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**Two-Asset Option Pricing**  
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# TWO-ASSET OPTION PRICING

Master's thesis

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## Abstract

This thesis delves into the pricing of four distinct two-asset options: basket options, correlation options, spread options and options on the minimum and maximum of two assets. Through numerical analysis, three pricing models are employed: A lattice model with five branches which involves an extension of the lattice binomial method by Cox, Ross, and Rubinstein to value options with only one asset, A Modification of the Binomial method and the Adaptive Binomial Lattice Method for time interval  $[T - \Delta t, T]$  with refinement level 1. These models are used to price both European and American two-asset options. The accuracy of the methods are shown by valuing the two-asset options and comparing with the exact prices or prices gotten from relevant papers. In the numerical experiments, all models perform quite well, but the adaptive binomial lattice method has better accuracy than other methods.

**CERCS research specialisation:** P160 Statistics, operations research, programming, financial and actuarial mathematics.

**Key Words:** Two-Asset Options, Binomial Model, Adaptive Binomial Lattice Model, Black-Scholes Model

# KAHE VARAOPTSIOONI HINNAKUJUNDUS

Magistritöö

Lilian Nnamdi

## Lühikokkuvõte

Käesolev magistritöö uurib nelja erineva kahe varaga optsiooni hinnakujundust: korvoptsioonid, korrelatsioonoptsioonid, hajutusoptsioonid ja optsioonid kahe vara miinimumi ja maksimumi kohta. Numbrilise analüüsi abil kasutatakse kolme hinnastamismudelit: viie haruga ruudustikumudel, milles laiendatakse ruudustiku binomiaalmeetodi kaudu hinna määramist ühe varaga optsioonidele, binomiaalmeetodi muudatus ning kohanduv ruudustiku binomiaalmeetodit ajavahemiku  $[T - \Delta t, T]$  jaoks, mille rafineerimistasand on 1. Neid mudeleid kasutatakse nii Euroopa kui ka Ameerika stiilis kahe vara optsioonide hinnastamiseks. Meetodite täpsust näidatakse kahe vara optsioonide väärtustamisega ja võrdlemisega täpsete hindade või vastavate artiklitest saadud hindadega. Numbrilistes katsetes toimivad kõik mudelid üsna hästi, kuid kohanduval ruudustiku binomiaalmeetodil on suurem täpsus kui teistel meetoditel.

**CERCS teaduseriala:** P160 Statistika, operatsioonianalüüs, programmeerimine, finants- ja kindlustusmatemaatika.

**Märksõnad:** Kahe vara optsioonid, binomiaalne mudel, kohanduv ruudustikuline binomiaalne mudel, Black-Scholesi mudel.

# Contents

<b>Introduction</b>	<b>4</b>
<b>1 Introduction to Option Theory</b>	<b>6</b>
1.1 Options and Option Price . . . . .	6
1.2 Behavior of the Stock Price . . . . .	9
1.3 Black-Scholes Model . . . . .	10
1.4 Binomial Model . . . . .	13
1.5 Trinomial Model . . . . .	16
<b>2 Multi Asset Options</b>	<b>20</b>
2.1 Types of Multi Asset Option Pricing . . . . .	20
2.1.1 Basket Options . . . . .	20
2.1.2 Spread Options . . . . .	21
2.1.3 Rainbow Options . . . . .	22
2.1.4 Correlation Options . . . . .	23
2.2 A Lattice Model with Five Branches for Pricing Two Assets . . . . .	24
2.3 A Modification of the Binomial Method for Two Assets . . . . .	32
2.4 Adaptive Binomial Lattice Method for Pricing Two Asset Options . . . . .	34
<b>3 Numerical Experiments</b>	<b>39</b>
3.1 Results of Options on the Maximum and Minimum of Two Assets . . . . .	39
3.2 Result of Spread Options . . . . .	41
3.3 Result of Correlation Options . . . . .	41
3.4 Result of Basket Options . . . . .	42
<b>Conclusions</b>	<b>44</b>
<b>References</b>	<b>46</b>

# Introduction

The realm of financial markets is a complex web of instruments and strategies that facilitate the management of risk and the pursuit of profit. Within this landscape, Options play a key role, offering investors the ability to speculate on the future movement of underlying assets or hedge against adverse market fluctuations. The pricing of options has been a subject of significant interest and research, driven by the intricate interplay between financial instruments, market dynamics, and mathematical modeling. The pricing of these options poses challenging issues since they may involve a complex bundle of embedded options whose payoff depend on multiple assets.

In this thesis, our focus revolves around the pricing intricacies of four distinct two-asset options: basket options, correlation options, spread options, and options on the minimum and maximum values of two assets. By addressing these diverse options, we endeavor to unravel the complexities underlying their valuation, bridging the gap between theoretical models and practical applications. Our goal is to contribute to the existing body of knowledge by presenting a systematic exploration of pricing methodologies and showcasing their effectiveness through numerical experimentation.

To achieve our objectives, we employ three pricing models as analytical tools throughout this study. The foundation is laid with a lattice-based approach, characterized by a five-branch binomial model. Building upon this, we introduce a modification to the binomial method, tailored specifically for pricing options involving two assets. Additionally, we harness the power of the Adaptive Binomial Lattice Method, a dynamic framework that enhances accuracy.

The thesis unfolds in a structured manner, commencing with an overview of Option Theory, where we establish the groundwork for our exploration. This is followed by an introduction to multi-asset options, explaining the various types of

options under consideration. Subsequently, we delve into the binomial lattice-based model with five branches and its application to the pricing of two-asset options. A modified binomial method is introduced as a strategic enhancement, leading us to the Adaptive Binomial Lattice Method.

Chapter 1 introduces the fundamental concepts of Option Theory, highlighting the role of options and the dynamics of stock prices. The Black-Scholes Model and Binomial Model are discussed to provide essential theoretical underpinnings. Chapter 2 delves into multi-asset options, explaining basket options, spread options, rainbow options, and correlation options. The three pricing models are introduced, setting the stage for their application in subsequent chapters.

Chapter 3 embarks on numerical experiments, presenting the results of our pricing methodologies across different categories of two-asset options. The accuracy and efficiency of the models are evaluated, and their performance is assessed against exact prices and relevant literature.

# 1 Introduction to Option Theory

Options are a type of derivative that gives the option buyer the right to buy or sell an underlying asset at an exercise price within a specific period (expiration date). The underlying asset can be a commodity, stock, currency or other financial instruments. The option buyer has the right to choose whether to exercise his right to buy or sell the option or not, but the option seller does not have the right to do same.

It is important to emphasize that an option buyer is not required to take any action and could allow the option to expire worthless. This is a key feature that makes options appealing. The option buyer can profit from a rise in the price of the underlying asset, but is not harmed by a fall in the price. The option seller must follow the option buyer's decision no matter what it is. The option seller carries the risk of the possibility of losing the premium paid and also the potential for unlimited losses in some cases [6].

## 1.1 Options and Option Price

A call option is a contract that grants the option buyer the right to buy the underlying asset at an exercise price on or before the expiration date. This means that if the price of the underlying asset goes up, the option buyer can exercise the call option and buy the asset at the lower exercise price, then sell it at the higher stock price to make a profit.

The opposite of a call option is a put option. A put option is a contract that grants the option buyer the right to sell the underlying asset at an exercise price on or before the expiration date. This means that if the price of the underlying asset goes down, the option buyer can exercise the put option and sell the asset at the higher exercise price, then buy it back at the lower stock price to make a profit.

The option buyer will be required to pay the option seller a premium in return for gaining this privilege. Call and put options can be used for various purposes such as hedging, speculation and generating income.

### **European Options versus American Options**

European and American options are two types of options based on when the option can be exercised. In contrast to the American option, which can be exercised at any time during the life of the option up to and including the expiration date, the European option only allows the option to be exercised at the expiration date. The premium for European options is much lower because the option buyer can only exercise it on the expiration date while the American option is more expensive because of its greater flexibility.

American options are in high demand and are therefore traded more frequently than their European counterparts, they also make up the majority of the options market. American options can be more profitable than European options when there are favorable price movements in the underlying asset before the expiration date, and the option buyer can benefit from exercising the option earlier. However, it is important to consider the cost of the option premium and the risks associated with options trading before making any investment decisions.

For example, if an investor holds an American call option on a stock with an exercise price of \$50, and the stock price rises to \$60 before the expiration date, the investor can exercise the American option and buy the stock at the lower exercise price of \$50, then sell it immediately in the market for \$60, realizing a profit of \$10 per share. With a European call option, the investor would have to wait until the expiration date to realize the profit.

## Payoff of Call Options

Generally, it is clear that the value of the option increases as the stock price goes higher. The option is virtually certainly going to be exercised when the stock price exceeds the exercise price. On the other hand, the option's value will be close to zero if the stock price is below the exercise price because it is very certain that the option will expire without being used.

If the option's expiration date is very far away, the value of the option will be about equal to the price of the stock since the price of an asset that pays the exercise price on the day of maturity will be relatively low. On the other hand, if the option's expiration date is relatively close, its value will be roughly equal to the stock price less the exercise price, or zero if the stock price is lower than the exercise price. If the stock price stays the same, an option's value typically decreases as its expiration date draws near [1]. The payoff of a European call option can be written mathematically as:

$$P_c = \max\{S_T - E, 0\},$$

where  $P_c$  is the call option payoff,  $S_T$  is the stock price with expiration date  $T$ , and  $E$  is the exercise price. This means that if  $S_T$  is greater than  $E$ , the call option payoff equals  $S_T - E$ ; else, it is zero [5]. The payoff of an American call option is

$$P_c = \max\{S_t - E, 0\}, \text{ if expiration time is } t.$$

## Payoff of Put Options

The buyer of a put option has the right, but not the obligation, to sell an underlying asset at an exercise price. The payoff of a European put option may be determined using the same reasoning as a European call option, and it can be expressed as

follows:

$$P_p = \max\{E - S_T, 0\},$$

where  $P_p$  is the put option payoff,  $S_T$  is the stock price with expiration date  $T$ , and  $E$  is the exercise price.

## 1.2 Behavior of the Stock Price

It is important to note that stock prices can be volatile and may fluctuate rapidly in response to external factors. The Geometric Brownian Motion (GBM) model, often known as the Wiener process, is a commonly used method for describing stock price behavior. It is based on the assumption that the logarithmic returns of a stock price follow a Brownian motion process, which is a stochastic process that describes the random movements of particles in a fluid [3].

Typically, researchers are more concerned in modelling returns than price alone. A return is a percentage that reflects a price change. As returns are scale-free, manipulating them is simpler than doing so with actual prices [4]. The GBM model is typically represented by the following equation:

$$d(\log(S_t)) = \mu dt + \sigma dW_t,$$

where  $S_t$  is the stock price at time  $t$ ,  $\mu$  is the drift,  $\sigma$  is the volatility of the underlying asset,  $t$  is the time, and  $W_t$  is a Wiener process or Brownian motion process, which represents the random fluctuations in the stock price over time. We can refer to a Wiener process as a stochastic process if these conditions are fulfilled:

- $W_0 = 0$ ;

- For  $0 \leq t_1 < t_2 \leq t_3 < t_4$ , the random variables increments,  $W_{t_2} - W_{t_1}$  and  $W_{t_4} - W_{t_3}$  are independent. It also has stationary increments;
- $W_t$  is continuous in  $t$  with a probability of 1;
- For  $t > 0$  and  $\Delta t > 0$ , the increment  $W_{t+\Delta t} - W_t \sim N(0, \Delta t)$ . This means that the increments of  $W$  have normal distribution with zero mean and variance  $\Delta t$ .

Using this equation, it is possible to simulate the future price movements of a stock over a given time period. By estimating the values of  $\mu$  and  $\sigma$  based on historical data, we can then use the GBM model to model the behavior of the stock price. It's important to carefully evaluate the assumptions and limitations of the GBM model when using it to make decisions [4].

Using Ito's formula gives:

$$dS_t = (\mu - \frac{1}{2}\sigma^2)S_t dt + \sigma S_t dW_t.$$

### 1.3 Black-Scholes Model

The paper [1] will be heavily referenced in this section. In order to determine the theoretical value of derivative financial instruments, one of the most widely used mathematical model is the Black-Scholes Model, which was created in 1973 by Fischer Black and Myron Scholes. The Black-Scholes model is used to price options and it is based on the assumption that the underlying asset follows a geometric Brownian motion. The model is widely used in the financial industry for option pricing and risk management. The key assumptions of the Black-Scholes model are:

- The risk-free interest rate and variance are constants or functions of time  $t$

and known at time  $t = 0$ .

- The stock price follows a continuous log process with equation  $d(\log(S_t)) = \mu dt + \sigma dW_t$ .
- No dividend payments are made.
- No transaction costs are incurred during the sale or purchase of the stock or the option.
- Any portion of the cost of a security may be borrowed in order to purchase or hold it at the risk-free interest rate.
- Short selling has no consequences. Short selling is used to profit from a decline in the price of an asset or security. In short selling, an investor borrows an asset (usually a stock or security) from a broker or another investor and sells it in the open market at the current market price. The investor aims to buy back the same asset later at a lower price and return it to the lender, thus making a profit from the difference between the initial selling price and the subsequent lower buying price.
- No arbitrage condition. A no-arbitrage condition refers to a state in financial markets where it is not possible to make risk-free profits by exploiting price discrepancies or inefficiencies in the market.

The Black-Scholes model has some limitations, for example, it assumes deterministic volatility, no dividends, and no transaction costs, which might not hold true in real-world scenarios. Additionally, it does not account for extreme market events observed in some financial markets, which can lead to inaccurate option pricing during highly volatile periods. To address some of the limitations of the original Black-Scholes model, various extensions and variations have been developed.

Assuming we have an option  $V = V(S_t, t)$  and its value depends on  $S_t$  and  $t$ , we can think of  $V$  as a simple call or put. Using Ito's formula, the Black-Scholes partial differential equation (PDE) is given as:

$$\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + r S_t \frac{\partial V}{\partial S_t} - rV + \frac{\partial V}{\partial t} = 0.$$

This PDE can have infinitely many solutions. To get an exact solution, we know that the option price is equal to the payoff at time  $T$ . In case of a European call option  $V_c = \max\{S_T - E, 0\}$  and  $V_p = \max\{E - S_T, 0\}$  for a European put option. If volatility  $\sigma$  and risk-free interest rate  $r$  are constants, then:

$$V_c(S_t, t) = S_t \mathcal{N}(d_1) - E \exp(-r(T-t)) \mathcal{N}(d_2),$$

$$d_1 = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$V_p(S_t, t) = E \exp(-r(T-t)) \mathcal{N}(-d_2) - S_t \mathcal{N}(-d_1),$$

where:

$V_c$  is the call option's price

$V_p$  is the put option's price

$S_t$  is the current stock price of the underlying asset at time  $t$

$\mathcal{N}()$  is the standard normal distribution function

## 1.4 Binomial Model

William Sharpe first put forth the Binomial Model in 1978, and it wasn't until Cox, Ross, and Rubinstein modified it the following year that it became the model that is currently extensively employed for pricing options. It provides a discrete approximation to option pricing. The Binomial model, as opposed to other models, examines the underlying price over a lengthy period of time as opposed to concentrating on a single moment in time.

The binomial model is based on the assumption that the price of the underlying asset can follow only two possible paths in each time period; up with value  $uS$  or down with value  $dS$ . The probability of an up move is represented by  $p$ , while the probability of a down move is represented by  $1 - p$ . The time period starts at time  $t = 0$  and ends at  $t = T$ . The expiration time  $T$  is divided by the number of time steps  $n$  to get the time steps  $\Delta t$ , such that  $\Delta t = T/n$  [15].

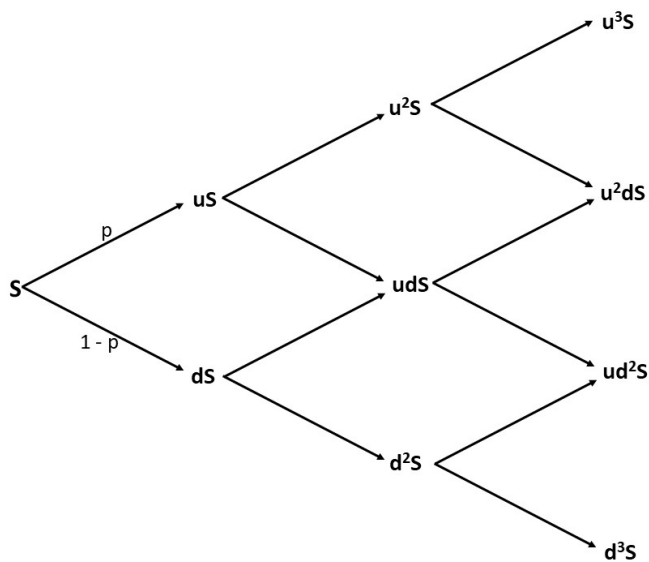


Figure 1: A Binomial Model with 3 time periods

The model is considered arbitrage-free if a risk-neutral probability measure  $Q$  can be established, ensuring that the expected return on the risky asset equals the

risk-free interest rate  $r$ . This condition eliminates the possibility of arbitrage by aligning the risky asset's return with that of a risk-free asset. As a result, the model's pricing of assets and derivatives becomes fair and consistent, adhering to the principles of an efficient and rational market. Also, for the model to be arbitrage free, the following condition  $u > r > d$  must be fulfilled. This means that there can never be an instance where the rate of change in the stock price due to an up movement  $u$  is less than the risk-free interest rate  $r$ , and vice versa for a down movement  $d$  [2]. Therefore,

$$E_Q\left[\frac{S(t + \Delta t)}{S(t)} \middle| S(t)\right] = \exp(r\Delta t),$$

Since  $E_Q\left[\frac{S(t + \Delta t)}{S(t)} \middle| S(t)\right] = \frac{E_Q[S(t + \Delta t) | S(t)]}{S(t)}$  and

$$E_Q[S(t + \Delta t) | S(t)] = (pu + (1 - p)d)S(t), \quad (1)$$

consequently,

$$pu + (1 - p)d = \exp(r\Delta t). \quad (2)$$

Using the equation above,  $p$  can be expressed as:

$$p = \frac{\exp(r\Delta t) - d}{u - d}.$$

Also, we need to ensure that the variance of the rate of return of the binomial model aligns with the historical rate of return of the risky asset.

$$\text{Var}_Q\left[\frac{S(t + \Delta t) - S(t)}{S(t)} \middle| S(t)\right] = \sigma^2 \Delta t + \mathcal{O}(\Delta t), \quad (3)$$

$\mathcal{O}(\Delta t)$  signifies that as  $\Delta t \rightarrow 0$  then  $\frac{\mathcal{O}(\Delta t)}{\Delta t} \rightarrow 0$ .

Using the formula for variance  $Var(x) = Ex^2 - (Ex)^2$  and equations (1) and (2) gives

$$Var_Q\left[\frac{S(t + \Delta t) - S(t)}{S(t)} \mid S(t)\right] = \frac{1}{S(t)^2} Var_Q[S(t + \Delta t) \mid S(t)],$$

$$Var_Q[S(t + \Delta t) \mid S(t)] = pu^2 S(t)^2 + (1 - p)d^2 S(t)^2 - (puS(t) + (1 - p)dS(t))^2.$$

From the two equations above and equation (3),

$$pu^2 + (1 - p)d^2 = \exp(2r\Delta t) + \sigma^2\Delta t + \mathcal{O}(\Delta t) \quad (4)$$

is gotten and it can be used to find  $u$  and  $d$ . As there is only one equation with two unknowns, it won't be simple to find  $u$  and  $d$ . One option is to take  $d = 1/u$ . It is possible to show that equation (4) is satisfied by:

$$u = \exp(\sigma\sqrt{\Delta t}).$$

We can find the value of the option at time  $t = 0$  and expiration date  $T$ . At time  $T$ , the binomial tree has the asset values  $S_{n,i} = S_0 u^i d^{n-i}$ ,  $i = 0, 1, \dots, n$ . Then the possible values at  $t = T$  with payoff  $P$  is

$$P_{n,i} = P(S_0 u^i d^{n-i}).$$

Since the option price  $V$  equals the payoff  $P$  at time  $T$ , we get

$$V_{n,i} = P_{n,i}, \quad i = 0, 1, \dots, n$$

Now, we can compute recurrently to get

$$V_{j,i} = \exp(-r\Delta t)(pV_{j+1,i+1} + (1 - p)V_{j+1,i}), \quad i = 0, 1, \dots, j, \quad j = n - 1, n - 2, \dots, 0$$

This represents the European options. At time  $t = 0$ , the price of the option is  $V_{0,0}$ . For American options, we use almost the same calculations as for its European counterpart but only one equation is changed as seen below:

$$V_{j,i} = \exp(-r\Delta t) \max\{pV_{j+1,i+1} + (1-p)V_{j+1,i}, P_{j,i}\}.$$

## 1.5 Trinomial Model

In the binomial model, we use a two jump process but there's also a possibility of a three jump process, this was introduced by Boyle (1986). The trinomial model improves upon the binomial model by allowing the price of the asset  $S$  to move up ( $uS$ ), down ( $dS$ ) or horizontally ( $S$ ) after a time period  $\Delta t$ . We can price both European and American options using the trinomial method. Also, because the trinomial model assumes movements in 3 directions, the number of time steps can be decreased and similar results compared to the Binomial model will still be gotten. One advantage this method has is its flexibility which comes in handy when pricing complex options.

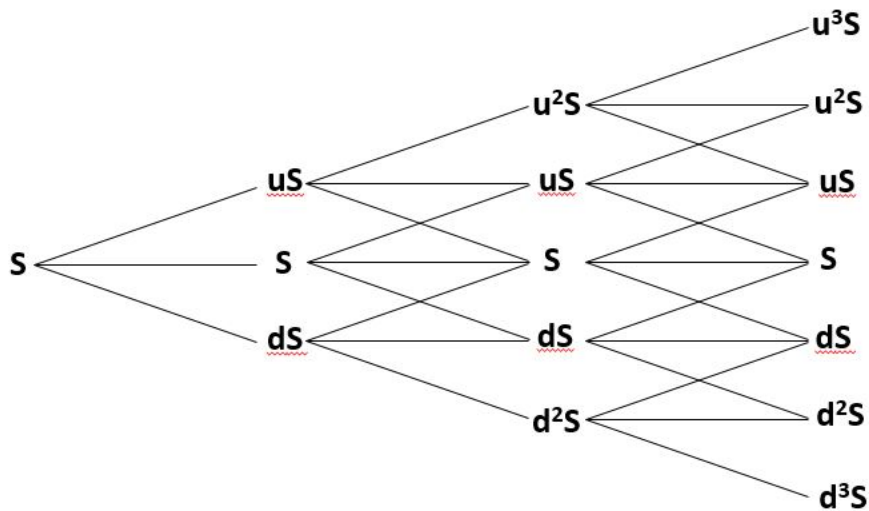


Figure 2: Trinomial Model with 3 time periods

Taking

$$S = \begin{cases} uS, & \text{with probability } p_u \\ S, & \text{with probability } 1 - p_u - p_d \\ dS, & \text{with probability } p_d. \end{cases}$$

Based on the no arbitrage condition, the mean and variance of the models distribution is as follows [13]:

$$E_Q[S(t + \Delta t)|S(t)] = \exp(r\Delta t)S(t),$$

$$Var_Q[S(t + \Delta t)|S(t)] = \Delta t S(t)^2 \sigma^2 + \mathcal{O}(\Delta t).$$

Here, we have made the assumption that the volatility  $\sigma$  is constant and the asset price  $S$  exhibits Geometric Brownian Motion. Also, the size of the upward move should be the inverse of the size of the downward move,

$$ud = 1.$$

For trinomial models, the general representation of its jump sizes are:

$$u = \exp(\sigma\sqrt{2\Delta t}), \quad d = 1/u,$$

while the general representation of its transition probabilities are given by:

$$p_u = \left( \frac{\exp\left(\frac{r\Delta t}{2}\right) - \exp\left(-\sigma\sqrt{\frac{\Delta t}{2}}\right)}{\exp\left(\sigma\sqrt{\frac{\Delta t}{2}}\right) - \exp\left(-\sigma\sqrt{\frac{\Delta t}{2}}\right)} \right)^2,$$

$$p_d = \left( \frac{\exp(\sigma\sqrt{\frac{\Delta t}{2}}) - \exp(\frac{r\Delta t}{2})}{\exp(\sigma\sqrt{\frac{\Delta t}{2}}) - \exp(-\sigma\sqrt{\frac{\Delta t}{2}})} \right)^2,$$

$$p_m = 1 - p_u - p_d.$$

If we apply the recursive algorithm, the formula for the value of an European option using the trinomial model gives

$$V_t = \exp(-r\Delta t)(p_u V_u + p_m V_m + p_d V_d),$$

where  $V_t$  is the value of the option,  $p_u$  is the probability of an up move,  $V_u$  is the value of the option when it makes an up move. Also,  $p_m$ ,  $p_d$ ,  $V_m$  and  $V_d$  has a similar representation.  $r$  is the risk free interest rate and  $\Delta t$  is the length of one time step. This can also be written as:

$$V_{i,j} = \exp(-r\Delta t)(p_u V_{i+1,j+1} + p_m V_{i+1,j} + p_d V_{i+1,j-1}).$$

Using the lattice approach as proposed by Boyle [9], the up and down jump sizes and the probabilities are as follows:

$$u = \exp(\lambda\sigma\sqrt{\Delta t}),$$

where  $\lambda$  is greater than 1.

$$d = 1/u.$$

If we denote

$$M := \exp(r\Delta t),$$

$$D := M^2[\exp(\sigma^2\Delta t) - 1],$$

then, the probability of an up jump/move is:

$$p_u = \frac{(D + M^2 - M)u - (M - 1)}{(u - 1)(u^2 - 1)},$$

and the probability of a down jump/move is:

$$p_d = \frac{u^2(D + M^2 - M) - u^3(M - 1)}{(u - 1)(u^2 - 1)}.$$

All probabilities must sum up to 1, therefore

$$p_m = 1 - p_u - p_d.$$

The value of the option can be determined recursively in the conventional manner after creating the asset price tree.

## 2 Multi Asset Options

Multi-asset options are financial derivatives that provide the option buyer with the right, but not the obligation, to buy or sell a basket of underlying assets (or multiple assets) at an exercise price on or before the expiration date. These options can be used to hedge against the risk of price movements in multiple assets, or to speculate on the future price movements of a portfolio of assets.

Today, a lot of payoffs depend on the performance of numerous assets. The linkages between the assets become crucial when an option's value is based on the prices of several other assets. Correlations are used to define these connections between the underlying assets. The degree and direction of a linear relationship between many underlying factors are shown via correlation. Because of the inter-dependencies between the underlying assets, pricing the options may be difficult.

### 2.1 Types of Multi Asset Option Pricing

#### 2.1.1 Basket Options

A basket option is a type of financial derivative that gives the option buyer the right, but not the obligation, to buy or sell a basket of underlying assets at an exercise price, on or before the expiration date. It pays off the greater of zero or the difference between the exercise price and the average return of the  $n$  distinct assets in the basket at expiration. The underlying assets can be stocks, bonds, currencies, commodities, or other financial instruments [7].

The payoff formula for a European basket call option is:

$$P_{c,b} = \max\left\{\sum_{i=1}^n w_i S_{T,i} - E, 0\right\},$$

where  $P_{c,b}$  is the basket call option payoff,  $w_i$  is a vector of weights,  $S_{T,i}$  is the price of the  $i$ th underlying asset in the basket at the expiration date of the option,  $E$  is the exercise price, and  $n$  is the number of underlying assets in the basket. The payoff of a European basket put option  $P_{p,b}$  is:

$$P_{p,b} = \max\left\{E - \sum_{i=1}^n w_i S_{T,i}, 0\right\}.$$

The key benefit of a basket option is that using one is more affordable than using a similar portfolio of plain vanilla options for portfolio insurance. A basket option does, in fact, take into account the imperfect correlation between the assets in the basket, and in addition, transaction costs are reduced because an investor only needs to purchase one option rather than several.

### 2.1.2 Spread Options

A spread option is an option whose payoff is based on the price difference or spread between two or more underlying assets. If the exercise price  $E$  is zero, a spread option becomes an exchange option, which involves the exchange of one asset for the other. Spread options are very popular between two correlated commodities. For instance, the two commodities could be crude oil and gasoline, where the producers will often be exposed to the spread between both commodities. The payoff of a European call spread option with two assets is:

$$P_{c,s} = \max\{S_{T,1} - S_{T,2} - E, 0\},$$

For a European put spread call option with two assets, the payoff is

$$P_{p,s} = \max\{E - S_{T,1} + S_{T,2}, 0\}.$$

### 2.1.3 Rainbow Options

A rainbow option is a basket option that pays, in its most popular form, a non-equally weighted average of the basket's assets based on how well they have performed. A rainbow call, for example, on three assets with weights of 45%, 40%, and 15%, pays 45% of the best return of the underlying assets at maturity, 40% of the second-best return, and 15% of the third-best.

Rubinstein (1991), who emphasized that this choice was based on a mixture of many assets like a rainbow is a combination of various colors, gave the term '*rainbow*' its name. Rainbow can take many different shapes, but the objective behind it all is to have a payoff that depends on the assets, ranked by their performance at expiration.

Option on the minimum of two assets is a type of rainbow option where the option holder has the right to buy or sell the asset that has the lower value between two underlying assets at the exercise price. It is only exercised when the value of the underlying asset falls below the exercise price. Option on the maximum of two assets is a type of rainbow option where the holder has the right to buy or sell the asset that has the higher value between two underlying assets at the exercise price. It is only exercised when the value of the underlying asset rises above the exercise price [8].

The payoff formula for the European options is given below. For call options:

$$P_{c,min} = \max\{\min(S_{T,1}, S_{T,2}) - E, 0\},$$

$$P_{c,max} = \max\{\max(S_{T,1}, S_{T,2}) - E, 0\},$$

For put options:

$$P_{p,min} = \max\{E - \min(S_{T,1}, S_{T,2}), 0\},$$

$$P_{p,max} = \max\{E - \max(S_{T,1}, S_{T,2}), 0\},$$

where  $S_{T,1}$  and  $S_{T,2}$  are the stock prices of asset 1 and asset 2 respectively and  $E$  is the exercise price.

#### 2.1.4 Correlation Options

A correlation option of two assets is an option that gets its value from the correlation between two underlying assets. The price of a correlation option is based on the correlation coefficient, which measures the degree to which two assets move together. The payoff of the European call correlation option with exercise prices  $E_1$  and  $E_2$  and prices  $S_{T,1}$  and  $S_{T,2}$  is:

$$P_{c,c} = \begin{cases} \max\{S_{T,2} - E_2, 0\}, & \text{if } S_{T,1} > E_1, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding payoff for the European put correlation option is:

$$P_{p,c} = \begin{cases} \max\{E_2 - S_{T,2}, 0\}, & \text{if } S_{T,1} > E_1, \\ 0, & \text{otherwise.} \end{cases}$$

The payoff of the European call correlation option will pay  $\max\{S_{T,2} - E_2, 0\}$  if  $S_{T,1} > E_1$  and will pay 0 otherwise. However, the payoff of the European put correlation option will pay  $\max\{E_2 - S_{T,2}, 0\}$  if  $S_{T,1} > E_1$  and will pay 0 otherwise. The European and American two asset correlation option are quite similar, the difference is that the American has the advantage of early exercise while the European can only be exercised on expiration.

## 2.2 A Lattice Model with Five Branches for Pricing Two Assets

The paper [9] is heavily referenced in this section. The section focuses on creating a formula for determining the value of options when there are two underlying assets involved. Here, the calculation of the probabilities of jumps in the model's lattice, as well as how to choose the size of these jumps will be explained. To differentiate between the two assets, the label '1' will refer to the first asset and '2' will refer to the second asset. The current value of asset 1 will be  $S_1$  while the current value of asset 2 will be  $S_2$ . Also, there will be  $\sigma_1$  for asset 1 and  $\sigma_2$  for asset 2.

Consider an European option with expiration date  $T$ . The time  $[0, T]$  will be split into  $n$  different intervals with step size  $\Delta t = T/n$ . If we consider the movement of the two assets in xy-coordinates and assume that the initial prices of the assets,  $S = (S_1, S_2)$ .

At the time  $t = \Delta t$ , the prices have five branches  $(S_1 u_1, S_2 u_2)$ ,  $(S_1 u_1, S_2 / u_2)$ ,  $(S_1 / u_1, S_2 / u_2)$ ,  $(S_1 / u_1, S_2 u_2)$ ,  $(S_1, S_2)$  with respective probabilities  $p_{uu}$ ,  $p_{ud}$ ,  $p_{dd}$ ,  $p_{du}$  and  $p_{mm}$ .

Here,  $u_i, i = 1, 2$  represents a movement in the up direction while  $1/u_i = d_i$  represents a movement downward. At the next time step,  $t = 2\Delta t$ , the asset price will have 13 branches as shown in the figure below.

The assumption is that the option reaches maturity at time  $T$  and has an exercise price of  $E$ . The risk-free interest rate is  $r$  per year, compounded continuously. To achieve an effective algorithm, it was determined that a five-point jump process was the optimal choice. With the current value of the asset pair being  $S_1$  and  $S_2$ , there are five unique results for the suggested jump process. The outcomes of the jump process can be described in the following manner:

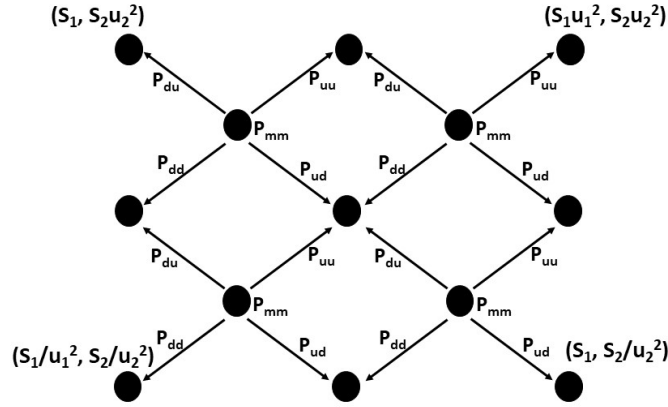


Figure 3: Two Period Lattice Model

Event	Probability of Event	Asset price after the event has occurred	
		Asset 1	Asset 2
Event 1	$p_{uu}$	$S_1 u_1$	$S_2 u_2$
Event 2	$p_{ud}$	$S_1 u_1$	$S_2/u_2$
Event 3	$p_{dd}$	$S_1/u_1$	$S_2/u_2$
Event 4	$p_{du}$	$S_1/u_1$	$S_2 u_2$
Event 5	$p_{mm}$	$S_1$	$S_2$

Table 1: Outcome of the five jump process

The factor  $u_1$  denotes the degree of stretch for the first asset, whereas  $u_2$  represents the stretch factor for the second asset. Recall that in the regular CRR model,

$$u = \exp(\sigma\sqrt{\Delta t}),$$

If this formula is used, some probabilities will turn out negative and that shouldn't happen. Therefore, to fix this problem,  $u$  will taken as:

$$u = \exp(\lambda\sigma\sqrt{\Delta t}).$$

where  $\lambda$  has to be greater than one. For the scope of this work,  $\lambda$  will be assigned a value of 1.1.

Assuming that the procedure begins with the asset values  $S_1$  and  $S_2$ . A three-dimensional depiction with time on the vertical axis can be visualized. After time  $\Delta t$ , the asset price pair can adopt one of the five values contingent upon the occurrence of each event.

These five points correspond to locations in the  $S_1$  to  $S_2$  space situated in a plane directly above the origin. The first four points are situated at the corners of a rectangle, and the fifth point is positioned in the center of the rectangle directly above the origin. When this process is repeated, a three-dimensional lattice configuration that resembles an inverted pyramid can be obtained.

To compute the probabilities and stretch factors, begin by computing the probabilities in terms of  $u_1$ ,  $u_2$ , and other relevant variables. After that, it is possible to obtain two equations by equating the mean of the discrete distribution to the continuous distribution's mean.

$$(p_{uu} + p_{ud})S_1u_1 + p_{mm}S_1 + (p_{dd} + p_{du})S_1/u_1 = S_1M, \quad (5)$$

$$(p_{uu} + p_{du})S_2u_2 + p_{mm}S_2 + (p_{ud} + p_{dd})S_2/u_2 = S_2M. \quad (6)$$

Two equations can be obtained by equating the variances as follows:

$$(p_{uu} + p_{ud})(S_1^2u_1^2 - S_1^2M^2) + p_{mm}(S_1^2 - S_1^2M^2) + (p_{dd} + p_{du})(S_1^2u_1^{-2} - S_1^2M^2) = S_1^2D_1, \quad (7)$$

$$(p_{uu} + p_{du})(S_2^2 u_2^2 - S_2^2 M^2) + p_{mm}(S_2^2 - S_2^2 M^2) + (p_{ud} + p_{dd})(S_2^2 u_2^{-2} - S_2^2 M^2) = S_2^2 D_2. \quad (8)$$

The sum of all the probabilities have to be 1, therefore:

$$p_{mm} = 1 - p_{uu} - p_{ud} - p_{dd} - p_{du}. \quad (9)$$

Dividing Equation (5) by  $S_1$  and substituting Equation (9) gives

$$(p_{uu} + p_{ud})u_1 + (1 - p_{uu} - p_{ud} - p_{dd} - p_{du}) + (p_{dd} + p_{du}) = M,$$

regrouping terms, we get

$$(p_{uu} + p_{ud})(u_1 - 1) + (p_{dd} + p_{du})\left(\frac{1}{u_1} - 1\right) = M - 1.$$

It is also convenient to divide Equation (7) by  $S_1^2$  and substitute Equation (9), this gives

$$(p_{uu} + p_{ud})(u_1^2 - M^2) + (1 - p_{uu} - p_{ud} - p_{dd} - p_{du})(1 - M^2) + (p_{dd} + p_{du})(u_1^{-2} - M^2) = D_1,$$

Regrouping the terms further gives,

$$(p_{uu} + p_{ud})(u_1^2 - 1) - (p_{dd} + p_{du})\left(\frac{1}{u_1^2} - 1\right) = D_1 + M^2 - 1.$$

Similarly, the following equation can be gotten for Equation (6) and Equation (8)

$$(p_{uu} + p_{du})(u_1 - 1) + (p_{ud} + p_{dd})\left(\frac{1}{u_1} - 1\right) = M - 1,$$

$$(p_{uu} + p_{du})(u_1^2 - 1) - (p_{ud} + p_{dd})\left(\frac{1}{u_1^2} - 1\right) = D_2 + M^2 - 1.$$

By considering  $(p_{uu} + p_{ud})$ ,  $(p_{dd} + p_{du})$ , and  $p_{mm}$  as new probabilities, solutions can be generated for these probabilities. The same approach will be used to solve for the probabilities  $(p_{uu} + p_{du})$ ,  $(p_{ud} + p_{dd})$ , and  $p_{mm}$ . This solution process represents the projection of the two-variable distribution onto the unconditional distribution of a single variable. So from (5) and (7), we get the following equations:

$$(p_{uu} + p_{ud})(u_1 - 1) = M - 1 - (p_{dd} + p_{du})\left(\frac{1}{u_1} - 1\right), \quad (10)$$

$$(p_{dd} + p_{du})\left(\frac{1}{u_1^2} - 1\right) = D_1 + M^2 - 1 - (p_{uu} + p_{ud})(u_1^2 - 1). \quad (11)$$

Regrouping terms in (11) and substituting  $(p_{dd} + p_{du})$  from (11) into (10):

$$(p_{uu} + p_{ud}) = M - 1 - \left(\frac{D_1 + M^2 - 1 - (p_{uu} + p_{ud})(u_1^2 - 1)}{\frac{1}{u_1} + 1}\right),$$

Solving further and putting like terms together:

$$(p_{uu} + p_{ud})\frac{u_1^2 - 1}{u_1} - (p_{uu} + p_{ud})(u_1^2 - 1) = \frac{M - 1}{u_1} - D_1 - M^2 + M,$$

Multiplying through by  $u_1$  gives:

$$-(p_{uu} + p_{ud})((u_1^2 - 1)(u_1 - 1)) = M - 1 - (D_1 + M^2 - M)u_1,$$

Dividing through by  $(u_1^2 - 1)(u_1 - 1)$  gives:

$$p_{uu} + p_{ud} = f_1 := \frac{(D_1 + M^2 - M)u_1 - (M - 1)}{(u_1 - 1)(u_1^2 - 1)}, \quad (12)$$

Substituting Equation (10) into Equation (11) gives:

$$(p_{dd} + p_{du})\left(\frac{1}{u_1} - 1\right) = (M - 1) - \frac{(D_1 + M^2 - M)u_1 - (M - 1)}{(u_1^2 - 1)},$$

Dividing through by the coefficient of  $(p_{dd} + p_{du})$  gives and grouping common multiples gives:

$$p_{dd} + p_{du} = g_1 := \frac{(D_1 + M^2 - M)u_1^2 - u_1^3(M - 1)}{(u_1 - 1)(u_1^2 - 1)}. \quad (13)$$

Solving similarly for  $(p_{uu} + p_{du})$  and  $(p_{ud} + p_{dd})$  and  $p_{mm}$ :

$$p_{uu} + p_{du} = f_2 := \frac{(D_2 + M^2 - M)u_2 - (M - 1)}{(u_2 - 1)(u_2^2 - 1)}, \quad (14)$$

$$p_{ud} + p_{dd} = g_2 := \frac{(D_2 + M^2 - M)u_2^2 - u_2^3(M - 1)}{(u_2 - 1)(u_2^2 - 1)}. \quad (15)$$

Based on the equation for  $p_{mm}$ , the equation below needs to be satisfied

$$f_1 + g_1 = f_2 + g_2.$$

With this, there is a relation between  $u_1$  and  $u_2$  that needs to be fulfilled. The expected value of the product of asset 1 and asset 2 can be found by utilizing the characteristics of the joint lognormal distribution.

$$E(S_1 S_2) = S_1 S_2 M^2 [\exp(\rho \sigma_1 \sigma_2 \Delta t)],$$

Let  $R = M^2 [\exp(\rho \sigma_1 \sigma_2 \Delta t)]$ ,

Then,

$$E(S_1 S_2) = S_1 S_2 R.$$

By utilizing the previous equation, the covariance between the two assets can be determined. As a result, the means and variance-covariance matrix for the bivariate lognormal distribution of the asset values at the conclusion of the time interval  $\Delta t$  have been gotten. To substitute this distribution with a discrete distribution, it has to be ensured that the approximating discrete distribution's first and second moments, including the covariance term, match those of the continuous distribution.

The last equation is derived by comparing the expected value of the product of the two assets based on the continuous distribution and the approximating discrete distribution, which leads to the following equation:

$$(p_{uu}u_1u_2 + p_{ud}u_1u_2^{-1} + p_{dd}u_1^{-1}u_2^{-1} + p_{du}u_1^{-1}u_2 + p_{mm})S_1S_2 = RS_1S_2,$$

dividing through by  $S_1S_2$  and eliminating  $p_{mm}$ :

$$p_{uu}(u_1u_2 - 1) + p_{ud}(u_1u_2^{-1} - 1) + p_{dd}(u_1^{-1}u_2^{-1} - 1) + p_{du}(u_2u_1^{-1} - 1) = R - 1. \quad (16)$$

Using (12), (13), (14), (15), and (16) we will get:

$$p_{uu} = \frac{u_1u_2(R - 1) - f_1(u_1^2 - 1) - f_2(u_2^2 - 1) + (f_2 + g_2)(u_1u_2 - 1)}{(u_1^2 - 1)(u_2^2 - 1)}. \quad (17)$$

Armed with this equation for  $p_{uu}$ , we can find  $p_{ud}$  using (12), (13), (14), (15)

$$p_{ud} = f_1 - p_{uu}.$$

Substituting (17) into the equation above and multiplying both sides by  $(u_1^2 - 1)(u_2^2 - 1)$

$$p_{ud}(u_1^2 - 1)(u_2^2 - 1) =$$

$$f_1(u_1^2 - 1)(u_2^2 - 1) - u_1u_2(R - 1) + f_1(u_1^2 - 1) + f_2(u_2^2 - 1) - (f_2 + g_2)(u_1u_2 - 1).$$

Factorizing  $f_1(u_1^2 - 1)$ , regrouping terms and dividing through by  $(u_1^2 - 1)(u_2^2 - 1)$  gives

$$p_{ud} = \frac{f_1(u_1^2 - 1)u_2^2 + f_2(u_2^2 - 1) - (f_2 + g_2)(u_1u_2 - 1) - u_1u_2(R - 1)}{(u_1^2 - 1)(u_2^2 - 1)}.$$

In the same way, we can apply (12), (13), (14), (15) to get

$$p_{dd} = \frac{u_1u_2(R - 1) - f_1(u_1^2 - 1)u_2^2 + g_2(u_2^2 - 1)u_1^2 + (f_2 + g_2)(u_1u_2 - u_2^2)}{(u_1^2 - 1)(u_2^2 - 1)},$$

$$p_{du} = \frac{f_1(u_1^2 - 1) + f_2(u_2^2 - 1)u_1^2 - (f_2 + g_2)(u_1u_2 - 1) - u_1u_2(R - 1)}{(u_1^2 - 1)(u_2^2 - 1)}.$$

The algorithm for finding the European option price using this method is as follows:

1. Calculate the up and down factors for both assets;  $u_1 = \exp(\lambda\sigma_1\sqrt{\Delta t})$ ,  $u_2 = \exp(\lambda\sigma_2\sqrt{\Delta t})$  where  $\lambda = 1.1$ .
2. Calculate the underlying asset prices at  $T = n\Delta t$ ;  $S_{i,j} = (S_1u_1^j, S_2u_2^i)$  for  $i, j = 0$  to  $n$ .
3. Calculate the five probabilities  $p_{uu}$ ,  $p_{ud}$ ,  $p_{du}$ ,  $p_{dd}$ , and  $p_{mm}$ .
4. Calculate option price  $V_{n,i}$  at time  $T$  which is equal to the payoff at  $T$  using the payoff formulas for the two asset types, for  $i, j = -n, -n+1, -n+2, \dots, n-1, n$ .
5. Starting at the nodes on the second-to-last level of the tree, work recursively to calculate option price at the nodes immediately above it using the formula

below:

$$V_{i,j} = \exp(-r\Delta t)(p_{uu}V_{i+1,j+1} + p_{ud}V_{i+1,j-1} + p_{du}V_{i-1,j+1} + p_{dd}V_{i-1,j-1} + p_{mm}V_{i,j}),$$

$$\text{for } m = n - 1, n - 2, \dots, 0 \text{ and } i, j = -m, -m + 1, \dots, 0.$$

6. The final option price  $V_{0,0}$  is the option price at the point  $t = 0$ .

## 2.3 A Modification of the Binomial Method for Two Assets

The paper [11] is heavily referenced in this section. Examine an European-style option that has an expiration date  $T$ . Initially, the time domain  $[0, T]$  is broken up into a series of  $N$  intervals, with step size  $\Delta t = T/N$ . The log asset price  $X = \log(S)$ . In the binomial tree, the asset price is allowed to move up ( $uS$ ) or down ( $dS$ ). In the same way, With each next time period, the price  $X$  moves to  $X + h$  where  $h = \log(u)$  and to  $X - h$  since  $d = 1/u$ .

The movement of two assets in xy-coordinates is then considered. The log price of the first asset,  $X_1$ , is represented on the horizontal axis, while the log price of the second asset,  $X_2$ , is represented on the vertical axis. The initial log asset prices are given by  $X^0 = (X_1^0, X_2^0)$ . When  $t = \Delta t$ , the initial log asset prices  $X^0$  have four branches:  $(X_1^0 + h_1, X_2^0 + h_2)$ ,  $(X_1^0 + h_1, X_2^0 - h_2)$ ,  $(X_1^0 - h_1, X_2^0 + h_2)$ ,  $(X_1^0 - h_1, X_2^0 - h_2)$ , each with respective probabilities,  $p_{uu}$ ,  $p_{ud}$ ,  $p_{du}$ , and  $p_{dd}$ , where  $h_i, i = 1, 2$  denotes the increment in each direction. These probabilities will be derived below.

Typically, at time  $t = k\Delta t$ , the asset price  $X^k = (X_{1,i}^k, X_{2,j}^k)$  located at  $(i, j)$  may transition to one of four potential states at the next time step,  $t = (k + 1)\Delta t$ . These four states are:  $(X_{1,i}^k + h_1, X_{2,j}^k + h_2)$ ,  $(X_{1,i}^k + h_1, X_{2,j}^k - h_2)$ ,  $(X_{1,i}^k - h_1, X_{2,j}^k + h_2)$ , and  $(X_{1,i}^k - h_1, X_{2,j}^k - h_2)$ . The figure displayed below illustrates the movement

of  $(X_{1,i}^n, X_{2,j}^n)$  at time  $t = n\Delta t$  on the binomial tree.

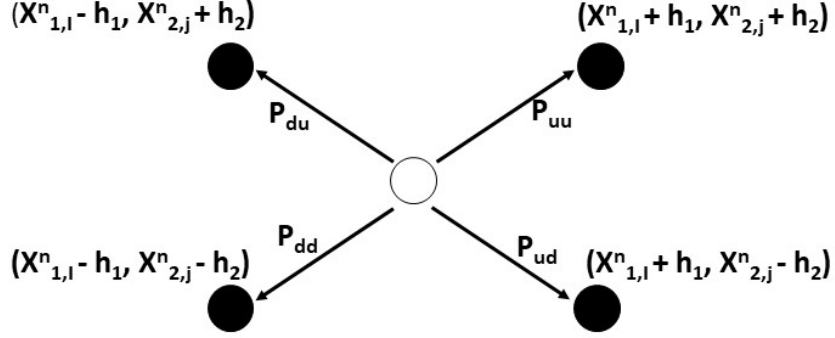


Figure 4: Binomial model with two assets and four probabilities at  $t = n\Delta t$

Next, we need to determine the values of the unknown probabilities  $p_{uu}, p_{ud}, p_{du}$ , and  $p_{dd}$  as well as the increments  $h_1$  and  $h_2$ . The increment of the process  $X^{k+1} - X^k$  is required to have a mean of 0 and a variance of  $(\sigma_1^2 \Delta t, \sigma_2^2 \Delta t)$  with covariance  $\rho \sigma_1 \sigma_2 \Delta t$ , which is derived from the definition of the continuous process  $X_l$ , where  $l = 1, 2$ . Additionally, the probabilities must sum up to 1. Therefore, we have six equations and six unknowns, which enable us to solve for these values.

$$p_{uu} = p_{dd} = \frac{1}{4}(1 + \rho),$$

$$p_{ud} = p_{du} = \frac{1}{4}(1 - \rho),$$

$$h_1 = \sigma_1 \sqrt{\Delta t},$$

$$h_2 = \sigma_2 \sqrt{\Delta t}.$$

We can then find the option price  $V$  using the following algorithm:

1. Get every potential value of the prices of the underlying assets,  $X^N$ , at the time  $T = N\Delta t$ ;  $S_{i,j} = (S_1 \exp(jh_1), S_2 \exp(ih_2))$  for  $i, j = -N, -N +$

2, ..., N - 2, N.

2. Calculate option price  $V_{i,j}^N$  at time  $T$  which is equal to the payoff at  $T$  using the payoff formulas for the two asset types, for  $i, j = -N, -N+2, \dots, N-2, N$ .
3. Calculate the price of the option for  $n = N - 1, \dots, 0$ . This can be achieved using the following formula.

$$V_{i,j}^n = \exp(-r\Delta t)(p_{uu}V_{i+1,j+1}^{n+1} + p_{ud}V_{i+1,j-1}^{n+1} + p_{du}V_{i-1,j+1}^{n+1} + p_{dd}V_{i-1,j-1}^{n+1}).$$

for  $i, j = -n, -n + 2, \dots, n - 2, n$

4. Find the price of the option  $V_{0,0}^0$  at time  $t = 0$ .

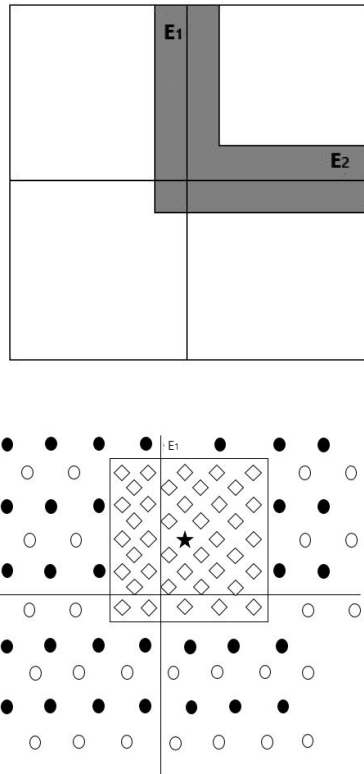
## 2.4 Adaptive Binomial Lattice Method for Pricing Two Asset Options

The papers [10] and [11] are heavily referenced in this section. The section proposes a novel method for developing a lattice-based valuation model, which permits the user to adjust the level of detail in different regions of the model. The majority of the lattice is constructed using a relatively coarse grid to expedite calculations, but a minor section of the lattice is created with a finer mesh where greater precision is essential.

In their work, Figlewski and Gao [1999] discuss two types of approximation errors that arise when using a lattice model. The first type, which is the *distribution error*, occurs because a discrete binomial or trinomial probability distribution is used to approximate the continuous lognormal distribution generated by a diffusion process. However, this error diminishes as the number of tree steps increases. The second type of error is the *nonlinearity error* and it arises when the option value function is highly nonlinear or discontinuous in certain areas, such as at the expiration date's exercise price. By adding one or more little sections of finely

detailed lattice onto a tree with coarser time and price steps, the adaptive mesh model greatly diminishes nonlinearity error.

Using an adaptive mesh model is a viable option for decreasing computational expenses while enhancing accuracy. The adaptive mesh model developed by Figlewski and Gao can be expanded to cover multivariate contingent claims. The fine mesh is incorporated in the region around the exercise prices of two assets between  $T - \Delta t$  and  $T$ . We minimize the price step sizes to  $h_1/2$  and  $h_2/2$  and the time steps to  $\Delta t/4$ . The fine mesh is included at all coarse mesh nodes in  $R_c$ , which is as follows:



Here,  $R_c =$  coarse mesh nodes at time  $T - \Delta t$  | their fine mesh paths end up in-the-money and out-of-the-money at  $T$ .

An illustration of this can be seen with the European two-asset correlation call option, where the payoff is determined by two stock prices,  $S_1$  and  $S_2$ .

$E_1$  and  $E_2$  are the exercise prices of  $S_1$  and  $S_2$ . If  $S_1 > E_1$  and  $S_2 > E_2$  at expiration  $T$ , the option is in-the-money. Conversely, if  $S_1 < E_1$  or  $S_2 < E_2$ , the option is out-of-the-money. We are given the asset price  $X^n = (X_{1,i}^n, X_{2,j}^n)$ , where  $i, j = -n, -n + 2, \dots, n - 2, n$ , at time  $t = n\Delta t = T$ . Let  $M_1$  be the smallest index  $i$  of  $X_{1,i}^{n-1}$  at  $t = T - \Delta t$ , for which one of its fine mesh paths exceeds the exercise price.

$$M_1 = \min(i : X_{1,i} + 2h_1 \geq E_1^1),$$

The strike price of  $X_1$ , after the transformation from  $S_1$  to  $X_1$ , is denoted by  $E_1^1$ . In a similar manner,  $M_2$  can be derived for  $X_2$ . If  $h_1$  is halved on the fine grid, the time step  $\Delta t$  is reduced to  $\Delta t/4$  to allow  $X_{1,i}^{n-1}$  to increase up to  $X_{1,i}^{n-1} + 2h_1$  during  $\Delta t$ . Following this, the coarse mesh nodes within the region  $R_c$  can be obtained.

$$(R_c = \text{mesh points near } E_2 \cup \text{mesh points near } E_1)$$

In order to compute the coarse mesh nodes in region  $R_c$ , the fine mesh should be used, as shown in the grey region. In this region, the fine price steps of  $h_1/2$  and  $h_2/2$  result in 41 fine mesh nodes for each coarse mesh node. For instance, in the figure above, the coarse grid point represented by a star has 41 diamonds inside the square that represent fine mesh nodes corresponding to the star. It should be noted that a different option would have a different region  $R_c$ . Since the price steps on the fine grids are halved, we call this refinement level 1.

The Adaptive Mesh Model (AMM) has the advantageous property of being isomorphic when refining levels. This means that once an AMM tree has been created, it is easy to add another level of even finer mesh with price steps of size  $h/4$ . The same process can then be applied as we have already described for the  $T - \Delta t/4$  to  $T$  period. The algorithm for finding the option price for refinement level 1 is as follows:

1. Get every potential value of the prices of the underlying assets,  $X^N$ , at

the time  $T = N\Delta t$ ;  $S_{i,j} = (S_1 \exp(jh_1), S_2 \exp(ih_2))$  for  $i, j = -N, -N + 2, \dots, N - 2, N$ .

2. Calculate option price  $V_{i,j}^N$  at time  $T$  which is equal to the payoff at  $T$  using the payoff formulas for the two asset types, for  $i, j = -N, -N + 2, \dots, N - 2, N$ .
3. For  $n = N - 1$ ,
  - (a) Select the coarse mesh nodes for  $R_c$ , and pick the corresponding fine mesh nodes.
  - (b) Calculate the prices of the coarse mesh nodes at time  $T = n\Delta t$  for  $i, j = -n, -n + 2, \dots, n - 2, n$ .
  - (c) Find the smallest index  $i$  of  $X_{1,i}^n$  at  $t = T - \Delta t$ , such that

$$M_1 = \min\{i : X_{1,i}^n + 2h_1 < K_1^1\} \quad (18)$$

and the smallest index  $j$  of  $X_{2,j}^n$  such that

$$M_2 = \min\{j : X_{2,j}^n + 2h_2 \geq K_2^1\} \quad (19)$$

- (d) Calculate the asset prices at each point in the fine mesh nodes at time  $T - \Delta t$  using the formula

$$S_{i_1, j_1} = (S_1 \exp(jh_1) \exp(j_1 h_1 / 2), S_2 \exp(ih_2) \exp(i_1 h_2))$$

for  $i_1, j_1 = -4, -2, \dots, 4$ .

- (e) Determine the the option value in the coarse mesh nodes and its complement which is equal to the payoff, denoted as  $F_{i_1, j_1}^m$ , for every node in the set of fine mesh nodes, at time  $T$ , considering the number of iterations in time required to reach  $T - \Delta t$  under the refinement level 1, denoted by m. Here,  $\Delta t$  is decreased to  $\Delta t/4$ , and m equals 4.

(f) Calculate the option price using the following formula

$$F_{i_1, j_1}^m = \exp(-r\Delta t/4)(p_{uu}F_{i_1+1, j_1+1}^{m+1} + p_{ud}F_{i_1+1, j_1-1}^{m+1} + p_{du}F_{i_1-1, j_1+1}^{m+1} + p_{dd}F_{i_1-1, j_1-1}^{m+1}).$$

for  $m = 3, 2, 1, 0$  and  $i_1, j_1 = -m, -m + 2, \dots, 0$

(g) Set the option values for the nodes in  $R_c$  by utilizing  $F_{i_1, j_1}^0$ .

4. Starting from  $n = N - 2$  down to 0, calculate the option price for each node in the range of  $i, j = -n, -n + 2, \dots, n - 2, n$ .
5. Find the price of the option  $V_{0,0}^0$

### 3 Numerical Experiments

Here, the results of the numerical experiments are presented. The time measurements were generated from a system with a Core i3 processor and 8GB RAM. All the algorithms are implemented on Python. Three methods are used to price the multi asset options: we will refer to the Lattice Model with five branches as Model 1, the Modification of the Binomial Model as Model 2, and the Adaptive Binomial Lattice Model as Model 3. We have results for four types of two-asset options: Maximum and Minimum Options, Spread Options, Correlation Options, and Basket Options.

#### 3.1 Results of Options on the Maximum and Minimum of Two Assets

Here, we assume the following values: the initial stock prices are  $S_1 = 40$ ,  $S_2 = 40$ , and the exercise price  $E = 40$  at expiration time  $T = 7/12 = 0.583333$ . The correlation is  $\rho = 0.5$  while the risk free interest rate is  $r = 0.048790$ . If the volatilities  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.3$ , and the assets do not pay any dividends, then it can be shown that the exact price of the European call options on the Maximum of Two Assets and the European put options on the Minimum of Two Assets are 5.488 and 3.798 respectively [9].

We have results for the European call options on the Maximum of Two Assets, European put options on the Minimum of Two Assets and American put options on the Minimum of Two Assets at  $n = (20,50,100,200)$ . Because there is an assumption of no dividend payments, the value of the European call option on the maximum of the two assets is the same as that of an American call option with identical parameters. For this particular set of values, the American put option is not significantly more valuable than the European put option.

Number of Steps	Option Price			Run time		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
20	5.4985	5.4770	5.4774	46.9 ms	22.9 ms	26.9 ms
50	5.4954	5.4865	5.4830	688 ms	169 ms	173 ms
100	5.4935	5.4869	5.4848	5.64 s	1.55 s	1.59 s
200	5.4920	5.4873	5.4875	41.3 s	12.4 s	12.7 s

Table 2: European call option on the Maximum of Two Assets with  $E = 40$

Number of Steps	Option Price			Run time		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
20	3.8024	3.7859	3.7922	50.7 ms	22.9 ms	31.2 ms
50	3.8015	3.7968	3.7950	906 ms	156 ms	199 ms
100	3.8007	3.7978	3.7955	6.95 s	1.45 s	1.73 s
200	3.8002	3.7980	3.7982	48.9 s	9.53 s	12.42 s

Table 3: European put option on the Minimum of Two Assets with  $E = 40$

Number of Steps	Option Price			Run time		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
20	3.8735	3.8416	3.8493	58.7 ms	22 ms	31.2 ms
50	3.8715	3.8551	3.8541	1.02 s	172 ms	213 ms
100	3.8704	3.8573	3.8554	8.11 s	1.16 s	1.53 s
200	3.8697	3.8581	3.8583	46.6 s	10.1 s	14.8 s

Table 4: American put option on the Minimum of Two Assets with  $E = 40$

As stated previously, the exact price of the European call on the Maximum of Two Assets is 5.488. The maximum difference is 0.011, this shows that the agreement between the numbers obtained from the models and the exact price is really good, especially for 200 time steps. For the European put on the Minimum of Two Assets, the exact price is 3.798. The maximum difference here is 0.0044, the agreements between the numbers are also really good. Model 1 takes longer to run because

there are five branches/ five probabilities. In terms of accuracy, Model 3 is the closest to the exact price and it is therefore the most accurate.

### 3.2 Result of Spread Options

Here, we assume the following values: the initial stock prices are  $S_1 = 110$ ,  $S_2 = 60$ , and the exercise price  $E = 50$  at expiration time  $T = 182/365$ . The correlation is  $\rho = 0.4$ , the risk free interest rate is  $r = 0.1$ , the volatilities  $\sigma_1 = 0.4$  and  $\sigma_2 = 0.2$  and the assets do not pay any dividends. We compare the results from our models to those obtained by the PDE discretization method, yielding a European put option price of 10.123873 at 200 time steps [14]. We have results for  $n = (20,50,100,200)$  as shown in the table below.

Number of Steps	Option Price			Run time		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
20	10.0223	10.1621	10.1588	31.2 ms	11.2 ms	36.9 ms
50	10.0427	10.1400	10.1394	819 ms	199 ms	212 ms
100	10.0601	10.1338	10.1335	4.88 s	936 ms	1.4 s
200	10.0763	10.1297	10.1296	56.9 s	6.96 s	9.1 s

Table 5: European spread put option

Our results are in good agreement with the price from the PDE discretization method. Model 1 takes longer to run as before and Model 3 has the best approximation to the PDE price.

### 3.3 Result of Correlation Options

Here, we assume the following values: the initial stock prices are  $S_1 = 52$ ,  $S_2 = 65$ , and their exercise prices are  $E_1 = 50$  and  $E_2 = 70$  at expiration time  $T = 0.5$ . The correlation is  $\rho = 0.75$  while the risk free interest rate is  $r = 0.1$ . If the volatilities  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.3$ , and the assets do not pay any dividends, then it can be shown

that the exact price of the European call correlation option is 4.707 [11]. We have results for  $n = (20,50,100,200)$  as shown in the table below.

Number of Steps	Option Price			Run time		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
20	4.7247	4.7921	4.7374	65.8 ms	24 ms	29.9 ms
50	4.6714	4.6621	4.6903	855 ms	186 ms	212 ms
100	4.6897	4.6835	4.6993	6.95 s	986 ms	1.38 s
200	4.6944	4.6911	4.7024	39.2 s	11.5 s	13 s

Table 6: European call correlation option

As stated previously, the exact price of the correlation option is 4.707 and the maximum difference is 0.085, this signifies that the numbers are in good agreement. Also Model 1 takes a longer time to run compared to Model 2 and Model 3 as in the other option types. In terms of accuracy, Model 3 performs best.

### 3.4 Result of Basket Options

Here, we assume the following values: the initial stock prices of the basket are  $S_1 = 100$ ,  $S_2 = 120$  with weights  $w_i = [-1,1]$ , and the exercise price  $E = 30$  at expiration time  $T = 1$ . The correlation is  $\rho = 0.9$ , the risk free interest rate is  $r = 0.05$ , the volatilities  $\sigma_1 = 0.2$  and  $\sigma_2 = 0.3$  and the assets do not pay any dividends. We compare the results from our models to those obtained by the GLN method-based binomial tree, yielding an American put option price as 14.0748 [12]. We have results for  $n = (20,50,100,200)$  as shown in the table below.

Number of Steps	Option Price			Run time		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
20	13.6568	14.2355	14.2355	60.9 ms	15.6 ms	23.1 ms
50	13.6169	14.2645	14.2645	819 ms	252 ms	286 ms
100	13.5903	14.2743	14.2743	4.88 s	1.47 s	2.04 s
200	13.5733	14.2782	14.2782	36.9 s	8.03 s	11.2 s

Table 7: American Basket Put option

Our values are quite close to the GLN method, though not as close as we had in other two-asset option types. Model 1 takes a longer time to run, also Model 2 and Model 3 performs remarkably better than Model 1.

## Conclusions

In the intricate world of financial markets, where different tools and strategies work together to manage risk and achieve profits, options play a significant role. This thesis has explored the complex topic of pricing options involving two assets. By combining theoretical knowledge and numerical analysis, we have gained a deeper understanding of this multifaceted area.

Our study focused on pricing four specific types of options that involve two assets: basket options, correlation options, spread options, and options related to the minimum and maximum of two assets. We realized that valuing these options is a challenge due to the complex interactions between different assets and market behavior. To tackle this, we used different mathematical models, such as the binomial lattice model with five branches, a modified version of the binomial approach, and the adaptive binomial lattice method. These models helped us find ways to determine prices for these intricate options.

Through numerical experiments, we validated the effectiveness of these methods. By comparing our results with exact prices and established research, we confirmed that our models are accurate tools for understanding option pricing. Notably, the adaptive binomial lattice method showed exceptional accuracy, suggesting its potential for practical use.

It's clear that the pricing of Two-Asset Options is a continuously evolving field. Our study contributes to the ongoing discussion about this topic by offering a comprehensive view of pricing techniques for a specific set of options. However, the complex nature of financial markets means that there is still much to explore and refine.

In conclusion, this master's thesis serves as a foundation for further exploration into Two-Asset Option Pricing. We hope that this work encourages more research

and provides guidance for those interested in understanding how to value two-asset options in the complex world of finance.

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