1(u) &= \lambda_1 \phi_1(u) \\
 &= \lambda_2 \phi_2(u) \quad \text{from commutativity} \\
 &= \lambda \theta_2 \phi_2(u).
 \end{aligned}$$

Hence  $(\theta_1 \phi_1(u), \theta_2 \phi_2(u)) \in \ker \lambda$ . Recalling that

$$R = \{(\theta_1 \phi_2(u), \theta_2 \phi_2(u)) : u \in U\}$$

we can assert that  $R \subseteq \ker \lambda$ . Since  $\ker \lambda$  is a congruence, we have

$$\rho = R^\# \subseteq \ker \lambda.$$

Define now

$$\beta : P \rightarrow T$$

by

$$\beta(\bar{w}) = \lambda(w) \quad w \in F.$$

The morphism  $\beta$  is well defined, as for every  $w_1, w_2 \in F$  we have

$$\begin{aligned}
 \bar{w}_1 = \bar{w}_2 &\Rightarrow (w_1, w_2) \in \rho \\
 &\Rightarrow (w_1, w_2) \in \ker \lambda \\
 &\Rightarrow \lambda(\bar{w}_1) = \lambda(\bar{w}_2).
 \end{aligned}$$

It can also be seen that  $\beta \circ \rho^\natural = \lambda$ , so we have

$$\beta \mu_i = \beta \rho^\natural \theta_i = \lambda \theta_i = \lambda_i \quad \forall i \in \{1, 2\}.$$

Hence the diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\phi_1} & S_1 \\
 \phi_2 \downarrow & & \downarrow \mu_1 \\
 S_2 & \xrightarrow{\mu_2} & P \\
 & \searrow \lambda_2 & \downarrow \beta \\
 & & T
 \end{array}$$

commutes as required. To see that  $\beta$  is unique let  $\beta'$  be another morphism satisfying the same conditions as  $\beta$ . Then observe that for every  $a_1 \in S_1$

$$\begin{aligned}
 \beta\mu_1(a_1) &= \lambda_1(a_1) \\
 &= \beta'\mu_1(a_1).
 \end{aligned}$$

That is for every  $\bar{a}_1 \in P$ ,  $a_1 \in S_1$ , we have  $\beta(\bar{a}_1) = \beta'(\bar{a}_1)$ . Similarly for all  $\bar{a}_2 \in P$ ,  $a_2 \in S_2$ , we have  $\beta(\bar{a}_2) = \beta'(\bar{a}_2)$ . Now for any  $w = \overline{a_1 a_2 \dots a_n} \in P$

$$\begin{aligned}
 \beta(w) &= \beta(\overline{a_1 a_2 \dots a_n}) = \beta(\bar{a}_1)\beta(\bar{a}_2) \dots \beta(\bar{a}_n) \\
 &= \beta'(\bar{a}_1)\beta'(\bar{a}_2) \dots \beta'(\bar{a}_n) \\
 &= \beta'(\overline{a_1 a_2 \dots a_n}) \\
 &= \beta'(w).
 \end{aligned}$$

So  $\beta = \beta'$  and hence  $P = S_1 *_U S_2$  is the required pushout.

### 3. Semigroup amalgams

In this chapter we consider semigroup amalgams and their embedding. We first relate the notion of pushouts, considered in the previous section, to the embedding of semigroup amalgams. Afterwards we give some results concerning the non-embeddability of the said amalgamas.

**Definition 3.1.** A semigroup amalgam is a list  $[U; S_1, S_2; \phi_1, \phi_2]$  where  $U$ ,  $S_1$  and  $S_2$  are semigroups and  $\phi_i: U \rightarrow S_i$ ,  $i \in \{1, 2\}$  are monomorphisms.

Instead of using a two elements indexing set to define semigroup amalgams one may also use bigger indexing sets (see for example [4]). For numerous practical reasons it however suffices to consider the above definition (see [3, Theorem 1.4]).

**Definition 3.2.** A semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  is said to be *embedded* into a semigroup  $T$  if there exist monomorphisms  $\psi_1, \psi_2$ , such that

(1) the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \psi_1 \\ S_2 & \xrightarrow{\psi_2} & T \end{array}$$

commutes and

(2) for any  $s_1 \in S_1$  and  $s_2 \in S_2$

$$\psi_1(s_1) = \psi_2(s_2) \implies s_1 = \phi_1(u), s_2 = \phi_2(u) \text{ for some } u \in U.$$

We say that  $[U; S_1, S_2; \phi_1, \phi_2]$  is *weakly embedded* if only condition (1) is fulfilled.

In 1927, Otto Schreier [6] showed that a group amalgam  $[V; G_1, G_2; \phi_1, \phi_2]$  (in which  $V$ ,  $G_1$  and  $G_2$  are groups) is always embeddable in a group. Formally, we say that the class of groups has the amalgamation property. A

counterexample showing that semigroups do not have this property was given by N. Kimura in 1957 in his doctoral thesis [5]. We reproduce this example from [1].

**Example 3.1.** Let  $U = \{u, v, w, z\}$  be a null semigroup in which all products are equal to  $z$ . Let  $S_1 = U \cup \{a\}$  where  $a \notin S_1$ ,  $au = ua = v$ , and all the other products in  $S_1$  are set equal to  $z$ . Let  $S_2 = U \cup \{b\}$  where  $b \notin S_1$ ,  $bv = vb = w$ , and all other products in  $S_2$  are set equal to  $z$ . Then a straightforward verification shows that both  $S_1$  and  $S_2$  are semigroups. Now if the semigroup amalgam  $[U; S_1, S_2; i_1, i_2]$  (where  $i_1$  and  $i_2$  are the embedding morphisms) is embeddable in some semigroup  $T$  via monomorphisms  $j_1$  and  $j_2$ , then one can calculate in  $T$

$$\begin{aligned}
j_2 i_2(w) &= j_2 i_2(bv) \\
&= j_2 i_2(b) j_2 i_2(v) \\
&= j_2 i_2(b) j_1 i_1(v) \\
&= j_2 i_2(b) j_1 i_1(ua) \\
&= j_2 i_2(b) j_1 i_1(u) j_1 i_1(a) \\
&= j_2 i_2(b) j_2 i_2(u) j_1 i_1(a) \\
&= j_2 i_2(bu) j_1 i_1(a) \\
&= j_2 i_2(z) j_1 i_1(a) \\
&= j_1 i_1(z) j_1 i_1(a) \\
&= j_1 i_1(za) \\
&= j_1 i_1(z) \\
&= j_2 i_2(z).
\end{aligned}$$

This implies that (even weak) embedding of  $[U; S_1, S_2, i_1, i_2]$  is not possible as  $w$  and  $z$  coincide in  $T$ .

In 1962, J. M. Howie proved (Theorem 2.4 of [2]) that a semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$ , in which  $S_1$  and  $S_2$  are groups, is embeddable if and only if  $U$  is also a group. This result is particularly interesting, as it shows that the class of unembeddable semigroup amalgams is infinite.

The following result, connecting pushouts and amalgamation, narrows down the question of embeddability for semigroup amalgams.

**Theorem 3.1** (cf. [4, page 264–266]). *A semigroup amalgam is embeddable if and only if it is naturally embedded in its pushout.*

*Proof.* Sufficiency is trivial. So let  $[U; S_1, S_2; \phi_1, \phi_2]$  be embedded in some semigroup  $T$ . This means that there exist monomorphisms  $\psi_1$  and  $\psi_2$  such

that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \psi_1 \\ S_2 & \xrightarrow{\psi_2} & T \end{array} .$$

No by Definition 2.2 there exists a (unique) morphism  $\lambda : S_1 *_{U} S_2 \rightarrow T$  such that

$$\begin{array}{ccccc} U & \xrightarrow{\phi_1} & S_1 & & \\ \phi_2 \downarrow & & \downarrow \mu_1 & \searrow \psi_1 & \\ S_2 & \xrightarrow{\mu_2} & S_1 *_{U} S_2 & \xrightarrow{\lambda} & T \\ & \searrow \psi_2 & & & \end{array}$$

is a commutative diagram. It is clear that  $\mu_1$  and  $\mu_2$  are monomorphisms, as  $\psi_1$  and  $\psi_2$  are such. Let  $\mu_1(s_1) = \mu_2(s_2)$  for some  $s_1 \in S_1$  and  $s_2 \in S_2$ . Then

$$\psi_1(s_1) = \lambda\mu_1(s_1) = \lambda\mu_2(s_2) = \psi_2(s_2).$$

Since  $[U; S_1, S_2; \phi_1, \phi_2]$  is embedded in  $T$  via  $\psi_1$  and  $\psi_2$  there exists  $u \in U$  such that  $s_1 = \phi_1(u)$  and  $s_2 = \phi_2(u)$ , which means that  $[U; S_1, S_2; \phi_1, \phi_2]$  is embedded in its pushout.  $\square$

Theorem 3.1 of [3] tells that a commutative semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  is embeddable if  $U$  is a regular semigroup. Our following theorem provides a sort of more general statement.

**Theorem 3.2.** *A semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  where  $S_1$  and  $S_2$  are commutative regular semigroups is embeddable if and only if  $U$  is a regular semigroup.*

*Proof.* Sufficiency follows from Theorem 3.1 of [3]. To prove the necessity, let  $S_1$  and  $S_2$  be commutative regular semigroups and let  $U$  be a semigroup that is not regular. Assume that the semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  is embeddable in some semigroup  $T$  via monomorphisms, say,  $\mu_1 : S_1 \rightarrow T$  and  $\mu_2 : S_2 \rightarrow T$  such that

- (1)  $\mu_1\phi_1 = \mu_2\phi_2$ ;
- (2) for any  $s_1 \in S_1$  and  $s_2 \in S_2$ ,

$$\mu_1(s_1) = \mu_2(s_2) \implies \phi_1(u) = s_1 \text{ and } \phi_2(u) = s_2 \text{ for some } u \in U.$$

Since  $U$  is not regular there exists  $u \in U$  for which there is no inverse in  $U$ . But since  $S_1$  and  $S_2$  are inverse semigroups there are inverses  $s_1^{-1}$  and  $s_2^{-1}$  for  $\phi_1(u) =: s_1$  and  $\phi_2(u) =: s_2$  in  $S_1$  and  $S_2$  respectively such that

$$\begin{aligned} s_1 s_1^{-1} s_1 &= s_1 & \text{and} & & s_1^{-1} s_1 s_1^{-1} &= s_1^{-1} & \text{in } S_1, \\ s_2 s_2^{-1} s_2 &= s_2 & \text{and} & & s_2^{-1} s_2 s_2^{-1} &= s_2^{-1} & \text{in } S_2. \end{aligned}$$

Note that these inverses have no preimages in  $U$ . Now from  $\mu_2(s_2) = \mu_2\phi_2(u) = \mu_1\phi_1(u) = \mu_1(s_1)$  we can calculate

$$\begin{aligned} \mu_2(s_2^{-1}) &= \mu_2(s_2^{-1} s_2 s_2^{-1}) \\ &= \mu_2(s_2^{-1}) \mu_2(s_2) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2^{-1}) \mu_1(s_1) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2^{-1}) \mu_1(s_1 s_1^{-1} s_1) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2^{-1}) \mu_1(s_1) \mu_1(s_1^{-1} s_1) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2^{-1}) \mu_2(s_2) \mu_1(s_1 s_1^{-1}) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2^{-1} s_2) \mu_1(s_1) \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2 s_2^{-1}) \mu_2(s_2) \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2 s_2^{-1} s_2) \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \\ &= \mu_2(s_2) \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \\ &= \mu_1(s_1) \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \\ &= \mu_1(s_1 s_1^{-1}) \mu_2(s_2^{-1}) \\ &= \mu_1(s_1^{-1} s_1) \mu_2(s_2^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_1(s_1) \mu_2(s_2^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2) \mu_2(s_2^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2 s_2^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2^{-1} s_2) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \mu_2(s_2) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \mu_1(s_1) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \mu_1(s_1 s_1^{-1} s_1) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \mu_1(s_1) \mu_1(s_1 s_1^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2^{-1}) \mu_2(s_2) \mu_1(s_1) \mu_1(s_1^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2^{-1} s_2) \mu_2(s_2) \mu_1(s_1^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2 s_2^{-1} s_2) \mu_1(s_1^{-1}) \\ &= \mu_1(s_1^{-1}) \mu_2(s_2) \mu_1(s_1^{-1}) \end{aligned}$$

$$\begin{aligned}
&= \mu_1(s_1^{-1})\mu_1(s_1)\mu_1(s_1^{-1}) \\
&= \mu_1(s_1^{-1}s_1s_1^{-1}) \\
&= \mu_1(s_1^{-1}).
\end{aligned}$$

Thus  $s_1^{-1}$  and  $s_2^{-1}$  ‘coincide’ in  $T$ , a contradiction. This means that the semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$ , in which  $S_1$  and  $S_2$  are commutative regular semigroups, is not embeddable if  $U$  is not a regular semigroup.  $\square$

We next give two results about the non-embeddability of semigroup amalgams. The following theorem generalizes Theorem 2.4 of [2] to the union of groups.

**Theorem 3.3.** *A semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  where  $S_1$  and  $S_2$  are completely regular semigroups is not embeddable if  $U$  is not completely regular.*

*Proof.* Let  $S_1$  and  $S_2$  be completely regular semigroups. Let  $U$  be a semigroup that is not completely regular. This means that  $U$  cannot be regular. Now assume that the semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  is embeddable in some semigroup  $T$ . Then there exist monomorphisms  $\mu_i : S_i \rightarrow T$ , for  $i \in \{1, 2\}$ . Let  $u \in U$  be a non regular element. Let us denote  $s_1 = \phi_1(u)$  and  $s_2 = \phi_2(u)$ . As  $S_1$  and  $S_2$  are completely regular semigroups, there exist inverses  $s_1^{-1} \in S_1$  and  $s_2^{-1} \in S_2$  of the elements  $s_1$  and  $s_2$  respectively. This means that

$$\begin{aligned}
s_1s_1^{-1}s_1 &= s_1 \text{ and } s_1^{-1}s_1s_1^{-1} = s_1^{-1}, \\
s_2s_2^{-1}s_2 &= s_2 \text{ and } s_2^{-1}s_2s_2^{-1} = s_2^{-1}.
\end{aligned}$$

Note that these inverses have no preimages in  $U$ . Also note that

$$\mu_1(s_1) = \mu_1\phi_1(u) = \mu_2\phi_2(u) = \mu_1(s_2).$$

Since  $S_i$  is a completely regular semigroup, we also have

$$s_i s_i^{-1} = s_i^{-1} s_i.$$

Now we can calculate

$$\begin{aligned}
\mu_1(s_1^{-1}) &= \mu_1(s_1^{-1}s_1s_1^{-1}) \\
&= \mu_1(s_1^{-1})\mu_1(s_1)\mu_1(s_1^{-1}) \\
&= \mu_1(s_1^{-1})\mu_2(s_2)\mu_1(s_1^{-1}) \\
&= \mu_1(s_1^{-1})\mu_2(s_2s_2^{-1}s_2s_2^{-1}s_2)\mu_1(s_1^{-1}) \\
&= \mu_1(s_1^{-1})\mu_2(s_2)\mu_2(s_2s_2^{-1}s_2^{-1})\mu_2(s_2)\mu_1(s_1^{-1}) \\
&= \mu_1(s_1^{-1})\mu_1(s_1)\mu_2(s_2)\mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1)\mu_1(s_1^{-1})
\end{aligned}$$

$$\begin{aligned}
&= \mu_1(s_1^{-1}s_1)\mu_1(s_1)\mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1s_1^{-1}) \\
&= \mu_1(s_1s_1^{-1}s_1)\mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1^{-1}s_1) \\
&= \mu_1(s_1)\mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1^{-1}s_1) \\
&= \mu_2(s_2)\mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1^{-1}s_1) \\
&= \mu_2(s_2s_2^{-1}s_2^{-1})\mu_1(s_1^{-1}s_1) \\
&= \mu_2(s_2^{-1}s_2^{-1}s_2)\mu_1(s_1^{-1}s_1) \\
&= \mu_2(s_2^{-1}s_2^{-1})\mu_2(s_2)\mu_1(s_1^{-1}s_1) \\
&= \mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1)\mu_1(s_1^{-1}s_1) \\
&= \mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1s_1^{-1}s_1) \\
&= \mu_2(s_2^{-1}s_2^{-1})\mu_1(s_1) \\
&= \mu_2(s_2^{-1}s_2^{-1})\mu_2(s_2) \\
&= \mu_2(s_2^{-1}s_2^{-1}s_2) \\
&= \mu_2(s_2^{-1}s_2s_2^{-1}) \\
&= \mu_2(s_2^{-1}).
\end{aligned}$$

This means that  $\mu_1(s_1^{-1}) = \mu_2(s_2^{-1})$ . This equality however implies that we can find an element  $v \in U$  such that  $\phi_1(v) = s_1^{-1}$  and  $\phi_2(v) = s_2^{-1}$ . This is a contradiction as we assumed that  $s_1^{-1}$  and  $s_2^{-1}$  do not have a preimages in  $U$ . This means that the semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  is not embeddable if  $U$  is not completely regular.  $\square$

The following corollary concerns the semilattices of groups (see Theorem 4.2.1 of [4]).

**Corollary 3.4.** *A semigroup amalgam  $[U; S_1, S_2; \phi_1, \phi_2]$  where  $S_1$  and  $S_2$  are Clifford semigroups is not embeddable if  $U$  is not a Clifford semigroup.*

*Proof.* Let  $S_1$  and  $S_2$  be Clifford semigroups. This means that they are completely regular and for all  $x, y \in S_i$   $i \in \{1, 2\}$

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

Suppose  $U$  is not a Clifford semigroup. Then  $U$  fails to be completely regular because if it does it must necessarily satisfy the above condition that makes it a Clifford semigroup. Now the corollary follows from the above theorem.  $\square$

# Kokkuvõte

## Poolrühma amalgaamidest

### Bakalaureusetöö

### Kristiina Rahkema

Antud lõputöö käsitleb poolrühma amalgaame. Poolrühma amalgaamide all mõistame listi  $[U; S_1, S_2; \phi_1, \phi_2]$ , kus  $U$ ,  $S_1$  ja  $S_2$  on poolrühmad ning  $\phi_i : U \rightarrow S_i$  ( $i \in \{1, 2\}$ ) on monomorfismid. Öeldakse, et poolrühma amalgaam  $[U; S_1, S_2; \phi_1, \phi_2]$  on sisestatav poolrühma  $W$ , kui esiteks leiduvad monomorfismid  $\mu_i : S_i \rightarrow W$  ( $i \in \{1, 2\}$ ) nii, et diagramm

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & S_1 \\ \phi_2 \downarrow & & \downarrow \mu_1 \\ S_2 & \xrightarrow{\mu_2} & W \end{array}$$

kommuteerub. Ja teiseks juhul, kui mingid kaks elementi poolrühmadest  $S_1$  ja  $S_2$  kujutatakse poolrühmas  $W$  samaks elemendiks, siis leidub neil ühine originaal poolrühmas  $U$ .

O. Schreier püstitas esimesena küsimuse, kas rühma amalgaamid on sisestatavad rühma. Aastal 1927 tõestas ta, et see tõesti nii on. Seega oli loomulik küsida, kas sellist omadust oleks võimalik edasi kanda ka teistele klassidele. Aastal 1957 konstrueeris N. Kimura oma doktoritöös kontranaite, mis tõestas, et kõik poolrühma amalgaamid ei ole sisestatavad poolrühma. J. M. Howie tõestas aastal 1962, et poolrühma amalgaam  $[U; S_1, S_2]$ , kus  $S_1$  ja  $S_2$  on rühmad, on sisestatav siis ja ainult siis kui  $U$  on samuti rühm. See tulemus oli eriti huvitav, kuna tänu sellele tekkis lõpmatu hulk poolrühma amalgaame, mis ei ole sisestatavad. T. E. Hall tõestas aastal 1975, et Schreieri tulemust on võimalik üldistada ning et analoogne tulemus kehtib ka inversete poolrühmade korral.

Antud lõputöö eesmärgiks oli lähemalt tutvuda poolrühma amalgaamidega ning tõestada mõned tulemused poolrühma amalgaamide sisestusest. Viimases peatükis on tõestatud, et poolrühma amalgaam  $[U; S_1, S_2]$ , kus  $S_1$  ja  $S_2$  on kommutatiivsed regulaarsed poolrühmad, on sisestatav parajasti siis, kui  $U$  on regulaarne. See tulemus üldistab teatud määral varem Howie poolt tõestatud tulemust, et kommutatiivne regulaarne amalgaam, kus ka  $U$  on regulaarne, on sisestatav. Samuti on ka tõestatud, et poolrühma amalgaam, kus  $S_1$  ja  $S_2$  on täielikult regulaarsed ei ole sisestatav, kui  $U$  ei ole täielikult regulaarne. Sama tulemus tõestatakse ka Cliffordi poolrühmade jaoks.

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**Lihtlitsents lõputöö reprodutseerimiseks ja lõputöö üldsusele kättesaadavaks tegemiseks**

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