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FUNCTIONAL ANALYSIS AND THEORY OF SUMMABILITY

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Matrix transformations of summability fields of regular perfect matrix methods

Ants Aasma

In this paper we shall prove theorems that give necessary and sufficient conditions for a matrix M to be a transformation of the summability field of regular perfect matrix method into the summability field of a triangular matrix. We shall also define the M-consistency of matrices and find necessary and sufficient conditions for it. This paper extends the author's research started in [1, 2, 8, 9]. The matrix transformations of the summability fields of reversible matrices have been studied also in $[3-6]$.

1. Notions and notations

Let $A = (\alpha_{nk})$ be a matrix with $\alpha_{nk} \in \mathbb{C}$. In the sequel we consider the following sequence spaces :

$$c = \{x = (x_n) \mid \lim_n x_n \text{ exists}\},$$

$$c^0 = \{x = (x_n) \mid x \in c \text{ and } \lim_n x_n = 0\},$$

$$cs = \{x = (x_n) \mid \text{the series } \textstyle\sum_k x_k \text{ converges}\},$$

$$s_A = \{x = (x_n) \mid A_n x = \textstyle\sum_k \alpha_{nk} x_k \ (n \in \mathbb{N}) \text{ exists}\},$$

$$c_A = \{x = (x_n) \mid x \in s_A \text{ and } (A_n x) \in c\}$$

and

$$c_A^0 = \{x = (x_n) \mid x \in s_A \text{ and } \lim_n A_n x = 0\}.$$

In addition, we put

$$\mathfrak{M}(c_A) = \{M = (m_{nk}) \mid m_{nk} \in \mathbb{C} \text{ and } M_n x = \sum_k m_{nk} x_k$$
$$\text{exists for each } n \in \mathbb{N} \text{ and } x = (x_k) \in c_A\}$$

and

$$(c_A, c_B) = \{M = (m_{nk}) \mid M \in \mathfrak{M}(c_A) \quad \text{and} \quad (M_n x) \in c_B$$

$$\text{for each} \quad x \in c_A\}$$

where $B = (\beta_{nk})$ is a matrix with $\beta_{nk} \in \mathbb{C}$. Let $M = (m_{nk})$ be an arbitrary matrix with $m_{nk} \in \mathbb{C}$. We say that A and B are M-consistent when

$$\lim_n A_n x = \lim_n \sum_k \beta_{nk} M_k x$$

for each $x \in c_A$. If $M = (\delta_{nk})$ where $\delta_{nk} = 1$ for $n = k$ and $\delta_{nk} = 0$ for $n \neq k$, M-consistency of matrices coincides with their ordinary consistency.

Let further $A = (\alpha_{nk})$ be a series-to-sequence method, $\mathfrak{A} = (a_{nk})$ a sequence-to-sequence method and $B = (\beta_{nk})$ a triangular matrix. In addition, let $e^k = (0, \ldots, 0, 1, 0, \ldots)$ with number 1 in k-th position, $e = (1, 1, \ldots)$ and $\Delta = \{e^0, e^1, \ldots\}$. We assume that A and $\mathfrak{A}$ are perfect and regular. It means that the sets Δ and $\Delta \cup \{e\}$ are fundamental sets for c_A and $c_{\mathfrak{A}}$ respectively, $\lim_n A_n x = \sum_n x_n$ for each $x = (x_n) \in cs$ and $\lim_n \mathfrak{A}_n x = \lim_n x_n$ for each $x = (x_n) \in c$. Besides, we assume that $c_{\mathfrak{A}}^0$ and c_A are BK-spaces, i.e. the Banach spaces where the coordinate-wise convergence holds. The norm is defined in c_A by the equality $\| x \|_{c_A} = \sup_n | A_n x |$ and in $c_{\mathfrak{A}}^0$ by $\| x \|_{c_{\mathfrak{A}}^0} = \sup_n | \mathfrak{A}_n x |$. We denote the topological conjugate spaces of c_A and $c_{\mathfrak{A}}^0$ by $(c_A)'$ and $(c_{\mathfrak{A}}^0)'$ respectively.

2. Auxiliary results

We shall here find necessary and sufficient conditions for M to belong to $\mathfrak{M}(c_A)$ or $\mathfrak{M}(c_{\mathfrak{A}})$.

Lemma 1. *Let $\mathfrak{A} = (a_{nk})$ be a regular perfect method such that $c_{\mathfrak{A}}^0$ is a BK-space. Then numbers ϵ_k are the convergence factors for $\mathfrak{A}$ if and only if there exist functionals $f_l \in (c_{\mathfrak{A}}^0)'$ such that*

$$1) \quad f_l(e^k) = \begin{cases} \epsilon_k, & \text{if } k \leq l, \\ 0, & \text{if } k > l, \end{cases}$$

and

$$2) \quad \| f_l \|_{(c_{\mathfrak{A}}^0)'} = O(1).$$

Proof. Necessity. Let numbers ϵ_k be convergence factors for $\mathfrak{A}$ and

$$f_l(x^0) = \sum_{k=0}^{l} \epsilon_k x_k^0 \tag{1}$$

for each $x^0 = (x_k^0) \in c_{\mathfrak{A}}^0$. Then $f_l \in (c_{\mathfrak{A}}^0)'$ and therefore condition 1) is fulfilled. Condition 2) is valid by the principle of uniform boundedness because $c_{\mathfrak{A}}^0$ is a BK-space and the finite limit $\lim_l f_l(x^0)$ exists for each $x^0 \in c_{\mathfrak{A}}^0$.

Sufficiency. Let all conditions of the lemma be fulfilled. We shall show that the numbers ϵ_k are the convergence factors for $\mathfrak{A}$. First we show that the equalities (1) hold for each $x^0 = (x_k^0) \in c_{\mathfrak{A}}^0$. For doing it, let us denote

$$H_l(x^0) = f_l(x^0) - \sum_{k=0}^{l} \epsilon_k x_k^0$$

for each $x^0 \in c_{\mathfrak{A}}^0$. Then $H_l \in (c_{\mathfrak{A}}^0)'$ and moreover, $H_l(e^k) = 0$ by condition 1). Thus $H_l(x^0) = 0$ on the fundamental set Δ of the space $c_{\mathfrak{A}}^0$. Hence $H_l(x^0) = 0$ for each $x^0 \in c_{\mathfrak{A}}^0$. Therefore (1) holds for each $x^0 \in c_{\mathfrak{A}}^0$. Further, $x - \xi e \in c_{\mathfrak{A}}^0$ for each $x = (x_k) \in c_{\mathfrak{A}}$ if $\xi = \lim_n \mathfrak{A}_n x$. Thus each $x \in c_{\mathfrak{A}}$ may be represented in the form

$$x = x^0 + \xi e \tag{2}$$

where $x^0 = (x_k^0) \in c_{\mathfrak{A}}^0$. Hence we have

$$\sum_{k=0}^{l} \epsilon_k x_k = f_l(x^0) + \xi \sum_{k=0}^{l} \epsilon_k \tag{3}$$

for each $x = (x_k) \in c_{\mathfrak{A}}$. In addition, $\lim_l f_l(e^k) = \epsilon_k$ by condition 1), i.e. the sequence (f_l) converges on the fundamental set Δ of $c_{\mathfrak{A}}^0$. Consequently, we have by condition 2) and the theorem of Banach–Steinhaus that there exists the finite limit $\lim_l f_l(x^0)$ for each $x^0 \in c_{\mathfrak{A}}^0$. As $c^0 \subset c_{\mathfrak{A}}^0$, it is easy to see that $(\epsilon_k) \in cs$. Therefore from (3) follows the convergence of the series $\sum_k \epsilon_k x_k$ for each $x = (x_k) \in c_{\mathfrak{A}}$. Thus, the numbers ϵ_k are convergence factors for $\mathfrak{A}$.

Lemma 2. *Let $A = (\alpha_{nk})$ be a regular perfect method such that c_A is a BK-space. Then numbers ϵ_k are the convergence factors for A if and* *only if there exist functionals* $f_l \in (c_A)'$ *such that condition* 1) *of Lemma* 1 *is fulfilled and*

$$\| f_l \|_{(c_A)'} = O(1).$$

Proof is similar to the proof of Lemma 1.

For $M = (m_{nk}) \in \mathfrak{M}(c_{\mathfrak{A}})$ and $M \in \mathfrak{M}(c_A)$ it is necessary and sufficient that the numbers m_{nk} for each $n \in \mathbb{N}$ are the convergence factors for $\mathfrak{A}$ and A respectively. Therefore the next results hold by Lemmas 1 and 2.

Lemma 3. *Let* $\mathfrak{A} = (a_{nk})$ *be a regular perfect method such that* $c^0_{\mathfrak{A}}$ *is a BK-space and* $M = (m_{nk})$ *be an arbitrary matrix. Then* $M \in \mathfrak{M}(c_{\mathfrak{A}})$ *if and only if there exist functionals* $f_{sl} \in (c^0_{\mathfrak{A}})'$ *such that*

$$\text{(I)} \quad f_{sl}(e^k) = \begin{cases} m_{sk}, & \text{if } k \le l, \\ 0, & \text{if } k > l \end{cases}$$

and

$$\text{(II)} \quad \| f_{sl} \|_{(c^0_{\mathfrak{A}})'} = O_s(1).$$

Lemma 4. *Let* $A = (\alpha_{nk})$ *be a regular perfect method such that* c_A *is a BK-space and* $M = (m_{nk})$ *be an arbitrary matrix. Then* $M \in \mathfrak{M}(c_A)$ *if and only if there exist functionals* $f_{sl} \in (c_A)'$ *such that conditions* (I) *and*

$$\text{(III)} \quad \| f_{sl} \|_{(c_A)'} = O_s(1)$$

are fulfilled.

3. Main results

For an arbitrary triangular matrix $B = (\beta_{nk})$ and an arbitrary matrix $M = (m_{nk})$ we put

$$g_{nk} = \sum_{s=0}^{n} \beta_{ns} m_{sk}. \tag{4}$$

Theorem 1. *Let* $\mathfrak{A} = (a_{nk})$ *be a regular perfect method such that* $c^0_{\mathfrak{A}}$ *is a BK-space,* $B = (\beta_{nk})$ *be a triangular matrix and* $M = (m_{nk})$ *be an arbitrary matrix. Then* $M \in (c_{\mathfrak{A}}, c_B)$ *if and only if*

(IV) *there exist finite limits* $\lim_n g_{nk} = g_k$,

(V) *there exists finite limit* $\lim_n \sum_k g_{nk} = g$ *and there exist functionals* $f_{sl} \in (c^0_{\mathfrak{A}})'$ *such that conditions* (I), (II) *and*

$$\text{(VI)} \quad \| F_n \|_{(c^0_{\mathfrak{A}})'} = O(1),$$

where functionals F_n *are defined on* $c^0_{\mathfrak{A}}$ *by the equalities*

$$F_n(x^0) = \sum_{s=0}^{n} \beta_{ns} f_s(x^0) \tag{5}$$

and

$$f_s(x^0) = \lim_l f_{sl}(x^0), \tag{6}$$

are fulfilled.

Proof. Necessity. Let $M \in (c_{\mathfrak{A}}, c_B)$. Then we have

$$\sum_{s=0}^{n} \beta_{ns} \sum_{k=0}^{\infty} m_{sk} x_k = \sum_k g_{nk} x_k = G_n x$$

for each $x = (x_k) \in c_{\mathfrak{A}}$. Thus $c_{\mathfrak{A}} \subseteq c_G$. As $\mathfrak{A}$ is a regular method, the method G is conservative. Consequently, conditions (IV) and (V) are fulfilled.

As $M \in \mathfrak{M}(c_{\mathfrak{A}})$ then there exist functionals $f_{sl} \in (c^0_{\mathfrak{A}})'$ so that conditions (I) and (II) are fulfilled. These functionals may be represented in the form

$$f_{sl}(x^0) = \sum_{k=0}^{l} m_{sk} x^0_k \tag{7}$$

on $c^0_{\mathfrak{A}}$. Consequently, $f_s(x^0) = \lim_l f_{sl}(x^0) = M_s x^0$ for each $x^0 \in c^0_{\mathfrak{A}}$ and moreover, $f_s \in (c^0_{\mathfrak{A}})'$. Hence the functionals F_n, defined by (5) on $c^0_{\mathfrak{A}}$ for each $n \in \mathbb{N}$, are continuous and linear on $c^0_{\mathfrak{A}}$. We also notice, that $F_n(x^0) = G_n x^0$ for each $x^0 \in c^0_{\mathfrak{A}}$. Now it is easy to see that the sequence of continuous linear functionals (F_n) converges everywhere on Banach space $c^0_{\mathfrak{A}}$. Therefore condition (VI) is fulfilled by the principle of uniform boundedness.

Sufficiency. Let the conditions of this theorem be fulfilled. We shall show that $M \in (c_{\mathfrak{A}}, c_B)$. First we notice that $M \in \mathfrak{M}(c_{\mathfrak{A}})$ and (7) holds on the fundamental set Δ of $c^0_{\mathfrak{A}}$ by Lemma 3. Therefore (7) holds everywhere on $c^0_{\mathfrak{A}}$, whence it follows that $f_s(x^0) = \lim_l f_{sl}(x^0) = M_s x^0$ for each $x^0 \in c^0_{\mathfrak{A}}$. Consequently, $f_s \in (c^0_{\mathfrak{A}})'$. Thus we have $f_s(e^k) = m_{sk}$, $F_n(e^k) = g_{nk}$, the functionals $F_n \in (c^0_{\mathfrak{A}})'$ and the equalities $F_n(x^0) = G_n x^0$ hold for each $x^0 \in c^0_{\mathfrak{A}}$. Hence the sequence (F_n) converges on the fundamental set Δ of $c^0_{\mathfrak{A}}$ by condition (IV). Accepting it, we have by (VI) and the theorem of Banach–Steinhaus, that there exists the finite limit $\lim_n F_n(x^0)$ for each $x^0 \in c^0_{\mathfrak{A}}$. Moreover, it follows from (2) that

$$G_n x = F_n(x^0) + \xi \sum_k g_{nk} \tag{8}$$

for each $x = (x_k) \in c_{\mathfrak{A}}$ where $\xi = \lim_n \mathfrak{A}_n x$ and $x^0 \in c^0_{\mathfrak{A}}$. Therefore there exists the finite limit $\lim_n G_n x$ for each $x \in c_{\mathfrak{A}}$ by (V). Thus $M \in (c_{\mathfrak{A}}, c_B)$.

Theorem 2. *Let $A = (\alpha_{nk})$ be a regular perfect method such that c_A is a BK-space, $B = (\beta_{nk})$ be a triangular matrix and $M = (m_{nk})$ be an arbitrary matrix. Then $M \in (c_A, c_B)$ if and only if condition* (IV) *is fulfilled and there exist functionals $f_{sl} \in (c_A)'$ such that conditions* (I), (III) *and*

$$\text{(VII)} \quad \| F_n \|_{(c_A)'} = O(1),$$

where the functionals F_n are defined on c_A by the equalities

$$F_n(x) = \sum_{s=0}^{n} \beta_{ns} f_s(x)$$

and

$$f_s(x) = \lim_l f_{sl}(x),$$

are fulfilled.

Proof is similar to the proof of Theorem 1.

As an essential special case of Theorem 1 we shall consider now the case when $\mathfrak{A}$ is a regular method such that $c^0_{\mathfrak{A}}$ is a BK-AK-space. It means that $c^0_{\mathfrak{A}}$ is simultaneously a BK-space and an AK-space, i.e. $\Delta \subset c^0_{\mathfrak{A}}$ and $\lim_n \| x^{[n]} - x \| = 0$ for each $x = (x_k) \in c^0_{\mathfrak{A}}$ where $x^{[n]} = (x_0, \ldots, x_n, 0, \ldots)$. It is equivalent to the weak convergence by the section in the $c^0_{\mathfrak{A}}$ (cf. [7], p. 176). Thus

$$\lim_n |f(x^{[n]}) - f(x)| = 0$$

for each $x = (x_k) \in c_{\mathfrak{A}}^0$ and $f \in (c_{\mathfrak{A}}^0)'$. In this case Δ is a fundamental set of $c_{\mathfrak{A}}^0$. As (2) holds for each $x \in c_{\mathfrak{A}}$ where $x^0 \in c_{\mathfrak{A}}^0$ and $\xi = \lim_n \mathfrak{A}_n x$, then $\Delta \cup \{e\}$ is a fundamental set for $c_{\mathfrak{A}}$. Consequently, $\mathfrak{A}$ is a perfect method. But for each regular perfect method $\mathfrak{A}$ the space $c_{\mathfrak{A}}^0$ is not necessarily an AK-space (cf., for example, [7], p. 214-215). Now we prove the result which is given without the proof in [8].

Theorem 3. *Let $\mathfrak{A} = (a_{nk})$ be a regular method such that $c_{\mathfrak{A}}^0$ is a BK-AK-space, $B = (\beta_{nk})$ be a normal matrix and $M = (m_{nk})$ be an arbitrary matrix. Then $M \in (c_{\mathfrak{A}}, c_B)$ if and only if conditions* (IV) *and* (V), *where $G = (g_{nk})$ is defined by* (4), *are fulfilled and there exist functionals $F_n \in (c_{\mathfrak{A}}^0)'$ such that conditions* (VI) *and*

$$\text{(VIII)} \quad g_{nk} = F_n(e^k)$$

hold.

Proof. Necessity. Let $M \in (c_{\mathfrak{A}}, c_B)$. Then all conditions of Theorem 1 are fulfilled because the method $\mathfrak{A}$ is perfect. In addition, it is easy to see that functionals F_n, defined on $c_{\mathfrak{A}}^0$ by (5) and (6), belong to $(c_{\mathfrak{A}}^0)'$ and satisfy conditions (VI) and (VIII).

Sufficiency. Let all conditions of the theorem be fulfilled. We shall show that $M \in (c_{\mathfrak{A}}, c_B)$. As $c_{\mathfrak{A}}^0$ is an AK-space, for each $x^0 = (x_k^0) \in c_{\mathfrak{A}}^0$ we have

$$G_n x^0 = \sum_k F_n(e^k) x_k^0 = \lim_l F_n(x^{0[l]}) = F_n(x^0)$$

by condition (VIII). Thus $c_{\mathfrak{A}}^0 \subset s_G$. Moreover, (F_n) converges on the fundamental set Δ of $c_{\mathfrak{A}}^0$ by (IV) and (VIII). Consequently, (F_n) converges also on $c_{\mathfrak{A}}^0$ by (VI) and the theorem of Banach–Steinhaus. In addition to it, the equalities (8) hold for each $x = (x_k) \in c_{\mathfrak{A}}$ where $\xi = \lim_n \mathfrak{A}_n x$ and $x^0 \in c_{\mathfrak{A}}^0$. Therefore $c_{\mathfrak{A}} \subset c_G$ by (V). Then obviously $c_{\mathfrak{A}} \subset s_G$, whence it follows that $M \in \mathfrak{M}(c_{\mathfrak{A}})$ by the normality of B. Hence

$$G_n x = \sum_{s=0}^{n} \beta_{ns} M_s x$$

for each $x \in c_{\mathfrak{A}}$. Thus we have $M \in (c_{\mathfrak{A}}, c_B)$.

Now we can find (by Theorems $1 - 3$) necessary and sufficient conditions for M-consistency of $\mathfrak{A}$ (or A) and B.

Corollary 1. *Let* $\mathfrak{A} = (a_{nk})$ *be a regular perfect method such that* $c^0_{\mathfrak{A}}$ *is a BK-space,* $B = (\beta_{nk})$ *be a triangular matrix and* $M = (m_{nk})$ *be an arbitrary matrix. Then* $\mathfrak{A}$ *and* B *are M-consistent if and only if conditions* (IV) *and* (V) *with* $g_k = 0$ *and* $g = 1$ *are fulfilled and there exist functionals* $f_{sl} \in (c^0_{\mathfrak{A}})'$ *such that conditions* (I), (II) *and* (VI) *hold.*

Proof. *Necessity.* Let $\mathfrak{A}$ and B be M-consistent. Then $M \in (c_{\mathfrak{A}}, c_B)$. Therefore all the conditions of Theorem 1 are fulfilled . As the method $\mathfrak{A}$ is regular, $\lim_n \mathfrak{A}_n e^k = 0$ and $\lim_n \mathfrak{A}_n e = 1$. Hence $g_k = 0$ and $g = 1$ by the M-consistency of $\mathfrak{A}$ and B .

Sufficiency. Let all conditions of the corollary be fulfilled. Then $M \in (c_{\mathfrak{A}}, c_B)$ by Theorem 1. Moreover, for each $x \in c_{\mathfrak{A}}$ the equalities (8), in which $\xi = \lim_n \mathfrak{A}_n x$, $F_n(x^0) = G_n x^0$ and $x^0 \in c^0_{\mathfrak{A}}$ is defined by (2), are valid. As the sequence of the continuous linear functionals (F_n) converges everywhere on the Banach space $c^0_{\mathfrak{A}}$, its limit $F \in (c^0_{\mathfrak{A}})'$ and $F(x^0) = 0$ on the fundamental set Δ of the space $c^0_{\mathfrak{A}}$ by $g_k = 0$. Consequently , $F(x^0) = 0$ for each $x^0 \in c^0_{\mathfrak{A}}$. Therefore the M-consistency of $\mathfrak{A}$ and B follows from (8) by $g = 1$.

As the proofs of next results are similar to the proof of Corollary 1 then we give these results without proofs.

Corollary 2. *Let* $A = (\alpha_{nk})$ *be a regular perfect method such that* c_A *is a BK-space,* $B = (\beta_{nk})$ *be a triangular matrix and* $M = (m_{nk})$ *be an arbitrary matrix. Then* A *and* B *are M-consistent if and only if condition* (IV) *with* $g_k \equiv 1$ *is fulfilled and there exist functionals* $f_{sl} \in (c_A)'$ *such that conditions* (I), (III) *and* (VII) *hold.*

Corollary 3. *Let* $\mathfrak{A} = (a_{nk})$ *be a regular method such that* $c^0_{\mathfrak{A}}$ *is a BK-AK-space,* $B = (\beta_{nk})$ *be a normal matrix and* $M = (m_{nk})$ *be an arbitrary matrix. Then* $\mathfrak{A}$ *and* B *are M-consistent if and only if conditions* (IV) *and* (V) *with* $g_k = 0$ *and* $g = 1$ *are fulfilled and there exist functionals* $F_n \in (c^0_{\mathfrak{A}})'$ *such that conditions* (VI) *and* (VIII) *hold.*

References

1. Aasma, A., *On matrix transformations of summability fields.* Abstracts of conference "Problems of pure and applied mathematics". Tartu, 1990,

122–124.

2. Aasma, A., *The characterization of matrix transformations of summability fields.* Acta et Comment. Univ. Tartuensis, 1991, **928**, 3–14.
3. Alpár, L., *Sur certains changements de variable des series de Faber.* Studia Sci. Math. Hungar., 1978, **13**, no. 1-2, 173–180 (1981).
4. Alpár, L., *Cesáro summability and conformal mapping.* Functions, series, operators, vol. I, II (Budapest, 1980), 101–125; Colloq. Math. Soc. János Bolyai, **35**, North-Holland, Amsterdam and New York, 1983.
5. Alpár, L., *On the linear transformations of series summable in the sense of Cesáro.* Acta Math. Hungar., 1982, **39**, no.1, 233–243.
6. Thorpe, B., *Matrix transformations of Cesáro summable series.* Acta Math. Hungar., 1986, **48**, no.3-4, 255–265.
7. Wilansky, A., *Summability through Functional Analysis.* North- Holland Mathematics Studies, 85. Notas de Matematica., **91**. North-Holland Publishing Co., Amsterdam and New York, 1984.
8. Аасма А., Описание преобразований полей суммируемости. Тезисы докладов конференции "Теоретические и прикладные вопросы математики" I. Тарту, 1985, 6–8.
9. Аасма А., Преобразования полей суммируемости. Уч. зап. Тарт. ун-та, 1987, **770**, 38–50.

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Regulaarsete perfektsete maatriksmenetluste summeeruvusväljade maatriksteisendused

Ants Aasma

Resümee

Olgu $A = (\alpha_{nk})$ selline regulaarne perfektne maatriksmenetlus, mille summeeruvusväli c_A (jada-jada teisendusega antud menetluse korral nulliks summeeruvate jadade ruum c_A^0) on BK-ruum, $B = (\beta_{nk})$

kolmnurkne maatriks üle $\mathbb{C}$ ning $M = (m_{nk})$ suvaline maatriks üle $\mathbb{C}$. Artiklis leitakse tarvilikud ja piisavad tingimused selleks, et maatriksteisendus

$$Y_n = \sum_k m_{nk} x_k$$

kujutaks ruumi c_A ruumi c_B. Samuti leitakse tarvilikud ja piisavad tingimused selleks, et A ja B oleksid M-kooskõlas, s.t.

$$\lim_n \sum_k \beta_{nk} \sum_l m_{kl} x_l = \lim_n \sum_k \alpha_{nk} x_k$$

iga $x = (x_k) \in c_A$ korral. Jada-jada teisendusega antud menetluse A korral vaadeldakse eraldi juhtu, kus c_A^0 on AK-ruum.

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The signed weak gliding hump property

Johann Boos and Toivo Leiger

D. Fleming and the first author [2] proved that the implication

$$Y \subset F \implies Y \subset S_F \qquad (*)$$

holds for each separable FK–space F whenever Y is a sequence space containing the space φ of all finite sequences and having the so–called *weak gliding hump property*. As a consequence they got that

$$Y \cap S_E \subset F \implies Y \cap S_E \subset S_F \qquad (\text{⁑})$$

is true if, in addition, E is any FK–space containing φ. Since the sequence space bs of all sequences having bounded partial sums does not have the weak gliding hump property, it is unknown whether (⁑) holds for $Y := bs$. The key for a positive answer is the so–called *signed weak gliding hump property* introduced by C. Stuart in [6]: He proved that $(*)$ remains true if Y has the signed weak gliding hump property and that bs has this property. In this paper we re–prove Stuarts result and show that (⁑) holds if Y has the signed weak gliding hump property.

Let ω, c and c_0 denote the linear space of all scalar (real or complex) sequences, the space of all convergent sequences and the space of all null sequences, respectively. By a sequence space E we shall mean any linear subspace of ω. A sequence space E endowed with a locally convex topology is called a K–space if the inclusion map $i : E \longrightarrow \omega$ is continuous where ω has the topology of coordinatewise convergence. A K–space E with a Fréchet topology is called an FK–space.

If E is any sequence space then the β–dual of E is given by

$$E^\beta := \left\{ x \in \omega \;\middle|\; \sum_k x_k y_k \quad \text{converges for each } y \in E \right\}.$$

For any $x = (x_k) \in \omega$ and $n \in \mathbb{N}$ the n^{th} section of x is

$$x^{[n]} := \sum_{k=1}^{n} x_k e^k$$

where $e^k := (\delta_{ik})_{i\in\mathbb{N}}$ is the k^{th} coordinate vector.

If (E,F) is a dual pair then $\sigma(E,F)$, $\tau(E,F)$ denotes the weak topology and the Mackey topology, respectively. For a sequence space E and a linear subspace F of E^β containing φ, the space of finitely non-zeros sequences, (E,F) is a dual pair under the natural bilinear form defined by

$$\langle x,y\rangle := \sum_k x_k y_k \qquad (x=(x_k)\in E\,,\ y=(y_k)\in F)\,.$$

If (E,τ_E) is a K-space containing φ, we set

$$\begin{aligned} L_E &:= \left\{x\in E \;\middle|\; \{x^{[n]}\mid n\in\mathbb{N}\} \text{ is bounded}\right\},\\ W_E &:= \left\{x\in E \;\middle|\; x^{[n]}\to x\ (\sigma(E,E'))\right\},\\ S_E &:= \left\{x\in E \;\middle|\; x^{[n]}\to x\ (\tau_E)\right\}\end{aligned}$$

where E' denotes the topological dual of (E,τ_E). A K-space E containing φ with $E=S_E$ is called an *AK-space.*

If $B=(b_{nk})$ is an infinite matrix with scalar entries the convergence domain

$$c_B := \left\{x\in\omega \;\middle|\; Bx := \left(\sum_k b_{nk}x_k\right)_{n=1}^{\infty}\in c\right\}$$

admits a natural FK-topology [8]. For $x\in c_B$ we write $\lim_B x := \lim Bx$.

If $\varphi\subset c_B$ let $b_k := \lim_n b_{nk}$ and define

$$I_B := \left\{x\in c_B \;\middle|\; \sum_k b_k x_k \text{ exists}\right\},$$

$\Lambda_B : I_B \longrightarrow \mathbb{K}$ by $\Lambda_B(x) := \lim_B x - \sum_k b_k x_k$ (where $\mathbb{K} := \mathbb{C}$ or $\mathbb{K} := \mathbb{R}$) and

$$\Lambda_B^{\perp} := \left\{x\in I_B \;\middle|\; \Lambda_B(x)=0\right\}.$$

Further if $\varphi\subset c_B$ we write L_B, W_B, S_B instead of $L_{c_B}, W_{c_B}, S_{c_B}$. In this case $W_B = L_B\cap\Lambda_B^{\perp}$ (see e. g. [8]).

Now we define several types of gliding hump properties.

Definition 1 . A sequence $(y^{(n)})$ in $\omega\setminus\{0\}$ is called a *block sequence* if there exists an index sequence (k_j) with $k_1=1$ such that $y_k^{(n)}=0$ for any $n,k\in\mathbb{N}$ with $k\notin[k_n,k_{n+1}[$ and it is called a 1*-block sequence* if furthermore $y_k^{(n)}=1$ for each $k\in[k_n,k_{n+1}[$ and $n\in\mathbb{N}$.

Let E be a sequence space containing φ.

- E has the *pointwise gliding hump property* (P_GHP) if for each $x \in E$, and any block sequence $(y^{(n)})$ satisfying $\sup_{n\in\mathbb{N}} \|y^{(n)}\|_{bv} < \infty$ there exists a subsequence $y^{(n_j)}$ of $y^{(n)}$ with $\sum_{j=1}^{\infty} xy^{(n_j)} \in E$ (pointwise sum). Thereby $\| \ \|_{bv}$ denotes the natural norm on the space of all sequences having bounded variation.
- E has the *pointwise signed weak gliding hump property* (SIGNED P_WGHP) if for each $x \in E$ and any subsequence $(y^{(j)})$ of each 1-block sequence there exists a subsequence $(y^{(n_j)})$ of $(y^{(j)})$ and a sequence (h_j) with $h_j = 1$ or $h_j = -1$ $(j \in \mathbb{N})$ such that $\sum_{j=1}^{\infty} h_j xy^{(m_j)} \in E$ (pointwise sum).
- E has the *pointwise weak gliding hump property* (P_WGHP) if the definition of the SIGNED P_GHP is fulfilled with $h_k = 1$ $(k \in \mathbb{N})$.

Remark 2. D. Noll [5] introduced the notion of the weak gliding hump property whereas the notion of the signed weak gliding hump property is due to C. Stuart [6]. On the base of Noll's definition and several kinds of gliding hump properties D. Fleming and the first author introduced the pointwise gliding hump property. By reason of the 'historical' definition of the gliding hump property they prefer to use the additive 'pointwise' in the above different definitions of gliding hump properties.

Obviously, we get

$$\text{P\_GHP} \Longrightarrow \text{P\_WGHP} \Longrightarrow \text{SIGNED P\_WGHP}.$$

C. Stuart [6] proved that the space bs of all sequences with bounded partial sums has the SIGNED P_WGHP. Thus, since bs does not have the P_WGHP (see [2]) the second inclusion is strict. Further in [2] the first inclusion is proved to be strict, too.

The following theorem generalizes the main result of the paper of D. Fleming and the first author in [2, Theorem 3.5].

Theorem 3. *Let Y be a sequence space containing φ and having the* SIGNED P_WGHP. *Then the implication*

$$Y \subset c_B \Longrightarrow Y \subset S_B$$

holds for any matrix B.

As we will see in Theorem 5, Theorem 3 holds even for separable FK–spaces F (instead of domains c_B). This is Stuart's result in [6, Theorem

3.10]. We re–prove it using alternative methods of proof:
We vary the method used by the authors in the proof of a theorem of Mazur–Orlicz type (see [4, Theorem 1] and also [3]). At first view this method looks more complicated as that of C. Stuart and C. Swartz, but in a joint paper of the authors and Dan Fleming it will be proved that this method is suitable to extend essentially the class of sequence spaces having the SIGNED P_WGHP into a general class of sequence spaces Y having weakly sequentially complete β–dual. First of all we prove a lemma corresponding to Lemma 3 in [4].

Lemma 4 . *Let Y be a sequence space containing φ and having the* SIGNED P_WGHP *and let B be a matrix with $\varphi \subset c_B$. Then for any $x \in Y \cap c_B$ each of the following properties implies the existence of a $z \in Y \backslash c_B$:*

(i) *There exists an index sequence (η_ν) such that* $\lim_\nu \sum_{k=1}^{\eta_\nu - 1} b_k x_k \neq \lim_B x$.

(ii) $\sup_\nu \left| \sum_{k=1}^{\nu} b_k x_k \right| = \infty$.

(iii) $x \in I_B \setminus S_B$.

On the base of that Lemma we can easily prove Theorem 3.

Proof of Theorem 3. For a proof of Theorem 3 we verify that the following implications are true:

(α) $Y \subset c_B \Longrightarrow Y \subset I_B$.

(β) $Y \subset I_B \Longrightarrow Y \subset S_B$.

In case of (α) we assume that there exists an $x \in c_B$ with $x \notin I_B$. Then (i) or (ii) in Lemma 4 is fulfilled; therefore we may choose a $z \in Y \backslash c_B$, that is $Y \not\subset c_B$. The implication (β) is equivalent to property (iii) in Lemma 4. ∎

Now, we are going to prove Lemma 4.

Proof of Lemma 4. First of all we make some considerations in advance. In each of the cases (i)–(iii) we show the existence of a $z \in Y \setminus c_B$ on the base of the following idea: For any index sequences (k_i) and (n_i) and any sequence $z \in \omega$ we use the notation

$$\sum_k b_{n_i k} z_k = A_i + A_i^* + B_i + C_i \qquad (i \in \mathbb{N})$$

where the convergence of $\sum_k b_{n_i} z_k$ is assumed,

$$A_i := \sum_{k=1}^{k_i-1} (b_{n_ik} - b_k) z_k\,, \qquad A_i^* := \sum_{k=1}^{k_i-1} b_k z_k\,,$$
$$B_i := \sum_{k=k_i}^{k_{i+1}-1} b_{n_ik} z_k \quad \text{and} \quad C_i := \sum_{k=k_{i+1}}^{\infty} b_{n_ik} z_k\,.$$

In all cases we will construct z and the index sequences (k_i) and (n_i) such that

$$(A_i) \in c_0 \quad \text{and} \quad (C_i) \in c_0\,.$$

By that it is easy to show that each of the following conditions implies $z \notin c_B$:

(a) $(A_i^*) \in c$ and $(B_i) \notin c$.

(b) $(A_{i_j}^*) \notin c$ and $(B_{i_j}) \in c_0$ where (i_j) is a suitable index sequence.

Let $x \in Y \cap c_B$ and let (η_ν) be any given index sequence. (Later on we will fix (η_ν).) We are going to construct index sequences (n_i), (k_i) and (ν_i) having certain properties:

For $\nu_1 := 1$ and $k_1 := \eta_{\nu_1}$ we may choose an $n_1 \in \mathbb{N}$ such that

$$\sum_{k=1}^{k_1-1} |b_{nk} - b_k|\,|x_k| < 2^{-1} \qquad (n \geq n_1)\,.$$

Then we may choose $\nu_2 > \nu_1$ such that we get for $k_2 := \eta_{\nu_2}$ the estimation

$$\left|\sum_{k=l}^{m} b_{nk} x_k\right| < 2^{-2} \qquad (n \leq n_1\,,\ k_2 \leq l < m)\,.$$

If we have chosen n_{i-1} and ν_i then for $k_i := \eta_{\nu_i}$ we determine $n_i > n_{i-1}$ with

$$\sum_{k=1}^{k_i-1} |b_{nk} - b_k|\,|x_k| < 2^{-i} \qquad (n \geq n_i)\,; \tag{1}$$

furthermore we choose $\nu_{i+1} > \nu_i$ such that for $k_{i+1} := \eta_{\nu_{i+1}}$ we have

$$\left|\sum_{k=l}^{m} b_{nk} x_k\right| < 2^{-(i+1)} \qquad (n \leq n_i\,,\ k_{i+1} \leq l < m)\,. \tag{2}$$

Using the constructed index sequences we consider the block sequence $(z^{(i)})$ defined by

$$z_k^{(i)} := \begin{cases} x_k & \text{if } \alpha_i \leq k < \beta_i \\ 0 & \text{otherwise} \end{cases}$$

and the subsequence $(z^{(2i)})$, where in the cases (i) and (ii), $\alpha_i := k_i$ and $\beta_i := k_{i+1}$ $(i \in \mathbb{N})$ and, in case (iii), (α_i) and (β_i) are suitable index sequences fulfilling $k_i \leq \alpha_i < \beta_i \leq k_{i+1}$ $(i \in \mathbb{N})$. Since Y has the SIGNED P_WGHP there exists both a subsequence $\left(z^{(l_\rho)}\right)$ of $(z^{(2i)})$ and a sequence (h_ρ) with $h_\rho = 1$ or $h_\rho = -1$ such that

$$z := \sum_{\rho=1}^{\infty} h_\rho z^{(l_\rho)} \in Y .$$

Using the notation introduced above we obviously get

$$|A_i| \le \sum_{k=1}^{k_i-1} |b_{n_ik} - b_k|\,|x_k| \overset{i\to\infty}{\longrightarrow} 0 \qquad \text{[on account of (1)]}$$

and

$$\begin{aligned} |C_i| &\le \left|\sum_{k=\eta_{\nu_i+1}}^{\infty} b_{n_ik} z_k\right| \le \sum_{j=\nu_i+1}^{\infty} \left|\sum_{k=\eta_j}^{\eta_{j+1}-1} b_{n_ik} x_k\right| \\ &< \sum_{j=\nu_i+1}^{\infty} 2^{-(j+1)} \overset{i\to\infty}{\longrightarrow} 0 \qquad \text{[because of (2)]}; \end{aligned}$$

in particular, the last estimation proves the convergence of $\sum_k b_{n_ik} z_k$. Now, we are going to fix (η_ν) dependent on (i)–(iii). In case of (i) we may choose (η_ν) such that

$$\alpha := \lim_\nu \sum_{k=1}^{\eta_\nu-1} b_k x_k \ne \lim_B x =: d;$$

furthermore, we may assume

$$\left|\sum_{k=\eta_\nu}^{\eta_{\nu+\mu}-1} b_k x_k\right| < 2^{-\nu} \qquad (\nu,\mu \in \mathbb{N}). \tag{3}$$

For any ρ and ν with $\nu \le \rho$ we obtain on account of (3) the estimation

$$\begin{aligned} &\left|\sum_{k=1}^{\eta_\rho-1} b_k z_k - \sum_{k=1}^{\eta_\nu-1} b_k z_k\right| \\ &\quad = \left|\sum_{k=\eta_\nu}^{\eta_\rho-1} b_k z_k\right| \le \sum_{j=\nu}^{\rho-1} \left|\sum_{k=\eta_j}^{\eta_{j+1}-1} b_k x_k\right| < \sum_{j=\nu}^{\infty} 2^{-(j+1)} \overset{\nu\to\infty}{\longrightarrow} 0 . \end{aligned}$$

By that we get the existence of

$$\lim_\nu \sum_{k=1}^{\eta_\nu-1} b_k z_k .$$

Consequently, $(A_i^*) \in c$. Now, we prove $(B_{l_\rho}) \notin c_0$. By the definition of B_i we get for $i := l_\rho$ the identity

$$\begin{aligned} B_i &:= \sum_{k=k_i}^{k_{i+1}-1} b_{n_ik} z_k = h_i \sum_{k=k_i}^{k_{i+1}-1} b_{n_ik} x_k \\ &= h_i \left(\sum_k b_{n_ik} x_k - \sum_{k=1}^{k_i-1} b_k x_k\right) - h_i \sum_{k=1}^{k_i-1} (b_{n_ik} - b_k)\, x_k \\ &\qquad - h_i \sum_{k=k_{i+1}}^{\infty} b_{n_ik} x_k . \end{aligned}$$

Since $\left(\sum_{k=1}^{\infty} b_{n_i k} x_k - \sum_{k=1}^{k_i - 1} x_k\right)_{i \in \mathbb{N}}$ converges to $d - \alpha \neq 0$ the first term cannot converge to 0. However, the second and third term converge to zero. Therefore, we have $(B_{l_\rho}) \notin c_0$. If $i := 2j - 1$ then $B_i = 0$ thus $(B_{2j-1}) \in c_0$. Altogether, we proved $(A_i^*) \in c$ and $(B_i) \notin c$, thus $z \notin c_B$ by (a).

In the case (ii) we may choose an index sequence (η_ν) such that

$$\left|\sum_{k=\eta_\nu}^{\eta_{\nu+1}-1} b_k x_k\right| \geq \nu + \sum_{k=1}^{\eta_\nu - 1} |b_k x_k| \qquad (\nu \in \mathbb{N}).$$

Considering $i := i_\rho := l_\rho + 1$ we get the statement $B_i = 0$ by the definition of z and $z^{(l_\rho)}$, thus $(B_{i_\rho}) \in c_0$, and also $(A_{i_\rho}^*) \notin c$ since

$$|A_i^*| = \left|\sum_{k=1}^{k_i - 1} b_k z_k\right| \geq \left|\left|\sum_{k=\eta_{\nu_{i-1}}-1}^{\eta_{\nu_i}-1} b_k x_k\right| - \sum_{j=1}^{\eta_{\nu_{i-1}}-1} |b_k x_k|\right| \geq i_\rho \overset{\rho\to\infty}{\longrightarrow} \infty .$$

Altogether we have $z \notin c_B$ by (b).

In case of (iii) we have $x \in I_B$ and $x \notin S_B$. The first statement gives us an index sequence (η_ν) such that

$$\left|\sum_{k=r}^{s} b_k x_k\right| < 2^{-\nu} \qquad (\nu, r, s \in \mathbb{N}\,;\ \eta_\nu \leq r < s). \tag{3*}$$

holds whereas on account of the second statement there exist $\varepsilon > 0$ and index sequences (α_j), (β_j) and (μ_j) with $\alpha_j < \beta_j < \alpha_{j+1}$ $(j \in \mathbb{N})$ and

$$\left|\sum_{k=\alpha_j}^{\beta_j - 1} b_{\mu_j k} x_k\right| \geq \varepsilon \qquad (j \in \mathbb{N}). \tag{4}$$

Without loss of generality we may assume that $n_i = \mu_i$ and $k_i \leq \alpha_i < \beta_i \leq k_{i+1}$ $(i \in \mathbb{N})$ (otherwise we switch over to a subsequence of (μ_j)). Using (3*) instead of (3) we can show $(A_i^*) \in c$ quite similar to case (i). If we know $(B_i) \notin c$ then we get $z \notin c_B$ from (a). First of all we state $B_i = 0$ for $i = 2j - 1$ (by the definition of z and $z^{(l_\rho)}$). Therefore, $(B_i) \notin c$ follows since for $i := l_\rho$ we have

$$|B_i| = \left|\sum_{k=k_i}^{k_{i+1}-1} b_{n_i k} z_k\right| = \left|\sum_{k=\alpha_i}^{\beta_i - 1} b_{n_i k} x_k\right| \geq \varepsilon > 0$$

on account of (4). ∎

The following theorem shows that Theorem 3 remains true if we replace the domain c_B by any separable FK–space. In this theorem the equivalence of (i) and (ii) is due to G. Bennett and N. J. Kalton (see [1, Theorem 6]) while the extension to the equivalence of (i)–(iii) was done by D. Fleming and the first author (see [2, Theorem 3.6]). The present extension to (iv) shows the close relationship of Theorem 3 (including the proof) and the proof method of C. Swartz and C. Stuart (use of Basic Matrix Theorems).

Theorem 5 . *Let Y be a sequence space containing φ. Then the following statements are equivalent:*

(i) $(Y,\tau(Y,Y^\beta))$ *is an AK–space and* Y^β *is* $\sigma(Y^\beta,Y)$*–sequentially complete.*

(ii) *If F is any separable FK–space with $Y \subset F$ then $Y \subset S_F$.*

(iii) *If B is any matrix with $Y \subset c_B$ then $Y \subset S_B$.*

(iv) Y^β *is* $\sigma(Y^\beta,Y)$*–sequentially complete and each* $\sigma(Y^\beta,Y)$*–Cauchy sequence* $(b^{(n)})$ *converges pointwisely, coordinatewisely, uniformly to the coordinatewise limit, that means*

(∗) $b_k^{(n)}x_k \longrightarrow b_kx_k$ *($n \to \infty$, uniformly in $k \in \mathbb{N}$)*

where $b_k := \lim_n b_k^{(n)}$.

Proof. For a proof of the equivalence of (i)–(iii) see [1, Theorem 6] and [2, Theorem 3.6].

(iii)⇒(iv): Let (iii) be valid and $(b^{(n)})$ be a $\sigma(Y^\beta,Y)$–Cauchy sequence in Y^β. We consider the matrix B having $b^{(n)}$ as n^{th} row. Obviously, $Y \subset c_B$, thus

$$Y \subset S_B \subset W_B = \Lambda_B^\perp \cap L_B \subset I_B$$

by (iii). Thereby $Y \subset W_B$ gives us

$$x \in I_B \quad \text{and} \quad \lim_B x = \sum_k b_k x_k \qquad (x \in Y)$$

that is

$$(b_k) \in Y^\beta \quad \text{and} \quad b^{(n)} \overset{n\to\infty}{\longrightarrow} (b_k) \text{ with respect to } \sigma(Y^\beta,Y)$$

(thus Y^β is $\sigma(Y^\beta,Y)$–sequentially complete), and $Y \subset S_B$ tells us

$$\forall\, x = (x_k) \in Y : \sum_k b_k^{(n)} x_k \text{ converges uniformly in } n \in \mathbb{N}$$

which is equivalent to (∗).

(iv)⇒(iii): Let (iv) be true and $B = (b^{(n)})$ be a matrix with $Y \subset c_B$. The $\sigma(Y^\beta,Y)$–sequential completeness of Y^β gives us $Y \subset W_B$ (see [1, Theorem 5]) while, in addition, (∗) applied on the $\sigma(Y^\beta,Y)$–Cauchy sequence $(b^{(n)})$ means $Y \subset S_B$. ∎

Remark and Definition 6 . D. Fleming and the first author (see [2, Theorem 3.3]) proved that S_E has the SP_WGHP whenever E is an FK-space containing φ. However, they proved a little bit more as one can easily check in the proof of that result: *S_E has the* ABSOLUTE SP_GHP. Thereby, a sequence space Y is said to have the *absolute pointwise gliding hump property* (ABSOLUTE P_GHP) if for each $x \in Y$ and any block sequence $(y^{(n)})$ satisfying $\sup_{n\in\mathbb{N}} \|y^{(n)}\|_{bv} < \infty$ there exists a subsequence $(y^{(n_j)})$ of $(y^{(n)})$ such that $\sum_{j=1}^{\infty} h_j x y^{(n_j)} \in Y$ (pointwise sum) where (h_j) is any sequence with $h_j = 1$ or $h_j = -1$. By definition Y has the *absolute strong pointwise gliding hump property* (ABSOLUTE SP_GHP) if $\sum_{j=1}^{\infty} h_j x y^{(\nu_j)} \in Y$ (pointwise sum) holds for any subsequence (y^{ν_j}) of (y^{n_j}) in the definition of the ABSOLUTE P_GHP and any sequence with $h_j = 1$ or $h_j = -1$.

Corollary 7 . *Let Y be a sequence space containing φ and having the* SIGNED P_WGHP *and let E be any FK-space containing φ. Then $Y \cap S_E$ has the* SIGNED P_WGHP, *thus the implication*

$$Y \cap S_E \subset F \Longrightarrow Y \cap S_E \subset S_F$$

(thus each corresponding statement in Theorem 5) holds for every separable FK-space F.

Proof. Since Y has the SIGNED P_WGHP and S_E has the ABSOLUTE SP_GHP by Remark 6, the intersection $Y \cap S_E$ has the SIGNED P_WGHP as we can easily verify and we may apply Theorem 3 and 5 in case of $Y \cap S_E$ instead of Y. ∎

In [5, Theorem 6] D. Noll proved that $(Y^\beta, \sigma(Y^\beta, Y))$ is sequentially complete if Y has the P_WGHP. This result has been generalized by C. Swartz (see [7]) to the general case of vector sequence spaces Y having the P_WGHP. Using his method C. Stuart showed in [6] that Noll's result remains true in the more general case of sequence spaces Y having the SIGNED P_WGHP. Moreover, he proved that even the stronger statement (i) in Theorem 5 (see [2, Example 5.1(6)]) holds in case of spaces having the SIGNED P_WGHP.

References

[1] Bennett, G. and Kalton, N. J., *Inclusion theorems for K-spaces*, Canad. J. Math., 1973, **25**, 511–524.

[2] Boos, J. and Fleming, D. J., *Gliding hump properties and some applications*, Int. J. Math. Math. Sci. (to appear).

[3] Boos, J. and Leiger, T., *Sätze vom Mazur–Orlicz–Typ*, Studia Math., 1985, **81**, 197–211.

[4] Boos, J. and Leiger, T., *General theorems of Mazur–Orlicz type*, Studia Math., 1989, **92**, 1–19.

[5] Noll, D., *Sequential completeness and spaces with the gliding humps property*, Manuscripta Math., 1990, **66**, 237–252.

[6] Stuart, C. E., *Weak sequential completeness in sequence spaces*, Thesis, New Mexico State University, Las Cruces, New Mexico, 1993.

[7] Swartz, C., *The gliding hump property in vector sequence spaces*, Mh. Math., 1993, **116**, 147–158.

[8] A. Wilansky, *Summability through functional analysis*, Notas de Matemática, vol. 85, North Holland, Amsterdam – New York – Oxford, 1984.

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Absolute Cesàro summability factors of infinite series

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1. Introduction

A sequence (c_n) of numbers is said to be δ-quasi-monotone, if $c_n \to 0$, $c_n > 0$ ultimately and $\Delta c_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [2]). Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series. We denote by t_n^α the n-th Cesàro mean of order $\alpha(\alpha > -1)$ of the sequence (na_n), i.e.

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu, \tag{1.1}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = O(n^\alpha), \; \alpha > -1, \; A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{1.2}$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, where $k \geq 1$ and $\alpha > -1$, if (see [5])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty \tag{1.3}$$

and it is said to be summable $\varphi\!-\!|C, \alpha|_k$, $k \geq 1$, if (see[1])

$$\sum_{n=1}^{\infty} n - k |\varphi_n t_n^\alpha|^k < \infty. \tag{1.4}$$

In the special case $\varphi_n = n^{1-1/k}$, $\varphi\!-\!|C, \alpha|_k$ summability is the same as $|C, \alpha|_k$ summability. We write

$$X_n = \sum_{\nu=1}^{n} \frac{1}{\nu}, \tag{1.5}$$

then (X_n) is a positive increasing sequence tending to infinity with n.

Mazhar [6] proved the following theorem for $|C,1|_k$ summability methods.

Theorem A. *Let $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $|\Delta\lambda_n| \leq |B_n|$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n}|t_n|^k = O(\log m) \quad as \quad m \longrightarrow \infty, \tag{1.6}$$

then the series $\sum a_n\lambda_n$ is summable $|C,1|_k$, $k \geq 1$.

2. The main result

The aim of this paper is to prove Theorem A for $\varphi\text{–}|C,\alpha|_k$ summability. Now we shall prove the following theorem.

Theorem. *Let $0 < \alpha \leq 1$ and let $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum B_nX_n$ is convergent and $|\Delta\lambda_n| \leq |B_n|$. If there exists an $\epsilon_n > 0$ such that the sequence $(n^{\epsilon-k}|\varphi_n|^k)$ is non-increasing and if the sequence (ω_n^α), defined by*

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha|, & (\alpha = 1) \\ \max\limits_{1\leq\nu\leq n} |t_\nu^\alpha|, & (0 < \alpha < 1) \end{cases} \tag{2.1}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(\omega_n^\alpha|\varphi_n|)^k = O(X_m) \quad as \quad m \longrightarrow \infty, \tag{2.2}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi\text{–}|C,\alpha|_k$, $k \geq 1$.

Remark. Since $X_m \sim \log m$, by (1.5), if we take $\epsilon = 1$, $\alpha = 1$ and $\varphi_n = n^{1-1/k}$ in this theorem, then we get Theorem A. Because in this case the condition (2.2) is reduced to the condition (1.6).

3. Needed lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1([4]). *If* $0 < \alpha \leq 1$ *and* $1 \leq \nu \leq n$, *then*

$$|\sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} a_p| \leq \max_{1\leq m\leq \nu} |\sum_{p=1}^{m} A_{m-p}^{\alpha-1} a_p|, \tag{3.1}$$

where A_n^α *is as in* (1.2).

Lemma 2([3]). *If* (B_n) *is* δ*-quasi-monotone with* $\sum nX_n\delta_n < \infty$ *and* $\sum B_n X_n$ *is convergent, then*

$$mX_m B_m = O(1) \quad as \quad m \longrightarrow \infty, \tag{3.2}$$

$$\sum_{n=1}^{\infty} nX_n |\Delta B_n| < \infty. \tag{3.3}$$

4. Proof of the Theorem

Let T_n^α be the n-th (C, α) mean of the sequence $(na_n\lambda_n)$, where $0 < \alpha \leq 1$. Then, by (1.1), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu \lambda_\nu.$$

Using Abel's transformation, we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n-1} \Delta\lambda_\nu \sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu,$$

so that making use of Lemma 1, we get that

$$|T_n^\alpha| \leq \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n-1} |\Delta\lambda_\nu| |\sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} p a_p| + \frac{|\lambda_n|}{A_n^\alpha} |\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu|$$

$$\leq \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n-1} A_\nu^\alpha \omega_\nu^\alpha |\Delta\lambda_\nu| + |\lambda_n| \omega_n^\alpha = T_{n,1}^\alpha + T_{n,2}^\alpha.$$

say. To complete the proof of the theorem by Minkowski's inequality for $k > 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}|\varphi_n T_{n,r}^{\alpha}|^k < \infty, \quad \text{for} \quad r = 1, 2, \tag{4.1}$$

by (1.4). Now, when $k > 1$, by applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, we have that

$$\sum_{n=2}^{m+1} n^{-k}|\varphi_n T_{n,1}^{\alpha}|^k$$

$$\leq \sum_{n=2}^{m+1} n^{-k}(A_n^{\alpha})^{-k}|\varphi_n|^k \sum_{v=1}^{n-1} A_{\nu}^{\alpha}(\omega_{\nu}^{\alpha})^k|B_{\nu}| \left(\sum_{\nu=1}^{n-1} A_{\nu}^{\alpha}|B_{\nu}|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} n^{-k-1}|\varphi_n|^k \sum_{\nu=1}^{n-1} \nu^{\alpha}(\omega_{\nu}^{\alpha})^k|B_{\nu}| \left(\sum_{\nu=1}^{n-1} |B_{\nu}|\right)^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m} \nu^{\alpha}(\omega_{\nu}^{\alpha})^k|B_{\nu}| \sum_{n=\nu+1}^{m+1} \frac{n^{\epsilon-k}|\varphi_n|^k}{n^{\alpha+\epsilon}}$$

$$= O(1)\sum_{\nu=1}^{m} \nu^{\alpha}(\omega_{\nu}^{\alpha})^k|B_{\nu}|\nu^{\epsilon-k}|\varphi_{\nu}|^k \int_{\nu}^{\infty} \frac{dx}{x^{\alpha+\epsilon}}$$

$$= O(1)\sum_{\nu=1}^{m} \nu|B_n|\nu^{-k}(\omega_{\nu}^{\alpha}|\varphi_{\nu}|)^k = O(1)\sum_{\nu=1}^{m-1} \Delta(\nu|B_{\nu}|)\sum_{r=1}^{\nu} r^{-k}(\omega_r^{\alpha}|\varphi_r|)^k$$

$$+O(1)m|B_m|\sum_{\nu=1}^{m-1} \nu^{-k}(\omega_{\nu}^{\alpha}|\varphi_{\nu}|)^k = O(1)\sum_{\nu=1}^{m-1} \nu X_{\nu}|\Delta B_{\nu}|$$

$$+O(1)\sum_{\nu=1}^{m-1} |B_{\nu+1}|X_{\nu} + O(1)m|B_m|X_m = O(1) \quad \text{as} \quad m \longrightarrow \infty,$$

by virtue of the hypotheses and Lemma 2. Again, we have that

$$\sum_{n=1}^{m} n^{-k}|\varphi_n T_{n,2}^{\alpha}|^k = \sum_{n=1}^{m} |\lambda_n||\lambda_n|^{k-1}n^{-k}(\omega_n^{\alpha}|\varphi_n|)^k$$

$$= O(1)\sum_{n=1}^{m} |\lambda_n|n^{-k}(\omega_n^{\alpha}|\varphi_n|)^k = O(1)\sum_{n=1}^{m} n^{-k}(\omega_n^{\alpha}|\varphi_n|)^k \sum_{\nu=n}^{\infty} |\Delta\lambda_{\nu}|$$

$$= O(1)\sum_{\nu=1}^{\infty} |\Delta\lambda\nu|\sum_{n=1}^{\nu} n^{-k}(\omega_n^{\alpha}|\varphi_n|)^k = O(1)\sum_{\nu=1}^{\infty} |B_{\nu}|X_{\nu} < \infty,$$

by virtue of the hypotheses of the theorem. Thereforė (4.1) holds. This completes the proof of the theorem.

Remark. It is natural to ask wheter our theorem is true with $\alpha > 1$. All that we can say with certanity is, that our proof fails if $\alpha > 1$, for estimate of $T^{\alpha}_{n,1}$ depens upon Lemma 1, and Lemma 1 is known to be false when $\alpha > 1$ (see [4] for details).

References

1. Balci, M. *Absolute φ-summability factors.* Comm. Fac. Sci. Univ. Ankara Sér. A_1 Math., 1980, **29**, 63–68.
2. Boas, R. P., *Quasi-positive sequences and trigonometric series.* Proc. London Math. Soc., 1965, **14A**, 38–46.
3. Bor, H., *On quasi-monotone sequences and their applications.* Bull. Austral. Math. Soc., 1991, **43**, 187–192.
4. Bosanquet, L. S., *A mean value theorem.* J. London Math. Soc., 1941, **16**, 146–148.
5. Flett, T. M., *On an extension of absolute summability and some theorems of Littlewood and Paley.* Proc. London Math. Soc., 1957, **7**, 113–141.
6. Mazhar, S. M., *On generalized quasi-convex sequence and its applications.* Indian J. Pure Appl. Math., 1977, **8**, 784–790.

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Matrix mappings between rate-spaces and spaces with speed

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1. Introduction *

Let $\rho = (\rho_n)$ be a sequence of positive numbers and E an FK-space. We shall consider the sets of sequences $x = (x_n)$

$$E_\rho := \{x \in \omega \mid (\frac{x_n}{\rho_n}) \in E\}.$$

The set E_ρ may be considered as FK-space. Following J. Sikk (see [9]), we shall call them as "rate-spaces". If $E = c$ then we get the rate space c_ρ. We shall demonstrate that the rate-spaces of this kind are very closely connected with spaces c^λ, i.e. the spaces of sequences convergent with speed λ. These spaces c^λ were introduced by G. Kangro in 1967 (see [illegible]). He used the following definition:

$$c^\lambda := \{x \in c \mid \exists \lim_n \lambda_n(x_n - \lim x)\},$$

where $\lambda = (\lambda_n)$, $0 < \lambda_n \leq \lambda_{n+1} \nearrow \infty$ and

$$\lim x := \lim_n x_n.$$

Some properties of these spaces have been studied in [6], where G.Kangro has considered also the space

$$m^\lambda := \{x \in c \mid \lambda_n(x_n - \lim x) = O(1)\},$$

the space of λ-bounded sequences.

We shall show that

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$$c^\lambda = c_{\lambda^{-1}} \oplus <e> \text{ and } m^\lambda = m_{\lambda^{-1}} \oplus <e>,$$

where $\lambda^{-1} := (1/\lambda_n)$ and $\lim \lambda = \infty$. These simple relations give us a possibility for a universal treatment for the ordinary and for the λ-summability. By these relations many things will be clearer and many proofs simpler to us. The consideration a treatment of this kind was suggested to the author by the paper of W. Beekmann and S.-C. Chang [1]. In this paper it was shown that for any matrix A there exists a matrix B such that the λ-summability field of A is the ordinary summability field of B. This means that the properties of both kind of summability have to be similar.

Let X, Y be sets of sequences. Then (X, Y) is the set of matrices $A = (a_{nk})$, $n, k \in \mathbb{N}$, such that $Ax \in Y$ for all $x \in X$. In the present paper we shall consider the spaces of the types $c_{0\rho}$, c_ρ, m_ρ, c^λ and m^λ as X and Y. In sections 2 and 3 we shall study the properties of these spaces. The subject of the next sections is to obtain the equivalent conditions for $A \in (X, Y)$. We can get these conditions by the next three classical theorems.

Theorem 1.1 (Kojima – Schur). *A matrix $A = (a_{nk}) \in (c, c)$ if and only if the following statements are true:*

$$\text{(i)} \quad \exists \lim_A e_k = \lim_n a_{nk} = a_k, \quad k \in \mathbb{N},$$

$$\text{(ii)} \quad \exists \lim_A e = \lim_n \sum_k a_{nk} = a,$$

$$\text{(iii)} \quad \sum_k | a_{nk} | = O(1).$$

If $x \in c$ then

$$\lim_A x := \lim_n \sum_k a_{nk} x_k = (a - \sum_k a_k) \lim x + \sum_k a_k x_k.$$

Remark. $A \in (c_0, c) \Leftrightarrow$ (i) and (iii) are true.

Theorem 1.2 (Schur). *A matrix $A = (a_{nk}) \in (m, c)$ if and only if the statements* (i), (iii) *and*

$$\text{(iv)} \quad \lim_n \sum_k | a_{nk} - a_k | = 0$$

are true. If $x \in m$ then

$$\lim_A x = \lim_n \sum_k a_{nk} x_k = \sum_k a_k x_k.$$

Theorem 1.3. *The following are equivalent:* (iii); $A \in (m,m)$; $A \in (c,m)$; $A \in (c_0,m)$.

From these theorems we shall get 16 "mapping-theorems" – the equivalent conditions to $A = (a_{nk}) \in (X,Y)$, where $X = c_\rho$, c^ν, m_ρ and m^ν and $Y = c_\pi, c^\lambda, m_\pi$ and m^λ. The summary of these mapping-theorems is given in the next table.

$X \setminus Y$	c_π	c^λ	m_π	m^λ
c_ρ	Th.4.3	Th.5.1	Th.6.1	Th.7.5
c^ν	Th.4.7	Th.5.4	Th.6.2	Th.7.8
m_ρ	Th.4.10	Th.5.8	Th.6.1	Th.7.1
m^ν	Th.4.12	Th.5.10	Th.6.2	Th.7.3

2. Spaces m_ρ, c_ρ and $c_{0\rho}$

Let $\rho = (\rho_n)$, $\pi = (\pi_n)$ be sequences of positive numbers i.e. $\rho_n > 0$, $\pi_n > 0$ $\forall n \in \mathbb{N}$. The set

$$m_\rho := \{x = (x_n) \in \omega \mid \frac{x_n}{\rho_n} = O(1)\}$$

is the bounded domain of diagonal matrix $\mathrm{diag}(1/\rho_n)$. This matrix is a normal matrix and so the set m_ρ is a *BK*-space with the norm

$$\| x \|_\rho := \sup_n \frac{| x_n |}{\rho_n}.$$

By ρ we shall also denote the diagonal matrix determined by sequence ρ^{-1} i.e.

$$\rho := \mathrm{diag}\,(1/\rho_n),$$

$$\lim_\rho x := \lim_n (x_n/\rho_n).$$

If $\rho = e = (1,1,\ldots)$ then $\lim_e x = \lim x$.

The set of all complex sequences is denoted by ω and the subset of all finitely non-zero sequences by ϕ.

We shall consider the next subspaces of m_ρ :

$$c_\rho := \{x \in m_\rho \mid \exists \lim_\rho x\},$$

$$c_{0\rho} := \{x \in c_\rho \mid \lim_\rho x = 0\}.$$

They are both *BK*-spaces with norm $\| \ \|_\rho$.

Every $f \in c_\rho'$, the continuous dual space of c_ρ , can be expressed in the form

$$f(x) = \sum_n \tau_n x_n + \mu \lim_\rho x, \tag{1}$$

where $\tau \cdot \rho = (\tau_n \rho_n) \in l$ and $\mu \in \mathbb{C}$.

Now we shall give some facts in connection with these spaces.

Proposition 2.1. $e_k \in c_{0\rho} \subset c_\rho \quad \forall k \in \mathbb{N}$.

Proposition 2.2. $\phi \subset c_{0\rho} \subset c_\rho$.

Proposition 2.3. $c_\rho = c \Leftrightarrow \exists \lim \rho \neq 0$.

Proposition 2.4. $\rho \in c_0 \Rightarrow c_\rho \subset c_0$.

Proposition 2.5. *$c_{0\rho}$ has AK (sectional convergence) i.e. $x^{[n]} \to x$ for each $x \in c_{0\rho}$.*

Proposition 2.6. *Every $x \in c_\rho$ can be expressed as*

$$x = \rho \lim_\rho x + \sum_k (x_k - \rho_k \lim_\rho x) e_k$$

i.e.

$$cl_{c_\rho}\{\rho, e_k \mid k \in \mathbb{N}\} = c_\rho.$$

Proposition 2.7. $c_\rho = c_{0\rho} \oplus <\rho>$.

Proposition 2.8. *If $\rho = (\rho_n)$ and $\pi = (\pi_n)$ are two sequences of positive numbers then*

$$\text{(i)} \quad c_\rho = c_\pi \Leftrightarrow \exists \lim_n \frac{\rho_n}{\pi_n} \neq 0.$$

$$\text{(ii)} \quad c_\rho \subsetneqq c_\pi \Leftrightarrow \lim_n \frac{\rho_n}{\pi_n} = 0.$$

$$\text{(iii)} \quad c_\rho \supsetneqq c_\pi \Leftrightarrow \lim_n \frac{\rho_n}{\pi_n} = \infty.$$

3. Spaces c^λ and m^λ

Let $\lambda = (\lambda_n)$ be a (real) sequence with

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1} \leq \lambda_n \to \infty$$

In 1967 G. Kangro [4] has defined the space

$$c^\lambda := \{x = (x_n) \in c \mid \exists \lim_n \lambda_n(x_n - \lim x)\}.$$

These spaces are called "space of λ-convergent sequences" or "space of sequences convergent with spaced λ ". In this section we shall consider some facts in connection with these spaces. We do not assume the monotony of sequence λ .

Proposition 3.1. *If* $\rho \in c_0$ *then*

$$x \in c_\rho \Leftrightarrow x \in c^\lambda \cap c_0,$$

where $\lambda := \rho^{-1} := (1/\rho_n)$. .

Theorem 3.2. $c^\lambda = c_\rho \oplus \langle e\rangle = c_{0\rho} \oplus \langle\rho\rangle \oplus \langle e\rangle$, *where* $\rho = \lambda^{-1} \in c_0$.

Proof. (i) $c^\lambda \subset c_\rho \oplus \langle e\rangle$.

$$\begin{aligned} x \in c^\lambda &\Leftrightarrow \exists \lim_n \lambda_n(x_n - \lim x) \Leftrightarrow z = (z_n) \in c,\ z_n = \lambda_n(x_n - \lim x) \\ &\Leftrightarrow x_n = \frac{z_n}{\lambda_n} + \lim x = y_n + \xi, \quad y_n = \frac{z_n}{\lambda_n} \quad \text{and} \quad \xi = \lim x \\ &\Leftrightarrow x = y + \xi e, \quad y \in c_\rho = c_{\lambda^{-1}} \end{aligned}$$

The sequence $y = (y_n) \in c_\rho$ since $\lim_n(y_n/\rho_n) = \lim_n \lambda_n y_n = \lim_n z_n$ exists. Thus $x \in c_\rho \oplus \langle e\rangle$.

(ii) $c_\rho \oplus \langle e\rangle \subset c^\lambda$.

$$x \in c_\rho \oplus \langle e\rangle \quad \Leftrightarrow \quad x = z + \xi e,\ z \in c_\rho \Leftrightarrow$$

$$x_n = z_n + \xi,\ \exists \lim_n(z_n/\rho_n) = \lim_n \lambda_n z_n \Leftrightarrow$$

$$(\lambda_n z_n) = (\lambda_n(x_n - \xi)) \in c.$$

The last condition implies that $\lim x = \xi$ exists and $x \in c^\lambda$. ■

We shall denote

$$\lambda^n(x) := \lambda_n(x_n - \lim x),$$
$$\lambda(x) := \lim_n \lambda^n(x) = \lim_n \lambda_n(x_n - \lim x).$$

Theorem 3.3. *Each continuous linear functional* f *on* c^{λ} *has the representation*

$$f(x) = \sum_n \tau_n(x_n - \lim x) + \mu\lambda(x) + \sigma \lim x, \tag{2}$$

where $\tau \cdot \lambda^{-1} \in l$ *and* $\mu, \sigma \in \mathbb{C}$.

Proof. In the proof of Theorem 3.2 we have got that for each $x \in c^{\lambda}$ we have $x = z + \xi e$, where $z \in c_{\rho}$, $\rho = \lambda^{-1}$ and $\xi = \lim x$. This implies that $z_n = x_n - \lim x$ i.e. $x = z + e \lim x$. The representation of f on c_{ρ} (see (1)) implies

$$\begin{aligned} f(x) &= f(z) + \xi f(e) = \\ &= \sum_k \tau_k z_k + \mu \lim{}_{\rho} z + \sigma \lim x, \end{aligned}$$

where $(\tau_k \lambda_k^{-1}) \in l$ and $\sigma = f(e)$. We get the representation (2) since

$$z_k = x_k - \lim x \quad \text{and} \quad \lim{}_{\rho} z = \lambda(x).$$

Theorem 3.4. *Every* $x \in c^{\lambda}$ *has the expressions*

$$\begin{aligned} x &= (\lim x) \cdot e + \lambda(x) \cdot \lambda^{-1} + \sum_k \Big(x_k - \lim x - \frac{\lambda(x)}{\lambda_k}\Big) e_k = \\ &= (\lim x) \cdot e + \lambda(x) \cdot \lambda^{-1} + \sum_k \frac{\lambda^k(x) - \lambda(x)}{\lambda_k} e_k. \end{aligned}$$

Proof. Since $c^{\lambda} = c_{\lambda^{-1}} \oplus \langle e \rangle$ then for each $x \in c^{\lambda}$ we have the expression $x = z + \xi e$, where $z = (x_n - \xi) \in c_{\lambda^{-1}}$ and $\xi = \lim x$ (see the proof of 3.2). Thus by 2.6

$$x = (\lim{}_{\lambda^{-1}} z) \cdot \lambda^{-1} + \sum_k (z_k - \frac{1}{\lambda_k} \lim{}_{\lambda^{-1}} z) e_k + (\lim x) \cdot e.$$

Since $\lim_{\lambda^{-1}} z = \lim_n \lambda_n (x_n - \xi) = \lambda(x)$ it follows, that our statement is true. ∎

We shall consider the matrix mappings $A = (a_{nk}) \in (X, c^{\lambda})$ i.e.

$$y_n = \sum_k a_{nk} x_k, \quad n \in \mathbb{N},$$

where

$$x = (x_k) \in X \quad \text{and} \quad y = (y_n) \in c^{\lambda}.$$

For these mappings there exist the functionals:

$$\lim\nolimits_A x := \lim_n \sum_k a_{nk} x_k,$$
$$\lambda_A^n(x) := \lambda_n(\sum_k a_{nk} x_k - \lim\nolimits_A x),$$
$$\lambda_A(x) := \lim_n \lambda_A^n(x).$$

Theorem 3.5. $m^{\lambda} = m_{\lambda^{-1}} \oplus \langle e \rangle$, *where* $\lambda^{-1} \in c_0$.

Proof. (i) $m^{\lambda} \subset m_{\lambda^{-1}} \oplus \langle e \rangle$.

$$\begin{aligned} x \in m^{\lambda} \quad &\Leftrightarrow \quad \exists \lim x = \xi, \ \lambda_n(x_n - \xi) = O(1) \\ &\Leftrightarrow \quad z = (z_n) \in m, \ z_n = \lambda_n(x_n - \xi) \\ &\Leftrightarrow \quad x_n = \frac{z_n}{\lambda_n} + \xi = y_n + \xi, \ y_n = \frac{z_n}{\lambda_n} \\ &\Leftrightarrow \quad x = y + \xi e, \ y = (y_n). \end{aligned}$$

Since $(\lambda_n y_n) = (z_n) \in m$ then $y = (y_n) \in m_{\lambda^{-1}}$ and thus $x \in m_{\lambda-1} \oplus \langle e \rangle$.

(ii) $m_{\lambda^{-1}} \oplus \langle e \rangle \subset m^{\lambda}$.

$$x \in m_{\lambda^{-1}} \oplus \langle e \rangle \quad \Leftrightarrow \quad x = z + \xi e, \ z \in m_{\lambda^{-1}} \quad \Leftrightarrow$$

$$x_n = z_n + \xi, \ \lambda_n z_n = O(1) \quad \Leftrightarrow \quad \lambda_n(x_n - \xi) = O(1).$$

Since $\lim \lambda = \infty$ then $\xi = \lim x$ and $x \in m^{\lambda}$. ■

A possibility of determining *BK*-topology on c^{λ} and m^{λ} is to do it by norm

$$\|x\|^{\lambda} := \sup\{|\lambda^n(x)|; \ |\lim x|\}.$$

This norm was used by G. Kangro (see[6]), who introduced these spaces for monotonic speed λ .

4. Mappings $A \in (X, c_\pi)$

In this section we shall obtain the equivalent conditions to $A \in (X, c_\pi)$, where X is $c_\rho, c^\lambda, m_\rho$ or m^λ. We get these conditions from Theorems 1.1 and 1.2 by next two lemmas.

Lemma 4.1. *Let $\rho = (\rho_n)$ and $\pi = (\pi_n)$ be two sequences of positive numbers and $A = (a_{nk})$ be an infinite matrix. Then the following statements are equivalent:*

$$\begin{aligned}
&\text{(i)} \quad A = (a_{nk}) \in (X_\rho, Y_\pi),\\
&\text{(ii)} \quad (\frac{a_{nk}}{\pi_n}) \in (X_\rho, Y),\\
&\text{(iii)} \quad (a_{nk}\rho_k) \in (X, Y_\pi),\\
&\text{(iv)} \quad (\frac{a_{nk}\rho_k}{\pi_n}) \in (X, Y).
\end{aligned}$$

Proof.

$$\frac{1}{\pi_n}\sum_k a_{nk}x_k = \sum_k \frac{a_{nk}}{\pi_n}x_k = \frac{1}{\pi_n}\sum_k a_{nk}\rho_k(\frac{x_k}{\rho_k}) = \sum_k \frac{a_{nk}\rho_k}{\pi_n}(\frac{x_k}{\rho_k}).\blacksquare$$

We shall use the next symbols:

$$c_{\pi A} := \{x \in \omega \mid Ax \in c_\pi\},$$
$$\lim_{\pi A} x := \lim_n \frac{1}{\pi_n}\sum_k a_{nk}x_k, \quad x \in c_{\pi A}.$$

If $\pi = e = (1, 1, \ldots)$ then $\lim_{eA} = \lim_A$.

Lemma 4.2. *Let X, Y and Z be FK-spaces, where $Z = X \oplus \langle u \rangle$, $u \in \omega$. Then the following statements are equivalent:*

$$\begin{aligned}
&\text{(i)} \quad A \in (Z, Y),\\
&\text{(ii)} \quad A \in (X, Y) \quad \textit{and} \quad Au \in Y.
\end{aligned}$$

Theorem 4.3. *A matrix $A = (a_{nk}) \in (c_\rho, c_\pi)$ if and only if it satisfies the following conditions:*

$$\begin{aligned}
&\text{(i)} \quad \exists \lim_{\pi A} e_k := \lim_n \frac{a_{nk}}{\pi_n} =: a_k^\pi, \quad k \in \mathbb{N},\\
&\text{(ii)} \quad \exists \lim_{\pi A} \rho := \lim_n \frac{1}{\pi_n}\sum_k a_{nk}\rho_k =: a^{\rho\pi},\\
&\text{(iii)} \quad \sum_k | a_{nk} | \rho_k = O(\pi_n).
\end{aligned}$$

Proof. This follows from 1.1 by 4.1.

Proposition 4.4. *If* $A \in (c_\rho, c_\pi)$ *then*

$$\text{(iv)} \quad \sum_k | a_k^\pi | \rho_k < \infty,$$

$$\text{(v)} \quad \sum_k | \frac{a_{nk}}{\pi_n} - a_k^\pi | \rho_k = O(1).$$

Proof.

$$\text{(iii)} \Longrightarrow \frac{1}{\pi_n} \sum_{k=1}^m | a_{nk} | \rho_k = O(1) \Longrightarrow \sum_{k=1}^m | \frac{a_{nk}}{\pi_n} | \rho_k = O(1)$$

$$\Longrightarrow \sum_k | a_k^\pi | \rho_k < \infty.$$

(iii) and (iv) imply (v).

Proposition 4.5. *If* $A \in (c_\rho, c_\pi)$ *and* $x \in c_\rho$ *then*

$$\lim_{\pi A} x = (a^{\rho\pi} - \sum_k a_k^\pi \rho_k) \lim_\rho x + \sum_k a_k^\pi x_k.$$

Proof. We apply 2.6, 4.3, 4.4 and the fact that $\lim_{\pi A} \in c'_\rho$.

Corollary 4.6. *A matrix* $A \in (c_\rho, c)$ *if and only if the following statements are true:*

$$\text{(i)} \quad \exists \lim_A e_k =: a_k, \quad k \in \mathbb{N},$$

$$\text{(ii)} \quad \exists \lim_A \rho =: a^{\rho 1},$$

$$\text{(iii)} \quad \sum_k | a_{nk} | \rho_k = O(1).$$

For every $x \in c_\rho$

$$\lim_A x = (a^{\rho 1} - \sum_k a_k \rho_k) \lim_\rho x + \sum_k a_k x_k.$$

Theorem 4.7. *A matrix* $A = (a_{nk}) \in (c^\nu, c_\pi)$ *if and only if the* *following statements are true:*

$$\text{(i)}\quad \exists \lim_{\pi A} e_k = a_k^{\pi}, \quad k \in \mathbb{N},$$

$$\text{(ii)}\quad \exists \lim_{\pi A} \nu^{-1} := \lim_n \frac{1}{\pi_n} \sum_k \frac{a_{nk}}{\nu_k} =: a^{\nu^{-1}\pi},$$

$$\text{(iii)}\quad \exists \lim_{\pi A} e := \lim_n \frac{1}{\pi_n} \sum_k a_{nk} =: a^{1\pi},$$

$$\text{(iv)}\quad \sum_k \frac{| a_{nk} |}{\nu_k} = O(\pi_n).$$

Proof. This follows from 4.3 by 4.2. ■

Applying the expression for $x \in c^{\nu}$ (see 3.4) we have the next

Proposition 4.8. *If* $A \in (c^{\nu}, c_{\pi})$ *and* $x \in c^{\nu}$ *then*

$$\lim_{\pi A} x = a^{1\pi} \lim x + \left(a^{\nu^{-1}\pi} - \sum_k \frac{a_k^{\pi}}{\nu_k}\right)\nu(x) + \sum_k \frac{a_k^{\pi}}{\nu_k}\nu^k(x).$$

Corollary 4.9. *Let* $\pi \in c_0$. *Then*

$$\lim_A \rho = 0 \quad \forall A \in (c_{\rho}, c_{\pi})$$

and

$$\lim_A \nu^{-1} = \lim_A e = 0 \quad \forall A \in (c^{\nu}, c_{\pi}).$$

Proof. The condition (ii) of 4.6 implies the first assertion. The second assertion follows from the conditions (ii) and (iii) of 4.7.

Theorem 4.10. *A matrix* $A \in (a_{nk}) \in (m_{\rho}, c_{\pi})$ *if and only if the following statements are true:*

$$\text{(i)}\quad \exists \lim_{\pi A} e_k = a_k^{\pi}, \quad k \in \mathbb{N},$$

$$\text{(ii)}\quad \sum_k | a_{nk} | \rho_k = O(\pi_n) \quad \left(or \sum_k | a_k^{\pi} | \rho_k < \infty\right),$$

$$\text{(iii)}\quad \lim_n \sum_k \left| \frac{a_{nk}}{\pi_n} - a_k^{\pi} \right| \rho_k = 0.$$

Proof. This follows from 1.2 by 4.1.

Proposition 4.11. *If* $A \in (m_\rho, c_\pi)$ *and* $x \in m_\rho$ *then*

$$\lim_{\pi A} x = \sum_k a_k^\pi x_k.$$

Proof. This follow from 1.2 by 4.1.

Theorem 4.12. *A matrix* $A = (a_{nk}) \in (m^\nu, c_\pi)$ *if and only if the following statements are true:*

$$\text{(i)} \quad \exists \lim_{\pi A} e_k = a_k^\pi, \quad k \in \mathbb{N},$$

$$\text{(ii)} \quad \sum_k \frac{|a_{nk}|}{\nu_k} = O(\pi_n) \quad \left(or \sum_k \frac{|a_k^\pi|}{\nu_k} < \infty\right),$$

$$\text{(iii)} \quad \lim_n \sum_k \left| \frac{a_{nk}}{\pi_n} - a_k^\pi \right| \frac{1}{\nu_k} = 0,$$

$$\text{(iv)} \quad \exists \lim_{\pi A} e = a^{1\pi}.$$

Proof. Theorem 4.10 implies this assertion by 3.5 and 4.2.

Proposition 4.13. *If* $A \in (m^\nu, c_\pi)$ *and* $x \in m^\nu$ *then*

$$\lim_{\pi A} x = a^{1\pi} \lim x + \sum_k \frac{a_k^\pi}{\nu_k} \nu^k(x).$$

Proof. $x \in m^\nu \Leftrightarrow x = z + \xi e,\ z \in m_{\nu^{-1}},\ \xi = \lim x$

$$\Leftrightarrow z = x - \xi e \Rightarrow \lim_{\pi A} z = \lim_{\pi A} x - \xi \lim_{\pi A} e \Leftrightarrow$$

$$\Leftrightarrow \lim_{\pi A} x = \lim_{\pi A} z + \xi \lim_{\pi A} e =$$

$$= \sum_k a_k^\pi (x_k - \xi) + a^{1\pi} \lim x = \sum_k \frac{a_k^\pi}{\nu_k} \nu^k(x) + a^{1\pi} \lim x.$$

5. Mappings $A \in (X, c^\lambda)$

The examination of the matrices $A \in (X, c^\lambda)$ has some additional complications. Lemma 4.2 is expediency, but because of the structure of the space c^λ (the space of images) the futher argumentations are necessary.

Theorem 5.1. *A matrix* $A = (a_{nk}) \in (c_\rho, c^\lambda)$ *if and only if the following statements are true:*

$$\text{(i)} \quad Ae_k \in c^\lambda, \quad k \in \mathbf{N},$$

$$\text{(ii)} \quad A\rho \in c^\lambda,$$

$$\text{(iii)} \quad \sum_k | a_{nk} | \rho_k = O(1) \quad (or \sum_k | a_k | \rho_k < \infty),$$

$$\text{(iv)} \quad \lambda_n \sum_k | a_{nk} - a_k | \rho_k = \sum_k | \lambda_A^n(e_k) | \rho_k = O(1).$$

Proof. By 4.2 and the equality $c_\rho = c_{0\rho} \oplus <\rho>$ it is true that

$$A \in (c_\rho, c^\lambda) \quad \Leftrightarrow \quad \begin{cases} A \in (c_{0\rho}, c^\lambda), \\ A\rho \in c^\lambda. \end{cases} \tag{ii}$$

Definition of the space c^λ implies that

$$A \in (c_{0\rho}, c^\lambda) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A x \quad \forall x \in c_{0\rho}, & (1) \\ \exists \lambda_A x \quad \forall x \in c_{0\rho}. & (2) \end{cases}$$

Corollary 4.9 implies that

$$(1) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A e_k = a_k, \quad k \in \mathbb{N}, & \text{(i)} \\ \sum_k | a_{nk} | \rho_k = O(1) & \text{(iii)} \end{cases}$$

and

$$(2) \quad \Leftrightarrow \quad \exists \lim_n \sum_k \lambda_n(a_{nk} - a_k)x_k \quad \forall x \in c_{0\rho} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} \exists \lim_n \sum_k \lambda_n(a_{nk} - a_k) = \lambda_A(e_k), \quad k \in \mathbb{N} & \text{(i)} \\ \sum_k \lambda_n | a_{nk} - a_k | \rho_k = O(1). & \text{(iv)} \end{cases}$$

Thus the theorem is proved.

Remark. It is possible to formulate Theorem 5.1 as follows.

The matrix $A = (a_{nk}) \in (c_\rho, c^\lambda)$ *if and only if the following statements are true:*

$$(i) \quad \sum_k | a_k | \rho_k < \infty,$$

$$(ii) \quad \mathfrak{B} = (\beta_{nk}) \in (c, c),$$

where

$$\beta_{nk} = \lambda_n(a_{nk} - a_k)\rho_k, \quad a_k = \lim_n a_{nk}.$$

Proposition 5.2. *If* $A \in (c_\rho, c^\lambda)$ *and* $x \in c_\rho$ *then*

$$\lambda_A(x) = (\lambda_A(\rho) - \sum_k \lambda_A(e_k)\rho_k)\lim_\rho x + \sum_k \lambda_A(e_k)x_k.$$

Proof. This expression for $\lambda_A(x) \quad \forall x \in c_\rho$ follows from 2.6 and 5.1. ■

Since $A \in (c_\rho, c^\nu) \subset (c_\rho, c)$ we get the next corollary.

Corollary 5.3. *If* $A \in (c_\rho, c^\lambda)$ *and* $x \in c_\rho$ *then*

$$\lim_A x = \sum_k a_k x_k.$$

Proof. The condition (iv) of 5.1 implies

$$\lim_A \rho = \sum_k a_k \rho_k.$$

Our assertion follows now from 4.6.

Theorem 5.4. *A matrix* $A = (a_{nk}) \in (c^\nu, c^\lambda)$ *if and only if it satisfies the following conditions:*

(i) $Ae_k \in c^\lambda, \quad k \in \mathbb{N}$,

(ii) $A\nu^{-1} \in c^\lambda$,

(iii) $Ae \in c^\lambda$,

(iv) $\sum_k \frac{| a_{nk} |}{\nu_k} = O(1) \quad (or \sum_k \frac{| a_k |}{\nu_k} < \infty)$,

(v) $\lambda_n \sum_k \frac{| a_{nk} - a_k |}{\nu_k} = \sum_k \frac{| \lambda_A^n(e_k) |}{\nu_k} = O(1)$.

Proof. This follows from 5.1 by 4.2 since $c^\nu = c_{\nu^{-1}} \oplus <e>$.

Remark. This theorem for monotonic speeds ν and λ was proved by G.Kangro in 1969 (see [5]). We could formulate Theorem 5.4 in the similar way as we did by 5.1.

A matrix $A = (a_{nk}) \in (c^\nu, c^\lambda)$ *if and only if the following statements are true:*

(i) $Ae \in c^\lambda$,

(ii) $\sum_k \frac{| a_k |}{\nu_k} < \infty, \quad a_k = \lim_n a_{nk}$,

(iii) $\mathfrak{A} = (\alpha_{nk}) \in (c, c)$, *where*

$$\alpha_{nk} := \frac{\lambda_n(a_{nk} - a_k)}{\nu_k}.$$

Proposition 5.5. *If* $A \in (c^\nu, c^\lambda)$ *and* $x \in c^\nu$ *then*

$$\lim_A x = a \lim x + \sum_k a_k(x_k - \lim x),$$

where $a := \lim_A e$.

Proof. This result follows from the expression of $x \in c^\nu$ by applying condition (iv) of 5.4.

Proposition 5.6. *If* $A \in (c^\nu, c^\lambda)$ *and* $x \in c^\nu$ *then*

$$\lambda_A(x) = \lambda_A(e) \lim x + \Big(\lambda_A(\nu^{-1}) - \sum_k \frac{\lambda_A(e_k)}{\nu_k}\Big)\nu(x) + \sum_k \frac{\lambda_A(e_k)}{\nu_k}\nu_k(x).$$

Proof is the same as the previous one applying only (v) instead of (iv) from 5.4.

Corollary 5.7. *If* $A \in (c^\nu, c^\lambda)$ *and* $x \in c_0 \cap c^\nu$ *then*

$$\lim_A x = \sum_k a_k x_k,$$

especially

$$\lim_A \nu^{-1} = \sum_k \frac{a_k}{\nu_k}.$$

Proof. This follows from 5.5 and from equality $\lim \nu^{-1} = 0$.

Theorem 5.8. *A matrix* $A = (a_{nk}) \in (m_\rho, c^\lambda)$ *if and only if it satisfies the following conditions:*

(i) $Ae_k \in c^\lambda, \quad k \in \mathbb{N}$,

(ii) $\sum_k \mid a_{nk} \mid \rho_k = O(1) \quad (or \sum_k \mid a_k \mid \rho_k < \infty)$,

(iii) $\sum_k \mid \lambda_A^n(e_k) \mid \rho_k = O(1) \quad (or \sum_k \mid \lambda_A(e_k) \mid \rho_k < \infty)$,

(iv) $\lim_n \sum_k \mid \lambda_A^n(e_k) - \lambda_A(e_k) \mid \rho_k = 0$.

Proof.

$$A \in (m_\rho, c^\lambda) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A x \quad \forall x \in m_\rho, & (1) \\ \exists \lambda_A(x) \quad \forall x \in m_\rho. & (2) \end{cases}$$

$$(1) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A e_k, \quad k \in \mathbb{N}, & \text{(i)} \\ \sum_k \mid a_{nk} \mid \rho_k = O(1), & \text{(ii)} \\ \lim_n \sum_k \mid a_{nk} - a_k \mid \rho_k = 0. & (\alpha) \end{cases}$$

Since $(m_\rho, c^\lambda) \subset (m_\rho, c)$ then by 1.2 and 4.1

$$\lim_A x = \sum_k a_k x_k \quad \forall x \in m_\rho.$$

Thus

$$(2) \Leftrightarrow \exists \lambda_A(x) = \lim_n \lambda_n \sum_k (a_{nk} - a_k) x_k = \lim_n \sum_k \lambda_A^n(e_k) x_k \quad \forall x \in m_\rho$$

$$\Leftrightarrow \quad (\lambda_A^n(e_k)) \in (m_\rho, c)$$

$$\Leftrightarrow \quad \begin{cases} \exists \lim_n \lambda_A^n(e_k) = \lambda(e_k), & \text{(i)} \\ \sum_k \mid \lambda_A^n(e_k) \mid \rho_k = O(1), & \text{(iii)} \\ \lim_n \sum_k \mid \lambda_A^n(e_k) - \lambda_A(e_k) \mid \rho_k = 0. & \text{(iv)} \end{cases}$$

Since $\lim \lambda = \infty$ then (iii) implies (α) and we get the assertion.

Remark. We give an another formulation for this theorem as we did by 5.1 and 5.4.

A matrix $A = (a_{nk}) \in (m_\rho, c^\lambda)$ *if and only if the following statements are true:*

(i) $\sum_k \mid a_{nk} \mid \rho_k < \infty, \quad a_k = \lim_n a_{nk},$

(ii) $\mathfrak{B} = (\beta_{nk}) \in (m, c),$

where β_{nk} *as by* 5.1.

Proposition 5.9. *If* $A \in (m_\rho, c^\lambda)$ *and* $x \in m_\rho$ *then*

$$\lim_A x = \sum_k a_k x_k$$

and

$$\lambda_A(x) = \sum_k \lambda_A(e_k) x_k.$$

Proof. The first assertion was shown in the proof of 5.8. For the second assertion we must verify that

$$\lim_n \lambda_A^n(x) = \lim_n \sum_k \lambda_A^n(e_k)x_k = \sum_k \lambda_A(e_k)x_k.$$

This follows from (iv) of 5.8.

Theorem 5.10. *A matrix* $A = (a_{nk}) \in (m^\nu, c^\lambda)$ *if and only if it satisfies the following conditions:*

(i) $Ae_k \in c^\lambda, \quad k \in \mathbb{N},$

(ii) $Ae \in c^\lambda,$

(iii) $\sum_k \frac{|a_{nk}|}{\nu_k} = O(1) \quad (or \sum_k \frac{|a_k|}{\nu_k} < \infty),$

(iv) $\sum_k \frac{|\lambda_A^n(e_k)|}{\nu_k} = O(1) \quad (or \sum_k \frac{|\lambda_A(e_k)|}{\nu_k} < \infty),$

(v) $\lim_n \sum_k \frac{|\lambda_A^n(e_k) - \lambda_A(e_k)|}{\nu_k} = 0.$

Proof. The assertion follows from 5.8 and 4.2.

Remark. We formulate theorem 5.10 in another way as we did by 5.4. *A matrix* $A = (a_{nk}) \in (m^\nu, c^\lambda)$ *if and only if the following statements are true:*

(i) $Ae \in c^\lambda,$

(ii) $\sum_k \frac{|a_k|}{\nu_k} < \infty,$

(iii) $\mathfrak{A} = (\alpha_{nk}) \in (m, c),$

where α_{nk} *as by* 5.4.

This formulation (and similar formulations for 5.1, 5.4 and 5.8) demonstrates the importance of matrix $\mathfrak{A}$ for studying of the matrix $A \in (c^\nu, c^\lambda)$ and $A \in (m^\nu, c^\lambda)$. This matrix $\mathfrak{A}$ was used already by G. Kangro in [4,6]. The same role has the matrix $\mathfrak{B}$ by matrices $A \in (c_\rho, c^\lambda)$ and $A \in (m_\rho, c^\lambda)$.

Proposition 5.11. *If* $A \in (m^\nu, c^\lambda)$ *and* $x \in m^\nu$ *then*

$$\lim_A x = \sum_k a_k(x_k - \lim x) + a \lim x,$$

where

$$a_k = \lim_n a_{nk} = \lim_A e_k, \quad a = \lim_n \sum_k a_{nk} = \lim_A e.$$

Proof. $x \in m^\nu \Rightarrow x = z + \xi e,\ z \in m_{\nu-1},\ \xi = \lim x \Leftrightarrow z = x - \xi e.$ By 5.9 we have

$$\lim_A z = \lim_A x - \xi \lim_A e = \sum_k a_k z_k$$

i.e.

$$\lim_A x = \sum_k a_k(x_k - \lim x) + a \lim x.$$

Corollary 5.12. *Let* $A \in (m^\nu, c^\lambda)$. *Then*

$$\lim_A x = \sum_k a_k x_k$$

if (a) $x \in m^\nu \cap c_0$ *or* (b) $x \in m^\nu$ *and* $\chi(A) = a - \sum_k a_k = 0.$

Proof. (a) It is clear.

(b) $\chi(A) = 0$ implies $\lim_A e = \sum_k a_k$. The assertion follows now from the relations $m^\nu \subset c$ and $x = u + \xi e,\ u \in c_0$.

Proposition 5.13. *If* $A \in (m^\nu, c^\lambda)$, $x \in m^\nu$ *and* $\lim_A x = \sum_k a_k x_k$ *then*

$$\lambda_A(x) = \sum_k \lambda_A(e_k) x_k.$$

Proof is the same as by 5.9 applying (v) of 5.10 instead of (iv) of 5.8.

6. Mappings $A \in (X, m_\pi)$

By Lemma 4.1 Theorem 1.3 implies immediately the next theorem.

Theorem 6.1. *The following are equivalent:*

(i) $A \in (m_\rho, m_\pi)$,

(ii) $A \in (c_\rho, m_\pi)$,

(iii) $A \in (c_{0\rho}, m_\pi)$,

(iv) $\sum_k | a_{nk} | \rho_k = O(\pi_n)$.

Since $m^\lambda = m_{\lambda^{-1}} \oplus \langle e \rangle$ and $c^\lambda = c_{\lambda^{-1}} \oplus \langle e \rangle$ then by Lemma 4.2 we get from 6.1 (taking $\rho = \nu^{-1}$) the next assertion.

Theorem 6.2. *A matrix* $A = (a_{nk}) \in (m^\nu, m_\pi) = (c^\nu, m_\pi)$ *if and only if it satisfies the following conditions:*

$$\text{(i)} \quad Ae \in m_\pi,$$

$$\text{(ii)} \quad \sum_k \frac{| a_{nk} |}{\nu_k} = O(\pi_n).$$

7. Mappings $A \in (X, m^\lambda)$

In this section we shall consider the mapping-theorems of type $(X, m^\lambda) : (m_\rho, m^\lambda)$ – Theorem 7.1; (m^ν, m^λ) – Theorem 7.3; (c_ρ, m^λ) – Theorem 7.5; (c^ν, m^λ) – Theorem 7.8. The representations for functional $\lim_A$ are given by Propositions 7.2, 7.4, 7.7 and 7.10.

Theorem 7.1. *A matrix* $A = (a_{nk}) \in (m_\rho, m^\lambda)$ *if and only if the following statements are true:*

$$\text{(i)} \quad \exists \lim_A e_k = a_k, \quad k \in \mathbb{N},$$

$$\text{(ii)} \quad \sum_k | a_{nk} | \rho_k = O(1) \quad (or \sum_k | a_k | \rho_k < \infty),$$

$$\text{(iii)} \quad \lambda_n \sum_k | a_{nk} - a_k | \rho_k = \sum_k | \lambda_A^n(e_k) | \rho_k = O(1).$$

Proof.

$$A \in (m_\rho, m^\lambda) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A x & \forall x \in m_\rho, \qquad (1) \\ \lambda_A^n(x) = O(1) & \forall x \in m_\rho. \qquad (2) \end{cases}$$

$$(1) \quad \Leftrightarrow \quad A \in (m_\rho, c) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A e_k, \quad k \in \mathbb{N}, & \text{(i)} \\ \sum_k | a_{nk} | \rho_k = O(1), & \text{(ii)} \\ \lim_n \sum_k | a_{nk} - a_k | \rho_k = 0. & (\alpha) \end{cases}$$

$$(2) \quad \Leftrightarrow \quad \lambda_A^n(x) = \lambda_n(\sum_k a_{nk}x_k - \lim_A x) =$$

$$= \quad \sum_k \lambda_n(a_{nk} - a_k)x_k = O(1) \quad \forall x \in m_\rho \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (\lambda_n(a_{nk} - a_k)) \in (m_\rho, m) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \lambda_n \sum_k | a_{nk} - a_k | \rho_k = \sum_k | \lambda_A^n(e_k) | \rho_k = O(1). \qquad \text{(iii)}$$

Since $\lim \lambda = \infty$ then (iii) implies (α) ■.

We can formulate this theorem as following.

A matrix $A \in (m_\rho, m^\lambda)$ if and only if the following statements are true:

$$\text{(i)} \quad \sum_k | a_k | \rho_k < \infty,$$

$$\text{(ii)} \quad \mathfrak{B} = (\beta_{nk}) \in (m, m),$$

where

$$\beta_{nk} = \lambda_n(a_{nk} - a_k)\rho_k = \lambda_A^n(e_k)\rho_k, \; a_k = \lim_n a_{nk}.$$

Proposition 7.2. *If $x \in m_\rho$ and $A \in (m_\rho, m^\lambda)$ then*

$$\lim_A x = \sum_k a_k x_k.$$

Proof. Since $(m_\rho, m^\lambda) \subset (m_\rho, c)$ then Proposition 4.12 $(\pi = e)$ implies the assertion.

Theorem 7.3. *A matrix $A = (a_{nk}) \in (m^\nu, m^\lambda)$ if and only if the following statements are true:*

$$\text{(i)} \quad \exists \lim_A e_k = a_k, \quad k \in \mathbb{N},$$

$$\text{(ii)} \quad \sum_k \frac{| a_{nk} |}{\nu_k} = O(1) \quad \left(or \sum_k \frac{| a_k |}{\nu_k} < \infty\right),$$

$$\text{(iii)} \quad \lambda_n \sum_k \frac{| a_{nk} - a_k |}{\nu_k} = \sum_k \frac{| \lambda_A^n(e_k) |}{\nu_k} = O(1),$$

$$\text{(iv)} \quad Ae \in m^\lambda.$$

Proof. The assertion follows from 7.1 by Lemma 4.2 and the equality

$$m^\lambda = m_{\lambda^{-1}} \oplus \langle e \rangle. \quad ■$$

This theorem for monotonic speed was given by G.Kangro (see [7]). We can give to this theorem the similar formulation as we did by 7.1.

A matrix $A \in (m^\nu, m^\lambda)$ if and only if the following statements are true:

$$\text{(i)} \quad Ae \in m^\lambda,$$

$$\text{(ii)} \quad \sum_k \frac{| a_k |}{\nu_k} < \infty, \quad a_k = \lim_n a_{nk},$$

$$\text{(iii)} \quad \mathfrak{A} = \left(\frac{\lambda_A^n(e_k)}{\nu_k}\right) \in (c, c).$$

Proposition 7.4. *If* $A \in (m^\nu, m^\lambda)$ *and* $x \in m^\nu$ *then*

$$\lim_A x = a \lim x + \sum_k \frac{a_k}{\nu_k} \nu^k(x).$$

Proof. This follows from 4.13 (by $\pi = e$), since $(m^\nu, m^\lambda) \subset (m^\nu, c)$.

Theorem 7.5. *A matrix* $A = (a_{nk}) \in (c_\rho, m^\lambda)$ *if and only if the following statements are true:*

$$\text{(i)} \quad \exists \lim_A e_k = a_k, \quad k \in \mathbb{N},$$

$$\text{(ii)} \quad \sum_k | a_{nk} | \rho_k = O(1) \quad (or \sum_k | a_k | \rho_k < \infty),$$

$$\text{(iii)} \quad \lambda_n \sum_k | a_{nk} - a_k | \rho_k = \sum_k | \lambda_A^n(e_k) | \rho_k = O(1).$$

Proof.

$$A \in (c_\rho, m^\lambda) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A x & \forall x \in c_\rho, \quad (1) \\ \lambda_A^n(x) = O(1) & \forall x \in c_\rho. \quad (2) \end{cases}$$

$$(1) \quad \Leftrightarrow \quad \begin{cases} \exists \lim_A e_k = a_k, \quad k \in \mathbb{N}, & \text{(i)} \\ \exists \lim_A \rho, & \\ \sum_k | a_{nk} | \rho_k = O(1). & \text{(ii)} \end{cases}$$

Since $(c_\rho, m^\lambda) \subset (c_\rho, c)$ then by 4.6

$$\lim_A x = \sum_k a_k x_k \quad \forall x \in c_{0\rho}.$$

Thus

$$\lambda_A^n(x) = \sum_k \lambda_n(a_{nk} - a_k)x_k \quad \forall x \in c_{0\rho}$$

and

$$(2) \Leftrightarrow \begin{cases} \lambda_A^n(\rho) = O(1), \\ \sum_k \lambda_n(a_{nk} - a_k)x_k = O(1) \quad \forall x \in c_{0\rho}, \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_A^n(\rho) = O(1), \\ (\lambda_n(a_{nk} - a_k)) \in (c_{0\rho}, m^\lambda), \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_A^n(\rho) = O(1), \\ \lambda_n \sum_k \mid a_{nk} - a_k \mid \rho_k = O(1). \end{cases} \qquad \text{(iii)}$$

Since $\lim \lambda = \infty$ then the last condition implies that exists

$$\lim_A \rho = \lim_n \sum_k a_{nk}\rho_k = \sum_k a_k \rho_k.$$

Hence

$$\lambda_A^n(\rho) = \lambda_n \sum_k (a_{nk} - a_k)\rho_k$$

and (iii) implies $\lambda_A^n(\rho) = O(1)$. The proof is completed.

Corollary 7.6. $(m_\rho, m^\lambda) = (c_\rho, m^\lambda)$.

Proposition 7.7. *If* $x \in c_\rho$ *and* $A \in (c_\rho, m^\lambda)$ *then*

$$\lim_A x = \sum_k a_k x_k.$$

Proof is the same as for 5.3. ■

As by 7.1 the Theorem 7.5 can be formulated in the another way.
A matrix $A \in (c_\rho, m^\lambda)$ *if and only if the following statements are true:*

$$\text{(i)} \quad \sum_k \mid a_k \mid \rho_k < \infty,$$

$$\text{(ii)} \quad \mathfrak{B} = (\beta_{nk}) \in (c, m),$$

where $\beta_{nk} = \lambda_n (a_{nk} - a_k)\rho_k$ *and* $a_k = \lim_n a_{nk}$.

Theorem 7.8. *A matrix* $A = (a_{nk}) \in (c^\nu, m^\lambda)$ *if and only if the following statements are true:*

(i) $Ae \in m^\lambda$,

(ii) $\exists \lim_A e_k = a_k, \quad k \in \mathbb{N}$,

(iii) $\sum_k \frac{\mid a_{nk} \mid}{\nu_k} = O(1) \quad (or \sum_k \frac{\mid a_k \mid}{\nu_k} < \infty)$,

(iv) $\lambda_n \sum_k \frac{\mid a_{nk} - a_k \mid}{\nu_k} = \sum_k \frac{\mid \lambda_A^n(e_k) \mid}{\nu_k} = O(1)$.

Proof. This follows from 7.5 by 4.2. ■

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For the last theorem we can use the similar formulation as we did by 7.3.

A matrix $A = (a_{nk}) \in (c^{\nu}, m^{\lambda})$ *if and only if the following statements are true:*

$$\text{(i)} \quad Ae \in m^{\lambda},$$

$$\text{(ii)} \quad \sum_k \frac{|a_k|}{\nu_k} = O(1), \quad a_k = \lim_n a_{nk},$$

$$\text{(iii)} \quad \mathfrak{A} = \left(\frac{\lambda_A^n(e_k)}{\nu_k}\right) \in (c, m).$$

Corollary 7.9. $(c^{\nu}, m^{\lambda}) = (m^{\nu}, m^{\lambda})$.

This assertion is given also in [8], p.138.

Proposition 7.10. *If* $x \in c^{\nu}$ *and* $A \in (c^{\nu}, m^{\lambda})$ *then*

$$\lim{}_A x = a \lim x + \sum_k \frac{a_k}{\nu_k} \nu^k(x),$$

where

$$a = \lim{}_A e = \lim_n \sum_k a_{nk}.$$

Proof. Since $(c^{\nu}, m^{\lambda}) \subset (c^{\nu}, c)$ then by 4.8 $(\pi = e)$

$$\lim{}_A x = a \lim x + \sum_k \frac{a_k}{\nu_k} \nu^k(x) + \left(\lim_n \sum_k \frac{a_{nk}}{\nu_k} - \sum_k \frac{a_k}{\nu_k}\right) \nu(x).$$

The condition (iv) of 7.8 implies $(\lim \lambda = \infty)$ that

$$\lim_n \sum_k \frac{a_{nk}}{\nu_k} = \sum_k \frac{a_k}{\nu_k}.$$

Proof is completed.

References

1. Beekmann, W. and Chang, S.-C., *λ-convergence and λ-conullity.* Z. Anal. Anwendungen., 1993, **12**, 179–182.
2. Boos, J., *Eine Erweiterung des Satzes von Schur.* Manuscripta Math., 1980, **31**, 111–117.
3. Boos, J., *Vergleich μ-beschränkter Wirkfelder mit Hilfe von Quotienten-darstellungen.* Math. Z., 1982, **181**, 71–81.

4. Kangro, G., *On some investigations on the theory of summability* (in Russian). Eesti NSV Tead. Akad. Toimetised Füüs.-Mat., 1967, **16**, 255–266.
5. Kangro, G., *On the summability factors of the Bohr-Hardy type for a given rapidity.* I (in Russian). Eesti NSV Tead. Akad. Toimetised Füüs.-Mat., 1969, **18**, 137–146.
6. Kangro, G., *On λ-perfecticity of summability methods and its applications.* I (in Russian). Eesti NSV Tead. Akad. Toimetised Füüs.-Mat., 1971, **20**, 111–120.
7. Kangro, G., *Summability factors for the series λ-bounded by the methods of Riesz and Cesàro.* Tartu Ülik. Toimetised, 1971, **277**, 136–154.
8. Leiger, T., *Methods of functional analysis in summability* (in Estonian). Tartu, 1992, pp. 202.
9. Sikk, J., *Matrix mappings for rate-spaces and K-multipliers in the theory of summability.* Tartu Ülik. Toimetised, 1989, **846**, 118–129.

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Maatriksteisendused järgu-ruumide ja kiirusega ruumide vahel

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Resümee

Käesolevas artiklis on vaadeldud maatriksteisendusi $y = Ax$, kus $A = (a_{nk})$, $x = (x_k) \in X$, $y = (y_n) \in Y$ ja

$$y_n = \sum_k a_{nk} x_k, \quad k, n \in \mathbb{N}.$$

Ruumideks X ja Y on järgu või kiirusega määratud jadaruumid (p.2 – 3). Kiirusega ruumide (c^λ ja m^λ) mõiste pärineb G.Kangrolt (1967). Järgu-ruume vaatles 1989.a. J. Sikk. Siinkasutatud definitsioon erineb mõnevõrra tema omast, kuid sisuliselt on mõlemad definitsioonid samaväärsed, kuigi tema oma on antud mõnevõrra üldisemana, lähtudes rakendustest. Käesolevas on vaadeldud järguruume c_ρ ja m_ρ. Kogu käsitluse aluseks on p.3 tõestatud lihtsad seosed (Th. 3.2 ja Th. 3.5) järgu-ruumide ja kiirusega ruumide vahel.

Olgu X ja Y mingid jadaruumid. Sümboliga (X,Y) on tähistatud nende maatriksite A klassi, mis kujutavad ruumi X ruumi Y.

J. Sikk [9] näitas, kuidas klassikalisi maatriksteisenduste kohta tuntud teoreeme üle kanda järgu-ruumide juhule. Käesolevas artiklis esitatud seosed järgu- ja kiirusega ruumide vahel võimaldavad neid klassikalisi teoreeme laiendada juhtudele, kus nii X kui ka Y on kas järguruumid (m_ρ või c_ρ) või kiirusega ruumid (m^λ või c^λ).

Kui kujutis $y = (y_n) \in c_\pi$ kus $\pi = (\pi_n)$ ning $\pi_n > 0$, siis on tema puhul oluliseks suuruseks

$$\lim_\pi y := \lim_n (y_n/\pi_n).$$

Kui aga on tegu kiirusega ruumiga c^λ, kus $\lambda = (\lambda_n)$ ning $\lambda_n > 0$, $\lambda_n \to \infty$, siis vastavaks oluliseks suuruseks on

$$\lambda(y) := \lim_n \lambda_n(y_n - \lim y).$$

Juhtudel $A \in (X, c_\pi)$ on leitud, kuidas avalduvad suurused

$$\lim_{\pi A} x := \lim_\pi y, \quad x \in X,$$

kus $y = Ax$. Juhtudel $A \in (X, c^\lambda)$ on aga leitud sama suuruste

$$\lim_A x := \lim y \quad \text{ja} \quad \lambda_A(x) := \lambda(y), \quad x \in X,$$

korral.

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Properties of domains of matrix mappings on rate-spaces and spaces with speed

E.Jürimäe

1. Introduction *

We shall consider the matrix mappings $y = Ax$ i.e.

$$y_n = \sum_k a_{nk} x_k, \quad k \in \mathbb{N},$$

where $A = (a_{nk})$, $x = (x_k) \in X$ and $y = (y_n) \in Y$. The purpose of this paper is to study properties of these mappings, where X and Y are rate-spaces or spaces with speed (see [5]).

Let $\pi = (\pi_n)$ be a sequence of positive numbers and ω be the set of all sequences of complex numbers. Then the sets

$$m_\pi := \{x = (x_n) \in \omega \mid (\frac{x_n}{\pi_n}) \in m\},$$
$$c_\pi := \{x \in m_\pi \mid \exists \lim_\pi x := \lim_n \frac{x_n}{\pi_n}\}$$

are BK-spaces with norm

$$\| x \|_\pi := \sup_n \mid \frac{x_n}{\pi_n} \mid .$$

We call them "rate-spaces" (spaces with rate π). These sets are closely connected with spaces c^λ and m^λ (see [5,6]):

$$m^\lambda := \{x = (x_n) \in c \mid (\lambda_n(x_n - \lim x)) \in m\},$$
$$c^\lambda := \{x = (x_n) \in c \mid (\lambda_n(x_n - \lim x)) \in c\},$$

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where $\lambda = (\lambda_n)$, $\lambda_n > 0$ and $\lim \lambda := \lim_n \lambda_n = \infty$. The connection between the rate-spaces and the spaces with speed is grounded on the equalities $c^\lambda = c_{\lambda^{-1}} \oplus <e>$ and $m^\lambda = m_{\lambda^{-1}} \oplus <e>$, where $e = (1, 1, \ldots)$ and $\lambda^{-1} = (1/\lambda_n)$.

The sets m^λ and c^λ are BK-spaces with norm

$$\| x \|^\lambda = \sup\{| \lambda^n(x) |, | \lim x | \mid n \in \mathbb{N}\},$$

where

$$\lambda^n(x) = \lambda_n(x_n - \lim x) \quad \text{and} \quad \lim x := \lim_n x_n.$$

These spaces are called "spaces with speed λ ". The properties of the rate-spaces and spaces with speed are considered in [5] (see also [6]).

In this paper we shall study matrix mappings connected with rate-spaces and spaces with speed. Some facts on topological structure of the sets, the domains of A ,

$$c_{\pi A} := \{x \in \omega \mid Ax \in c_\pi\},$$
$$c_A^\lambda := \{x \in \omega \mid Ax \in c^\lambda\}$$

are presented in section 2. A definition of conullity for general summability methods was given in [2]. A similar definition is used in section 3. The necessary and sufficient conditions for the different kind of conullity, which are connected with different classes of matrices $A \in (X, Y)$ i.e. $y = Ax \in Y$ for any $x \in X$, are also given there.

It is a well-known fact (theorem of Steinhaus) that $A \in (m, c)$ implies $\chi(A) = 0$ i.e. A is conull. In the last section 4 the similar facts are obtained for $A \in (X, Y)$, where $X = m_\rho$ or m^ν and $Y = c_\pi$ or c^λ .

2. Domains $c_{\pi A}$ and c_A^λ

In this section we consider the topological properties of $c_{\pi A}$, c_A^λ and their continuous duals. Many of the mentioned facts follow immediately from the general theory of K.Zeller (see [9], §§4-5 or [8], section 4).

For any $x \in c_{\pi A}$ there exists the functional

$$\lim_{\pi A} x := \lim_n \frac{1}{\pi_n} \sum_k{}' a_{nk} x_k.$$

Proposition 2.1. *Domain* $c_{\pi A}$ *is an FK-space with seminorms*

$$p_0(x) = \sup_n \frac{1}{\pi_n} \mid \sum_k a_{nk}x_k \mid,$$
$$p_{2n}(x) = \mid x_n \mid, \quad n \in \mathbb{N},$$
$$p_{2n-1}(x) = \sup_m \mid \sum_{k=1}^{m} a_{nk}x_k \mid, \quad n \in \mathbb{N}.$$

Proposition 2.2. *Every* $f \in (c_{\pi A})'$ *has the representation*

$$f(x) = \sum_k t_k x_k + \sum_n \tau_n \sum_k a_{nk}x_k + \mu \lim_{\pi A} x, \tag{1}$$

where

$$(\tau_n \pi_n) \in l, \quad (t_n) \in (c_{\pi A})^{\beta} \quad \text{and} \quad \mu \in \mathbb{C}.$$

Definition 2.3. *A matrix* A *is called* c_π*-reversible if for each* $y \in c_\pi$ *there is a unique* x *such that* $Ax = y$.

Proposition 2.4. *If* A *is* c_π*-reversible then* $c_{\pi A}$ *is a BK-space with norm* $p_0(x)$ *and every* $f \in (c_{\pi A})'$ *has a representation* (1), *where* $t_k = 0 \quad \forall k \in \mathbb{N}$.

The matrix $M = (m_{nk})$, where

$$m_{nk} = \begin{cases} \tau_k, & k < n, \\ \mu, & k = n, \\ 0, & k > n, \end{cases}$$

and $\mu \neq 0$, $(\tau_k) \in l$, is called Mazur matrix. It is a well-known fact that $c_M = c$.

Theorem 2.5. *If* $M = (m_{nk})$ *is a Mazur matrix and*

$$Q = (\frac{\pi_n m_{nk}}{\rho_k})$$

then $c_{\pi Q} = c_\rho$.

Proof.

$$\lim_{\pi Q} x = \lim_n \frac{1}{\pi_n} \sum_k \frac{\pi_n m_{nk}}{\rho_k} x_k = \lim_n \sum_k m_{nk} \frac{x_k}{\rho_k}.$$

Theorem 2.6. *For each* $f \in (c_{\pi A})'$ *and for any rates* π *and* ρ *there exists a matrix* B *with* $c_{\rho B} \supset c_{\pi A}$ *and* $\lim_{\rho B} x = f(x) \quad \forall x \in c_{\pi A}$. *If* f *has a representation* (1) *with* $\mu \neq 0$ *then there exists a matrix* B *with* $c_{\rho B} = c_{\pi A}$ *and* $\lim_{\rho B} x = f(x) \quad \forall x \in c_{\pi A}$.

Proof. We consider the matrix $M = (m_{nk})$, where (τ_n) and μ are from (1). By this matrix M we determine the matrix $D = (m_{nk}/\pi_n)$ and then $C = DA = (c_{nk})$. We shall get the required matrix $B = (b_{nk})$ by taking

$$b_{nk} = \begin{cases} \rho_1 t_k, & n = 1, \\ \rho_n(t_k + c_{n-1,k}), & n > 1, \end{cases}$$

where (t_k) is from (1).

The second part of the statement follows from the fact that by $\mu \neq 0$ the matrix M is Mazur matrix and thus $c_M = c$.

Corollary 2.7. *For every* $f \in (c_{\pi A})'$ *there exist matrices* B *and* D *such that* f *has the representations*

$$f(x) = \lim_{\pi B} x \quad \forall x \in c_{\pi A},$$
$$f(x) = \lim_{D} x \quad \forall x \in c_{\pi A}.$$

Proof. For the first case we take $\rho = \pi$ in Theorem 2.6 and for the second case $\rho = e$.

Corollary 2.8. *Let* ρ *be a rate. Then for any* $f \in (c_\pi)'$ *there exists a matrix* B *such that*

$$f(x) = \lim_{\rho B} x \quad \forall x \in c_\pi.$$

If $\mu \neq 0$ *in the representation of* f *then there exists a corresponding* B *with* $c_{\rho B} = c_\pi$.

Now we consider the domains c_A^λ . For any $x \in c_A^\lambda$ there exist the functionals

$$\lim_A x := \lim_n \sum_k a_{nk} x_k,$$

$$\lambda_A(x) := \lim_n \lambda_n (\sum_k a_{nk} x_k - \lim_A x).$$

Proposition 2.9. *The domain* c_A^λ *is an FK-space with the seminorms*

$$q_0(x) = \sup\{|\lambda_A^n(x)|, |\lim_A x| \mid n \in \mathbb{N}\}$$

and $p_{2n}(x), p_{2n-1}(x), n \in \mathbb{N}$.

Proposition 2.10. *Every* $f \in (c_A^\lambda)'$ *has a representation*

$$f(x) = \sum_k t_k x_k + \sum_n \tau_n \lambda_A^n(x) + \mu\lambda_A(x) + \sigma \lim_A x, \tag{2}$$

where

$$\tau \in l, \quad t \in (c_A^\lambda)^\beta \quad \text{and} \quad \mu, \sigma \in \mathbb{C}.$$

In [3] the next assertion was proved.

Proposition 2.11. *Let* λ *be a monotonic speed i.e.* $\lambda_{n+1} \geq \lambda_n$ $\forall n \in \mathbb{N}$. *Then for every* $f \in (c_A^\lambda)'$ *there exists a matrix* B *with* $c_B^\lambda \supset c_A^\lambda$, $\lambda_B(x) = f(x)$ $\forall x \in c_A^\lambda$. *If* f *has a representation with* $\mu \neq 0$ *then there exists a matrix* B *with* $c_B^\lambda = c_A^\lambda$ *and* $\lambda_B(x) = f(x)$ $\forall x \in c_A^\lambda$.

Questions: *1. Is the last assertion true without assuming of monotony?*

2. Does there exist for every $f \in (c_A^\lambda)'$ *and for a given speed* ν *a matrix* B *with* $c_B^\nu \supset c_A^\lambda$ *and* $\nu_B(x) = f(x)$ $\forall x \in c_A^\lambda$ *?*

In [10] K. Zeller has shown that for every unbounded sequence λ there exists a regular normal matrix $D = (d_{nk})$ such that $c_D = c \oplus <\lambda>$ and $\lim_D \lambda = 0$. Applying this result W. Beekmann and S.-C. Chang (see [1]) have shown that for each matrix A and speed λ there exists a matrix $E = (e_{nk})$ such that $c_E = c_A^\lambda$. These matrices A and E are connected with the equality

$$E = D \cdot diag(\lambda_k) \cdot A,$$

consequently,

$$A = diag(1/\lambda_k) \cdot D^{-1} \cdot E.$$

Every $f \in (c_E)' = (c_A^\lambda)'$ has the representations

$$f(x) = \sum_k \bar{t}_k x_k + \sum_n \bar{\tau}_n \sum_k e_{nk} x_k + \bar{\mu} \lim_E x =$$
$$= \sum_k t_k x_k + \sum_n \tau_n \lambda_A^n(x) + \mu\lambda_A(x) + \sigma \lim_A x.$$

It can be realized so that

$$\mu = \bar{\mu}, \quad \sigma = \sum_n \bar{\tau}_n \sum_k d_{nk}\lambda_k,$$

$$\tau_k = \sum_n \bar{\tau}_n d_{nk} \quad \text{and} \quad t_k = \bar{t}_k.$$

From this facts we get the next propositions.

Proposition 2.12. *For every* $f \in (c_A^\lambda)'$ *and for every rate* π *there exists a matrix* B *with* $c_{\pi B} \supset c_A^\lambda, \lim_{\pi B} x = f(x) \quad \forall x \in c_A^\lambda$. *If* f *has a representation with* $\mu \neq 0$ *then there exists a matrix* B *with* $c_{\pi B} = c_A^\lambda$ *and* $\lim_{\pi B} x = f(x)\ \forall x \in c_A^\lambda$.

Proof. 1) Let $\pi = e$. Then the assertion follows immediately from the facts given above.

2) Let π be an arbitrary rate. Then we take $B = Q \cdot E$, where $Q = (\pi_n m_{nk})$ (see Theorem 2.5). By 2.5 (if $\rho = e$) we get the statement.

Proposition 2.13. *For every* $f \in (c_{\pi A})'$ *and for every monotonic speed* λ *there exists a matrix* B *with* $c_B^\lambda \supset c_{\pi A}$. $\lambda_B(x) = f(x)$ $\forall x \in c_{\pi A}$. *If* f *has a representation with* $\mu \neq 0$ *then there exists a matrix* B *with* $c_B^\lambda = c_{\pi A}$ *and* $\lambda_B(x) = f(x) \quad \forall x \in c_{\pi A}$.

Proof. This statement follows from the above-mentioned facts and Proposition 2.11.

3. Conullity of matrix mappings

The notion and the importance of "conullity" for conservative matrices are well-known (see [8]). It must be pointed out that the notion of conullity for conservative matrices is connected with the summability domain as an *FK*-space. Our notion here is connected with the mapping $A : X \to Y$ i.e. it is depending on the both rooms X and Y . This means that conullity of a given matrix A is not determined only by the properties of the domain of this matrix (cf. Theorems 3.5 and 3.6). A given matrix A can be conull for one type of mapping but coregular (i.e. not conull) for the another type. We shall consider only the cases, where X and Y are spaces c_π or c^λ. So we shall study only four types of conullity, though the definition gives us much more possibilities.

Let $A \in (X,Y)$, where X is a space of type c_ρ or c^λ. We denote

$$Y_A := \{x \in \omega \mid Ax \in Y\}.$$

By $x^{[n]}$ we denote the section of sequence x i.e.

$$x^{[n]} := (x_1, \ldots, x_n, 0, 0, \ldots).$$

Definition 3.1. *Let $A \in (X,Y)$ and $X = c_\rho$ or $X = c_\rho \oplus <u>$. Then a matrix A is called (X, Y)-conull, respectively, (X, Y)-coregular if $\rho^{[n]} \to \rho$ weakly in Y_A, respectively $\rho^{[n]} \not\to \rho$ weakly in Y_A.*

Corollary 3.2. *If $X = c$ (i.e. $\rho = e$) and $Y = c$ then we get the definition of the ordinary conull matrix. It is wellknown that A is conull (in our terms (c,c)-conull) if and only if*

$$\chi(A) := a - \sum_k a_k = \lim_A e - \sum_k \lim_A e_k = 0.$$

Corollary 3.3. *If $X = c^\lambda$ (i.e. $\rho = \lambda^{-1} \in c_0$) and $Y = c^\lambda$ then we get the definition of the λ-conull matrix* (see [4,6]). *It is known that A is λ-conull (in our terms (c^λ, c^λ)-conull) if and only if*

$$\Psi(A) := \lambda_A(\lambda^{-1}) - \sum_k \frac{\lambda_A(e_k)}{\lambda_k} = 0.$$

These constants $\chi(A)$ and $\Psi(A)$ are called as the characteristics of the given matrix A. We shall consider that kind of characteristics also in other cases. These two special cases indicate that the characteristic is expressed by functional $\lim_{\pi A}$ when $Y = c_\pi$, and by λ_A when $Y = c^\lambda$. The following discussion shows that it really is so.

Lemma 3.4. *If $A \in (c_\rho, c_\pi)$, respectively, $A \in (c^\lambda, c_\pi)$ then for any $f \in (c_{\pi A})'$ and for any $x \in m_\rho \cap c_{\pi A}$, respectively, for any $x \in m_{\lambda^{-1}} \cap c_{\pi A}$*

$$f(x) = \sum_k s_k x_k + \mu \lim_{\pi A} x,$$

where $\mu \in \mathbb{C}$ and $s = (s_k) \in (c_{\pi A} \cap m_\rho)^\beta$, respectively, $s \in (c_{\pi A} \cap m_{\lambda^{-1}})^\beta$.

Proof. We use Theorems 4.3 and 4.7 from [5]. The condition $x \in c_{\pi A} \cap m_\rho$ implies that the second member in the representation (1) of f (see 2.2) is absolute convergent and we can change the order of summation. So the statement for the case $A \in (c_\rho, c_\pi)$ follows immediately. For the case $A \in (c^\lambda, c_\pi)$ we get the same taking $\rho = \lambda^{-1}$.

Theorem 3.5. *A matrix* $A \in (c_\rho, c_\pi)$ *is* (c_ρ, c_π)*-conull if and only if*

$$\chi_{c_\rho}^{c_\pi}(A) := \lim\nolimits_{\pi A}\rho - \sum_k \rho_k \lim\nolimits_{\pi A} e_k = 0.$$

Proof. By Definition 3.1 and Lemma 3.4 matrix A is (c_ρ, c_π)-conull if and only if

$$\lim_m \Big(\sum_{k>m} s_k \rho_k + \mu \Big(\lim\nolimits_{\pi A}\rho - \sum_{k=1}^{m} \rho_k \lim\nolimits_{\pi A} e_k \Big) \Big) = 0$$

since $\rho \in c_{\pi A} \cap m_\rho$. The case $\mu \neq 0$ implies the statement.

Theorem 3.6. *A matrix* $A \in (c^\lambda, c_\pi)$ *is* (c^λ, c_π)*-conull if and only if*

$$\chi_{c^\lambda}^{c_\pi}(A) := \lim\nolimits_{\pi A}\lambda^{-1} - \sum_k \frac{\lim_{\pi A} e_k}{\lambda_k} = 0.$$

Proof is the same as for 3.5.

By 5.3 from [5] the next lemma is true.

Lemma 3.7. *If* $A \in (c_\rho, c^\lambda)$ *then the series* $\sum_k a_k \rho_k$ *is convergent and*

$$\lim\nolimits_A \rho = \sum_k a_k \rho_k.$$

Theorem 3.8. *A matrix* $A \in (c_\rho, c^\lambda)$ *is* (c_ρ, c^λ)*-conull if and only if*

$$\chi_{c_\rho}^{c^\lambda}(A) := \lambda_A(\rho) - \sum_k \rho_k \lambda_A(e_k) = 0.$$

Proof. According to Definition 3.1 the assertion of this theorem is equivalent to the statement

$$\lim_m f(\rho - \sum_{k=1}^{m} \rho_k e_k) = 0 \quad \forall f \in (c_A^\lambda)'.$$

Applying condition (iv) from 5.1 of [5] we get by 2.10 and 3.7 that

$$f\Big(\rho-\sum_{k=1}^{m}\rho_k e_k\Big)=\sum_{k>m}t_k\rho_k+\sum_{k>m}\sum_{n}\tau_n\lambda_A^n(e_k)\rho_k+$$

$$+\mu\Big(\lambda_A(\rho)-\sum_{k=1}^{m}\lambda_A(e_k)\rho_k\Big)+\sigma\sum_{k>m}a_k\rho_k.$$

Our statement follows from this equality by $\mu\neq 0$.

Theorem 3.9. *A matrix* $A\in(c^\nu,c^\lambda)$ *is* (c^ν,c^λ)*-conull if and only if*

$$\chi_{c^\nu}^{c^\lambda}(A):=\lambda_A(\nu^{-1})-\sum_k\frac{\lambda_k(e_k)}{\nu_k}=0.$$

Proof. This statement follows from Theorem 3.8 and Definition 3.1 since

$$c^\nu=c_{\nu^{-1}}\oplus\langle e\rangle. \quad \blacksquare$$

Next we shall consider some properties of the conullity in connection with different rates and speeds. Let X,Y and Z be rooms of type c_ρ or $c_\rho\oplus\langle e\rangle$. If a matrix $A\in(X,Y)$ then we call X as "domain-room" and Y as "range-room".

Theorem 3.10. *If* $A\in(X,Y)$ *and* $Z\subsetneqq X$ *then* A *is* (Z,Y)*-conull.*

Proof. Let $Z=c_\kappa$ (or $Z=c_\kappa\oplus\langle e\rangle$) and $X=c_\rho$ (or $X=c_\rho\oplus\langle e\rangle$). Then $Z\subsetneqq X$ implies that $\lim_n(\kappa_n/\rho_n)=0$ (see [5], Proposition 2.8) i.e. $\kappa\in c_{0\rho}$. So κ has AK in $c_{0\rho}$. The (Z,Y)-conullity follows now immediately from Definition 3.1 and the relation $c_{0\rho}\subset X\subset Y_A$.

Theorem 3.11. *If* A *is* (X,Y)*-conull and* $Z\supset Y$ *then* A *is* (X,Z)*-conull.*

Proof. The assertion follows from the fact $c_\rho\subset Y_A\subset Z_A$. $\blacksquare$

Let $\rho=(\rho_n)$ and $\kappa=(\kappa_n)$ be two different rates. We say that ρ is greater than κ if $\lim_n(\kappa_n/\rho_n)=0$ i.e. $c_\kappa\subsetneqq c_\rho$. In this case we write $\kappa\prec\rho$.

In view of properties of the rate-spaces (see [5]) we can formulate theorems 3.10 and 3.11 as follows.

If $A \in (X, Y)$ then the decrease of the rate (or the increase of the speed) of the "domain-room" turns the matrix A into conull of the corresponding type.

If A is (X, Y)-conull then the increase of the rate (or the decrease of the speed) of the "range-room" does not change the conullity.

Examples. 1. *Let $A \in (c, c)$. Then A is (c^λ, c)-conull for any speed λ and (c_ρ, c)-conull for any rate ρ with $\lim \rho = 0$.*

2. *Let A be λ-conull i.e. (c^λ, c^λ)-conull. Then A is (c^λ, c^μ)-conull for any speed $\mu \prec \lambda$.*

4. Theorem of Steinhaus type

In 1911 Steinhaus proved that any regular matrix cannot sum all bounded sequences. This fact was generalized by A. Wilansky. He has shown that the relation $c_A \supset m$ can be true only for conull matrix A (see [7]). A very simple and impressive proof for this theorem was given by G. Kangro in his lectures. We use the similar proofs to show that the theorem of Steinhaus is true also for another type of mappings.

Theorem 4.1. *The following statements are true:*

$$\begin{aligned}
&\text{(i)} \quad A \in (m_\rho, c_\pi) \implies \chi_{c_\rho}^{c_\pi}(A) = 0,\\
&\text{(ii)} \quad A \in (m^\nu, c_\pi) \implies \chi_{c^\nu}^{c_\pi}(A) = 0,\\
&\text{(iii)} \quad A \in (m_\rho, c^\lambda) \implies \chi_{c_\rho}^{c^\lambda}(A) = 0,\\
&\text{(iv)} \quad A \in (m^\nu, c^\lambda) \implies \chi_{c^\nu}^{c^\lambda}(A) = 0.
\end{aligned}$$

Proof. (i) $A \in (m_\rho, c_\pi) \implies A \in (c_\rho, c_\pi)$. Then by Theorem 4.10 from [5] the matrix A satisfies the condition

$$\lim_n \sum_k \mid \frac{a_{nk}}{\pi_n} - \lim_{\pi A} e_k \mid \rho_k = 0.$$

This implies that

$$\lim_{\pi A} \rho := \lim_n \sum_k \frac{a_{nk}}{\pi_n} \rho_k = \sum_k \rho_k \lim_{\pi A} e_k.$$

Theorem 3.5 implies the assertion (i).

For remaining cases (ii), (iii) and (iv) the proof is the same applying Theorems 4.12, 5.8, 5.10 from [5] and the definitions of the corresponding characteristics from section 3.

References

1. Beekmann, W., Chang, S.-C., *λ-convergence and λ-conullity.* Z. Anal. Anwendungen., 1993, **12**, 179–182.
2. Jürimäe, E., *Conservative methods of summation* (in Russian). Eesti NSV Tead. Akad. Toimetised, Füüs.-Mat. ja Tehn., 1960, **9**, 257–267.
3. Jürimäe, E., *Zeller's theorem for the λ-summability* (in Russian). Tartu Ülik. Toimetised, 1989, **846**, 160–165.
4. Jürimäe, E., *Replaceability for λ-summability.* Israel Math. Conference Proc., 1991, **4**, 157–162.
5. Jürimäe, E., *Matrix mappings between rate-spaces and spaces with speed.* Ibidem, pp. 29–52
6. Kangro, G., *On λ-perfecticity of summability methods and its applications.* I (in Russian). Eesti NSV Tead. Akad. Toimetised, Füüs.-Mat., 1971, **20**, 111–120.
7. Wilansky, A., *An application of Banach linear functionals to summability.* Trans. Amer. Math. Soc., 1949, **67**, 59–68.
8. Wilansky, A., *Summability through Functional Analysis.* North. Holland, Amsterdam–New York–Oxford, 1984.
9. Zeller, K., *Allgemeine Eigenschaften von Limitierungsverfahren.* Math. Z., 1951, **53**, 463–487.

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Järgu- ja kiirusega ruumide vaheliste maatriksteisenduste väljade omadusi

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Resümee

Käesolevas artiklis on vaadeldud maatriksteisendusi $y = Ax$, kus $x = (x_k) \in X$, $y = (y_n) \in Y$ ning

$$y_n = \sum_k a_{nk}x_k, \quad n,k \in \mathbb{N}.$$

Kui iga $x \in X$ puhul $y = Ax \in Y$, siis kirjutame $A \in (X,Y)$. Ruumidena X ja Y on vaadeldud järgu-ruume c_π (lk. 53) või siis kiirusega ruume c^λ (lk. 53). Kui maatriksi A korral $Y = c_\pi$, siis kõneldakse väljast $c_{\pi A}$ (lk. 54), kui aga $Y = c^\lambda$, siis väljast c^λ_A (lk. 54). Neid välju on vaadeldud kui FK-ruume, milles omakorda on vaadeldud pidevate lineaarsete funktsionaalide erinevaid esitusi (p. 2). P. 3 on pühendatud konullilisuse mõiste käsitlemisele erinevate maatriksteisenduste korral.

Definitsioon. *Olgu $A \in (X,Y)$, kus $X = c_\rho$ või $X = c_\rho \oplus \langle u \rangle$. Maatriksit A nimetatakse (X,Y)-konulliliseks, kui jada ρ puhul kehtib nõrk lõikekoonduvus vaadeldavas väljas.*

Osutub, et väljade $c_{\pi A}$ puhul on konullilisus iseloomustatav funktsionaali $\lim_{\pi A}$ (lk. 54) väärtuste abil (teoreemid 3.5 ja 3.6), väljade c^λ_A konullilisus aga funktsionaali λ_A (lk. 56) väärtuste abil (teoreemid 3.8 ja 3.9).

Aastast 1911 on teada fakt, mida tuntakse Steinhausi teoreemina. Kaasaegset terminoloogiat kasutades on see formuleeritav järgmiselt.

Kui maatriks A teisendab kõik tõkestatud jadad koonduvaiks, siis on ta konulliline.

Käesolevas töös (p. 4) on analoogilised väited tõestatud kõikvõimalike kombinatsioonide korral, kus nii originaalide ruum kui ka kujutiste ruum on järgu- või siis kiirusega ruumid.

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Inclusion theorems for some sequence spaces defined by a sequence of moduli

Enno Kolk

1. Introduction

Ruckle [5] and Maddox [3] used the idea of modulus function to construct new sequence spaces.

Definition 1. *A function* $f : [0, \infty) \to [0, \infty)$ *is called a modulus if*

(a) $f(t) = 0$ *if and only if* $t = 0$,

(b) $f(t + u) \leq f(t) + f(u)$ *for all* $t \geq 0, u \geq 0$,

(c) f *is increasing,*

(d) f *is continuous from the right of 0.*

It immediately follows from (b) and (d) that f is continuous everywhere on $[0, \infty)$. A modulus may be unbounded or bounded. For example, $f(t) = t^p$ $(0 < p \leq 1)$ is unbounded but $f(t) = t \,/\, (1 + t)$ is bounded.

For a certain sequence space X of real or complex numbers and for a modulus f , Ruckle and Maddox considered a new sequence space

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}.$$

The extension of this definition was given in [2] (see also [1]) by replacing one modulus with a sequence of moduli. Thus for a sequence space X and a sequence of moduli $F = (f_k)$, we define

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}. \tag{1}$$

It is not difficult to see that if X is a normal sequence space (i.e. $(y_k) \in X$ whenever $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$ for some $(x_k) \in X$) then $X(F)$ is also a normal sequence space. For example, the spaces m and c_0 of all bounded and of all null sequences, respectively, are normal. So

$$m(F) = \{x = (x_k) : \sup_k f_k(|x_k|) < \infty\},$$

$$c_0(F) = \{x = (x_k) : \lim_k f_k(|x_k|) = 0\}$$

are normal sequence spaces.

In the particular case $f_k(t) = t^{p_k}$ $(0 < p_k \leq 1)$ the spaces $m(F)$ and $c_0(F)$ are reduced to $m(p)$ and $c_0(p)$, respectively, where $p = (p_k)$ (see [8,4]).

Let $\lambda = (\lambda_k)$ be a real sequence with $\lambda_k \neq 0$ $(k \in \mathbb{N})$. For the sequence space X Sikk [6] introduced rate-space

$$X_\lambda = \{x = (x_k) : (\lambda_k x_k) \in X\}.$$

If X is here normal, then $(\lambda_k x_k) \in X$ is equivalent to $(|\lambda_k x_k|) \in X$ and so the rate-space X_λ can be considered as the space $X(F)$, where $f_k(t) = |\lambda_k| t$.

In [2,1] the necessary and sufficient conditions for the inclusions $m \subseteq m(F)$ and $c_0 \subseteq c_0(F)$ were given. In this paper we shall examine all other inclusion relations between X and $Y(F)$, where X and Y are one of the spaces m and c_0. At that we use the following characteristics of a sequence of moduli $F = (f_k)$:

(M1) $\sup_k f_k(t) < \infty \quad (t > 0)$,

(M2) $\lim_{t \to 0+} \sup_k f_k(t) = 0$,

(M3) $\inf f_k(t) > 0$,

(M4) $\lim_{t \to \infty} \underline{\lim}_k f_k(t) = \infty$,

(M5) $\lim_k f_k(t) = 0 \quad (t > 0)$,

(M6) $\lim_k f_k(t) = \infty \quad (t > 0)$.

At this we shall regard that (M4) is satisfied also for $\underline{\lim}_k f_k(t) = \infty \quad (t > 0)$.

2. Preliminary results

First we formulate the two lemmas proved in [1].

Lemma A. *The condition* (M1) *is fulfilled if and only if there is a point* $t_0 > 0$ *such that* $\sup_k f_k(t_0) < \infty$.

Lemma B. *The condition* (M3) *is fulfilled if and only if there exists a point* $t_0 > 0$ *such that* $\inf_k f_k(t_0) > 0$.

Along with a modulus we introduce the notion of a premodulus.

Definition 2. *A continuous function* $f : [0,\infty) \to [0,\infty)$ *is called a premodulus if the conditions* (a) *and* (c) *of Definition 1 are satisfied.*

It is clear that every modulus is premodulus and there exist premoduli which are not moduli. For example $f(t) = t^p$ is a premodulus for all $p > 0$ but it is not a modulus for $p > 1$.

If a premodulus f is strictly increasing and unbounded, then it is obviously invertible and so admits inverse function f^{-1} which is also a premodulus.

Let $F = (f_k)$ be a sequence of strictly increasing unbounded premoduli and $G = (g_k)$ a sequence of arbitrary premoduli. For two sequence spaces X, Y we consider the inclusion

$$X(F) \subseteq Y(G), \tag{2}$$

where the spaces $X(F)$ and $Y(G)$ are defined by (1). If $y = (y_k)$ with $y_k = f_k(|x_k|)$ then $|x_k| = f_k^{-1}(y_k) = f_k^{-1}(|y_k|)$ and so (2) is true when

$$y \in X \Longrightarrow (g_k f_k^{-1}(|y_k|)) \in Y.$$

Thus (2) holds if $X \subset Y(GF^{-1})$ where $F^{-1} = (f_k^{-1})$.

Conversely, since for every $z = (z_k) \in X$ we have $|z_k| = f_k f_k^{-1}(|z_k|)$ with $(f_k^{-1}(|z_k|)) \in X(F)$ then (2) implies $(g_k f_k^{-1}(|z_k|)) \in Y$, i.e. $z \in Y(GF^{-1})$. So $X \subseteq Y(GF^{-1})$ is also necessary for the inclusion (2). In fact, we have proved

Proposition 1. *Let* X, Y *be normal sequence spaces and* $G = (g_k)$ *a sequence of premoduli. For a sequence* $F = (f_k)$ *of strictly increasing unbounded premoduli the inclusion* $X(F) \subseteq Y(G)$ *holds if and only if* $X \subseteq Y(GF^{-1})$.

Analogously we can prove

Proposition 2. *Let* X, Y *be normal sequence spaces and* $F = (f_k)$ *a sequence of premoduli. If* $G = (g_k)$ *is a sequence of strictly increasing unbounded premoduli, then* $X(F) \subseteq Y(G)$ *if and only if* $X(FG^{-1}) \subseteq Y$.

Propositions 1 and 2 show that in the investigation of inclusion (2), the inclusions $X \subseteq Y(H)$ and $X(H) \subseteq Y$ where $H = (h_k)$ is the well-defined sequence of premoduli play an essential role. In sections 3 and 4 we consider the previous inclusions where H is a sequence of moduli and $X, Y \in \{m, c_0\}$. It should be noted that these inclusions are trivial for one modulus since $c_0(h) = c_0$ for every modulus h, $m(h) = m$ for unbounded modulus h, and $m(h)$ is the space ω of all sequences if h is a bounded modulus.

3. The space $m(F)$

In [2,1] was proved

Theorem A. *The condition* (M1) *is necessary and sufficient for the inclusion* $m \subseteq m(F)$.

Here we complement Theorem A.

Theorem 1. *The following statements are equivalent for a sequence of moduli* $F = (f_k)$:

(a) $m \subseteq m(F)$;

(b) $c_0 \subseteq m(F)$;

(c) (M1) *is satisfied.*

Proof. (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c). Let $c_0 \subset m(F)$. If we suppose that (M1) is not satisfied, then by Lemma A $\sup_k f_k(t) = \infty$ for all $t > 0$. Thus, there is an index sequence (k_i) such that

$$f_{k_i}(1/i) > i \quad (i \in \mathbb{N}). \tag{3}$$

Define $x_k = 1/i$ for $k = k_i$ $(i \in \mathbb{N})$ and $x_k = 0$ otherwise. Then $x = (x_k)$ belongs to c_0. But by (3) we get $x \notin m(F)$, contrary to $c_0 \subseteq m(F)$. Therefore, (M1) must be satisfied.

(c) $\Longrightarrow$ (a) follows from Theorem A. ■

The necessary and sufficient conditions for the inverse inclusions are contained in the following two theorems.

Theorem 2. *The inclusion* $m(F) \subseteq c_0$ *holds if and only if* (M6) *is satisfied.*

Proof. Let $m(F) \subseteq c_0$. If (M6) is not fulfilled, then there is a number

$t_0 > 0$ and an index sequence (k_i) such that

$$f_{k_i}(t_0) \leq M < \infty.$$

In addition we can assume that $\mathbb{N} \setminus \{k_i\}$ is infinite. Now the sequence $x = (x_k)$, where $x_k = t_0$ for $k = k_i$ $(i \in \mathbb{N})$ and $x_k = 0$ otherwise, belongs to $m(F)$. But $x \notin c_0$. So (M6) is necessary for the inclusion $m(F) \subseteq c_0$.

For the converse, let (M6) be satisfied and let $x \in m(F)$, i.e. $f_k(|x_k|) \leq M < \infty$ $(k \in \mathbb{N})$. If $x \notin c_0$, then for some number $\varepsilon_0 > 0$ and index k_0, $|x_k| \geq \varepsilon_0$ $(k \geq k_0)$. Thus

$$f_k(\varepsilon_0) \leq f_k(|x_k|) \leq M < \infty \quad (k \geq k_0),$$

contrary to (M6). Hence $x \in c_0$. This completes the proof.

Theorem 3. *The inclusion* $m(F) \subseteq m$ *is valid if and only if* (M4) *is fulfilled.*

Proof. Let $m(F) \subseteq m$. If (M4) fails to hold, then the function $h(t) = \underline{\lim}_k f_k(t)$ must be finite and bounded. Similarly to Lemma B we can show that either $h(t) = 0$ $(t > 0)$ or $h(t) > 0$ $(t > 0)$. In both cases there exists an index sequence (n_i) and a number $H \geq 0$ such that $h(t) = \lim_i f_{n_i}(t) \leq H$ $(t > 0)$. Thus for fixed $\varepsilon > 0$ we can choose by induction an index subsequence (k_i) of (n_i) with $f_{k_i}(i) \leq H + \varepsilon$ $(i \in \mathbb{N})$. Define $x_k = i$ for $k = k_i$ $(i \in \mathbb{N})$ and $x_k = 0$ otherwise. Then $x = (x_k)$ belongs to $m(F)$. But $x \notin m$, contrary to $m(F) \subseteq m$. Consequently, (M4) must be satisfied.

Conversely, let (M4) hold If $x \in m(F)$ and $h(t) = \underline{\lim}_k f_k(t)$ is finite, then there is a number $P > 0$ and an index k_0 such that

$$h(|x_k|) \leq f_k(|x_k|) \leq P \quad (k \geq k_0).$$

By the increase of h we have $|x_k| \leq M$ $(k \geq k_0)$ where $M = \sup\{t : h(t) = P\}$. Hence $x \in m$.

In case $\underline{\lim}_k f_k(t) = \infty$ $(t > 0)$, the condition (M6) is satisfied and the inclusion $m(F) \subseteq m$ follows from Theorem 2. The proof is completed.

Let $p = (p_k)$ with $0 < p_k \leq 1$ and $f_k(t) = t^{p_k}$. Since

$$t^{\underline{\lim}_k p_k} = \underline{\lim}_k t^{p_k} \leq \max\{1, t\}$$

for all $t > 0$, then (M1) is always satisfied and (M4) holds if and only if $\underline{\lim}_k p_k > 0$ or, equivalently, $\inf_k p_k > 0$. So, from Theorems 1 and 2 it follows (see [8], Theorem 9)

Corollary 1. *Let $p = (p_k)$ with $0 < p_k \leq 1$. Then $m(p) = m$ if and only if $\inf_k p_k > 0$.*

Let $\lambda = (\lambda_k)$ be a sequence of real numbers with non-zero elements. For $f_k(t) = |\lambda_k|t$ $(k \in \mathbb{N})$ the conditions (M1), (M4) and (M6) are equivalent to $\sup_k |\lambda_k| < \infty$, $\underline{\lim}_k |\lambda_k| > 0$ and $\lim_k |\lambda_k| = \infty$, respectively. So, using Propositions 1 and 2 from Theorems 1–3 for rate-spaces we conclude (cf. [7], Theorem 8)

Corollary 2. *Let $\lambda = (\lambda_k)$, $\mu = (\mu_k)$ be two sequences with non-zero elements. Then*

(a) $m_\lambda \subseteq m_\mu \Leftrightarrow m \subseteq m_{\mu\lambda^{-1}} \Leftrightarrow \sup_k |\mu_k \lambda_k^{-1}| < \infty,$

(b) $m_\lambda \subseteq m_\mu \Leftrightarrow m_{\lambda\mu^{-1}} \subseteq m \Leftrightarrow \underline{\lim}_k |\lambda_k \mu_k^{-1}| > 0,$

(c) $(c_0)_\lambda \subseteq m_\mu \Leftrightarrow c_0 \subseteq m_{\mu\lambda^{-1}} \Leftrightarrow \sup_k |\mu_k \lambda_k^{-1}| < \infty.$

(d) $m_\lambda \subseteq (c_0)_\mu \Leftrightarrow m_{\lambda\mu^{-1}} \subseteq c_0 \Leftrightarrow \lim_k |\lambda_k \mu_k^{-1}| = \infty,$

where $\lambda^{-1} = (\lambda_k^{-1})$ and $\mu\lambda = (\mu_k \lambda_k)$.

4. The space $c_0(F)$

The following was proved in [2,1].

Theorem B. *The condition* (M2) *is necessary and sufficient for the inclusion $c_0 \subseteq c_0(f)$.*

Here we consider the inclusions $m \subseteq c_0(F)$, $c_0(F) \subseteq c_0$ and $c_0(F) \subseteq m$.

Theorem 4. *The inclusion $m \subseteq c_0(F)$ is true if and only if* (M5) *is satisfied.*

Proof. Let $m \subseteq c_0(F)$. If (M5) is not satisfied, then $\lim_k f_k(t_0) = 0$ fails to hold for some $t_0 > 0$. Thus the constant (and hence bounded) sequence $x = (x_k)$ with $x_k = t_0$ $(k \in \mathbb{N})$ does not belong to $c_0(F)$. So (M5) must hold.

Conversely, if (M5) is fulfilled and $x \in m$ then $|x_k| \leq M < \infty$ $(k \in \mathbb{N})$. So $f_k(|x_k|) \leq f_k(M)$ $(k \in \mathbb{N})$ and $x \in c_0(F)$ immediately follows from (M5).

Theorem 5. *The following statements are equivalent for a sequence of moduli* $F = (f_k)$:

(a) $c_0(F) \subseteq c_0$;

(b) $c_0(F) \subseteq m$;

(c) (M3) *is fulfilled.*

Proof. (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c). Let $c_0(F) \subseteq m$. If (M3) fails to hold, then by Lemma B

$$\inf_k f_k(t) = 0 \quad (t > 0).$$

Thus by induction we can choose an index sequence (k_i) such that $f_{k_i}(i) < 1/i$ $(i \in \mathbb{N})$. Now the sequence $x = (x_k)$, where $x_k = i$ for $k = k_i$ $(i \in \mathbb{N})$ and $x_k = 0$ otherwise, belongs to $c_0(F)$. But $x \notin m$, contrary to $c_0(F) \subseteq m$. Hence (M3) must be satisfied.

(c) $\Longrightarrow$ (a). Let (M3) hold and let $x \in c_0(F)$, i.e. $\lim_k f_k(|x_k|) = 0$. If we suppose that $x \notin c_0$, then for some number $\varepsilon_0 > 0$ and index k_0 we have $|x_k| \geq \varepsilon_0$ $(k \geq k_0)$. Thus

$$f_k(\varepsilon_0) \leq f_k(|x_k|) \quad (k \geq k_0)$$

which implies $\lim_k f_k(\varepsilon_0) = 0$, contrary to (M3). Consequently, $x \in c_0$. The theorem is proved.

In case $f_k(t) = t^{p_k}$ with $0 < p_k \leq 1$ $(k \in \mathbb{N})$, the condition (M2) reduces to $\inf_k p_k > 0$. Hence from Theorem B we get (cf. [4], Lemma 1)

Corollary 3. *Let* $p = (p_k)$ *with* $0 < p_k \leq 1$. *Then* $c_0 \subset c_0(p)$ *if and only if* $\inf_k p_k > 0$.

For $f_k(t) = |\lambda_k|t$, $\lambda_k \neq 0$ $(k \in \mathbb{N})$, the conditions (M2), (M3) and (M5) are equivalent to $\sup_k |\lambda_k| < \infty$, $\inf_k |\lambda_k| > 0$ and $\lim_k |\lambda_k| = 0$, respectively. Thus, from Propositions 1, 2 and Theorems B, 4 and 5 it follows

Corollary 4. *Let* $\lambda = (\lambda_k)$, $\mu = (\mu_k)$ *be the sequences with non-zero* *elements. Then*

$$
\begin{array}{lllll}
\text{(a)} & (c_0)_\lambda \subseteq (c_0)_\mu & \Leftrightarrow \quad c_0 \subseteq (c_0)_{\mu\lambda^{-1}} & \Leftrightarrow & \sup_k |\mu_k \lambda_k^{-1}| < \infty, \\
\text{(b)} & (c_0)_\lambda \subseteq (c_0)_\mu & \Leftrightarrow \quad (c_0)_{\lambda\mu^{-1}} \subseteq c_0 & \Leftrightarrow & \inf_k |\lambda_k \mu_k^{-1}| > 0, \\
\text{(c)} & m_\lambda \subseteq (c_0)_\mu & \Leftrightarrow \quad m \subseteq (c_0)_{\mu\lambda^{-1}} & \Leftrightarrow & \lim_k |\mu_k \lambda_k^{-1}| = 0.
\end{array}
$$

References

1. Kolk, E., *On strong boundedness and summability with respect to a sequence of moduli.* Acta et Comment. Univ. Tartuensis, 1993, **960**, 41–50.
2. Kolk, E., *Sequence spaces defined by a sequence of moduli.* Abstracts of conference "Problems of pure and applied mathematics". Tartu, 1990, 131–134.
3. Maddox, I.J., *Sequence spaces defined by a modulus.* Math. Proc. Camb. Phil. Soc., 1986, **100**, 161–166.
4. Maddox, I.J., *Spaces of strongly summable sequences.* Quart. J. Math. Oxford Ser. (2), 1967, **18**, 345–355.
5. Ruckle, W.H., *FK spaces in which the sequence of coordinate vectors is bounded.* Canad. J. Math., 1973, **25**, 973–978.
6. Sikk, J., *Matrix mappings for rate-spaces and K-multipliers in the theory of summability.* Acta et Comment. Univ. Tartuensis, 1989, **846**, 118–129.
7. Sikk, J., *The rate-spaces* $m(\lambda)$, $c(\lambda)$, $c_0(\lambda)$ *and* $l^p(\lambda)$ *of sequences.* Ibidem, pp. 87–96.
8. Simons, S., *The sequence spaces* $l(p_\nu)$ *and* $m(p_\nu)$. Proc. London Math. Soc., 1965, **15**, 422–436.

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Inclusion between the cores concerning summability methods (R, p_k), (J, p_k) and (J_α, p_k)

Leiki Loone

Let $U_-(\tau_0)$ be an arbitrary fixed left-hand neighbourhood of a number $\tau_0 \in \mathbb{R}$. Suppose that for every $\tau \in U_-(\tau_0)$ there is a matrix $A(\tau) = (a_{nk}(\tau))$ such that

$$\sup_n \sum_k | a_{nk}(\tau) | < \infty \quad \forall \tau \in U_-(\tau_0). \tag{1}$$

Definition 1. *It is said that a sequence $x = (\xi_k)$ is summable by a semicontinuous sequential summability method $(A(\tau))$ (in short $\alpha(\tau)$ - summable) to a number a if*

$$\lim_{\tau \to \tau_0 -} \sum_k a_{nk}(\tau)\xi_k = a$$

uniformly in n.

A semicontinuous sequential summability method $(A(\tau))$ is called regular if every convergent sequence is $\alpha(\tau)$-summable to the previous limit. In the special case of

$$a_{nk}(\tau) := a_k(\tau) \quad \forall n \in \mathbb{N} \tag{2}$$

the $\alpha(\tau)$-summability method $(A(\tau))$ turns into ordinal semicontinuous summability method $(a_k(\tau))$.

Let W be the set of all sequences $(\tau_m) \subset U_-(\tau_0)$ which are convergent to τ_0. It means that

$$W := \{w = (\tau_m) : \tau_m \longrightarrow \tau_0, \quad \tau_m \in U_-(\tau_0) \quad \forall m \in \mathbb{N}\}.$$

Let $w = (\tau_m)$ be an arbitrarily fixed element from W and let us define the α-method (A_m) where $a_{mnk} = a_{nk}(\tau_m)$. If a sequence x is α-summable by this α-method (A_m) we say in short that it is w-summable. The set which defines the core for the w-summability is denoted by K_w.

It is known that a sequence $x = (\xi_k)$ is $\alpha(\tau)$-summable to a iff it is w-summable to a for every $w \in W$ (see [5]).

Definition 2. *The core for the $\alpha(\tau)$-method $(A(\tau))$ is the core defined by the set*

$$K := \text{clco} \cup \{K_w \mid w \in W\}. \tag{3}$$

The set of all $\alpha(\tau)$-summable sequences coincides with the set of all sequences x for which the core

$$K(x) := \{f(x) \mid f \in K\}$$

is a singleton (see [5]). The necessary and sufficient conditions for the regularity of an $\alpha(\tau)$-method $(A(\tau))$ are as follows

$$1^0 \quad \lim_{\tau \to \tau_0 -} \sup_n |a_{nk}(\tau)| = 0 \quad \forall k = 0, 1, \dots, \tag{4}$$

$$2^0 \quad \lim_{\tau \to \tau_0 -} \sum_k a_{nk}(\tau) = 1 \text{ uniformly in } n, \tag{5}$$

$$3^0 \quad \sup_n \sum_k |a_{nk}(\tau)| < M \text{ for every } \tau \in U_-(\tau_0). \tag{6}$$

Let $\mathbf{m}$ be the set of all bounded sequences

$$\mathbf{m} := \{x = (\xi_k) \mid \sup_k |\xi_k| < \infty\}$$

and let $K^0(x)$ be the Knopp's core of x.

The inclusion

$$K(x) \subset K^0(x) \quad \forall x \in \mathbf{m} \tag{7}$$

holds iff the $\alpha(\tau)$-method is regular and

$$\lim_{\tau \to \tau_0 -} \sup_n \sum_k |a_{nk}(\tau)| = 1. \tag{8}$$

(see [5]). The method with the property (7) is called core-regular.

Let $L(x)$ be the set of Banach limits of a sequence x. This set is the core of almost convergence of x (see [4]). The inclusion

$$K(x) \subset L(x) \quad \forall x \in \mathbf{m} \tag{9}$$

holds iff the inclusion (7) holds and

$$\lim_{\tau\to\tau_0-}\sup_n\sum_k|a_{nk}(\tau)-a_{nk+1}(\tau)|=0. \tag{10}$$

Suppose throughout that (p_k) is a sequence of real numbers with $p_k>0$ for all $k=0,1,2,\dots,$ where $p_0=1$ and

$$P_m:=\sum_{k=0}^m p_k\longrightarrow\infty \quad\text{as}\quad m\longrightarrow\infty. \tag{11}$$

Let τ_0 be the radius of convergence of the power series

$$\sum_k p_k\tau^k$$

and let

$$p(\tau):=\sum_k p_k\tau^k\longrightarrow\infty \quad\text{as}\quad \tau\longrightarrow\tau_0-. \tag{12}$$

It follows from (11) that $\tau_0\le 1$. If $\tau_0\le 1$, then the power series

$$\sum_k P_k\tau^k \quad\text{and}\quad \sum_k p_k\tau^k$$

have the same radius of convergence, i.e.

$$\limsup_k\sqrt[k]{p_k}=\limsup_k\sqrt[k]{P_k} \tag{13}$$

(see [3]).

The weighted means summability method (R,p_k) is defined by Riesz matrix $P=(a_{mk})$, where

$$a_{mk}=\begin{cases}\frac{p_k}{P_k}, & \text{if} \quad k\le m,\\ 0, & \text{if} \quad k>m.\end{cases}$$

As $P_m\to\infty$, this method is regular (see[1]).

Let $\mathbf{c}$ and $\mathbf{c_P}$ be the set of all convergent sequences and the set of all (R,p_k)-summable sequences respectively, i.e.

$$\mathbf{c}:=\{x=(\xi_k)\mid\exists\lim_k\xi_k\},$$

$$\mathbf{c_P}:=\{x=(\xi_k)\mid\exists\lim_m\sum_{k=0}^m\frac{p_k}{P_k}\xi_k\}.$$

Lemma 1. *For the matrix* $P = (R, p_k)$ *the following statements hold:*

$$1. \quad \liminf_k \sqrt[k]{\frac{P_k}{p_k}} = 1. \tag{14}$$

2. *If* $\tau_0 = 1$, *then* P *is not Mercerian, i.e.* $\mathbf{c_P} \neq \mathbf{c}$.

3. *If* $\tau_0 < 1$, *then the sequence* $(\frac{P_k}{p_k})$ *has a bounded subsequence.*

Proof. 1. If (14) fails then there must exists $C > 1$ and $k_0 \in \mathbb{N}$ such that

$$\sqrt[k]{\frac{P_k}{p_k}} > C \quad \forall k > k_0.$$

If so, then

$$P_k > C^k p_k \quad \forall k > k_0$$

and

$$\limsup_k \sqrt[k]{P_k} \geq C \limsup_k \sqrt[k]{p_k}.$$

Using (13) we find that $C \leq 1$, i.e. it is impossible for (14) to fail.

2. We shall use the inequality

$$\liminf_k \frac{P_{k+1}}{P_k} \leq \limsup_k \sqrt[k]{P_k}.$$

In the case of $\tau_0 = 1$ this yields

$$\liminf_k (1 + \frac{p_{k+1}}{P_k}) \leq 1.$$

Hence

$$\liminf_k \frac{p_{k+1}}{P_k} = 0$$

and consequently the sequence $(\frac{P_k}{p_{k+1}})$ is not bounded, i.e. P is not Mercerian (see [1]).

3. In the case of $\tau_0 < 1$ we use the inequality

$$\limsup_k \sqrt[k]{P_k} \leq \limsup_k \frac{P_{k+1}}{P_k}.$$

This yields

$$1 < \frac{1}{\tau_0} = \limsup_k \sqrt[k]{P_k} \leq \limsup_k (1 + \frac{p_{k+1}}{P_k}).$$

It means that

$$\limsup_k \frac{p_{k+1}}{P_k} > 0$$

and therefore the sequence $(\frac{P_k}{p_k})$ has a bounded subsequence. ■

Let

$$P^r := (R, (\frac{r_0}{r})^k p_k) \quad \forall r \in (0,1]. \tag{15}$$

Since

$$\frac{1}{\limsup_k \sqrt[k]{(\frac{r_0}{r})^k p_k}} = r,$$

the radius of convergence of the power series

$$\sum_k (\frac{r_0}{r})^k p_k \tau^k$$

is equal to r. It is obvious that

$$P^{r_0} = P.$$

Let $\mathbf{P}$ be the set of Riesz matrices P^r, generated by $P = (R, p_n)$ using the formula (15), i.e.

$$\mathbf{P} := \{P^r \mid r \in (0,1]\}.$$

One can easily check that every member of the set $\mathbf{P}$ generates the same $\mathbf{P}$.

Theorem 2. *If* $0 < a < r_0 < b < 1$, *then*

$$K^0(P^1 x) \subset K^0(P^b x) \subset K^0(Px) \subset K^0(P^a x) \quad \forall x \in \mathbf{m}. \tag{16}$$

Proof. Corollary 1.1 in [7] asserts that if for two arbitrary positive Riesz matrices $T = (R, t_k)$ and $Q = (R, q_k)$

$$\frac{t_{k+1}}{t_k} \le \frac{q_{k+1}}{q_k} \quad \forall k \ge k_0, \tag{17}$$

then

$$K^0(Tx) \subset K^0(Qx) \quad \forall x \in \mathbf{m}. \tag{18}$$

This fact yields the inclusion (16) because of

$$\frac{p_{k+1}}{p_k} r_0 \le \frac{p_{k+1}}{p_k} \frac{r_0}{b} \le \frac{p_{k+1}}{p_k} \le \frac{p_{k+1}}{p_k} \frac{r_0}{a} \quad \forall k \ge 0. \quad ■$$

Let P_* be the method of arithmetical means, i.e. $P_* = (R, 1)$.

Corollary 2.1. *If* $0 < a < b \le 1$ *then there exists such* $x_0 \in \mathbf{m}$ *that*

$$K^0(P_*^b x_0) \neq K^0(P_*^a x_0).$$

Proof. Let $T := P_*^b$ and $Q := P_*^a$. By Theorem 2

$$K^0(Tx) \subset K^0(Qx) \quad \forall x \in \mathbf{m}.$$

Lemma 1 in [7] asserts that for the inverse inclusion it is necessary that

$$\limsup_m \frac{T_m}{Q_m} \cdot \frac{q_m}{t_m} < 1.$$

In our case

$$\frac{T_m}{Q_m} \cdot \frac{q_m}{t_m} = \frac{1 - b^{m+1}}{1 - b} \cdot \frac{1}{b^m} \cdot \frac{1 - a}{1 - a^{m+1}} \cdot a^m \cdot \frac{b^m}{a^m} =$$
$$= \frac{1 - a}{1 - b} \cdot \frac{(1 - b^{m+1})}{(1 - a^{m+1})} \longrightarrow \frac{1 - a}{1 - b} > 1,$$

and therefore the statement of Corollary 2.1. is true. ■

Theorem 3. *If the method* $P = (R, p_n)$ *is such, that there exists*

$$\lim_k \frac{p_{k+1}}{p_k}, \tag{19}$$

then for every $r \in (0, 1)$ *the following equality holds*

$$K^0(P^r x) = K^0(P_*^r x) \quad \forall x \in \mathbf{m}. \tag{20}$$

Proof. If (19), then

$$\lim_k \frac{p_{k+1}}{p_k} = \frac{1}{r_0}. \tag{21}$$

Let $Q = (R, q_k) =: P^r$, i.e.

$$q_k = \left(\frac{r_0}{r}\right)^k p_k.$$

Consequently,

$$\lim_k \frac{q_{k+1}}{q_k} = \lim_k \frac{r_0^{k+1} p_{k+1} r^k}{r^{k+1} r_0^k \, p_k} = \frac{1}{r} > 1.$$

By Corollary 4.1 in [7] for the arbitrary $Q=(R,q_k)$, the condition

$$\lim_k \frac{q_{k+1}}{q_k} := q > 1$$

yields

$$K^0(Qx) = K^0(P_*^{\frac{1}{q}}x) \quad \forall x \in \mathbf{m},$$

and therefore (20) is valid. ■

Theorem 4. *If the method* $P=(R,p_k)$ *is such that*

$$\frac{p_k}{p_{k-1}} \leq \frac{p_{k+1}}{p_k} \quad \forall k > k_0, \tag{22}$$

then for every given $r \in (0,1)$ *the inclusion*

$$K^0(P^1x) \subset L(x) \subset K^0(P^rx) \quad \forall x \in \mathbf{m}. \tag{23}$$

holds.

Proof. Following the Corollary 3.2 from [7] the inclusion

$$K^0(Qx) \subset L(x) \quad \forall x \in \mathbf{m} \tag{24}$$

holds if

$$1^0. \quad \lim_m \frac{q_k}{Q_k} = 0, \tag{25}$$

$$2^0. \quad \lim_k \frac{q_{k+1}}{q_k} = 1. \tag{26}$$

In this case $q_k = p_k r_0^k$ and on account of the inequality

$$\liminf_n \frac{p_{k+1}}{p_k} \leq \frac{1}{r_0} \leq \limsup_n \frac{p_{k+1}}{p_k}$$

it follows from (22) that (26) and the existence of $k_0 \in \mathbb{N}$, such that

$$\frac{p_{k+1}}{p_k} \leq \frac{1}{r_0} \quad \forall k > k_0.$$

Consequently

$$p_k r_0^k \leq r_0^{k_0} p_{k_0} \quad \forall k > k_0.$$

The assertion (12) gives us that

$$P_k^1 = \sum_{i=0}^{k} p_i r_0^i \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty$$

and this yields (25) and therefore (24) is true.

The right part of the inclusion (23) follows from Theorem 3 and from the fact that for every $r \in (0,1)$ the following inclusion holds

$$L(x) \subset K^0(P^r_* x) \quad \forall x \in \mathbf{m}$$

(see Remark in [7]).

Corollary 4.1. *Let* $P = P^1$ *be such that* (22) *holds and let* Q *be such that*

$$q(\tau) = \frac{d}{d\tau} p(\tau),$$

i.e., $Q = (R, (k+1)p_{k+1})$. *Then*

$$K^0(Px) \subset K^0(Qx) \subset L(x) \quad \forall x \in \mathbf{m}.$$

Proof. In this case $\tau_0 = 1$. The proof of Theorem 4 gives us that (26) holds and there exists $k_0 \in \mathbb{N}$ such that

$$p_{k+1} \le p_k \quad \forall k > k_0.$$

Therefore,

$$\frac{q_m}{Q_m} = \frac{(m+1)p_{m+1}}{\sum_{k=1}^{m+1} k p_k} \le \frac{(m+1)p_{m+1}}{p_{m+1}\sum_{k=1}^{m+1} k} = \frac{2m}{(m+2)(m+1)} \longrightarrow 0$$

as $m \to \infty$, i.e. (25) is true. Corollary 3.2 from [7] which was cited in our proof of Theorem 4 asserts that

$$K^0(Qx) \subset L(x) \quad \forall x \in \mathbf{m}.$$

Because of (22) we have that

$$\frac{p_{k+1}}{p_k} \le \frac{p_{k+2}}{p_{k+1}} \le \frac{(k+2)p_{k+2}}{(k+1)p_{k+1}} = \frac{q_{k+1}}{q_k}$$

and due to Corollary 1.1 in [7] cited in the proof of Theorem 2, holds the inclusion

$$K^0(Px) \subset K^0(Qx) \quad \forall x \in \mathbf{m}. \quad \blacksquare$$

Let now $Q = (R, q_k)$ be such a Riesz matrix that $q(\tau) = \frac{d^n}{d\tau^n} p(\tau)$, i.e.

$$q_k = (k+n)(k+n-1)\dots(k+1)p_{k+n} \quad \forall k = 0, 1, \dots.$$

Let us denote this method by $P^{(n)}$.

Corollary 4.2. *Let* $P = P^1$ *be such that* (22) *holds. Then for every* $n \in \mathbf{N}$ *the inclusion*

$$K^0(P^{(n)}(x)) \subset K^0(P^{(n+1)}(x)) \subset L(x) \quad \forall x \in \mathbf{m}$$

is true.

Proof is analogous to the proof of Corollary 4.1 and is based on the same two corollaries from [7], namely on Corollary 3.2 and on Corollary 1.1. ■

The semicontinuous summability method (J, p_k) is defined by the semicontinuous matrix $(a_k(\tau))$, where

$$a_k(\tau) = \frac{p_k}{p(\tau)}\tau^k \quad \forall \tau \in (0, \tau_0).$$

It means that for every weighted means method (R, p_k) there is a corresponding semicontinuous method (J, p_k) . One can easily check that one and the same method (J, p_k) corresponds to every member $P^r \in \mathbf{P}$. Therefore, we will always define (J, p_k) with respect to the method P^1 , i.e. while defining (J, p_k) we will consider only such (p_k) for which $\tau_0 = 1$. Let $\mathbf{c_J}$ denote the set of all (J, p_k)-summable sequences. For every $P = P^1$ it holds that $\mathbf{c_P} \subset \mathbf{c_J}$ and it is possible that $\mathbf{c_P} = \mathbf{c_J}$ (see [2]) . Let $K_J(x)$ denote the core of $x \in \mathbf{m}$ determined by the method (J, p_k) (see [6]). It is known that the method (J, p_k) is core-regular and that

$$K_J(x) \subset K^0(P^1 x) \quad \forall x \in \mathbf{m} \tag{27}$$

(see [6]). From (27) and Theorem 2 the inclusion

$$K_J(x) \subset K^0(P^r x) \quad \forall r \in (0, 1] \quad \forall x \in \mathbf{m}$$

follows. This result can be strengthened as follows:

Theorem 5. *Let* $Q = (R, q_k)$ *be such that*

$$\limsup_k \sqrt[k]{\frac{Q_k p_k}{q_k}} = 1. \tag{28}$$

The inclusion

$$K_J(x) \subset K^0(Qx) \quad \forall x \in \mathbf{m} \tag{29}$$

holds iff

$$\lim_{\tau \to 1-} \frac{1}{p(\tau)} \sum_k Q_k \mid \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \tau \mid \tau^k = 1. \tag{30}$$

Proof. Let Q^{-1} be the inverse matrix to the Riesz matrix Q and let

$$G(\tau) = (g_k(\tau)) := (a_k(\tau))Q^{-1}.$$

Using the form of Q^{-1} one can easily check that

$$g_k(\tau) = \frac{Q_k}{p(\tau)} \Big(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \tau \Big) \tau^k$$

(for Q^{-1} see [1]). Due to (28) we have that for the method $G(\tau)$ the condition (1) is valid. Indeed, for every $\tau \in (0,1)$

$$\sum_k \mid g_k(\tau) \mid \leq \frac{1}{p(\tau)} \sum_k \frac{Q_k p_k}{q_k} \tau^k + \frac{1}{p(\tau)} \sum_k \frac{Q_k p_{k+1}}{q_{k+1}} \tau^{k+1} < \infty$$

because of the convergence of power series

$$\sum_k \frac{Q_k p_k}{q_k} \tau^k \quad \forall \tau \in (0,1).$$

Consequently (29) holds iff the method $G(\tau)$ is core-regular, i.e., if the conditions (4), (5) and (8) are satisfied. In this case (4) turns into the condition

$$\lim_{\tau \to 1-} g_k(\tau) = 0 \quad \forall k = 0, 1, \ldots$$

which is valid due to (12). Futhermore, for every $\tau \in (0,1)$

$$\sum_{k=0}^{\infty} g_k \tau = \frac{1}{p(\tau)} \Big[\sum_{k=0}^{\infty} \frac{Q_k}{q_k} p_k \tau^k - \sum_{k=1}^{\infty} \frac{Q_k}{q_k} p_k \tau^k + \sum_{k=1}^{\infty} p_k \tau^k \Big] =$$

$$= \frac{1}{p(\tau)} \Big[\frac{Q_0}{q_0} p_0 \tau^0 + \sum_{k=1}^{\infty} p_k \tau^k \Big] = 1,$$

and therefore (5) is satisfied. In case of $G(\tau)$, condition (8) turns into condition (30). This completes the proof. ∎

Theorem 6. *The inclusion*

$$K_J(x) \subset L(x) \quad \forall x \in \mathbf{m} \tag{31}$$

holds iff

$$\lim_{\tau \to 1-} \frac{1}{p(\tau)} \sum_k | p_k - p_{k+1}\tau | \tau^k = 0. \quad (32)$$

Proof follows from the facts that (J, p_k) is core-regular and that the condition (10) turns in this case into (32).

The semicontinuous sequential summability method (J_α, p_k) is defined by the family of matrices $A(\tau) = (a_{nk}(\tau))$, where $\tau \in (0, \tau_0)$ and

$$a_{nk}(\tau) = \begin{cases} \frac{p_{k-n}\tau^{k-n}}{p(\tau)}, & \text{if } k \geq n, \\ 0, & \text{if } 0 \leq k < n. \end{cases}$$

It means that $x = (\xi_k)$ is (J_α, p_k)-summable to a number a if

$$\lim_{\tau \to \tau_0} \sum_{k=n}^{\infty} \frac{p_{k-n}}{p(\tau)} \tau^{k-n} \xi_k = a$$

uniformly in n. Analogously to the case of (J, p_k) we may consider only such (p_k) for which $\tau_0 = 1$. It is easy to check that (1), (4), (5) and (8) are satisfied and therefore

$$K(x) \subset K^0(x) \quad \forall x \in \mathbf{m}.$$

Here $K(x)$ denotes the core of x determined by the $\alpha(\tau)$-method (J_α, p_k). It follows from Definition 2 that

$$K_J(x) \subset K(x) \quad \forall x \in \mathbf{m},$$

(see also [5]).

Theorem 7. *Let* $P = P^1$ *and let* Q *be such that* (28) *holds. The inclusion*

$$K(x) \subset K^0(Qx) \quad \forall x \in \mathbf{m} \quad (33)$$

holds iff

1^0 Q *is Mercerian, i.e.* $\mathbf{c}_Q = \mathbf{c}$, (34)

$$2^0 \quad \lim_{\tau \to 1-} \sup_n \sum_{k=n}^{\infty} \frac{Q_k}{p(\tau)} \left| \frac{p_{k-n}}{q_k} - \frac{p_{k-n+1}}{q_{k+}}\tau \right| \tau^{k-n} = 1. \quad (35)$$

Proof. Let

$$(g_{nk}(\tau)) := (a_{nk}(\tau))Q^{-1}.$$

One would obtain that

$$g_{nk}(\tau) = \begin{cases} \frac{Q_k}{p(\tau)}\Big(\frac{p_{k-n}}{q_k} - \frac{p_{k-n+1}}{q_{k+1}}\tau\Big)\tau^{k-n} & \text{if} \quad k \geq n, \\ -\frac{1}{p(\tau)} \cdot \frac{Q_k}{q_{k+1}} \quad \text{if} \quad k = n-1, \\ 0, \qquad \text{if} \quad 0 \leq k < n-1. \end{cases}$$

Due to (28) we have that for the method $(g_{nk}(\tau))$ the condition (1) is valid and therefore (33) holds iff $(g_{nk}(\tau))$ is core-regular. The necessary and sufficient conditions for the core-regularity are (4), (5) and (8). It is easy to check that for $(g_{nk}(\tau))$ the conditions (4) and (5) are satisfied and (8) turns into (34) and (35). ■

Theorem 8. *The inclusion*

$$K(x) \subset L(x) \quad \forall x \in \mathbf{m} \tag{36}$$

holds iff the inclusion (31) *holds.*

Proof. Method (J_α, p_k) is core-regular and, for it the condition (10) turns into the condition (32). Indeed,

$$\lim_{\tau\to 1-} \sup_n \Big(| a_{nk}(\tau) | + \sum_{k=n}^{\infty} | a_{nk}(\tau) - a_{nk+1}(\tau) |\Big) =$$

$$= \lim_{\tau\to 1-} \sup_n \Big(\frac{p_0}{p(\tau)} + \sum_{k=n}^{\infty} | \frac{p_{k-n}\tau^{k-n}}{p(\tau)} - \frac{p_{k+1-n}\tau^{k+1-n}}{p(\tau)} |\Big) =$$

$$= \lim_{\tau\to 1-} \frac{1}{p(\tau)} \sup_n \sum_{k=0}^{\infty} | p_k\tau^k - p_{k+1}\tau^{k+1} | =$$

$$= \lim_{\tau\to 1-} \frac{1}{p(\tau)} \sum_k | p_k - p_{k+1}\tau | \tau^k. \quad ■$$

References

1. Baron, S., *Introduction to the theory of summability of series.* Tallinn, 1977 (in Russian).
2. Borwein, D., Meier, A., *A Tauberian theorem concerning weighted means and power series.* Math. Proc. Cambr. Phil. Soc., 1987, **101**, 283-286.
3. Hardy, G. H., *Divergent Series,* Oxford, 1949.
4. Loone, L., *Knopp's core and almost convergency core in space* **m**. Acta et Comment. Univ. Tartuensis, 1975, **355**, 148-156 (in Russian).
5. Loone, L., *On cores of semicontinuous sequential summability methods.* Acta et Comment. Univ. Tartuensis, 1991, **928**, 61-68.
6. Loone, L., *Inclusion between the cores concerning weighted means and power series.* Acta et Comment. Univ. Tartuensis, 1991, **928**, 67-72.
7. Loone, L., Tohver, E., *On cores summability methods generated by weighted means.* Acta et Comment. Univ. Tartuensis, 1993, **960**, 51-66.

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Tuumade sisalduvusest summeerimismeetodite (R,p_k), (J,p_k) ja (J_α,p_k) korral

Leiki Loone

Resümee

Käesolevas töös võrreldakse ühe ja sellesama positiivsete reaalarvude jada (p_k) abil defineeritud kolme erineva menetlusega määratud tuumade vahekordi. Nendeks menetlusteks on klassikaline Rieszi kaalutud keskmiste menetlus (R,p_k), poolpidev summeerimismenetlus $(J,p_k) =$

$(a_k(\tau))$, kus

$$a_k(\tau) = \frac{p_k}{p(\tau)}\tau^k \quad \forall \tau \in (0, \tau_0)$$

ja kus $p(\tau)$ on määratud seosega (12), ning jadaline poolpidev menetlus $(J_\alpha, p_k) = (a_{nk}(\tau))$, mille korral

$$a_{nk}(\tau) = \begin{cases} \frac{p_{k-n}\tau^{k-n}}{p(\tau)}, & \text{kui} \quad k \geq n, \\ 0, & \text{kui} \quad 0 \leq k < n. \end{cases}$$

Rieszi kaalutud keskmiste menetlusega (R, p_k) seostatakse menetluste klass $\mathbf{P}$, kus

$$\mathbf{P} := \{P^r \mid r \in (0, 1]\}$$

ja kus P^r on antud seosega (15) ning uuritakse selle klassi poolt määratud elementide tuumade vahelisi seoseid (vt.(16)). Vaadeldakse ka tuumasisalduvust

$$K^0(Px) \subset K^0(Qx) \quad \forall x \in \mathbf{m},$$

kus $P = (R, p_k)$ ja $Q = (R, q_k)$ ning

$$q_k = (k+n)(k+n-1)\ldots(k+1)p_{k+n} \quad \forall k = 0, 1, \ldots,$$

s.t.

$$q(\tau) = \frac{d^n}{d\tau^n}p(\tau).$$

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The rate-spaces $m(\lambda)$, $c(\lambda)$, $c_0(\lambda)$ and $l^p(\lambda)$ of sequences

Jaak Sikk

1. Introduction

In [1] we introduced the notions of abstract rate-spaces $X(\lambda)$ and $X_c(\lambda)$, studied their matrix mappings and K-multipliers. Using these results we shall consider the rate-spaces type $m(\lambda)$, $c(\lambda)$, $c_0(\lambda)$, $l^p(\lambda)$ and their inclusion relations.

The standard notions of sequence spaces $m = l^\infty, c, c_0, l^p$ and ω are used in this paper (see [2]).

Rates $\lambda, \mu, \ldots$ are real sequences with nonzero elements only. Thus, $\lambda = (\lambda_k)$ is a rate iff $\lambda \in \omega$ and $\lambda_k \neq 0$ for all k. For a real vector space of sequences X we introduced the rate-spaces

$$X(\lambda) = \{x : (\lambda_k x_k) \in X\}$$

and

$$X_c(\lambda) = \{x : x \in c \quad \text{and} \quad (\lambda_k(x_k - x')) \in X, \quad where \quad \lim_k x_k = x'\}.$$

We call X a basic space if the rate-spaces are introduced for it.

Given a matrix $A = (a_{nk})$ and a sequence $x = (x_k)$, we write $y = Ax$, to mean that for each n

$$y_n = (Ax)_n = \sum_k a_{nk} x_k. \tag{1}$$

Let X and Y be basic spaces. If for every $x \in X(\lambda)$ the sequence $y \in Y(\mu)$ then A is a matrix mapping $X(\lambda)$ into $Y(\mu)$ and we write $A \in (X(\lambda) : Y(\mu))$. Analogously, if for every $x \in X$ the sequence $y \in Y(\mu)$ then A is a matrix mapping X into $Y(\mu)$ and we write $A \in (X : Y(\mu))$. The matrix mappings $(X(\lambda) : Y_c(\mu))$, $(X_c(\lambda) : Y(\mu))$,

$(X(\lambda):Y)$ etc. are defined analogously. In [1] we investigated these mappings and worked out the method to obtain mapping conditions. We proved

Lemma 1. *Let* $A=(a_{nk})$, $A(\lambda^{-1},\mu)=(a_{nk}\lambda_k^{-1}\mu_n)$, $A(\lambda^{-1},1)=(a_{nk}\lambda_k^{-1})$ *and* $A(1,\mu)=(a_{nk}\mu_n)$ *then*

1. $A\in(X(\lambda):Y(\mu))$ *iff* $A(\lambda^{-1},\mu)\in(X:Y)$,
2. $A\in(X:Y(\mu))$ *iff* $A(1,\mu)\in(X:Y)$,
3. $A\in(X(\lambda):Y)$ *iff* $A(\lambda^{-1},1)\in(X:Y)$.

(see [1], Theorem 1).

This Lemma 1 shows how matrix mappings for rate-spaces are linked with corresponding mappings of basic spaces. Using Lemma 1 and well-known results about matrix mappings we deduced the necessary and sufficient conditions for rate-space mappings. For example, we proved the following result (see [1], Example 1.1).

Lemma 2. *Matrix* $A\in(l^p(\lambda):m(\mu))$ *iff*

$$\sum_n |\mu_n|^p \sum_k |a_{nk}\lambda_k^{-1}|^p<\infty. \tag{2}$$

The sequence x is called $AX_c(\lambda)$-summable if the sequence $y=Ax\in X_c(\lambda)$. The sequence x is called $AX(\lambda)$-summable if $y\in X(\lambda)$. The series $\sum u_k$ is called $AX(\lambda)$-summable if the sequence of partial sums of $\sum u_k$ is $AX(\lambda)$-summable. The series $\sum u_k$ is called $AX_c(\lambda)$-summable if the sequence of partial sums of $\sum u_k$ is $AX_c(\lambda)$- summable. The sequence $\varepsilon=(\varepsilon_k)$ is a K-multiplier of class $(AX_c(\lambda);BY(\mu))$ if for every $AX_c(\lambda)$-summable series $\sum u_k$ the series $\sum \varepsilon_k u_k$ is $BY(\mu)$-summable. The classes of K-multipliers $(AX_c(\lambda);\ BY_c(\mu),(AX(\lambda);\ BY(\mu),\ (AX(\lambda);\ BY_c(\mu))$ are defined analogously. We proved the following result which gives the necessary and sufficient conditions for a large class of K-multipliers (see [1], Theorem 5).

Lemma 3. *Let* $\tilde{X}$ *stand for a space* X *or* $X(\lambda)$ *or* $X_c(\lambda)$ *and* $\tilde{Y}$ *for a space* Y *or* $Y(\mu)$ *or* $Y_c(\mu)$ *and let* $A=(a_{nk})$ *be a triangle with* $A^{-1}=(a'_{nk})$, $B=(b_{nk})$ *a triangular and* $C=(c_{nk})$ *a matrix with* *elements*

$$c_{nk} = \sum_{\nu=k}^{n} b_{n\nu}\varepsilon_\nu a'_{\nu k},$$

then

$$\varepsilon \in (A\tilde{X};\ B\tilde{Y}) \quad \textit{iff} \quad C \in (\tilde{X} : \tilde{Y}).$$

It is easy to see that all summability factors can be deemed special cases of the K-multipliers and it is possible to use Lemma 3 for those cases. For example $(A, B_0) = (Ac(1);\ Bm(1))$, $(A, B) = (Ac(1);\ Bc(1))$, $(A_0, B) = (Am(1);\ Bc(1))$, $(A_0, B_0) = (Am(1);\ Bm(1))$. In special cases, if we consider only positive, monotone and increasing rates we will get summability factors $(A_0^\lambda,\ B_0^\mu) = (Am(\lambda);\ Bm(\mu))$, $(A^\lambda,\ B_0^\mu) = (Ac(\lambda);\ Bm(\mu))$, $(A^\lambda, B^\mu) = (Ac(\lambda);\ Bc(\mu))$ etc., investigated by Kangro (about the concept of summability factors see [3]).

2. The rate-spaces $l^p(\lambda)$, $c_0(\lambda)$, $c(\lambda)$, $m(\lambda)$ and their inclusions

The intent of this paper is to generate new sequence spaces in ω which have (in some sense) the same structure as the given basic-space. The results of this section will demonstrate that the rate-spaces are just such a type of sequence spaces. We will see that the rate-spaces are isometric with their basic spaces and that for every sequence $x \in \omega$ there exists rate λ so, that $x \in X(\lambda)$. We will also investigate the inclusion relations between the rate-spaces.

Let X be Banach space and $X(\lambda)$ its rate-space. Since for every $x = (x_k) \in X(\lambda)$ corresponds $(\lambda_k x_k) \in X$, the rate λ determines a mapping $L : X(\lambda) \to X$. The mapping $L : X(\lambda) \to X$ is one to one, linear and onto. Therefore the space $X(\lambda)$ becomes a Banach space which is equivalent with X with the identification norm

$$\| x \|_{X(\lambda)} = \| \lambda x \|_X \tag{3}$$

where $\lambda x = (\lambda_k x_k)$. Hence we have

Theorem 1. *Banach space X with norm $\| \cdot \|_X$ and the rate λ determine a Banach space $X(\lambda)$ with norm* (3). *The spaces X and $X(\lambda)$ are isometric.*

The spaces c_0, c and $l^\infty = m$ are Banach spaces with norm $\| x \|_\infty = \sup | x_k |$ and thus, the rate spaces $c_0(\lambda), c(\lambda)$ and $m(\lambda)$ are also Banach spaces with the induced norm

$$\| x \|_{\infty\lambda} = \sup | \lambda_k x_k | .$$

The space l^p is a Banach space with norm $\| \cdot \|_p$ and so $l^p(\lambda)$ is also a Banach space with norm

$$\| x \|_{p\lambda} = (\sum | \lambda_k x_k |^p)^{1/p}.$$

Next we shall consider the dual space $X(\lambda)'$, i.e. the space of all linear, continuous functional on $X(\lambda)$.

Theorem 2. *Let $c_0(\lambda)'$ be a dual of $c_0(\lambda)$, then $f \in c_0(\lambda)'$ iff $f(x) = \sum a_k x_k$ with $a \in l(\lambda^{-1})$, where λ^{-1} is a rate (λ_k^{-1}).*

Proof. For $x \in c_0(\lambda)$ we seek a dual having a form

$$f(x) = \sum a_k x_k = \sum \frac{a_k}{\lambda_k}(\lambda_k x_k)$$

where $(\lambda_k x_k) \in c_0$. Therefore f is continuous linear functional on $c_0(\lambda)$ if and only if

$$\sum \frac{a_k}{\lambda_k} y_k,$$

where $(y_k) \in c_0$, determines a continuous linear functional on c_0. Thus $(\lambda_k^{-1} a_k) \in l$ and consequently $a \in l(\lambda^{-1})$, which completes the proof.

Similarly to this proof one can prove the following

Theorem 3. *Let $p \geq 1$, $p^{-1} + q^{-1} = 1$, then $f \in l^p(\lambda)'$ iff*

$$f(x) = \sum a_k x_k$$

with $a \in l^q(\lambda^{-1})$ and

$$l^p(\lambda)' = l^q(\lambda^{-1}).$$

Let $1 \leq p_1 < p < \infty$, then the spaces l^p, l^{p_1}, c_0, c and m are related by the well-known chain of inclusions

$$l^{p_1} \subset l^p \subset c_0 \subset c \subset m. \tag{4}$$

It follows from (4) and from the definition of the rate-space that for fixed rate λ the corresponding rate-spaces are related by the same type of inclusion relation. Therefore we have

Theorem 4. *Let* $1 \leq p_1 < p < \infty$, *then*

$$l^{p_1}(\lambda) \subset l^p(\lambda) \subset c_0(\lambda) \subset c(\lambda) \subset m(\lambda) \tag{5}$$

and for every $x \in l^{p_1}(\lambda)$

$$\| x \|_{\infty\lambda} \leq \| x \|_{p\lambda} \leq \| x \|_{p_1\lambda} . \tag{6}$$

Example. Let $z = (z_k) \in \omega$, we shall show that there exists a rate $\lambda = (\lambda_k)$ such that $z \in l^p(\lambda)$. For that purpase we shall consider a sequence $\bar{z} = (\bar{z}_k)$ with

$$\bar{z}_k = \begin{cases} z_k \text{ for all } z_k \neq 0 \\ 1 \text{ for all } z_k = 0. \end{cases}$$

Let now $\lambda_k = \bar{z}_k^{-1}\alpha_k$, where $\alpha = (\alpha_k) \in l^p$ and $\alpha_k \neq 0$. By the definition of rate-space it follows that $z \in l^p(\lambda)$. One can use the same construction to generate the desired rate-space in the case of c_0, c or m instead of l^p.

Let us consider the K-multipliers classes

$$(IX;\ IY) \text{ and } (IX(\lambda);\ IY(\lambda))$$

where $I = (\delta_{nk})$ is identity matrix. The sequence $\varepsilon = (\varepsilon_k)$ is a K-multiplier of class $(IX;\ IY)$ if for every $x = (x_k) \in X$ is true $(\varepsilon_k x_k) \in Y$. We write $(X;\ Y)$ instead of $(IX;\ IY)$, also $(X(\lambda);\ Y(\mu))$ instead of $(IX(\lambda);\ IY(\mu))$.

Theorem 5. *The classes of multipliers* $(X;\ Y)$ *and* $(X(\lambda);\ Y(\lambda))$ *are identical.*

Proof. By Theorem 1 the spaces X and $X(\lambda)$ are isometric, the spaces Y and $Y(\lambda)$ are also isometric. Therefore $\varepsilon \in (X;\ Y)$ iff $\varepsilon \in (X(\lambda);\ Y(\lambda))$ which gives the desired identity of classes.

Lemma 1A. *Let* λ *and* μ *be rates and* X *and* Y *be sequence spaces, then*

1. $X(\lambda) \subset Y(\mu)$ *iff* $(\delta_{nk}\lambda_k^{-1}\mu_n) \in (X : Y)$,

2. $X \subset Y(\mu)$ *iff* $(\delta_{nk}\mu_n) \in (X : Y)$,

3. $X(\lambda) \subset Y$ *iff* $(\delta_{nk}\lambda_k^{-1}) \in (X : Y)$.

Proof. Our statement is an immediate consequence of Lemma 1 if $A = I$.

Theorem 6. *Let* λ *and* μ *be rates and* X *and* Y *sequence spaces, then*

1. $X(\lambda) \subset Y(\mu)$ *iff* $(\mu_k / \lambda_k) \in (X; Y)$,

2. $X \subset Y(\mu)$ *iff* $(\mu_k) \in (X; Y)$,

3. $X(\lambda) \subset Y$ *iff* $(\lambda_k^{-1}) \in (X; Y)$.

Proof. The matrix $(\delta_{nk}\lambda_k^{-1}\mu_n)$ is a diagonal triangle. Its diagonal is a sequence $(\mu_k\lambda_k^{-1})$. By definitions of K-multipliers $(X; Y)$ and matrix mappings $(X : Y)$

$$(\frac{\mu_k}{\lambda_k}) \in (X; Y)$$

iff

$$(\delta_{nk}\lambda_k^{-1}\mu_n) \in (X : Y).$$

Now proof follows from Lemma 1A.

Definition. *Let* λ *and* μ *be rates and* X *and* Y *sequence spaces. We say that* μ *is (X, Y)-stronger than* λ *if*

$$(\frac{\mu_k}{\lambda_k}) \in (X; Y). \tag{7}$$

We denote the set of all such λ *by* $\bar{\mu}(X, Y)$.

Corollary 6.1. a) *Let* μ *be (X, Y)-stronger than* λ, *i.e.* $\lambda \in \bar{\mu}(X, Y)$ *then*

$$X(\lambda) \subset Y(\mu); \tag{8}$$

b) *let* μ *be (X,X)-stronger than* λ, *i.e.* $\lambda \in \bar{\mu}(X, X)$ *then*

$$X(\lambda) \subset X(\mu). \tag{9}$$

Partial order in the space of rates is determined by the notion "*(X,X)*-stronger". It is easy to see that the general relation "*(X, Y)*-stronger" determines a partial order if for every $\varepsilon, \varepsilon' \in (X; Y)$ is valid $\varepsilon \cdot \varepsilon' \in (X; Y)$.

Now we shall demonstrate possibilities which are opened up by the Lemma 1A and Theorem 6. By linking the results about matrix mappings, K-multipliers and rate-space inclusions we shall get the conditions for rate-space inclusions. What follows, explicates the meaning of the concepts of rate, rate-space and "order".

We shall need the following well-known results about matrix mappings.

Lemma 4. *A matrix* $A \in (c : c)$ *iff*

a) $\lim_n a_{nk} = a_k$ *exists,*

b) $\lim_n \sum_k a_{nk} = a$ *exists,*

c) $\sum_k \mid a_{nk} \mid = O(1)$

(see [2], p.5).

Lemma 5. $A \in (m : m)$ *also* $A \in (c : m)$ *and* $A \in (c_0 : m)$ *iff* $\| A \| < \infty$, *where*

$$\| A \| = \sup_n \sum_k \mid a_{nk} \mid$$

(see [2], p.5).

Lemma 6. *Let* $p \geq 1$, $A \in (l,\ l^p)$ *iff*

$$\sup_k \sum_n \mid a_{nk} \mid^p < \infty \tag{10}$$

(see [2], p.126).

Lemma 7. $A \in (l^p : m)$, $p > 1$, *iff*

$$\sup_n \sum_k \mid a_{nk} \mid^q < \infty$$

(see [2], p.129).

Theorem 7. *Let* $p \geq 1$, *then*

1. $l(\lambda) \subset l^p(\mu)$ *iff* $(\mu_k \,/\, \lambda_k) \in l^p$,
2. $l \subset l^p(\mu)$ *iff* $(\mu_k) \in l^p$,
3. $l(\lambda) \subset l^p$ *iff* $(\lambda_k^{-1}) \in l^p$.

The proof is an immediate consequence of Theorem 6 and Lemma 6.

By Theorem 7 one can say that μ is $(l,\ l^p)$-stronger than λ iff

$$(\frac{\mu_n}{\lambda_n}) \in l^p.$$

The relation "$(l,\ l^p)$-stronger" does not determine a partial order because of its non-reflexivity. It follows from the fact that

$$\mu \bar{\in} \bar{\mu}(l,\ l^p).$$

Let us consider rates $\mu = (k^{\alpha})$ and $\lambda = (k^{\beta})$ where $k \in \mathbb{N}$ and a sequence $\mu\lambda^{-1} = (k^{\alpha-\beta})$. The sequence $\mu\lambda^{-1} \in l^p$ iff

$$\sum k^{(\alpha-\beta)p} < \infty,$$

it means that $\alpha - \beta < -\frac{1}{p}$. Consequently by theorem 7 $l(\lambda) \subset l^p(\mu)$. One can easily check that the same inclusion is true if μ and λ will satisfy the following condition

$$(\frac{\mu_k}{\lambda_k}) = O(k^{\nu}), \tag{11}$$

where $\nu < -\frac{1}{p}$. Therefore we have

Corollary 7.1. *Let $p \geq 1$ and $\nu < -\frac{1}{p}$ and let (11) be satisfied. Then μ is (l, l^p)-stronger than λ and*

$$l(\lambda) \subset l^p(\mu). \tag{12}$$

For arbitrary fixed μ Corollary 7.1. determines a class of rates for which (12) is satisfied. It is obvious that there exists a vast class of pairs μ and λ satisfying (11) and tending together to infinity or tending together to zero.

Corollary 7.2. *Let $p \geq 1$ then $c_0(\lambda) \subset l^p(\mu)$, $c(\lambda) \subset l^p(\mu)$ and $m(\lambda) \subset l^p(\mu)$ iff $(\mu_k\lambda_k^{-1}) \in l^p$.*

Proof. It is known that $A \in (c_0 : l^p) = (c : l^p) = (m : l^p)$ iff

$$\sup\{\sum | \sum_{k \in K} a_{nk} |^p : K \text{ a finite set of positive integers } \} < \infty \tag{13}$$

(see [2], p.131). Let $A = (a_{nk})$ be such that

$$a_{nk} = \delta_{nk}\frac{\mu_k}{\lambda_k}.$$

Therefore the condition (13) is equal to (10). Consequently, our statement follows from the Theorem 7 by replaceing l with c_0 or c or m.

Examples. 1) Let $\mu = (k^{-0,51})$, then

$$m \subset l^2(\mu),$$

2) let $\mu = (k^{-\frac{1}{100}})$, then $l \subset l^{100}(\mu)$;

3) let $\lambda = (k^{0,51})$, then $l(\lambda) \subset l^2$, $c_0(\lambda) \subset l^2$, $c(\lambda) \subset l^2$ and $m(\lambda) \subset l^2$;

4) let $\lambda = (k)$ and $\mu = (\sqrt{k})$ then

$$l(\lambda) \subset l^4(\mu).$$

Let us now consider the inclusion relations $m(\lambda) \subset m(\mu)$, $c(\lambda) \subset m(\mu)$ and $c_0(\lambda) \subset m(\lambda)$. By Theorem 6 and Lemma 5 we have

Theorem 8. *Let X be one of the spaces m or c or c_0, then*

1. $X(\lambda) \subset m(\mu)$ *iff*

$$(\frac{\mu_k}{\lambda_k}) \in m, \tag{14}$$

2. $X \subset m(\mu)$ *iff* $(\mu_k) \in m$,
3. $X(\lambda) \subset m$ *iff* $(\lambda_k^{-1}) \in m$.

Now one can say that μ is *(m,m)*-stronger than λ iff (14) is satisfied. What follows is a detailed examination of the condition (14).

a) Let $(\mu_k \lambda_k^{-1}) \in c_0$, then by Theorem 8 $m(\lambda) \subset m(\mu)$, $c(\lambda) \subset m(\mu)$ and $c_0(\lambda) \subset m(\mu)$. The position that basic space will take in the chain of inclusions depends on rates.

If $\mu_k \to \infty$ and $\lambda_k \to \infty$ then

$$c_0(\lambda) \subset m(\lambda) \subset m(\mu) \subset c_0.$$

If $\mu, \lambda \in c_0$ then

$$m \subset m(\lambda) \subset m(\mu).$$

If $\mu \in c_0$ and $\lambda_k \to \infty$ then

$$c_0(\lambda) \subset c(\lambda) \subset m(\lambda) \subset c \subset m \subset m(\mu).$$

b) Let $(\mu_k \lambda_k^{-1}) \in m$ and $(\lambda_k \mu_k^{-1}) \in m$ then by Theorem 8 $m(\mu) \subset m(\lambda)$, $c(\mu) \subset m(\lambda)$ and $c_0(\mu) \subset m(\lambda)$. Therefore $m(\lambda) = m(\mu)$ and $c_0(\lambda) = c_0(\mu)$. Consequently, there exists a class of rates every element of which determines one and the same rate-space for basic space m (or c_0). Let us consider the stronger condition

$$(\frac{\mu_k}{\lambda_k}) \in c \setminus c_0.$$

By Lemma 4 then $c(\lambda) = c(\mu)$ as $\lim \mu_k \lambda_k^{-1}$ exists.

Definition. *Let X be basic space. All rates λ which determine one and the same rate-space for X we call X-equipotent.*

Now we have

Theorem 9. a) *Rates λ and μ are m-equipotent and c_0-equipotent if*

$$(\frac{\mu_k}{\lambda_k}) \in m \quad and \quad (\frac{\lambda_k}{\mu_k}) \in m.$$

b) *Rates λ and μ are m-equipotent, c-equipotent and c_0-equipotent if*

$$(\frac{\mu_k}{\lambda_k}) \in c \setminus c_0.$$

References

1. Sikk, J., *Matrix mappings for rate-spaces and K-multipliers in the theory of summability.* Acta et Comment. Univ. Tartuensis, 1989, **846,** 118 – 128 .
2. Wilansky, A., *Summability through Functional Analysis.* Volume 85 of Note de Matematica, North Holland, Amsterdam–New York–Oxford, 1984.
3. Кангро Г. , Множители суммируемости для рядов, λ-ограниченных методами Риса и Чезаро. Уч. зап. Тарт. ун-та, 1971, **277,** 136 – 154 .

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Strong almost convergence in Banach spaces

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1. Introduction

In this paper the notion of strong almost convergence of sequence in Banach spaces is introduced.

Let m denote the space of all bounded complex-valued sequences $x = (\xi_k)$. A Banach limit L is a continuous linear functional on m satisfying the conditions

1^0 $L \geq 0$,

2^0 $L(e) = 1$, where $e = (1, 1, \ldots)$,

3^0 $L(Sx) = L(x)$, where $S(x) = (\xi_{k+1})$.

Definition 1. *The bounded sequence $x = (\xi_k)$ of complex numbers is called almost convergent to l if $L(x) = l$ for each Banach limit L.*

The notion of almost convergence was introduced by Lorentz [3]. He characterized a sequence $x = (\xi_k)$ as almost convergent to l if

$$\lim_n \frac{1}{n+1} \sum_{k=i}^{i+n} \xi_k = l \tag{1}$$

uniformly in i.

We denote by c, f and f_0 the spaces of convergent, almost convergent and almost convergent to zero sequences respectively.

Definition 2. *The sequence $x = (\xi_k)$ of complex numbers is called strongly almost convergent to l if $(|\ \xi_k - l\ |) \in f_0$.*

The notion of strong almost convergence was introduced by Maddox [4]. We denote the set of all strongly almost convergent sequences by $[f]$. Then f, f_0 and $[f]$ are closed subspaces of m (with the usual supremum norm) and with strict inclusions we have

$$c \subset [f] \subset f \subset m.$$

Kurtz [2] extended the results of Lorentz to bounded sequences of elements in a Banach space. He treated almost convergence as the generalization of weak convergence. Suppose that X is a Banach space. Let $\omega(X)$ be the set of all sequences $u = (x_k)$, $x \in X$. We denote by $m(X)$, $c(X)$ and $c_0(X)$ the spaces of X-valued bounded, convergent and null sequences respectively, i.e.

$$m(X) = \{u = (x_k) \in \omega(X), \quad \sup_k \|x_k\| < \infty\},$$
$$c(X) = \{u = (x_k) \in \omega(X), \quad \exists \lim_k x_k = l\},$$
$$c_0(X) = \{u \in c(X), \quad l = 0\}.$$

Let X' be the conjugate space of X .

Definition 3. *The sequence* $u = (x_k)$, $x_k \in X$ *is called almost convergent to* $l \in X$ *if* $(x^*(x_k - l)) \in f_0$ *for each* $x^* \in X'$.

It is easy to see that if $(x^*(x_k - l)) \in f_0$, then $u = (x_k) \in m(X)$. Indeed, it is known that $f \subset m$ and hence the sequence $u = (x_k)$ is weakly bounded in the Banach space X and consequently also norm-bounded i.e. $u \in m(X)$.

Let now $u = (x_k) \in m(X)$ and $\Lambda(u)(x^*) = L(x^*(x_k))$ for $x^* \in X'$ and for Banach limit L . Then $\Lambda(u) \in X''$. If we suppose that X is a reflexive, we may assume that $\Lambda(u) \in X$. Then the correspondence $u \longmapsto \Lambda(u)$ defines operators $\Lambda : m(X) \to X$ which have analogical properties to Banach limits (see [2]).

Let $U(X)$ be the class of sequences $u = (x_k) \in m(X)$ which have conditionally compact range.

Theorem 1. [2] *If* $u = (x_k) \in U(X)$, *then* u *is almost convergent to* $l \in X$ *iff*

$$\lim_n \| \frac{1}{n+1} \sum_{k=i}^{i+n} x_k - l \| = 0 \quad \textit{uniformly in } i. \tag{2}$$

2. Strong almost convergence in Banach spaces

We shall now introduce the notion of strong almost convergence of the sequences $u = (x_k)$, $x_k \in X$.

Definition 4. *The sequence* $u = (x_k)$ *is called strongly almost convergent to* $l \in X$ *if* $(| x^*(x_k - l) |) \in f_0$ *for each* $x^* \in X'$.

The sets of all almost convergent and strongly almost convergent X−valued sequences are denoted by $f(X)$ and $[f(X)]$ respectively.

Now we may establish

Theorem 2. *If* $u = (x_k) \in U(X)$, *then* u *is strongly almost convergent to* $l \in X$ *iff*

$$\lim_n \frac{1}{n+1} \sum_{k=i}^{i+n} \| x_k - l \| = 0 \quad \textit{uniformly in } i. \tag{3}$$

Proof. If $u = (x_k)$ is strongly almost convergent to l, then for each $\epsilon > 0$ there exists $N > 0$ so that for each $n > N$, $x^* \in X'$ and $i = 0, 1, \ldots$ we have

$$\frac{1}{n+1} \sum_{k=i}^{i+n} | x^*(x_k - l) | < \frac{\epsilon}{2}. \tag{4}$$

If $u = (x_k) \in U(X)$, then it is clear that $(x_k - l) \in U(X)$. Hence for each $\epsilon > 0$ there exist functionals $x_j^* \in X'$, $1 \leq j \leq r$ so that

$$\sup_{1 \leq j \leq r} | x_j^*(x_k - l) > \| x_k - l \| - \frac{\epsilon}{2} \tag{5}$$

for all $k = 0, 1, \ldots$ (see [2], lemma 2.1.1).

Now it follows from (4) and (5) that for each $n > N$ and $i = 0, 1, \ldots$

$$\frac{1}{n+1}\sum_{k=i}^{i+n} \| x_k - l \| < \epsilon.$$

i.e. (3) is valid.

For the converse, the sufficiency of (3) implies from the inequality

$$\frac{1}{n+1}\sum_{k=i}^{i+n} | x^*(x_k - l) | \leq \| x^* \| \frac{1}{n+1}\sum_{k=i}^{i+n} \| x_k - l \| .$$

This completes the proof.

Remark. Condition (3) is always sufficient for $u \in [f(X)]$.

If X is a finite dimensional Banach space, then every sequence $u \in m(X)$ has conditionally compact range and hence, in this case almost convergence and strong almost convergence in X are respectively characterized by (2) and (3). This does not hold in general. Deeds [1] and Kurtz [2] gave examples of sequences not in $U(X)$ which are weakly convergent to zero, hence are almost convergent to zero, but for which (2) is not fulfilled.

3. Matrix methods from $c(X)$ to $[f(Y)]$

Let a matrix method A be determined by an operator matrix $A = (a_{nk})$ where a_{nk}, n, $k = 0, 1, \ldots$ are bounded linear operators from X into Y. Then for $u = (x_k)$ we have $v = Au = (\sum_k a_{nk}x_k)$. Suppose that E and F are nonempty subsets of $\omega(X)$ and $\omega(Y)$ respectively. We define the matrix class (E, F) by saying that $A \in (E, F)$ if and only if for every $u = (x_k) \in E$ the series $\sum_k a_{nk}x_k$ converge in the norm of Y for each n and the sequence $v = (\sum_k a_{nk}x_k)$ belongs to Y. If A is a bounded linear operator in $m(X)$, then $v \in U(Y)$ for each $u \in U(X)$. Then almost convergence and strong almost convergence of the sequence v are characterized by conditions (2) and (3) respectively. Hence, using standard methods, we may find conditions for $A \in (L(X), f(Y))$ and for $A \in (L(X), [f(Y)])$, where $L(X)$ is a subspace of $m(X)$ such that $L(X) \subset U(X)$ (for example in theorem 3 $L(X) = c(X)$).

In the proof of the next theorem the following lemma will be used.

Lemma 2. *Let $\{T_{ni}\}$ be a set of bounded sublinear functionals on a Banach space E. If the conditions*

1^0 *there exists $K > 0$ so that $\sup_{n,i} \| T_{ni} \| \leq K$,*

2^0 $\lim_n T_{ni}(u) = 0$ *uniformly in i for each u in a fundamental set of E,*

are fulfilled, then

$$\lim_n T_{ni}(u) = 0 \quad uniformly\ in\ i$$

for each $u \in E$.

Lemma 2 is an analogue of the well-known Banach-Steinhaus theorem and we omit the proof.

Theorem 3. *$A \in (c(X), [f(Y)])$ and the sequence $v = (\sum_k a_{nk}x_k)$ is strongly almost convergent to $l = \lim_k x_k$ for each $u = (x_k) \in c(X)$ if and only if*

1^0 $\| \sum_{k=0}^{m} a_{nk}x_k \| \leq M \sup_k \| x_k \|$
for each m, $n = 0, 1, \ldots$ and $u = (x_k) \in m(X)$,

2^0 $(\| a_{nk}x \|) \in f_0$ *for each $k = 0, 1, \ldots$ and $x \in X$,*

3^0 $(\| \sum_k a_{nk}x - x \|) \in f_0$ *for each $x \in X$.*

Proof. 1) Assume that $A \in (c(X), f[(Y)])$. Since $[f(X)] \subset m(X)$, then $A(c(X)) \subset m(X)$ and condition 1^0 must be valid (see [6]). The necessity of conditions 2^0 and 3^0 follows by considering the sequences $(0, \ldots 0, x, 0, \ldots)$ and $(x, x, \ldots)$, $x \in X$ respectively.

2) Let $u = (x_k)$ and $v = (\sum_k a_{nk}x_k)$. If condition 1^0 holds, then $A : u \to v$ is a bounded linear operator on $c(X)$ and $v \in U(Y)$ for each $u \in c(X) \subset U(X)$. Hence we may describe almost convergence of the sequence v by condition (3), i.e. $v \in [f(Y)]$ iff

$$\lim_n \frac{1}{n+1} \sum_{\nu=i}^{i+n} \| \sum_k a_{\nu k}x_k - l \| = 0$$

uniformly in i .

We may write

$$\| \sum_k a_{\nu k} x_k - l \| \quad \leq \quad \| \sum_k a_{\nu k} t_k \| \quad + \quad \| \sum_k a_{\nu k} l - l \|$$

where $t_k = x_k - l$ and $w = (t_k) \in c_0(X)$. Since $(\| \sum_k a_{\nu k} l - l \|) \in f_0$ by condition 3^0 , it is sufficient to show that $(\| \sum a_{nk} t_k \|) \in f_0$ for each $w = (t_k) \in c_0(X)$. Let

$$T_{ni}(w) = \frac{1}{n+1} \sum_{\nu=i}^{i+n} \| \sum_k a_{\nu k} t_k \|$$

for $w = (t_k) \in c_0(X)$. Then T_{ni} are sublinear functionals on $c_0(X)$ and it follows from 1^0 that condition 1^0 of Lemma 2 is fulfilled. As the set of sequences $u_k(x) = (0, \dots 0,\ x,\ 0, \dots)$ is the fundamental set in $c_0(X)$, the condition 2^0 of Lemma 2 follows from 2^0 . By Lemma 2 we have that $\lim_n T_{ni}(w) = 0$ uniformly in i for each $w \in c_0(X)$. This completes the proof.

Let now $l(X) = \{u = (x_k), \quad x_k \in X \mid \sum_k \| x_k \| < \infty\}$.

Then $l(X) \subset U(X)$ and the next theorem is valid

Theorem 4. $A \in l(X), [f(Y)]$ *if and only if*

1^o $\quad \exists M > 0, \quad \| a_{nk} \| < M, \quad n, k = 0, 1, \dots,$

2^o *for each* $x \in X$ *and* $k = 0, 1, \dots$ *there exist* $l_k = l_k(x) \in Y$ *such that* $(\| a_{nk} x - l_k \|) \in f_o$.

Proof is analogous to the proof of Theorem 3 and we omit it. Let now $A = (a_{nk})$ be a matrix of complex numbers and $X = Y = \mathbb{C}$. Then Theorem 4 has the form $(l = l(\mathbb{C}))$

Corollary 4.1. $A \in (l, [f])$ *if and only if*

1^o $\quad \exists M > 0, \quad | a_{nk} | < M, \quad n, k = 0, 1, \dots,$

2^o $\quad (a_{nk}) \in [f]$ *for each* $k = 0, 1, \dots$.

4. Almost convergence in some Banach space

Suppose that X be a BK-space. We shall now need the following conditions

$$\text{(C1)} \quad \exists M > 0, \quad \| x_k \| < M, \quad k = 0, 1, \ldots,$$

$$\text{(C2)} \quad (\xi_j^{(k)}) \in f, \quad j = 0, 1, \ldots,$$

$$\text{(C3)} \quad (\xi_j^{(k)}) \in [f], \quad j = 0, 1, \ldots .$$

Theorem 5. *Let* $u = (x_k)$, $x_k = (\xi_j^{(k)}) \in X$. *Then*

$1^0 \quad u \in f(X) \Longrightarrow \text{(C1)}, \text{(C2)}$

and

$2^0 \quad u \in [f(X)] \Longrightarrow \text{(C1)}, \text{(C3)}.$

Proof. 1^0. Condition (C1) means that $f(X) \subset m(X)$ and this inclusion is proved in section 1 for an arbitrary Banach space X . Let now $\pi_j(x) = \xi_j$ for each $x = (\xi_j) \in X$. Since X is a BK-space then $\pi_j \in X'$ and $(\xi_j^{(k)}) = (\pi_j(x_k)) \in f$ must hold. The proof of 2^0 is analogous. ■

Let now X be a BK-AK-space. Then every $x^* \in X'$ may be presented in the form

$$x^*(x) = \sum_j \alpha_j \xi_j \quad \text{for each} \quad x = (\xi_j) \in X,$$

i.e. we may consider that $x^* = (\alpha_k) \in X'$. For $x_k = (\xi_j^{(k)}) \in X$, $k = 0, 1, \ldots$ we have

$$(x^*(x_k)) = (\sum_j \alpha_j \xi_j^{(k)}) = (\sum_j a_{kj} \alpha_j),$$

where $a_{kj} = \xi_j^{(k)}$. Consequently, the following theorem is valid.

Theorem 6. *Let* X *be a* BK-AK*-space. Then*

$1^0 \quad u = (x_k) \in f(X)$ *iff* $A \in (X', f)$

and

$2^0 \quad u = (x_k) \in [f(X)]$ *iff* $A \in (X', [f])$.

If $X = c_0, l^p\,(p > 0)$, then $(c_0)' = l$ and $(l^p)' = l^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Conditions for $A \in (l^q, f)$, $q \geq 1$ are founded in [5] and in the case

$A = (\xi_j^{(k)})$ we obtain conditions (C1) and (C2). Consequently, if $X = c_0, l^p$, then the conditions (C1), (C2) are sufficient for $u \in f(X)$. Applying corollary 4.1, we get sufficiency of (C1), (C3) for $u \in [f(X)]$. Therefore we have proved

Theorem 7. *Let* $X = c_0$ *or* $X = l^p$, $p > 1$, *then*

1^0 $u = (x_k) \in f(X)$ *iff conditions* (C1) *and* (C2) *are fulfilled*

and

2^0 $u = (x_k) \in [f(X)]$ *iff conditions* (C1) *and* (C3) *are fulfilled.*

5. The multipliers of the set $f(X)$

Let X be a Banach algebra and $E \subset \omega(X)$. The multipliers of the set E are defined by the set

$$M(E) = \{v \in w(X) \mid uv \in E \quad \forall u \in E\},$$

where $uv = (x_k y_k)$ for each $u = (x_k)$ and $v = (y_k)$.

It is known that $M(f) = [f]$. For the space $f(X)$ such equality must not be valid. For example, let X be a BK-space and $v = (e, e, \ldots)$ where $e = (1, 1, \ldots)$. Then it is obvious that $v \in M(f(X))$ but in the case $e \notin X$ we have $v \notin [f(X)]$. For $M(f(X))$ we may state the following results

Theorem 8. *Let X be a BK-space. If almost convergence of the sequence* $u = (x_k)$, $(\xi_j^{(k)})$ *is described by conditions* (C1) *and* (C2) *then every sequence* $v = (y_k)$, $y_k = (\eta_j^{(k)})$ *satisfying conditions* (C1), (C3) *is a multiplier of the set* $f(X)$.

Proof. Suppose that the sequence $v = (y_k)$, $y_k = (\eta_j^{(k)}) \in X$ satisfies conditions (C1), (C3). Then it is obvious that the sequence $(\xi_j^{(k)} \eta_j^{(k)})$ satisfies condition (C1) and it follows from $M(f) = [f]$ that $(\xi_j^{(k)}) \in f$. This completes the proof.

Theorem 9. *If the sequence* $v = (y_k)$ *satisfies condition* (3), *then* $v \in M(f(X))$.

Proof. For each $s \in X$ and $x^* \in X'$ we define functionals x_s^* as follows

$$x_s^*(x) = x^*(xs).$$

It is easy to see that $x_s^* \in X'$. Suppose now that $u = (x_k)$ is almost convergent to $s \in X$ and $v = (y_k)$ satisfies the condition (3), i.e. $(\| y_k - l \|) \in f_0$. Therefore,

$$|\frac{1}{n+1}\sum_{k=i}^{i+n} x^*(x_k y_k - sl)| \leq \frac{1}{n+1}\sum_{k=i}^{i+n} |x^*[x_k(y_k - l)]| +$$

$$+|x^*[\frac{1}{n+1}\sum_{k=i}^{i+n}(x_k - s)\cdot l]| \leq \| x^* \| \sup_k \| x_k \| \frac{1}{n+1}\sum_{k=i}^{i+n} \| y_k - l \| +$$

$$+ | \frac{1}{n+1}\sum_{k=i}^{i+n} x_l^*(x_k - s) | \longrightarrow 0, \quad n \longrightarrow \infty$$

uniformly in i. This completes the proof.

References

1. Deeds, J., *Summability of vector sequences.* Studia Math., 1968, **30**, 361–372.
2. Kurtz, J. C., *Almost convergent vector sequences.* Tohoku Math. J., 1970, **22**, 493–498.
3. Lorentz, G. G., *A contribution to the theory of divergent series.* Acta Math., 1948, **80**, 167–190.
4. Maddox, I. J., *A new type of convergence.* Math. Proc. Cambridge Phil. Soc., 1978, **83**, 61–64.
5. Nanda, S., *Matrix transformations and almost boundedness.* Glas. Mat., 1979, **14**, 99–107.

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Resümee

Olgu X Banachi ruum. Artiklis on defineeritud jada $u = (x_k)$, $x_k \in X$ tugev peaaegu koonduvus analoogselt J. Kurtzi (vt. [2]) poolt defineeritud jada peaaegu koonduvusega Banachi ruumis. Töös on näidatud, et teatud eeldustel on jada tugev peaaegu koonduvus Banachi ruumis kirjeldatav nagu arvjada tugev peaaegu koonduvus (vt. [3], [4]). On uuritud ka peaaegu koonduvate jadade hulga multiplikaatoreid.

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Some equivalent forms for convexity conditions for a family of normal matrix methods

Anne Tali

In a recent paper [6] the author found the necessary and sufficient conditions for convexity of a family of normal matrix methods A_α for summation complex (or real) valued sequences (see [6], Theorems 1.3, 1.5-1.7). The above mentioned convexity conditions have a constructive character and are quite difficult to control. Therefore, it will be useful to know the different equivalent forms for them. The main idea of the present paper is to get some new equivalent forms for these convexity conditions as well as to transfer them to the summation of sequences in locally convex spaces (Theorems 3, 4 and 5).

1. Preliminaries

Let $\mathcal{E}$ be [1] a locally convex space over the field $\mathbb{K}$ where topology is given by the set of seminorms $\mathcal{P} = \{p\}$. In this paper we deal with summation of sequences $x = (\xi_n)$ with $\xi_n \in \mathcal{E}$ for $n = 0, 1, 2, \ldots$. Let A be, in general, a summability method given by sequence-to-sequence transformation of [2] $x \in \omega(\mathcal{E})_A$ into $Ax = (\eta_n)$ where $\eta_n \in \mathcal{E}$. In the sequel we will use the notations $\omega(\mathcal{E})$, $m(\mathcal{E})$, $c(\mathcal{E})$ and $c_0(\mathcal{E})$ for the sets of all sequences, all bounded sequences, all convergent sequences and

[1] A locally convex space $\mathcal{E}$ is supposed to be separated and $\mathbf{K} = \mathbf{C}$ or $\mathbb{K} = \mathbf{R}$ everywhere.

[2] We denote the transformation of $x \in \omega(\mathcal{E})_A$ into $Ax = (\eta_n)$ also by A. The notation $\omega(\mathcal{E})_A$ is used here for the set of all sequences x where the transformation A is applied.

all zero-sequences in $\mathcal{E}$, respectively. We also denote:

$$m(\mathcal{E})_A = \{x \in \omega(\mathcal{E})_A \mid Ax \in m(\mathcal{E})\},$$

$$c(\mathcal{E})_A = \{x \in \omega(\mathcal{E})_A \mid Ax \in c(\mathcal{E})\},$$

$$c_0(\mathcal{E})_A = \{x \in \omega(\mathcal{E})_A \mid Ax \in c_0(\mathcal{E})\}.$$

In the major part of the paper we deal with matrix methods $A = (a_{nk})$ where $a_{nk} \in \mathbb{K}$ for $n, k = 0, 1, 2, \ldots$. A matrix method A (and a matrix A also) is called normal if $a_{nk} = 0$ for all $k > n$ and $a_{nn} \neq 0$ $(n = 0, 1, 2, \ldots)$. A matrix method A is said to be regular in $\mathcal{E}$ if $c(\mathcal{E}) \subset c(\mathcal{E})_A$ and $\lim_n \eta_n = \lim_n \xi_n$ for every $x \in c(\mathcal{E})$. It is well-known that matrix method A is regular in $\mathcal{E} = \mathbb{K}$ if and only if the following conditions are fulfilled:

$$\lim_n a_{nk} = 0 \quad (k = 0, 1, 2, \ldots), \tag{1}$$

$$\lim_n \sum_k a_{nk} = 1, \tag{2}$$

$$\sum_k | a_{nk} | = O(1). \tag{3}$$

We call a matrix A satisfying the conditions (1)-(3) a T-matrix. If matrix A satisfies the conditions (1) and (3) then we call it a T_0-matrix. In the sequel we need the next two propositions that immediately follow from Propositions 1 and 2 of paper [4].

Proposition 1. *Let $\mathcal{E}$ be a sequentially complete locally convex space and* [3] *A be a matrix. Then the following statements are valid.*

1) *Matrix method A is regular in $\mathcal{E}$ if and only if A is a T-matrix.*

2) *Matrix transformation A is a $c_0(\mathcal{E}) \longrightarrow c_0(\mathcal{E})$ transformation if and only if matrix A is a T_0-matrix.*

3) *Transformation A is a $m(\mathcal{E}) \longrightarrow m(\mathcal{E})$ transformation if and only if the condition* (3) *is fulfilled.*

Proposition 2. *Let $\mathcal{E}$ be a locally convex space and A be a row-finite matrix. Then the statements* 1)-3) *from Proposition 1 are valid.*

[3] The elements of matrices belong to $\mathbb{K}$ everywhere.

2. Convex families of summability methods

Let A_α be a family of summability methods given by transformations of $x \in \omega(\mathcal{E})_{A_\alpha}$ into $A_\alpha x = (\eta_n^\alpha)$, where $\eta_n^\alpha \in \mathcal{E}$ $(n = 0, 1, 2, \ldots)$ and α is a continuous parameter with values $\alpha > \alpha_0$. Next we will formulate the central notion of our paper (see [5]).

Definition. *The family of summability methods A_α is said to be convex if for every $\alpha < \beta$ and for every $\alpha < \gamma < \beta$ the conditions*

$$m(\mathcal{E})_{A_\alpha} \subset m(\mathcal{E})_{A_\beta}, \quad c(\mathcal{E})_{A_\alpha} \subset c(\mathcal{E})_{A_\beta} \tag{4}$$

and

$$c(\mathcal{E})_{A_\gamma} \supset m(\mathcal{E})_{A_\alpha} \cap c(\mathcal{E})_{A_\beta} \tag{5}$$

hold. The family A_α is said to be zero-convex (0-convex) if the conditions (4) *and* (5) *hold with c_0 instead of c in them.*

The proofs of the convexity theorems can be simplified by the following trivial lemmas.

Lemma 1. *If for every $\alpha > \alpha_0$ and $0 < \delta < 1$ the conditions*

$$m(\mathcal{E})_{A_\alpha} \subset m(\mathcal{E})_{A_{\alpha+\delta}}, \quad c(\mathcal{E})_{A_\alpha} \subset c(\mathcal{E})_{A_{\alpha+\delta}} \tag{6}$$

and

$$c(\mathcal{E})_{A_{\alpha+\delta}} \supset m(\mathcal{E})_{A_\alpha} \cap c(\mathcal{E})_{A_{\alpha+1}} \tag{7}$$

hold, then the family A_α is convex.

Lemma 2. *If for every $\alpha > \alpha_0$ and $0 < \delta < 1$ the conditions*

$$m(\mathcal{E})_{A_\alpha} \subset m(\mathcal{E})_{A_{\alpha+\delta}}, \quad c_0(\mathcal{E})_{A_\alpha} \subset c_0(\mathcal{E})_{A_{\alpha+\delta}} \tag{8}$$

and

$$c_0(\mathcal{E})_{A_{\alpha+\delta}} \supset m(\mathcal{E})_{A_\alpha} \cap c_0(\mathcal{E})_{A_{\alpha+1}} \tag{9}$$

hold, then the family A_α is 0-convex.

Completing this section, we will formulate two propositions.

Proposition 3. *Let A_α be linear transformations for every $\alpha > \alpha_0$ transforming each stationary sequence $x' = (\xi'_n)$ with $\xi'_n = \xi \in \mathcal{E}$ $(n = 0, 1, 2, \ldots)$ into sequences $A_\alpha x' = (\eta_n'^\alpha)$ where*

$$\lim_n \eta_n'^\alpha = a_\alpha \xi \quad (0 \neq a_\alpha \in \mathbb{K}). \tag{10}$$

If the family A_α *is 0-convex, then it is convex.*

The proof of this proposition is trivial and will be therefore omitted. In particular, for matrix transformations $A_\alpha = (a^\alpha_{nk})$ the condition (10) is:

$$\lim_n \sum_k a^\alpha_{nk} = a_\alpha \neq 0.$$

Proposition 4. *Suppose that for every* $\alpha > \alpha_0$ *and* $0 < \delta < 1$ *the conditions* (8) *hold and*

1) *For each* $\varepsilon > 0$ *there exsists a* $c_0(\mathcal{E}) \longrightarrow c_0(\mathcal{E})$ *matrix* $Q_{\alpha\delta\varepsilon} = (q^{\alpha\delta\varepsilon}_{nk})$ *and a row-finite matrix* $R_{\alpha\delta\varepsilon} = (r^{\alpha\delta\varepsilon}_{nk})$ *satisfying*

$$\limsup_n \sum_k |r^{\alpha\delta\varepsilon}_{nk}| < \varepsilon \tag{11}$$

and

$$p(\eta^{\alpha+\delta}_n) \leq K_p \cdot p\left(\sum_k q^{\alpha\delta\varepsilon}_{nk} \eta^{\alpha+1}_k\right) + L_p \cdot p\left(\sum_k r^{\alpha\delta\varepsilon}_{nk} \eta^\alpha_k\right) \tag{12}$$

for every $x \in m(\mathcal{E})_{A_\alpha} \bigcap c_0(\mathcal{E})_{A_{\alpha+1}}$, $n = 1, 2, \ldots$ *and* $p \in \mathcal{P}$, *where* K_p *and* L_p *are real constants depending on* p.

Then the family A_α *is 0-convex.*

Proof. Let us fix an $x \in m(\mathcal{E})_{A_\alpha} \bigcap c_0(\mathcal{E})_{A_{\alpha+1}}$. Taking for the starting point the inequality (12) we conclude from condition 1) that

$$\begin{aligned}
\limsup_n p(\eta^{\alpha+\delta}_n) &\leq K_p \cdot \limsup_n p\left(\sum_k q^{\alpha\delta\varepsilon}_{nk} \eta^{\alpha+1}_k\right) \\
&\quad + L_p \cdot \limsup_n p\left(\sum_k r^{\alpha\delta\varepsilon}_{nk} \eta^\alpha_k\right) \\
&\leq L_p \cdot \sup_n p(\eta^\alpha_n) \cdot \limsup_n \sum_k |r^{\alpha\delta\varepsilon}_{nk}| < \varepsilon \cdot L_p \cdot \sup_n p(\eta^\alpha_n),
\end{aligned}$$

and thus $\lim_n p(\eta^{\alpha+\delta}_n) = 0$ for every $p \in \mathcal{P}$. Therefore, $x \in c_0(\mathcal{E})_{A_{\alpha+\delta}}$ and inclusion (9) holds for all $\alpha > \alpha_0$ and $0 < \delta < 1$. The 0-convexity of the family A_α follows now from Lemma 2.

We note that in particular if $\mathcal{E}$ is sequentially complete then the condition of row-finity of matrix $R_{\alpha\delta\varepsilon}$ can be omitted in Proposition 4. Theorem 1 of paper [5] follows from Proposition 4 as an immediate corollary (for the case $\mathcal{E} = \mathbb{K}$).

3. On the Quotient Theorem of Baumann

The proofs of the convexity theorems in paper [6] were based on the following quotient theorem of H. Baumann (see [1], Theorem 1), here formulated in the notations of this paper.

Theorem 1. *Let A and B be T-matrices and $\mathcal{E} = \mathbf{K}$. Then the following statements are equivalent.*

a) $c(\mathcal{E})_B \supset m(\mathcal{E}) \bigcap c(\mathcal{E})_A$.

b) *For every $\varepsilon > 0$ there exists a row-finite and column-finite T-matrix $Q_\varepsilon = (q^\varepsilon_{nk})$ and a matrix $R_\varepsilon = (r^\varepsilon_{nk})$ satisfying*

$$B = Q_\varepsilon A + R_\varepsilon \tag{13}$$

and

$$\limsup_n \sum_k |r^\varepsilon_{nk}| < \varepsilon. \tag{14}$$

The Baumann Theorem 1 was refined by J. Boos in papers [2] and [3] (see [2], Theorem 4 and [3], Theorem) where some new statements equivalent to the statement a) were proved. It is easy to see, that Theorem 1 remains true for any sequentially complete locally convex space $\mathcal{E}$. In the sequel we make use of the following variant of the Baumann Theorem.

Theorem 2. *Let $A = (a_{nk})$ and $B = (b_{nk})$ be row-finite T-matrices and $\mathcal{E}$ be a locally convex space. Then the following statements are equivalent.*

a) $c(\mathcal{E})_B \supset m(\mathcal{E}) \bigcap c(\mathcal{E})_A$ *and A and B are consistent on* $m(\mathcal{E}) \bigcap c(\mathcal{E})_A$.

a*) $c(\mathcal{E})_B \supset m(\mathcal{E}) \bigcap c(\mathcal{E})_A$.

b) *For every $\varepsilon > 0$ there exists a row-finite and column-finite T-matrix Q_ε and a matrix R_ε satisfying* (13) *and* (14).

c) *For every $\varepsilon > 0$ there exists a row-finite T_0-matrix Q_ε and a matrix R_ε satisfying* (13) *and* (14).

d) *For every $\varepsilon > 0$ there exists a row-finite T_0-matrix $Q_\varepsilon = (q^\varepsilon_{nk})$ and a row-finite matrix $R_\varepsilon = (r^\varepsilon_{nk})$ satisfying the conditions* (14) *and*

$$p\left(\sum_k b_{nk}\xi_k\right) \le K_p \cdot p\left(\sum_k q^\varepsilon_{nk} \sum_\nu a_{k\nu}\xi_\nu\right) + L_p \cdot p\left(\sum_k r^\varepsilon_{nk}\xi_k\right)$$

for each $n = 0, 1, 2, \ldots,$ $p \in \mathcal{P}$ *and* $x \in m(\mathcal{E}) \bigcap c_0(\mathcal{E})_A$.

e) *Statement* d) *is fulfilled with a* $c_0(\mathcal{E}) \to c_0(\mathcal{E})$ *matrix* Q_ε *instead of a row-finite* T_0*-matrix* Q_ε.

f) $c_0(\mathcal{E})_B \supset m(\mathcal{E}) \bigcap c_0(\mathcal{E})_A$.

If in addition the matrix A *is normal then we have further statements being equivalent to* a).

g) *For every* $\varepsilon > 0$ *there exist matrices* Q_ε *and* R_ε *satisfying the conditions* (13), $\sup_n \sum_k |q^\varepsilon_{nk}| < \infty$ *and*

$$\sup_n \sum_k |r^\varepsilon_{nk}| < \varepsilon. \tag{15}$$

b*) *Statement* b) *is fulfilled with* (15) *instead of* (14).

c*) *Statement* c) *is fulfilled with* (15) *instead of* (14).

d*) *Statement* d) *is fulfilled with* (15) *instead of* (14).

e*) *Statement* e) *is fulfilled with* (15) *instead of* (14).

Additional Remark. *In particular, if* $\mathcal{E}$ *is sequentially complete, then the statements* a) - f) *without the condition of row-finity of matrix* $R_\mathcal{E}$ *in* d) *and* e) *are equivalent for all T-matrices* A *and* B .

Proof. Obviously, a) $\Longrightarrow$ a*) is valid. Implication a*) $\Longrightarrow$ b) is also valid, because it is true for $\mathcal{E} = \mathbb{K}$ as one direction of the equivalence in Theorem 1 and because the Banach space $\mathbb{K}$ is equivalent to any one-dimensional subspace $\mathcal{E}_\xi = \{\lambda\xi \in \mathcal{E} \mid \lambda \in \mathbb{K}\}$ of space $\mathcal{E}$. Implications b) $\Longrightarrow$ c) $\Longrightarrow$ d) are trivially fulfilled. Implication d) $\Longrightarrow$ e) is fulfilled by Proposition 2. Furthermore, both e) and e*) imply f) and f) implies a) since A and B are T-matrices. These considerations will also prove the validity of Additional Remark, if we use Proposition 1 instead of Proposition 2 in them.

In particular, let A be a normal matrix. Then implications a) $\Longrightarrow$ g) $\Longrightarrow$ b*) are valid because for $\mathcal{E} = \mathbb{K}$ they are contained in Theorem 4 of [2]. The implications b*) $\Longrightarrow$ c*) $\Longrightarrow$ d*) $\Longrightarrow$ e*) are obviously valid. Consequently, we have proved our theorem.

We note that in case of a sequentially complete space $\mathcal{E}$ a $c_0(\mathcal{E}) \longrightarrow c_0(\mathcal{E})$ matrix Q_ε is just a T_0-matrix in statements e) and e*).

As we can see from Theorem 2, the validity of inclusion $c(\mathcal{E})_B \supset m(\mathcal{E}) \cap c(\mathcal{E})_A$ does not depend on space $\mathcal{E}$ depending only on row-finite T-matrices A and B .

Proposition 5. *Let A and B be T-matrices. If the inclusion $c(\mathcal{E})_B \supset m(\mathcal{E}) \cap c(\mathcal{E})_A$ holds in a given locally convex space $\mathcal{E}$, then it holds in any sequentially complete locally convex space $\mathcal{E}$, and also in any locally convex space $\mathcal{E}$ if in particular matrices A and B are row-finite.*

4. The necessary and sufficient conditions for the convexity of a family of normal matrix methods

Suppose now that $A_\alpha = (a^\alpha_{nk})$ with $\alpha > \alpha_0$ are normal matrix methods. Then there exists the inverse matrix A_α^{-1} for every matrix A_α. Let us denote by $D_{\alpha\delta}$ the product of matrices $A_{\alpha+\beta}$ and A_α^{-1} where $\delta > 0$ so that $D_{\alpha\delta} = A_{\alpha+\delta}A_\alpha^{-1}$.

Theorem 3. *The family of normal matrix methods A_α (where $\alpha > \alpha_0$) is 0-convex in locally convex space $\mathcal{E}$ if and only if matrix $D_{\alpha\delta} = A_{\alpha+\delta}A_\alpha^{-1}$ satisfies the following conditions for every $\alpha > \alpha_0$ and $0 < \delta < 1$.*

1) *$D_{\alpha\delta}$ is a T_0-matrix.*

2) $c_0(\mathcal{E})_{D_{\alpha\delta}} \supset m(\mathcal{E}) \cap c_0(\mathcal{E})_{D_{\alpha 1}}$.

Proof. Since $A_{\alpha+\delta} = D_{\alpha\delta}(A_\alpha x)$ for each $x \in \omega(\mathcal{E})$ and methods A_α are normal, then the condition 1) is equivalent to the condition (8) by Proposition 2. The relations $x \in m(\mathcal{E})_{A_\alpha} \cap c_0(\mathcal{E})_{A_{\alpha+1}}$ and $A_\alpha x \in m(\mathcal{E}) \cap c_0(\mathcal{E})_{D_{\alpha 1}}$ are equivalent in the same way as the relations $x \in c_0(\mathcal{E})_{A_{\alpha+\delta}}$ and $A_\alpha x \in c_0(\mathcal{E})_{D_{\alpha\delta}}$ are equivalent. Therefore, the condition 2) is equivalent to the condition (9). Our statement follows now from Lemma 2.

The following theorem together with the additional remark to it forms the main result of our paper.

Theorem 4. *The family of normal matrix methods A_α (where $\alpha > \alpha_0$) is convex and methods A_α are pairwise consistent in locally convex space $\mathcal{E}$ if and only if matrix $D_{\alpha\delta} = A_{\alpha+\delta}A_\alpha^{-1}$ satisfies the following conditions for every $\alpha > \alpha_0$ and $0 < \delta < 1$.*

1) *$D_{\alpha\delta}$ is a T-matrix.*

2) $c(\mathcal{E})_{D_{\alpha\delta}} \supset m(\mathcal{E}) \cap c(\mathcal{E})_{D_{\alpha 1}}$ *(with consistency).*

Additional Remark. *Applying Theorem 2 to T-matrices* $A = D_{\alpha 1}$ *and* $B = D_{\alpha\delta}$ *we get as immediate corollaries the following statements each of which is equivalent to* 2) *(if the statement* 1) *is fulfilled for every* $\alpha > \alpha_0$ *and* $0 < \delta < 1$ *).*

2a*) $c(\mathcal{E})_{D_{\alpha\delta}} \supset m(\mathcal{E}) \cap c(\mathcal{E})_{D_{\alpha 1}}$.

2b) *For every* $\varepsilon > 0$ *there exists a row-finite and column-finite T-matrix* $Q_{\alpha\delta\varepsilon} = (q_{nk}^{\alpha\delta\varepsilon})$ *and a matrix* $R_{\alpha\delta\varepsilon} = (r_{nk}^{\alpha\delta\varepsilon})$ *satisfying* (11) *and*

$$D_{\alpha\delta} = Q_{\alpha\delta\varepsilon} D_{\alpha 1} + R_{\alpha\delta\varepsilon}. \tag{16}$$

2c) *For every* $\varepsilon > 0$ *there exists a row-finite* T_0*-matrix* $Q_{\alpha\delta\varepsilon}$ *and a matrix* $R_{\alpha\delta\varepsilon}$ *satisfying* (11) *and* (16).

2d) *For every* $\varepsilon > 0$ *there exists a row-finite* T_0*-matrix* $Q_{\alpha\delta\varepsilon}$ *and a row-finite matrix* $R_{\alpha\delta\varepsilon}$ *satisfying* (11) *and*

$$p\Big(\sum_k d_{nk}^{\alpha\delta}\xi_k\Big) \;\le\; K_p \cdot p\Big(\sum_k q_{nk}^{\alpha\delta\varepsilon} \sum_\nu d_{k\nu}^{\alpha 1}\xi_\nu\Big) \;+\; L_p \cdot p\Big(\sum_k r_{nk}^{\alpha\delta\varepsilon}\xi_k\Big)$$

for each $n = 0, 1, 2, \ldots,$ $p \in \mathcal{P}$ *and* $x \in m(\mathcal{E}) \cap c_0(\mathcal{E})_{D_{\alpha 1}}$.

2e) *Statement* 2d) *holds with a* $c_0(\mathcal{E}) \longrightarrow c_0(\mathcal{E})$ *matrix* $Q_{\alpha\delta\varepsilon}$ *instead of a row-finite* T_0*-matrix* $Q_{\alpha\delta\varepsilon}$.

We here omit the formulations of statements 2f), 2g), 2b*), 2c*) *and* 2d*) *that are analogous to the same statements from Theorem 2, and formulate the statement*

2e*) *Statement* 2e) *is fulfilled with*

$$\sup_n \sum_k | r_{nk}^{\alpha\delta\varepsilon} | < \varepsilon$$

instead of (11).

If in addition $\mathcal{E}$ *is sequentially complete, then the condition of row-finity of matrix* $R_{\alpha\delta\varepsilon}$ *can be omitted in statements* 2d), 2e), 2d*) *and* 2e*).

Proof of Theorem 4. The condition 1) is equivalent to the condition (6) by Proposition 2 since $A_{\alpha+\delta} = D_{\alpha\delta}(A_\alpha x)$ for each $x \in \omega(\mathcal{E})$ and methods A_α are normal. Furthermore, the condition 2) is equivalent to the condition (7) because the relations $x \in m(\mathcal{E})_{A_\alpha} \cap c(\mathcal{E})_{A_{\alpha+1}}$ and $A_\alpha x \in m(\mathcal{E}) \cap c(\mathcal{E})_{D_{\alpha 1}}$ are equivalent and the relations $x \in c(\mathcal{E})_{A_{\alpha+\delta}}$ and $A_\alpha x \in c(\mathcal{E})_{\mathcal{D}_{\alpha\delta}}$ are equivalent. Obviously, our considerations keep consistency of the methods. Theorem 4 follows now from Lemma 1.

In particular, if $\mathcal{E} = \mathbb{K}$, then Theorem 4 and also the equivalence of statements 2), 2b), 2c), 2d) are proved in [6] (see [6], Theorem 1.3 and 1.5–1.7. The next result follows immediately from Theorem 4 with the help of Additional Remark to it.

Proposition 6. *If the family of normal matrix methods* A_α *(where* $\alpha > \alpha_0$ *) is convex and the methods* A_α *are pairwise consistent in a given locally convex space* $\mathcal{E}$, *then the family* A_α *is convex (with consistency) in any locally convex space* $\mathcal{E}$.

5. The sufficient conditions for the convexity of a family of summability methods

The restrictions on the methods A_α can be weakened so that the conditions 1) and 2) (for every $\alpha > \alpha_0$ and $0 < \delta < 1$) of Theorems 3 and 4 remain sufficient for the 0-convexity and convexity of the family A_α , respectively. The methods A_α need not be normal, not even matrix methods. Let the methods A_α be given by the transformations of $x \in \omega(\mathcal{E})_{A_\alpha}$ into $A_\alpha x = (\eta_n^\alpha)$ where $\eta_n^\alpha \in \mathcal{E}$ $(n = 0, 1, 2, \ldots)$.

Theorem 5. *Let* $\mathcal{E}$ *be a locally convex space and let the summability methods* A_α *and* $A_{\alpha+\delta}$ *for every* $\alpha > \alpha_0$ *and* $0 < \delta \leq 1$ *be connected by the row-finite matrix* $D_{\alpha\delta}$ *so that* $A_{\alpha+\delta} = D_{\alpha\delta}(A_\alpha x)$ *for each* $x \in \omega(\mathcal{E})_{A_\alpha}$.

If the matrix $D_{\alpha\delta}$ *for every* $\alpha > \alpha_0$ *and* $0 < \delta < 1$ *satisfies the conditions* 1) *and* 2) *of Theorem 3, then the family* A_α *is* 0*-convex in* $\mathcal{E}$. *If in addition* $D_{\alpha\delta}$ *is a T-matrix, then the family* A_α *is convex and the methods* A_α *are pairwise consistent in* $\mathcal{E}$.

Additional Remark. *If the matrix* $D_{\alpha\delta}$ *defined in Theorem 5 satisfies the condition* 2e) *from the Additional Remark to Theorem 4, then the condition* 2) *of Theorem 3 is satisfied.*

The proof of Theorem 5 coincides with one direction (sufficiency) of proof of Theorem 3 and of proof of Theorem 4 in additional case. We note only that the condition 2) of Theorem 3 implies the condition 2) of Theorem 4 for T-matrices $D_{\alpha\delta}$ and $D_{\alpha 1}$.

We notice that the implications $2b^*) \Longrightarrow 2b) \Longrightarrow 2c) \Longrightarrow 2d) \Longrightarrow 2e)$ and $2b^*) \Longrightarrow 2c^*) \Longrightarrow 2d^*) \Longrightarrow 2e^*) \Longrightarrow 2e)$ from the Additional Remark to Theorem 4 are valid for connection matrices $D_{\alpha\delta}$ and $D_{\alpha 1}$. If we replace the condition 2) by condition 2e) in Theorem 5, then we get the result that is an immediate corollary from Proposition 4.

A method for constructing the quotient representations (15) satisfying 2c) for certain class of connection matrices $D_{\alpha\delta}$ was built up in [6], and special convex families (in case of $\mathcal{E} = \mathbb{K}$) were also found (see [6], sections 2 and 3).

References

1. Baumann, H., *Quotientensätze für Matrizen in der Limitierungstheorie*. Math. Z., 1967, **100**, 147–162.
2. Boos, J., *Zwei-Normen-Konvergenz und Vergleich von beschränkten Wirkfeldern*. Math. Z., 1976, **148**, 265–294.
3. Boos, J., *The comparison of bounded convergence domains of regular matrices*. Math. Z., 1985, **193**, 11–13.
4. Menihes, L., *Summability in topological spaces*. Proc. Ural Univ., 1975, **9(2)**, 65–76 (in Russian).
5. Tali, A., *On zero-convex families of summability methods*. Acta et Comment. Univ. Tartuensis, 1981, **504**, 48–57 (in Russian).
6. Tali, A., *Convexity conditions for families of summability methods*. Acta et Comment. Univ. Tartuensis, 1993, **960**, 117–138.

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On the integrability and L^1-convergence of cosine series

Margus Tynnov

1. Introduction

We study the cosine series the coefficients of which form summability factors. We are concerned with the following problems: the sum of the series is integrable; the series is the Fourier series of its sum; the series converges in L^1-norm. The following theorem of Kolmogorov [5] is well-known for cosine series.

If (a_k) *is a quasiconvex null sequence, then the cosine series*

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx \tag{1}$$

converges, except possibly at $x=0$, *to an integrable function* $f(x)$, *is the Fourier series of* f, *and the partial sums converge in* $L^1(0,\pi)$*-norm to* f *if and only if* $a_k\ln k\to 0$ *as* $k\to\infty$.

In this paper we will extend this result. We will show that the conditions of Kolmogorov can be replaced by the conditions of summability factors.

2. Summability factors

Let $T=(\tau_{nk})$ be a triangular matrix of real numbers and let ω be the space of all real valued sequences $x=(x_k)$. We denote the summability field of T by

$$c_T=\{x\in\omega:\lim_n\sum_{k=0}^{n}\tau_{nk}x_k \quad exist\},$$

the boundedness domain of T by

$$m_T=\{x\in\omega:\sup_n|\sum_{k=0}^{n}\tau_{nk}x_k|<\infty\}$$

and the set of summability factors by

$$(m_T, c_T) = \{(a_k) \in \omega : (a_k x_k) \in c_T \quad \text{for every} \quad (x_k) \in m_T\}.$$

Let T be the matrix of the series-sequence Cesàro method C^α of order $\alpha > 0$ by

$$\tau_{nk} = \frac{A^\alpha_{n-k}}{A^\alpha_n}, \quad A^\alpha_n = \frac{(n+\alpha)(n+\alpha-1)\dots(\alpha+1)}{n!}$$

or Riesz method P by

$$\tau_{nk} = 1 - \frac{P_{k-1}}{P_n}, \quad P_n = p_0 + \dots + p_n,$$

where

$$\lim_{n\to\infty} | P_n | = \infty,$$

$$\frac{1}{| P_n |} \sum_{k=0}^{n} | p_k | \le M \quad (n = 0, 1, 2, \dots),$$

$$\frac{1}{| P_n |} \int_0^\pi | \frac{1}{2} + \sum_{k=1}^{n} (P_n - P_{k-1}) \cos kx \mid dx =$$

$$= \frac{1}{| P_n |} \int_0^\pi | \sum_{k=0}^{n} p_k D_k(x) \mid dx \le K \quad (n = 1, 2, \dots), \tag{2}$$

$$D_k(x) = \frac{1}{2} + \sum_{\nu=1}^{k} \cos \nu x = \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

The methods C^α $(\alpha > 0)$ and P are regular methods (see for example [1] or [4]). Bohr [2] and Kangro [4] showed that

$$(m_{C^\alpha}, c_{C^\alpha}) = \{(a_k) \in c_0 : \sum_{k=0}^{\infty} (k+1)^\alpha \mid \Delta^{\alpha+1} a_k \mid < \infty\}; \tag{3}$$

$$(m_P, c_P) = \{(a_k) \in c_0 : \sum_{k=0}^{\infty} | P_k \Delta \frac{\Delta a_k}{p_k} | < \infty, \quad \lim_k \frac{P_k}{p_k} \Delta a_k = 0\}, \tag{4}$$

where

c_0 – space of null sequences,

$$\Delta^{\alpha+1} a_k = \Delta^\alpha a_k - \Delta^\alpha a_{k+1}.$$

3. Convergence of cosine series

Theorem. *If* $(a_k) \in (m_T, c_T)$ *, then the cosine series* (1) *converges, except possibly at* $x = 0$ *, to an integrable function* $f(x)$ *, is the Fourier series of* $f(x)$ *, and the partial sums converge in* L^1*-norm to* f *if and only if* $a_k \ln k \to 0$ *as* $k \to \infty$.

Proof. By

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \sum_{k=0}^{n} (\sum_{\nu=k}^{n} \tau_{\nu k}^{-1} a_k) K_k(x),$$

where

$$K_k(x) = \frac{\tau_{n0}}{2} + \sum_{\nu=1}^{n} \tau_{k\nu} \cos \nu x,$$

for $C^\alpha \, (\alpha > 0)$

$$\begin{aligned} S_n &= \sum_{k=0}^{n} A_k^\alpha \sum_{\nu=k}^{n} A_{\nu-k}^{-(\alpha+2)} a_\nu K_\nu^\alpha(x) = \\ &= \sum_{k=0}^{n-(\alpha+1)} A_k^\alpha (\Delta^{\alpha+1} a_k) K_k^\alpha(x) + A_{n-\alpha}^\alpha (\Delta^\alpha a_{n-\alpha}) K_{n-\alpha}^\alpha(x) + \\ &+ A_{n-(\alpha-1)}^{\alpha-1} (\Delta^{\alpha-1} a_{n-(\alpha-1)}) K_{n-(\alpha-1)}^{\alpha-1}(x) + \\ &\ldots\ldots\ldots\ldots\ldots \\ &+ (\Delta a_{n-1}) K_{n-1}^1(x) + \\ &+ a_n D_n(x), \end{aligned} \tag{5}$$

where

$$K_\nu^\alpha(x) = \frac{1}{A_\nu^\alpha} \sum_{k=0}^{\nu} A_{\nu-k}^{\alpha-1} D_k(x),$$

$$\Delta^{\alpha+1} a_k = \Delta^\alpha a_k - \Delta^\alpha a_{k+1},$$

$$\Delta^{\alpha+1} a_k = \sum_{\nu=k}^{k+\alpha+2} A_{\nu-k}^{-(\alpha+2)} a_\nu .$$

Since

$$A_k^\alpha \sim \frac{k^\alpha}{\alpha!} \quad \text{as} \quad k \to \infty, \quad (a_k) \in (m_{C^\alpha}, c_{C^\alpha}),$$

then by (3)

$$\sum_{k=0}^{\infty} (k+1)^\alpha \mid \Delta^{\alpha+1} a_k \mid < \infty, \tag{6}$$

and

$$n^\alpha \mid \Delta^\alpha a_n \mid = n^\alpha \mid \sum_{\nu=n-\alpha}^{\infty} \Delta^{\alpha+1} a_\nu \mid \leq n^\alpha \sum_{\nu=n-\alpha}^{\infty} \frac{(1+\nu)^\alpha}{(1+\nu)^\alpha} \mid \Delta^{\alpha+1} a_\nu \mid \leq$$

$$\leq \frac{n^\alpha}{(1+n-\alpha)^\alpha} \sum_{\nu=n-\alpha}^{\infty} (1+\nu)^\alpha \mid \Delta^{\alpha+1} a_\nu \mid \to 0 \quad \text{as} \quad n \to \infty. \tag{7}$$

By [7, p.158]

$$\mid K_n^\alpha(x) \mid \leq M_\alpha\left(\frac{1}{n^\alpha} \cdot \frac{1}{x^{\alpha+1}} + \frac{1}{n} \cdot \frac{1}{x^2}\right) \quad (0 < x \leq \pi)$$

and by (5), (6), (7) series

$$\sum_{k=0}^{\infty} A_k^\alpha (\Delta^{\alpha+1} a_k) K_k^\alpha(x)$$

converges to an integrable function $f(x)$ for $0 < x \leq \pi$. Clearly,

$$\begin{aligned} S_n - f(x) = - \sum_{k=n-\alpha}^{\infty} & A_k^\alpha (\Delta^{\alpha+1} a_k) K_k^\alpha(x) + \\ & + A_{n-\alpha}^\alpha (\Delta^\alpha a_{n-\alpha}) K_{n-\alpha}^\alpha(x) + \\ & \ldots\ldots\ldots\ldots\ldots \\ & + \Delta a_{n-1} K_{n-1}^1(x) + \\ & + a_n D_n(x) \end{aligned} \tag{8}$$

and $S_n \to f(x)$ as $n \to \infty$ for $0 < x \leq \pi$. Using the integrability [7, p.157]

$$\int_0^\pi \mid K_n^\alpha(x) \mid dx \leq M_\alpha \quad (n = 0, 1, \ldots)$$

and (6), (7), (8) we have

$$\begin{aligned} \int_0^\pi \mid S_n - f(x) \mid dx \leq \sum_{k=n-\alpha}^{\infty} & A_k^\alpha \mid \Delta^{\alpha+1} a_k \mid \int_0^\pi \mid K_k^\alpha(x) \mid dx + \\ & + A_{n-\alpha}^\alpha \mid \Delta^\alpha a_{k-\alpha} \mid \int_0^\pi \mid K_{n-\alpha}^\alpha(x) \mid dx + \\ & \ldots\ldots\ldots\ldots\ldots \\ & + \mid \Delta a_{n-1} \mid \int_0^\pi \mid K_{n-1}^1(x) \mid dx + \\ & + \mid a_n \mid \int_0^\pi \mid D_n(x) \mid dx \to 0 \end{aligned}$$

as $n \to \infty$, and

$$| a_n | \int_0^\pi | D_n(x) | dx \le \int_0^\pi | S_n - f(x) | dx +$$

$$+ \sum_{k=n-\alpha}^{\infty} A_k^\alpha | \Delta^{\alpha+1} a_k | \int_0^\pi | K_k(x) | dx +$$

$$+ A_{n-\alpha}^\alpha | \Delta^\alpha a_{n-\alpha} | \int_0^\pi | K_{n-\alpha}^\alpha(x) | dx +$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$+ | \Delta a_{n-1} | \int_0^\pi | K_{n-1}^1(x) | dx \to 0$$

as $n \to \infty$. Hence,

$$\int_0^\pi | S_n - f(x) | dx \to 0$$

if and only if

$$a_n \int_0^\pi | D_n(x) | dx \to 0$$

as $n \to \infty$ and this is equivalent [7, p.115] to $a_n \ln n \to 0$ as $n \to \infty$.

For Riesz matrix (see for example [4] or [1], p.116) we have

$$\tau_{nn}^{-1} = \frac{P_n}{p_n}, \quad \tau_{n,n-1}^{-1} = -\frac{P_{n-1}}{p_n} - \frac{P_{n-1}}{p_{n-1}},$$

$$\tau_{n,n-2}^{-1} = \frac{P_{n-2}}{p_{n-1}}, \quad \tau_{n,k}^{-1} = 0, \quad k \le n-2.$$

Hence,

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx =$$

$$= \sum_{k=0}^{n-2} \left[P_k \left(\frac{a_k}{p_k} - \frac{a_{k+1}}{p_k} \right) - P_k \left(\frac{a_{k+1}}{p_{k+1}} - \frac{a_{k+2}}{p_{k+1}} \right) \right] K_k(x) +$$

$$+ \frac{P_{n-1}}{p_{n-1}} (\Delta a_{n-1}) K_{n-1}(x) + \frac{P_n}{p_n} a_n K_n(x) - \frac{P_{n-1}}{p_n} a_n K_{n-1}(x) =$$

$$= \sum_{k=0}^{n-2} \left(P_k \Delta \frac{\Delta a_k}{p_k} \right) K_k(x) + \frac{P_{n-1}}{p_{n-1}} a_{n-1} K_{n-1}(x) +$$

$$+ \frac{a_n}{p_n} (P_n K_n(x) - P_{n-1} K_{n-1}(x)),$$

where

$$K_n(x) = \frac{1}{2} + \frac{1}{P_n}\sum_{k=1}^{n} P_{n-k}\cos kx = \frac{1}{P_n}\sum_{k=0}^{n} p_k D_k(x).$$

Since

$$P_nK_n(x) - P_{n-1}K_{n-1}(x) = P_n\sum_{k=0}^{n}\frac{p_k}{P_n}D_k(x) - P_{n-1}\sum_{k=0}^{n-1}\frac{p_k}{P_{n-1}}D_k(x) =$$
$$= p_nD_n(x),$$

then

$$S_n = \sum_{k=0}^{n-2}\left(P_k\Delta\frac{\Delta a_k}{p_k}\right)K_k(x) + \frac{P_{n-1}}{p_{n-1}}\Delta a_{n-1}K_{n-1}(x) + a_nD_n(x).$$

By (2) and (4) series

$$\sum_{k=0}^{\infty} P_k\Delta\frac{\Delta a_k}{p_k}K_k(x)$$

converges to an integrable function $f(x)$ for $0 < x \leq \pi$. We have

$$\int_0^\pi \left|\sum_{k=n+1}^{\infty} a_k\cos kx\right| dx = \int_0^\pi |S_n - f(x)|\, dx \leq$$
$$\leq \sum_{k=n-1}^{\infty}\left|P_k\Delta\frac{\Delta a_k}{p_k}\right|\int_0^\pi |K_k(x)|\, dx + \left|\frac{P_{n-1}}{p_{n-1}}\Delta a_{n-1}\right|\int_0^\pi |K_{n-1}(x)|\, dx +$$
$$+ |a_n|\int_0^\pi |D_n(x)|\, dx$$

and

$$|a_n|\int_0^\pi |D_n(x)|\, dx \leq \int_0^\pi |S_n - f(x)|\, dx +$$
$$+ \sum_{k=n-1}^{\infty}\left|P_k\Delta\frac{\Delta a_k}{p_k}\right|\int_0^\pi |K_k(x)|\, dx + \left|\frac{P_{n-1}}{p_{n-1}}\Delta a_{n-1}\right| . \int_0^\pi |K_{n-1}(x)|\, dx.$$

By (2) and (4) hence $S_n \to f(x)$ in L^1-norm as $n \to \infty$, if and only if

$$a_n\int_0^\pi |D_n(x)|\, dx \longrightarrow 0,$$

it is if and only if $a_n \ln n \to 0$, as $n \to \infty$.

If $(a_k) \in (m_T, c_T)$, then cosine series is a Fourier series investigated by G. Goes [3] and the author [6].

References

1. Baron, S., *Introduction to the theory of summability of series.* "Valgus", Tallinn, 1977 (in Russian).
2. Bohr, H., *Sur la serie de Dirichlet.* C.R.Acad.Sci., 1909, **148**, 75–80.
3. Goes, G., *Carakterisierung von Fourierkoeffizienten mit einem Summierbarkeitsfaktorentheorem und multiplikatoren.* Studia Math., 1960, **19**, 133–148.
4. Kangro, G., *Summability factors.* Tartu Ülik. Toimetised, 1955, **37**, 191–232 (in Russian).
5. Kolmogorov, A.N., *Sur l'ordre de grandeur des coefficients de la sèrie de Fourier-Lebesgue.* Bull. Internat. Acad. Polon. Sci. Letters Ser. (A) Sci. Math., **1923**, 83–86.
6. Tynnov, M., *Summability factors, Fourier coefficients and multipliers.* Tartu Ülik. Toimetised, 1966, **192**, 82–87 (in Russian).
7. Zygmund, A., *Trigonometric Series,* vol.I. Moscow, 1965 (in Russian).

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Koosinusridade integreeruvusest ja L^1-koonduvusest

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Resümee

A. Kolmogorov [5] tõestas 1923.a., et kui koosinusrea kordajad moodustavad kvaasikumera nulljada, siis see koosinusrida on Fourier rida ja tema osasummade jada koondub L^1-normi järgi parajasti siis, kui

$a_k \ln k \longrightarrow 0$. Kolmogorovi teoreemi eeldus tähendab aga seda, et kordajate jada on klassist (3), kui $\alpha = 1$, see on summeeruvustegurite klass Cesàro menetluse C^1 korral.

Käesolevas artiklis tõestatakse, et Kolmogorovi teoreemis võib eelduse asendada kordajate jada kuulumisega klassi (3) või (4), see tähendab kordajate jada kuulumisega summeeruvustegurite klassi α–järku ($\alpha > 0$) Cesàro menetluse suhtes või summeeruvustegurite klassi Rieszi menetluse suhtes. Väite esimese poole, et nimetatud eeldusel on tegemist Fourier reaga, on tõestanud Cesàro menetluse C^α ($\alpha > 0$) korral G. Goes [3] ja üldjuhul autor [6].

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Open problems and some results on strongly closed subalgebras of $B(X)$

W. Żelazko [1] [2]

We give here a motivation for the study of strongly closed subalgebras of $B(X)$, announce some results, and pose several open questions.

Let X be a real or complex Banach space and let $B(X)$ denote the algebra of all its bounded endomorphisms. The strong operator topology on $B(X)$ is the topology of pointwise convergence of nets of operators. A basis of (open) neighbourhoods of the origin for this topology is given by the sets $U(\varepsilon; x_1, \dots, x_n) = \{T \in B(X) : \|Tx_i\| < \varepsilon, i = 1, \dots, n\}$, where ε is a positive number and $x_1, \dots, x_n$ are linearly independent elements of X. It is well-known that the closure in the strong operator topology of a subalgebra of $B(X)$ is again such a subalgebra, and that every strongly closed subalgebra of B(X) is also uniformly closed (closed in the norm topology). A subalgebra A of $B(X)$ is said to be a maximal strongly closed algebra (m.s.c.a.) if it is a strongly closed proper subalgebra of $B(X)$, and for any subalgebra A_1 of $B(X)$ satisfying $A \subset A_1 \subset B(X)$ we have either $A = A_1$, or A_1 is strongly dense in $B(X)$.

Let X_o be a closed linear subspace of X satisfying $(0) \neq X_o \neq X$

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[1]) The author's presentation given at the conference devoted to the 80th anniversary of Professor Gunnar Kangro (Tartu, 20-22 November, 1993)

[2]) supported by the KBN grant No 2.2007 92 03

and put

$$\mathcal{A}(X_o) = \{T \in B(X) : TX_o \subset X_o\}. \tag{1}$$

It is easy to see, that $\mathcal{A}(X_o)$ is a strongly closed proper subalgebra of $B(X)$.

Proposition 1. *Every algebra of the form* (1) *is an m.s.c.a.*

Proposition 2. *If the dimension* $\dim X_o$ *of* X_o *is finite, then the algebra* $\mathcal{A}(X_o)$ *is maximal in* $B(X)$ *in the sense that if* A_1 *is a subalgebra of* $B(X)$ *satisfying* $\mathcal{A}(X_o) \subset A_1 \subset B(X)$, *then either* $A_1 = \mathcal{A}(X_o)$ *or* $A_1 = B(X)$.

Proposition 3. *If the codimension* $\operatorname{codim} X_o$ *is finite, then the conclusion of Proposition 2 also holds true.*

A partial converse of the Propositions 2 and 3 is the following

Proposition 4. *Let* H *be an infinite dimensional separable Hilbert space and let* H_o *be a closed subspace of* H *with* $\dim H_o = \operatorname{codim} H_o = \infty$. *Then there exists a proper uniformly closed subalgebra* A *of* $B(H)$ *satisfying*

$$\mathcal{A}(H_o) \subset A \subset B(H) \text{ with } \mathcal{A}(H_o) \neq A \neq B(H).$$

The proofs of the above results and of the following Proposition 5 will appear elsewhere.

As a motivation for the study of strongly closed algebras we mention the following well-known problems.

I. The Problem of Fell and Doran ([1], p. 321, Problem II, see also [5]).

Let X be a topological vector space, $L(X)$ — the algebra of all its continuous endomorphisms, and A — an algebra over the same field of scalars as X. A representation T of A on X is a homomorphism $a \to T_a$ of A into $L(X)$. We assume that if A has the unity e, then $T_e = I$ — the identity operator on X. A representation T is said to be irreducible, if there is no proper closed subspace X_o of X which is invariant with respect to all operators T_a of the representation T or, equivalently, if each orbit

$$\mathcal{O}(T;x_o) = \{T_a x_o : a \in A\}, \quad 0 \neq x_o \in X$$

is dense in X (one can easily see that the closure of the above orbit is an invariant subspace for all operators of the representation T). Similarly, we call T to be n-fold irreducible ($n \in \mathbf{N}$) if for any n -tuple $x_1, \ldots, x_n$ of linearly independent elements of X the orbit $\mathcal{O}(T;x_1,\ldots,x_n) = \{(T_a x_1, \ldots, T_a x_n) \in X^n : a \in A\}$ is dense in X^n provided with the Cartesian product topology. A representation T is said to be totally irreducible, if it is n -fold irreducible for all n . The Problem of Fell and Doran reads as follows. Let X be a complex locally convex space and suppose that T is an irreducible representation on X of a complex algebra A, such that the commutant $T' = \{S \in L(X) : ST_a = T_a S, \forall a \in A\}$ consists only of scalar multiples of the identity operator. Does it follow that T is totally irreducible? The Problem makes sense for arbitrary topological vector spaces and also for the real spaces and algebras. The Problem is open even for Hilbert spaces. If T is a totally irreducible representation on a Banach space X, then obviously the algebra $\{T_a \in B(X) : a \in A\}$ is strongly dense in $B(X)$. Thus, if we are looking for a counterexample on a Banach space, we must find there a strongly closed proper subalgebra of $B(X)$ with a trivial commutant and with no proper closed subspace which is invariant with respect to all operators in the algebra in question. It is believed, that such a counterexample should exist.

II. The Transitive Algebra Problem (see [3], Chapter 8).

An algebra A of operators on a vector space X is said to be algebraically transitive if for each non-zero element x in A and each y in A there is an operator T in A with $Tx = y$. If X is a topological vector space, then A is said to be transitive, if for each non zero element x in X , each y in X and each neighbourhood U of y there is a T in A with $Tx \in U$. Thus A is transitive on a t.v.s. X if all orbits $\mathcal{O}(A;x_o) = \{Tx_o : T \in A\}$, $x_o \neq 0$ are dense in X or, equivalently, if there is no proper closed subspace X_o of x which is invariant for all operators T in A . The Transitive Algebra Problem is the question whether for a complex Hilbert space H a strongly closed transitive subalgebra of $B(H)$ must coincide with $B(H)$ .If $\dim H < \infty$, then the positive solution of this problem follows from the classical Burnside theorem (see [2],p.276). Again, in order to solve this problem, we have to study strongly closed subalgebras of $B(H)$.

Let X be a Banach space, and call a subspace $\mathcal{M}$ of the Cartesian product X^n to be in a general position if it contains a point with linearly independent coordinates. Let $\mathcal{M}$ be a closed subspace of X^n and put

$$\mathcal{A}(\mathcal{M}) = \{T \in B(X) : (x_1, \ldots, x_n) \in \mathcal{M} \Rightarrow (Tx_1, \ldots, Tx_n) \in \mathcal{M}\}.$$

It is easy to see, that $\mathcal{A}(\mathcal{M})$ is a strongly closed subalgebra of $B(X)$, in order to have it differ from $B(X)$, we must assume that X is in a general position.We have the following

Proposition 5. *Let X be a Banach space and let A be a proper strongly closed subalgebra of $B(X)$. Then there is a natural number n and a closed subspace $\mathcal{M}$) of X^n which is in a general position, such that*

$$A \subset \mathcal{A}(\mathcal{M}).$$

Corollary. *Every m.s.c.a. is of the form $\mathcal{A}(\mathcal{M})$.*

Let A be a proper strongly closed subalgebra of $B(X)$. We say that A is of order n if $A \subset \mathcal{A}(\mathcal{M})$ for some subspace $\mathcal{M}$ of X^n , which is in a general position, and $A \not\subset \mathcal{N}$ for any such a subspace $\mathcal{N}$ of X^k with $k < n$. Thus A is of order 1 if and only if there is a proper closed subspace $X_0 \subset X$ which is invariant with respect to all operators T in A . Proposition 5 implies that every proper strongly closed subalgebra of $B(X)$ has some positive order.Proposition 1 implies that every algebra of order 1 is contained in some m.s.c.a. of order 1.

Problem 1. *Let A be a proper strongly closed subalgebra of $B(X)$, is it contained in some m.s.c.a. (or in some m.s.c.a. of the same order) ?*

If we had a positive answer for this problem, we could have an m.s.c.a. of order 2 on an infinite dimensional Banach space (the author knows only one example of an m.s.c.a. of order 2 in $B(R^4)$). To this end we should take the commutant of an operator T without a closed invariant subspace (see [4]), it is a strongly closed algebra of order 2 (it equals to $\mathcal{A}(\mathcal{M})$, where $\mathcal{M}$ is the graph of T).

Problem 2. *Does there exist a Banach space X such that $B(X)$ has subalgebras of arbitrarily high orders ?*

A weaker question is

Problem 3. *Does there exist a Banach space X for each natural number n such that $B(X)$ has a subalgebra of order n ?*

A still weaker question is

Problem 4. *Does there exist a Banach space X such that $B(X)$ has a subalgebra of order 3 ?*

A positive answer to this question solves in negative the Problem of Fell and Doran. In fact, if A is not of order 1, then there is no proper closed subspace of X which is invariant with repect to all operators in A , so that the identity map of A onto itself is an irreducible representation. Since A is not of order 2, algebra A has a trivial commutant (if T is an operator in the commutant of A and T is not a scalar multiple of the identity, then the graph of T is a subspace of X^2 which is in a general position and A is contained in $\mathcal{A}(\mathcal{M})$, where $\mathcal{M}$ is the graph of T). On the other hand, A is not strongly dense in $B(X)$, so that the representation in question is not totally irreducible.

References

1. Fell, J.M.G. and Doran, R.S., *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles.* Pure Appl. Math.125 and 126, Academic Press, 1988.
2. Jacobson, N., *Lectures in abstract algebra,* vol.II. Van Nostrand, 1953.

3. Radjavi, H. and Rosenthal, P., *Invariant subspaces.* Springer Verlag, 1973.
4. Read, C.J., *A solution to the invariant space problem on the space* l_1. Bull. London Math. Soc., 1985, **17**, 305–317.
5. Żelazko, W., *On the problem of Fell and Doran.* Coll. Math., 1991, **62**, 31–37.

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