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A Multi-State Model for Post-Retirement Long-Term Care for the Risk of Cognitive Impairment

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A Multi-State Model for Post-Retirement Long-Term Care for the Risk of Cognitive Impairment

Master's Thesis

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Abstact. We study the lifetime cost of cognitive impairment in post-retirement long-term care. The contribution of this thesis is to design a model for various levels of cognitive impairment in post-retirement period, using time-continuous Markov process, with focus on the actuarial present value of costs associated with the levels of cognitive impairment in the model.

CERCS research specialisation: P160 Statistics, operations research, programming, actuarial mathematics.

Keywords: constant transition rates, actuarial present value of cost of care, quality of care.

Mitme olekuga mudel pensionijärgseks pikaajaliseks hoolduseks kognitiivse kahjustuse riskile

Magistritöö

Afua Durowaa-Boateng.

Lühikokkuvõte. Me uurime kognitiivsete häirete elukestvaid kulusid pensionijärgses pikaajalises hoolduses. Antud töö panus on kujundada mudel erinevatele kognitiivsete häirete tasemele pensionijärgsel perioodil, kasutades ajas pidevaid Markovi protsesse, keskendudes kognitiivse kahjustuse tasemega seotud kulude oodatavatele nüüdiväärtusele.

CERCS teaduseriala: P160 Statistika, operatsioonianalüüs, programmeerimine, finantsja kindlustusmatemaatika.

Märksõnad: hoolduse kvaliteet, hooldustasu oodatavad nüüdisväärtused, konstansed üleminekumäärad

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Contents

Introduction

The problem of knowing the cost of care and types of care needed after retirement has become a household issue. After a certain age, retired individuals may need assistance at least once in their life, in performing basic tasks that are common in everyday life [22]. These basic tasks include activities of daily living (ADLs) and instrumental activities of daily living (IADLs) [27]. Activities of daily living are regular physical activities which include ability to bath, eat, go to the bathroom and ability to control bladder and bowel movements [15]. Instrumental activities of daily living refer to activities that are associated with independent living, these include preparing meals, going to the store, and cleaning [27].

In a study by [25], 83 countries are estimated to have at least 20% of their population over the age of 65 by 2050. As the number of older individuals in populations is projected to grow, some major concerns for governments and families are the prevalence of age-related health issues [19], provision of care (long-term care), corresponding costs of care and quality of care provided. Dementia and cognitive impairment are some the major incidences that affects the physical and social well being of older individuals [24]. Cognitive impairment can be as a result of Alzheimer's disease [20]. Though there is a concern for cognitively impaired older individuals, care needs to be given to cognitively intact older individuals as well. We say a person is "cognitively intact if there is no evidence for dementia or cognitive impairment after clinical examination" [29]. In 2015, 47.5 million people globally were living with dementia [4]. The number of people living with dementia is estimated to likely double every 20 years [12].

The important issue for us concerns which types of long-term care, quality of care, and the costs of care to be provided for the case of cognitive impairment among retirees, as the higher the level of cognitive impairment, the higher the cost of care (e.g., [1]). It was projected by [1] that, the cost of various care for people above age 65 with some sort of cognitive impairment in 2019 is USD$290 billion for United States using 2018 dollars. The projected 2019 annual average cost of care for long-term care services and health care services for a cognitively intact individual above age 65 was USD$13,976 using 2018 dollars. However, for an individual above age 65 with some cognitive impairment, the annual average cost of care for long-term care and health care services was projected to be USD$48,977 for 2019 using 2018 dollars [1].

The differences in the cost of long-term care services for individuals above 65 with and without cognitive impairment are very significant. Thus, it is important that individuals who are 65 and above have some knowledge of the expected costs likely to be incurred during their lifetime [10]. Using the method of actuarial present value, the total costs likely to be incurred in the future can be estimated. Actuarial present value (also called the expected present value) is found by discounting the future payments at a determined rate of interest and multiplying by the probabilities of the payments occurring [11, p. 72]. With the knowledge of actuarial present values, elderly adults and their families can prepare adequately to fund the cost of care that comes with old age.

In this thesis the main objective is to study and define a multi-state model for post-retirement long-term care. We pay more attention to retirees aged 65 and over with risk of cognitive impairment. [27] used IADLs and ADLs to define states in their research, however, [4] defined states using a retiree's performance on the so-called Mini-Mental State Examination (MMSE). For our study, we define five unique health states using a combination of MMSE scores and ADLs. Given these states, we construct a multi-state Markov model transitions among the various states and study possible costs that may arise from these transitions, and calculate the actuarial present value of direct costs over their lifetime taking into account the quality of care.

Chapter 1

Modelling Post-Retirement Long-Term Care

In this chapter, we introduce the basic theory of multi-state models, definitions and notations. We use multi-state models to describe long-term care for post-retirement. The risk of focus is cognitive impairment within long-term care. The evolution of risk is a sequence of events which determine the cash flows of costs associated with post-retirement, examples of such events include dementia, Alzheimer's disease, stroke and severely cognitive impairment. We also explain some of the basic care associated with the aging process of retired persons and some of the costs associated with aging if long-term care is required. Finally we develop formulae to valuate the actuarial present values (APV) of the future costs associated with multi-sate model.

1.1 Multi-State Models

We assume that the "evolution of a risk can be described in terms of the presence of the risk itself, at every point of time, in a certain state belonging to a specified set of states, or state space." [14, Section 1.1, p. 1].

We follow Equations (1.1) and (1.2) in [14, p. 2] and denote the state space by $\mathscr{S}$. We assume that $\mathscr{S}$ is a finite set. Denoting the states by integral numbers,

$$\mathscr{S} = \{1, 2, \dots, N\}.$$

Let $\mathscr{T}$ denote the set of direct transitions from state i to state j. In general, $\mathscr{T}$ is a subset of the set of pairs (i, j) where i, $j \in \mathscr{S}$ then,

$$\mathscr{T} \subseteq \{(i,j) | i \neq j, i, j \in \mathscr{S}\}.$$

The pair $(\mathscr{S}, \mathscr{T})$ is called a multi-state model. The multi-state model $(\mathscr{S}, \mathscr{T})$ describes the uncertainty which is the "possibilities pertaining to an insured risk, as far as evolution is concerned" [14, p. 2].

Example 1. We consider a four-state model to illustrate the theory of sets and transitions. In this example, we describe state A as Healthy, state B as Ill, state C as Severely Ill, and state D as Dead. Thus, the state space is given by,

$$\mathscr{S} = \{\text{A,B,C,D}\}.$$

Naturally, a healthy individual can move to ill, severely ill or dead. It is common for people to recover from severe illness and illnesses in general to healthy, however, for Example 1, we do not allow recovery from severely Ill. It is unnatural for individuals to return from the dead, as such, in our Example 1, we do not allow transition from dead. The set of transitions for our Example 1 is given by,

$$\mathscr{T} = \{(\text{A,B}),(\text{A,C}),\ (\text{A,D}),\ (\text{B,A}),\ (\text{B,C}),\ (\text{B,D}),\ (\text{C,D})\}.$$

The multi-state model in Example 1 is the pair,

$$(\mathscr{S}, \mathscr{T}) = (\{\text{A,B,C,D}\}\,, \{(\text{A,B}),(\text{A,C}),\ (\text{A,D}),\ (\text{B,A}),\ (\text{B,C}),\ (\text{B,D}),\ (\text{C,D})\}).$$

The same model can be defined graphically. The states and transitions are given in Figure 1.1.

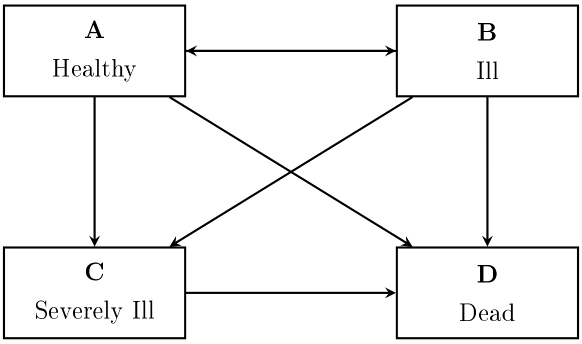

Figure 1.1: An Example of A Four-State Model State Model.

The arrows in Figure 1.1 indicates the direction in which transitions are allowed. Intuitively a state can be either **transient, strictly transient** or **an absorbing state** [14, p. 1–2].

- S1 States that have the possibility to exit and return are called **transient**.
- S2 We say a state is **strictly transient** if it does not have the possibility to reenter once the state has been left.
- S3 **Absorbing** states are defined as states that do not have the possibility to exit once it has been entered.

The formal definition of transient, strictly transient and absorbing states are given in Section 1.1.2 Following the various states defined in S1, S2 and S3, in our Example 1, states A and B are transient states, state C is a strictly transient state and state D is an absorbing state.

1.1.1 Time-Continuous Markov Model

Let $Y_x(t)$ denote a random state occupied by a person aged x, $x \geq 0$ at time $t \geq 0$. For example, $Y_x(0)$ is a given state, we can assume $Y_x(0) = 1$ [14, p. 2]. This means that an individual currently aged x is in state 1 at time 0. We let

$$Y_x(t),\ t \geq 0 \tag{1.1}$$

be a time-continuous stochastic process, with values in the set $\mathscr{S}$ [14, p. 2]. We follow Equation (1.5) in [14, Subsection 1.4.2, p. 13] and we consider the time-continuous stochastic process $Y_x(t)$, $t \geq 0$. We say that $Y_x(t)$, $t \geq 0$ is a **time-continuous Markov process** if for any n and each finite set of times $(0 \leq)\, t_0 < t_1 < \cdots < t_n < t$ and there exists a set of states $i_0, i_1, \ldots, i_n$, j in $\mathscr{S}$ so that

$$\Pr[Y_x(t_0) = i_0 \wedge \cdots \wedge Y_x(t_{n-1}) = i_{n-1}, Y_x(t_n) = i_n \wedge Y_x(t) = j] > 0,$$

and the following Markov property is satisfied,

$$\begin{aligned}\Pr[Y_x(t) = j | Y_x(t_0) = i_0 \wedge \cdots \wedge Y_x(t_{n-1}) = i_{n-1} \wedge Y_x(t_n) = i_n] = \\ = \Pr[Y_x(t) = j | Y_x(t_n) = i_n].\end{aligned}$$

The Markov property is considered memory-less, that is, it does not take into account previous information (transitions), rather it focuses only on the present or current information or transition.

1.1.2 Transition Probabilities

We follow Equation (1.9) in [14, p. 15] and we define the **transition probabilities** as

$$_tp_x^{ij} = \Pr[Y_x(t) = j | Y_x(0) = i],$$

for $t \geq 0$, $i, j \in \mathscr{S}$ and $x \geq 0$. Following Equation (1.11) in [14, p. 15] we define the **occupancy probabilities** as

$$\Pr[Y_x(s) = i \text{ for all } s \in [0, t] | Y_x(0) = i]$$

for $i \in \mathscr{S}$, $t \geq 0$ and $x \geq 0$. Naturally transition probabilities satisfy the probability conditions,

$$0 \leq {_tp_x^{ij}} \leq 1, \qquad \text{for all } i, j \in \mathscr{S}, \qquad t \geq 0.$$

For any finite multi-state model, it holds that,

$$\sum_{j \in \mathscr{S}} {_tp_x^{ij}} = 1, \qquad \text{for all } i \in \mathscr{S}, \qquad t \geq 0.$$

Following [11, p. 238], we say that for multi-state models, ${}_tp_x^{\overline{ii}}$ is included in ${}_tp_x^{ij}$. Thus,

$${}_tp_x^{\overline{ii}} \leq {}_tp_x^{ij}, \qquad \text{for all } i,j, \qquad t \geq 0.$$

For all states $i, j \in \mathscr{S}$, and for all $x \geq 0$, we assume that ${}_tp_x^{ij}$ is a differentiable function of t.

Let an $N \times N$ matrix $\mathbf{P}$(x,x+t) form the matrix of transition probabilities of an N state model. A time-dependent matrix, is written as,

$$\mathbf{P(x,x+t)} = \begin{bmatrix} {}_tp_x^{11} & {}_tp_x^{12} & \cdots & {}_tp_x^{1N} \\ {}_tp_x^{21} & {}_tp_x^{22} & \cdots & {}_tp_x^{2N} \\ {}_tp_x^{31} & {}_tp_x^{32} & \cdots & {}_tp_x^{3N} \\ \cdots & \cdots & \cdots & \cdots \\ {}_tp_x^{N1} & {}_tp_x^{N2} & \cdots & {}_tp_x^{NN} \end{bmatrix}.$$

In particular, we say that a state i is **transient** [14, Equation 1.15, p. 16] if,

$$\lim_{t\to\infty} {}_tp_x^{ii} = 0.$$

We say that a state i is a **strictly transient** state if [14, Equation 1.16, p. 16],

$${}_tp_x^{ii} = {}_tp_x^{\overline{ii}} < 1 \qquad (t \geq 0).$$

A state i is an **absorbing** state if [14, Equation 1.14, p. 16],

$${}_tp_x^{\overline{ii}} = 1 \qquad (t \geq 0).$$

Given the transition probability from state i to state j by a person aged x, ${}_tp_x^{ij}$, we consider the complete expected future lifetime of the individual aged x at time 0. We follow Equation (66) in [27, p. 40] and we denote the expected number of years spent in j if a person aged x is in state i at time 0 by $\mathring{e}_x^{ij}$, defined as,

$$\mathring{e}_x^{ij} = \int_0^\infty {}_tp_x^{ij}\,dt \qquad i,j \in \mathscr{S}. \tag{1.2}$$

We let complete future lifetime of an individual aged x in any state i at time 0 be denoted by $\mathring{e}_x^{i}$, defined as,

$$\mathring{e}_x^{i} = \sum_j \mathring{e}_x^{ij} \qquad i,j \in \mathscr{S}. \tag{1.3}$$

The complete future lifetime $\mathring{e}_x^{i}$ estimates how long in total, normally in years, an individual aged x spends in all $j \in \mathscr{S}$ if the individual was in state i at time 0.

1.1.3 Transition Intensities

We follow [14, Subsection 1.4.3, p. 17] in defining transition intensities. For each state $i \neq j$, the force of transition or transition intensity between states i and j in $\mathscr{S}$ for a person aged x, $x \geq 0$, at time $t \geq 0$ is defined as:

$$\mu_{x+t}^{ij} = \lim_{h \to 0^+} \frac{{}_{t+h}p_x^{ij}}{h} \text{ , for } i \neq j. \tag{1.4}$$

We denote force of transition by μ_{x+t}^{ij}. The limits in Equation (1.4) is "assumed to exist for all relevant t and $i \neq j$, and the intensities are assumed to be integrable on compact intervals" [14, p. 17]. The expression $\mu_{x+t}^{ij}dt$ is interpreted as "the conditional probability that x transitions from state i into state j occurs over the infinitesimal interval $[x+t, x+t+dt)$ given that the risk is in state i at time t" [14, p. 17]. Following equation (1.18) in [14, p. 17], we define total intensity, as "(total) intensity of decrement from state i" [14, p. 17]. The total intensity of state i is denoted by, $\mu_{x+t}^{i\cdot}$ and defined as,

$$\mu_{x+t}^{i\cdot} = \sum_{j:j \neq i} \mu_{x+t}^{ij} \qquad i, j \in \mathscr{S},\ x \geq 0,\ t \geq 0. \tag{1.5}$$

Let a $N \times N$ matrix $\mathbf{M}$(x+t) denote the matrix of transition intensities of N states, $\mathbf{M}(x+t) = (\mu_{x+t}^{ij})$, $i, j \in \mathscr{S}$

$$\mathbf{M}(x+t) = \begin{bmatrix} -\mu_{x+t}^{1\cdot} & \mu_{x+t}^{12} & \mu_{x+t}^{13} & \cdots & \mu_{x+t}^{1j} \\ \mu_{x+t}^{21} & -\mu_{x+t}^{2\cdot} & \mu_{x+t}^{23} & \cdots & \mu_{x+t}^{2j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{x+t}^{N1} & \mu_{x+t}^{N2} & \mu_{x+t}^{N3} & \cdots & -\mu_{x+t}^{N\cdot} \end{bmatrix}.$$

The relationship between transition intensities and probabilities gives us the understanding that transition intensities are fundamental in determining everything about a multi-state model [11, p. 239].

1.1.4 Finding Transition Probabilities

Following Equation (1.12) in [14, p. 15], transition probabilities satisfy the Chapman-Kolmogorov equation.

The Chapman-Kolmogorov equation describes the path of x starting in i at time 0, and gets to state j at time t, but visits state k at an arbitrary intermediate time u [14, p. 15]. The Chapman-Kolmogorov general equation is,

$$ {}_tp_x^{ij} = \sum_{k \in \mathscr{S}} {}_up_x^{ik}\, {}_{t-u}p_{x+u}^{kj}, \qquad 0 \leq u \leq t. \tag{1.6}$$

Chapman-Kolmogorov Forward Equation

Following equation (1.19) in [14, p. 17], we define the Chapman-Kolmogorov forward equation. According to [14, p. 17-18], the Chapman-Kolmogorov forward equation can be derived starting from the Chapman-Kolmogorov equations in equation (1.6).

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x^{ij} = \sum_{k:k\neq j} ({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}), \qquad x, t \geq 0. \tag{1.7}$$

Chapman-Kolmogorov Backward Equation

We define the Chapman-Kolmogorov backward equation based on equation (1.20) in [14, p. 18].

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x^{ij} = {}_tp_x^{ij}\mu_{x+t}^{i\cdot} - \sum_{k:k\neq i} {}_tp_x^{kj}\mu_{x+t}^{ik}, \qquad x, t \geq 0. \tag{1.8}$$

Transition probabilities are often found when the Chapman-Kolmogorov equations are solved through numerical methods.

The problem of finding transition probabilities should be approached in more general terms. Transition probabilities can be found using methods [14, p. 41-44], based on the definition of transition intensities (i.e, constant or time-dependent). We briefly introduce how transition probabilities can be found in the simplest case, i.e., with constant transition intensities.

We assume $\mu_{x+t}^{ij} = \mu^{ij}$ for all t and for all $i, j \in \mathscr{S}$. Let $\mathbf{M}$ denote the matrix form of constant transition intensities, $\mathbf{M}(x+t) = \mathbf{M} = (\mu^{ij})$.

$$\mathbf{M} = \begin{bmatrix} -\mu^{1\cdot} & \mu^{12} & \mu^{13} & \dots & \mu^{1j} \\ \mu^{21} & -\mu^{2\cdot} & \mu^{ij} & \dots & \mu^{2j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu^{N1} & \mu^{N2} & \mu^{N3} & \dots & -\mu^{N\cdot} \end{bmatrix}.$$

Following Equation (1.77) in [14, p. 17], the transition probability can be solved by,

$$\mathbf{P}(x, x+t) = e^{\mathbf{M}t} = \mathbf{I} + \mathbf{M}t + \frac{\mathbf{M}^2t^2}{2!} + \dots \tag{1.9}$$

There are ways in which Equation (1.9) can be solved. [21] provides guidance on efficient ways of calculating a matrix exponential.

In general, the Chapman-Kolmogorov forward and backward equations can be used to find transition probabilities when transition intensities are not constant. There are various numerical methods that can be used to find the transition probability matrix given non-constant transition intensities. For example, [23] proposes the Euler and trapezium methods and discusses these methods in details, [11] suggests Euler's method, while [18] suggests that Runge-Kutta methods can be employed in solving the differential equations for transition probabilities.

1.2 Post-Retirement Long-Term Care

In this section, we explain some of the basic care associated with the aging process of retired persons and some of the costs associated with aging if long-term care is required. Most ideas in this chapter are based on ideas from [22], [10], [15], [27] and [14].

Post-retirement long-term care is defined as care provided to retirees who are not able to physically take care of themselves [27]. According to [22] long-term care does not try to restore the individual to a healthy state, however, it provides the individual with support to live in that condition by taking care of the individual. Individuals who maybe in need of post-retirement long-term care may need assistance with activities of daily living (ADLs), ADLs can be in six main forms [15, p. 17].

- Bathing
- Eating
- Dressing
- Ability to manage personal hygiene
- Bladder control and bowel functions
- Getting in and out of bed.

The number of ADLs that an individual cannot perform, determines the type of long-term care services to use. Post-retirement long-term care services can be administered at home, in a nursing home, an assisted living home or a hospice facility [31].

The following are some types of long-term care services for post-retirement. These services differ one from another in their functions.

- **Home health care** services involve assistance in the home of the individual. Home health care services can be unpaid, most unpaid caregivers are family members and friends [31]. There are also paid services where a certified nurse, care taker or therapist visits the home of the person in need of the services. These services are mostly on a regular basis.

- **Adult day care** services are not home based. The services are housed in a facility for retirees. These facilities organizes activities that stimulate the minds of the service seekers. Most people who visit adult day care services have been diagnosed with dementia [31]. Adult day care services are designed to give home caregivers some relief during the day. The services provide support and companionship for older adults through social activities, meals, recreational activities, etc [27].

- **Custodial care** is a type of long-term care where the retirees receive assisted living [10], including but not limited to assistance in eating, changing of bedding, continence, etc.

- **Nursing facilities** provide both medical and rehabilitation services [31]. The certified nurses in these nursing facilities assist the residents of the facility in both activities of daily living and taking medications.

- **Assisted living facilities** are similar to nursing facilities, however, they do not provide extensive medical services. They are facilities that provide assistance for adults who need support in activities of daily living which include aid in bedding, bathing, eating, dressing up and other activities [31].

- **Hospice care** is purposefully designed for retirees who are at the last stages of their lives. This care provides companionship for the individual and does not seek to increase the individual's life span [31]. This type of care can be given at home, in a nursing facility, or a retirement community.

Since each type of care is unique, it is best to know which services can cater for the retiree's needs. There could many more types of care for the post-retirement period. According to results from [2], preparing for old age improves psychological and physical health among other factors in old age. Though results by [3] suggests that single women have low death rates, they were also estimated to have the risk of getting poorer as they got older thus, there is the need for creating awareness even for younger individuals to start planning for life after retirement (post-retirement) and they should be encouraged to make provisions to foot the cost of care in their old age. [31] suggests having such a plan or product is planning for a successful retirement. However, to plan for post-retirement long-term care requires some knowledge of the cost of care.

1.3 Various Types of Costs of the Care

We will now take a look at the costs associated with post-retirement long-term care. We consider costs associated with long-term care services as well as medical costs. We at the basic definitions and notations of actuarial present values for cost, unless otherwise stated, all equations and notations are based on [11], [14] and [16]. We refer to cost of care as the amount paid to cover any and all sorts of care while in a particular health risk state [27, p. 21], [4] and [1]. This cost includes, but is not limited to,

- medical bills for normal medical conditions due to aging,

- assistance with ADLs,

- long-term care services,
- rehabilitation facilities,
- mandated physical exam,
- medical certification for transitions between health states, and
- home visits.

We assume that the costs are incurred during the lifetime of the retiree once a health risk state has been entered and the costs depend on the health state [27]. Multi-state models are a natural way to model costs for long term policies [11, p. 247]. The health risks can be viewed as a sequence of events which determines costs [14, p. 1]. To further describe the costs of post-retirement long-term care, we develop formulae for the valuation of costs at each health state. We assume there are three types of costs:

- annuity types of costs that are paid continuously until death and these costs depend on the health risk state,
- lump sum costs incurred for medical bills and medical certification costs, paid only when the certification is needed to verify that a person has moved to a different health risk state, and
- mandated physical exam costs that are incurred at specific times in the retiree's lifetime for routine physical exams.

We use actuarial present values to valuate the future costs associated with each state. For our cost valuations, we assume a deterministic constant force of interest $\delta > 0$. We then derive the actuarial present values of all the possible costs that may arise. For our valuation formulae, we assume that a person aged x is in state i at time 0, that is, $Y_x(0) = i$, where $Y_x(t)$ is given by (1.1). For evaluating actuarial present value of total costs incurred, we use for $Y_x(0) = i$ (a person aged x in state i at time 0), the transition intensity, μ_{x+t}^{ij} defined by Equation (1.4) and the transition probability ${}_tp_x^{ij}$ defined in Subsection 1.1.2.

1. Consider a continuous cost incurred at rate $c^j(t) > 0$, $j \in \mathscr{S}$. Then, following Equation (1.100) and (1.101) in [14, p. 48] the random present value of rate $c^j(t)$ at time $t \geq 0$, if $Y_x(t) = j$ is $c^j(t)\mathrm{e}^{-\delta t} I_{\{Y_x(t)=j\}}\mathrm{d}t$. Thus, the random present value over time interval $[0, \infty)$ is,

$$\int_0^\infty c^j(t)\mathrm{e}^{-\delta t} I_{\{Y_x(t)=j\}}\mathrm{d}t.$$

Assuming the conditional event $Y_x(0) = i$, the corresponding actuarial present value of the continuous cost rate for, $c^j(t)$, $t \geq 0$, during period $[0, \infty)$ is defined as,

$$\int_0^\infty \mathrm{e}^{-\delta t}{}_tp_x^{ij} c^j(t) dt, \qquad i, j, \in \mathscr{S}. \tag{1.10}$$

Following Equations (1.106) and (1.107) in [14, p. 49], the actuarial present value of a unit-level continuous cost, $c^j(t)$=1, $t \geq 0$, for the period $[0, \infty)$ (if a person aged x in state i at time 0, moves to state j at time t) is denoted by $\bar{a}_x^{ij}$, defined as,

$$\bar{a}_x^{ij} = \int_0^\infty \mathrm{e}^{-\delta t}{}_tp_x^{ij} dt, \qquad i, j, \in \mathscr{S}. \tag{1.11}$$

2. We consider a continuous lump sum cost, $C^{ij}(t) > 0$, $t \geq 0$. We assume that the individual aged x is in state i at time 0, then, at some point within time interval $(0, t)$, the individual transfers to state k and then finally moves to state j at time t. The lump sum cost is incurred immediately at time t if transition from k to j occurs immediately at time t. The actuarial present value of the lump sum cost $C^{ij}(t) > 0$, $t \geq 0$, incurred over the time period $[0, \infty)$, as given in Equation (1.110) in [14, p. 49], is defined as,

   $$\int_0^\infty e^{-\delta t}{}_tp_x^{ik} \mu_{x+t}^{kj} C^{ij}(t) dt, \qquad k \neq j, \;\; i, k, j \in \mathscr{S}. \tag{1.12}$$

   Equation (1.12) occurs every time upon transition and there is no specified time at which transition should occur.
   For the case of $C^{ij}(t) = 1$, $t \geq 0$, (unit transition lump sum cost), a special notation, $\bar{A}_x^{ikj}$, is introduced.

   $$\bar{A}_x^{ikj} = \int_0^\infty e^{-\delta t}{}_tp_x^{ik} \mu_{x+t}^{kj} dt, \qquad k \neq j, \;\; i, k, j \in \mathscr{S}. \tag{1.13}$$

   Following Equation (1.112) in [14, p. 49], the actuarial present value of the total transition cost for future transfer to state j, given that x is currently in state i, is denoted by, $\bar{A}_x^{ij}$, defined as,

   $$\bar{A}_x^{ij} = \sum_{j:k \neq j} \bar{A}_x^{ikj}. \tag{1.14}$$

3. Consider the lump sum cost $C^j(t)$, incurred at a fixed time $t \geq 0$ if x is in state j at time $t \geq 0$, that is, $Y_x(t) = j$. The actuarial present value associated with lump sum cost of care $C^j(t)$ at time $t \geq 0$, is defined as,

   $$e^{-\delta t}{}_tp_x^{ij} C^j(t) \qquad i, j \in \mathscr{S}. \tag{1.15}$$

   Following Equation (1.114) in [14, p. 50], the actuarial present value associated with lump sum cost of care $C^j(t) = 1$ at time $t \geq 0$, is denoted by, ${}_tE_x^{ij}$, defined as,

   $${}_tE_x^{ij} = e^{-\delta t}{}_tp_x^{ij} \qquad i, j \in \mathscr{S}. \tag{1.16}$$

After finding the actuarial present values of the individual types of costs that may incur, the lifetime total cost incurred for any state can be found. Let the actuarial present value of lifetime total cost incurred for state i over period $[0,\infty)$ be denoted by $\mathscr{C}^i(0,\infty)$,

$$\mathscr{C}^i(0,\infty) = \sum_j \int_0^\infty \mathrm{e}^{-\delta t} {}_tp_x^{ij} c^j(t)dt + \sum_j \int_0^\infty e^{-\delta t} {}_tp_x^{ik} \mu_{x+t}^{kj} C^{ij}(t)dt + \sum_{t:t\geq 0} e^{-\delta t} {}_tp_x^{ij} C^j(t), \tag{1.17}$$

where $i,k,j \in \mathscr{S}$, $k \neq j$ and $t \geq 0$.

In a special case of all costs being constant, that is, $c^j(t) = c^j$, $C^{ij}(t) = C^{ij}$ and $C^j(t) = C^j$. Then, the actuarial present value of lifetime total cost if the individual is aged x at time 0 is given as,

$$\mathscr{C}^i(0,\infty) = \sum_j c^j \bar{a}_x^{ij} + \sum_j C^{ij} \bar{A}_x^{ikj} + \sum_{t:t\geq 0} \sum_j C^j {}_tE_x^{ij} \qquad i,k,j \in \mathscr{S},\, k \neq j. \tag{1.18}$$

We sum up all the actuarial present values of direct costs, over all possible transitions over the lifetime of the retiree to get the actuarial present value of lifetime total costs incurred for state i that is, the retiree is in state i at time 0.

Chapter 2

A Multi-State Model for Post-Retirement Long-Term Care for the Risk of Cognitive Impairment

In this chapter, we design a post-retirement multi-state model and define the cost of care for each defined health state. This model proposal is based on the works and results of [27], and [4]. These works emphasize the need for preparation towards long-term care.

Type of Retiree: For the purpose of this thesis, we define post-retirement as the period after which the individual has ceased working for regular income. Retirement age differs from country to country but generally ranges between ages 60 and 70. In this model, we take the retirement age to be age 65.
We consider all types of health shocks that occur at random points in time and we assume these health shocks can affect the health of the retiree such that the extent of medical care needed ranges from unsupervised home care to long term medical care.

2.1 States and Transitions of the Model

The states of the proposed model are defined using retiree's inability to perform ADLs and degree of cognitive impairment using scores on from an MMSE test (e.g., [4], [15], [27]). The MMSE examines levels of cognitive impairment (mental alertness) of an individual. The test examines 30 items through questionnaires. The test features the following categories, orientation to time, orientation to place, registration, execution of complex commands, language skills, recollection abilities, repetition of words, attention and calculation. The test scores 30 points with high scores indicating better cognition [4]. We define five health states for the proposed model as follows,

- **State 1**: retiree is cognitively intact (scores MMSE ≥ 24),

- **State 2**: retiree is mildly cognitively impaired (scores MMSE 18-23) and cannot perform at least one ADL,
- **State 3**: retiree is moderately cognitively impaired (scores MMSE 10-17) and cannot perform at least one ADL,
- **State 4**: retiree is severely cognitively impaired, the retiree scores MMSE $\leq$ 9 (cannot perform any ADLs) and
- **State 5**: retiree is dead.

The multi-state model has the state space, $\mathscr{S}^* = \{1,2,3,4,5\}$ and the set of transitions,

$$\begin{aligned}\mathscr{T}^* = \{\, &(1,2),(1,3),(1,4),(1,5),(2,1),(2,3),(2,4),(2,5),(3,1),\\ &(3,2),(3,4),(3,5),(4,3),(4,5)\,\} \ .\end{aligned}$$

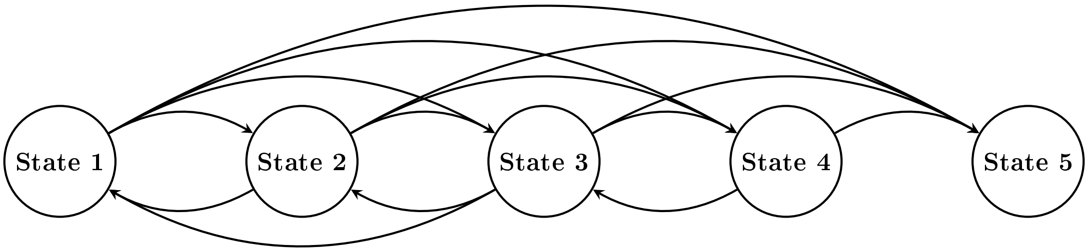

Figure 2.1: The Multi-State Model for Post-Retirement Long-Term Care for the Risk of Cognitive Impairment.

The proposed Markov multi-state model is the pair,

$$\begin{aligned}(\mathscr{S}^*, \mathscr{T}^*) = (&\{1,2,3,4,5\}\,, \{(1,2),(1,3),(1,4),(1,5),(2,1),(2,3),(2,4),(2,5),\\ &(3,1),(3,2),(3,4),(3,5),(4,3),(4,5)\}) \ .\end{aligned}$$

Quality of Care:We assume the long term market consists of a variety of long-term care service providers. These services differ one from another in quality of care, where quality of care is a publicly known parameter α. We associate a five star quality of care system to any and all long-term care facilities and services. For example, [8] in the United States uses a five-star rating system to rate long-term care facilities. We assume $\alpha \in [1,5]$ with higher α representing higher quality of care and $\bar{\alpha}$ is the standard (average) quality of care ($\bar{\alpha} = 3$). We assume that all transition intensities and long-term care service costs depend on α, that is, the higher the α, the higher the cost of long-term care and the higher the quality of care, the lower the transition intensity rates [27].

To determine the effect of quality of care on transition intensities, given that retirees are of the same age, with same medical conditions, and receiving different levels of quality of care, retirees receiving higher levels of quality of care, A1 have lower tendency to transition to higher cognitively impaired states (lower probability to get worse) and

A2 have a higher tendency to transition to lower cognitively impaired stated (higher recovery probabilities).

The quality-based transition intensity rates are defined as,

$$\mu_{x+t}^{ij}(\alpha) = \varphi_{ij}(\alpha)\mu_{x+t}^{ij} \qquad i,j \in \mathscr{S}^*, \tag{2.1}$$

where μ_{x+t}^{ij} is given by Equation (1.4), and φ is a function determining the impact of quality of care. The transition intensity $\mu_{x+t}^{ij}(\alpha)$ changes for each $\alpha \in [1,5]$. When defining the quality of care function φ, then the quality-based transition intensities should follow the following assumptions,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mu_{x+t}^{ij}(\alpha) = \begin{cases} 0 & \text{for } i=1 \text{ and } j = i+1, i+2, \ldots, 5 \\ \leq 0 & \text{for } i=2,3,4 \text{ and } j = i+1, i+2, \ldots, 5 \\ \geq 0 & \text{for } i=2,3,4 \text{ and } j = 1,2,\ldots,i-1 \\ \text{ambiguous} & \text{for } i=j \text{ and } j=1,2,3,4. \end{cases}$$

Let the transition probability from state i to state j given quality of care $\alpha \in [1,5]$ be denoted by ${}_tp_x^{ij}(\alpha)$, where ${}_tp_x^{ij}$ is given by Equation (1.9).

To show how quality of care affects complete future lifetime of the retiree, we use Equations (1.2) and (1.3). We denote the quality-based expected number of years spent in j if retiree is aged 65 in state i at time 0 by $\mathring{e}_{65}^{ij}(\alpha)$, defined as,

$$\mathring{e}_{65}^{ij}(\alpha) = \int_0^\infty {}_tp_{65}^{ij}(\alpha)dt, \qquad i,j = 1,2,3,4 \text{ and } \alpha = 1,2,3,4,5. \tag{2.2}$$

We denote quality-based complete future lifetime of the retiree aged 65 in any state i by $\mathring{e}_{65}^{i}(\alpha)$, defined as,

$$\mathring{e}_{65}^{i}(\alpha) = \sum_{j=1}^{4} \mathring{e}_{65}^{ij}(\alpha), \qquad i = 1,2,3,4 \text{ and } \alpha = 1,2,3,4,5. \tag{2.3}$$

2.2 Long-Term Care related to the Model

Following the theory in Chapter 1.2, we allocate the type of care for each state in our model.

We assume that a retiree who is in state 1 (cognitively intact), does not need any type of long-term care services, we assume that the retiree can perform all ADLs and does not need any assistance.

A retiree in state 2 however receives home health care without the need for a constantly supervised medical long-term care institution.

However, a retiree in states 3 and 4 requires constant formal medical long-term care. We assume that the retiree needs constant supervision and care from a certified long-term care facility.

Below is a table allocating types of care needed in each of the 5 states. We follow Table 11 in [1] and allocate types of services needed for each state.

We denote the various types of care with the following abbreviations,

- Inpatient Hospital (IH),
- Medical Provider (MP),
- Skilled Nursing Facility (SNF),
- Nursing Home (NH),
- Hospice care (HC),
- Home Health Care (HHC),
- Prescription Medications (PM).

In Table 2.1, we allocate the type of care for each state. We apply positive signs (+) to inclusive care and negative signs (-) indicate the type of care is not included in the health state.

Table 2.1: Types of Long Term Services for the Model.

States	**IH**	**MP**	**SNF**	**NH**	**HC**	**HHC**	**PM**
State 1	-	+	-	-	-	-	+
State 2	-	+	-	-	-	+	+
State 3	-	+	+	-	-	-	+
State 4	-	+	-	+	+	-	+
State 5	-	-	-	-	-	-	-

We establish that there is no long term facility needed for state 1, however, we assign transition costs to state 1 in the event of transitions from other states to state 1. Transitions to state 1 requires the routine prescription medication care allocated for transitions. This care essentially consists of confirmation by a medical professional for the transition. We demand in the model at time $t = 1, 2, \dots$ upon transition to state 1 a physical exam conducted by a medical provider.

In state 2, we assume the retiree can move around and perform certain tasks, however, there is the need for occasional home visits from a certified health professional. Our model requires certification upon transition to state 2 from other states (prescription medications) and at time $t = 1, 2, \ldots,$ a physical exam (medical provider).

Individuals in state 3 need constant care in our model, thus, we assign the retiree in state 3 to a skilled nursing facility, where the retiree gets constant support and care, nonetheless, transition costs are accrued upon transition to state 3.

In state 4, we assume the retiree needs lots of physical care, as such, we assign the retiree to a certified nursing home with hospice care. However in state 4, transition costs are still incurred in the event of transition to state 4 from other health states.

In this model, our focus and interest is on a retiree who is state 1 at time $t = 0$. That is, the retiree is healthy at age 65. However, there is possibility that the retiree is in state $i = 2, 3, 4$ at time $t = 0$. The model seeks to give the retiree an understanding of the expected cost incurred if he or she starts age 65 in any of the health states defined in Figure 2.1. For each state $i = 1, 2, 3, 4$ we define all costs incurred over the life time of the retiree over the period $[0, \infty)$. As in theory and our proposed model, we assume that three types of costs are incurred in every state, regular costs incurred by long-term care costs, lump sum costs incurred immediately upon transition and lump sum costs incurred at a specified time after transition. For our numerical illustration, we assume that quality of care directly affects regular costs (annuities), however, transition costs are not affected directly by quality of care.

2.3 An Illustrative Example

In our illustrative example, we look at two groups of retirees, males and females aged 65 at time 0. We assume the that the retiree of focus (aged 65 at time 0) is in state $i = 1, 2, 3, 4$ at time 0. We study the impact of quality of care α on lifetime total costs. We consider the impact of quality of care on transition intensities and transition probabilities at large.

2.3.1 Hypothetical Quality of Care Based Intensities

For our numerical calculations, we assume constant intensities for both males and females aged 65, that is, $\mu^{ij}_{65+t} = \mu^{ij}_{65}$. We also assume, that the transition intensities depend on the quality of care $\alpha = 1, 2, 3, 4, 5$.

We define our basic hypothetical transition intensities for males aged 65 at time 0 in Table 2.2.

Table 2.2: Transition Intensities for Males Aged 65 at Time 0.

	State j				
State i	State 1	State 2	State 3	State 4	State 5
State 1	-0.19355	0.15760	0.00748	0.00372	0.02475
State 2	0.07880	-0.21798	0.09456	0.00898	0.03564
State 3	0.01576	0.04728	-0.32699	0.22064	0.04331
State 4	0.00000	0.00000	0.00150	-0.12401	0.12251
State 5	0.00000	0.00000	0.00000	0.00000	0.00000

We assume that males and females have different transition rates. For the basic transition intensities of females aged 65 at time $t = 0$, we make a hypothetical assumption,

$$\text{female } \mu_{65}^{ij} = \begin{cases} 0.9\,\text{x male}\,\mu_{65}^{ij} & \text{if}\, i = i+1, i+2, \ldots, 5 \\ 1.0\,\text{x male}\,\mu_{65}^{ij} & \text{if}\, i = i-1, i-2, \ldots, 1. \end{cases}$$

This hypothetical assumption implies that, for transitions to higher cognitively impaired states, female's rates are slower (smaller) than the male's rates. However, both males and females are assumed to have the same recovery rates (from higher cognitive impairment to lower cognitive impairment). The transition intensities for females is given in Table 2.3.

Table 2.3: Transition Intensities for Females Aged 65 at Time 0.

	State j				
State i	State 1	State 2	State 3	State 4	State 5
State 1	-0.174195	0.141840	0.006732	0.003348	0.022275
State 2	0.078800	-0.204062	0.085104	0.008082	0.032076
State 3	0.015760	0.047280	-0.300595	0.198576	0.038979
State 4	0.000000	0.000000	0.001500	-0.111759	0.110259
State 5	0.000000	0.000000	0.000000	0.000000	0.000000

Based on Tables 2.2 and 2.3, transitions are allowed from state 1 to states 2, 3, 4 and 5. From state 2, we allow transitions to states 1, 3, 4 and 5. While the retiree is in state 3, permission is given to transition to states 1, 2, 4 and 5. In state 4 however, transitions are limited to states 3 and 5. There are no allowed transitions from state 5.

For our numerical example, we follow [27] and in Equation (2.1) set $\varphi_{ij}(\alpha)$ as follows

$$\varphi_{ij}(\alpha) = \begin{cases} 1 & \text{for } i = 1 \text{ and } j = i+1, i+2, \dots, 5 \\ e^{-g(1)(\alpha - \bar{\alpha})} & \text{for } i = 2,3,4 \text{ and } j = i+1, i+2, \dots, 5 \\ e^{g(2)(\alpha - \bar{\alpha})} & \text{for } i = 2,3,4 \text{ and } j = 1, 2, \dots, i-1 \\ \dfrac{\mu_x^{i\cdot}(\alpha)}{\mu_x^{i\cdot}} & \text{for } i = j \text{ and } j = 1,2,3,4 \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

where $\bar{\alpha} = 3$ is the standard (average) quality of care, and $g^{(1)} = 0.14$ and $g^{(2)} = 0.12$ are non-negative constants such that for males starting in health state 1 at time 0 the expected lifetime, given in Equation (2.3), of a male aged 65 is approximately 25. From 2.4, the transition intensities chances for each $\alpha = 1, 2, 3, 4, 5$. The different transition intensities for $\alpha = 1, 2, 3, 4, 5$ are given in Appendix A for both males and females aged 65 at time 0. For time $t = 1$ and time $t = 20$, the transition probability for both males and females aged 65 at time 0 for each $\alpha = 1, 2, 3, 4, 5$ and states $i = 1, 2, 3, 4, 5$ are given in Appendices B and C.

For each $\alpha = 1, 2, 3, 4, 5$ and assuming that the retiree aged 65 is in each state $i = 1, 2, 3, 4$ at time 0, then using Equations 2.2 and 2.3, we calculate the complete future lifetime for both males and females. If a retiree is initially at state $i = 1$, then the complete future lifetime is found as,

$$\mathring{e}^{1}_{65}(\alpha) = \int_0^\infty {}_tp^{11}_{65}(\alpha)dt + \int_0^\infty {}_tp^{12}_{65}(\alpha)dt + \int_0^\infty {}_tp^{13}_{65}(\alpha)dt + \int_0^\infty {}_tp^{14}_{65}(\alpha)dt.$$

If a retiree is initially at state $i = 2$, then the complete future lifetime is found as,

$$\mathring{e}^{2}_{65}(\alpha) = \int_0^\infty {}_tp^{21}_{65}(\alpha)dt + \int_0^\infty {}_tp^{22}_{65}(\alpha)dt + \int_0^\infty {}_tp^{23}_{65}(\alpha)dt + \int_0^\infty {}_tp^{24}_{65}(\alpha)dt.$$

Suppose a retiree is in state $i = 3$ at time 0, the complete future lifetime is found as,

$$\mathring{e}^{3}_{65}(\alpha) = \int_0^\infty {}_tp^{31}_{65}(\alpha)dt + \int_0^\infty {}_tp^{32}_{65}(\alpha)dt + \int_0^\infty {}_tp^{33}_{65}(\alpha)dt + \int_0^\infty {}_tp^{34}_{65}(\alpha)dt.$$

If a retiree is initially at state $i = 1$, the complete future lifetime is found as,

$$\mathring{e}^{4}_{65}(\alpha) = \int_0^\infty {}_tp^{41}_{65}(\alpha)dt + \int_0^\infty {}_tp^{42}_{65}(\alpha)dt + \int_0^\infty {}_tp^{43}_{65}(\alpha)dt + \int_0^\infty {}_tp^{44}_{65}(\alpha)dt.$$

In Table 2.4, the complete future lifetime of the male retiree, if he is in state $i = 1, 2, 3, 4$ at time 0, is given.

Table 2.4: Complete Expected Future Lifetime For Males.

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	16.089	17.870	19.932	22.306	25.007
State 2	12.854	14.946	17.367	20.148	23.307
State 3	8.735	10.250	12.070	14.261	16.892
State 4	6.188	7.126	8.210	9.465	10.923

In Table 2.4, it is seen that, if the retiree is in state $i = 1, 2, 3, 4$ at time 0, for each state $i = 1, 2, 3, 4$, as α increases, complete future lifetime increases.
We consider Table 2.5, where the complete expected future lifetime for females, if the retiree aged 65 is in state $i = 1, 2, 3, 4$ at time 0, is given.

Table 2.5: Complete Expected Future Lifetime For Females.

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	18.037	20.054	22.390	25.073	28.116
State 2	14.474	16.844	19.585	22.728	26.284
State 3	9.808	11.533	13.613	16.121	19.131
State 4	6.878	7.923	9.131	10.531	12.160

In Table 2.5, it is seen that, if the retiree is in state $i = 1, 2, 3, 4$ at time 0, for each state $i = 1, 2, 3, 4$, as α increases, complete future lifetime for females increases.

2.3.2 Hypothetical Quality of Care Based Cost of Care

We assume that for state 1, the cost of long-term care incurred is 0. This is because, in our model, we assumed that the retiree in state 1 does not need any sort of long-term care. However, since the retiree has possibility to transition back to state 1 from other states, we assign transition costs. We assume that in order to return to cognitively intact state, there is a need for a medical exam, in our model, we assigned this type of care as prescription medications in Table 2.1. After returning to state 1, there is a need for an annual physical exam, in our model, this care is called medical provider in Table 2.1.

For state 2, in our model proposal, we suggested that the retiree receives home health care services. For home health care services, we assume that there is a difference in the quality of care received. The cost for this service differs along with the quality of care. We assume that for the highest or best home health care service, $\alpha = 5$, the annual cost of home health care services at time 0 is $\$75,000$, however, for the lowest quality of care, the annual cost of home health care services at time 0 is $\$30,000$. We assume that home health care services are paid regularly over the lifetime of the retiree aged 65. For transition into state 2 from other states, there are transition costs incurred that do not depend on quality of care. We assume that there is a need for a medical exam to confirm this transition. This cost is incurred once only upon transition into state 2. As with our model, upon transition, we allow one time costs (lump sum) after every year. For our numerical calculations, we name this cost as costs for annual physical exam upon transition.

For state 3, we assumed that the retiree receives care from a skilled nursing facility. Since skilled nursing facility is a type of long-term care service, we assume quality of care is implied. For the highest quality of care, $\alpha = 5$, the annual cost of care at time 0 is $\$110,000$, while the annual cost of care for the lowest quality, $\alpha = 1$, at time 0 is $\$110,000$. As with other long-term care services defined in our proposed model, the costs incurred are regular over the lifetime of the retiree. Following our model, there are two types of transition costs incurred for state 3 that do not depend directly on quality of care. These costs are, transition cost incurred immediately upon transition and transition cost incurred annually after transition. We assume that these costs are incurred only upon transitions. We assign a transition medical exam cost (prescription medications in our model) as the one time payment immediately upon transition. We allow a physical exam upon transition to state 3 (medical provider in our model) every year.

We consider the costs incurred for state 4. In our model we assigned the retiree in state 4 to nursing facility with hospice care. For this type of care, we assume that there are differences in the quality of care. With the highest quality of care, $\alpha = 5$, we assume an annual cost of $\$150,000$ at time 0. For the lowest quality of care, $\alpha = 1$, we assume an annual cost of $\$80,000$ at time 0. As with other long-term care costs, we assume that these costs are incurred over the lifetime of the retiree. Following our proposed model, we assign transition costs, medical exam (prescription medication in model) and annual physical exam (medical provider in model).

We follow the example of values from [7] and we assign costs to states $1, 2, 3 \, \text{and} \, 4$ at time 0. For long-term care services, we set the highest possible cost for the highest quality of care, $\alpha = 5$, and the lowest possible cost for the lowest quality of care, $\alpha = 1$. We assume that there are no costs incurred for state 5.

The annual cost of care at any time t under this numerical example depends of quality of care α, this cost is denoted by $c^j(t, \alpha)$, and $c^j(t, \alpha) = c^j(0, \alpha)\mathrm{e}^{\gamma t}$, where γ is the annual continuous inflation rate. In Table 2.6, $c^j(0, \alpha)$ is the cost incurred at time 0 if the retiree aged 65 at time 0 moves to state $j = 1, 2, 3, 4$. We define the individual costs at time 0 in Table 2.6.

Table 2.6: Illustrative Data with Relevant Parameter Values at time $t = 0$.

	State j			
	State 1	State 2	State 3	State 4
Annual Cost of long-term care, $c^j(0, \alpha = 1)$	0	$\$30,000$	$\$50,000$	$\$80,000$
Annual Cost of long-term care, $c^j(0, \alpha = 5)$	0	$\$75,000$	$\$110,000$	$\$150,000$
Transition Medical Exam, $C^{ij}(0)$	$250	$500	$700	$\$1,000$
Annual Physical Exam, $C^j(0)$	$200	$200	$300	$400

At time $t \geq 0$ we take $C^{ij}(t) = C^{ij}(0)\mathrm{e}^{\gamma t}$, and $C^{j}(t) = C^{j}(0)\mathrm{e}^{\gamma t}$, where γ is the annual continuous inflation rate, which indicates that costs are increasing with time.

The values in Table 2.6 are assumed to be the current cost of care (costs at time $t = 0$). Every time the retiree transitions to any state, we require a medical exam from a certified health practitioner to confirm that indeed the retiree has transitioned. The transition exams are incurred once, every time there is a transition.
Though annual physical exam and transition medical exam are both lump sums incurred upon transitions, in annual physical exam, the timing is specific. For physical exam cost, the retiree is required to visit the medical practitioner for a yearly routine checkup. This cost is incurred irrespective of the health state of the retiree.
The values we use for this numerical example are arbitrary and are not the exact costs of long-term care in any country or institution.

With each α, we use a linear interpolation method to find the annual cost of long-term care at time 0. We follow Equation (70) in [27]. Let $c^{j}(0, \alpha)$ denote the annual cost of long-term care for each $\alpha = 1, 2, 3, 4, 5$, for state j at time 0, for $1 \leq \alpha \leq 5$. Then for each level of α, annual cost of long-term care at time 0 is given by,

$$c^{j}(0, \alpha) = \left(\frac{5-\alpha}{4}\right) c^{j}(0, 1) + \left(\frac{\alpha - 1}{4}\right) c^{j}(0, 5).$$

where $c^{j}(0, 1)$ is defined in Table 2.6 as Annual Cost of long-term care, $c^{j}(t = 0, \alpha = 1)$ and $c^{j}(0, 5)$ is defined in Table 2.6 as Annual Cost of long-term care, $c^{j}(t = 0, \alpha = 5)$. These costs only apply to long-term care services, and is not related to transition lump sum costs. Following our linear interpolation method to find the cost of care for each $\alpha = 2, 3, 4$ at time 0, $c^{j}(0, \alpha)$, we arrive at the figures in Table 2.7 below for states 2, 3 and 4. In Table 2.6, the cost of care for time 0 was given for $\alpha = 1$, and 5.

Table 2.7: Annual Cost of long-term care.

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
$c^{2}(0, \alpha)$	\$30,000	\$41,250.00	\$52,500.00	\$63,750.00	\$75,000.00
$c^{3}(0, \alpha)$	\$50,000.00	\$65,000.00	\$80,000.00	\$95,000.00	\$110,000.00
$c^{4}(0, \alpha)$	\$80,000.00	\$97,500.00	\$115,000.00	\$132,500.00	\$150,000.00

We assume that all transition lumps sum costs do not depend directly on quality of care, α, as such there are no special functions for transition costs.
In calculating the actuarial present values of lifetime cost incurred if the retiree is in state $i = 1, 2, 3, 4$ at time 0, we assume that if the retiree chooses a quality of care α, their transition intensities are immediately influenced by α. As such, even though transition lump sum costs do not depend directly on α, we calculate the expected lifetime transition

lump sum costs for various α using the transformed transition intensities for $\alpha = 1, 2, 3, 4$ for both males and females aged 65.

2.3.3 Actuarial Present Values of the Cost of Care

Assume that the retiree aged 65 is in one of the states $i = 1, 2, 3, 4, 5$ at time $t \geq 0$. Then, given that the retiree is in $i = 1, 2, 3, 4$ at time 0, depending on quality of care, $\alpha = 1, 2, 3, 4, 5$, the actuarial present value of lifetime total costs can be found for each state $i = 1, 2, 3, 4$. For the formulas of actuarial present value of costs, we denote $C^{ij}(0) = C^{ij}$, $c^j(0, \alpha) = c^j(\alpha)$ and $C^j(0) = C^j$. For calculations, we assume an annual force of interest, δ' and an annual force of inflation of γ. Hereby, the calculation in Equation (1.18) are done at real annual force of interest $\delta = \delta' - \gamma$. Following Equation (1.18), the actuarial present value of lifetime total cost given that retiree aged 65 is in state i at time 0 is given as,

$$\mathscr{C}^i(0, \infty, \alpha) = \sum_{j=1}^{4} c^j(\alpha)\bar{a}_{65}^{ij}(\alpha) + \sum_{j=1}^{4} C^{ij}\bar{A}_{65}^{ij}(\alpha) + \sum_{t=1}^{\infty}\sum_{j=1}^{4} C^j {}_tE_{65}^{ij}(\alpha) \tag{2.5}$$

where,

$$\bar{A}_{65}^{ij}(\alpha) = \sum_{j:j\neq k} \bar{A}_{65}^{ikj}(\alpha).$$

For $i, k, j \in \{1, 2, 3, 4, 5\}$, the formulae for $\bar{a}_x^{ij}(\alpha)$, $\bar{A}_x^{ikj}(\alpha)$, $\bar{A}_x^{ij}(\alpha)$ and ${}_tE_{65}^{ij}(\alpha)$ are given by Equations (1.11), (1.13), (1.14) and (1.16) respectively. The parameter α considers the impact of quality of care on transition intensities given by 2.1 with $\varphi_{ij}(\alpha)$ given by 2.4.

The actuarial present value of lifetime total costs depending on transition intensities in Tables 2.2 and 2.3 as well as costs given in Table 2.6 can be expressed (for both males and females aged 65) for state $i = 1$ and for each $\alpha = 1, 2, 3, 4, 5$ as,

$$\begin{aligned}\mathscr{C}^1(0, \infty, \alpha) =& c^2(\alpha)\bar{a}_{65}^{12}(\alpha) + c^3(\alpha)\bar{a}_{65}^{13}(\alpha) + c^4(\alpha)\bar{a}_{65}^{14}(\alpha) \\ &+ C^{12}\bar{A}_{65}^{12}(\alpha) + C^{13}\bar{A}_{65}^{13}(\alpha) + C^{14}\bar{A}_{65}^{14}(\alpha) \\ &+ \sum_{t:t=1}^{\infty}(C^1 {}_tE_{65}^{11}(\alpha) + C^2 {}_tE_{65}^{12}(\alpha) + C^3 {}_tE_{65}^{13}(\alpha) + C^4 {}_tE_{65}^{14}(\alpha)),\end{aligned} \tag{2.6}$$

where,

$$\begin{aligned}\bar{A}_{65}^{12}(\alpha) &= \bar{A}_{65}^{112}(\alpha) + \bar{A}_{65}^{132}(\alpha) + \bar{A}_{65}^{142}(\alpha) \\ \bar{A}_{65}^{13}(\alpha) &= \bar{A}_{65}^{113}(\alpha) + \bar{A}_{65}^{123}(\alpha) + \bar{A}_{65}^{143}(\alpha) \\ \bar{A}_{65}^{14}(\alpha) &= \bar{A}_{65}^{114}(\alpha) + \bar{A}_{65}^{124}(\alpha) + \bar{A}_{65}^{134}(\alpha).\end{aligned}$$

Assuming the retiree aged 65 (time 0) is in state 2, then by Equation (2.5), the actuarial present value of lifetime total cost, $\mathscr{C}^2(0,\infty,\alpha)$, is,

$$\begin{aligned}\mathscr{C}^2(0,\infty,\alpha) =& c^2(\alpha)\bar{a}^{22}_{65}(\alpha) + c^3(\alpha)\bar{a}^{23}_{65}(\alpha) + c^4(\alpha)\bar{a}^{24}_{65}(\alpha)\\ &+ C^{21}\bar{A}^{21}_{65}(\alpha) + C^{23}\bar{A}^{23}_{65}(\alpha) + C^{24}\bar{A}^{24}_{65}(\alpha)\\ &+ \sum_{t:t=1}^{\infty}(C^1{}_tE^{21}_{65}(\alpha) + C^2{}_tE^{22}_{65}(\alpha) + C^3{}_tE^{23}_{65}(\alpha) + C^4{}_tE^{24}_{65}(\alpha)),\end{aligned} \tag{2.7}$$

where,

$$\begin{aligned}\bar{A}^{21}_{65}(\alpha) &= \bar{A}^{221}_{65}(\alpha) + \bar{A}^{231}_{65}(\alpha) + \bar{A}^{241}_{x}(\alpha)\\ \bar{A}^{23}_{65}(\alpha) &= \bar{A}^{213}_{65}(\alpha) + \bar{A}^{223}_{65}(\alpha) + \bar{A}^{243}_{65}(\alpha)\\ \bar{A}^{24}_{65}(\alpha) &= \bar{A}^{214}_{65}(\alpha) + \bar{A}^{224}_{65}(\alpha) + \bar{A}^{234}_{65}(\alpha).\end{aligned}$$

Suppose the retiree (male and female) aged 65 (time 0) is in state 3, then by Equation (2.5), the actuarial present value of lifetime total cost, $\mathscr{C}^3(0,\infty,\alpha)$, is,

$$\begin{aligned}\mathscr{C}^3(0,\infty,\alpha) =& c^2(\alpha)\bar{a}^{32}_{65}(\alpha) + c^3(\alpha)\bar{a}^{33}_{65}(\alpha) + c^4(\alpha)\bar{a}^{34}_{65}(\alpha)\\ &+ C^{31}\bar{A}^{31}_{65}(\alpha) + C^{32}\bar{A}^{32}_{65}(\alpha) + C^{34}\bar{A}^{34}_{65}(\alpha)\\ &+ \sum_{t:t=1}^{\infty}(C^1{}_tE^{31}_{65}(\alpha) + C^2{}_tE^{32}_{65}(\alpha) + C^3{}_tE^{33}_{65}(\alpha) + C^4{}_tE^{34}_{65}(\alpha)),\end{aligned} \tag{2.8}$$

where,

$$\begin{aligned}\bar{A}^{31}_{65}(\alpha) &= \bar{A}^{321}_{65}(\alpha) + \bar{A}^{331}_{65}(\alpha) + \bar{A}^{341}_{65}(\alpha)\\ \bar{A}^{32}_{65}(\alpha) &= \bar{A}^{312}_{65}(\alpha) + \bar{A}^{332}_{65}(\alpha) + \bar{A}^{342}_{65}(\alpha)\\ \bar{A}^{34}_{65}(\alpha) &= \bar{A}^{314}_{65}(\alpha) + \bar{A}^{324}_{65}(\alpha) + \bar{A}^{334}_{65}(\alpha).\end{aligned}$$

Given that the retiree aged 65 (time 0) is in state 4, then by Equation (2.5), the actuarial present value of lifetime total cost, $\mathscr{C}^4(0,\infty,\alpha)$, is,

$$\begin{aligned}\mathscr{C}^4(0,\infty,\alpha) =& c^2(\alpha)\bar{a}^{42}_{65}(\alpha) + c^3(\alpha)\bar{a}^{43}_{65}(\alpha) + c^4(\alpha)\bar{a}^{44}_{65}(\alpha)\\ &+ C^{41}\bar{A}^{41}_{65}(\alpha) + C^{42}\bar{A}^{42}_{65}(\alpha) + C^{43}\bar{A}^{43}_{65}(\alpha)\\ &+ \sum_{t:t=1}^{\infty}(C^1{}_tE^{41}_{65}(\alpha) + C^2{}_tE^{42}_{65}(\alpha) + C^3{}_tE^{43}_{65}(\alpha) + C^4{}_tE^{44}_{65}(\alpha)),\end{aligned} \tag{2.9}$$

where,

$$\begin{aligned}\bar{A}^{41}_{65}(\alpha) &= \bar{A}^{421}_{65}(\alpha) + \bar{A}^{431}_{65}(\alpha) + \bar{A}^{441}_{65}(\alpha)\\ \bar{A}^{42}_{65}(\alpha) &= \bar{A}^{412}_{65}(\alpha) + \bar{A}^{432}_{65}(\alpha) + \bar{A}^{442}_{65}(\alpha)\\ \bar{A}^{43}_{65}(\alpha) &= \bar{A}^{413}_{65}(\alpha) + \bar{A}^{423}_{65}(\alpha) + \bar{A}^{443}_{65}(\alpha).\end{aligned}$$

In our calculations, to solve the transition probabilities for constant transition intensities, given by Equations (1.8), we employ `R` [26] package `expm` by [32] that provides a function to compute the matrix exponential of a real, square matrix. To find the actuarial present values (APV) for lifetime annual costs and transition medical exam costs, we use the numerical integration function `integrate` available in [26]. For calculations, we assume an annual force of interest, $\delta' = 5\%$, and, following [27], we suggest an annual force of inflation of $\gamma = 3.5\%$. Hereby, calculation in Equations (4.3)-(4.6) are done at real annual force of interest $\delta = \delta' - \gamma = 1.5\%$.

APVs of Lifetime Annual Cost

Following the first lines of Equations (2.6), (2.7), (2.8) and (2.9), the actuarial present values of lifetime annual long term costs are found.

Recall that the corresponding costs at time t were defined as $c^j(t, \alpha) = c^j(0, \alpha)\mathrm{e}^{\gamma t}$, with $c^j(0, \alpha)$, $j = 2, 3, 4$, $\alpha = 1, 2, 3, 45$, where the constant costs at $t = 0$ are given in Table 2.7. The actuarial present values of lifetime annual long term costs, are given in Table 2.8 (for males) and Table **??** (for Females), assuming that the retiree aged 65 at time $t = 0$ is in state $i = 1, 2, 3$ or 4 at time t=0.

Table 2.8: Actuarial Present Value of Lifetime Annual Cost for Males

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	\$373,226	\$528,752	\$708,451	\$910,760	\$1,131,765
State 2	\$459,855	\$652,149	\$873,697	\$1,125,045	\$1,392,322
State 3	\$492,194	\$690,423	\$927,550	\$1,205,793	\$1,524,716
State 4	\$451,886	\$625,592	\$836,655	\$1,090,731	\$1,393,744

In Table 2.8, we see that as quality of care, α, gets higher, the actuarial present value of lifetime annual long term cost gets higher. This is explained by the effect of α on transition intensities and in effect, on transition probabilities.

For starting in State 1, the lifetime annual costs for long-term care for each α is the sum of the lifetime annual costs of long-term care starting at time $t = 0$ to ∞. We add the lifetime annual cost of long-term care incurred while the retiree moves from State 1 at time $t = 0$ to States $j = 2, 3, 4$.

Transition probabilities of moving to a particular health state is vital in determining the actuarial present value of costs. The results in Table 2.8 show that, the probability of moving between states ${}_tp_x^{ij}(\alpha)$ and the value of annual cost of care for quality of care $\alpha = 1, 2, 3, 4, 5$ are important factors in determining the actuarial present value of long term costs, higher transition probabilities (chances of staying or moving to a health state) and higher cost of care produces higher actuarial present values of lifetime costs.

In Table 2.9, we have the actuarial present value of lifetime annual long term cost of care if a female retiree aged 65 at time 0 is in state $i = 1, 2, 3$ or 4 at time 0.

Table 2.9: Actuarial Present Value of Lifetime Annual Cost for Females

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	\$401,387	\$566,363	\$755,113	\$965,060	\$1,191,053
State 2	\$498,069	\$703,411	\$937,582	\$1,196,945	\$1,474,579
State 3	\$538,992	\$754,233	\$1,010,021	\$1,307,558	\$1,644,722
State 4	\$497,399	\$687,708	\$918,366	\$1,195,228	\$1,524,303

In Table 2.9, there are similar patterns in the actuarial present value of lifetime annual long-term care costs for females as with the males. As with actuarial present value of the cost of annual lifetime long-term care for males, the results of the female actuarial present value of annual lifetime long-term care costs are greatly influenced by transition probabilities and costs of care. The expected costs for both males and females increases as α increases.

In Figure 2.2, the actuarial present value of lifetime annual long term cost of care for males and females when the retiree aged 65 at time 0 is in state $i = 1$ at time 0 is visually represented.

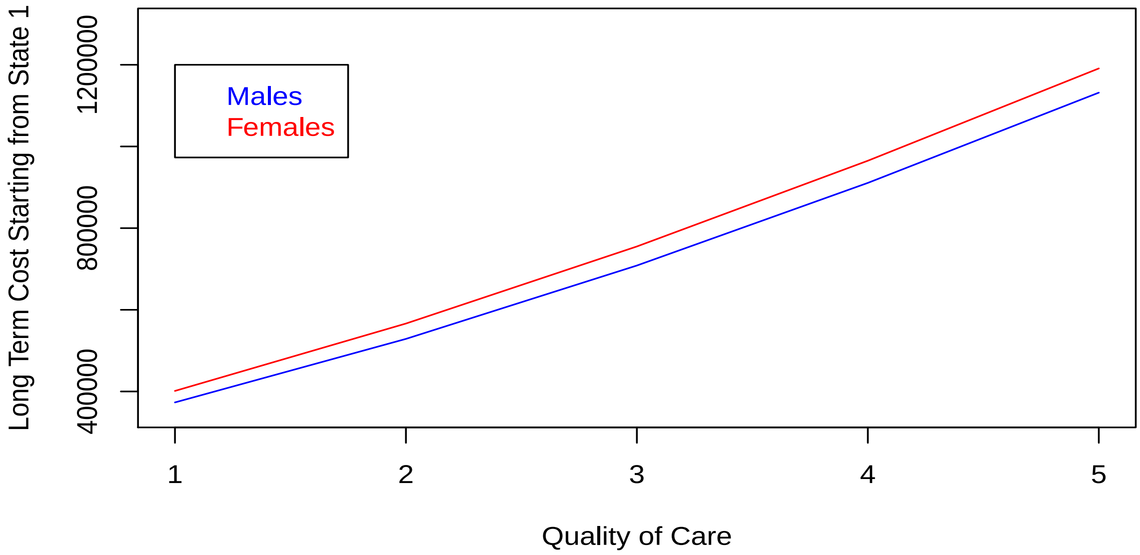

Figure 2.2: Actuarial Present Value of Annual Lifetime Costs of Care For Different Levels of Quality of Care.

In Figure 2.2, the values for females is greater than the values for males. This is attributed to the hypothetical relationship that was derived between males and females.

Since transition intensity from lower cognitive impaired states is lower in females, then moving within states is much slower for females than for males, since transitions are generally slower, then females tend to stay relatively longer in a state than males, thus accruing more costs. Higher probability of transitioning into a particular health state, higher cost care at a particular time, coupled with higher probabilities of staying in the state results in higher costs (the longer the stay in an expensive facility, the higher the cost to be incurred).

APVs of Lifetime Transition Medical Exam Cost

We consider lump sum costs accrued due to transition medical exam. These costs are not directly influenced by quality of care, however, the transition intensities used in calculating the actuarial present values are affected by quality of care. The actuarial present value of lifetime transition medical exam costs are found using the second lines of Equations (2.6), (2.7), (2.8) and (2.9). We use the transition cost given in Table 2.6 of the destination state for our calculations. In Table 2.10 are actuarial present values of lifetime transition costs incurred due to medical exam for males. We consider costs when the retiree aged 65 at time 0 is in state $i = 1, 2, 3$ or 4 at time $t = 0$ for all $\alpha = 1, 2, 3, 4, 5$.

Table 2.10: Actuarial Present Value of Lifetime Transition Medical Exam Cost for Males

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	$\$1,299$	$\$1,312$	$\$1,330$	$\$1,352$	$\$1,379$
State 2	$\$1,049$	$\$1,047$	$\$1,046$	$\$1,047$	$\$1,073$
State 3	$859	$869	$899	$933	$980
State 4	$6	$8	$10	$14	$19

From Equation (1.12), the actuarial present values of transition exam costs are affected by both the rate of transition $\mu_{x+t}^{ij}(\alpha)$ and transition probabilities ${}_tp_x^{ij}(\alpha)$. In Table 2.10, we assume that the retiree is in state $i = 1, 2, 3, 4$ at time $t = 0$. We estimate the actuarial present value of lifetime transition costs that is likely to be incur suppose the retiree who is aged 65 at time 0 and in state $i = 1, 2, 3, 4$ at time $t = 0$ transitions to other health states $j = 1, 2, 3, 4$ over his lifetime. In Table 2.10 it is noticed that as α progresses, the actuarial present value of transition exam cost generally increases. This is explained by the effect of α on the transition intensity matrix and transition probabilities. quality of care α causes a decreases in transition rates from lower cognitive to higher cognitive impairment but increases transition rates from higher cognitive impairment to lower cognitive impairment (higher α means speedy recovery from cognitive impairment). As such, frequent transitions over the lifetime produces higher expected value of costs since every transition incurs lump sum transition exam costs. Thus irrespective of the time, multiple transitions within a year will accrue more costs.

We consider the actuarial present value of lifetime transition medical exam costs for females aged 65 at time 0 in state $i = 1, 2, 3$ or 4 at time 0 in Table 2.11.

Table 2.11: Actuarial Present Value of Lifetime Transition Medical Exam Cost for Females

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	$1,287	$1,300	$1,317	$1,339	$1,366
State 2	$1,040	$1,037	$1,036	$1,035	$1,065
State 3	$862	$880	$907	$944	$996
State 4	$6	$8	$12	$16	$22

There are similar patterns in Table 2.11 as in Table 2.14. Since the female transition intensity was formed based on the male transition intensity, it is expected that the actuarial present value of lifetime medical exam cost for males and females follow similar patterns. If the retiree is in state 4 at time 0, the overall actuarial present value of lifetime transition medical exam cost is smaller in Tables 2.10 and 2.11. This is explained by the very slow transition rate from state 4 to state 3. Since frequent transitions incur higher costs, very minimum transitions incur very minimum costs.

In Figure 2.3, there is a clear representation of how much the actuarial present value of lifetime medical costs for males differ from the actuarial present value of lifetime medical costs for females. In Figure 2.3, we assume that the retiree aged 65 at time 0 is in state $i = 1$ at time 0.

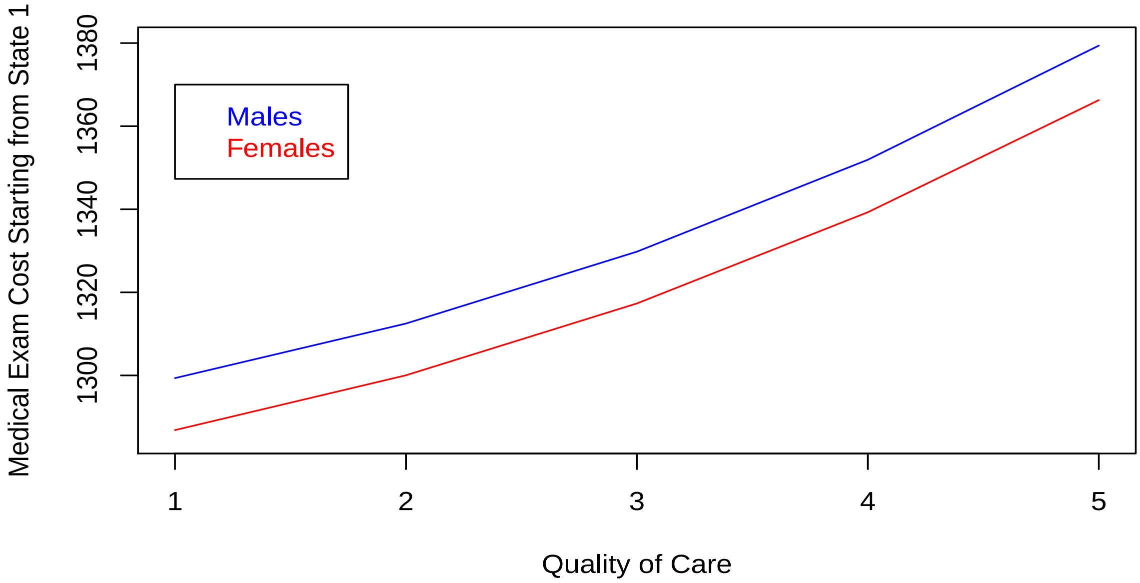

Figure 2.3: Actuarial Present Value of Lifetime Medical Exam Costs For Different Levels of Quality of Care.

In Figure 2.3, there is an increase in actuarial present value of lifetime medical cost as α increased for both males and females. However, in Figure 2.3, it is seen that, the actuarial present value of cost of lifetime medical exam for males is higher than the actuarial present values of lifetime medical cost for females. The actuarial present value of lifetime medical cost is affected by both transition probabilities and transition intensities. Transition intensities from State 1 to other states is higher for males than for females, as such, males tend to move more frequently to other states than females, since every transitions incur costs, then frequent transitions incur frequent costs.

APVs of Lifetime Annual Physical Exam Cost

We use the third lines of Equations (2.6), (2.7), (2.8) and (2.9) to calculate the actuarial present value of lifetime annual physical exam for males and females aged 65. In Table 2.12, we give the actuarial present value of lifetime annual physical exam cost for males if the retiree aged 65 at time 0 is in state $i = 1, 2, 3, 4$ at time 0.

Table 2.12: Actuarial Present Value of Lifetime Annual Physical Exam Cost for Males

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	$\$3,220$	$\$3,515$	$\$3,832$	$\$4,165$	$\$4,508$
State 2	$\$2,842$	$\$3,209$	$\$3,601$	$\$4,012$	$\$4,433$
State 3	$\$2,534$	$\$2,897$	$\$3,300$	$\$3,742$	$\$4,217$
State 4	$\$2,067$	$\$2,374$	$\$2,719$	$\$3,104$	$\$3,532$

The dynamics in Table 2.12 are largely explained by the transition probability matrix at time t. In Table 2.12, if the retiree aged 65 at time 0 is in state $i = 1, 2, 3, 4$ at time 0, the actuarial present value of lifetime annual physical physical exam cost increases generally as α increases.

In Table 2.13, the actuarial present value of lifetime annual physical exam costs for females aged 65 at time 0 is given, if the retiree aged 65 at time 0 is in state $i = 1, 2, 3, 4$ at time 0.

Table 2.13: Actuarial present value of Lifetime Annual Physical Exam Cost for Females

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	$\$3,529$	$\$3,843$	$\$4,176$	$\$4,523$	$\$4,877$
State 2	$\$3,130$	$\$3,523$	$\$3,938$	$\$4,370$	$\$4,806$
State 3	$\$2,804$	$\$3,198$	$\$3,633$	$\$4,105$	$\$4,608$
State 4	$\$2,295$	$\$2,630$	$\$3,004$	$\$3,421$	$\$3,884$

As with other actuarial present value of costs, in Table 2.13, we notice similar patterns in the values of actuarial present value of lifetime annual physical costs if the female retiree aged 65 at time 0 is in state $i = 1, 2, 3, 4$ at time 0 and the actuarial present value of lifetime annual physical exam cost if the male retiree aged 65 at time 0 is in state $i = 1, 2, 3, 4$ at time 0. This, as established is due to the nature of the relationship between male and female transition intensities.

In Figure 2.4, the difference in actuarial present value of lifetime annual physical exam cost between males and females if the retiree aged 65 at time 0is in state $i = 1$ at time 0, is more clear.

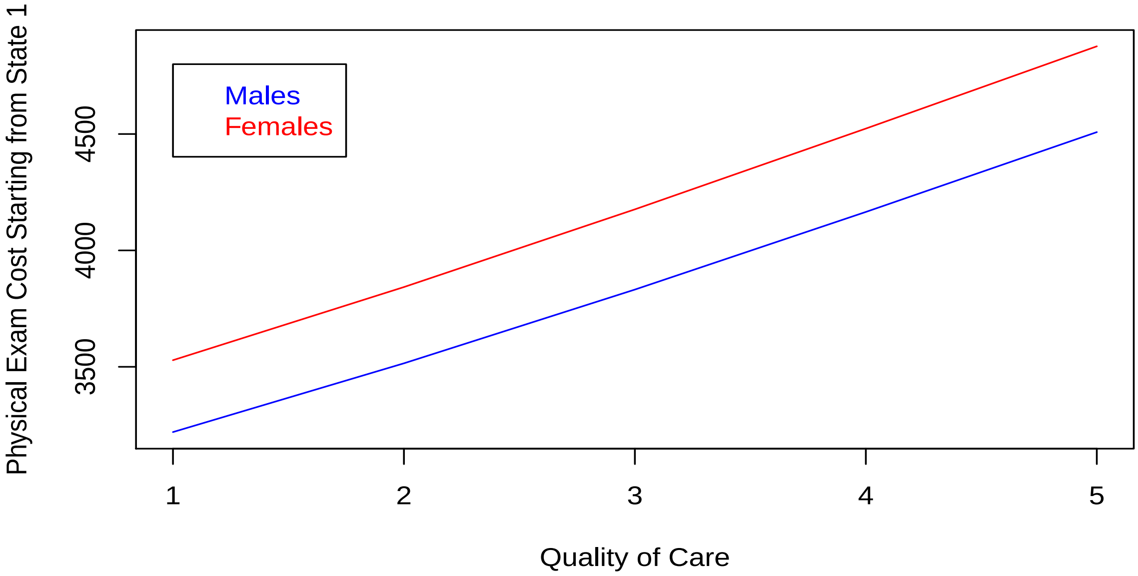

Figure 2.4: Actuarial Present Value of Lifetime Annual Physical Exam Costs For Different Levels of Quality of Care.

It is noticeable in Figure 2.4 that the females actuarial present value of lifetime annual physical exam cost is higher than the actuarial present value of lifetime annual physical exam for males. Since the Equation for finding the actuarial present value for physical exam costs, Equation (1.15), depends mostly on transition probabilities ${}_tp_x^{ij}(\alpha)$, the value of the transition probabilities greatly influences actuarial present value of costs. actuarial present value of lifetime annual physical exam cost increases along α because, as α increases, probability to stay in a particular state increases for state 1, however, the physical exam cost is incurred yearly irrespective of the state. Since females have relatively higher probabilities of staying in a particular state and lower transition rates, the costs incurred would be higher than the actuarial present value of cost incurred for males.

APVs of Lifetime Total Costs

We sum up all actuarial present value of lifetime costs as shown in Equation (2.6). We find the actuarial present value of lifetime total costs if the male retiree aged 65 at time

0 is in state $i = 1, 2, 3, 4$ at time 0 and $\alpha = 1, 2, 3, 4, 5$ in the Table 2.14.

Table 2.14: Actuarial Present Value of Lifetime Total Costs For Males

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	$\$377,745$	$\$533,580$	$\$713,612$	$\$916,277$	$\$1,137,652$
State 2	$\$463,746$	$\$656,404$	$\$878,343$	$\$1,130,103$	$\$1,397,827$
State 3	$\$495,587$	$\$694,189$	$\$931,749$	$\$1,210,434$	$\$1,529,913$
State 4	$\$453,959$	$\$627,974$	$\$839,385$	$\$1,093,849$	$\$1,397,296$

From Table 2.14, the actuarial present value of lifetime total cost increases as quality of care gets higher for each state $i = 1, 2, 3, 4$, thus supporting the work of [27]. The actuarial present value of lifetime total costs for each α is the costs that the retiree will accrue over his lifetime if he is in state $i = 1, 2, 3, 4$ at time $t = 0$. For example, for State 1, actuarial present value of lifetime total cost for $\alpha = 2$, $\$533,560$ means that, the retiree is likely to accrue $\$533,560$ over his entire lifetime after retirement if he is cognitively intact (State 1) at age 65 (at time 0) and if he chooses a long-term care service of quality $\alpha = 2$. This cost includes all the possible long-term care services with quality of care, $\alpha = 2$ and all transition costs over his entire lifetime.
We consider the actuarial present value of lifetime total cost for females aged 65 at time 0 suppose the retiree aged 65 at time 0 is in state $i = 1, 2, 3, 4$ at time 0 in Table 2.15.

Table 2.15: Actuarial Present Value of Lifetime Total Costs For Females

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
State 1	$\$406,202$	$\$571,505$	$\$760,606$	$\$970,923$	$\$1,197,296$
State 2	$\$502,240$	$\$707,971$	$\$942,556$	$\$1,202,350$	$\$1,480,450$
State 3	$\$542,658$	$\$758,312$	$\$1,014,561$	$\$1,312,570$	$\$1,650,327$
State 4	$\$499,701$	$\$690,346$	$\$921,382$	$\$1,198,666$	$\$1,528,208$

Similar to Table 2.14, in Table 2.15, as quality of care, α increases, the actuarial present value of lifetime total cost increases. These results are largely influenced by the actuarial present value of lifetime annual costs. It is seen that values from Tables 2.8 and 2.9 greatly influences values in Tables 2.14 and 2.15 respectively.
In Figure 2.5, we focus on the actuarial present value of lifetime total cost for males and females suppose they are aged 65 at time 0 and in state $i = 1$ at time 0.

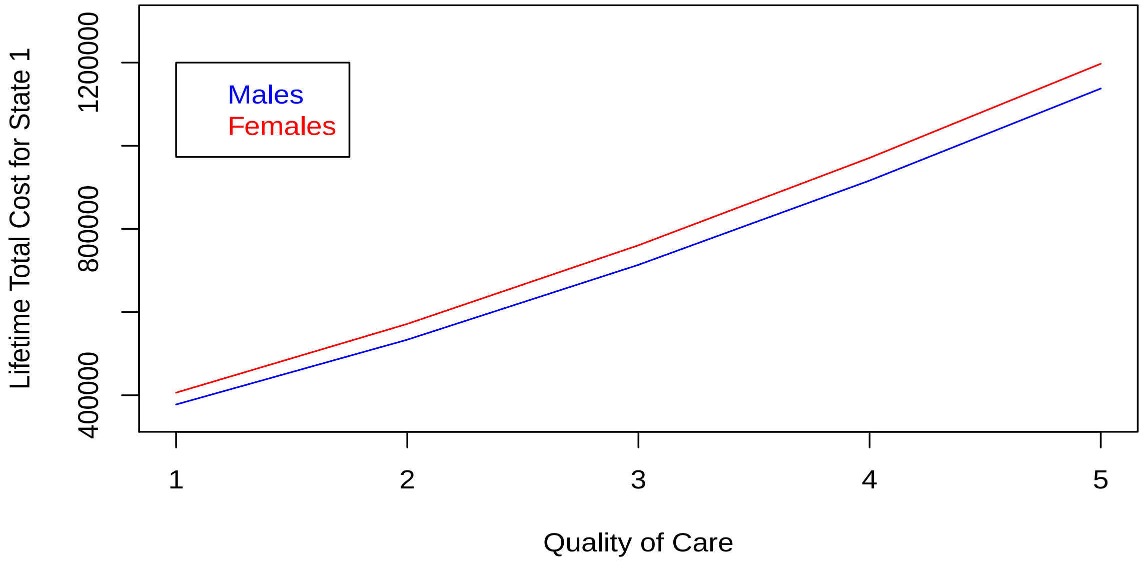

Figure 2.5: Actuarial Present Value Costs For Lifetime Total Costs For Different Levels of Quality of Care.

In Figure 2.5, it is noticeable that the pattern is similar to the pattern in Figure 2.2. This is because the actuarial present value of long term costs greatly influences the actuarial present value of lifetime total cost. Since long-term care cost is the most expensive of all the costs, taking into account that the payments are continuous over the lifetime of the retiree, it is implied that majority of the total cost incurred by the retiree would be due to cost from long-term care services.

2.4 Discussion

The transition intensities and costs of care used in this numerical example are all hypothetical and do not reflect the current conditions and situation for long-term care products, however, our results strongly suggests that the rate of transition, cost of the long-term care and the quality of care provided greatly influences the actuarial present value of lifetime total cost of care incurred for the retiree aged 65 at time 0. Though our data was generated hypothetically, the results showed that quality of care does play an important role in the overall total cost of care. It is understood that home health care facilities costs are per hour, while other facilities costs are per month [31]. As such, the longer you stay in a long term facility, the higher the costs to be incurred.

In our results, it was seen that quality of care influences the complete expected future lifetime of the retiree. As such quality of care is an important parameter in determining the type of long-term care service to use.

Conclusion

In this thesis, we focused on allocating types of care, costs associated with types of care as well as quality of care to defined states in a multi-state model. The results from our numerical analysis supports the necessity of quality of care. The results confirms that quality of care plays a vital role in the selection of long-term care products. Retirees should have all necessary information to make better decisions as to which quality of care they prefer. Our results supports the claim that with higher quality of care gives higher costs [27].

This thesis supports [10] and [31] that retirees need to know which types of long-term care products are available to them. With these results, it can be noticed that costs for long-term care is not very affordable. The contribution of this thesis is to give the retiree a fair knowledge of the possibilities of long-term care services, possible costs and quality of care.

For future research, investigations can be made into defining more states for all possible health states, especially for cognitively intact retirees. Studies can also be made on risk of cognitively impairment for all ages between 60 and 70. In this thesis, quality of care directly affected only the annual cost of long-term care, however, in future researches, the impact of quality of care on transition lump sum costs can be studied.

Appendix A

Constant Intensities For the Levels of the Quality of Care

Table A.1: Males (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 1$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.194	0.158	0.007	0.004	0.025
State 2	0.062	-0.246	0.125	0.012	0.047
State 3	0.012	0.037	-0.399	0.292	0.057
State 4	0.000	0.000	0.001	-0.163	0.162
State 5	0.000	0.000	0.000	0.000	0.000

Table A.2: Females (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 1$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.174	0.142	0.007	0.003	0.022
State 2	0.062	-0.228	0.113	0.011	0.042
State 3	0.012	0.037	-0.364	0.263	0.052
State 4	0.000	0.000	0.001	-0.147	0.146
State 5	0.000	0.000	0.000	0.000	0.000

Table A.3: Males (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 2$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.194	0.158	0.007	0.004	0.025
State 2	0.070	-0.230	0.109	0.010	0.041
State 3	0.014	0.042	-0.360	0.254	0.050
State 4	0.000	0.000	0.001	-0.142	0.141
State 5	0.000	0.000	0.000	0.000	0.000

Table A.4: Females (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 2$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.174	0.142	0.007	0.003	0.022
State 2	0.070	-0.214	0.098	0.009	0.037
State 3	0.014	0.042	-0.329	0.228	0.045
State 4	0.000	0.000	0.001	-0.128	0.127
State 5	0.000	0.000	0.000	0.000	0.000

Table A.5: Males (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 3$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.194	0.158	0.007	0.004	0.025
State 2	0.079	-0.218	0.095	0.009	0.036
State 3	0.016	0.047	-0.327	0.221	0.043
State 4	0.000	0.000	0.002	-0.124	0.123
State 5	0.000	0.000	0.000	0.000	0.000

Table A.6: Females (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 3$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.174	0.142	0.007	0.003	0.022
State 2	0.079	-0.204	0.085	0.008	0.032
State 3	0.016	0.047	-0.301	0.199	0.039
State 4	0.000	0.000	0.002	-0.112	0.110
State 5	0.000	0.000	0.000	0.000	0.000

Table A.7: Males (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 4$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.194	0.158	0.007	0.004	0.025
State 2	0.089	-0.210	0.082	0.008	0.031
State 3	0.018	0.053	-0.301	0.192	0.038
State 4	0.000	0.000	0.002	-0.108	0.107
State 5	0.000	0.000	0.000	0.000	0.000

Table A.8: Females (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 4$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.174	0.142	0.007	0.003	0.022
State 2	0.089	-0.198	0.074	0.007	0.028
State 3	0.018	0.053	-0.278	0.173	0.034
State 4	0.000	0.000	0.002	-0.098	0.096
State 5	0.000	0.000	0.000	0.000	0.000

Table A.9: Males (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 5$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.194	0.158	0.007	0.004	0.025
State 2	0.100	-0.205	0.071	0.007	0.027
State 3	0.020	0.060	-0.280	0.167	0.033
State 4	0.000	0.000	0.002	-0.094	0.093
State 5	0.000	0.000	0.000	0.000	0.000

Table A.10: Females (Aged 65 at $t = 0$) Constant Transition Intensities For Quality of Care $\alpha = 5$.

	State 1	State 2	State 3	State 4	State 5
State 1	-0.174	0.142	0.007	0.003	0.022
State 2	0.100	-0.195	0.064	0.006	0.024
State 3	0.020	0.060	-0.260	0.150	0.029
State 4	0.000	0.000	0.002	-0.085	0.083
State 5	0.000	0.000	0.000	0.000	0

Appendix B

Males Probabilities at $t = 1$ for the Levels of Quality of Care

Table B.1: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 1$ For Quality of Care $\alpha = 1$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.828	0.127	0.013	0.005	0.026
State 2	0.050	0.787	0.091	0.024	0.047
State 3	0.010	0.028	0.673	0.221	0.068
State 4	0.00001	0.00002	0.001	0.849	0.150
State 5	0.000	0.000	0.000	0.000	1

Table B.2: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 1$ For Quality of Care $\alpha = 2$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.829	0.128	0.012	0.005	0.026
State 2	0.057	0.801	0.081	0.020	0.041
State 3	0.012	0.032	0.700	0.198	0.058
State 4	0.00001	0.00002	0.001	0.868	0.131
State 5	0.000	0.000	0.000	0.000	1

Table B.3: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 1$ For Quality of Care $\alpha = 3$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.829	0.129	0.012	0.005	0.026
State 2	0.065	0.811	0.072	0.016	0.036
State 3	0.014	0.037	0.723	0.177	0.050
State 4	0.00001	0.00003	0.001	0.884	0.115
State 5	0.000	0.000	0.000	0.000	1

Table B.4: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 1$ For Quality of Care $\alpha = 4$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.830	0.129	0.011	0.005	0.025
State 2	0.073	0.818	0.064	0.013	0.031
State 3	0.016	0.043	0.742	0.157	0.042
State 4	0.00001	0.00004	0.001	0.898	0.101
State 5	0.000	0.000	0.000	0.000	1

Table B.5: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 1$ For Quality of Care $\alpha = 5$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.831	0.130	0.010	0.004	0.025
State 2	0.083	0.823	0.057	0.011	0.027
State 3	0.018	0.049	0.758	0.139	0.036
State 4	0.00002	0.00005	0.002	0.910	0.088
State 5	0.000	0.000	0.000	0.000	1

Appendix C

Males Probabilities at $t = 20$ for the Levels of Quality of Care

Table C.1: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 20$ For Quality of Care $\alpha = 1$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.068	0.085	0.037	0.110	0.700
State 2	0.036	0.048	0.023	0.094	0.799
State 3	0.007	0.010	0.005	0.062	0.915
State 4	0.0001	0.0001	0.0002	0.039	0.960
State 5	0.000	0.000	0.000	0.000	1

Table C.2: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 20$ For Quality of Care $\alpha = 2$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.082	0.106	0.045	0.121	0.646
State 2	0.050	0.069	0.031	0.114	0.736
State 3	0.012	0.017	0.008	0.089	0.874
State 4	0.0002	0.0003	0.0005	0.060	0.939
State 5	0.000	0.000	0.000	0.000	1

Table C.3: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 20$ For Quality of Care $\alpha = 3$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.099	0.129	0.051	0.127	0.593
State 2	0.069	0.093	0.039	0.130	0.668
State 3	0.019	0.027	0.013	0.118	0.823
State 4	0.0003	0.0005	0.001	0.086	0.912
State 5	0.000	0.000	0.000	0.000	1

Table C.4: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 20$ For Quality of Care $\alpha = 4$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.119	0.153	0.057	0.129	0.542
State 2	0.091	0.121	0.047	0.139	0.601
State 3	0.030	0.040	0.018	0.148	0.764
State 4	0.001	0.001	0.001	0.118	0.879
State 5	0.000	0.000	0.000	0.000	1

Table C.5: Males (Aged 65 at $t = 0$) Transition Probabilities at Time $t = 20$ For Quality of Care $\alpha = 5$.

	State 1	State 2	State 3	State 4	State 5
State 1	0.142	0.176	0.060	0.126	0.496
State 2	0.117	0.149	0.054	0.142	0.538
State 3	0.044	0.057	0.025	0.175	0.699
State 4	0.001	0.001	0.002	0.155	0.840
State 5	0.000	0.000	0.000	0.00	1

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