DISSERTATIONES PHYSICAE UNIVERSITATIS TARTUENSIS

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On the second order relativistic deviation equation and its applications

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Contents

List of publications

This Thesis is based on the following papers, which will be referred to in the text by their Roman numerals.

I T.Mullari, R.Tammelo, *On the relativistic tidal effects in the second approximation*, Classical and Quantum Gravity, **23**, p. 4047-4067 (2006)

II R.Tammelo, T.Mullari, *On the pressure of gravitational waves* , General Relativity and Gravitation, **38** (1), p. 1-13 (2006)

III T.Mullari, R.Tammelo, *Some applications of relativistic deviation equations.* Hadronic Journal, **22**(4), p. 373 - 389 (1999)

IV T.Mullari, R.Tammelo, *An alternative way to derive the geodesic deviation equation for rapidly diverging geodesics.* Acta et Commentationes Universitatis Tartuensis de Mathematica, **4**, p. 1 - 10 (2000)

Other publications by the applicant:

V Ü.Kotta, T.Mullari, *Equivalence of different realization methods for higher order nonlinear input-output differential equations.* European Journal of Control, 11(3), p. 185 - 193 (2005).

VI T.Mullari, Ü.Kotta, *Simplification of the generalized state equations.* Kybernetika, 42(5), p. 617 - 628 (2006)

VII T.Mullari, Ü.Kotta, S.Nõmm, M.Tõnso, *Realization of nonlinear composite systems.* Control and Cybernetics, 35(4), p. 905 - 922 (2006).

VIII Ü.Kotta, T.Mullari, *Equivalence of realizability conditions for nonlinear control systems.* Proceedings of the Estonian Academy of Sciences, Phys. Math, 55(1), p. 24 - 42 (2006).

IX Ü.Kotta, T.Mullari, R.Pearson, *Design of observers for a class of nonlinear systems in associative observer form.* AT&P journal PLUS, 2, p. 28 - 43 (2007).

X Ü.Kotta, T.Mullari, A.S.I.Zinober, *State space realization of bilinear continuous-time input-output equations.* International Journal of Control, 80(10), p. 1607 - 1615 (2007).

XI T.Mullari, Ü.Kotta, *Transformation the nonlinear system into the observer form: simplification and extension.* European Journal of Control, 15(2), p. 177 - 183 (2009).

XII T.Mullari, Ü.Kotta, P.Kotta, M.Tõnso, A.S.I.Zinober, *Removing the input derivatives in the generalized bilinear state equations.* Proceedings of the Estonian Academy of Sciences, 58(2), p. 98 - 107 (2009).

XIII Ü.Kotta, T.Mullari, *Realization of nonlinear systems described by input/output differential equations: equivalence of different methods.* In: ECC'03 : European Control Conference : 1-4 September 2003, Cambridge, UK: Stevenage, UK.: IEE, p. 1194 - 1197 (2003).

XIV Ü.Kotta, T.Mullari, P.Kotta, A.S.I.Zinober, *On classical state space realizability of bilinear input-output differential equations.* In: 14th Mediterranean Conference on Control and Automation : Ancona, Italy - June, 28-30 2006: [S.l.]: IEEE; Mediterranean Control Association, p. 6 (2006).

XV Ü.Kotta, T.Mullari, *Equivalence of different realizability conditions for nonlinear MIMO differential equations.* In: Proceedings of the 16th IFAC World Congress : Prague, Czech Republic, July 3-8, 2005: (Toim.) Piztek, Pavel. [Oxford]: Elsevier, 2006, (Elsevier-IFAC publications), p. 133 - 138 (2006).

XVI T.Mullari, Ü. Kotta, M.Tõnso, *The connection between different static state feedback linearizability conditions of discrete time nonlinear control systems.* In: ECC '07 European Control Conference : July 2-5, 2007, Kos, Greece, Proceedings: Kos, Greece: EUCA, p. 4268 - 4275 (2007).

XVII J.Belikov, Ü.Kotta, T.Mullari, S.Nõmm, M.Tõnso, *Discretization of continuous-time nonlinear control systems with computer algebra system Mathematica.* In: Kybernetika a Informatika : Medzina'rodna' konferencia SSKI, 10.-14.februa'r 2008, ẐDIAR, Slovenska' republika, conference preprints: Bratislava: Slovenska' technicka' univerzita v Bratislave, p. 10 (2008).

XVIII V.Kaparin, Ü.Kotta, T.Mullari, M.Tõnso, *Transformation of nonlinear control system into observer form with computer algebra system Mathematica.* In: Kybernetika a Informatika : Medzina'rodna' konferencia SSKI, 10.-14.februa'r 2008, ẐDIAR, Slovenska' republika, conference preprints: Bratislava: Slovenska' technicka' univerzita v Bratislave, p. 10 (2008).

XIX T.Mullari, Ü. Kotta, *Transformation of nonlinear control systems into the observer form: necessary conditions.* In: Proceedings of the 16th Mediterranean Conference on Control and Automation, June 25-27, 2008, Ajaccio, Corscia, France: IEEE, p. 475 - 480 (2008).

Author's contribution

Publication I. The author is responsible for the derivation of the second order relativistic deviation equation of rapidly diverging world lines of accelerated point masses in case of their arbitrary parametrization, and the formula relating the proper times of the clocks moving along these world lines. He has derived the exact formulae to calculate the components of the metric tensor in the Fermi coordinates of an accelerated observer and calculated the acceleration of free fall in the second approximation in frame of reference moving at the relativistic 3-velocity relative to the source of gravitational field, with the special attention to the radially accelerated motion in the Schwarzschild spacetime.

Publication II. The author is responsible for the derivation of an alternative formula to calculate the pressure of a plane monochromatic weak gravitational wave on the Weber oscillator using the conservation laws, and for derivation of a formula for pressure of a gravitational wave in case of a

Weber oscillator moving at the relativistic 3-velocity relative to the source of the gravitational wave.

Publication II. The author is responsible for the calculating the relativistic effects in the motion of two freely falling point masses whose relative velocity is close to the velocity of the light. In the case of the Schwarzschild spacetime while one point mass falls radially, the relative motion of the point masses can decrease the tidal forces between the point masses or force them to change the sign. In the case of the field of the plane gravitational wave the relative motion of point masses can cause also the longitudinal tidal forces between them in the first approximation.

Publication IV. The author is responsible for the finding a method to derive the first order geodesic deviation equation for the rapidly diverging geodesics, using the third order approximation of their separation vector.

1 Introduction

1.1 General background

The geodesic deviation equation, derived by Levi-Civita [1] and Synge [2] in 1926 was first used by a variety of geometric problems, see e.g. [3]-[8]. In the papers [9] Pirani and [10] Weber begin the use of the geodesic deviation equation researching the relative motion of the test particles both in an arbitrary Riemann space, see e.g. [11]-[17] and in the certain background spaces, see e.g. [18]-[20]. In the early seventies of the 20th century started the study of the geodesic deviation equation in the second and higher approximations with respect to the deviation vector. Plebanski [21] and Bazanski [22]-[26] examined the second order geodesic deviation equation. Considering the relative motion, Hodgkinson [27] derived the deviation equation containing also the terms of the higher order of smallness. Using the exponential map described in [28], [29]. Pyragas and Aleksandrov derived the exact non-local deviation equation, see [30], [32]. That was a grand step forward, because this more general equation found a large application in the general relativity, for example in papers [33]-[35], and also by researcing of the nonlinear oscillation of the test particles, see [36]-[38]. The independent approaches to the higher order are given by Li, Zimmermann and Ni, see [39]-[41] and by DeFacio and Retzloff, see [42]-[45].

One of the most significant questions of the contemporary gravitational theory is the problem of the gravitational waves, which has been widely researched both theoretically and experimentally. If Einstein in [46] using the linearized relativistic equations of the gravitational field predicted the existence of the gravitational waves, hr concluded, that the energy transfered by the gravitational waves must be very small and their experimental detection must be impossible. Consequently, during the following decades the gravitational waves have been researched only theoretically from the point of view of their quantization, of the velocity of propagation and the conservation laws for the physical processes where the gravitational waves are present. First Pirani [9] in 1956, using the geodesic deviation equation derived by Levi-Civita [1] and Synge [2] proposed to measure the relative acceleration of the test particles in order to determine the curvature of the spacetime, including the field of the gravitational waves. In 1958 Weber recommended to use the phenomenon of the mechanical resonance for the amplification the oscillations of a detector and concluded, that the gravitational radiation can be experimentally discovered, see [10]. Already in 1969 Weber reported the experimental detection of the gravitational waves [47], but the more precise measurements did not confirm his results and the astrophysical theory could not find the sources of the gravitational waves, powerful enough to confirm the Weber's measurements. This means, the first generation of the Weber detectors could not confirm the existence of the gravitational waves. The discovery of the binary pulsar PSR 1913+16 and its observations [48] proved the necessity to research the gravitational waves and at the same time started an epoch of the gravitational astronomy. One could explain the disturbances in the pulsar's oscillating period as caused by the gravitational waves, because they were in accordance with the quadrupol formula describing the emission of energy by a binary system in the form of the gravitational waves. The quadrupol formula has been discussed in [49]. At the same time the first detectors belonging to the second generation came into the use, see e.g. [50]. The other experiments using the various systems of test particles ar the extended test bodies like the Weber detector are described in [51], [52], [53], [54] and [55]. So the theory of the gravitational waves changed from a pure mathematical discipline more and more into the experimental one.

The geodesic deviation equation derived by Levi-Civita [1] and Synge [2] allows us to calculate the relative acceleration of two freely falling point masses moving along *almost parallel* geodesic world lines and thereby estimate the magnitude of the tidal forces in an extended body. This equation is linear with respect to the components of the deviation vector (the position vector of one point mass relative to the other), i.e. it allows us to calculate only the linear terms in the Taylor expansion of the relative acceleration of freely falling point masses.

A generalization of the ordinary geodesic deviation equation to the case of non-parallel world lines, i.e. to arbitrary relative velocities, was derived half a century later by Hodgkinson [27], Mashhoon [56] and Ciufolini [57]. It was named the *relativistic geodesic deviation equation* by Mashhoon [56]. In Paper IV [58] has given an alternative, more transparent and shorter than the earlier ones. Like the non-relativistic geodesic deviation equation [1], [2], its relativistic counterpart [27], [56], [57], [58] takes into account only the *linear* terms with respect to the deviation vector in the Taylor expansion.

Its applications have been considered, for example, in following papers. The relative acceleration of two point masses moving along two orthogonal great circles on the sphere is examined by Giufolini [57]. Paper III [59] describes the 3-acceleration of the test particles in the frame of reference of an observer falling freely in the radial direction in the Schwarzschild spacetime, and of a stationary observer in the field of a linearly polarized, weak and monochromatic gravitational wave. The frame of reference of a radially moving observer in the Schwarzschild spacetime has also been considered by Chicone and Mashhoon, who examined the accelerations of free fall of a swarm of test particles moving at relativistic velocities relative to the observer, see [60] and [61], and generalize the Jacobi equation, which describes the behaviour of a swarm of particles with the nearby parallel world lines, also for the case of relativistic relative motion of the particles, see [62] . The papers [59, 60] consider only the case of an observer moving at a much lower velocity than the velocity of light; therefore they do not reveal the dependence of the behaviour of the test particles on the observer's velocity.

This leads us to another aspect of the relative motion concerning the phenomena of the observer's motion. When studying the 3-acceleration of test particles and tidal forces in the frame of reference of an observer, we must pay special attention to the cases where the observer moves at a relativistic velocity with respect to the sources of the gravitational field; or these sources are also in motion (for example a rotating black hole). Then the *relativistic tidal forces* will appear, (see e.g. Ref. [54]). In the case of the observer's relativistic motion relative to the stationary sources, the dependence of tidal forces on the observer's velocity appears. This is the case, if we apply, for example, the nonrelativistic deviation equation in the frame of reference of an observer moving in the Schwarzschild spacetime at a nonzero component of his velocity, perpendicular to the radial direction. If the velocity approaches the velocity of light, it leads to the unlimited growth of the tidal forces and the breakdown of the theory.

Let us point out that in certain situations the linear deviation equation is inadequate to describe the dependence of tidal forces on the observer's velocity. Therefore it is necessary to take into account also some higher order terms, because the dependence of tidal forces on the observer's velocity may be "hidden" in the higher order small terms. These terms will become significant if the observer moves at a relativistic speed relative to the source of gravitational field. If the observer's velocity becomes comparable to the velocity of light, the higher order terms begin to dominate over the linear ones. Consequently, the deviation equation as the Taylor expansion does not converge and is not applicable in the case of the observer's hyperrelativistic velocities.

This leads us to the necessity to derive an exact deviation equation of rapidly diverging world lines in the general case (non-gravitationally accelerated observer and particle, arbitrary parametrization of the world lines and correspondence between them) and study its second approximation to determine the dependence of the tidal forces and the relative acceleration of a test particle on the observer's velocity. Special attention has been paid to the observer moving in the Schwarzschild spacetime.

In the current Thesis also some applications of the second order approximations of the deviation equation have been examined.

First – there is considered the acceleration of free fall in the frame of reference of a radially accelerated observer in the Schwarzschild spacetime, moving at a relativistic 3-velocity relative to the black hole. The effects caused by the observer's own motion appear at first in the second order small terms and if the observer's 3-velocity approaches to the velocity of light, the second order terms become comparable with the first order ones and can not be ignored. In case of the observer's hyperrelativistic 3-velocity they may begin to dominate over the first order terms and consequently the deviation equation as the Taylor expansion will not converge. Therefore also the conditions for the convergence have been determined.

Second – the pressure of a plane monochromatic gravitational wave on an elastic body has been studied. The momentum obtained by the Weber oscillator from gravitational waves has been calculated using the second approximation of deviation equation. A special attention has been devoted to the case, when the Weber oscillator moves towards the source of the gravitational waves with the relativistic 3-velocity.

Third – the tidal forces and acceleration of free fall have been examined also in the higher approximations to study the effects caused by the observer's motion. To avoid very complicated and uncompendious formulae we have confined oneself to the case of the 2-dimensional spacetime.

1.2 Outline of the original papers

Papers I-II are devoted to the second approximation of the relativistic deviation equation in general case and in the Fermi coordinates of a moving observer. Paper III gives an alternative and more simple and transparent way to derive the first order geodesic deviation equation and Paper IV examines its applications in the Schwarzschild spacetime and in the field of the weak monochromatic plane gravitational wave.

In Paper I, using the concept of the world function, an exact deviation equation has been derived, which describes the relative motion of two accelerated point masses and is valid in the case of their arbitrary 4-velocities, 4-accelerations, parametrizations of their world lines, and arbitrary correspondence between the points of the world lines. The second approximation of the exact deviation equation, with respect to the components of deviation vector, has been elaborated in general coordinates and in the Fermi coordinates of an observer, comoving with one of the point masses. The 3-acceleration of a freely falling test particle in the frame of reference of an accelerated observer in the Schwarzschild spacetime has been calculated in the second approximation. It turns out that the quadratic terms involve the observer's 3-velocity relative to the black hole and become significant compared to the linear terms, if the observer moves at a relativistic 3-velocity. Moreover, in most cases, if the observer's 3-velocity approaches the velocity of light, the deviation equation as the Taylor expansion does not converge and is no more applicable. It has been established that if a freely falling observer starts its motion from spacelike infinity at a nonrelativistic initial velocity, the problem of convergence does not appear. A special attention has been paid to the motion in the Schwarzschild spacetime, if the observer's hyperrelativistic velocity relative to the source of gravitational field leads to enormous tidal forces, predictable already using the first approximation of deviation equation and able to destroy any extended body comoving with the observer.

In Paper II, a system of coupled charged point masses under the influence of gravitational and electromagnetic waves propagating in the same direction is considered. By means of the geodesic deviation equation as the equation of motion it is shown, taking into account the second order small terms, that there exist forces which cause the acceleration of the system in the longitudinal direction. The longitudinal force is due to the fact that simultaneously with energy the momentum is also absorbed from waves. It is proved directly on the basis of the equations of motion of the point masses that the energy and momentum absorbed by the test system obey the special relativistic relationship of a zero rest mass particle. The case when the Weber oscillator moves at a relativistic speed with respect to the source of gravitational waves is also examined. In this case, the absorption of energy and momentum by the Weber oscillator is considerably enhanced compared to the case when the distance between the detector and the source of waves is stationary.

Paper III considers the relativistic deviation of two geodesics, i.e. a system consisting of two test particles whose relative velocity is close to the velocity of light. Two particular cases have been addressed. First, the case of two free falling point masses in the gravitational field of a black hole, one of them moving in the radial direction. Has been established that at certain values of the relative velocity of the point masses the tidal forces between them can decrease, increase or even change a sign. Second, the case of freely falling point masses in the field of a weak plane monochromatic gravitational wave. Has been demonstrated, that under certain conditions the tidal forces between the point masses have the longitudinal components already in the first approximation with respect to the deviation vector, and the tidal forces can occur even if the deviation vector is longitudinal.

Paper IV introduces an alternative way to derive the first order relativistic geodesic deviation equation, describing the relative acceleration of two freely falling point masses in the curved spacetime. Using the general equation of a geodesic and its prolongations, the components of the deviation vector have been found in the third approximation with respect to the coordinates of the point masses. Calculating their second order covariant time derivatives gives the first order geodesic deviation equation under the assumption that the geodesics are adjacent in some neigbourhood, but rapidly diverging. The resulting modified equation is nonlinear, but it reduces to the ordinary linear geodesic deviation equation in the non-relativistic case, if the geodesics are almost parallel. This derivation is straightforward, but shorter and more transparent than the earlier ones. Some consequences of the modified geodesic deviation equation are also discussed.

2 On the relativistic tidal effects in the second approximation

2.1 Deviation of world lines in a Riemannian spacetime

Let us consider two point masses in a four-dimensional Riemannian spacetime and label their world lines and 4-velocities respectively by $L : u^i = u^i(t)$ and $\bar{L} : u^a = u^a(\bar{t})$, where $i, a = 0, ...3$, and $X := x^i \partial_i$ and $\bar{X} := x^a \partial_a$. Here u^i and u^a stand for the coordinates, t and $\bar{t}$ for the proper times of the first and second point mass, respectively. The index notation used here and further will be explained in Appendix A. In the tangent vector space T_u, where u is the location of the first point mass, we define a vector $H := \eta^i \dfrac{\partial}{\partial u^i}$ so that the exponential image of its endpoint is a point $\bar{u}$ on the second world line $\bar{L}$. This vector is called the *deviation vector.* Its components with respect to the coordinate basis at the point u are, in the first approximation, the differences of the corresponding curvilinear coordinates $\bar{u}^i$ and u^i of the two points $\bar{u}$ and u, i.e.,

$$\eta^i = \delta^i_a u^a - u^i + \mathcal{O}_2,$$

where the Landau symbol $\mathcal{O}_2 := \mathcal{O}\left(|\eta^i|^2\right)$ means the second order small quantities with respect to $|\eta^i|$. We denote the 4-acceleration vectors of the considered point masses by $K := \kappa^i \partial_i$ and $\bar{K} := \kappa^a \partial_a$. The first point mass will be regarded as an *observer* and the second one as a *test particle.*

If the observer and the test particle are falling freely, i.e. their world lines are geodesics, then in the case of almost parallel geodesics, when the rate of separation of geodesics $\dfrac{DH}{dt}$ is of the same order of smallness than the deviation vector H itself, we can use the ordinary geodesic deviation equation

$$\frac{D^2 \eta^i}{dt^2} = -R^i_{jkl} x^j \eta^k x^l. \tag{1}$$

Here x^i and R^i_{jkl} stand, respectively, for the components of the unit tangent vector X to the first (geodesic) world line and for the Riemann curvature tensor at the point u:

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{pk}\Gamma^p_{jl} - \Gamma^i_{pl}\Gamma^p_{jk}. \tag{2}$$

A generalised, i.e. relativistic deviation equation holds in the case of non-parallel geodesics whose rate of separation is arbitrary, i.e. the relative velocity of the test particle with respect to the observer is relativistic, and has the form

$$\frac{D^2 \eta^i}{dt^2} = -R^i_{jkl} x^j \eta^k x^l - 2R^i_{jkl} \frac{D\eta^j}{dt} \eta^k x^l - \frac{2}{3} R^i_{jkl} \frac{D\eta^j}{dt} \eta^k \frac{D^2 \eta^l}{dt}. \tag{3}$$

If the rate of separation of the geodesic world lines is small, this equation reduces, naturally, to the ordinary geodesic deviation equation (1). The equations (1) and (3) describe the behaviour of the deviation vector in the case of so-called natural correspondence, when the vector H connects a pair of corresponding points on the geodesic world lines L and $\bar{L}$ with the same value of the affine parameters t and $\bar{t}$. For the Fermi coordinates related to an accelerated observer the *deviation equation* is derived by Mashhoon, (see in Ref. [56]). In the case of a freely falling observer this reduces also to equation (3).

Formula (3) takes into account only the first order small quantities, but in general, the terms of higher order smallness may in certain conditions cause some important effects, see e.g. Ref. [63]. Therefore the aim of this Section is to derive an exact deviation equation, which will allow us to calculate the relative acceleration of test particle with respect to observer in arbitrary approximation. Later the exact equation is used to calculate the deviation equation in the second approximation. Some of its applications in special cases will be also considered.

An exact deviation equation was also derived by Alexandrov and Pyragas [32], but this equation describes only the relative motion of freely falling point masses in the case of natural correspondence, when the deviation vector connects the points on world lines of observer and test particle with the same values of proper times. The deviation equation, derived in the present Section, is valid for arbitrary parametrizations of world lines, for an arbitrary correspondence between them and for arbitrary, i.e., non-geodesic world lines.

To derive the exact deviation equation we define a spacelike *connecting geodesic* $\Lambda : u^i = u^i(s)$ between the observer and test particle, where s is the natural parameter along Λ. If $N = n^i \partial_i$ is the unit tangent vector to Λ at the observer's location and Δs is the geodesic arc length between the observer and test particle, then the deviation vector $H = \Delta s N$, determines the location of test particle: $\bar{u} = \exp(\Delta s N)\, u = \exp(H)\, u$.

We use the *world function* of Synge σ [64], determined by the locations of the observer and test particle, defined as

$$\sigma(u, \bar{u}) = \frac{1}{2} g(H, H) = \frac{\Delta s^2}{2}. \tag{4}$$

The components of the deviation vector can be calculated (see Ref. [64], page 59), as the derivatives of σ with respect to the observer's coordinates u^i with a minus sign:

$$\eta^i = -\sigma^i = -\sigma_j g^{ij}. \tag{5}$$

Here the lower and upper case indices at the symbol of world function label respectively its covariant and contravariant derivatives with respect to the corresponding coordinates. The meaning of indices from the first and second half of the Latin alphabet is explained in appendix A. The covariant derivative of the deviation vector with respect to the observer's proper time has the following components:

$$\frac{D\eta^i}{dt} = -\frac{D\sigma^i}{dt} = -\sigma^i_j x^j - \sigma^i_a x^a \frac{d\bar{t}}{dt}. \tag{6}$$

Contracting formula (6) with $\overset{-1}{\sigma}{}^a_i$ as the inverse of the matrix σ^i_a, the components of the 4-velocity of the test particle can be written as

$$x^a \frac{d\bar{t}}{dt} = \mathcal{K}^a_i x^i - \mathcal{H}^a_i \frac{D\sigma^i}{dt}, \tag{7}$$

where

$$\begin{aligned} \mathcal{H}^a_i &= -\overset{-1}{\sigma}{}^a_i, \\ \mathcal{K}^a_i &= -\overset{-1}{\sigma}{}^a_j \, \sigma^j_i \end{aligned} \tag{8}$$

are the *Jacobi propagators*, see Ref. [65].
Remark. In the Minkowski spacetime, where $\mathcal{H}^a_i = \mathcal{K}^a_i = \delta^a_i$, we obtain

$$x^a \frac{d\bar{t}}{dt} = \delta^a_i \left(x^i - \frac{D\sigma^i}{dt} \right).$$

This allows us to calculate the components of the 3-velocity of the test particle:

$$\begin{aligned} v^\alpha &= \frac{\delta^\alpha_\lambda x^\lambda}{x^{\bar{0}}} = \frac{\delta^\alpha_\lambda \left(x^\lambda - \dfrac{D\sigma^\lambda}{dt} \right)}{x^0 - \dfrac{D\sigma^0}{dt}} = \delta^\alpha_\lambda \frac{x^\lambda + \dfrac{D\eta^\lambda}{dt}}{x^0 + \dfrac{D\eta^0}{dt}}, \\ \lambda &= 1, 2, 3. \end{aligned} \tag{9}$$

Here $\bar{0}$ denotes the global time coordinate at the location of the test particle. In the case of an observer, moving on the $u^0 u^1$-plane, we have

$$x^i = \frac{(1, v, 0, 0)}{\sqrt{1 - |v|^2}}.$$

The components of the time derivative of the deviation vector in the observer's Fermi coordinates are according to Subsection 3

$$\frac{D\eta^{(i)}}{dt} = (-\vec{\kappa} \cdot \vec{\eta}, \beta^{(\lambda)}), \quad \lambda = 1, 2, 3,$$

where $\beta^{(\lambda)}$ are the components of the 3-velocity of the test particle in the observer's Fermi coordinates, and the arrow over a symbol denotes a 3-vector. Using the Lorentz transformations from the observer's Fermi coordinates into the so-called global coordinates, we obtain from formula (9) for the components of the 3-velocity of the test particle the two following formulae:

$$\begin{aligned}\bar{v}^1 &= \frac{v(1-\vec{\kappa}\cdot\vec{\eta})+\beta^{(1)}}{(1-\vec{\kappa}\cdot\vec{\eta})+\beta^{(1)}v},\\ \bar{v}^2 &= \frac{\beta^{(1)}\sqrt{1-|v|^2}}{(1-\vec{\kappa}\cdot\vec{\eta})+\beta^{(1)}v}.\end{aligned} \tag{10}$$

Consequently, in the case of Minkowski spacetime formula (7) reduces to the well known law of addition of velocities of the special theory of relativity.
Taking the second covariant derivative of σ^i with respect to t, we obtain

$$\begin{aligned}\frac{D^2\sigma^i}{dt^2} &= \sigma^i_{jk}x^j x^k + 2\sigma^i_{ja}x^j x^a\frac{d\bar{t}}{dt} + \sigma^i_j\kappa^j + \\ &+ \sigma^i_{ab}x^a x^b\left(\frac{d\bar{t}}{dt}\right)^2 + \sigma^i_a\kappa^a\left(\frac{d\bar{t}}{dt}\right)^2 + \sigma^i_a x^a\frac{d^2\bar{t}}{dt^2}.\end{aligned} \tag{11}$$

Substituting the 4-velocity of the test particle from formula (7) into the last formula, we obtain the exact deviation equation, valid for arbitrary 4-velocities and 4-accelerations of the observer and test particle and in the case of arbitrary correspondence between their world lines:

$$\begin{aligned}\frac{D^2\sigma^i}{dt^2} &= \sigma^i_{jk}x^j x^k + 2\sigma^i_{ja}\mathcal{K}^a_k x^j x^k - 2\sigma^i_{ja}\mathcal{H}^a_k x^j\frac{D\sigma^k}{dt} + \sigma^i_j\kappa^j\\ &+\sigma^i_{ab}\mathcal{K}^a_j\mathcal{K}^b_k x^j x^k - \sigma^i_{ab}\mathcal{K}^a_j\mathcal{H}^b_k x^j\frac{D\sigma^k}{dt} - \sigma^i_{ab}\mathcal{H}^a_j\mathcal{K}^b_k\frac{D\sigma^j}{dt}x^k\\ &+\sigma^i_{ab}\mathcal{H}^a_j\mathcal{H}^b_k\frac{D\sigma^j}{dt}\frac{D\sigma^k}{dt} + \sigma^i_a\kappa^a\left(\frac{d\bar{t}}{dt}\right)^2 + \sigma^i_a\mathcal{K}^a_j x^j\frac{dt}{d\bar{t}}\frac{d^2\bar{t}}{dt^2}\\ &-\sigma^i_a\mathcal{H}^a_j\frac{D\sigma^j}{dt}\frac{dt}{d\bar{t}}\frac{d^2\bar{t}}{dt^2}.\end{aligned} \tag{12}$$

Let us emphasize that the obtained formula (12) is valid also in the case, when t and $\bar{t}$ do not denote the proper times of observer and test particle, but are arbitrary parameters on the world lines. Here the components x^i and $\bar{x}^a$ are the first whereas κ^i and κ^a the second covariant derivatives of u^i and u^a with respect to t and $\bar{t}$.

Formula (7) allows us to calculate also the ratio $\dfrac{d\bar{t}}{dt}$ of the proper times of the test particle and the observer. Contracting formula (7) as an expression of a vector defined at the location of the test particle with the 1-form $g_{ab}x^b$, we obtain due to $\bar{g}(\bar{X},\bar{X})=1$ the following equation:

$$\left(\frac{d\bar{t}}{dt}\right)^2 = g_{ab}\mathcal{K}^a_i x^i\mathcal{K}^b_j x^j + g_{ab}\mathcal{H}^a_i\frac{D\sigma^i}{dt}\mathcal{H}^b_j\frac{D\sigma^j}{dt} - 2g_{ab}\mathcal{K}^a_i x^i\mathcal{H}^b_j\frac{D\sigma^i}{dt}. \tag{13}$$

Or, alternatively, we can shift the vector $X\dfrac{d\bar{t}}{dt}$ into the observer's location via the parallel displacement along the connecting geodecics and then calculate the square of its modulus:

$$\left(\frac{d\bar{t}}{dt}\right)^2 = g_{ij}\overset{-1}{P^i_a}(H)\mathcal{K}^a_k x^k\,\overset{-1}{P^j_b}(H)\mathcal{K}^b_l x^l + g_{ij}\overset{-1}{P^i_a}(H)\mathcal{H}^a_k\frac{D\sigma^k}{dt}\,\overset{-1}{P^j_b}(H)\mathcal{H}^b_l\frac{D\sigma^l}{dt}$$
$$-2g_{ij}\overset{-1}{P^i_a}(H)\mathcal{K}^a_k x^k\,\overset{-1}{P^j_b}(H)\mathcal{H}^b_l\frac{D\sigma^l}{dt}. \tag{14}$$

2.2 Relativistic deviation equation in the second approximation

The explicit form of the world function is known only for some metrics, e.g. the de Sitter universe [66], the Bianchi type I universe [67] and a special class of Robertson-Walker metrics [68]. If the explicit form of the world function is not known, one can nevertheless calculate the Taylor expansion of equation (12), containing the components of the deviation vector in an arbitrary approximation. Here we calculate the second approximation assuming, that the velocity of the test particle relative to the observer can be relativistic. In Appendix B we will perform some Taylor expansions useful for deriving the second order relativistic deviation equation from the exact equation (12).

The partial derivatives of the world function σ are intimately related to the tangent vectors of the connecting geodesic as it follows from the results of differentiation of the definition formula (4) (see Ref. [64]):

$$\begin{aligned} \sigma^i &= \frac{\partial \sigma}{\partial u^j} g^{ij} = -\eta^i, \\ \sigma^a &= \frac{\partial \sigma}{\partial u^b} g^{ab} = \eta^a = P^a_i(H)\eta^i. \end{aligned} \tag{15}$$

In the last formula $P^a_i(H)$ are the components of the operator of parallel displacement, which transports a vector, defined at the observer's location, parallel to itself into the location of the test particle. The components of its inverse are $\overset{-1}{P^i_a}(H)$.

Using the Taylor expansions (211) and (212), the components of operators (214) and (215) in the fourth approximation, and taking into account that the Christoffel symbols and their derivatives depend only on the observer's coordinates, one can find the following expansions of the second order derivatives of the world function in the third approximation:

$$\begin{aligned} \sigma^i_r &= -\frac{D\eta^i}{\partial u^r} = \delta^i_r - \frac{1}{3} R^i_{jrk}\eta^j\eta^k - \frac{1}{12} R^i_{jrk;l}\eta^j\eta^k\eta^l + \mathcal{O}_4, \\ \sigma^i_a &= -\frac{D\eta^i}{\partial \bar{u}^a} = -\overset{-1}{P^i_a}(H) - \frac{1}{6} R^i_{jqk}\eta^j\eta^k \overset{-1}{P^q_a}(H) - \frac{1}{12} R^i_{jpk;l}\eta^j\eta^k\eta^l\delta^p_a + \mathcal{O}_4, \\ \sigma^a_r &= \frac{D\bar{\eta}^a}{\partial u^r} = -P^a_r(H) - \frac{1}{6} P^a_q(H) R^q_{jrk}\eta^j\eta^k - \frac{1}{12}\delta^a_p R^p_{jrk;l}\eta^j\eta^k\eta^l + \mathcal{O}_4. \end{aligned} \tag{16}$$

Taking into account, that $\sigma^i_a \overset{-1}{\sigma^a_r} = \delta^i_r$ the expansion of the two-point tensor $\overset{-1}{\sigma^a_r}$ must have in the third approximation the following components:

$$\overset{-1}{\sigma^a_r} = -P^a_r(H) - \frac{1}{6} P^a_q(H) R^q_{jrk}\eta^j\eta^k + \frac{1}{12}\delta^a_p R^p_{jrk;l}\eta^j\eta^k\eta^l + \mathcal{O}_4 \,. \tag{17}$$

Covariant differentiation of the expansions (16) yields

$$
\begin{aligned}
\sigma^i_{rs} &= \frac{1}{3}\Big(R^i_{srj}+R^i_{jrs}\Big)\eta^j-\frac{1}{4}R^i_{jrk;s}\eta^j\eta^k+\frac{1}{12}\Big(R^i_{srj;k}+R^i_{jrs;k}\Big)\eta^j\eta^k+\mathcal{O}_3,\\
\sigma^i_{ar} &= -\frac{1}{3}\left(R^i_{jrq}+R^i_{qrj}\right)\eta^j\overset{-1}{P}{}^q_a(H)\\
&\quad -\frac{1}{12}\left(R^i_{jrk;q}+R^i_{qrj;k}+R^i_{jrq;k}\right)\eta^j\eta^k\delta^q_a+\mathcal{O}_3,\\
\sigma^i_{ab} &= -\frac{1}{2}R^i_{qrj}\overset{-1}{P}{}^q_a(H)\overset{-1}{P}{}^r_b(H)\eta^j-\frac{1}{3}R^i_{pbj;k}\delta^p_a\eta^j\eta^k\\
&\quad -\frac{1}{6}\left(R^i_{jqr}+R^i_{rqj}\right)\overset{-1}{P}{}^q_a(H)\overset{-1}{P}{}^r_b(H)\eta^j\\
&\quad -\frac{1}{12}\left(R^i_{rpj;k}+R^i_{jpr;k}+R^i_{jpk;r}\right)\delta^p_a\delta^r_b\eta^j\eta^k+\mathcal{O}_3.
\end{aligned}
\tag{18}
$$

We also used here the following components of the covariant derivatives of the operators $P(H)$ and $\overset{-1}{P}(H)$:

$$
\begin{aligned}
\overset{-1}{P}{}^i_{a;b}(H) &= \frac{1}{2}R^i_{qrj}\overset{-1}{P}{}^q_a(H)\overset{-1}{P}{}^r_b(H)\eta^j+\frac{1}{3}R^i_{prj;k}\delta^p_a\delta^r_b\eta^j\eta^k+\mathcal{O}_3,\\
P^a_{j;k}(H) &= \frac{1}{3}P^a_q(H)R^q_{jlk}\eta^l+\frac{1}{6}\delta^a_r R^r_{jlk;p}\eta^l\eta^p+\mathcal{O}_3.
\end{aligned}
\tag{19}
$$

Taking into account formulae (16) and (17), the Jacobi propagators (8) can be written as

$$
\begin{aligned}
\mathcal{H}^a_i &= P^a_i(H)-\frac{1}{6}P^a_q(H)R^q_{jik}\eta^j\eta^k-\frac{1}{12}\delta^a_p R^p_{jik;l}\eta^j\eta^k\eta^l+\mathcal{O}_4,\\
\mathcal{K}^a_i &= P^a_i(H)-\frac{1}{2}P^a_q(H)R^q_{jik}\eta^j\eta^k-\frac{1}{6}\delta^a_p R^p_{jik;l}\eta^j\eta^k\eta^l+\mathcal{O}_4.
\end{aligned}
\tag{20}
$$

Substituting the obtained components of the Jacobi propagators into formula (7), shifting the vector $\bar{X}\dfrac{d\bar{t}}{dt}$ by means of the operator $\overset{-1}{P}(H)$ into the observer's location and calculating the square of its modulus, we have

$$
\begin{aligned}
\left(\frac{d\bar{t}}{dt}\right)^2 &= \frac{1}{\bar{g}(\bar{X},\bar{X})}\left[1+2g\left(X,\frac{DH}{dt}\right)+g\left(\frac{DH}{dt},\frac{DH}{dt}\right)\right.\\
&\quad -g_{ij}\left(x^i+\frac{D\eta^i}{dt}\right)R^j_{klp}\eta^k\left(x^l+\frac{1}{3}\frac{D\eta^l}{dt}\right)\eta^p\\
&\quad \left.-\frac{1}{3}g_{ij}\left(x^i+\frac{D\eta^i}{dt}\right)R^j_{klp;q}\,\eta^k\left(x^l+\frac{1}{2}\frac{D\eta^l}{dt}\right)\eta^p\eta^q\right]+\mathcal{O}_4.
\end{aligned}
\tag{21}
$$

If t and $\bar{t}$ are the proper times of the observer and test particle, respectively, then of course $g(X,X)=\bar{g}(\bar{X},\bar{X})=1$.

Substituting now expressions (15)-(20) into equation (12), we obtain in the second approximation the following deviation equation:

$$
\begin{aligned}
\frac{D^2\eta^i}{dt^2} &= -\kappa^i + \frac{1}{3}R^i_{jkl}\eta^j\kappa^k\eta^l \\
&- \left[R^i_{jlk}\eta^l + \frac{1}{2}\left(R^i_{lpj;k} - R^i_{jkl;p}\right)\eta^l\eta^p\right]x^j x^k \\
&- \left[2R^i_{klj}\eta^l + \left(\frac{1}{3}R^i_{lpk;j} - R^i_{kjl;p}\right)\eta^l\eta^p\right]x^j\frac{D\eta^k}{dt} \\
&- \left[\frac{2}{3}R^i_{jlk}\eta^l + \frac{1}{12}\left(R^i_{lpj;k} - 5R^i_{jkl;p}\right)\eta^l\eta^p\right]\frac{D\eta^j}{dt}\frac{D\eta^k}{dt} \\
&+ \overset{-1}{P}{}^i_a(H)\bar{\kappa}^a g_{jk}\left(x^j + \frac{D\eta^j}{dt}\right)\left[x^k + \frac{D\eta^k}{dt} - R^k_{lpq}\eta^l\left(x^p + \frac{D\eta^p}{dt}\right)\eta^q\right] \\
&+ \frac{1}{6}g\left(X + \frac{DH}{dt}, X + \frac{DH}{dt}\right)R^i_{jkl}\eta^j\bar{\kappa}^k\eta^l \\
&+ \left(x^i + \frac{D\eta^i}{dt} - \frac{1}{3}R^i_{jkl}\eta^j x^k\eta^l\right)\frac{dt}{d\bar{t}}\frac{d^2\bar{t}}{dt^2} + \mathcal{O}_3. \qquad (22)
\end{aligned}
$$

The right hand side of the obtained equation involves an unknown quantity $\frac{dt}{d\bar{t}}\frac{d^2\bar{t}}{dt^2}$. To eliminate it, we must know the correspondence between the world lines. In the next Subsection we will consider the orthogonal correspondence, that is, the deviation vector H and the observer's 4-velocity X are orthogonal along the observer's world line.

If we are working in the global coordinates, the motion of the observer and test particle is described respectively by the components of the 4-velocities X and $\bar{X}$ and the 4-accelerations K and $\bar{K}$. However, in the observer's frame of reference the relative motion is described by the quantities $\frac{D\eta^i}{dt}$, which can be expressed via the velocities X and $\bar{X}$ by means of formula (6). Taking into account the Taylor expansions of the second order derivatives (16) and the Jacobi propagators (20), we obtain the components of the vector $\frac{DH}{dt}$ in the third approximation

$$
\begin{aligned}
\frac{D\eta^i}{dt} &= \overset{-1}{P}{}^i_a(H)x^a\frac{d\bar{t}}{dt} - x^i + \frac{1}{3}R^i_{jkl}\eta^j\left(x^k + \frac{1}{2}\overset{-1}{P}{}^k_a(H)x^a\frac{d\bar{t}}{dt}\right)\eta^l \\
&+ \frac{1}{12}R^i_{jkl;p}\eta^j\left(-x^k + \delta^k_a x^a\frac{d\bar{t}}{dt}\right)\eta^l\eta^p + \mathcal{O}_4. \qquad (23)
\end{aligned}
$$

Substituting the components of $\frac{D\eta^i}{dt}$ from formula (6) into formula (21), we can derive a quadratic equation for determining the quantity $\frac{d\bar{t}}{dt}$.

2.3 Acceleration of free fall and the metric tensor in the frame of reference of a uniformly accelerated observer

In this Subsection we derive relevant formulae, needed later on. One of the aims of this Subsection is to test our formalism, reobtaining some results already known. We consider the frame of reference of an accelerated observer in a curved spacetime, using the observer's Fermi coordinates, that is, its proper time t and orthogonal space coordinates. According to Synge (see Ref. [64], p. 80) the Fermi coordinates of the test particle are the observer's proper time and the corresponding components of the position vector of the test particle in observer's frame of reference, namely

$$\begin{aligned} u^{\bar{0}} &= t, \\ u^{\alpha} &= \delta^{\alpha}_{\lambda}\eta^{\lambda}, \end{aligned} \tag{24}$$

and the zeroth vector of the corresponding ON-basis the observer's 4-velocity, the deviation vector and the observer's 4-acceleration have the following components:

$$\begin{aligned} X &= (1,0,0,0), \\ H &= (0,\eta^{\lambda}) = (0,\vec{\eta}), \\ K &= (0,\kappa^{\lambda}) = (0,\vec{\kappa}); \quad \lambda = 1,2,3. \end{aligned} \tag{25}$$

The only nonvanishing components of the metric tensor and Christoffel symbols at the observer's location are

$$\begin{aligned} g_{00} &= -g_{\lambda\lambda} = 1, \\ \Gamma^{\lambda}_{00} &= \Gamma^{0}_{\lambda 0} = \Gamma^{0}_{0\lambda} = \kappa^{\lambda} = -\kappa_{\lambda}. \end{aligned} \tag{26}$$

The 4-velocity of the test particle has in the observer's Fermi coordinates the following components:

$$\frac{D\eta^i}{dt} = \left(-\vec{\kappa}\cdot\vec{\eta},\ \ \bar{\beta}^{\lambda}\right), \tag{27}$$

where $\bar{\beta}^{\lambda} = \delta^{\lambda}_{\alpha}\dfrac{u^{\alpha}}{dt}$ denotes the component of the 3-velocity of the test particle. Due to formulae (26) and (27), the acceleration of free fall of the test particle reads

$$\frac{D^2\eta^i}{dt^2} = \left(-\frac{D\vec{\kappa}}{dt}\cdot\vec{\eta} - 2\vec{\kappa}\cdot\vec{\bar{\beta}},\ \frac{d\bar{\beta}^{\lambda}}{dt} - \kappa^{\lambda}(\vec{\kappa}\cdot\vec{\eta})\right). \tag{28}$$

We substitute now expressions (25), (27) and (28), into the deviation equation (22). In the case of a freely falling test particle, $\bar{\kappa} = 0$, we obtain the following formula for the acceleration of free fall in the Fermi coordinates of an accelerated observer:

$$
\begin{aligned}
\frac{d\bar{\beta}^{\lambda}}{dt} \;=\;& -\kappa^{\lambda}\left(1-\vec{\kappa}\cdot\vec{\eta}\right)+\left[\frac{1}{3}R^{\lambda}_{\rho\mu\sigma}\kappa^{\mu}+2\kappa_{\sigma}R^{\lambda}_{0\rho0}\right]\eta^{\rho}\eta^{\sigma}\\
&-\left[R^{\lambda}_{0\rho0}\eta^{\rho}+\frac{1}{2}\left(R^{\lambda}_{\rho\sigma0;0}-R^{\lambda}_{00\rho;\sigma}\right)\eta^{\sigma}\eta^{\rho}\right]\\
&-\left[2R^{\lambda}_{\mu\rho0}\eta^{\rho}+\left(\frac{1}{3}R^{\lambda}_{\rho\sigma\mu;0}-R^{\lambda}_{\mu0\rho;\sigma}-\frac{2}{3}\kappa_{\sigma}R^{\lambda}_{\mu\rho0}-\frac{2}{3}\kappa_{\sigma}R^{\lambda}_{0\rho\mu}\right)\eta^{\rho}\eta^{\sigma}\right]\bar{\beta}^{\mu}\\
&-\left[\frac{2}{3}R^{\lambda}_{\mu\rho\nu}\eta^{\rho}+\frac{1}{12}\left(R^{\lambda}_{\rho\sigma\mu;\nu}-5R^{\lambda}_{\mu\nu\rho;\sigma}\right)\eta^{\rho}\eta^{\sigma}\right]\bar{\beta}^{\mu}\bar{\beta}^{\nu}\\
&+\left(\bar{\beta}^{\lambda}-\frac{1}{3}R^{\lambda}_{\rho0\sigma}\eta^{\rho}\eta^{\sigma}\right)\frac{dt}{d\bar{t}}\frac{d^{2}\bar{t}}{dt^{2}}+\mathcal{O}_{3}. \qquad (29)
\end{aligned}
$$

The last term contains the factor $\dfrac{dt}{d\bar{t}}\dfrac{d^2\bar{t}}{dt^2}$ to be determined. Contracting formula (22) in the case $\bar{\kappa}=0$ with $g_{ij}x^i$ and taking into account the antisymmetry of the curvature tensor with respect to its first two indices, we obtain in the observer's Fermi coordinates

$$
\begin{aligned}
\frac{dt}{d\bar{t}}\frac{d^{2}\bar{t}}{dt^{2}} \;=\;& -\left(1-\vec{\kappa}\cdot\vec{\eta}\right)\frac{D\kappa_{\rho}}{dt}\eta^{\rho}+\left(\frac{1}{2}R^{0}_{\rho\sigma0;0}-\frac{1}{3}R^{0}_{\rho\mu\sigma}\kappa^{\mu}\right)\eta^{\sigma}\eta^{\mu}\\
&-2\kappa_{\mu}\bar{\beta}^{\mu}\left[1+\vec{\kappa}\cdot\vec{\eta}+\left(\vec{\kappa}\cdot\vec{\eta}\right)^{2}\right]\\
&+\left[2R^{0}_{\mu\rho0}\eta^{\rho}+\left(\frac{4}{3}\kappa_{\sigma}R^{0}_{\mu\rho0}+\frac{1}{3}R^{0}_{\rho\sigma\mu;0}-\frac{2}{3}\kappa_{\mu}R^{0}_{\rho0\sigma}\right.\right.\\
&\left.\left.-\;R^{0}_{\mu0\rho;\sigma}\right)\eta^{\rho}\eta^{\sigma}\right]\bar{\beta}^{\mu}+\left[\frac{2}{3}R^{0}_{\mu\rho\nu}\eta^{\rho}\right.\\
&\left.+\left(\frac{2}{3}\kappa_{\sigma}R^{0}_{\mu\rho\nu}+\frac{1}{12}R^{0}_{\rho\sigma\mu;\nu}-\frac{5}{12}R^{0}_{\mu\nu\rho;\sigma}\right)\eta^{\rho}\eta^{\sigma}\right]\bar{\beta}^{\mu}\bar{\beta}^{\nu}+\mathcal{O}_{3}.
\end{aligned}
\qquad (30)
$$

Formulae (29) and (30) give us the following equation for free fall in the frame of reference of an accelerated observer in the second approximation, which really coincides with the results obtained by Li and Ni, (see Ref. [40]), and by Retzloff, De Facio and Dennis, (see Ref. [44]), but obtained via calculating the Christoffel symbols in observer's Fermi coordinates and replacing them into the equation of 3-acceleration of a freely falling body:

$$
\begin{aligned}
\frac{d\bar{\beta}^{\lambda}}{dt} \;=\;& -\kappa^{\lambda}\left(1-\vec{\kappa}\cdot\vec{\eta}\right)\left[\frac{1}{3}R^{\lambda}_{\rho\mu\sigma}\kappa^{\mu}+2\kappa_{\sigma}R^{\lambda}_{0\rho0}\right]\eta^{\rho}\eta^{\sigma}\\
&-\left[R^{\lambda}_{0\rho0}\eta^{\rho}+\frac{1}{2}\left(R^{\lambda}_{\rho\sigma0;0}-R^{\lambda}_{00\sigma;\rho}\right)\eta^{\sigma}\eta^{\mu}\right]\\
&-\left[2R^{\lambda}_{\mu\rho0}\eta^{\rho}+\left(\frac{1}{3}R^{\lambda}_{\rho\sigma\mu;0}-R^{\lambda}_{\mu0\rho;\sigma}-\frac{2}{3}\kappa_{\sigma}R^{\lambda}_{\mu\rho0}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{3}\kappa_\mu R^\lambda_{\rho 0\sigma} - \frac{2}{3}\kappa_\sigma R^\lambda_{0\rho\mu}\Big) \eta^\rho \eta^\sigma\Big] \bar{\beta}^\mu \\
&-\left[\frac{2}{3} R^\lambda_{\mu\rho\nu}\eta^\rho + \frac{1}{12}\left(R^\lambda_{\rho\sigma\mu;\nu} - 5R^\lambda_{\mu\nu\rho;\sigma}\right)\eta^\rho\eta^\sigma\right]\bar{\beta}^\mu\bar{\beta}^\nu \\
&-\bar{\beta}^\lambda\left(1+\vec{\kappa}\cdot\vec{\eta}\right)\frac{D\kappa_\sigma}{dt}\eta^\sigma + \bar{\beta}^\lambda\left(\frac{1}{12}R^0_{\rho\sigma 0;0} - \frac{1}{3}R^0_{\rho\mu\sigma}\kappa^\mu\right)\eta^\rho\eta^\sigma \\
&-2\bar{\beta}^\lambda\kappa_\mu\bar{\beta}^\mu\left[1+\vec{\kappa}\cdot\vec{\eta}+(\vec{\kappa}\cdot\vec{\eta})^2\right] \\
&+\bar{\beta}^\lambda\Big[2R^0_{\mu\rho 0}\eta^\rho + \Big(\frac{4}{3}\kappa_\sigma R^0_{\mu\rho 0} + \frac{1}{3}R^0_{\rho\sigma\mu;0} - \frac{2}{3}\kappa_\mu R^0_{\rho 0\sigma} \\
&-R^0_{\mu 0\rho;\sigma}\Big)\eta^\rho\eta^\sigma\Big]\bar{\beta}^\mu + \bar{\beta}^\lambda\Big[\frac{2}{3}R^0_{\mu\rho\nu}\eta^\rho \\
&+\left(\frac{2}{3}\kappa_\rho R^0_{\mu\nu\sigma} + \frac{1}{12}R^0_{\rho\sigma\mu;\nu} - \frac{5}{12}R^0_{\mu\nu\rho;\sigma}\right)\eta^\rho\eta^\sigma\Big]\bar{\beta}^\mu\bar{\beta}^\nu + \mathcal{O}_3. \quad (31)
\end{aligned}
$$

Comparing the equations (22) and (31) confirms again, that by transforming an equation from the global coordinates into a local coordinate system system comoving with the observer, makes the equations in the general case more complicated, but gives them a more clear physical content and makes them easierto interprete. However, there is another opposite approach to use the similar coordinate transformation to make the equations simpler. For example, the author of the present theses has shown in [69]. There a physical system is described by the state equations, where the changing rates of the state equations of the system depend on the states themselves, certain input parameters and the higher order time derivatives of the inputs. Writing the equations in the observer's comoving frame of reference, where some basis vector fields in the global coordinates are replaced by their changing rates relative to the moving observer, allows the eliminating of the higher order time derivatives of the input parameters from the equations and consequently leads to the simplification of the state equations.

We also calculate the quotient of the infinitesimal proper time intervals of the test particle and observer. Due to formulae (25)-(27), equation (21) yields

$$
\begin{aligned}
\left(\frac{d\bar{t}}{dt}\right)^2 &= (1-\vec{\kappa}\cdot\vec{\eta})^2 - |\bar{\beta}|^2 - \left(1-\frac{4}{3}\vec{\kappa}\cdot\vec{\eta}\right)R^0_{\mu 0\nu}\eta^\mu\eta^\nu - \frac{1}{3}R^0_{\mu 0\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma \\
&- \Big[\frac{1}{3}\left(1-\vec{\kappa}\cdot\vec{\eta}\right)R^0_{\mu\lambda\nu}\eta^\mu\eta^\nu + \left(1-\frac{1}{3}\vec{\kappa}\cdot\vec{\eta}\right)R_{\lambda\mu 0\nu}\eta^\mu\eta^\nu \\
&+ \frac{1}{3}R_{\lambda\mu 0\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma + \frac{1}{6}R^0_{\mu\lambda\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma\Big]\bar{\beta}^\lambda \\
&+ \left[\frac{1}{3}R_{\lambda\mu\rho\nu}\eta^\mu\eta^\nu + \frac{1}{6}R_{\lambda\mu\rho\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma\right]\bar{\beta}^\lambda\bar{\beta}^\rho + \mathcal{O}_4. \qquad (32)
\end{aligned}
$$

In the Minkowski spacetime our formulae (29) and (30) reduce, of course, to the following well known formulae (see e.g. Ref. [70], p. 245):

$$\begin{aligned}
\frac{D\bar{\beta}^\lambda}{dt} &= -\kappa^\lambda\,(1-\vec{\kappa}\cdot\vec{\eta}) - \frac{2\bar{\beta}^\lambda}{1-\vec{\kappa}\cdot\vec{\eta}}\,\vec{\kappa}\cdot\vec{\bar{\beta}}, \\
\left(\frac{d\bar{t}}{dt}\right)^2 &= (1-\vec{\kappa}\cdot\vec{\eta})^2 - |\bar{\beta}|^2.
\end{aligned} \tag{33}$$

Because in the Minkowski spacetime $\mathcal{K}^i_a = \mathcal{H}^i_a = P^i_a(H) = \delta^i_a$, the last expression for the quotient $(d\bar{t}/dt)^2$ reduces to the square of modulus of the vector $X + DH/dt$.

At the end of this Subsection we express the components of the metric tensor in the observer's Fermi coordinates. Formulae (24) and (27) yield

$$\begin{aligned}
\frac{D\eta^\lambda}{dt}dt &= \delta^\lambda_\alpha du^\alpha, \\
\frac{D\eta^0}{dt}dt &= (\vec{\kappa}\cdot\vec{\eta})dt.
\end{aligned} \tag{34}$$

Multiplying of formula (14) by dt^2, we have

$$\begin{aligned}
d\bar{t}^2 &= g_{ab}du^a du^b = g_{ij}\hat{\mathcal{K}}^i_0\hat{\mathcal{K}}^j_0 dt^2 - g_{ij}\hat{\mathcal{K}}^i_0\hat{\mathcal{H}}^j_0\vec{\eta}\cdot\vec{\kappa}dt^2 \\
&\quad + g_{ij}\hat{\mathcal{K}}^i_0\hat{\mathcal{H}}^j_\lambda\delta^\lambda_\alpha du^\alpha dt + g_{ij}\hat{\mathcal{H}}^i_0\hat{\mathcal{H}}^j_0(\vec{\kappa}\cdot\vec{\eta})^2 dt^2 \\
&\quad - 2g_{ij}\hat{\mathcal{H}}^i_0\hat{\mathcal{H}}^j_\lambda\delta^\lambda_\alpha du^\alpha dt + g_{ij}\hat{\mathcal{H}}^i_\lambda\hat{\mathcal{H}}^j_\mu\delta^\lambda_\alpha\delta^\mu_\beta du^\alpha du^\beta,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
\hat{\mathcal{K}}^i_j &= \overset{-1}{P^i_a}(H)\mathcal{K}^a_j = \delta^i_j - \frac{1}{2}R^i_{kjl}\eta^k\eta^l - \frac{1}{6}R^i_{kjl;p}\eta^k\eta^l\eta^p + \mathcal{O}_4, \\
\hat{\mathcal{H}}^i_j &= \overset{-1}{P^i_a}(H)\mathcal{H}^a_j = \delta^i_j - \frac{1}{6}R^i_{kjl}\eta^k\eta^l - \frac{1}{12}R^i_{kjl;p}\eta^k\eta^l\eta^p + \mathcal{O}_4.
\end{aligned} \tag{36}$$

Thus, formula (35) provides the exact expressions for the components of the metric tensor at the location of the test particle in the observer's Fermi coordinates:

$$\begin{aligned}
g_{\bar{0}\bar{0}} &= \eta_{ij}\hat{\mathcal{K}}^i_0\hat{\mathcal{K}}^j_0 - 2\vec{\kappa}\cdot\vec{\eta}\eta_{ij}\hat{\mathcal{K}}^i_0\hat{\mathcal{H}}^j_0 + (\vec{\kappa}\cdot\vec{\eta})^2\eta_{ij}\hat{\mathcal{H}}^i_0\hat{\mathcal{H}}^j_0, \\
g_{\bar{0}\alpha} &= \left(\frac{1}{2}\eta_{ij}\hat{\mathcal{K}}^i_0\hat{\mathcal{H}}^j_\lambda - \eta_{ij}\hat{\mathcal{H}}^i_0\hat{\mathcal{H}}^j_\lambda\right)\delta^\lambda_\alpha, \\
g_{\alpha\beta} &= \eta_{ij}\hat{\mathcal{H}}^i_\lambda\hat{\mathcal{H}}^j_\mu\delta^\lambda_\alpha\delta^\mu_\beta,
\end{aligned} \tag{37}$$

where the quantities η_{ij} are the components of the Minkowski metric tensor. Next we apply the general formulae to the Fermi coordinates in order to compare them with the results known already. Substituting, for example, the

Taylor series for Jacobi propagators from equations (20) into equations (37), we obtain in the third approximation

$$
\begin{aligned}
g_{\bar{0}\bar{0}} &= (1-\vec{\kappa}\cdot\vec{\eta})^2 - \left(1-\frac{4}{3}\vec{\kappa}\cdot\vec{\eta}\right) R_{0\mu 0\nu}\eta^\mu\eta^\nu - \frac{1}{3}R_{0\mu 0\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma + \mathcal{O}_4, \\
g_{\bar{0}\alpha} &= -\left[\frac{1}{6}(1-\vec{\kappa}\cdot\vec{\eta})R_{0\mu\lambda\nu}\eta^\mu\eta^\nu + \frac{1}{2}\left(1-\frac{1}{3}\vec{\kappa}\cdot\vec{\eta}\right) R_{\lambda\mu 0\nu}\eta^\mu\eta^\nu + \right. \\
&\quad \left. + \frac{1}{6}R_{\lambda\mu 0\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma + \frac{1}{12}R_{0\mu\lambda\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma\right]\delta^\lambda_\alpha + \mathcal{O}_4, \qquad (38) \\
g_{\alpha\beta} &= -\delta_{\alpha\beta} + \left[\frac{1}{3}R_{\lambda\mu\rho\nu}\eta^\mu\eta^\nu + \frac{1}{6}R_{\lambda\mu\rho\nu;\sigma}\eta^\mu\eta^\nu\eta^\sigma\right]\delta^\lambda_\alpha\delta^\rho_\beta + \mathcal{O}_4.
\end{aligned}
$$

The last expressions reproduce in a compact form the metric tensor for non-rotationg observer found earlier by Li and Ni in Ref. [41] and also lead to the same result as the integral formulae for calculating the components of the metric tensor, found by Nesterov in Ref. [71], page 4.

2.4 Acceleration of free fall in the frame of reference of a radially accelerated observer in the Schwarzschild spacetime

As an application of equation (31) we will examine the acceleration of free fall in the frame of reference of a radially accelerated observer (called in what follows in this Subsection simply "observer"), moving in the gravitational field of a black hole. The necessary components of the Riemann curvature tensor in observer's Fermi coordinates are calculated in Appendix C (225). Equation (31) also generalizes the formula for acceleration of free fall in the radially moving observer's frame of reference, given by Chicone and Mashhoon, (see Ref. [60], p.13), to the case of a relativistically moving observer, and enables us to treat also the case of a nonradially moving observer. We will not present this formula for the general case, because it will be very long and uncomprehensive, see for example the corresponding formula for static observer, derived by Li and Ni in Ref. [40]. So we will only consider some interesting special cases.

First we examine an observer and a test particle, both moving in the radial direction on the hypersurface $\theta = \pi/2$. As in this Subsection we work only in the observer's Fermi coordinates, we will omit the parentheses around the indices. If the 3-velocity of the test particle relative to the observer is zero, it follows from equation (31) that the 3-acceleration of the test particle has the following components:

$$\begin{aligned}
\frac{d\bar{\beta}^1}{dt} &= -\kappa(1+\kappa\eta^1) - R^1_{010}\eta^1 + \frac{1}{3}R^1_{212}\kappa(\eta^2)^2 - \frac{1}{2}R^1_{010;1}(\eta^1)^2 - \\
&\quad -2\kappa R^1_{010}(\eta^1)^2 + \mathcal{O}_3, \qquad (39)\\
\frac{d\bar{\beta}^2}{dt} &= -R^2_{020}\eta^2 - \frac{1}{2}R^2_{020;1}\eta^1\eta^2 + \frac{1}{3}\kappa R^2_{112}\eta^1\eta^2 - 2\kappa R^2_{020}\eta^1\eta^2 + \mathcal{O}_3,
\end{aligned}$$

where κ is the modulus of the observer's 4-acceleration. The limit $\rho \to \infty$ gives us the Minkowski spacetime limits, i.e. $\frac{d\bar{\beta}^1}{dt} = -\kappa(1+\kappa\eta^1)$ and $\frac{d\bar{\beta}^2}{dt} = 0$. As we see from the formulae (39) the tidal effects, caused by the curvature of spacetime, and the effects, caused by the observer's nongravitational acceleration, are in the first approximation independent, but in the second approximation this "principle of superposition" will not hold any more. Substitution of the components of curvature tensor (225) and their covariant derivatives into the formulae (39), yields

$$\begin{aligned}
\frac{d\bar{\beta}^1}{dt} &= -\kappa(1+\kappa\eta^1) + \frac{2\gamma M}{\rho^3}\eta^1 - \frac{3\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2-\beta^2_\rho}}(\eta^1)^2 - \\
&- \kappa\frac{2\gamma M}{\rho^3}\left[\frac{(\eta^2)^2}{3} - 2(\eta^1)^2\right] + \mathcal{O}_3, \qquad (40)
\end{aligned}$$

$$\frac{d\bar{\beta}^2}{dt} = -\frac{\gamma M}{\rho^3}\eta^2 + \frac{3}{2}\frac{\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2-\beta^2_\rho}}\eta^1\eta^2 - \frac{2}{3}\kappa\frac{\gamma M}{\rho^3}\eta^1\eta^2 + \mathcal{O}_3, \qquad (41)$$

where $\beta_\rho = \dfrac{d\rho}{d\tau}$ is the radial component of the observer's 3-velocity, τ and ρ are the proper time of an infinitely far stationary observer and the radial coordinate, respectively. Examining now equations (40) and (41), we reach the following important consequences, namely, in the frame of reference of an observer, accelerated radially in the Schwarzschild spacetime, the following is true:

(i) The acceleration of free fall of a test particle with $\bar{\beta} \ll 1$ in the radial direction depends also on the component η^2 of deviation vector, i.e. depends also on the particle's distance perpendicular to the radial direction. It means, that according to the last term in formula (40), the test particles with larger distance perpendicular to the radial direction will have larger radial components of the acceleration of free fall.

(ii) The last term in formula (41) means, that in the case $\beta_\rho \approx 0$ and $\eta^1 > 0$, i.e. the radial coordinate of test particle is greater than the observer's one, the test particle will obtain, compared with the case of a freely falling observer, an extra tidal acceleration perpendicular to the radial direction, directed towards the observer. In case $\eta^1 < 0$ this acceleration is directed away from the observer.

For example, formula (41) yields in the case $\beta_\rho \approx 0$ for the acceleration of free fall perpendicular to the radial direction

$$\frac{d\bar{\beta}^2}{dt} = \frac{\gamma M}{\rho^3}\eta^2 \left(1 - \frac{2}{3}\kappa\eta^1\right) + \mathcal{O}_2.$$

Consequently, the component of the tidal 3-acceleration of a freely falling test particle perpendicular to the radial direction depends also on the radial coordinate of the test particle. If the observer's nongravitational acceleration and the radial coordinate of the test particle have the same (opposite) sign, the freely falling test particle has in the direction, perpendicular to the radial direction, the lesser (larger) component of tidal 3-acceleration as in the case of a freely falling observer. The test particle, moving at a non-relativistic 3-velocity relative to the observer and having the radial coordinate

$$\eta^1 = \frac{3}{2\kappa},$$

will have no 3-acceleration perpendicular to the radial direction.
If we use the first approximation of formula (40), we have

$$\frac{d\bar{\beta}^1}{dt} = -\kappa(1 + \kappa\eta^1) + \frac{2\gamma M}{\rho^3}\eta^1 + \mathcal{O}_2.$$

The acceleration of free fall of a test particle, moving at a nonrelativistic 3-velocity relative to the observer, is a second order small quantity, if the radial component of the deviation vector is

$$\eta^1 = \frac{\kappa}{\frac{2\gamma M}{\rho^3} - \kappa^2}. \tag{42}$$

It means, one can eliminate the radial tidal forces at a certain value of radial coordinate with a suitable choice of 4-acceleration.

Another important consequence is, as we see from formulae (40) and (41), that the dependence of the acceleration of free fall of a test particle on the observer's 3-velocity β_ρ appears in the second approximation. Namely:

(i) Due to the third term on the right hand side of formula (40), the apparent tidal 3-acceleration of a freely falling test particle, moving in parallel with the observer's world line, diminishes in the observer's frame of reference, if the observer moves in the radial direction at a relativistic 3-velocity. It means, the observer's radial motion "reduces" the effects of the black hole. This result is counterintuitive, because in the frame of reference of a radially moving observer the mass of the black hole is larger than its rest mass M and the apparent distance to its center is less than ρ due to the relativistic contraction of the length; thus, according to the common logic, these two relativistic effects should increase the apparent acceleration of the test particle, caused by the gravitational field of the black hole.

(ii) At the same time, the orbital component of the tidal 3-acceleration of the test particle also depends on the observer's motion in the radial direction. If the radial component η^1 of its position vector relative to the observer is less than zero (the test particle is nearer to the black hole than the observer), its tidal 3-acceleration in the orbital direction is greater than in the case of a stationary observer. If the radial component of its position vector is larger than zero, its 3-acceleration in the orbital direction is less than in the case of a stationary observer.

It must be pointed out here, that these effects are valid only if the observer does not move at a hyperrelativistic 3-velocity. If β_ρ approaches the velocity of light (219), the factor $\sqrt{c^3\left(c^2-\beta_\rho^2\right)^{-1}}$ grows rapidly and the terms, involving β_ρ, will become significant. The third term in formula (40) and the second term in formula (41) begin to dominate over the linear terms. Consequently, the Taylor expansions (40) and (41) do not converge and the geodesic deviation equation is not applicable anymore. It may lead to enormous tidal forces, not predictable relying only on the first order deviation equation. This case will considered in the next Subsection.

Now we consider the situation, where the test particle moves in the observer's frame of reference at a relativistic 3-velocity. To avoid very complicated formulae, we will here also restrict ourselves to some simple particular cases. Let us examine a two-dimensional case, when both the observer and particle move in the radial direction and the deviation vector also has in the observer's frame of reference only the radial component. This means, the nonzero components of the deviation vector and 3-velocity of the test particle are

$$\eta^1 = \Delta r, \quad \bar{\beta}^1 = \bar{\beta}. \tag{43}$$

Here r is the radial coordinate in the observer's frame of reference. Substituting the components of the curvature tensor (225) into formulae (31), we obtain for the acceleration of free fall of the test particle the following expression:

$$\begin{aligned}\frac{d\bar{\beta}}{dt} &= -\kappa(1+\kappa\Delta r) + \frac{2\bar{\beta}^2\kappa}{1+\kappa\Delta r} + \frac{2\gamma M}{\rho^3}(1-2\bar{\beta}^2)\Delta r \\ &\quad + \frac{4\gamma M}{\rho^3}\kappa(1-\bar{\beta}^2)\Delta r^2 - \frac{3\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}(1-2\bar{\beta}^2)\Delta r^2 \\ &\quad + \frac{6\gamma M}{\rho^4}\sqrt{\frac{c}{c^2-\beta_\rho^2}}\bar{\beta}\beta_\rho\Delta r^2 + \mathcal{O}_3.\end{aligned} \tag{44}$$

As we see, if the observer moves with a constant 4-acceleration, its radial 3-velocity in the global coordinates is $\beta_\rho \ll c$, and the 3-velocity of the test particle has a critical value

$$\bar{\beta}_{cr} = \frac{\sqrt{2}}{2}\left(1+\kappa\Delta r - \frac{\gamma M}{\rho^3}\Delta r^2\right) \tag{45}$$

at which the acceleration of free fall turns out to be a second order small quantity (only the last term in formula (44) does not vanish), which in case $\bar{\beta} > \bar{\beta}_{cr}$ changes its sign. The quantity in the brackets in formula (45) is the velocity of light $\bar{c}$ in the observer's frame of reference. The same result was obtained by Mould (see Ref. [70], p. 246) for the frame of reference of an accelerated observer in the 2-dimensional Minkowski spacetime, and by the authors of the present Thesis for an observer, falling freely in the Schwarzschild spacetime in the radial direction (see Ref. [59]). The last case was later considered also by Chicone and Mashhoon (see Refs. e.g. [76], [94] and [60]) where the autors discuss the physical significance of the critical 3-velocity $\bar{c}/\sqrt{2}$ of the test particle relative to observer accelerated in the Schwarzchild spacetime. As one can conclude from the above-mentioned papers of Chicone and Mashhoon, the critical value of 3-velocity $\bar{c}/\sqrt{2}$ appears only if the metric in observer's Fermi coordinates is stationary. If the observer is moving at the nonrelativistic velocity β_ρ, his metric remains stationary in the first approximation, and in case $\bar{\beta} = \bar{\beta}_{cr}$ the zeroth and first order terms in formula (44) vanish. In the second approximation the only nonvanishing term is the last one, containing also the observer's 3-velocity β_ρ. This term becomes significant if the observer moves at the relativistic 3-velocity, we can no longer consider the metric in his frame of reference as the stationary one and consequently, the acceleration of free fall of the test particle differs from zero. A further examination of equation (44) shows, that if the test particle moves at the velocity $\bar{c}/\sqrt{2}$ relative to the observer ($\bar{\beta} < 0$) and the observer approaches the black hole, the acceleration of free fall of the test particle will have the sign opposite to its relative 3-velocity $\bar{\beta}$ and the particle deccelerates. And, if the observer moves away from the black hole, the test particle accelerates. Formula (45) is a generalization of both cases – the accelerated motion in the Minkowski spacetime and the radial free fall in the Schwarzschild spacetime. Here we point out again, that the formulae (44) and (45) are not applicable in the case of the observer's hyperrelativistic 3-velocity, because β_ρ approaching the radial velocity of light leads to unlimited growth of quadratic terms in formula (44) and the Taylor expansion does not converge.

We also consider the special case, when the observer's 3-velocity has in the global coordinates also a component β_ϕ, perpendicular to the radial direction. If the observer and test particle are freely falling and their world lines are initially parallel ($\beta^\lambda = 0\,\forall\lambda$), formulae (31) and (225) give us in the first approximation

$$\begin{aligned}
\frac{d\bar{\beta}^1}{dt} &= -R^1_{010}\eta^1 + \mathcal{O}_2 = \frac{2\gamma M}{\rho^3}\left(1 + \frac{3}{2}\frac{c\rho^2\beta_\phi^2}{c^2 - \beta_\rho^2 - c\rho^2\beta_\phi^2}\right)\eta^1 + \mathcal{O}_2,\\
\frac{d\bar{\beta}^2}{dt} &= -R^2_{020}\eta^2 + \mathcal{O}_2 = -\frac{\gamma M}{\rho^3}\eta^2 + \mathcal{O}_2,\\
\frac{d\bar{\beta}^3}{dt} &= -R^3_{030}\eta^3 + \mathcal{O}_2 = -\frac{\gamma M}{\rho^3}\left(1 + 3\frac{c\rho^2\beta_\phi^2}{c^2 - \beta_\rho^2 - c\rho^2\beta_\phi^2}\right)\eta^3 + \mathcal{O}_2.
\end{aligned} \tag{46}$$

Already in the first approximation one can see, that if the observer moves in the radial direction at a 3-velocity $\beta_\rho \approx c$ and has also a small perpendicular component of the 3-velocity, it causes enormous tidal forces in the radial direction and in direction, perpendicular to hypersurface $\theta = \pi/2$.

We illustrate the last result (46) with a more specialised example, when an observer moves around a black hole along a circular orbit. The angular velocity which secures free motion on a circular orbit is

$$\beta_\phi = \sqrt{\frac{\gamma M}{\rho^3}}. \tag{47}$$

According to formula (46), the radial 3-acceleration of a free test particle, moving in parallel with the observer and placed on the straight line connecting the observer and the center of black hole, in the observer's frame of reference reads

$$\frac{d\bar{\beta}^1}{dt} = \frac{2\gamma M}{\rho^3}\left(1 + \frac{3\gamma M}{2\rho}\frac{1}{1 - \dfrac{3\gamma M}{\rho}}\right)\eta^1. \tag{48}$$

In the case $\rho \to 3\gamma M$, the obtained 3-acceleration $\dfrac{d\bar{\beta}^1}{dt}$ approaches infinity. It means, that any body, extended in the radial direction and moving along a circular orbit around the center of the black hole near the radial coordinate $\rho = 3\gamma M$ (i.e. a circular orbit of a photon), will be destroyed by the tidal forces.

Remark. In the case of an observer, moving freely around the black hole along a circular orbit, we conclude, by substituting the angular velocity (47) into the third equation of system (46), that also the component of the tidal force perpendicular to the radial direction and the orbit approaches infinity in case $\rho \to 3\gamma M$, the orbital component of tidal forces does not depend on the observer's motion. In this case the quadratic terms will not produce any effects.

2.5 Free fall in the radial direction at relativistic 3-velocity: the problem of convergence

In this Subsection we consider the frame of reference of an observer, falling freely in the Schwarzschild spacetime in the radial direction at a relativistic 3-velocity β_ρ. We examine, when the geodesic deviation equation (40) as the Taylor expansion is applicable to determine the 3-acceleration of a freely falling test particle in the observer's Fermi coordinates. We restrict ourselves to the case $\bar{\beta} = 0$. Equations (40) and (41) take the form

$$\frac{d\bar{\beta}^1}{dt} = \frac{2\gamma M}{\rho^3} Q\eta^1 + \mathcal{O}_3, \tag{49}$$

$$\frac{d\bar{\beta}^2}{dt} = -\frac{\gamma M}{\rho^3} Q\eta^2 + \mathcal{O}_3, \tag{50}$$

where

$$Q = 1 - \frac{3}{2\rho}\sqrt{\frac{c^3}{c^2 - \beta_\rho^2}}\eta^1. \tag{51}$$

As already mentioned in the previous Subsection, if the observer's 3-velocity β_ρ approaches the velocity of light, the factor $\left(c^2 - \beta_\rho^2\right)^{-\frac{1}{2}}$ approaches infinity, the quadratic terms in Taylor expansion become larger than the linear terms, and the Taylor expansion diverges. Consequently, the geodesic deviation equation is not applicable to calculate the 3-velocity of freely falling test particles, having in the observer's Fermi coordinates a nonvanishing radial component of the deviation vector:

$$|\eta^1| \geq \frac{3\rho}{2}\sqrt{\frac{c^2 - \beta_\rho^2}{c^3}}. \tag{52}$$

Chicone and Mashhoon introduce (see Ref. [60], p.5) the radius $\mathcal{R}$ of the "cylinder of convergence". For the test particles outside of this cylinder the geodesic deviation equation is not applicable. In our 2-dimensional case the radius $\mathcal{R}$ is on the right hand side of equation (52). Its magnitude depends, as we see, also on the observer's radial 3-velocity β_ρ. If we use the geodesic deviation equation in the Scwarzschild spacetime, we assume that the position vector H of the test particle relative to the observer is much less than the observer's radial coordinate ρ. Ergo, for such a test particle the deviation equation is not applicable only then, if the observer moves at the hyperrelativistic 3-velocity.

Let us scrutinize now when the above case occurs. If the observer is falling freely, the energy per unit rest mass as a constant of motion reads

$$e = g_{\tau\tau}\frac{d\tau}{dt}. \tag{53}$$

Taking into account also the square of the differential of the proper time

$$dt^2 = (g_{\tau\tau} + g_{\rho\rho}\beta_\rho^2)d\tau^2 \quad (54)$$

and the formula $c = 1 - \dfrac{2\gamma M}{\rho}$, we obtain

$$e = \sqrt{\frac{c^3}{c^2 - \beta_\rho^2}}. \quad (55)$$

Substituting the last expression into formula (51) yields

$$Q = 1 \pm \frac{3e\eta^1}{2\rho}. \quad (56)$$

In the case of an observer, which starts free fall from infinitely far at the initial 3-velocity $\beta_\rho = 0$ and $e = 1$, the last formula reads

$$Q = 1 \pm \frac{3\eta^1}{2\rho}. \quad (57)$$

Because in the Taylor expansion must be $|\eta^1| \ll \rho$, the second term is much less than unity and in this case the geodesic deviation equation is applicable, and the enormous tidal forces will not appear on the Schwarzschild sphere. Taking into account formula (56) and the condition $|\eta^1| \ll \rho$ we see, that the Taylor expansion will not converge only if $e \gg 1$, i.e., if the observer moves at a hyperrelativistic 3-velocity β_ρ. Consequently the second term in formula (56) becomes significant if the radial component of the deviation vector becomes comparable with the quantity ρ/e. In this case the Taylor expansion will not converge and the geodesic deviation equation is no more applicable to estimate the magnitude of the tidal forces. Consequently, if the 3-velocity of an observer moving radially in the Schwarzschild spacetime is comparable with the radial velocity of light, the deviation equation for parallel geodesics is in the observer's frame of reference applicable only for the freely falling test particles with

$$|\eta^1| \ll \frac{\rho}{e}. \quad (58)$$

The divergence of the Taylor expansion has the following reason. Namely, for the convergence the spatial distance between the observer and the black hole must be much larger than the magnitude of the deviation vector. If the observer's radial 3-velocity β_ρ is relativistic, the spatial distance of the black hole relative to the observer is much less than relative to a stationary observer with the same radial coordinate, due to the relativistic contraction of the length. It means that in the frame of reference of the freely falling observer the magnitude of the deviation vector is in this case comparable with the spatial distance of the black hole.

For example, if an observer starts the free fall at the initial 3-velocity $\beta_{\rho 0}$ at the radial coordinate $\rho_0 \gg \gamma M$, then at the observer's initial location $c \approx 1$ and $e \approx (1-\beta_{\rho 0}^2)^{-\frac{1}{2}}$. The condition (58) reads in this case

$$|\eta^1| \quad \ll \quad \frac{\rho}{\sqrt{1-\beta_{\rho 0}^2}}. \tag{59}$$

Here $(1-\beta_{\rho 0}^2)^{-\frac{1}{2}}$ is the Lorentz contraction factor and the right hand side of the last inequality is the spatial distance of the black hole relative to observer. If the observer is placed on a body with the radial length comparable to the quantity $\rho(1-\beta_{\rho 0}^2)^{-\frac{1}{2}}$, the geodesic deviation equation is no longer applicable for estimating the magnitude of the tidal forces.

2.6 Conclusions

In the present Section we derived the exact deviation equation describing the relativistic relative motion of particles moving on accelerated world lines whose parametrization is arbitrary. Using the exact expression of the relation of the proper times of both point masses, we also derived the exact formulae for calculating the components of the metric tensor in the observer's frame of reference, namely, in the observer's Fermi coordinates. Next we derived the second approximation (with respect to the deviation vector) of the exact deviation equation in a general coordinate system and in the Fermi coordinates of an accelerated observer.

The second order deviation equation is used in the Fermi coordinates of an observer moving in the Schwarzschild spacetime. If the observer is falling freely in the radial direction, the radial component of acceleration of free fall of a test particle depends only on the radial component of its position vector. If the observer is nongravitationally accelerated radially, the radial component of acceleration of free fall will depend also on the components of the position vector perpendicular to the radial direction. At the same time, the component of the acceleration of free fall, perpendicular to the radial direction, also depends on the radial component of the position vector, but again only in the case of an accelerated observer.

Also a counterintuitive result has been found that the observer's radial motion reduces the effects of the black hole like the tidal forces. The acceleration of free fall of a test particle is in the observer's frame of reference smaller than in the case of the stationary observer and decreases if the observer's radial 3-velocity increases.

The abovementioned effects appear only if we use the second approximation of the deviation equation and can be considered only if the observer's 3-velocity is not hyperrelativistic. If the observer moves at a hyperrelativistic 3-velocity, the deviation equation as the Taylor series for calculating the acceleration of free fall does not converge and therefore the deviation equation is not applicable. It has been shown that if the freely falling observer starts its motion from spacelike infinity at a nonrelativistic initial velocity, no convergence problems appear.

Special attention has been paid to the observer's motion in the 2-dimensional Schwarzschild space. Two results are compared. First – the result of Mould that in the frame of reference of an accelerated observer, moving on the 2-dimensional Minkowski plane, the acceleration of free fall of a test particle turns to be zero (changes its sign) if its 3-velocity equals (is greater than) $\bar{c}/\sqrt{2}$. This has been compared with our analogous result for the frame of reference of a freely falling observer in the 2-dimensional Schwarzschild spacetime. A generalization of both results says that in the frame of reference of the accelerated observer in the 2-dimensional Schwarzschild spacetime the acceleration of free fall also turns to be a second order small quantity if the 3-velocity of the test particle is $\bar{c}/\sqrt{2}$, but this is not the case if the observer moves at the relativistic 3-velocity.

The effects predictable with the help of the deviation equation already in the first approximation have also been considered. Namely, if the radial component of the observer's 3-velocity is comparable to the velocity of light, also the small component of its 3-velocity, perpendicular to the radial direction, will rapidly increase the tidal forces. If this perpendicular component of 3-velocity is also relativistic, enormous tidal forces occur which are able to destroy any extended body. This makes free circular motion of an extended body impossible near the 1.5 Schwarzschild radius.

3 On the pressure of gravitational waves

3.1 Introduction

The interest in the studies of the effects of gravitational waves is significantly grown in the last few years (see e.g. [77]), as the new detectors (LISA, LIGO, VIRGO, TAMA etc.) became continuously operational, [78], [79]. In order to register the gravitational waves, the transversal displacements of parts of detectors, the apparent changes in photon velocity, and the variation in the clock speed caused by gravitational waves are usually considered [80].

The above three quantities enable us to directly calculate the energy absorbed from gravitational waves. Likewise it is of interest to directly calculate on the basis of equations of motion the momentum of gravitational waves absorbed by detector, which is one of the main aims of this section.

The well known ordinary (first oder) geodesic deviation equation [1], [64], describes relative motion of freely falling point masses in the linear approximation in their relative position vector. It is applicable to the so called "non-relativistic" systems, where the change rate of separation of world lines is of the same order of smallness as the separation itself [27], [56], [57]. To this order of accuracy (i) plane gravitational waves evoke only transverse tidal forces which in the case of periodic waves are oscillatory and (ii) the center of mass of point masses follows a geodesic in an arbitrary spacetime [9]. The second order geodesic deviation equation which describes relative motion of point masses in the quadratic approximation, was first derived in Refs. [27] and [22]. One of the interesting physical conclusions of the second order deviation equation is, that in a curved space the center of mass of freely falling point masses does not move along a geodesic [63]. In this approximation a plane gravitational wave also produces a longitudinal relative acceleration between the freely falling point masses whose transverse separation is nonvanishing. When there exists a non-gravitational interaction between the point masses and dissipative forces are present in the system, the power is absorbed from a gravitational wave. The phenomenon is coincident with a net longitudinal force, directed towards the propagation of the wave, which can be identified as the pressure of gravitational wave. The magnitude of gravitational radiation pressure can be determined by means of the second order geodesic deviation equation as the equation of motion of the point masses, or, alternatively, on the basis of the energy-momentum conservation law of the absorbed gravitons. We consider in this Section a system of coupled point masses with constant rest masses which interact with an impinging gravitational wave. The gravitational attraction forces between the point masses are neglected. We express the equation of motion of the point masses as the deviation equations and substitute the obtained separations and velocities of the point masses into the equation of motion of the center of mass, written in the second approximation. The result gives us the nonvanishing longitudinal acceleration of the considered system.

The fact that gravitational waves of type N, alongside with the transverse tidal acceleration also evoke a longitudinal tidal field has been mentioned earlier in few articles (see e.g. Ref. [82] and the papers referred to in Ref. [63]). Nevertheless the fact can not be considered as a matter of general knowledge. This is so, perhaps, because the underlying physical mechanism has not been made clear enough.

As it is well known, in the field of a N type gravitational wave propagating in the direction of the u^1 axis the tidal forces in the linear approximation in the separation vector η^ρ ($\rho = 1, 2, 3$) are purely transverse:

$$
\begin{aligned}
F^1 &= 0 + \mathcal{O}(|\eta^\rho|^2) \ , \\
F^N &= R^N_{00M}\eta^M + \mathcal{O}(|\eta^\rho|^2) \ ; \quad (N, M = 2, 3) \ .
\end{aligned}
$$

However, in the quadratic approximation in the separation vector the gravitational wave gives rise to a longitudinal tidal force [63]:

$$
\begin{aligned}
Q^1 &= -\frac{1}{2}\eta^M\eta^N\frac{d}{dt}R^1_{MN0} - 2\frac{d\eta^M}{dt}\eta^N R^1_{MN0} + \mathcal{O}\left(|\eta^\rho|^3\right) \ , \\
Q^M &= R^M_{00N}\eta^N - \eta^N\frac{d}{dt}R^M_{1N0} - 2\eta^N\frac{d\eta^1}{dt}R^M_{1N0} + \mathcal{O}\left(|\eta^\rho|^3\right) \ ,
\end{aligned}
$$

where t is the local time coordinate (e.g. the proper time of the Fermi frame).

In the physical sense the most fundamental effect caused by the longitudinal tidal force is the gravitational radiation pressure, i.e. the time-averaged longitudinal tidal force $< Q^1 >$ whose magnitude is determined by the power ΔE absorbed (or emitted) per unit mass of the receiver so that $\Delta E =< Q^1 >$. (We use units in which the velocity of light $c = 1$.) This effect provides the physical foundation of the longitudinal tidal force and demonstrates that if the longitudinal tidal force were absent the quanta of gravitational field would not adhere to the special relativistic relationship between energy and momentum $\Delta E = \Delta p$.

Finally we note that the quadratic tidal forces Q^ρ are completely transverse in the sense that the two particles whose separation is in the direction of propagation of the gravitational wave (the u^1-direction: $\eta^1 \neq 0$, $\eta^2 = \eta^3 = 0$) have constant coordinate separation as well as constant proper separation.

We use here the non-relativistic second order deviation equation, derived in Ref. [63], which describes the relative acceleration of two point masses with almost parallel world velocities:

$$
\frac{D^2\eta^i}{dt^2} = -R^i_{jkl}\frac{d\bar{u}^j}{dt}\eta^k\frac{d\bar{u}^l}{dt} - 2R^i_{jkl}\frac{D\eta^j}{dt}\eta^k\frac{d\bar{u}^l}{dt} +
$$

$$+\frac{1}{2}\Big(R^i_{jlk;m}-R^i_{lmj;k}\Big)\frac{d\bar{u}^k}{dt}\frac{d\bar{u}^j}{dt}\eta^l\eta^m+\frac{f^i}{m}-\frac{D^2\bar{u}^i}{dt^2}+\mathcal{O}_3\ . \tag{60}$$

where $\bar{u}^i$ are the coordinates of the first point mass, and η^i are the components of their separation vector, f^i are the components of non-gravitational forces acting on the second point mass and t is the proper time of the clock, co-moving with the first point mass.

3.2 Pressure of gravitational waves

In this Subsection the pressure exerted by a linearly polarized, monochromatic and weak gravitational plane wave is examined. We label the coordinates of the spacetime by u^i, $i = 0, ..., 3$, and consider a wave which propagates in u^1-direction. The disturbances h_{ij} of the components of the metric tensor $g_{ij} = \eta_{ij} + h_{ij}$ caused by this wave are

$$h_{22} = -h_{33} = A\sin[\omega(u^0 - u^1)] \ , \quad A << 1. \tag{61}$$

In this situation, the components of the curvature tensor in the TT-gauge we use in our calculations are

$$\begin{aligned} R_{0202} = R_{1220} &= (A/2)\omega^2 \sin(\omega(t - u^1)) \\ R_{0203} &= (A/2)\omega^2 \cos(\omega(t - u^1)) \ , \end{aligned} \tag{62}$$

(see e.g. Ref. [83], p. 218.)

To investigate the pressure of gravitational radiation, exerted on a system of coupled point masses, we write all the necessary equations in the frame of reference of an accelerated observer (further called simply as "observer"), moving along a world line $\mathcal{O} : u^i = u^i(t)$ of the center of mass of the mentioned system. This means, we use the Fermi normal coordinates, built along the world line $\mathcal{O}$, where t is the observer's proper time. The Latin indices denote here the 4-coordinates $(i, j, .. = 0, ..., 3)$, the Greek indices indicate the coordinates of 3-space. The subscripts A, B, etc. label the point masses of the considered system and the summation convention does not apply to these subscripts. A point over a symbol means here the covariant differentiation along the world line $\mathcal{O}$ with respect to t. The Fermi coordinate system is described, for example, in Ref. [81] §13.6. The system is defined only in the closest neighbourhood of the observer's world line, i.e. within the corresponding world tube. This is the best approximation for a system of Newtonian laboratory coordinates, possible to define in a curved spacetime. The coordinates, four-velocity and four-acceleration of the observer read in his Fermi coordinates as follows, see Ref. [81]:

$$u^i = (t, 0, 0, 0), \quad \dot{u}^i = (1, 0, 0, 0), \quad \ddot{u}^i = (0, \ddot{u}^\rho), \tag{63}$$

where $\ddot{u}^\rho$ is the acceleration of the observer relative to an instantaneous inertial frame of reference. The metric in the Fermi coordinates is along the world line described by the Minkowskian metric tensor

$$g_{00} = -g_{\rho\rho} = 1, \quad g_{ij} = 0 \quad \forall i \neq j. \tag{64}$$

In the case of a non-rotating accelerated observer the only nonvanishing Christoffel symbols at the observer's location are

$$\Gamma^\rho_{00} = \Gamma^0_{\rho 0} = \Gamma^0_{0\rho} = \ddot{u}^\rho. \tag{65}$$

Their derivatives, determined in the Fermi coordinates by the curvature of spacetime and the observer's 3-acceleration also do not vanish along its world line. One can find the components of the metric tensor near this world line in the Fermi coordinates, the Christoffel symbols etc. in the fourth approximation in Ref. [85]. Because in the present Section all tensors and covariant derivatives are calculated along the world line $\mathcal{O}$, the metric and Christoffel symbols only along this world line are needed.

We consider now a system of test particles or point masses m_A, $A = 1, ..., N$. As the relative motion of point masses is non-relativistic, i.e. their relative velocities with respect to the observer are much less than the velocity of light, we can suppose that $m_A = \text{const}\ \forall A$ and use the ordinary Newtonian definition of the center of mass. The position vectors, or separation vectors of all point masses have in the observer's frame of reference only the spatial components, as well the nongeometrical forces acting on the point masses and the observer's acceleration.

Let a weak gravitational field and a field of phenomenological nongeometrical forces $f^i(u^j)$ act on the considered system. We suppose, that the geodesic separation vectors H_A with the components η^ρ_A and the velocities $\dot{H}_A$ with the components $\dot{\eta}^\rho_A$ of point masses m_A with respect to the observer, and also the observer's acceleration in the instantaneous inertial frame of reference and the nongeometrical forces f^ρ_A, acting on the point masses m_A, are quantities of the first order of smallness:

$$\eta^i_A \sim \frac{D\eta^\rho_A}{dt} \sim \ddot{u}^\rho \sim f^\rho_A \leq \mathcal{O}\left(\frac{H}{\lambda}\right). \tag{66}$$

Here λ is the linear measure, characteristic for a gravitational field (in our case the length of gravitational wave). Then, due to the last formula and formula (65), we have

$$\Gamma^\rho_{00} = \Gamma^0_{\rho 0} = \Gamma^0_{0\rho} \leq \mathcal{O}\left(\frac{H}{\lambda}\right) . \tag{67}$$

Thus the replacement of the covariant derivatives by the ordinary ones within the second order small terms causes an error of the third order of smallness.

One can show, that under the above-mentioned assumptions the equations of motion for a point mass m_A have in the Fermi normal coordinates the following form (see Ref. [63]):

$$\ddot{\eta}^\rho_A \;=\; T^\rho_A + L^\rho_A + \frac{f^\rho_A}{m_A} - \ddot{u}^\rho + \mathcal{O}\left(\frac{H}{\lambda}\right)^3 . \tag{68}$$

Let us repeat, that all the quantities here are calculated along the observer's world line $\mathcal{O}$. The minus sign in front of $\ddot{u}^\rho$ means, that this force is the inertial one. Due to formula (60), the nonvanishing components of the tidal forces in the first and second approximations in the Fermi coordinates read

$$
\begin{aligned}
T^\rho_A &= R^\rho_{0\mu 0}\eta^\mu_A \,, & (69)\\
L^\rho_A &= 2R^\rho_{\mu\nu 0}\dot{\eta}^\mu_A\eta^\nu_A + \frac{1}{2}\left(R^\rho_{0\mu 0,\nu}\eta^\nu_A - R^\rho_{\mu\nu 0,0}\eta^\nu_A\right)\eta^\mu_A \,. & (70)
\end{aligned}
$$

Here the components of the curvature tensor and their derivatives are calculated along the observer's world line $\mathcal{O}$. Taking now into account, that the velocities of the test particles are non-relativistic, we can use the ordinary definition of the center of mass, familiar in the Newtonian mechanics:

$$\sum_A \eta^\rho_A m_A \quad = \quad 0 \quad \forall A \,. \tag{71}$$

On the assumption that $m_A = \text{const}\ \forall A$, we get from formula (68), using equations (69) - (71), that in the Fermi coordinates based on the world line of the center of mass, the equations of motion of the center of mass itself take the following form:

$$\ddot{u}^\rho \sum_A m_A = \sum_A m_A L^\rho_A + \sum_A f^\rho_A + \mathcal{O}\left(\frac{H}{\lambda}\right)^3 \,. \tag{72}$$

It must be pointed out, that the transforming of the equation (60) from the global coordinates into the Fermi coordinates gives to every term a clear physical meaning and makes the content of the equation easier to understand. On the other hand, in general the equations become in the observer's Fermi coordinates more complicated. This is not always the case, a good counterexample in in Publication III [69], where the state equations describing a physical system have just been simplified using the coordinate system comoving with the observer instead the stationary and global coordinate system. Taking instead some basis vector fields their Lie derivatives with respect to observer's velocity vector field allows even to reduce the number of the parameters necessary to describe the changing rates of the states and therefore simplifies the system of the state equations.

In what follows our aim is to find the magnitude of the pressure, caused by the gravitational waves, on an extended body (see, e.g. Ref [84]). In order to simplify the calculations, we model the extended body by the Weber oscillator, i.e. an oscillator made of two test particles with equal rest massest on the ends of an elastic spring which also dissipates energy. We will calculate directly, on the basis of the equations of motion of the two test particles, the pressure exerted by gravitational waves on the Weber oscillator. Let us emphasize that we will consider the motion of the oscillator not only in the linear approximation in the separation vector, as is usually done, but also in the second approximation. As the masses of the particles are equal

$$m_1 = m_2 := m \,, \tag{73}$$

it follows from the definition of the center of mass (71) , that $\eta^{\rho}_{A=1} = -\eta^{\rho}_{A=2}$. In this case formula (70) yields

$$L^{\rho}_{1} = L^{\rho}_{2} : = L^{\rho}. \tag{74}$$

Consequently, the equation of motion for the center of mass (72) takes the following form:

$$\ddot{u}^{\rho} \quad = \quad L^{\rho} + \frac{1}{2m}\sum_{A} f^{\rho}_{A} + \mathcal{O}\left(\frac{H}{\lambda}\right)^{3}. \tag{75}$$

We suppose now, that the sum of all nongeometrical forces equals to zero

$$\sum_{A} f^{\rho} := 0\ . \tag{76}$$

Then the equation of motion of the center of mass (75) reduces to

$$\ddot{u}^{\rho} \quad = \quad L^{\rho} + \mathcal{O}\left(\frac{H}{\lambda}\right)^{3}. \tag{77}$$

Note that the obtained equations of motion of the center of mass involve the second order tidal forces L^{ρ}, which have a nonvanishing longitudinal component.

Substituting the last equation into equation (68), we obtain the final form of the equation of relative motion of the test particles with respect of the center of mass

$$\ddot{\eta}^{\rho}_{A} \quad = \quad T^{\rho}_{A} + f^{\rho}_{A} + \mathcal{O}\left(\frac{H}{\lambda}\right)^{3}. \tag{78}$$

Note that the last equations are valid up to the second order terms included, however they contain only the first order terms while the second order terms are missing.

In this case from formulae (63), (65) and (75) follows that at the observer's location

$$\Gamma^{\rho}_{00} = \mathcal{O}\left(\frac{H}{\lambda}\right)^{2}. \tag{79}$$

This means that now the replacement of the covariant derivatives by the ordinary ones within the first order small terms causes only an error of the third order of smallness.

As said, the gravitational wave is supposed to propagate along the u^1-axis. Let the test particles be placed on the u^2-axis, i.e., perpendicularly to the propagation direction of gravitational wave. We can write

$$\eta^{1}_{1} = \eta^{1}_{2} = \eta^{3}_{1} = \eta^{3}_{2} = 0, \quad \eta^{2}_{1}(t) = -\eta^{2}_{2}(t) = \eta^{2}(t). \tag{80}$$

Then formulae (77) and (78), taking into account also formulae (69) and (70), reduce to

$$\begin{aligned} \ddot{\eta}^2 &= T_A^2 + f^2 + \mathcal{O}\left(\frac{H}{\lambda}\right)^3 = -R^2_{020}\eta^2 + f^2 + \mathcal{O}\left(\frac{H}{\lambda}\right)^3, \qquad (81) \\ \ddot{u}^1 &= L_A^1 + \mathcal{O}\left(\frac{H}{\lambda}\right)^3 \\ &= -2R^1_{220}\dot{\eta}^2\eta^2 - \frac{1}{2}R^1_{220,0}(\eta^2)^2 + \mathcal{O}\left(\frac{H}{\lambda}\right)^3. \qquad (82) \end{aligned}$$

Denoting the spring and damping constant by k and γ, respectively, and considering the component $\eta^\rho(t)$ of the separation vector H as the sum of constant η^ρ and infinitely small quantity $\xi^\rho(t)$, we write the equation of motion (81) in the following form;

$$\ddot{\xi}^2 + \frac{\gamma}{m}\dot{\xi}^2 + \frac{k}{m}\xi^2 = \frac{A}{2}\omega^2\eta^2\sin(\omega t) + \mathcal{O}\left(\frac{|\xi^2|}{|\eta^2|}\right), \qquad (83)$$

where

$$\eta^2(t) = \eta^2 + \xi^2(t), \quad |\xi^2| << |\eta^2|. \qquad (84)$$

The general solution of equation (83) is:

$$\begin{aligned} \xi^2(t) &= \xi^2_{max}e^{-\frac{t}{\tau}}\sin\left(\sqrt{km^{-1} - \tau^{-2}} \quad t\right) \\ &+ \frac{A}{2}\omega^2\eta^2 \cdot \frac{1}{\sqrt{(km^{-1} - \omega^2)^2 + 4\omega^2\tau^{-2}}}\sin(\omega t + \alpha^*), \end{aligned} \qquad (85)$$

here

$$\alpha^* = -\arctan\frac{\gamma\omega m^{-1}}{km^{-1} - \omega^2}, \quad \tau = \frac{2m}{\gamma}. \qquad (86)$$

where α^* denotes the phase difference between the oscillation of m and the gravitational wave. After a time interval, much longer than the relaxation time τ, we can write the last equation as

$$\eta^2(t) = \eta^2\left[(1 + a\sin(\omega t + \alpha^*)\right], \qquad (87)$$

where

$$a = \frac{1}{2}\frac{A\omega^2}{\sqrt{(km^{-1} - \omega^2)^2 + 4\omega^2\tau^{-2}}}. \qquad (88)$$

Differentiating equation (87) with respect to t gives the velocity and acceleration of m relative to the center of mass. Substituting the components of the separation vector (87) and its time derivatives into equation (82), we will obtain after short calculations the following expression for the longitudinal acceleration of considered system:

$$\begin{aligned}\ddot{u}^1 \;=\; & -\frac{1}{2}\left[\frac{A}{2}\omega^3(\eta^2)^2\cos(\omega t) + \frac{A}{2}\omega^3 a(\eta^2)^2\sin(2\omega t)\cos\alpha^* + \right. \\ & \left. + A\omega^3 a(\eta^2)^2\cos(2\omega t)\sin\alpha^* - A\omega^3 a(\eta^2)^2\sin^2(\omega t)\sin\alpha^*\right]. \end{aligned} \tag{89}$$

Averaging the last formula over one period $\dfrac{2\pi}{\omega}$ we obtain the mechanical momentum, absorbed by the system from gravitational waves during one period

$$\Delta p = -Am\pi\omega^2 a(\eta^2)^2\sin\alpha^*.$$

From equations (86) and (88) we have

$$\Delta p = \frac{1}{2}\cdot\frac{A^2\pi\omega^5(\eta^2)^2\gamma}{\left(km^{-1}-\omega^2\right)^2 + 4\omega^2\tau^{-2}}. \tag{90}$$

On principle, by deriving the last formula we used only the non-relativistic geodesic deviation equation as equation of motion of the test particles taking into account also the second order small terms.

Using this method the following question may arise. The components of the Riemann curvature tensor are calculated in the TT-gauge, but the geodesic deviation equation containing them is written in the frame of reference of a local observer. According to Rakhmanov, see Ref. [80], the calculations in the TT- and local coordinates may lead to different results. This is not the case, if we use the first approximation and the condition $\lambda >> |H|$ is satisfied, i.e. the length of the gravitational wave is much greater than the modulus of the separation vector. Consequently, our considerations are valid until formula (88), included. Formula (89) as the second order approximation is written also in the frame of reference of a local observer, but contains the components of curvature tensor of TT-gauge, i.e. can lead to wrong results. To prove the validity of our considerations, we obtain the same result using an alternative method.

Namely, this result can be derived directly from the definition of kinetic energy, using only the terms with the first order of smallness. Because the rest mass of graviton equals to zero, then the momentum, given to the system, must be equal to the energy, absorbed from gravitational waves during the same time. Because the energy of transverse oscillation of the system is much greater than its kinetic energy of longitudinal movement, we can say, that the energy, transferred to the system during one period, equals in the first approximation to the work, done against the dissipative forces during the same time. For two point masses, this work is

$$\mathcal{A} = 8\int_0^{\xi^2_{max}} \gamma\dot{\xi}^2 d\xi^2 = 2\pi\gamma(\eta^2)^2\omega a^2,$$

and, due to (88) we will get

$$\mathcal{A} = \Delta p. \tag{91}$$

Formula (89) gives us after averaging over one period the average longitudinal acceleration

$$\bar{\ddot{u}}^1 = \frac{1}{8m}\frac{A^2\omega^6(\eta^2)^2\gamma}{\left(km^{-1}-\omega^2\right)^2+4\omega^2\tau^{-2}}. \tag{92}$$

We obtained the same result using the second approximation of the geodesic deviation equation in the observer's frame of reference, but containing the components of Riemann curvature tensor of TT-gauge. Consequently, in this case the possible discrepancy due to use of the two different coordinate systems does not occur.

It is obvious from the last formula, that the longitudinal acceleration of an extended system of point masses caused by gravitational waves, depends quadratically on the transverse extension of the system and on the amplitude of the waves. In the case of resonance, when $\omega = \sqrt{\frac{k}{m}}$, it depends also on the fourth power of the frequency. The presence of dissipative forces is necessary, to change the phase of oscillations of the point masses. We used here the Weber oscillator to simplify our calculations, but the result is valid also in the case of an arbitrary extended physical body, included the interferometrical detectors.

3.3 The relativistic case

In the previous section we studied a system of coupled point masses m_1 and m_2, which moves at a non-relativistic speed with respect to the source of gravitational waves. Similarly, a source of gravitational waves, which moves at the relativistic speed with respect to the earth, is conceivable. However, unfortunately, at present we are not able to provide any particular example. Thus in this Section we will examine a more complicated case when the system moves along the direction of propagation of waves and its 3-velocity β is relativistic. Then the 4-velocity of the center of mass has in global coordinates the following components

$$\dot{u}^i = \frac{(1,\beta,0,0)}{\sqrt{1-\beta^2}} . \tag{93}$$

Let the point masses be again placed on the u^2-axis, then the components of the separation vectors of point masses m_A with respect to the center of mass are

$$\vec{\eta}_A = (0,0,\pm\eta_A(t),0) , \tag{94}$$

and the change rates of the separation vectors read

$$\frac{d\vec{\eta}_A}{dt} = \left(0,0,\frac{d\eta_A(t)}{dt},0\right) , \tag{95}$$

where t is now the proper time of an observer comoving with the center of mass. We denote the global time by u^0.

If the condition $\frac{d\eta}{dt} << 1$ is satisfied, the following considerations are valid. We do not need to distinguish the proper times of the center of mass and the particles m_A. Due to formulae (62) and (95) we obtain from the deviation equation (60) in the first approximation the relative acceleration of the point masses m_A with respect to the center of mass

$$\frac{d^2\eta_A}{dt^2} = \frac{A}{2}\frac{1-\beta}{1+\beta}\eta_A\omega^2\sin\left[\omega(u^0-u^1)\right] + \frac{f_A^2}{m} + \mathcal{O}_2 . \tag{96}$$

Taking into account the well known relation between the global time u^0 and the proper time of the center of mass

$$u^0 = \frac{t}{\sqrt{1-\beta^2}} ,$$

and also the formula $u^0-u^1 = u^0(1-\beta)$ we obtain for the relative acceleration of the point mass m_A with respect to the center of mass the following equation:

$$\frac{d^2\eta_A}{dt^2} = \frac{A}{2}\frac{1-\beta}{1+\beta}\eta_A\omega^2\sin\left(\omega t\sqrt{\frac{1-\beta}{1+\beta}}\right) + \frac{f_A^2}{m} + \mathcal{O}_2 . \tag{97}$$

Consequently – due to the Doppler effect, the frequency of gravitational wave in the frame of reference, comoving with the center of mass, depends on its 3-velocity β. If the system of point masses moves towards the source of gravitational waves, $\beta < 0$ and the relative acceleration of point masses m_A is greater than in the case of stationary system; if the system moves away from source, $\beta > 0$ and the relative acceleration is less than in the case of stationary system. Analogously to equation (83), the equation of relative motion with respect to the center of mass reads in the case of moving system as follows:

$$\ddot{\xi}_A + \frac{\gamma}{m}\dot{\xi}_A + \frac{k}{m}\xi_A = \frac{A}{2}\frac{1-\beta}{1+\beta}\omega^2\eta_A \sin\left(\omega t\sqrt{\frac{1-\beta}{1+\beta}}\right) , \tag{98}$$

where again the u^2-component of the separation vector of m_A consists of a constant term η_A and the time-dependent term $\xi_A(t)$, that is

$$\eta_A(t) = \eta_A + \xi_A(t), \quad |\xi_A| << |\eta_A| .$$

We denote the "apparent" frequency of gravitational waves for an observer comoving with the center of mass by

$$\Omega \;=\; \omega\sqrt{\frac{1-\beta}{1+\beta}} , \tag{99}$$

and use the scheme, described in the previous section, see formulae (83)-(92), in the Fermi coordinates attached to the center of mass. Thus we obtain analogously to formula (87) for the case of moving system

$$\eta_A(t) \;=\; \eta_A\left[1 + \hat{a}\sin(\Omega t + \hat{\alpha}^*)\right] , \tag{100}$$

where

$$\begin{aligned} \hat{a} &= \frac{A}{2}\frac{\Omega^2}{\sqrt{(km^{-1}-\Omega^2) + 4\Omega^2\tau^{-1}}} , \\ \tau &= \frac{2m}{\gamma} , \\ \hat{\alpha}^* &= -\arctan\frac{\gamma\Omega m^{-1}}{km^{-1}-\Omega^2} . \end{aligned} \tag{101}$$

Then the 3-velocity of the point mass m_A relative to the center of mass is

$$\frac{d\eta_A}{dt} \;=\; \hat{a}\Omega\eta_A\cos(\Omega t + \hat{\alpha}^*) . \tag{102}$$

In the case of resonance, if $\Omega^2 = \dfrac{k}{m}$, we have

$$\begin{aligned} \eta_A(t) &= \eta_A\left[1 + \frac{A}{4}\Omega\tau\cos(\Omega t)\right] , \\ \frac{d\eta_A}{dt} &= -\frac{A}{4}\omega^2\tau\frac{1-\beta}{1+\beta}\eta_A\sin(\Omega t) . \end{aligned} \tag{103}$$

As we see, under the influence of a weak gravitational wave and small η_A, the 3-velocities of the point masses m_A relative to the center of mass are non-relativistic even in the case, if for example the fraction $\dfrac{1-\beta}{1+\beta} = 10^3$. Consequently, if the 3-velocity β of a system in global coordinates approaches to the velocity of light, we can consider the quantities $\dfrac{d\eta_A}{dt}$ infinitely small and do not need to distinguish the proper times of the point masses m_A and the center of mass.

Using now formula (100), we can calculate analogously to formula (91) the work, done by gravitational waves during one period by shifting point masses m_A against the dissipative forces:

$$\mathcal{A} \;=\; \frac{1}{2}\frac{A^2\pi\Omega^5\eta^2\gamma}{(km^{-1}-\Omega^2)+4\Omega^2\tau^{-2}}\,. \qquad (104)$$

As already mentioned in the previous section, it equals the momentum, absorbed by the system from gravitational waves during one period. Taking into account formula (99) we see, that the last expression for $\mathcal{A}$ contains the factor $\sqrt{\left(\dfrac{1-\beta}{1+\beta}\right)^5}$, i.e. if the system moves at a relativistic 3-velocity towards the source of gravitational waves, the work done during one period against the dissipative forces will be much greater than in the case of stationary system.

For example, if the system of point masses moves towards the source of gravitational waves at a relativistic 3-velocity $\beta = -0.99$, then due to formula (99) the frequency of gravitational wave in a comoving frame of reference will be

$$\Omega \;\approx\; 14\omega\,, \qquad (105)$$

i.e. is about 14 times greater than the proper frequency of the source. Although the wavelength is also 14 times shorter, the condition $\lambda >> H$ still holds. In formula (97) the first term on the right hand side is nearby 200 times greater than in case of a stationary system of point masses. It means – the tidal forces caused by gravitational waves are 200 times greater. In the case of resonance, the system (103) implies, that if a system of point masses moves at the 3-velocity $\beta = -0.99$ the amplitude of oscillating point masses m_A is 14 times greater and their velocity relative to the center of mass is 200 times greater than in the case of a stationary system. Because A as an amplitude of gravitational waves is very small, the velocity of point masses m_A relative to the center of mass is still non-relativistic. According to the formulae (104) and (105), the work done by gravitational waves against the dissipative forces during one period (and also the momentum absorbed by the system of point masses during one period) is 2800 times greater than in the case of a stationary system.

If the system of point masses approaches to the source with an arbitrary relativistic 3-velocity β, we will get

$$\Omega \approx \sqrt{\frac{2}{1-|\beta|}}\,, \tag{106}$$

the amplitude of oscillations of point masses m_A will be $\sqrt{\frac{2}{1-|\beta|}}$ times and their velocity relative to the center of mass will be $\frac{2}{1-|\beta|}$ times greater than in the case of a stationary system. Again, due to the smallness of A we see, that even if $\beta \approx 1$ the velocities of m_A relative to center of mass are non-relativistic and it simplifies our considerations. In the case of resonance the work $\mathcal{A}$ done by gravitational waves during one period against the dissipative forces (also the momentum p absorbed by a system during one period) will be approximately $\sqrt{\left(\frac{2}{1-|\beta|}\right)^3}$ times greater as in case of a stationary system. Now the average longitudinal force in the frame of reference comoving with the center of mass reads

$$F = \frac{p}{T} = \frac{\Omega p}{2\pi}\,. \tag{107}$$

This is $\frac{4}{(1-|\beta|)^2}$ times greater than in the case of a stationary system.

Consequently – if the system of point masses approaches to the source of gravitational waves at a relativistic 3-velocity, the effects caused by absorption of gravitational waves will be much greater than in case of a stationary system. We need only to remember, that the resonance frequency of a stationary system of point masses must equal to the proper frequency of the source ω, the resonance frequency of a moving system must be $\Omega \approx \sqrt{\frac{2}{1-|\beta|}}$.

3.4 The superposition of gravitational and electromagnetic waves

In this section we consider a situation, when simultaneously with a gravitational wave, a linearily polarized monochromatic electromagnetic wave is propagating in the same direction. The equations of motion of the system of coupled point masses, obtained in subsection 4.2, will be modified for the case of charged point masses. The pressure of electromagnetic waves on the charged test particles is added to the pressure of gravitational waves on the test particles, and additionally some extra "mixed" terms will appear, as the interference between the movements induced by the simultaneously acting electromagnetic and gravitational fields will take place. Here we consider only the case, if the system of point masses moves at the nonrelativistic 3-velocity.

An electromagnetic wave, propagating in the same direction as the gravitational wave, is described by the electromagnetic field tensor with the following non-zero components:

$$\Phi_2^0 = \Phi_0^2 = \Phi_2^1 = -\Phi_1^2 = \Phi \sin[\tilde{\omega}(x^0 - x^1)] \ . \tag{108}$$

We consider here the same system of point masses as in the previous section. Now the point masses m_1 and m_2 have the charges q and $-q$, respectively. Their electrical attraction in the state of equilibrium is balanced by the elastic forces of the spring. In a point mass deviates from the point of equilibrium, ξ^2 an extra electric force appears, expressed in the first approximation as

$$-\frac{\partial}{\partial \eta^2} \frac{q^2}{16\pi\varepsilon_0 (\eta^2)^2} \xi^2 = \frac{q^2}{8\pi\varepsilon_0 (\eta^2)^3} \xi^2 \ ,$$

and is directed away form the point of equilibrium. So we can define the *corrected spring coefficient* of the spring as

$$\bar{k} = k - \frac{q^2}{8\pi\varepsilon_0 (\eta^2)^3} \ . \tag{109}$$

The last formula tells us, that the proper frequency of oscillations of an electric dipol must be greater as the one of the system of neutral point masses (we examine here only the case $\bar{k} > 0$). Then we will obtain the equation of relative motion for the test masses, corresponding to the deviation $\xi^2 << \eta^2$:

$$\ddot{\xi}^2 + \frac{\gamma}{m}\dot{\xi}^2 + \frac{\bar{k}}{m}\xi^2 \ = \ \frac{A}{2}\omega^2(\eta^2)\sin(\omega t) + \frac{q}{m}\Phi \sin(\tilde{\omega} t) \ , \tag{110}$$

and its solution

$$\xi^2(t) \ = \ \xi^2_{max} e^{-\frac{t\gamma}{2m}} \sin\left(\sqrt{\bar{k}m^{-1} - \tau^{-2}} \quad t\right) +$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\frac{A\omega^2\eta^2}{\sqrt{\left(\bar{k}m^{-1}-\omega^2\right)^2+4\omega^2\tau^{-2}}}\sin(\tilde{\omega}t+\alpha^*) \\
&+ \quad \frac{qm^{-1}\Phi}{\sqrt{\left(\bar{k}m^{-1}-\tilde{\omega}^2\right)^2+4\tilde{\omega}^2\tau^{-2}}}\sin(\omega t+\alpha^{**}) ,
\end{aligned} \tag{111}
$$

where

$$
\alpha^* = -\arctan\frac{\gamma\omega m^{-1}}{\bar{k}m^{-1}-\omega^2} , \quad \alpha^{**} = -\arctan\frac{\gamma\tilde{\omega}m^{-1}}{\bar{k}m^{-1}-\tilde{\omega}^2} . \tag{112}
$$

We also introduce the following notations:

$$
a = \frac{1}{2}\frac{A\omega^2}{\sqrt{\left(\bar{k}m^{-1}-\omega^2\right)^2+4\tau^{-2}\omega^2}} , \tag{113}
$$

$$
b = \frac{qm^{-1}\Phi}{\sqrt{\left(\bar{k}m^{-1}-\tilde{\omega}^2\right)^2+4\tau^{-2}\tilde{\omega}^2}} . \tag{114}
$$

Taking into account, that the first term in formula (111) vanishes quickly, the separation vector can be written as

$$
\eta^2(t) = \eta^2 + a\eta^2\sin(\omega t+\alpha^*) + b\sin(\tilde{\omega}t+\alpha^{**}) . \tag{115}
$$

The electromagnetic terms in the last system are usually much larger than the gravitational terms. Consequently, we need to take into account the quadrics of electromagnetic terms, and the formula for a longitudinal acceleration (72) of the system takes the following form

$$
\begin{aligned}
\ddot{u}^1 = &-\frac{1}{2}\Big[4R^1_{220}\dot{\eta}^2(t)\eta^2(t) + R^1_{220,0}\left(\eta^2(t)\right)^2 - \\
&-2\frac{q}{m}\Phi^1_2\dot{\eta}^2(t) - 2\frac{q}{m}\Phi^1_0\left(1+\frac{1}{2}(\dot{\eta}^2(t))^2\right)\Big] + \mathcal{O}\left(\frac{H}{\lambda}\right)^3 .
\end{aligned} \tag{116}
$$

By means of equations (61), (62), (108) and (115) we will obtain a cumbersome formula for the longitudinal acceleration, which depends on time and various parameters as indicated below:

$$
\kappa^1 = \Psi(t;\omega,\tilde{\omega},\alpha^*,\alpha^{**},a,b,\eta^2) .
$$

We pay a special attention to the case $\omega = \tilde{\omega}$, i.e. the frequency of gravitational wave equals to that of electromagnetic wave. Then due to equations

(112) also $\alpha^* = \alpha^{**}$ and averaging the last formula over period $\frac{2\pi}{\omega}$ gives us the momentum, imparted to the system by simultaneous action of eletromagnetic and gravitational waves during one period.

$$\begin{aligned}\Delta p^1 &= -m\left(A\eta^2\omega^2 + 2\frac{q}{m}\Phi\right)(a\eta^2{+}b)\,\omega\,\sin\alpha^*\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t)dt = \\ &= \frac{\pi\gamma\omega}{\left(\bar{k}m^{-1} - \omega^2\right)^2 + 4\tau^{-2}\omega^2}\times \\ &\quad \times\left[\frac{A^2}{2}(\eta^2)^2\omega^4 + 2\frac{q^2}{m^2}\Phi^2 + A\Phi\frac{q}{m}\omega^2\eta^2\right]\ .\end{aligned} \tag{117}$$

The first term here is the momentum, given to the system by the gravitational wave, which equals to the momentum, obtained from equation (91). Only instead the spring coefficient k one needs to take the corrected spring coefficient $\bar{k}$ from equation (109). The second term is the momentum, given only by the electromagnetic wave. The last term describes the "combined" oscillation, caused by both electromagnetic and gravitational waves.

For the charged point mass, usually, the acceleration, caused by electromagnetic wave, is much greater than the acceleration, caused by gravitational waves. It means, that we can in this case drop the first term in the brackets on the right side of (117). The mechanical momentum, caused by "combined" influence of the both waves, must be then much larger than the momentum, caused by "pure" gravitational wave. Consequently, it possible to amplify the gravitational waves, using the electromagnetic waves with the same frequency. However, there remains the problem, how to distinguish the pure electromagnetic momentum and the "combined" momentum, when the last one is much smaller than the first one.

Averaging the last formula over one period $\frac{\omega}{2\pi}$, and taking into account equations (113) and (114), gives us for the mechanical momentum

$$\begin{aligned}\Delta p^1 &= 8\int_0^{\xi_{max}}\gamma\dot{\xi}^2 d\xi \\ &= 2\gamma\omega^2(a\eta^2{+}b)^2\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t)dt = 2\pi\gamma\omega(a\eta^2{+}b)^2.\end{aligned} \tag{118}$$

The last result is obtained again from the non-relativistic geodesic deviation equation as equation of motion of the test particles, whereby the second order small terms are included. We can reobtain the same result directly from the definition of kinetic energy, as in the previous section. The momentum of graviton or photon equals to its energy, i.e. the energy, given to the system during one period, must be also calculated by means of equation (117). We can conclude, that this energy equals to the work, done against dissipative forces during one period.

$$\begin{aligned}
\mathcal{A} &= 8\int_0^{\xi_{max}} \gamma\dot{\xi}^2 d\xi = 2\gamma\omega^2(a\eta^2+b)^2\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t)dt \\
&= 2\pi\gamma\omega(a\eta^2+b)^2.
\end{aligned}$$

It gives us again

$$\Delta p^1 = \mathcal{A}.$$

3.5 Conclusions

In the present Section the pressure of a weak plane monochromatic gravitational wave on an elastic body is examined. The momentum, imparted to the body, is calculated by means of the nonrelativistic geodesic deviation equation, using the second approximation with respect to the deviation vector. It is found that the work done by gravitational waves against the dissipative forces of the body is equal to the obtained momentum. This result reconfirms that the rest mass of gravitons equals zero. Next we consider the case when the Weber oscillator moves along the direction of propagation of gravitational waves and its velocity is relativistic. The effects caused by absorption of gravitational waves are considerably amplified by comparison with a stationary system. Finally, the influence of electromagnetic and gravitational wave with the same frequency and propagation direction on an elastic body was examined. As could be expected, calculations showed that the rest mass of gravitons and photons is zero.

4 Tuning an "accordion rod" the curvature of spacetime

4.1 Introduction

The aim of the present Section is to study the effects generated by accelerated motion of an extended body in curved spacetimes. This problem is in the general case mathematically and physically quite intricate: it requires elaborating and solving of non-linear intergro-differential equations of motion. Under certain conditions, the equations of motion can be expressed as multipole expansions of the energy-momentum tensors of the extended body and of the acting fields. Truncating then the multipole series at a certain order one can find the solution with the required complexity and precision. For example, to describe the nonspherical deformations of an extended body one has at least to consider the quadrupole approximation which is still quite involved [87].

To bypass somewhat the mathematical complexity of the problem of motion of an extended body, some simplified models have been proposed and examined. For example, the test rigid rod by Faulkner and Flannery [89], where the authors use the d'Alembert principle to determine the tidal forces in the rigid rod moving in a curved spacetime, and the accordion rod by Mould [70], accelerated in the direction parallel to itself in the Minkowski spacetime.

In the present Section, to avoid the uncomprehensive formulae, we mainly confine ourselves to 2-dimensional curved spacetimes. The 4-dimensional case is briefly treated in the last Section. To study the motion of an extended body, we use the model of an accordion rod introduced by Mould [70] and Rindler [92], generalizing it to a curved spacetime. We find the 4-accelerations of the different points of the rod necessary to avoid the deformations, and the rates of the comoving clocks, placed at the different points of the rod. Writing the obtained formulae as Taylor series, we study the dependence of the 4-accelerations and the clock rates on the curvature of spacetime and also on rod's own motion.

Next we establish with the help of the abovementioned series the metric and connections in the Fermi coordinates of an accelerated observer. In our treatment the observer is placed on the rod so that in his frame of reference both ends of the rod are at the equal distance from the observer. We calculate the fifth order expansion of the metric and the fourth order expansion of the connections in the observer's Fermi coordinates – the observer's proper time and the proper length of the rod. This is necessary to study the dependence of the metric and connection on the curvature of spacetime and the observer's 4-acceleration, but also on the observer's 3-velocity relative to the sources of the gravitational field. As an example we use the derived formulae to establish the metric and connection in the frame of reference of an observer, moving in the 2-dimensional Schwarzschild spacetime. The abovementioned results are a continuation of papers [90] and [91], where the authors examined the acceleration of free fall and the tidal forces of an observer, moving at the relativistic 3-velocity relative to the sources of gravitational field, using the relativistic deviation equation in the second approximation. In this Section the effects caused by the observer's 3-velocity will be considered in the higher approximations.

Further, the calculated metric and connections allow us to examine the free fall of a test particle from the point of view of an accelerated observer. We pay special attention to the so-called $c\sqrt{2}/2$ phenomenon, considered in Refs [70], [92], [59], [76]. According to this phenomenon, the acceleration of free fall of a test particle turns to zero if the test particle's 3-velocity exceeds a certain value. Using the energy conservation law we show, that this phenomenon appears only if the metric is stationary.

The fall of an accordion rod in the case of absence of the nongravitational external forces is also studied. We use again the comoving observer placed at the "center" of the rod, calculate on the basis of the energy conservation law the observer's 3-acceleration, and compare it with the acceleration of free fall of a point mass. Here we consider only the case, when the observer's 3-velocity is not hyperrelativistic. We demonstrate that due to the tidal forces the rod moves at a lesser 3-acceleration than the point mass with the same 3-velocity. Caused by the relativistic length contraction, the difference of 3-acceleration is smaller, if the rod and point mass move at the greater 3-velocity relative to the source of gravitational field.

4.2 Orthogonal basis of an accelerated observer

In order to investigate accelerated motion of an extended body in *curved* spacetimes, we generalize some results of Mould [70] and Rindler [92] who have examined the properties of an uniformly accelerated coordinate system in the two-dimensional Minkowski spacetime. Mould has used the so-called accordion model of a rigid rod [70], p. 222, keeping its length constant relative to the comoving frame of reference. A comoving observer is placed at a certain point of the rod, which moves at the uniform acceleration κ. In order to avoid deformations, each section of the rod must be accelerated selectively. The necessary acceleration of the section at the spatial distance Δr relative to the observer must be [70]

$$\bar{\kappa} = \frac{\kappa}{1+\kappa\Delta r}. \tag{119}$$

Here and in what follows we denote the values of the components of the tensors at the observer's location without a bar. A barred symbol stands for the value of the tensor at a general point of the rod (i.e., not at the observer's location).

The rate of a comoving clock placed at the observer's location differs from the rate of a comoving clock at the spatial distance Δr. Labeling the observer's proper time by t, one can calculate the interval of the proper time of a clock at Δr by means of the following formula [70]:

$$\Delta\bar{t} = (1+\kappa\Delta r)\Delta t. \tag{120}$$

Consequently, at an arbitrarily selected point of the rod the rate of a comoving clock and its acceleration depend only on the spatial distance from the observer and on the observer's acceleration. Due to the symmetry properties of the Minkowski spacetime, the abovementioned quantities can not depend on other factors, for example on the observer's velocity. We will generalize the results of Mould and Rindler for the case of a two-dimensional curved spacetime, where the clock rates and accelerations at different points of the rod will depend also on the curvature of spacetime. Taking into account the higher order small terms, we will find also the dependence on the observer's velocity. In the present Section the accordion rod is constructed in such way, that its both ends are at the equal distances from the observer, measured in the observer's frame of reference. That is, the observer is placed at the centre of the rod.

Our first aim is to calculate the 4-acceleration $\bar{\kappa}$ of each point of the rod, necessary to keep them at the equal distances from the observer; and compare the rates of the comoving clocks, placed at these points, with the rate of the observer's clock.

To start we determine the orthogonal basis in the observer's Fermi coordinates – the observer's proper time t and the rod's proper length r. We define along the rod as the spatial geodesic the tangent unit vector field $\bar{Y}$

which is self-parallel. Because all points of the rod must remain at the constant distance measured by the observer, their 4-velocities $\bar{X}$ can not have the components along the rod. Consequently – the vector field $\bar{X}$ must be orthogonal with the vector field $\bar{Y}$ and we can define the orthogonal basis for the observer's Fermi coordinates as

$$\frac{d\bar{t}}{dt}\bar{X} = \frac{\partial}{\partial t}, \qquad \bar{Y}\frac{\partial}{\partial r}. \tag{121}$$

Here $\bar{t}$ is the proper time of an arbitrary point of the rod. Due to the orthogonality of the 4-acceleration and the 4-velocity the 4-acceleration vector field $\bar{K}$ must be directed along the rod:

$$\bar{K} = \nabla_{\bar{X}}\bar{X} = \bar{\kappa}\bar{Y}. \tag{122}$$

According to the definition of an orthogonal basis we have

$$\bar{g}(\bar{X},\bar{X}) = -\bar{g}(\bar{Y},\bar{Y}) = 1, \qquad \bar{g}(\bar{X},\bar{Y}) = 0. \tag{123}$$

The differential prolongation of system (123), with taking into account the self-parallelity of the vector fields $\bar{X}$ and $\bar{Y}$ along the rod and their orthogonality, yields:

$$\nabla_{\bar{X}}\bar{Y} = \bar{\kappa}\bar{X}, \qquad \nabla_{\bar{X}}\bar{X} = \bar{\kappa}\bar{Y} = \bar{K}, \qquad \nabla_{\bar{Y}}\bar{X} = \nabla_{\bar{Y}}\bar{Y} = 0, \tag{124}$$

where $\nabla_{\bar{X}}$ is the operator of covariant differentiation along the vector field $\bar{X}$. Denoting by $[\,,\,]$ the Lie brackets of two vector fields, we have due to the commutability of partial derivative operators

$$\left[\frac{d\bar{t}}{dt}\bar{X},\bar{Y}\right] \equiv 0 \quad \Rightarrow \quad \frac{d\bar{t}}{dt}\,[\bar{X},\bar{Y}] = \left(\nabla_{\bar{Y}}\frac{d\bar{t}}{dt}\right)\bar{X}. \tag{125}$$

Calculating the Lie brackets by means of formulae (124)

$$\frac{d\bar{t}}{dt}\bar{\kappa}\bar{X} = \left(\nabla_{\bar{Y}}\frac{d\bar{t}}{dt}\right)\bar{X}, \tag{126}$$

and contracting the result with the 1-form $g(\bar{X})$ we obtain

$$\nabla_{\bar{Y}}\frac{d\bar{t}}{dt} \;=\; \bar{\kappa}\frac{d\bar{t}}{dt}. \tag{127}$$

The last formula is relevant as its differential prolongations allow one to calculate the accelerations of the points of the rod and the rates of the comoving clocks in an arbitrary approximation.

Remark. Let us note that formula (127) can be written as

$$d\bar{t} \;=\; (1+\kappa\Delta r + \mathcal{O}_2)dt, \tag{128}$$

which coincides with the one, obtained by Mould [70] for a rod accelerated on a Minkowski plane. Consequently, the clock rates at different points of the rod in the first approximation do not depend on the curvature of spacetime.

4.3 Acceleration and proper time of different points of the accelerated rod

In this Section we calculate the accelerations of different points of the rod, which are necessary to avoid deformations, and the rates of the comoving clocks placed at these points. We use the Ricci identity

$$\nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{\Xi} - \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{\Xi} \equiv \bar{R}(\bar{\Xi}, \bar{Y}, \bar{X}) - \nabla_{[\bar{X},\bar{Y}]}\bar{\Xi}, \tag{129}$$

where $\bar{\Xi}$ is an arbitrary vector field. Taking $\bar{\Xi} = \bar{X}$, gives us due to formulae (124) and (127) the following relationship:

$$\frac{\partial \bar{\kappa}}{\partial r}\bar{Y} = \bar{R}(\bar{X}, \bar{Y}, \bar{X}) - \frac{dt}{d\bar{t}}\left(\frac{\partial}{\partial r}\frac{d\bar{t}}{dt}\right)\bar{\kappa}\bar{Y}. \tag{130}$$

Contracting the last equation with the covector $\bar{g}(\bar{Y})$ and taking into account formula (127), we obtain

$$\frac{\partial \bar{\kappa}}{\partial r} = -\bar{\kappa}^2 - \bar{\psi}, \tag{131}$$

where $\bar{\psi}$ is a scalar function defined as follows

$$\bar{\psi} = \bar{g}\left[\bar{Y}, \bar{R}(\bar{X}, \bar{Y}, \bar{X})\right]. \tag{132}$$

The differential prolongations of formula (127) yield, due to formula (131), the following expressions:

$$\begin{aligned}
\frac{\partial}{\partial r}\frac{d\bar{t}}{dt} &= \bar{\kappa}\frac{d\bar{t}}{dt}, \quad \frac{\partial^2}{\partial r^2}\frac{d\bar{t}}{dt} = -\bar{\psi}\frac{d\bar{t}}{dt}, \quad \frac{\partial^3}{\partial r^3}\frac{d\bar{t}}{dt} = -\left(\bar{\kappa}\bar{\psi} + \frac{\partial\bar{\psi}}{\partial r}\right)\frac{d\bar{t}}{dt},\\
\frac{\partial^4}{\partial r^4}\frac{d\bar{t}}{dt} &= -\left(2\bar{\kappa}\frac{\partial\bar{\psi}}{\partial r} - \bar{\psi}^2 + \frac{\partial^2\bar{\psi}}{\partial r^2}\right)\frac{d\bar{t}}{dt},\\
\frac{\partial^5}{\partial r^5}\frac{d\bar{t}}{dt} &= \left(\bar{\kappa}\bar{\psi}^2 + 4\bar{\psi}\frac{\partial\bar{\psi}}{\partial r} - 3\bar{\kappa}\frac{\partial^2\bar{\psi}}{\partial r^2} - \frac{\partial^3\bar{\psi}}{\partial r^3}\right)\frac{d\bar{t}}{dt}.
\end{aligned} \tag{133}$$

Taking the derivatives in formula (133) at the observer's location, where $(d\bar{t}/dt)|_{r=0} = 1$, we obtain for the rate of a comoving clock at the distance r from the observer the following Taylor expansion in the fifth approximation

$$\begin{aligned}
\frac{d\bar{t}}{dt} &= 1 + \kappa r - \frac{1}{2}\psi r^2 - \frac{1}{6}\left(\kappa\psi + \frac{\partial\psi}{\partial r}\right)r^3 + \frac{1}{24}\left(\psi^2 - 2\kappa\frac{\partial\psi}{\partial r} - \frac{\partial^2\psi}{\partial r^2}\right)r^4 +\\
&+ \frac{1}{120}\left(\kappa\psi^2 + 4\psi\frac{\partial\psi}{\partial r} - 3\kappa\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^3\psi}{\partial r^3}\right)r^5 + \mathcal{O}_6.
\end{aligned} \tag{134}$$

Here it must be mentioned, that the further differential prolongations of formula (133) yield easily some coefficients of the higher order terms in the Taylor expansion (134). Nevertheless in some special cases the calculations may be complicated, as we will see later. We also point out, that the observer's acceleration κ and the value of the scalar ψ are in general time dependent.

As the next step we calculate the 4-acceleration of a certain fixed point of the accordion rod, necessary to avoid the deformations, i.e. to keep the spatial distance of the selected point in the observer's frame of reference constant. The first equation of system (133) can be written as

$$\bar{\kappa} = \frac{dt}{d\bar{t}} \left(\frac{\partial}{\partial r} \frac{d\bar{t}}{dt} \right). \tag{135}$$

Taking also into account formula (134), we obtain

$$\begin{aligned}
\bar{\kappa}(r) \;\; &= \;\; \kappa - \left(\kappa^2 + \psi\right) r + \left(\kappa^3 + \kappa\psi - \frac{1}{2}\frac{\partial\psi}{\partial r}\right) r^2 - \\
&- \;\; \left(\kappa^4 + \frac{4}{3}\kappa^2\psi + \frac{1}{3}\psi^2 - \frac{1}{3}\kappa\frac{\partial\psi}{\partial r} + \frac{1}{6}\frac{\partial^2\psi}{\partial r^2}\right) r^3 + \\
&+ \;\; \left(\kappa^5 + \frac{5}{3}\kappa^3\psi - \frac{5}{12}\kappa^2\frac{\partial\psi}{\partial r} + \frac{2}{3}\kappa\psi^2 + \right. \\
&+ \;\; \left. \frac{1}{12}\kappa\frac{\partial^2\psi}{\partial r^2} - \frac{1}{4}\psi\frac{\partial\psi}{\partial r} - \frac{1}{24}\frac{\partial^3\psi}{\partial r^3}\right) r^4 + \mathcal{O}_5.
\end{aligned} \tag{136}$$

We see, that even if the point of the rod where the observer dwells is not accelerated, the other points must in general move at a nonzero 4-acceleration in order to avoid deformations. So the last formula reconfirms the fact, that in general the points on an extended body can not fall freely in the curved spacetime.

4.4 Metric and connection in the frame of reference of an accelerated observer

4.4.1 The general case 2-dimensional spacetime

The aim of this Section is to study the effects generated by acceleration if an extended body moves in a curved spacetime. This problem is mathematically and physically very complicated, because it supposes in the general case solving of the nonlinear differential equations with the partial derivatives. Therefore some simplified models have been used, for example the approximation of spinning spheres of Mathisson, see Ref. [88].

If we want to describe the motion of an extended body with a nonspherical symmetry, we need to use the quadrupol approximation, which is extremely complicated. However, to bypass the mathematical complexity some simple rod models have been proposed. For example – the test rigid rod by Faulkner and Flannery [89], where the authors use the d'Alembert principle to determine the tidal forces in the rigid rod moving in a curved spacetime; the accordion rod by Mould [70], accelerated in parallel to itself in the Minkowski spacetime.

To avoid the extremely complicated uncomprehensive formulae, we mainly confine ourselves to the 2-dimensional curved spacetime. Therefore we calculate, using the formulae derived in the previous Section, the components of the metric tensor and the Christoffel symbols in the fifth and fourth approximations, respectively, and consider the terms involving the observer's 3-velocity.

According to the Fermi theorem the metric along the observer's world line can be taken in the first approximation as the Minkowski metric [81]. It means, the metric in the frame of reference of an accelerated observer does not depend on the curvature of the spacetime in the first approximation with respect to the components of the position vector. Ni and Zimmermann [39] derived the second approximation expansion of the metric. They established that the inertial effects caused by the observer's acceleration and appearing in the first approximation, and the gravitational effects caused by the curvature of the spacetime and appearing in the second approximation, are clearly separable and there are no coupled terms. To study the coupled inertial and gravitational effects, Li and Ni [40] derived the third approximation expansion of the metric. In this order there occur the purely gravitational and also the coupled (inertial-gravitational) terms. The results of the previous Section allow us to calculate the metric in the fifth approximation in the 2-dimensional case. So we can study in greater detail the dependence of the gravitational and the coupled terms on the observer's motion.

In this Section special attention has been paid to the dependence of the gravitational and the coupled terms on the observer's 3-velocity relative to the sources of the gravitational field. In their earlier papers (see Ref. [90] and [91]) the autors have examined the influence of the observer's motion to the tidal forces and to the acceleration of free fall in the comoving frame of reference. The terms containing the observer's 3-velocity appear in the second approximation of the relativistic deviation equation and the authors have considered their effects in case of the observer moving at the relativistic 3-velocity in the field of a weak monochromatic gravitational wave [91] and in the Schwarzschild spacetime [90]. This Section is the expansion of the abovementioned results, the dependence of higher order approximations of the comoving metric on observer's motion will be investigated.

To calculate the components of the metric tensor in the frame of reference of an accelerated observer, where the coordinates are the observer's proper time t and the proper length r, we use the formula

$$d\bar{t}^2(r) = \bar{g}_{tt}dt^2 \Rightarrow \bar{g}_{tt} = \left(\frac{d\bar{t}}{dt}\right)^2, \tag{137}$$

where $\bar{t}$ is the proper time of a clock comoving with the point of the rod at the spatial distance r. Due to formula (134) we have

$$\begin{aligned}
\bar{g}_{tt} &= 1 + 2\kappa r + (\kappa^2 - \psi)r^2 - \frac{1}{3}\left(4\kappa\psi + \frac{\partial\psi}{\partial r}\right)r^3 + \\
&+ \left(\frac{1}{3}\psi^2 - \frac{1}{12}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{2}\kappa\frac{\partial\psi}{\partial r} - \frac{1}{3}\kappa^2\psi\right)r^4 + \\
&+ \left(\frac{4}{15}\kappa\psi^2 - \frac{1}{6}\kappa^2\frac{\partial\psi}{\partial r} + \frac{7}{30}\psi\frac{\partial\psi}{\partial r} - \frac{2}{15}\kappa\frac{\partial^2\psi}{\partial r^2} - \frac{1}{60}\frac{\partial^3\psi}{\partial r^3}\right)r^5 + \mathcal{O}_6, \\
\bar{g}_{rr} &= -1.
\end{aligned} \tag{138}$$

According to the definition formula of the Christoffel symbols $2\bar{\Gamma}^i_{jk} = \bar{g}^{il}\left(\bar{g}_{lj,k} + \bar{g}_{lk,j} - \bar{g}_{jk,l}\right)$ we get the following nonvanishing Christoffel symbols in the fourth approximation:

$$\begin{aligned}
\bar{\Gamma}^t_{tt} &= \frac{\partial\kappa}{\partial t}r - \left(\kappa\frac{\partial\kappa}{\partial t} + \frac{1}{2}\frac{\partial\psi}{\partial t}\right)r^2 + \\
&\quad + \left(\kappa^2\frac{\partial\kappa}{\partial t} + \frac{1}{3}\kappa\frac{\partial\psi}{\partial t} + \frac{1}{3}\frac{\partial\kappa}{\partial t}\psi - \frac{1}{6}\frac{\partial^2\psi}{\partial r\partial t}\right)r^3 + \\
&\quad + \left(-\kappa^3\frac{\partial\kappa}{\partial t} - \frac{1}{3}\kappa^2\frac{\partial\psi}{\partial r} - \frac{2}{3}\kappa\frac{\partial\kappa}{\partial t}\psi + \frac{1}{12}\kappa\frac{\partial^2\psi}{\partial r\partial t} - \right. \\
&\quad \left. - \frac{1}{12}\frac{\partial\kappa}{\partial t}\frac{\partial\psi}{\partial r} - \frac{1}{6}\psi\frac{\partial\psi}{\partial t} - \frac{1}{24}\frac{\partial^3\psi}{\partial r^2\partial t}\right)r^4 + \mathcal{O}_5, \\
\bar{\Gamma}^t_{rt} &= \kappa - (\kappa^2 + \psi)r + \left(\kappa^3 + \kappa\psi - \frac{1}{2}\frac{\partial\psi}{\partial r}\right)r^2 + \\
&\quad + \left(-\kappa^4 - \frac{4}{3}\kappa^2\psi - \frac{1}{3}\psi^2 + \frac{1}{3}\kappa\frac{\partial\psi}{\partial r} - \frac{1}{6}\frac{\partial^2\psi}{\partial r^2}\right)r^3 +
\end{aligned}$$

$$
\begin{aligned}
& + \left(\kappa^5 + \frac{5}{3}\kappa^3\psi + \frac{2}{3}\kappa\psi^2 - \frac{5}{12}\kappa^2\frac{\partial\psi}{\partial r} - \frac{1}{4}\psi\frac{\partial\psi}{\partial r} + \right. \\
& \left. + \frac{1}{12}\kappa\frac{\partial^2\psi}{\partial r^2} - \frac{1}{24}\frac{\partial^3\psi}{\partial r^3}\right) r^4 + \mathcal{O}_5, \\
\bar{\Gamma}^r_{tt} \;=\; & \kappa + (\kappa^2 - \psi) r - \left(2\kappa\psi + \frac{1}{2}\frac{\partial\psi}{\partial r}\right) r^2 + \\
& + \left(\frac{2}{3}\psi^2 - \frac{1}{6}\frac{\partial^2\psi}{\partial r^2} - \kappa\frac{\partial\psi}{\partial r} - \frac{2}{3}\kappa^2\psi\right) r^3 + \\
& + \left(\frac{2}{3}\kappa\psi^2 - \frac{5}{12}\kappa^2\frac{\partial\psi}{\partial r} + \frac{7}{12}\psi\frac{\partial\psi}{\partial r} - \frac{1}{3}\kappa\frac{\partial^2\psi}{\partial r^2} - \frac{1}{24}\frac{\partial^3\psi}{\partial r^3}\right) r^4 + \mathcal{O}_5.
\end{aligned}
\tag{139}
$$

The further differential prolongations of system (133) allow us to calculate relatively simply the components of the metric tensor and the Christoffel symbols in the frame of reference of an accelerated observer also in the further approximations. For the 2-dimensional case these formulae take a very simple form compared to the results of Refs [40] and [44].

According to the definition formula (132) the scalar function ψ depends on the curvature of the spacetime at the observer's location. Firstly, formula (138) confirms us the already known results, the curvature of the spacetime appears in the second approximation of the metric and the coupled terms in the third approximation. Secondly, because the partial derivative operators with respect to the observer's proper time t and the rod's proper length r are the observer's 4-velocity X and the unit vector Y orthogonal to X, see formula (123), the terms containing the partial derivatives of scalar ψ depend also on the observer's 3-velocity. Consequently, the Taylor expansion of the metric (138) contains the third and higher order gravitational and the fourth and higher order coupled terms dependent on the observer's 3-velocity. As we will see in the next Subsections, the velocity dependent terms become comparable with the lower order ones, if the observer's 3-velocity approaches to the velocity of the light and therefore they can not be neglected.

4.4.2 2-dimensional Schwarzschild spacetime.

As an example we calculate the Christoffel symbols in the frame of reference of an observer, accelerated in the 2-dimensional Schwarzschild spacetime. Using the proper time τ and the radial coordinate ρ as the global coordinates of an infinitely far observer we obtain for the differential of the observer's proper time the following formula

$$
dt^2 = c d\tau^2 - \frac{d\rho^2}{c}, \qquad \text{where} \qquad c = 1 - \frac{2\gamma M}{\rho}. \tag{140}
$$

Here the quantity c is the radial velocity of light. At present the scalar function (132) reads

$$\psi = \frac{2\gamma M}{\rho^3}. \tag{141}$$

In the case of an accelerated observer, moving in the radial direction at the 3-velocity $\beta_\rho = d\rho/d\tau$, the partial derivative operators $X = \partial/\partial t$ and $Y = \partial/\partial r$ have in the global coordinates the following form:

$$X = \frac{\partial}{\partial t} = \sqrt{\frac{c}{c^2 - \beta_\rho^2}}\left(\frac{\partial}{\partial \tau} + \beta_\rho \frac{\partial}{\partial \rho}\right), \quad Y = \frac{\partial}{\partial r} = \frac{\left(\beta_\rho \frac{\partial}{\partial \tau} + c^2 \frac{\partial}{\partial \rho}\right)}{\sqrt{c(c^2 - \beta_\rho^2)}}. \tag{142}$$

It means, the first order partial derivatives of the scalar function ψ relative to the observer's Fermi coordinates read

$$\frac{\partial \psi}{\partial t} = \nabla_X \psi = -\frac{6\gamma M}{\rho^4}\beta_\rho \sqrt{\frac{c}{c^2 - \beta_\rho^2}}, \quad \frac{\partial \psi}{\partial r} = \nabla_Y \psi = -\frac{6\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2 - \beta_\rho^2}}. \tag{143}$$

To calculate the higher order partial derivatives of ψ we must at first find the partial derivatives of the quantities c and β_ρ with respect to the observer's Fermi coordinates. We remind, that in the 2-dimensional Schwarzschild space-time only the following Christoffel symbols in the global coordinates differ from zero:

$$\Gamma^\tau_{\rho\tau} = -\Gamma^\rho_{\rho\rho} = \frac{1}{c}\frac{\gamma M}{\rho^2}, \qquad \Gamma^\rho_{\tau\tau} = c\frac{\gamma M}{\rho^2}. \tag{144}$$

Taking into account the definition of the covariant derivative and formula (142), we may write the last two formulae of equation (122) as follows:

$$\frac{DX}{dr} = \nabla_Y X = \Theta Y = 0, \qquad \frac{DY}{dr} = \nabla_Y Y = \Theta X = 0, \tag{145}$$

where

$$\Theta = \sqrt{\frac{c}{(c^2 - \beta_\rho^2)^3}}\left[\beta_\rho \frac{\partial \beta_\rho}{\partial \tau} - c^2 \frac{\partial \beta_\rho}{\partial \rho} - \frac{\gamma M}{c\rho^2}\beta_\rho(\beta_\rho^2 + c^2)\right]. \tag{146}$$

Consequently, the expression in the brackets in formula (146) must be equal to zero:

$$\beta_\rho \frac{\partial \beta_\rho}{\partial \tau} - c^2 \frac{\partial \beta_\rho}{\partial \rho} - \frac{\gamma M}{c\rho^2}\beta_\rho(\beta_\rho^2 + c^2) \;\; = \;\; 0. \tag{147}$$

Using now formulae (124), we get in the same way, that

$$\kappa = \left[c\left(\frac{\partial \beta_\rho}{\partial \tau} + \frac{\partial \beta_\rho}{\partial \rho}\beta_\rho\right) + \frac{\gamma M}{\rho^2}(c^2 - 3\beta_\rho^2)\right]\sqrt{\frac{c}{(c^2 - \beta_\rho^2)^3}}. \tag{148}$$

Solving the system of equations (147) and (148), we obtain the partial derivatives of β_ρ with respect to the global coordinates

$$\frac{\partial\beta_\rho}{\partial\tau}=\kappa\sqrt{c(c^2-\beta_\rho^2)}-\frac{\gamma M}{c\rho^2}(c^2-\beta_\rho^2),\quad \frac{\partial\beta_\rho}{\partial\rho}=\beta_\rho\left(\frac{2\gamma M}{c\rho^2}-\kappa\sqrt{\frac{c^2-\beta_\rho^2}{c^3}}\right). \quad (149)$$

Remark. The last system enables us to calculate the 3-acceleration of an accelerated point mass, moving in the 2-dimensional Schwarzschild spacetime:

$$\frac{d\beta_\rho}{d\tau}=\frac{\partial\beta_\rho}{\partial\tau}+\frac{\partial\beta_\rho}{\partial\rho}\beta_\rho \;=\kappa\sqrt{\frac{(c^2-\beta_\rho^2)^3}{c^3}}-\frac{\gamma M}{c\rho^2}(c^2-3\beta_\rho^2). \quad (150)$$

In the case of a freely falling point mass, $\kappa=0$, the last formula reduces to the one obtained by Mould [70] for the 3-acceleration.

System (149) and its prolongations allow us also to determine the 3-velocities of the other points of the rod. Because the position vector of an arbitrary point of the rod relative to the observer is Yr, its 3-velocity in the first approximation is

$$\beta_\rho(r)=\beta_\rho+\left(\frac{\partial\beta_\rho}{\partial\tau}y^\tau+\frac{\partial\beta_\rho}{\partial\rho}y^\rho\right)r=\beta_\rho\left(1+\frac{\gamma M}{\rho^2}\frac{c^2+\beta_\rho^2}{\sqrt{c^3(c^2-\beta_\rho^2)}}r\right)+\mathcal{O}_2. \quad (151)$$

Thus the point of the rod placed at r must move at the 3-velocity (151) in order to avoid deformations. We point out, that this approximation can not be used in the case of an observer, moving at a hyperrelativistic 3-velocity, because it leads to unlimited growth of the second order term in the parentheses and the Taylor expansion (151) may not converge.

Returning now to the partial derivatives of the quantities c and β_ρ, we obtain due to formulae (140), (142) and (149), that

$$\begin{aligned}
\frac{\partial c}{\partial t} &= \frac{\partial c}{\partial\rho}x^\rho=\frac{2\gamma M}{\rho^2}\beta_\rho\sqrt{\frac{c}{c^2-\beta_\rho^2}},\\
\frac{\partial c}{\partial r} &= \frac{\partial c}{\partial\rho}y^\rho=\frac{2\gamma M}{\rho^2}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}},\\
\frac{\partial\beta_\rho}{\partial t} &= \frac{\partial\beta_\rho}{\partial\tau}x^\rho+\frac{\partial\beta_\rho}{\partial\rho}x^\rho=\frac{\kappa\left(c^2-\beta_\rho^2\right)}{c}-\frac{\gamma M}{\rho^2}\frac{c^2-3\beta_\rho^2}{\sqrt{c\left(c^2-\beta_\rho^2\right)}},\\
\frac{\partial\beta_\rho}{\partial r} &= \frac{\partial\beta_\rho}{\partial\tau}y^\rho+\frac{\partial\beta_\rho}{\partial\rho}y^\rho=\frac{\gamma M}{\rho^2}\frac{\beta_\rho\left(c^2+\beta_\rho^2\right)}{\sqrt{c^3\left(c^2-\beta_\rho^2\right)}}.
\end{aligned} \quad (152)$$

This allows us to calculate the further differential prolongations of formulae (143):

$$\frac{\partial^2\psi}{\partial t\partial r} = \nabla_X\nabla_Y\psi=\frac{24\gamma M}{\rho^5}\frac{\beta_\rho c^2}{c^2-\beta_\rho^2}-\frac{6\gamma M}{\rho^4}\kappa\beta_\rho\sqrt{\frac{c}{c^2-\beta_\rho^2}},$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial r^2} &= \nabla_Y^2\psi = \frac{24\gamma M}{\rho^5}\frac{c^3}{c^2-\beta_\rho^2} - \frac{6\gamma^2 M^2}{\rho^6}, \\
\frac{\partial^3\psi}{\partial r^3} &= -\frac{120\gamma M}{\rho^6}\sqrt{\frac{c^9}{\left(c^2-\beta_\rho^2\right)^3}} + \frac{84\gamma^2 M^2}{\rho^7}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}, \\
\frac{\partial^3\psi}{\partial t\partial^2 r} &= \frac{48\gamma M}{\rho^5}\kappa\beta_\rho\frac{c^2}{c^2-\beta_\rho^2} - \frac{120\gamma M}{\rho^6}\beta_\rho\sqrt{\frac{c^7}{\left(c^2-\beta_\rho^2\right)^3}} + \\
&\quad + \frac{12\gamma^2 M^2}{\rho^7}\beta_\rho\sqrt{\frac{c}{(c^2-\beta_\rho^2)}}.
\end{aligned} \tag{153}
$$

It can be seen from system (153), that the expressions for the components of the metric tensor (138), the Christoffel symbols (139) and the acceleration of an arbitrary point of the accordion rod depend also on the observer's 3-velocity β_ρ. On the Minkowski plane this is not the case, due to the symmetry properties of the (pseudo-)Euclidian space. In the case of curved spacetime, the observer's 3-velocity is "hidden" in the terms, containing the derivatives of the scalar function ψ. The just-mentioned terms are higher order small quantities, which in some situations can carry significant information. Obviously, if the observer's 3-velocity approaches c, the unlimited growth of the partial derivatives (153) occurs. The terms in formulae (138) and (139), containing the abovementioned quantities, will begin to dominate over the other terms and approach infinity. It means that we can use the formulae (138) and (139) only, if the following conditions hold:

$$
\nabla_X\psi\Delta t << \psi, \qquad \nabla_Y\psi\Delta r << \psi. \tag{154}
$$

If these conditions are not satisfied, the Taylor expansions (138) and (139) may not converge and the formulae are not applicable in the frame of reference of an observer, moving at the hyperrelativistic 3-velocity in a curved spacetime.

4.4.3 2-dimensional Schwarzschild spacetime: connections in the frame of reference of an accelerated observer.

At first we calculate the rate of a comoving clock, placed on the rod at the distance r from the observer. Substituting the partial derivatives (143) and (153) into formula (138) and taking into account formula (137), we obtain

$$
\begin{aligned}
\frac{d\bar{t}}{dt} &= 1 + \kappa r - \frac{\gamma M}{\rho^3}r^2 + \left(-\frac{1}{3}\kappa\frac{\gamma M}{\rho^3} + \frac{\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}\right) r^3 + \\
&\quad + \left(\frac{5}{12}\frac{\gamma^2 M^2}{\rho^6} + \frac{1}{2}\kappa\frac{\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}} - \frac{\gamma M}{\rho^5}\frac{c^3}{c^2-\beta_\rho^2}\right) r^4 +
\end{aligned}
$$

$$+\left(\frac{11}{60}\kappa\frac{\gamma^2M^2}{\rho^6}-\frac{11}{10}\frac{\gamma^2M^2}{\rho^7}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}+\frac{\gamma M}{\rho^6}\sqrt{\frac{c^9}{\left(c^2-\beta_\rho^2\right)^3}}-\right.$$
$$\left.-\frac{3}{5}\kappa\frac{\gamma M}{\rho^5}\frac{c^3}{c^2-\beta_\rho^2}\right)r^5+\mathcal{O}_6. \tag{155}$$

Inserting now the partial derivatives of the function ψ occuring in system (153) into formulae (139), we obtain the following nonvanishing Christoffel symbols:

$$\begin{aligned}
\bar{\Gamma}^t_{tt} &= \frac{\partial\kappa}{\partial t}r-\left(\kappa\frac{\partial\kappa}{\partial t}-\frac{3\gamma M}{\rho^4}\beta_\rho\sqrt{\frac{c}{c^2-\beta_\rho^2}}\right)r^2+\\
&+\left(\kappa^2\frac{\partial\kappa}{\partial t}-\frac{\gamma M}{\rho^4}\kappa\beta_\rho\sqrt{\frac{c}{c^2-\beta_\rho^2}}+\frac{2\gamma M}{3\rho^3}\frac{\partial\kappa}{\partial t}-\frac{4\gamma M}{\rho^5}\frac{\beta_\rho c^2}{c^2-\beta_\rho^2}\right)r^3+\\
&+\left(-\kappa^3\frac{\partial\kappa}{\partial t}-\frac{4\gamma M}{3\rho^3}\kappa\frac{\partial\kappa}{\partial t}+\frac{3\gamma M}{2\rho^4}\kappa^2\beta_\rho\sqrt{\frac{c}{c^2-\beta_\rho^2}}+\right.\\
&\quad+\frac{\gamma M}{2\rho^4}\frac{\partial\kappa}{\partial t}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}+\frac{3\gamma^2M^2}{2\rho^7}\beta_\rho\sqrt{\frac{c}{c^2-\beta_\rho^2}}+\\
&\quad\left.+\frac{5\gamma M}{\rho^6}\beta_\rho\sqrt{\frac{c^7}{\left(c^2-\beta_\rho^2\right)^3}}\right)r^4+\mathcal{O}_5,\\
\bar{\Gamma}^t_{rt} &= \kappa-\left(\kappa^2-\frac{2\gamma M}{\rho^3}\right)r+\\
&+\left(\kappa^3+\frac{2\gamma M}{\rho^3}\kappa+\frac{3\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}\right)r^2+\\
&+\left(-\kappa^4-\frac{8\gamma M}{3\rho^3}\kappa^2-\frac{\gamma^2M^2}{3\rho^6}-\frac{2\gamma M}{\rho^4}\kappa\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}-\right.\\
&\quad\left.-\frac{4\gamma M}{\rho^5}\frac{c^3}{c^2-\beta_\rho^2}\right)r^3+\\
&+\left(\kappa^5+\frac{10\gamma M}{3\rho^3}\kappa^3+\frac{5\gamma M}{2\rho^4}\kappa^2\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}+\right.\\
&\quad+\frac{2\gamma M}{\rho^5}\kappa\frac{c^3}{c^2-\beta_\rho^2}+\frac{13\gamma^2M^2}{6\rho^6}\kappa+\frac{5\gamma M}{\rho^6}\sqrt{\frac{c^9}{\left(c^2-\beta_\rho^2\right)^3}}-\\
&\quad\left.-\frac{\gamma^2M^2}{2\rho^7}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}\right)r^4+\mathcal{O}_5,
\end{aligned}$$

$$
\begin{aligned}
\bar{\Gamma}^r_{tt} &= \kappa + \left(\kappa^2 - \frac{2\gamma M}{\rho^3}\right) r - \left(\frac{4\gamma M}{\rho^3}\kappa - \frac{3\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2 - \beta_\rho^2}}\right) r^2 + \\
&+ \left(\frac{11\gamma^2 M^2}{3\rho^6} - \frac{4\gamma M}{\rho^5}\frac{c^3}{c^2 - \beta_\rho^2} + \frac{6\gamma M}{\rho^4}\kappa\sqrt{\frac{c^3}{c^2 - \beta_\rho^2}} - \right. \\
&\quad \left. - \frac{4\gamma M}{3\rho^3}\kappa^2\right) r^3 + \\
&+ \left(\frac{5\gamma M}{2\rho^4}\kappa^2\sqrt{\frac{c^3}{c^2 - \beta_\rho^2}} - \frac{8\gamma M}{\rho^5}\kappa\frac{c^3}{c^2 - \beta_\rho^2} + \frac{14\gamma^2 M^2}{3\rho^6}\kappa + \right. \\
&+ \left. \frac{5\gamma M}{\rho^6}\sqrt{\frac{c^9}{\left(c^2 - \beta_\rho^2\right)^3}} - \frac{21\gamma^2 M^2}{2\rho^7}\sqrt{\frac{c^3}{c^2 - \beta_\rho^2}}\right) r^4 + \mathcal{O}_5.
\end{aligned}
\tag{156}
$$

The obtained Taylor expansions of the Christoffel symbols (156) generalize the results of Ni and Zimmermann [39]. In the lower approximation they confirm the results of Ref. [39], that the effects of the inertial terms, caused by the observer's acceleration, and the gravitational terms, caused by the curvature of spacetime, are in the first approximation independent. According to the Einstein equivalence principle, in the comoving frame the gravitational terms are infinitely small if compared with the inertial ones. It can be seen, that the inertial terms enter into the formulae (156) in the zeroth and the gravitational terms in the first approximation.

The first order terms do not depend on the observer's 3-velocity relative to the black hole. Therefore, if the observer moves at a small 3-velocity, the effects caused by the observer's motion are negligibly small. If the observer's acceleration also is constant, Γ^t_{tt} is a second order small quantity and consequently, we can consider the metric in the Fermi coordinates of a comoving observer as the stationary one.

The Fermi metric can not more considered as a stationary one, if the observer moves at a relativistic 3-velocity. The gravitational terms dependent on observer's motion as the second order small ones contain the factor $\left(c^2 - \beta_\rho^2\right)^{-\frac{1}{2}}$. Due to this factor, the abovementioned terms will grow, if the observer's 3-velocity approaches the velocity of light, and the effects caused by the observer's motion can no more be neglected. It must be also pointed out, that the observer's hyperrelativistic 3-velocity causes the unlimited growth of these second order terms, they will dominate over the first order ones and the Taylor expansions may not more converge.

Li and Ni have shown [40] that in the Taylor series of the Christoffel symbols, in the second approximation the coupled terms also appear. They depend at the same time on the observer's acceleration and on the curvature of the spacetime. Because the dependence on the observer's motion relative to the central body appears first in the same approximation, we can not examine the coupled terms without taking into account the observer's 3-velocity even in the nonrelativistic case. We have shown, that the third order coupled terms involve also the factor $\left(c^2-\beta_\rho^2\right)^{-\frac{1}{2}}$ and will grow if the observer's 3-velocity increases. It means, if the observer's 3-velocity approaches the velocity of light, the Taylor series of Christoffel symbols may not converge also due to the coupled terms. This means, in both gravitational and coupled terms the dependence on the observer's motion appears in one order higher approximation. At the same time, the purely inertial terms depending on the observer's 4-acceleration do not depend on the observer's 3-velocity even in the higher approximations. The reason is, that if the curvature of the spacetime is zero, the gravitational and coupled terms vanish, but the inertial terms remain. Due to the symmetry properties of the Minkowski spacetime they can not depend on observer's 3-velocity.

Because the coupled terms are small compared with the inertial and gravitational ones, then one can better examine their effects in the case $\kappa^2 = \left(2\gamma M\right)/\rho$, when the first order terms in formulae (156) for Γ^t_{rt} and Γ^r_{tt} vanish. We point out, that although in formulae (155) and (156) the higher order terms do not play any significant role in the most cases and can be dropped in the calculations, then in case of an observer moving at the relativistic 3-velocity these terms become comparable with the lower order ones and must be taken into account.

The fourth order terms in the Taylor expansions of the Christoffel symbols (156) do not cause any new effects.

4.4.4 2-dimensional Schwarzschild spacetime: accelerating the points of the rod

Formula (136) gives us the accleleration of a point of the rod at the distance r from the observer, necessary to avoid the deformations. Using the values of a scalar function ψ and its derivatives, see formulae (143) and (153), we obtain

$$\begin{aligned}
\bar{\kappa}(r) \;=\; & \kappa - \left(\kappa^2 + \frac{2\gamma M}{\rho^3}\right) r + \left(\kappa^3 + \frac{2\gamma M}{\rho^3}\kappa + \frac{3\gamma M}{\rho^4}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}\right) r^2 - \\
& - \left(\kappa^4 + \frac{8\gamma M}{3\rho^3}\kappa^2 + \frac{\gamma^2 M^2}{3\rho^6} - \frac{2\gamma M}{\rho^4}\kappa\sqrt{\frac{c^3}{c^2-\beta_\rho^2}} + \right. \\
& \left. + \frac{4\gamma M}{\rho^5}\frac{c^3}{c^2-\beta_\rho^2}\right) r^3 + \\
& + \left(\kappa^5 + \frac{10\gamma M}{3\rho^3}\kappa^3 + \frac{5\gamma M}{2\rho^4}\kappa^2\sqrt{\frac{c^3}{c^2-\beta_\rho^2}} + \frac{13\gamma^2 M^2}{6\rho^6}\kappa + \right. \qquad (157)
\end{aligned}$$

$$+\frac{2\gamma M}{\rho^5}\kappa\frac{c^3}{c^2-\beta_\rho^2}+\frac{\gamma^2 M^2}{2\rho^7}\sqrt{\frac{c^3}{c^2-\beta_\rho^2}}+$$
$$\left.+\frac{5\gamma M}{\rho^6}\sqrt{\frac{c^9}{\left(c^2-\beta_\rho^2\right)^3}}\right)r^4+\mathcal{O}_5.$$

Similarly to the Christoffel symbols (156) the zeroth order approximation is inertial, in the first approximation there appear the gravitational terms and in the second approximation the coupled ones. The gravitational terms in the second approximation and the coupled ones in the third approximation depend on the observer's 3-velocity. The higher order terms do not add any new effects.

Now we examine formula (134) in the case of an accelerated observer in the 2-dimensional Schwarzschild spacetime, moving at a non-relativistic 3-velocity, i.e. $\beta_\rho << c$. Then the convergence of the Taylor expansion (134) is guaranteed and only the first and second order terms must be considered. Due to formula (141), formula (134) reads in the second approximation

$$\frac{d\bar{t}}{dt} = 1+\kappa r-\frac{\gamma M}{\rho^3}r^2+\mathcal{O}_3, \tag{158}$$

the terms depending on the observer's 3-velocity can be neglected in formulae (155) and (156). Equation (158) shows, that there exists a certain critical value of the radial coordinate in the observer's frame of reference, namely

$$r_{\mathrm{cr}}=\frac{\kappa\rho^3}{\gamma M}, \tag{159}$$

which divides the effects, related to the clock rates, into three groups:

1. $r<r_{\mathrm{cr}} \Rightarrow \Delta\bar{t}>\Delta t$, the clock comoving with the point at r is fast with respect to the observer's clock.
2. $r=r_{\mathrm{cr}} \Rightarrow \Delta\bar{t}=\Delta t$, the clocks have equal rates,
3. $r>r_{\mathrm{cr}} \Rightarrow \Delta\bar{t}<\Delta t$, the clock at r is slow with respect to the observer's clock.

The cases 1. and 2. are possible only if

$$\kappa << \frac{1}{\sqrt{c}}\frac{\gamma M}{\rho^2}, \tag{160}$$

i.e., the accelerating force per unit mass is much smaller than the force necessary to prevent the observer from falling into the black hole. If the length of the rod is critical $r_{\rm cr}$, the clocks at the end and initial points have equal rates, the clock at the center of the rod having the maximal rate.

In the case of the rod at rest with respect to the source of gravitational field

$$\kappa = \frac{1}{\sqrt{c}}\frac{\gamma M}{\rho^2} > \frac{\gamma M}{\rho^3} r \;\Rightarrow\; \Delta\bar{t} > \Delta t, \tag{161}$$

i.e., the clock at the end point is faster. This result reconfirms the fact, that the clocks at rest near the sources of gravitational field are slow compared with the clocks at rest far from the sources of gravitational fields.

Next we consider the acceleration of free fall of a test particle, dropped at the distance r from the observer. A point mass freely falling in a two-dimensional spacetime of stationary metric has the following integral of motion (see Ref. [74]):

$$\dot{t} = e(r) = \bar{g}_{tt}\frac{dt}{d\bar{t}}, \tag{162}$$

where e is the energy per unit mass. If the 3-velocity of this point mass is zero relative to observer, we get

$$d\bar{t}^2 \quad = \quad \bar{g}_{tt} dt^2 \quad \Rightarrow \quad \bar{e} = \frac{d\bar{t}}{dt},$$

where the expression for $d\bar{t}/dt$ can be obtained from formula (158).

If the observer's 3-velocity is small compared with the velocity of light, we can consider the metric in the observer's frame of reference in our approximation as the stationary one. Then the "gravitational force" can be expressed as

$$f = -\mathrm{grad} e = -\kappa + \frac{2\gamma}{\rho^3} r. \tag{163}$$

Therefore a test particle dropped from the point with the radial coordinate r will behave as follows:

1. if $r < \frac{r_{\rm cr}}{2}$ moves towards the observer,
2. if $r = \frac{r_{\rm cr}}{2}$ "hangs" over the observer,
3. if $r > \frac{r_{\rm cr}}{2}$ moves away from the observer.

If the observer moves at a relativistic 3-velocity, we need to take into account also the higher order terms in the Taylor expansion (158) depending on the observer's 3-velocity and the formulae will be more complicated.

4.5 Free fall and the phenomenon of $\bar{c}\sqrt{2}/2$

This Section is devoted to the free fall of a test particle in the 2-dimensional spacetime. We pay special attention to the so-called phenomenon of $\bar{c}\sqrt{2}/2$. According to the $\bar{c}\sqrt{2}/2$ phenomenon the acceleration of free fall of a test particle turns to zero if its 3-velocity takes a certain critical value. If its 3-velocity exceeds the critical value, the acceleration of free fall changes the sign. In the 2-dimensional Minkowski spacetime this phenomenon has been examined by Mould [70] and Rindler [92] in the comoving frame of reference of an accelerated observer. The critical value of 3-velocity is $\bar{c}\sqrt{2}/2$ in this case. Mould considers in Ref. [70] also the freely falling test particle in the 2-dimensional Schwarzschild spacetime in the frame of reference of an infinitely far stationary observer, where the critical value of the 3-velocity is $\bar{c}\sqrt{3}/3$. On the other hand, the authors of present Thesis have shown [59], that also in the frame of reference of a freely falling observer in the Schwarzschild spacetime one can eliminate the acceleration of free fall of a test particle in the first approximation with a "suitable choice" of the 3-velocity, the corresponding value is in the radial direction $\bar{c}\sqrt{2}/2$ again. Later this phenomenon has been examined by Chicone and Mashhoon, see Refs. [60], [76]. In these papers the acceleration of free fall vanishes if the 3-velocity of the test particle reaches $\bar{c}\sqrt{2}/2$. In the higher approximations the above-mentioned phenomenon does not appear. The complete elimination of the acceleration of free fall via giving to the test particle an appropriate 3-velocity is possible only in the case of the stationary metrics. In this case the time is the cyclic coordinate and the corresponding component of the 4-momentum

$$\bar{e} \;=\; \bar{g}_{tt}\frac{d\bar{t}}{dt}, \tag{164}$$

as the energy of the test particle of the unit mass is constant. Because the stationary 2-dimensional metric is always diagonal, we get the square of the modulus of the test particle's 3-velocity $\left(\bar{g}_{tt}+\bar{g}_{rr}\bar{\beta}^2\right)\left(dt/d\bar{t}\right)^2 = 1$. At the same time, the equation of the photon's world line $\bar{g}_{tt}dt^2+\bar{g}_{tt}dr^2 = 0$ yields $\bar{c}=\sqrt{-(\bar{g}_{tt})/(\bar{g}_{rr})}$. Consequently

$$\bar{e} \;=\; \bar{c}\sqrt{\frac{\bar{g}_{tt}}{\bar{c}^2-\bar{\beta}^2}}. \tag{165}$$

Its time derivative

$$\frac{d\bar{e}}{dt} = \frac{1}{2\sqrt{\bar{g}_{tt}(\bar{c}^2-\bar{\beta}^2)^3}}\left[\bar{c}\frac{d\bar{g}_{tt}}{dt}(\bar{c}^2-\bar{\beta}^2) - 2\frac{d\bar{c}}{dt}\bar{g}_{tt}\bar{\beta}^2 + 2\bar{g}_{tt}\frac{d\bar{\beta}}{dt}\bar{\beta}\bar{c}\right]. \tag{166}$$

is zero in the case of the stationary metric. It means, the expression in brackets must equal to zero and we get for the 3-acceleration of a freely falling test particle the following formula:

$$\frac{d\bar{\beta}}{dt} \quad = \quad \frac{1}{\bar{g}_{tt}\bar{c}}\left[\frac{\partial\bar{c}}{\partial r}\bar{g}_{tt}\bar{\beta}^2 - \frac{1}{2}\bar{c}\frac{\partial\bar{g}_{tt}}{\partial r}(\bar{c}^2 - \bar{\beta}^2)\right]. \tag{167}$$

To "eliminate" the 3-acceleration, the test particle must have the 3-velocity

$$\bar{\beta}_{cr} \quad = \quad \sqrt{\frac{\bar{g}_{tt,r}\bar{c}^3}{2\bar{g}_{tt}\dfrac{\partial\bar{c}}{\partial r} + \bar{g}_{tt,r}\bar{c}}}. \tag{168}$$

We test the last formula (168) now for two special cases, examined also by Mould [70].

(i) If an observer moves at a constant 4-acceleration on the Minkowski plane, the velocity of light in the observer's frame of reference is $\bar{c}(r) = \bar{g}_{tt} = 1 + \kappa r$, and the critical value of the 3-velocity of a test particle is according to formula (168) indeed $\bar{\beta}_{cr} = \bar{c}\sqrt{2}/2$, as in Ref. [70], p. 239-245.

(ii) If a freely falling test particle moves radially in the Schwarzschild spacetime, the velocity of light in the radial direction is $\bar{c} = \bar{g}_{\tau\tau} = 1 - (2\gamma M)/\rho$, and according to formula (168) we get $\bar{\beta}_{cr} = \bar{c}\sqrt{3}/3$, as demonstrated in Ref. [70], p. 329. Formula (168) generalizes the results (i) and (ii) for an arbitrary two-dimensional spacetime with stationary diagonal metric.

We emphasize, that if the observer moves in a curved spacetime, then the metric in the comoving frame of reference is only in the first approximation stationary. If the observer's 3-velocity approaches the velocity of light, the dependence of metric on time can not more be neglected and the phenomenon of $\bar{c}\sqrt{2}/2$ does not appear.

Remark. One can generalize the last result also for the case of a freely falling test particle of unit rest mass, moving in the Schwarzschild spacetime in a non-radial direction. Without loss of generality we can take $\theta = \pi/2$ then

$$|X|^2 = c\dot{\tau}^2 - \frac{\dot{\rho}^2}{c} - \rho^2\dot{\phi}^2 = 1. \tag{169}$$

The generalized momentum corresponding to the cyclic coordinate ϕ as the angular momentum of the test particle per unit mass

$$L = -\rho^2\beta_\phi\dot{\tau} \tag{170}$$

is a constant integral of motion. Substituting it into formula (169) we obtain

$$\frac{d\tau}{dt} = \sqrt{\frac{c\Lambda}{c^2 - \beta_\rho^2}}, \qquad \text{where} \qquad \Lambda = 1 + \frac{L^2}{\rho^2}. \tag{171}$$

The energy of the test particle per unit mass according to formula (162) is

$$e = \sqrt{\frac{c^3\Lambda}{c^2 - \beta_\rho^2}}. \tag{172}$$

Now we take the partial derivative of formula (172) with respect to the global time τ. As $\bar{e}$ is constant in the case of the stationary metrics and the following formulae hold:

$$\frac{dc}{d\tau} = \frac{2\gamma M}{\rho^2}\beta_\rho \qquad \text{and} \qquad \frac{d\Lambda}{d\tau} = -\frac{2L^2}{\rho^3}\beta_\rho, \tag{173}$$

the 3-acceleration in the radial direction reads

$$\frac{d\beta_\rho}{d\tau} = -\frac{\gamma M}{c\rho^2}(c^2 - 3\beta_\rho^2) + \frac{L^2(c^2 - \beta_\rho^2)}{\rho^3\Lambda}. \tag{174}$$

The radial 3-acceleration vanishes, if

$$\beta_{\rho(\mathrm{cr})} = c\sqrt{1 - 2\left(3 - \frac{cL^2}{\gamma M \rho \Lambda}\right)^{-1}}. \tag{175}$$

In the limiting case $\beta_\phi = 0$ we obtain, of course, the Mould's result $\bar{\beta}_{\mathrm{cr}} = c\sqrt{3}/3$. For another special case, if the test particle is moving freely along the circular orbit around the black hole, $\beta_\rho = 0$ and the necessary angular component of the 3-velocity must be the first Kepler velocity $\beta_\phi = \sqrt{(\gamma M)/\rho^3}$. Inserting β_ϕ into formula (170) and taking into account formulae (171) and (174) we obtain

$$\frac{d\beta_\rho}{d\tau} = 0, \tag{176}$$

as it indeed must be in the case of circular motion. We can use formula (175) only in if the function under the square root is greater than zero. If this function is smaller than zero, the centrifugal force dominates over the gravitational one and the radial 3-acceleration is positive (directed away from the black hole) at the arbitrary radial component of the 3-velocity. The similar fact was established in another context in papers [95] and [96].

4.6 Comparing the motion of accelerated rod and pointlike particle

In this Section we consider the motion of an accordion rod falling in the radial direction in the gravitational field of a black hole. As already mentioned, in general the points of an extended body can not fall freely in the curved spacetime, consequently the 3-acceleration of a comoving observer must differ from the 3-acceleration of a freely falling point mass. To avoid the extremely complicated calculations while determining this 3-acceleration, we restrict ourselves to the case, when the observer's 3-velocity is not hyperrelativistic.

We denote the rod's proper length by l and its line density as the rest mass of a line element of the unit proper length by ξ. A comoving observer is again placed on the rod so that the distances of both ends relative to him are equal.

Due to formula (122) the 4-acceleration $\bar{K}$ of each point of the rod must be in observer's Fermi coordinates a purely spatial vector

$$\bar{K} = (0, \bar{\kappa}). \tag{177}$$

It means, its zeroth component must be equal to zero

$$\frac{D^2 t}{d\bar{t}^2} = \ddot{t} + 2\bar{\Gamma}^t_{rt}\dot{r}\dot{t} + \bar{\Gamma}^t_{tt}\dot{t}^2 = 0. \tag{178}$$

In this Section a dot over a symbol denotes the ordinary derivative with respect to $\bar{t}$, the proper time of a clock placed at Δr. Making use of the definition formula of the Christoffel symbols formula (178) takes the form

$$\bar{g}_{tt}\ddot{t} + \frac{1}{2}\bar{g}_{tt,t}\dot{t}^2 = 0. \tag{179}$$

The energy of an element of the rod of the unit rest mass as the zeroth component of its 4-momentum reads

$$\bar{e} = \bar{g}_{tt}\dot{t}. \tag{180}$$

Its change rate measured with the clock comoving with this element of the rod,

$$\dot{\bar{e}} = \bar{g}_{tt,t}\dot{t}^2 + \bar{g}_{tt}\ddot{t}, \tag{181}$$

is caused only by the curvature of spacetime. Due to formulae (179) and (181) we can write

$$\dot{\bar{e}} = \frac{1}{2}\bar{g}_{tt,t}\dot{t}^2. \tag{182}$$

As $\dot{r} = 0$, the square of a differential of the proper time

$$d\bar{t}^2 = \bar{g}_{tt}dt^2 \;\Rightarrow\; \dot{t} = \bar{g}^{tt}, \tag{183}$$

and consequently presents

$$\dot{\bar{e}} = \bar{\Gamma}^t_{tt}. \tag{184}$$

Remark. Equation (184) provides the physical meaning of the Christoffel symbol $\bar{\Gamma}^t_{tt}$. In the case of a test particle of the unit rest mass, at rest in observer's frame of reference, $\bar{\Gamma}^t_{tt}$ is the derivative of its energy with respect to the proper time of the clock at the location of this test particle.

Then, according to the observer's clock, the change rate of $\bar{e}$ reads

$$\frac{d\bar{e}}{dt} = \bar{\Gamma}^t_{tt}\frac{d\bar{t}}{dt}. \tag{185}$$

To express the factors on the right hand side, we use formulae (155) and (156). So far, as we supposed the observer's 3-velocity $\beta_\rho = d\rho/d\tau$ not to be hyperrelativistic, it is sufficient to take into account only the first two terms in the Taylor expansions of $\bar{\Gamma}^t_{tt}$ and $d\bar{t}/dt$. Consequently

$$\frac{d\bar{e}}{dt} = -\frac{\partial \kappa}{\partial t}r + \frac{1}{2}\frac{\partial \psi}{\partial t}r^2 + \mathcal{O}_3. \tag{186}$$

The change rate of the total energy of the rod relative to the observer can be found as follows:

$$\frac{dE}{dt} = 2\xi\int_{-l/2}^{l/2}\left(\frac{\partial \psi}{\partial t}r^2 - \frac{\partial \kappa}{\partial t}r\right)dr = \frac{\partial \psi}{\partial t}\frac{\xi l^3}{12} \tag{187}$$

According to formula (143), in the Schwarzschild spacetime

$$\frac{\partial \psi}{\partial t} = -\frac{6\gamma M}{\rho^4}\beta_\rho\sqrt{\frac{c}{c^2 - \beta_\rho^2}}. \tag{188}$$

Due to formulae (187) and (188), the total energy of the rod relative to the observer changes with the following rate:

$$\frac{dE}{dt} = -\frac{\gamma M \xi l^3}{4\rho^4}\beta_\rho\sqrt{\frac{c}{c^2 - \beta_\rho^2}}. \tag{189}$$

We replace the line density ξ by the mass of the rod in the observer's frame of reference, which in global coordinates τ and ρ would mean the rest mass of the rod as a whole. Taking into account formulae (158), (162) and (180) allows to calculate the energy of the line element of the proper length dr:

$$\xi\bar{e}dr = \xi\left(1 + \kappa r - \frac{1}{2}\psi r^2\right)dr. \tag{190}$$

The total energy of the rod as its mass in observer's frame of reference is then

$$m = \xi\left(l - \frac{\psi l^3}{24}\right) + \mathcal{O}_4. \tag{191}$$

Due to formula (140), it reads

$$m = \xi \left(l - \frac{\gamma M l^3}{12\rho^3} \right) + \mathcal{O}_4. \tag{192}$$

Substituting it into formula (189), we obtain

$$\frac{dE}{dt} = -\frac{\gamma M m l^2}{2\rho^4} \beta_\rho \left(1 + \frac{\gamma M l^2}{12\rho^3} \right) \sqrt{\frac{c}{c^2 - \beta_\rho^2}}. \tag{193}$$

Because the observer's proper time t and the global time τ are related by the formula

$$\frac{dt}{d\tau} = \sqrt{\frac{c^2 - \beta_\rho^2}{c}}, \tag{194}$$

the change rate of the internal energy of the rod as whole is in the global coordinates

$$\frac{dE}{d\tau} = -\frac{\gamma M m l^2}{2\rho^4} \beta_\rho \left(1 + \frac{\gamma M l^2}{12\rho^3} \right) \tag{195}$$

We point out once again, that m is the mass of the rod in observer's frame of reference and l is its proper length. The mechanical energy of the rod in the global coordinates as the zeroth component of its 4-momentum is

$$\varepsilon = g_{\tau\tau} \frac{d\tau}{dt} = m \sqrt{\frac{c^3}{c^2 - \beta_\rho^2}}, \tag{196}$$

its derivative with respect to the global time τ reads

$$\frac{d\varepsilon}{d\tau} = m\beta_\rho \sqrt{\frac{c}{(c^2 - \beta_\rho^2)^3}} \left[\frac{\gamma M}{\rho^2} \left(c^2 - 3\beta_\rho^2 \right) + c \frac{d\beta_\rho}{d\tau} \right]. \tag{197}$$

Due to the energy conservation law,

$$\frac{dE}{d\tau} = -\frac{d\varepsilon}{d\tau}. \tag{198}$$

Taking into account also formulae (195) and (197), we get the observer's 3-acceleration

$$\frac{d\beta_\rho}{d\tau} = -\frac{\gamma M}{\rho^2 c} (c^2 - 3\beta_\rho^2) + \frac{\gamma M l^2}{2\rho^4} \left(1 + \frac{\gamma M l^2}{12\rho^3} \right) \sqrt{\frac{(c^2 - \beta_\rho^2)^3}{c^3}}. \tag{199}$$

The first term on the right hand side is the 3-acceleration of a freely falling point mass. The second one as the "correction term" is related to the extension of the rod and is caused by the change of the internal energy of the rod while it is moving in a curved spacetime. As we see, the correction term is the greater, the greater is the proper length of the rod and the smaller is the oberver's 3-velocity.

4.7 On the 4-dimensional case

In this Section we consider the motion of an accelerated accordion rod in a 4-dimensional curved spacetime. Each point of the rod must be accelerated selectively so that in the frame of reference of an observer comoving with the "center" of the rod at the 4-velocity X and 4-acceleration K, the position vectors H of all points of the rod must remain constant, i.e., they must move along the observer's world line according to the Fermi-Walker transport:

$$\frac{DH}{dt} = -g(H, K)X. \tag{200}$$

Here we note, that the observer's 4-acceleration K must not be directed along the rod as it was the case in a 2-dimensional spacetime. At the same time, to avoid the longitudinal deformations, the world lines of the points of the rod must be parallel to the observer's worldline, i.e. their 4-velocities $\bar{X}$ are obtained via parallel displacement of the observer's 4-velocity X along the rod as the spacelike geodesic:

$$\bar{X} = P(H)X. \tag{201}$$

Here $P(H)$ means the operator of the parallel displacement along the vector H as a two point tensor, having the components

$$\begin{aligned} P_i^a(H) \quad = \quad & \delta_i^a - \delta_l^a \Gamma^l_{ij}\eta^j + \frac{1}{2}\delta_r^a \left(-\Gamma^r_{ij,k} + \Gamma^r_{kp}\Gamma^p_{ij} + \Gamma^r_{ip}\Gamma^p_{jk}\right)\eta^j\eta^k \\ & + \frac{1}{6}\delta_k^a \left(-\Gamma^k_{ji,lp} + 2\Gamma^k_{lq,p}\Gamma^q_{ji} + 2\Gamma^k_{iq,p}\Gamma^q_{jl} + \Gamma^k_{ip,q}\Gamma^q_{jl} + \Gamma^k_{pq}\Gamma^q_{ji,l}\right. \\ & + \Gamma^k_{iq}\Gamma^q_{jl,p} - 2\Gamma^k_{iq}\Gamma^q_{pr}\Gamma^r_{jl} - \Gamma^k_{pq}\Gamma^q_{ir}\Gamma^r_{jl} - \Gamma^k_{pq}\Gamma^q_{lr}\Gamma^r_{ji} \\ & \left. - \Gamma^k_{qr}\Gamma^q_{ip}\Gamma^r_{jl} - \Gamma^k_{qr}\Gamma^q_{lp}\Gamma^r_{ji}\right)\eta^j\eta^l\eta^p + \mathcal{O}_4 \end{aligned} \tag{202}$$

At first we examine the rates of clocks, comoving with the different points of the rod, comparing them with the rate of the observer's clock. We recall here formula (7)

$$x^a \frac{d\bar{t}}{dt} = \mathcal{H}^a_j \frac{D\eta^i}{dt} + \mathcal{K}^a_j x^j. \tag{203}$$

Shifting the vector $\bar{X}$ to the observer's location by means of the inverse $\overset{-1}{P^i_a}(H)$ of the operator of parallel displacement $P^a_i(H)$ along the vector H, we obtain, taking into account also formula (201)

$$x^i \frac{d\bar{t}}{dt} = \overset{-1}{P^i_a}(H) \left(\mathcal{H}^a_j \frac{D\eta^i}{dt} + \mathcal{K}^a_j x^j\right). \tag{204}$$

Contracting now formula (204) with the covector $g(X)$ gives the exact formula for the differentials of the proper times, taking into account formula (201):

$$\frac{d\bar{t}}{dt} = g_{ik}x^k \overset{-1}{P^i_a}(H) \left[\mathcal{K}^a_j - g(H, K)\mathcal{H}^a_j\right] x^j. \tag{205}$$

The Taylor series of Jacobi propagators are (20). Ergo, we get for the clock rates on the observer's location and at an arbitrary point of the rod the following expression:

$$\begin{aligned}\frac{d\bar{t}}{dt} &= 1 - g(H,K) - \frac{1}{2}g\left(X, R(H,X,H)\right) \\ &+ \frac{1}{6}g(H,K)g\left(X, R(H,X,H)\right) - \frac{1}{6}g\left(X, (\nabla_H R)(H,X,H)\right) + \mathcal{O}_4.\end{aligned} \tag{206}$$

In order to avoid the longitudinal deformations of the rod, all its points must be accelerated selectively. The necessary 4-acceleration $\bar{K}$ of an arbitrary point can be obtained by differential prolongation of formula (201), viz.

$$\bar{K} = \left(\frac{dP(H)}{dt}X + P(H)K\right)\frac{dt}{d\bar{t}}, \tag{207}$$

or otherwise

$$\bar{K} = \left\{[\nabla_X P(H)]X + P(H)K\right\}\frac{dt}{d\bar{t}} + [\nabla_{\bar{X}} P(H)]X.$$

We use now the expressions for the covariant derivatives of the operator of parallel displacement with respect to the coordinates of the observer and an arbitrary point of the rod, see formulae (19) which have been derived in [90], and also formulae (201) and (206), we get the following result

$$\begin{aligned}\bar{K} &= P(H)\Big\{K[1 + g(H,K) + g^2(H,K)] + Kg[X, R(H,X,H)] \\ &+ R(X,H,X) + \frac{1}{2}g(H,K)R(X,H,X) + \frac{1}{2}(\nabla_H R)(X,H,X)\Big\} + \mathcal{O}_3.\end{aligned} \tag{208}$$

Let us recall, that the modulus of a vector does not change under the parallel displacement. It means, if we want to calculate only the modulus of $\bar{K}$, we do not need to shift the vector into the location of the point of the rod. We point out again, that if the observer moves at the hyperrelativistic 3-velocity relative to the sources of gravitational field, the Taylor expansion (208) may not converge due to the last term.

4.8 Conclusions

In the present Section we examined the frame of reference of an observer, accelerated in a curved spacetime. Generalizing the accordion model of a rigid rod, introduced by Mould [70], we calculated first the accelerations of the elements of the rod, necessary to keep the rod's length constant in the comoving observer's frame of reference. Then we compared the rates of comoving clocks placed on the rod at various spatial distances from the observer. We have shown, that the accelerations and clock rates depend on observer's acceleration and the curvature of spacetime.

These two factors are in lower approximations independent, but if the observer moves in a curved spacetime at a relativistic 3-velocity, some higher order small "mixed" terms begin to dominate and the approximation breaks down. The abovementioned terms contain also the observer's 3-velocity. Next it has been shown, that the metric in the observer's Fermi coordinates is in the third approximation time-dependent. This circumstance causes, that the 3-acceleration of a freely falling test particle, moving relative to the observer at the 3-velocity $\bar{c}\sqrt{2}/2$, will no more be zero as it is the case in Minkowski spacetime or if the observer moves in a curved spacetime at a non-relativistic 3-velocity. The acceleration of free fall has been studied in the case of an arbitrary stationary and orthogonal metric in a two-dimensional spacetime.

Then the 3-acceleration of the rod, moving without energy exchange with the environment in the case of absence of non-gravitational forces, was considered. Compared with a freely falling point mass, a correction term in the formula of 3-acceleration of rod's mass centre appears. Consequently the rod's 3-acceleration is less than the 3-acceleration of the point mass. This correction term is the greater if the rod's length is greater and the rod's 3-velocity is lesser.

A The index notation

We adopt the metric signature $(+, -, -, -)$ with the Latin indices running and summing from 0 to 3. The Greek letters stand for the spatial indices, running from 1 to 3. We apply the technique of two-point tensors (or bitensors) and use the following notation and index convention. To distinguish the tensor indices which refer to the observer's location from those referring to the location of the test particle, we use the indices from the second half of the Latin and Greek alphabets like $i, j, k, ...$ and $\lambda, \mu, \nu, ...$ at the observer's location, and the indices from the first half of the Latin and Greek alphabets like $a, b, ...$ and $\alpha, \beta, ...$ at the location of the test particle. The quantities without indices like the proper time or the modulus of a vector, defined at the location of the test particle, are denoted with the bar (like $\bar{t}$ or $\bar{\kappa}$) to distinguish them from those defined at the observer's location.

The lower case indices at the symbol of world function σ from the second half of the Latin alphabet like $i, j, k, ...$ at σ mean its covariant derivatives with respect to the observer's coordinates, the lower case indices from the first half like $a, b, c, ...$ mean the covariant derivatives with respect to the corresponding coordinates of the test particle.

B Calculating the necessary Taylor expansions

Supposing that the observer and test particle are connected with the geodesic Λ, we can write at the observer's instantaneous location

$$\eta^i = \frac{du^i}{ds}\Delta s, \quad \nabla_H H = 0. \tag{209}$$

Using the differential prolongations of the last equation, we get the coordinate difference between the test particle and observer as a Taylor expansion

$$\Delta u^i = \bar{u}^i - u^i = \sum_{K=0}^{\infty} \frac{d^K u^i}{ds^K}\Delta s^K \tag{210}$$

in the following form:

$$\begin{aligned}
\Delta u^i = {} & \eta^i - \frac{1}{2}\Gamma^i_{jk}\eta^j\eta^k + \frac{1}{6}\left(2\Gamma^i_{pl}\Gamma^p_{jk} - \Gamma^i_{jk,l}\right)\eta^j\eta^k\eta^l \\
& + \frac{1}{24}\left(-\Gamma^i_{jk.lp} + 4\Gamma^i_{lq,p}\Gamma^q_{jk} + 2\Gamma^i_{pq}\Gamma^q_{jk,l} + \Gamma^i_{lp,q}\Gamma^q_{jk}\right. \\
& \left. - 2\Gamma^i_{rq}\Gamma^q_{lp}\Gamma^r_{jk} - 4\Gamma^i_{pq}\Gamma^q_{lr}\Gamma^r_{jk}\right)\eta^j\eta^k\eta^l\eta^p + \mathcal{O}_5
\end{aligned} \tag{211}$$

and its inverse

$$\begin{aligned}
\eta^i = {} & \Delta u^i + \frac{1}{2}\Gamma^i_{jk}\Delta u^j\Delta u^k + \frac{1}{6}\left(\Gamma^i_{jk,l} + \Gamma^i_{lp}\Gamma^p_{jk}\right)\Delta u^j\Delta u^k\Delta u^l \\
+ {} & \frac{1}{24}\left(\Gamma^i_{jk,lp} + \Gamma^i_{lp,q}\Gamma^q_{jk} + 2\Gamma^i_{p,q}\Gamma^q_{jk,l} + 2\Gamma^i_{pq}\Gamma^q_{jk,l}\right)\Delta u^j\Delta u^k\Delta u^l\Delta u^p + \mathcal{O}_5.
\end{aligned} \tag{212}$$

Now we write the components $P^a_i(H)$ of the *operator of parallel displacement* $P(H)$ as a two point tensor, enabling to shift an arbitrary vector Z, defined at the observer's location, parallel to itself into the location of test particle:

$$\begin{aligned}
P^a_i(H)z^i = {} & \left[\delta^a_i - \delta^a_l\Gamma^l_{ij}\eta^j + \frac{1}{2}\delta^a_r\left(-\Gamma^r_{ij,k} + \Gamma^r_{kp}\Gamma^p_{ij} + \Gamma^r_{ip}\Gamma^p_{jk}\right)\eta^j\eta^k\right. \\
& + \frac{1}{6}\delta^a_k\left(-\Gamma^k_{ji,lp} + 2\Gamma^k_{lq,p}\Gamma^q_{ji} + 2\Gamma^k_{iq,p}\Gamma^q_{jl} + \Gamma^k_{ip,q}\Gamma^q_{jl} + \Gamma^k_{pq}\Gamma^q_{ji,l}\right. \\
& + \Gamma^k_{iq}\Gamma^q_{jl,p} - 2\Gamma^k_{iq}\Gamma^q_{pr}\Gamma^r_{jl} - \Gamma^k_{pq}\Gamma^q_{ir}\Gamma^r_{jl} - \Gamma^k_{pq}\Gamma^q_{lr}\Gamma^r_{ji} \\
& \left.\left. - \Gamma^k_{qr}\Gamma^q_{ip}\Gamma^r_{jl} - \Gamma^k_{qr}\Gamma^q_{lp}\Gamma^r_{ji}\right)\eta^j\eta^l\eta^p + \mathcal{O}_4\right] z^i.
\end{aligned} \tag{213}$$

The last expression can be derived using the higher order differential prolongations of equation (209) and equation

$$\nabla_Y Z = 0. \tag{214}$$

The components of covectors, as well known, are shifted by means of $P(H)$ "backward", i.e. along the geodecic Λ parallel to itself from the location of the test particle into the observer's location. An operator $\overset{-1}{P}(H)$ as the inverse of $P(H)$ has the components

$$\begin{aligned}
\overset{-1}{P}{}^{i}_{a}(H)\bar{z}^{a} \;=\; & \Big[\delta^{i}_{a} + \Gamma^{i}_{kj}\eta^{j}\delta^{k}_{a} + \frac{1}{2}\left(\Gamma^{i}_{jl,k} + \Gamma^{i}_{kq}\Gamma^{q}_{jl} - \Gamma^{i}_{lq}\Gamma^{q}_{jk}\right)\delta^{l}_{a}\eta^{j}\eta^{k} \\
& + \frac{1}{6}\Big(\Gamma^{i}_{jp,kl} + \Gamma^{i}_{kq,l}\Gamma^{q}_{jp} - 2\Gamma^{i}_{pq,l}\Gamma^{q}_{jk} - \Gamma^{i}_{pl,q}\Gamma^{q}_{jk} + 2\Gamma^{i}_{lq}\Gamma^{q}_{jp,k} \\
& - \Gamma^{i}_{pq}\Gamma^{q}_{jk,l} + 2\Gamma^{i}_{pq}\Gamma^{q}_{lr}\Gamma^{r}_{jk} - 2\Gamma^{i}_{lq}\Gamma^{q}_{pr}\Gamma^{r}_{jk} - \Gamma^{i}_{qr}\Gamma^{q}_{pl}\Gamma^{r}_{jk} \\
& - \Gamma^{i}_{lq}\Gamma^{q}_{kr}\Gamma^{r}_{jp}\Big)\eta^{j}\eta^{k}\eta^{l}\delta^{p}_{a} + \mathcal{O}_{4}\Big]\,\bar{z}^{a}, \qquad (215)
\end{aligned}$$

where $\bar{Z}$ is an arbitrary vector, defined at the location of the test particle.

C Components of the curvature tensor in the Fermi coordinates of a radially accelerated observer in the Schwarzschild spacetime

In this Appendix we will calculate the components of the Riemann curvature tensor in frame of reference of a radially accelerated observer in the Schwarzschild spacetime, needed in Section 5. We will work with the so-called global coordinates, namely, the proper time τ of an infinitely far observer and the radial (areal) coordinate $\rho = \sqrt{\dfrac{S}{4\pi}}$, where S is the area of a sphere around the black hole. The polar angles ϕ and θ are labelled by ordinary superscripts, i.e., $(u^0, u^1, u^2, u^3) = (\tau, \rho, \phi, \theta)$. The square of the line element then reads

$$ds^2 = \left(1 - \frac{2\gamma M}{\rho}\right) d\tau^2 - \frac{d\rho^2}{1 - \dfrac{2\gamma M}{\rho}} - \rho^2(\sin^2\theta d^2\phi + d^2\theta).$$

The observer's Fermi coordinates will be indicated by superscripts in parentheses, i.e., $u^{(i)}$. To use formula (31) in the observer's frame of reference, we must first calculate the components of curvature tensor in the coordinates $u^{(i)}$. We use here the method, described in Application B of Ref. [72].

The observer's 4-velocity X, which is at the same time the zeroth vector of the ON-basis, comoving with the observer, can be written as

$$X = \alpha\left(\partial_\tau + \beta_\rho \partial_\rho + \beta_\phi \partial_\phi\right), \tag{216}$$

where

$$\alpha = \sqrt{\frac{c}{c^2 - \beta_\rho^2 - c\beta_\phi^2 \rho^2}} \tag{217}$$

and

$$\beta_\rho = \frac{d\rho}{d\tau}, \quad \beta_\phi = \frac{d\phi}{d\tau}. \tag{218}$$

Without loss of generality, we can take in the present case $\theta \equiv \dfrac{\pi}{2}$ and $\dfrac{d\theta}{dt} = 0$, i.e., we can consider the motion to take place on the equatorial plane. Here the quantity c means the radial velocity of light relative to an infinitely far observer

$$c = 1 - \frac{2\gamma M}{\rho}. \tag{219}$$

The first basis vector reads

$$Y = \psi(\beta_\rho \partial_\tau + c^2 \partial_\rho), \tag{220}$$

where

$$\psi = \frac{1}{\sqrt{c(c^2-\beta_\rho^2)}}.$$

To calculate the components of the second basis vector

$$Z = z^\tau\partial_\tau + z^\rho\partial_\rho + z^\phi\partial_\phi,$$

we use the system of equations which defines the ON-basis, viz.,

$$g(Z,Z) = -1, \quad g(X,Z) = 0, \quad g(Y,Z) = 0.$$

Solving the system and taking the third basis vector W to be a unit vector, orthogonal to the hypersurface $\theta = \pi/2$, we get finally the following ON-basis, comoving with the observer in the Schwarzschild spacetime:

$$\begin{aligned}
X &= \alpha(\partial_\tau + \beta_\rho\partial_\rho + \beta_\phi\partial_\phi), \\
Y &= \psi(\beta_\rho\partial_\tau + c^2\partial_\rho), \\
Z &= \zeta\left(\beta_\phi\partial_\tau + \beta_\phi\beta_\rho\partial_\rho + \frac{c^2-\beta_\rho^2}{\rho^2 c}\partial_\phi\right), \\
W &= \frac{1}{\rho}\partial_\theta,
\end{aligned} \tag{221}$$

where

$$\zeta = \frac{c\rho}{\sqrt{(c^2-\beta_\rho^2)(c^2-\beta_\rho^2-c\rho^2\beta_\phi^2)}}. \tag{222}$$

System (221) determines the transformation matrix (for the basis vectors and the components of 1-forms) from the global coordinates into the observer's frame of reference

$$A^i_{(j)} = \begin{pmatrix} \alpha & \psi\beta_\rho & \zeta\beta_\phi & 0 \\ \alpha\beta_\rho & \psi c^2 & \zeta\beta_\phi\beta_\rho & 0 \\ \alpha\beta_\phi & 0 & \zeta\dfrac{c^2-\beta_\rho^2}{\rho^2 c} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\rho} \end{pmatrix}. \tag{223}$$

Its inverse, allowing to transform the coordinate differentials and the components of vector fields from the global coordinates into the observer's frame of reference, reads

$$\overset{-1}{A}{}^{(i)}_{j} = \begin{pmatrix} \alpha c & -\dfrac{\alpha\beta_\rho}{c} & -\rho^2\beta_\phi\alpha & 0 \\ -c\beta_\rho\psi & c\psi & 0 & 0 \\ -c\zeta\beta_\phi & \dfrac{\zeta\beta_\rho\beta_\phi}{c} & \dfrac{\zeta(c^2-\beta_\rho^2)}{c} & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix}. \tag{224}$$

As known (see e.g. Ref. [97], [98]), the Riemann curvature tensor has in the Schwarzschild spacetime the following nonvanishing components (and the ones obtainable by symmetry):

$$\begin{aligned}
2R^1_{212} = 2R^1_{313} = 2R^0_{220} = 2R^0_{330} = R^3_{232} = R^2_{323} &= \frac{2\gamma M}{\rho}, \\
R^1_{010} = -R^1_{001} = -2R^2_{020} = -2R^3_{030} &= -\frac{2c\gamma M}{\rho^3}, \\
2R^2_{112} = 2R^3_{113} = R^0_{101} = -R^0_{110} &= \frac{2\gamma M}{c\rho^3}.
\end{aligned}$$

Using the transformation matrices (223) and (224), we can calculate the non-vanishing components of the Riemann curvature tensor in the observer's Fermi coordinates (and the ones obtainable by symmetry):

$$\begin{aligned}
R^{(0)}_{(1)(0)(1)} = R^{(1)}_{(0)(0)(1)} = R^{(2)}_{(3)(2)(3)} &= R^{(3)}_{(2)(3)(2)} = \\
&= \frac{2\gamma M}{\rho^3}\left(1 + \frac{3}{2}\frac{c\rho^2\beta_\phi^2}{c^2 - \beta_\rho^2 - c\rho^2\beta_\phi^2}\right), \\
R^{(0)}_{(2)(0)(2)} = R^{(2)}_{(0)(0)(2)} = R^{(3)}_{(1)(3)(1)} &= R^{(1)}_{(3)(1)(3)} = -\frac{\gamma M}{\rho^3}, \\
R^{(2)}_{(1)(2)(1)} = R^{(1)}_{(2)(1)(2)} = R^{(0)}_{(3)(0)(3)} &= R^{(3)}_{(0)(0)(3)} = \\
&= -\frac{\gamma M}{\rho^3}\left(1 + \frac{3c\rho^2\beta_\phi^2}{c^2 - \beta_\rho^2 - c\rho^2\beta_\phi^2}\right), \\
R^{(2)}_{(1)(0)(1)} = -R^{(2)}_{(1)(1)(0)} = R^{(1)}_{(0)(1)(2)} &= R^{(0)}_{(1)(1)(2)} = \\
= R^{(3)}_{(2)(3)(0)} = R^{(2)}_{(3)(3)(0)} = R^{(3)}_{(0)(2)(3)} &= R^{(0)}_{(3)(2)(3)} = \\
&= \frac{3\gamma M}{\rho^2}\beta^\phi \frac{\sqrt{c(c^2 - \beta_\rho^2)}}{c^2 - \beta_\rho^2 - c\rho^2\beta_\phi^2}\,. \qquad (225)
\end{aligned}$$

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Summary in English

With the help of the concept of the world function has beeb derived the exact generalized deviation equation is, describing the relative motion of two accelerated point masses in the curved spacetime. The obtained equation is valid in case of the arbitrary velocities and accelerations of the point masses and also of the arbitrary parametrization of their world lines. The Taylor expansion of the deviation equation relative to the components of the deviation vector is presented in the general coordinates and in the Fermi coordinates of an accelerated observer.

The second approximation of the relativistic deviation equation is examined in two cases – the radially accelerated observer in the Schwarzschild spacetime and the Weber oscillatori in the field of a weak monochromatic plane gravitational wave.

Using the second order deviation equation in the Schwarzschild spacetime, there is calculated the acceleration of free fall of a test particle in the Fermi coordinates of an observer, accelerated in the radial direction. The second order terms in the Taylor expansion of the acceleration of free fall contain also the observer's 3-velocity relative to the central mass as the source of the gravitational field. If the observer's 3-velocity approaches to the velocity of light, it causes the unlimited growth of the abovementioned second order terms. Ergo, if the observer's motion is relativistic, the second order terms become comparable with the first order ones and their contribution can not be neglected as in the case of an observer moving at the nonrelativistic 3-velocity. Then one can take into account only the first order terms independent on the observer's motion relative to the central mass.

In the second order terms are "hidden" some relativistic effects, not predictable with the first order equation. For example a counterintuitive result, that the observer's motion in the radial direction causes the *decreasing* of the tidal forces. Intuitively, taking into account the relativistic effects like the increasing of the mass and the shortening of the length, we should get the contrary result. At the same time, if the radial component of the observer's 3-velocity is relativistic and also the nonradial component differs from zero, it causes the enormous tidal forces able to destroy an arbitrary extended body. But if the obserfer moves at the hyperrelativistic 3-velocity relative to the central body, the second order terms in the deviation equation begin to dominate over the first order terms. Then the equation as the Taylor expansion does not more converge and the theory is not applicable.

Using the second order deviation equation is also derived the formula to calculate the longitudinal pressure of the gravitational wave on the Weber detector. The same result is obtained with the alternative method – according to the energy-momentum conservation law, the work per unit time, done by gravitational waves against the dissipative forces in order to keep the detector

oscillating, must be equal to the momentum obtained by detector per unit time. The result confirms also the well-known fact, that the rest mass of a graviton equals to zero. Additionally has been shown, that if the detector approaches the source of the gravitational waves at the relativistic 3-velocity, then the longitudinal pressure of the gravitational waves will be considerably amplified.

In the 2-dimensional spacetime also the higher approximations of the tidal forces and the acceleration of free fall in the observer's frame of reference have been examined. Here the Mould concept of the accordion rod is generalized to the case of the curved spacetime. Each point of the rod must be separately accelerated in order to keep their distances constant relative to an observer, comoving with the certain point of the rod. The necessary 4-acceleration of an arbitrary point of the rod and the rate of the clock comoving with this point are calculated in the fourth and the fifth approximations correspondingly. Using the obtained results, the components of the metric tensor in the fifth and the connection coefficients in the fourth approximation in the Fermi coordinates of an accelerated observer have been calculated. The dependence of the abovementioned quantities on the curvature of the spacetime, on the observer's acceleration and also on the observer's velocity has been analyzed. The calculations are made for the general case, but also in case of the 2-dimensional Schwarzschild spacetime. According to the results, the terms conteining the observer's 3-velocity depend in the first approximation only on the curvature of the spacetime, in the third and higher approximation also on the observer's 4-acceleration.

The special attention has been paid on the so-called "phenomena of $c/\sqrt{2}$" in the frame of reference of an observer accelerated in the 2-dimensional Schwarzschild spacetime. According to this phenomena, the acceleration of free fall of a test particle equals to zero, if the test particle moves at the 3-velocity $c/\sqrt{2}$ relative to the observer. If the 3-velocity exceeds $c/\sqrt{2}$, the acceleration of free fall changes the sign. Using the energy conservation law has been shown, that this phenomena appears only in the spacetime with the stationary metrics. If the observer himself moves at the relativistic 3-velocity relative to the central mass, then due to the higher order terms, depending on the observer's 3-velocity, then the metric in the comoving Fermi coordinates is not more stationary and the "phenomena of $c/\sqrt{2}$" is not more valid.

With the help of the energy conservation law is also compared the free fall of a point mass and the fall of an extended rod in absence of the nongravitational exterior forces in the 2-dimensional Schwarzschild spacetime. Due to the tidal forces, the 3-acceleration of the rod is smaller than the acceleration of free fall of the point mass. The difference is the smaller, the greater is the 3-velocity of the rod.

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Kokkuvõte

Käesolevas väitekirjas "Teist järku relativistlik hälbevõrrand ja selle võrrandi rakendusi" on maailmafunktsiooni mõistet kasutades tuletatud geodeetilise hälbe täpne üldistatud võrrand, mis kirjeldab kahe kiirendatava punktmassi suhtelist liikumist kõveras aegruumis nende osakeste meelevaldsete kiirenduste, kiiruste ja maailmajoonte suvalise parametrisatsiooni korral. Nimetatud võrrandi teist järku lähend rittaarenduses hälbevektori komponentide suhtes on esitatud nii üldistes koordinaatides kui ka ühe punktmassiga kaasa liikuva vaatleja Fermi koordinaatides.

Hälbevõrrandi teist järku lähendit kiireneva vaatleja Fermi koordinaatides on põhjalikumalt uuritud kahel erijuhul – radiaalsihis kiirenev vaatleja Schwarzschildi raskusjõu väljas ja Weberi ostsillaator nõrga monokromaatilise gravitatsioonilise tasalaine väljas.

Schwarzschildi välja juhul on hälbevõrrandit kasutades arvutatud proovi-osakese vaba langemise kiirendus radiaalsihis kiireneva vaatlejaga seotud taust-süsteemis. Vaba langemise kiirenduse Taylori reaksarenduses sisaldavad teist järku liidetavad ka vaatleja 3-kiirust tsentraalkeha kui gravitatsioonivälja allika suhtes ning vaatleja kiiruse lähenemine valguse kiirusele toob kaasa nende liidetavate piiramatu kasvu. Seega muutuvad vaatleja relativistlikul liikumisel teist järku liidetavad võrreldavateks esimest järku liidetavatega. Järelikult ei saa nende mõju jätta arvestamata nii nagu mitterelativistliku suhtelise kiirusega liikuva vaatleja korral, kus tähtsust omavad vaid esimest järku liidetavad, mis teatavasti ei sõltu vaatleja kiirusest tsentraalkeha suhtes. Teist järku liidetavatega kaasnevad mitmed uut tüüpi relativistlikud efektid, mida esimest järku võrrand ei kajasta. Nii näiteks on teist lähendit arvestades saadud intuitsioonile vastu rääkiv tulemus, et vaatleja liikumine radiaalsihis põhjustab temaga kaasa liikuvas taustsüsteemis loodejõudude vähenemist, kuigi massi relativisliku suurenemise ja kauguse relativistliku vähenemise arvestamine peaks viima hoopis vastupidisele tulemusele. Samas aga, kui relativistliku kiirusega vaatleja 3-kiirus omab ka mitteradiaalset komponenti, võib see põhjustada ülitugevaid loodejõude, mis on võimelised purustama igasuguse ulatuvusega keha. Kui vaatleja liigub tsentraalkeha suhtes hüperrelativistliku kiirusega, hakkavad hälbevõrrandis teist järku liidetavad domineerima esimest järku liidetavate üle. Siis võrrand kui Taylori reaksarendus enam ei koondu ja teooriat ei saa nimetatud juhule rakendada.

Gravitatsioonilise tasalaine juhul on teist järku hälbevõrrandit Weberi ostsillaatori näitel kasutades tuletatud valem gravitasioonilaine poolt ostsillaatorile avaldatava pikisuunalise rõhu arvutamiseks. Samale tulemusele viib ka alternatiivne meetod, kus energia-impulsi jäävuse seadust arvestades on lähtutud eeldusest, et gravitatsioonilainete poolt hõõrdejõudude vastu ajaühikus tehtud töö, mis on vajalik ostsillaatori võnkumise säilitamiseks, võrdub ostsillaatori pikisuunalise impulsi juurdekasvuga ajaühikus. Saadud tulemus kinnitab ka varemtuntud fakti, et gravitoni seisumass võrdub nulliga. Lisaks on näidatud, et kui Weberi ostsillaator läheneb gravitatsioonilainete allikale relativistliku kiirusega, siis gravitatsioonilainete pikisuunaline rõhk võimendub oluliselt. Kui vaatleja kiirus läheneb valguse kiirusele, siis gravitatsioonilainete rõhk kasvab piiramatult.

Lihtsamal, kahemõõtmelise kõvera aegruumi juhul, on uuritud liikuva vaatlejaga seotud taustsüsteemis loodejõude ja vaba langemise kiirendust ka kõrgemaid lähendusi arvestades. Siin on kasutatud Mouldi lõõtsvarda kontseptsiooni, kus Minkowski aegruumis pikisuunalise kiirendusega liikuva lõõtsvarda iga punkti kiirendatakse selliselt, et varda iga punkti kaugus mingis kindlas punktis paikneva kaasaliikuva vaatleja suhtes jääks muutumatuks. Kontseptsiooni on üldistatud kõverale aegruumile, neljandas lähenduses on arvutatud varda suvalise punkti kiirendus ning viiendas lähenduses selle punktiga kaasaliikuva kella käigukiirus vaatleja kella suhtes. Saadud tulemusi kasutades on tuletatud valemid vaatleja Fermi koordinaatides meetrilise tensori komponentide arvutamiseks viiendas ja seostuse kordajate arvutamiseks neljandas lähenduses. On analüüsitud nimetatud suuruste sõltuvust nii aegruumi kõverusest kui ka vaatleja kiirusest gravitatsioonivälja allika suhtes ja tema kiirendusest. Arvutused on tehtud üldjuhul ja kahemõõtmelise Schwarzschildi raskusvälja juhul. Tulemustest järeldub, et need vaatleja 3-kiirusest sõltuvad liidetavad, mis kiiruse lähenemisel valguse kiirusele piiramatult kasvama hakkavad, sisaldavad teises lähenduses ainult aegruumi kõverust, kolmandas ja kõrgemates lähendustes ka vaatleja 4-kiirendust.

Eraldi on käsitletud niinimetatud "$c/\sqrt{2}$ fenomeni" Schwarzschildi kahemõõtmelises aegruumis kiireneva vaatleja taustsüsteemis. Selle kohaselt muutub prooviosakese vaba langemise kiirendus nulliks, kui prooviosake liigub vaatleja suhtes kiirusega $c/\sqrt{2}$, suurema kiiruse korral aga muutub vaba langemise kiirendus vastandmärgiliseks. Energia jäävuse seadust kasutades on näidatud, et nimetatud fenomen leiab aset ainult statsionaarse meetrika korral. Kui aga vaatleja liigub tsentraalkeha suhtes relativistliku kiirusega, ei saa meetrikat Taylori rekasarenduses vaatleja kiirust sisaldavate kõremat järku liidetavate tõttu enam meetrikat statsionaarsena vaadelda ning "$c/\sqrt{2}$ fenomen" enam ei toimi.

Samuti on energia jäävuse seadust kasutades võrreldud Schwarzschildi kahemõõtmelises aegruumis punktmassi ja lõpliku ulatusega varda langemist väliste mittegravitatsiooniliste jõudude puudumise korral. Loodejõududest tingituna langeb varras väiksema 3-kiirendusega, kuid erinevus punktmassi 3-kiirendusest on seda väiksem, mida suurem on varda radiaalsihiline kiirus.

Attached original publications

Curriculum Vitae

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1998 - 2002	University of Tartu, Department of Physics, doctoral studies
1996 - 1998	University of Tartu, Department of Physics, MSc degree of Physics. MSc Thesis: "The alternative ways to derive the generalized geodesic deviation equation and the applications of this equation"
1982 - 1989	University of Tartu, Department of Physics, baccalaureate studies

Employment:

2000-present	Tallinn University of Technology, Department of Physics, Chair of Theoretical Physics; Assistant
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1995 - 2000	University of Tartu, Faculty of Mathematics and Computer Science, Institute of Pure Mathematics, Chair of Geometry and Topology; Secretary

Scientific interest:	Natural Sciences and Engineering
Physics and Technical Physics:	Relativistic gravitation
Mathematics:	Mathematical methods in control theory
Mechanical Engineering and Automation Technology and	
Nonlinear control systems:	modeling, analysis and synthesis

Curriculum Vitae

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Haridus:

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Teenistuskäik:

2000-...	Tallinna Tehnikaülikool, füüsikaosakond, teoreetilise füüsika õppetool, assistent
2000-...	Tallinna Tehnikaülikool, Küberneetika Instituut, teadur
1995 - 2000	Tartu Ülikool, matemaatika-informaatikateaduskond, puhta matemaatika instituut, geomeetria ja topoloogia õppetool; sekretär

Uurimisvaldkonnad:	Loodusteadused ja tehnika
Füüsika:	relativistlik gravitatsioon
Matemaatika:	juhtimisteooria matemaatilised alused
Mehhanotehnika, automaatika ja tööstustehnoloogia:	mittelineaarsed juhtimissüsteemid, modelleerimine, analüüs, süntees

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