

DÉBORA AGUIAR GOMES

Theoretical and astrophysical
aspects of extended general
relativity



DISSERTATIONES PHYSICAE UNIVERSITATIS TARTUENSIS

137

DÉBORA AGUIAR GOMES

Theoretical and astrophysical
aspects of extended general
relativity



UNIVERSITY OF TARTU

Press

This study was carried out at the University of Tartu.

The dissertation was admitted on 07.05.2025 in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Physics, and was allowed for defence by the Council of the Institute of Physics, University of Tartu.

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The author was supported by the following grants: Gauge gravity: unification, extensions and phenomenology (PRG356), The Dark Side of the Universe (TK133) and Foundations of the Universe (TK202). Additional financial funding for visits and conferences was provided by Cost Actions CA18108 and CA21136, as well as the Erasmus Plus Traineeship Program.

ISSN 1406-0647 (print)

ISBN 978-9916-27-888-8 (print)

ISSN 2806-2523 (pdf)

ISBN 978-9916-27-889-5 (pdf)

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University of Tartu Press

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List of publications

The thesis is based on the following three publications.

- I D. A. Gomes, A. Wojnar. “Early evolution of fully convective stars in scalar-tensor gravity ,” *Eur.Phys.J.C* **83**, 492 (2023)
See Chapter 5, and arXiv:2206.04464 [INSPIRE]
- II D. A. Gomes, J. B. Jiménez, T. S. Koivisto. “General parallel cosmology,” *JCAP* **12**, 010 (2023)
See Chapter 6, and arXiv:2309.08554 [INSPIRE]
- III D. A. Gomes, J. B. Jiménez, A. J. Cano, T. S. Koivisto. “Pathological Character of Modifications to Coincident General Relativity: Cosmological Strong Coupling and Ghosts in $f(\mathbb{Q})$ Theories,” *Phys.Rev.Lett.* **132**, 141401 (2024)
See Chapter 7, and arXiv:2311.04201 [INSPIRE]

Author’s contribution

I, Débora Aguiar Gomes, have calculated and checked most of the calculations in each paper. In particular, in paper II, I checked some of the main derivations by following an independent formalism which was reported in appendices A-I. I was the corresponding author in paper I. I was also responsible for a significant part of the writing and implementation of peer review suggestions in this paper. I partially wrote papers II and III, but was involved in the discussions and research development.

I thank my supervisors, Tomi Koivisto and Aneta Wojnar, for their guidance and feedback. I also thank Dr. José Beltrán Jiménez for the valuable discussions when I visited him in Salamanca.

I presented the results of paper I in the “Tartu-Tuorla Cosmology seminar” (4-6 May 2022) and at the “8th Conference of the Polish Society on Relativity” (Warsaw, 19 -23 September 2022). Paper II was presented in a talk I gave at the Complutense University of Madrid on 31 October 2023 and at the Tartu University Theoretical Physics Journal Club. The three papers were collectively presented in the Theoretical Physics Seminar at the University of Tartu on 11 June 2024.

Chapter 1

Introduction

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1.1 Motivation

General Relativity (GR) [1, 2] has been the standard theory of gravity, thanks to its remarkable agreement with numerous tests. These tests include Mercury’s anomalous perihelion precession, gravitational redshift, and the deflection of light by massive objects [3]. Additionally, GR predicted the existence of gravitational waves and black holes, both of which have been observationally confirmed.

The first gravitational wave detection, made by the Laser Interferometer Gravitational-wave Observatory (LIGO) in 2015 and named GW150914 [4], was emitted by a binary black hole event (BBH). Other significant events include binary neutron star (BNS) mergers (e.g., GW170817 [5]) and neutron star-black hole (NSBH) mergers (e.g., GW200105 and GW200115 [6]). To date, over 180 events have been detected [7], and this number is increasing rapidly. With the launch of third-generation observatories – such as the Einstein Telescope (ET) [8] and the Cosmic Explorer (CE) [9] – the number of observed binary black hole (BBH) mergers is expected to rise significantly. For instance, the Einstein Telescope is expected to detect around 10^6 BBH events during a ten-year observation period [10].

Furthermore, the first direct image of a black hole shadow, captured by the Event Horizon Telescope (EHT) in 2019 [11], has further validated GR’s predictions. Having successfully described and predicted several phenomena, GR has established itself as one of the most outstanding modern theories in Physics.

Despite its undeniable successes, GR is not considered the ultimate theory of gravity due to several unresolved issues. For example, it faces significant challenges on both galactic and cosmological scales. Observations of galaxy rotation curves [12] and anisotropies in the cosmic microwave background (CMB) [13] suggest the presence of an additional gravitational component, commonly referred to as dark matter (DM). Although the existence of DM is a reasonable conclusion drawn from these observed phenomena, its true nature remains unknown, and it has yet to be directly detected.

On cosmological scales, observations indicate that the Universe is expanding at an accelerated rate [14], leading to the introduction of Dark Energy (DE). However, the true nature of DE remains unknown. In the Λ CDM model, which is the standard theory of cosmology, DE is represented by a cosmological constant Λ [15, 16]. While the cosmological constant is the most straightforward candidate for DE, its observed value is several orders of magnitude smaller than theoretical predictions. This discrepancy gives rise to the well-known cosmological constant problem, as discussed in reviews [17, 18].

In addition to those issues, GR introduces singularities – such as the black hole and Big Bang singularities [19, 20] – which suggests the necessity for a more fundamental theory of gravity. Additionally, the late-universe measurements of the Hubble constant H_0 do not match its early-universe measured values, an observational discrepancy called H_0 tension [21]. All of these issues indicate the

potential limitations of GR, and for this reason, there is an increasing interest in exploring alternative theories of gravity.

Considering this, we can view GR as a theory of gravity that is valid within specific energy regimes, serving as the limiting case of a broader theory. It is also a widely accepted notion that GR functions as an effective field theory [22, 23]. The category of theories designed to extend the concepts of GR is commonly referred to as Modified Gravity (MG) theories (for further details, see the reviews [24–26], for instance). The exact nature of an MG theory stems from a particular motivation, which could be to develop an effective quantum theory of gravity or to solve observational tensions or mathematical inconsistencies.

An MG theory should be mathematically consistent and observationally testable. It should also reproduce well-established theories under specific conditions: when gravity is turned off, Special Relativity should be recovered, and in cases of weak fields and low velocities (much less than the speed of light), Newtonian Gravity should be recovered¹. Ultimately, to be considered a viable GR successor, an MG theory should replicate the successes of GR while also addressing its limitations. However, developing a theory that meets all these criteria is a significant challenge.

As previously discussed, MG theories aim to address both observational and theoretical challenges in GR. While cosmology is the primary area where MG theories are applied, their potential extends beyond large-scale structures. Notably, the stellar and galactic regimes have received comparatively less attention [30], despite the wealth of observational data from missions such as the Event Horizon Telescope [11], Gaia [31], and the James Webb Space Telescope [32]. In astrophysics, MG has mainly been explored in the context of compact objects, especially in the strong-field regime. Nevertheless, its applications are vast, covering areas such as astroseismology [33, 34], among others.

While GR has been remarkably successful, its limitations indicate that a more extensive theory of gravity may be necessary. MG provides a potential solution to these shortcomings, offering new insights into astrophysical and cosmological phenomena. By exploring these theories and their predictions, we can improve our understanding of gravity.

1.2 Aim of the thesis

This thesis examines the astrophysical and theoretical aspects of MG theories, assessing their influence on cosmology and stellar evolution and their fundamental viability as alternative descriptions of gravity. The studied theories include scalar-tensor and teleparallel theories, which share the common feature of having additional degrees of freedom.

¹However, in low-acceleration regimes, neither GR nor Newtonian gravity fully describes observed phenomena, a discrepancy known as the Low-Acceleration Gravitational Anomaly [27–29].

On the astrophysical side, the goal is to provide evidence of deviations from standard stellar theory arising from MG. In particular, this work investigates how low-mass pre-main-sequence stars evolve in the framework of scalar-tensor theories, focusing on key observational features such as their contraction and lithium depletion phases. The analysis of these effects enables the identification of potential astrophysical signatures of MG that could be tested against observational data.

On the theoretical side, this thesis examines the internal consistency and stability of MG models. A key objective is to determine whether $f(Q)$ theories of gravity [35] – which extend the symmetric teleparallel theory [36] – are free from ghostly instabilities and strong coupling issues. Additionally, we explore how the canonical frame is determined by gravitational dynamics, providing a broader perspective on viable alternatives to GR. In both cases, these investigations are done in cosmological frameworks.

This dual approach enables us to connect astrophysical applications with theoretical consistency, enhancing our understanding of the effect of MG on stellar evolution and cosmological models and the viability of these theories.

These objectives can be summarised as follows:

- To investigate how degenerate higher-order scalar-tensor (DHOST) gravity [37] affects the stellar structure, specifically focusing on which observable stellar properties (such as mass-radius relationship, luminosity, and age) are most sensitive to these changes.
- To assess the theoretical consistency of $f(Q)$ gravity as a viable alternative to GR by investigating whether it introduces unphysical (ghost) modes. Additionally, we aim to explore how the choice of the function $f(Q)$ affects the stability and well-posedness of the theory.
- To obtain cosmological solutions in the General Teleparallel Equivalent of GR (GTEGR) [38], formulated in the canonical frame, where essential observables such as energy and momentum can be consistently calculated.

These objectives highlight the importance of considering both theoretical consistencies and observational consequences.

1.3 Statements

The research statements below summarise the main contributions and implications of the thesis:

- Alternative theories of gravity, such as DHOST theories, alter the internal structure and evolution of low-mass stars, influencing their pre-main-sequence development, lithium depletion timescales, and other observational characteristics.

- These changes modify the Hayashi tracks in the Hertzsprung-Russell (HR) diagram and suggest that lithium abundance in young stellar clusters could serve as a test for MG observations.
- GTEGR includes local symmetries that enable the application of the canonical frame, which is essential for calculating energy and momentum.
- Not all cosmological solutions in GTEGR are simultaneously compatible with the canonical frame and the concept of an evolving Universe.
- $f(\mathcal{Q})$ theories have inherent ghost-like instabilities, raising concerns about their viability as alternative gravitational models.

1.4 Structure of the thesis

This thesis is structured into two main parts. The first part consists of an introduction, where the topics that are crucial for understanding the subsequent papers are reviewed. The second part contains the articles that form the core of the thesis.

The introductory part is divided into three chapters. Chapter 2 focuses on Gravity Theories. It begins with a review of GR in section 2.1 and subsequently explores modified theories of gravity. Section 2.2 discusses scalar-tensor theories, placing particular emphasis on Horndeski-like theories. In section 2.3, we address fundamental concepts of metric-affine theories. Section 2.4 delves into one of the metric-affine sectors: teleparallel theories. In this section, we separately examine the Metric and Symmetric Teleparallel versions of GR as well as the General Teleparallel Equivalent of GR, which integrates both theories. We conclude by presenting extensions of these theories, specifically the General Teleparallel, $f(\mathcal{T})$ and $f(\mathcal{Q})$ theories.

Chapter 3 explores some of the methods and mathematical concepts employed in the papers. We start by discussing the energy-momentum pseudotensor in section 3.1. We then describe the scalar-vector-decomposition used in perturbation theory in section 3.2, followed by a discussion on instabilities in section 3.3.

Lastly, chapter 4 highlights the main applications of this thesis in the fields of Astrophysics and Cosmology. We begin with section 4.1, which provides an overview of stellar physics. This section concludes with a discussion of how specific stellar characteristics are modified in the context of alternative theories of gravity and the implications of these changes. In section 4.2, we introduce fundamental concepts and equations of cosmology and explore how they are affected by $f(\mathcal{T})$ and $f(\mathcal{Q})$ theories. We conclude by examining the canonical frame in cosmology.

The second part is divided into three chapters, with each chapter dedicated to a specific paper. Chapter 5 presents the paper titled “Early Evolution of Fully

Convective Stars in Scalar-Tensor Gravity.” Chapter 6 contains the paper “General Parallel Cosmology.” Finally, Chapter 7 features the publication “Pathological Character of Modifications to Coincident General Relativity: Cosmological Strong Coupling and Ghosts in $f(\mathbb{Q})$ Theories.” At the end, the author’s CV is provided in both English and Estonian.

The conventions used in this work are as follows: we adopt a mostly positive signature for the metric, which is represented as $(-, +, +, +)$. Spatial coordinate indices are denoted by Latin letters such as i, j, k, \dots , while spacetime coordinate indices are represented by Greek letters, such as $\alpha, \beta, \gamma, \dots$.

Below, we provide a list of the acronyms utilised throughout the text.

DHOST	Degenerate Higher-Order Scalar-Tensor (Theories)
FLRW	Friedmann-Lamaître-Robertson-Walker (Metric)
GR	General Relativity
GTEGR	General Teleparallel Equivalent of General Relativity
HR	Hertzsprung-Russell (Diagram)
MG	Modified Gravity
MS	Main Sequence
STEGR	Symmetric Teleparallel Equivalent of General Relativity
SVT	Scalar-vector-tensor (Decomposition)
TEGR	Teleparallel Equivalent of General Relativity

Additionally, we present a list of the most frequently used symbols.

$\eta_{\mu\nu}$	Minkowski metric
$g_{\mu\nu}$	Spacetime metric
$\bar{g}_{\mu\nu}$	Background metric
$h_{\mu\nu}$	Metric perturbation
g	Metric determinant
$\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu}$	Levi-Civita connection
$\Gamma^\alpha{}_{\mu\nu}$	General affine connection
∂_μ	Partial derivative
$\overset{\circ}{\nabla}{}_\mu$	Levi-Civita covariant derivative (with respect to $\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu}$)
∇_μ	General covariant derivative (with respect to $\Gamma^\alpha{}_{\mu\nu}$)
$\overset{\circ}{R}{}^\alpha{}_{\mu\nu\rho}$	Curvature tensor
$\overset{\circ}{R}{}_{\mu\nu}$	Ricci tensor with respect to the Levi-Civita connection
$\overset{\circ}{R}$	Levi-Civita curvature scalar (Ricci scalar)
$R^\alpha{}_{\mu\nu\rho}$	General curvature tensor
$R_{\mu\nu}$	General Ricci tensor
R	General curvature scalar
$\overset{\circ}{G}{}_{\mu\nu}$	Einstein tensor
$\Theta_{\mu\nu}$	Energy-momentum tensor
$t_{\mu\nu}$	Gravitational energy-momentum pseudo-tensor
$\bar{G}_{\mu\nu}$	Metrical energy-momentum
$\bar{t}_{\mu\nu}$	Canonical energy-momentum
$h^\alpha{}_{\mu\nu}$	Energy-momentum superpotential
$M^\alpha{}_{\mu\nu}$	Distortion tensor
$K^\alpha{}_{\mu\nu}$	Contortion tensor
$L^\alpha{}_{\mu\nu}$	Disformation tensor
$T^\alpha{}_{\mu\nu}$	Torsion tensor
T^α	Torsion scalar
\mathcal{T}, T	TEGR torsion scalar, general torsion scalar
$S^\alpha{}_{\mu\nu}$	Torsion conjugate
$t_{\mu\nu}$	Symmetric tensor defined with respect to the torsion scalar
$Q^\alpha{}_{\mu\nu}$	Non-metricity tensor
$Q^\alpha, \tilde{Q}^\alpha$	Non-metricity scalars
\mathcal{Q}, Q	STEGR non-metricity scalar, general non-metricity scalar
$P^\alpha{}_{\mu\nu}$	Non-metricity conjugate
$q_{\mu\nu}$	Symmetric tensor defined with respect to the non-metricity scalar
\mathcal{G}	GTEGR scalar

Chapter 2

Gravity Theories

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In this chapter, we will discuss various theories of gravity, starting with GR. We will then explore alternative theories, such as scalar-tensor theories and teleparallel theories. Some of them also present extra degrees of freedom, a common aspect among MG theories. However, a key feature shared by these theories is their treatment of gravity through spacetime geometry, although they differ in how they describe this geometry, which we will explore throughout this chapter.

2.1 General Relativity

GR is based on Riemannian geometry, which is characterised by a symmetric tensor known as the *metric*, denoted by $g_{\mu\nu}$. This tensor is used to measure distances, vector magnitudes, and angles between vectors. It also allows the determination of the shortest path between two points and serves as the gravitational potential in GR. The metric tensor should be defined at every point in spacetime and possess the property of being non-degenerate, i.e., $g = \det |g_{\mu\nu}| \neq 0$. The metric non-degeneracy enables the definition of an inverse metric tensor $g^{\mu\rho}$, satisfying $g_{\mu\nu}g^{\mu\rho} = \delta_{\nu}^{\rho}$.

In addition to the concept of distance, which is enabled by the metric, we need a way to connect different tangent spaces, which can be achieved with the introduction of a non-tensor quantity called *affine connection*. The metric and connection are generally independent as they represent different aspects of geometry. However, in GR, these two notions are related since the connection used is the Levi-Civita connection, defined as:

$$\mathring{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda} (\partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\nu}). \quad (2.1)$$

This connection is symmetric in the last two indices and depends only on the metric and its first derivatives.

The connection is essential for parallel transport, a concept we will explore later, and covariant derivatives. In the latter case, the connection allows the definition of a covariant derivative operator:

$$\mathring{\nabla}_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + \mathring{\Gamma}^{\nu}_{\mu\lambda}A^{\lambda}, \quad (2.2)$$

$$\mathring{\nabla}_{\mu}B_{\nu} = \partial_{\mu}B_{\nu} - \mathring{\Gamma}^{\lambda}_{\mu\nu}B_{\lambda} \quad (2.3)$$

The covariant derivative obeys the usual derivative properties, such as the Leibniz product rule and linearity. Moreover, its main advantage is that it produces covariant quantities, i.e., the covariant derivative of a tensor is also a tensor.

A fundamental concept in the development of GR is the *Equivalence Principle*. This principle states that the behaviour of objects in a non-inertial frame is indistinguishable from that of objects in an inertial frame with a gravitational

field [39]. This equivalence reflects the idea that gravity is not considered a force in the traditional sense but rather a result of spacetime geometry.

In GR, gravity manifests as a consequence of spacetime curvature, which is represented by the Riemann-Christoffel curvature tensor (Riemann tensor for short), also known as Levi-Civita curvature. This tensor is defined as:

$$\mathring{R}^\alpha{}_{\mu\nu\rho} = 2\partial_{[\nu}\mathring{\Gamma}^\alpha{}_{\rho]\mu} + 2\mathring{\Gamma}^\alpha{}_{[\nu|\lambda}\mathring{\Gamma}^\lambda{}_{\rho]\mu}. \quad (2.4)$$

The Riemann tensor provides essential information about curvature, vanishing in flat spacetimes. Additionally, it exhibits several important symmetry properties, listed below [40]:

- Symmetry in the exchange of the first and second pairs of indices:

$$\mathring{R}_{\beta\mu\nu\rho} = \mathring{R}_{\nu\rho\beta\mu}, \quad (2.5)$$

where we lowered the first index using the metric, $\mathring{R}_{\beta\mu\nu\rho} = g_{\alpha\beta}\mathring{R}^\alpha{}_{\mu\nu\rho}$.

- Antisymmetry in the exchange of indices either in the first or second pair of indices:

$$\mathring{R}_{\beta\mu\nu\rho} = -\mathring{R}_{\mu\beta\nu\rho} = -\mathring{R}_{\beta\mu\rho\nu} = \mathring{R}_{\mu\beta\rho\nu}. \quad (2.6)$$

- Cyclicity of the last three indices:

$$\mathring{R}_{\beta\mu\nu\rho} + \mathring{R}_{\beta\rho\mu\nu} + \mathring{R}_{\beta\nu\rho\mu} = 0 \quad (2.7)$$

In addition to these properties, the Riemann tensor also obeys the Bianchi identity:

$$\mathring{R}_{\beta\mu\nu\rho;\eta} + \mathring{R}_{\beta\mu\eta\nu;\rho} + \mathring{R}_{\beta\mu\rho\eta;\nu} = 0. \quad (2.8)$$

The only non-vanishing contraction of the Riemann tensor produces the Ricci tensor:

$$\mathring{R}_{\mu\nu} = \mathring{R}^\alpha{}_{\mu\alpha\nu}. \quad (2.9)$$

We can note that the Ricci tensor is symmetric $\mathring{R}_{\mu\nu} = \mathring{R}_{\nu\mu}$, which follows from the symmetry property of the Riemann tensor (2.5). Moreover, the antisymmetry property (2.6) guarantees that this is the sole second-rank tensor that can be derived from the Riemann tensor, as other possible combinations are either equivalent to the Ricci tensor or null. The only curvature scalar that can be formed from the Riemann tensor is the Ricci scalar, defined as $\mathring{R} = g^{\mu\nu}\mathring{R}_{\mu\nu}$.

2.1.1 Geodesics and Auto-parallels

Our goal in this subsection is to derive the equations describing the motion of a free particle under the influence of gravity. These equations can be obtained by using the principle of least action, which states that the path followed by the particle extremises an integral known as action.

Freely falling particles follow curves called *geodesics*, which are defined as the shortest path connecting two points. In Euclidean spaces, the geodesic is simply a straight line. However, in Riemannian spaces, this curve takes a more complex form. In this case, the shortest curves correspond to the curves with maximum proper time¹ [39], which is defined as:

$$\tau = \int_a^b ds = \int_a^b \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda, \quad (2.10)$$

where λ parametrises the curve.

After applying the principle of least action to τ , we get the geodesic equations [40]:

$$\frac{d^2 x^\alpha}{d\tau^2} + \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (2.11)$$

where $\frac{dx^\mu}{d\tau}$ is the four-velocity, which satisfies $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$. We recall that, besides being the shortest curve between two points, the geodesic also represents the path followed by a test particle of mass m in a gravitational field. Thus, $\frac{d^2 x^\alpha}{d\tau^2}$ can be identified as the particle's four-acceleration while the quantity $-m \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$ can be interpreted as the gravitational force acting on the particle.

Thus, the geodesic curves can be regarded as the GR version of the straight paths followed by freely falling particles in Newtonian theory. In fact, the Newtonian limit of equation (2.11) is given by [40]:

$$\frac{d^2 \vec{x}}{dt^2} + \nabla \Phi = 0, \quad (2.12)$$

where $\Phi = -Gm/r$ is the gravitational potential at a distance r from a source mass m and \vec{x} is the position vector. The metric and the gravitational potential can be related by the equation $g_{00} = -(1 + 2\Phi)$, indicating that the metric assumes the role of gravitational field potential in GR.

Equation (2.11) also describes another type of curve, which can be thought of as the straightest path between two points (at least locally). In Euclidean geometry, these curves are simply straight lines. In Riemannian geometry, they are known as *auto-parallels* and involve the concept of parallel transport. Before proceeding, let us first define parallel transport.

¹In this context, we assume the curve is time-like. The procedure is similar for space-like curves, but with a positive sign inside the square root.

Parallel transport is the process in which a vector is translated along a curve while kept parallel to itself. In Euclidean space, transporting a vector in this way is straightforward – it involves moving the vector while keeping its components constant. However, in curved spaces, the process is not as straightforward, and the final result will depend on the path taken [39]. The deviation between the original and the parallel-transported vectors is measured by the curvature tensor.

In general, when a vector is parallel transported, its directional covariant derivative along the curve should vanish [41]:

$$\frac{D}{D\lambda} V^\mu \equiv \frac{dx^\alpha}{d\lambda} \nabla_\alpha V^\mu = 0, \quad (2.13)$$

where $\frac{dx^\alpha}{d\lambda}$ is the vector tangent to the curve and λ parametrizes this curve. The above equation is known as the equation of parallel transport and can be easily extended for tensors. It is now evident that the concept of parallel transport depends on the connection, and each choice of connection will produce different outcomes.

Auto-parallels are characterised by the parallel transport of their tangent vectors. Thus, the equations defining this type of curve can be directly derived from equation (2.13), resulting in:

$$\frac{D}{D\lambda} \left(\frac{dx^\alpha}{d\lambda} \right) = 0 \Rightarrow \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \quad (2.14)$$

If we require that the covariant derivative of the metric vanishes and that the connection is symmetric, we will obtain the Levi-Civita connection. Hence, the equation (2.14) will coincide with the geodesic equation (2.11). Therefore, geodesic and auto-parallel curves are equivalent in GR. However, in general, they represent different types of curves. Geodesics, being the shortest curves, depend on the concept of distance, which is determined by the metric tensor. In contrast, auto-parallels are associated with parallel transport, which is determined by the connection.

2.1.2 Einstein Equations

To establish the relationship between gravity and geometry, we should use the Einstein field equations, which are based on the assumption that the metric tensor describes the properties of the gravitational field. These equations follow the principle of covariance, which is achieved by using tensorial equations. Additionally, we need to ensure that energy and momentum are conserved, and this conservation should arise from the field equations. Finally, the field equations should also satisfy the *principle of correspondence*, which means that Newtonian gravity should be recovered in the limit of weak fields and small velocities.

The principle of least action can be applied to obtain the Einstein field equations, but a few adjustments are necessary. Firstly, the action should consist of two parts:

\mathcal{S}_g , which contains the geometry-related fields and \mathcal{S}_m , which is composed of the matter fields. Hence, the total action is given by $\mathcal{S} = \mathcal{S}_g + \mathcal{S}_m$. Secondly, in order to have an action that is invariant under transformation laws, we need to use $\sqrt{-g}d^4x$ as the four-dimensional volume element.

The gravitational action \mathcal{S}_g is formulated using a curvature scalar that, at most, involves second-order derivatives of the metric. A suitable candidate for this purpose is the Ricci tensor. On the other hand, the matter action \mathcal{S}_m is typically assumed to depend on the metric and, at most, its first derivatives, along with the matter fields. Therefore, the GR action, commonly known as the Einstein-Hilbert action, is defined as follows:

$$\mathcal{S}_{\text{EH}} = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa} \mathring{R} + \mathcal{L}_m \right), \quad (2.15)$$

where \mathcal{L}_m is the matter Lagrangian.

The variation of the above action with respect to the metric leads to the Einstein equations [40]:

$$\mathring{G}_{\mu\nu} = 8\pi\Theta_{\mu\nu}, \quad (2.16)$$

where $\mathring{G}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $\Theta_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}_m}{\delta g^{\mu\nu}}$ is the energy-momentum tensor. The equations (2.16) represent a set of second-order non-linear differential equations that can only be solved exactly in a few specific cases. These equations establish a profound connection between gravity and geometry. However, as we will discuss later, they are not the only equations that illustrate this relationship.

If we take the covariant derivative of both sides of the Einstein equations, we obtain $\mathring{\nabla}_\mu\Theta^{\mu\nu} = 0$, because $\mathring{\nabla}_\mu\mathring{G}^{\mu\nu}$ vanishes identically. The first equation represents the energy-momentum conservation (locally), which comes from the field equations, as required.

Although the Einstein equations form the foundation of GR, alternative formulations can offer different perspectives on gravity. Before exploring these alternative theories, let us first discuss how we can modify GR. According to Lovelock's theorem, the Einstein equations are the only solutions in four dimensions that can be derived from an action involving up to second-order derivatives of the metric [42, 43]. Hence, Lovelock's theorem implies that if we want to develop theories that produce field equations different from Einstein's, we must relax at least one of the assumptions outlined in Lovelock's theorem. Below, we list some of the ways to accomplish this [25]:

- Allow the gravitational action to include higher than second-order derivatives of the metric;
- Introduce extra fields (scalar, vector, tensor) in the action;

- Work in dimensions higher than four;
- Explore geometric quantities beyond just curvature.

Figure 2.1 illustrates the different Modified Theory classes, which are based on relaxing the assumptions of Lovelock’s theorem.

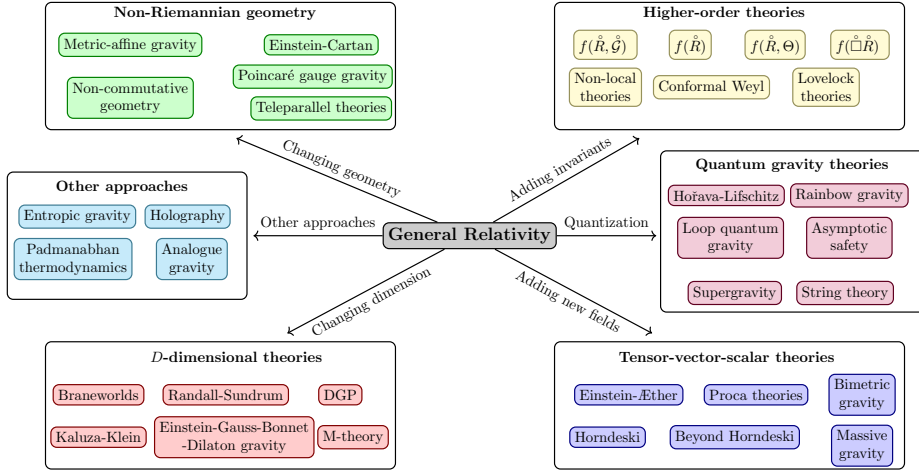


Figure 2.1: Classes of MG theories [44]. Examples of theories from each class are also provided.

2.2 Scalar-tensor Theories

This section reviews the main features of scalar-tensor theories of gravity, which are among the most extensively studied modifications to GR [3, 45, 46]. Scalar-tensor theories accommodate all theories of gravity in which a scalar field, denoted as ϕ , is non-minimally coupled to gravity. This class of theories is particularly interesting because they include the possibility of an expanding Universe while preserving its isotropy. For instance, it is argued that a scalar field known as the inflaton can explain the inflation of the Universe [47]. Additionally, with the introduction of an extra scalar field, a new perspective on gravity emerges: gravity is now influenced not only by the curvature of spacetime but also by the scalar field itself. Hence, the term “scalar-tensor” highlights the contributions of both the scalar field and the metric tensor in the description of gravitational phenomena.

Scalar-tensor theories of gravity have a long history, with the combination of gravity and scalar fields existing even before GR. For instance, in 1912, G. Nordström proposed a conformally flat scalar-tensor theory [48–50]. A scalar-tensor combination also appeared in Kaluza-Klein theory [51, 52], which unifies

gravity and electromagnetism. In Kaluza-Klein theory, there are 15 independent equations: 10 corresponding to Einstein's equations, four corresponding to Maxwell's equations and 1 for the scalar field. However, the foundation for modern scalar-tensor theories was established by the work of Jordan in 1955 [53] and of Brans and Dicke in 1961 [54].

Several theories fall in the scalar-tensor category, ranging from Jordan-Brans-Dicke theory to $f(\mathring{R})$ theories². The $f(\mathring{R})$ theories refer to modifications of GR in which the Levi-Civita curvature in the Einstein-Hilbert action is replaced by a non-linear function of that curvature [55, 56]. It is also important to note that, in this context, the curvature is a dynamical quantity. As a consequence, $f(\mathring{R})$ theories involve higher than first-order curvature terms. Moreover, these theories can be mapped to scalar-tensor theories [57]. In particular, $f(\mathring{R})$ theories are equivalent to Jordan-Brans-Dicke theory for either $\omega = 0$ (in the metric formalism) or $\omega = -3/2$ (in the Palatini formalism) [58]. Notably, in the latter case, the scalar field is not dynamic.

Beyond these classical scalar-tensor models, a broader class of theories known as Horndeski theories further generalises this framework while maintaining second-order equations of motion. The action for Horndeski theories is described by [59, 60]:

$$\mathcal{S}_H = \int d^4x \sqrt{-g} (\mathcal{L} + \mathcal{L}_m), \quad (2.17)$$

where \mathcal{L}_m is the matter Lagrangian density and

$$\begin{aligned} \mathcal{L} = & G_2(\phi, X) + G_3(\phi, X)\square\phi + G_4(\phi, X)\mathring{R} \\ & + G_{4,X}(\phi, X) [(\square\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)] + G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ & - \frac{1}{6}G_{5,X}(\phi, X) [(\square\phi)^3 - 3(\square\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) \\ & + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)], \end{aligned}$$

with $X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$.

To illustrate the generality of the Horndeski action, we will present some specific examples of the functions G_i and the corresponding theories.

- *General Relativity*: by setting $G_4 = 1/2\kappa$, $G_2 = G_3 = G_5 = 0$, we recover the Einstein Hilbert action

$$\mathcal{S}_{EH} = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa} \mathring{R} + \mathcal{L}_m \right). \quad (2.18)$$

²It is essential to clarify that the symbol \mathring{R} refers to the Levi-Civita Ricci scalar, which is derived solely from the metric. In contrast, later in the text, we will encounter another curvature scalar, denoted by R , which is based on an affine connection.

- *Brans-Dicke theory*: the action is given by

$$\mathcal{S}_{\text{BD}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(\phi \dot{R} - \frac{\omega_{\text{BD}}}{\phi} \partial_\mu \phi \partial^\mu \phi - 2V(\phi) \right) + \int d^4x \sqrt{-g} \mathcal{L}_m. \quad (2.19)$$

This corresponds to the choice $G_2 = \frac{\omega_{\text{BD}}}{\phi} X - V(\phi)$, $G_3 = G_5 = 0$, $G_4 = \frac{\phi}{2}$.

- $f(\dot{R})$ theories: the action

$$\mathcal{S}_{f(\dot{R})} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\dot{R}) + \int d^4x \sqrt{-g} \mathcal{L}_m, \quad (2.20)$$

is reproduced by the choice $G_2 = -\frac{1}{2}(\dot{R} \partial_{\dot{R}} f - f)$, $G_3 = G_5 = 0$, $G_4 = \frac{1}{2} \partial_{\dot{R}} f$.

Horndeski theories are characterised by the presence of second-order derivatives in the equations of motion, a feature which prevents the presence of instability ghosts. While this condition is sufficient to avoid such instabilities, it is not mandatory. Consequently, it is possible to formulate higher-order scalar-tensor theories, known as beyond Horndeski theories, that successfully evade Ostrogradski instabilities [37, 61, 62]. Moreover, these theories maintain the correct number of degrees of freedom [63, 64]. A deciding factor in the stability of beyond Horndeski theories is the degeneracy of their Lagrangian. For this reason, higher-order Horndeski theories that possess degenerate Lagrangians are labelled as degenerate higher-order scalar-tensor (DHOST) theories [37, 62, 65].

Scalar-tensor theories often introduce a *fifth force*, which arises from the non-minimal coupling of the scalar field to matter or gravity [66]. This additional force complements the four fundamental interactions known in nature, earning it the designation of “fifth force”. The fifth force is expected to operate over long ranges, such as on cosmological scales. Since this force has not been observed within the solar system, there must be a mechanism to suppress its influence on shorter ranges. The methods used to mitigate the effects of the fifth force are referred to as screening mechanisms (see, for example, [67, 68]).

An intriguing effect occurs when we consider scalar-tensor theories in astrophysics, particularly in the context of beyond Horndeski theories. Within non-relativistic matter sources, the Vainshtein screening is partially broken [69]. This broken screening mechanism results in a modified hydrostatic equilibrium equation, prompting modifications in the structures of stars and galaxies [70].

In addition to the tight solar system constraints imposed on scalar-tensor theories, gravitational waves further restrict the possible parameters of a theory [71–74]. Observations from the GW170817 event [5] constrain the speed difference between gravitational waves and light to less than 10^{-15} . As a result, G_4 must be

independent of X and G_5 should be independent of both X and ϕ . Hence, the form of the Horndeski Lagrangian is adjusted to be

$$\mathcal{L} = G_2(\phi, X) + G_3(\phi, X)\square\phi + G_4(\phi, X)\overset{\circ}{R}. \quad (2.21)$$

This restriction rules out any couplings between the scalar field and the Gauss-Bonnet term, which is defined as $\mathbb{G} = \overset{\circ}{R}^2 - 4\overset{\circ}{R}_{\alpha\beta}\overset{\circ}{R}^{\alpha\beta} + \overset{\circ}{R}_{\alpha\beta\mu\nu}\overset{\circ}{R}^{\alpha\beta\mu\nu}$. It also excludes functions of the Gauss-Bonnet term, denoted as $f(\mathbb{G})$. However, GR and $f(\overset{\circ}{R})$ theories remain viable. For a comprehensive list of both viable and non-viable theories, we encourage the reader to refer to [73].

2.3 Metric-affine Theories

In the previous sections, we discussed the Levi-Civita connection and the Riemann curvature tensor, both of which are solely based on the metric. However, by relaxing some assumptions about spacetime, we can introduce more general connections which are independent of the metric. This approach leads to a more general type of geometry, known as *metric-affine* geometry, where both the metric and the connection define the spacetime structure. A new connection can be constructed by adding some tensor $M^\alpha{}_{\mu\nu}$ to the Levi-Civita connection:

$$\Gamma^\alpha{}_{\mu\nu} = \overset{\circ}{\Gamma}^\alpha{}_{\mu\nu} + M^\alpha{}_{\mu\nu}. \quad (2.22)$$

There are three main geometric quantities that we can construct with the new connection (2.22): torsion, non-metricity and curvature. We start with the torsion tensor, defined as

$$T^\alpha{}_{\mu\nu} := 2\Gamma^\alpha{}_{[\mu\nu]} = \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu}. \quad (2.23)$$

The definition above shows that the torsion tensor is antisymmetric with respect to its last two indices and vanishes when the connection is symmetric in those indices. Therefore, it has only one independent trace: $T_\beta := T^\alpha{}_{\alpha\beta} = -T^\alpha{}_{\beta\alpha}$.

Furthermore, torsion is a proper tensor because it is the difference between two connections. This occurs because the inhomogeneous parts of the connections cancel out, leaving only the tensorial part. Geometrically, the torsion tensor measures the non-closure of parallelograms formed by two vectors that have been parallel transported along one another, as shown in Figure 2.2.

Next, we introduce another fundamental quantity: the non-metricity. The new connection creates a different covariant derivative operator, denoted as ∇ . In this case, the covariant derivative of the metric is generally not zero, which defines the non-metricity tensor:

$$Q_{\alpha\mu\nu} := \nabla_\alpha g_{\mu\nu} \neq 0. \quad (2.24)$$

By definition, the non-metricity is symmetric in its last two indices. Therefore, it has two independent traces: $Q_\alpha := Q_{\alpha\beta}{}^\beta$ and $\tilde{Q}_\alpha := Q^\beta{}_{\beta\alpha}$. In this case, the

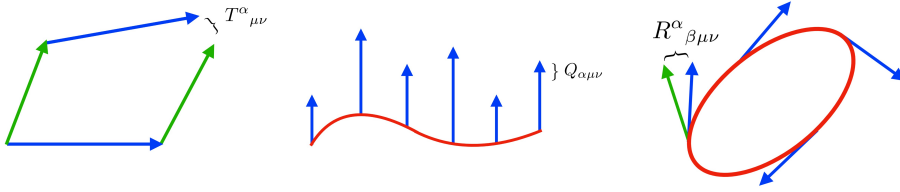


Figure 2.2: Geometrical interpretations of torsion, non-metricity and curvature. Adapted from [75].

magnitude of a vector changes when it is parallel transported along a curve, as shown in Figure 2.2.

To fully characterise the geometry of spacetime, we must also examine how vectors change after being parallel transported along a closed loop. This change, illustrated in Figure 2.2, is quantified by curvature. The general curvature is based on the affine connection (2.22) and given by:

$$R^\alpha{}_{\mu\nu\rho} = 2\partial_{[\nu}\Gamma^\alpha{}_{\rho]\mu} + 2\Gamma^\alpha{}_{[\nu|\lambda}\Gamma^\lambda{}_{\rho]\mu}. \quad (2.25)$$

It is important to note that this general curvature tensor depends solely on the connection. Additionally, the curvature tensor is antisymmetric in its last two indices only ($R^\alpha{}_{\mu\nu\rho} = -R^\alpha{}_{\rho\nu\mu}$), unlike the Riemann tensor, which is also antisymmetric in the first pair of indices. Consequently, the curvature has two independent traces: the Ricci tensor, defined as $R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu}$, and the homothetic tensor, given by $H_{\mu\nu} := R^\alpha{}_{\alpha\mu\nu}$.

The Riemann curvature cyclicity property, as presented in equation (2.7), as well as the Bianchi identity (2.8), can now be generalized to include torsion:

$$R^\alpha{}_{[\beta\mu\nu]} - \nabla_{[\beta}T^\alpha{}_{\mu\nu]} + T^\lambda{}_{[\beta\mu}T^\alpha{}_{\nu]\lambda} = 0 \quad (2.26)$$

$$\nabla_{[\gamma}R^\alpha{}_{|\beta|\mu\nu]} - T^\lambda{}_{[\gamma\mu}R^\alpha{}_{|\beta|\nu]\lambda} = 0 \quad (2.27)$$

The above equations are referred to as first and second Bianchi identities, respectively.

The three tensors defined above – torsion, non-metricity, and curvature – encode crucial geometric features of spacetime. Together, these quantities represent a more general structure of spacetime than that found in purely Riemannian geometry. In the most general metric-affine geometries, for instance, all three quantities appear. However, we can postulate, for instance, that the non-metricity vanishes:

$$Q_{\alpha\mu\nu} = 0. \quad (2.28)$$

The condition mentioned above is known as the *metricity condition*. When a connection, such as the Levi-Civita connection, meets this condition, it is called a

metric-compatible connection. Spacetimes that include such a connection are also described as metric-compatible. Consequently, non-metricity indicates the extent to which a connection deviates from metric compatibility.

Similarly, we define the *flatness condition* as follows:

$$R^\alpha{}_{\mu\nu\rho} = 0, \quad (2.29)$$

and the *torsion-free condition* as:

$$T^\alpha{}_{\mu\nu} = 0. \quad (2.30)$$

A spacetime with a connection that meets the flatness condition is called *flat*, while a spacetime with a connection that fulfils the torsion-free condition is termed *torsionless*.

The various combinations of these conditions lead to different subclasses of Metric-Affine Theories of Gravity, which are represented in Figure 2.3. A comprehensive classification of metric-affine theories can be found in [44], which details various geometric conditions and their implications. However, we will not explore this further here. Our focus will be on specific subclasses: Riemannian geometry, which includes GR and teleparallel geometry, which includes the Metric and Symmetric Teleparallel Theories of Gravity.

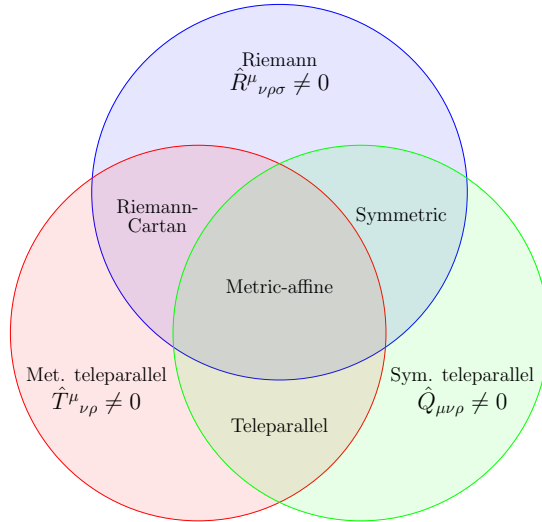


Figure 2.3: Metric affine geometries [44]. Each circle represents a non-vanishing tensor quantity, while their intersections indicate the presence of either two or three non-vanishing quantities. The corresponding geometries are also presented.

In GR, spacetime is curved, torsionless, and metric-compatible. In this framework, the metric serves as the crucial link between gravity and geometry.

Additionally, the metric is the only dynamic field present since we employ the Levi-Civita connection, which relies solely on the metric itself.

In teleparallel geometry, where the spacetime is flat, but torsion and non-metricity are non-vanishing, we can construct theories such as the General Teleparallel Equivalent of GR (GTEGR) [38, 76]. This geometry further subdivides as:

- Metric Teleparallel Geometry ($R^\alpha_{\mu\nu\rho} = 0$, $Q_{\alpha\mu\nu} = 0$, $T^\alpha_{\mu\nu} \neq 0$): The spacetime is flat and metric compatible. As a result, torsion assumes the function of a mediator of gravity. In this geometry, we can construct the Teleparallel Equivalent of GR (TEGR) [77, 78] and its extensions, such as the $f(\mathcal{T})$ theories [79–81].
- Symmetric Teleparallel Geometry ($R^\alpha_{\mu\nu\rho} = 0$, $Q_{\alpha\mu\nu} \neq 0$, $T^\alpha_{\mu\nu} = 0$): The spacetime is flat and torsionless, while non-metricity holds information about gravity. The connection is symmetric, allowing for the construction of theories such as the Symmetric Teleparallel Equivalent of GR (STTEGR) [35, 36, 75] and its extensions, including the $f(Q)$ theories [35, 78].

We will discuss this in more detail in the next section.

In the context of metric-affine gravity, the tensor $M^\alpha_{\mu\nu}$ is called *distortion* and can be decomposed as follows:

$$M^\alpha_{\mu\nu} = K^\alpha_{\mu\nu} + L^\alpha_{\mu\nu}, \quad (2.31)$$

where $K^\alpha_{\mu\nu}$ and $L^\alpha_{\mu\nu}$ denote the contortion and the disformation, respectively. This decomposition makes the distinction between the effects of torsion and non-metricity more evident. To illustrate this point, let us examine the definitions of contortion and disformation:

$$K^\alpha_{\mu\nu} = \frac{1}{2}T^\alpha_{\mu\nu} + \frac{1}{2}T_\mu^\alpha{}_\nu + \frac{1}{2}T_\nu^\alpha{}_\mu, \quad (2.32)$$

and

$$L^\alpha_{\mu\nu} = \frac{1}{2}Q^\alpha_{\mu\nu} - \frac{1}{2}Q_\mu^\alpha{}_\nu - \frac{1}{2}Q_\nu^\alpha{}_\mu. \quad (2.33)$$

The definitions provided clearly show that contortion arises exclusively from torsion, while disformation results solely from non-metricity. Therefore, we can recover the Levi-Civita connection by requiring that spacetime is both torsion-free and metric-compatible.

Now, we explore the physical interpretation of the contortion and disformation. If we set the disformation to zero, $L^\alpha_{\mu\nu} = 0$, then the contortion becomes the difference of the general and the Levi-Civita connections: $K^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \overset{\circ}{\Gamma}^\alpha_{\mu\nu}$. Consequently, the geodesic equation (2.11) becomes:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = K^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \quad (2.34)$$

Similarly, by setting the contortion to zero, $K^\alpha{}_{\mu\nu} = 0$, we obtain

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = L^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \quad (2.35)$$

We will now express the geodesic equation considering the presence of a Lorentz force acting on a charge q with mass m [39]:

$$\frac{d^2 x^\alpha}{d\tau^2} + \mathring{\Gamma}^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{q}{m} F^\alpha{}_\nu \frac{dx^\nu}{d\tau}. \quad (2.36)$$

The equation presented above clearly resembles equations (2.34) and (2.35). This observation enables us to conclude that the effects of contortion and disformation on a test particle are analogous to those produced by a Lorentz force.

2.4 Teleparallel Theories of Gravity

The equivalence principle suggests that gravity should be interpreted from a geometric perspective. Traditionally, gravity is understood in terms of spacetime curvature, but we can also analyse it through other geometric properties, such as torsion and non-metricity. The consideration of these alternative interpretations leads to the development of new theories of gravity.

With this in mind, we begin this section by examining the Metric and Symmetric Teleparallel Theories of Gravity, which are based on torsion and non-metricity, respectively. We also discuss the General Teleparallel Theory, which includes both torsion and non-metricity. Notably, these frameworks produce theories equivalent to GR and serve as a foundation for several MG models, which are also discussed in this section.

To formulate these theories, we define scalars that are quadratic in terms of torsion or non-metricity. Their explicit forms are given by:

$$\mathcal{T} = \frac{1}{8} T_{\alpha\mu\nu} T^{\alpha\mu\nu} + \frac{1}{4} T_{\alpha\mu\nu} T^{\mu\alpha\nu} - \frac{1}{2} T_\alpha T^\alpha \quad (2.37)$$

and

$$\mathcal{Q} = \frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} - \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} - \frac{1}{4} Q_\alpha Q^\alpha + \frac{1}{2} Q_\alpha \tilde{Q}^\alpha. \quad (2.38)$$

It will be useful to relate these scalars to the general curvature scalar, R . For this purpose, we express the general curvature tensor $R^\alpha{}_{\mu\nu\rho}(\Gamma)$ in terms of the Riemann curvature tensor $\mathring{R}^\alpha{}_{\mu\nu\rho}(g)$. This relationship follows from the decomposition given in Equation (2.31). By applying equations (2.4), (2.25), (2.32), and (2.33), we derive the following relation:

$$\begin{aligned} R^\alpha{}_{\mu\nu\rho}(\Gamma) &= \mathring{R}^\alpha{}_{\mu\nu\rho}(g) + T^\lambda{}_{\nu\rho} K^\alpha{}_{\lambda\mu} + 2\overset{\circ}{\nabla}_{[\nu} K^\alpha{}_{\rho]\mu} + T^\lambda{}_{\nu\rho} L^\alpha{}_{\lambda\mu} + 2\overset{\circ}{\nabla}_{[\nu} L^\alpha{}_{\rho]\mu} \\ &\quad + 2K^\alpha{}_{[\nu|\lambda} K^\lambda{}_{\rho]\mu} + 2L^\alpha{}_{[\nu|\lambda} K^\lambda{}_{\rho]\mu} + 2K^\alpha{}_{[\nu|\lambda} L^\lambda{}_{\rho]\mu} + 2L^\alpha{}_{\nu[\lambda} L^\lambda{}_{\rho]\mu}. \end{aligned} \quad (2.39)$$

Using this result, we can express the general Ricci scalar R in terms of the Levi-Civita Ricci scalar \mathring{R} :

$$R(\Gamma) = \mathring{R}(g) + \mathcal{T} + \mathcal{Q} + \mathring{\nabla}_\mu \left(2T^\mu + Q^\mu - \tilde{Q}^\mu \right), \quad (2.40)$$

where we used the contortion and the disformation traces $K^\beta{}_{\beta\alpha} = -T_\alpha$ and $L^\beta{}_{\beta\alpha} = -Q_\alpha/2$. From this relation, we identify two key subcases: one where non-metricity is absent (Teleparallel Gravity) and one where torsion is absent (Symmetric Teleparallel Gravity). In the former case, we have:

$$R(\Gamma) = \mathring{R}(g) + \mathcal{T} + 2\mathring{\nabla}_\mu (T^\mu), \quad (2.41)$$

while in the latter case, we obtain:

$$R(\Gamma) = \mathring{R}(g) + \mathcal{Q} + \mathring{\nabla}_\mu \left(Q^\mu - \tilde{Q}^\mu \right). \quad (2.42)$$

In the absence of both torsion and non-metricity, the theory reduces to GR, where the general connection and curvature coincide with their Levi-Civita counterparts.

Having defined the relationship of torsion and non-metricity scalars with the curvature scalar, we now explore the implications of the flatness postulate $R^\alpha{}_{\mu\nu\rho}(\Gamma) = 0$. In this case, the connection is referred to as a flat connection. Suppose that, in a given coordinate system, the connection that satisfies the flatness postulate is $\Gamma^\alpha{}_{\mu\nu} = 0$. Since the curvature is a tensor, the flatness condition remains valid in any coordinate system. However, the connection will transform inhomogeneously under a change of coordinates $x^\mu \rightarrow x'^\mu$ as follows:

$$\Gamma'^\alpha{}_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \underbrace{\Gamma^\rho{}_{\beta\gamma}}_{=0} + \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\nu} = \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\nu}. \quad (2.43)$$

Therefore, the condition $R'^\alpha{}_{\mu\nu\rho} = 0$ will be satisfied for any connection of the following form:

$$\Gamma'^\alpha{}_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\beta \partial_\mu \Lambda^\beta{}_\nu, \quad (2.44)$$

where the matrix $\Lambda^\beta{}_\nu \in GL(4, \mathbb{R})$ corresponds to the $\frac{\partial x^\beta}{\partial x'^\nu}$ in (2.43). Although we have shown that (2.44) is valid when the matrix Λ corresponds to a Jacobian, the flatness condition actually holds for any matrix $\Lambda \in GL(4, \mathbb{R})$, reflecting the freedom in choosing reference frames. Furthermore, the flat connection described in (2.44) exhibits the symmetry $\Lambda \rightarrow A\Lambda$ for any constant matrix $A \in GL(4, \mathbb{R})$.

Having discussed the properties of flat connections, we can now explore the teleparallel theories that are equivalent to GR.

2.4.1 Metric Teleparallel Equivalent of GR

In Metric Teleparallel Theories of gravity, torsion is the only significant quantity involved. In addition to the flatness condition, the connection must satisfy metric compatibility, given by $Q^\alpha{}_{\mu\nu} = 0$. This condition leads to

$$g^{\lambda(\mu} \partial_\alpha \Lambda^{\nu)\rho} (\Lambda^{-1})^\rho{}_\lambda = \frac{1}{2} \partial_\alpha g^{\mu\nu}. \quad (2.45)$$

Moreover, torsion can now be explicitly expressed in terms of the matrix $\Lambda^\mu{}_\nu$:

$$T^\alpha{}_{\mu\nu} = 2 (\Lambda^{-1})^\alpha{}_\gamma \partial_{[\mu} \Lambda^\gamma{}_{\nu]}. \quad (2.46)$$

The torsion tensor now depends on $\Lambda^\mu{}_\nu$, establishing it as the fundamental field of the theory. It is important to note that there can be multiple connections that satisfy both (2.44) and (2.45).

We now focus on the TEGR. Given the flatness of spacetime ($R(\Gamma) = 0$), equation (2.41) simplifies to:

$$\overset{\circ}{R}(g) = -\mathcal{T}(\Lambda) - 2\overset{\circ}{\nabla}_\mu (T^\mu(\Lambda)), \quad (2.47)$$

where \mathcal{T} is given by (2.37). In TEGR, the torsion scalar \mathcal{T} takes the place of the Ricci scalar $\overset{\circ}{R}$ in the Einstein-Hilbert action. The additional term on the right-hand side of the equation (2.47) is a total derivative and does not contribute to the field equations. This shift indicates that the matrix $\Lambda \in GL(4, \mathbb{R})$ has replaced the metric as the fundamental field of the theory. As a result, there will be 16 field equations in total: ten corresponding to the Einstein equations from GR and six associated with local Lorentz transformations.

We can define the TEGR action as [77]:

$$\mathcal{S}_{\text{TEGR}} = -\frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} \mathcal{T} + \mathcal{S}_m. \quad (2.48)$$

In the present case, the variational principle should be applied using the Palatini formalism, where the metric and connection are considered to be independent variables. This approach enables the consideration of more general connections, as the conditions of symmetry and compatibility can be relaxed. By varying the TEGR action (2.48) with respect to both the metric and the connection, we obtain two sets of field equations which are provided below [82]:

$$\begin{aligned} (\nabla_\alpha + T_\alpha) S_{(\mu\nu)}^\alpha + t_{\mu\nu} - \frac{1}{2} \mathcal{T} g_{\mu\nu} &= \kappa \Theta_{\mu\nu}, \\ (\nabla_\alpha + T_\alpha) [\sqrt{-g} S_{[\mu}{}^\alpha{}_{\nu]}] &= 0. \end{aligned} \quad (2.49)$$

Here, $\Theta_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$ is the energy-momentum tensor, $S^\alpha{}_{\mu\nu}$ is torsion conjugate, defined as

$$S_\alpha{}^{\mu\nu} = \frac{\partial \mathcal{T}}{\partial T^\alpha{}_{\mu\nu}} = \frac{1}{4} T_\alpha{}^{\mu\nu} + \frac{1}{2} T^{[\mu}{}_{\alpha}{}^{\nu]} - \delta_\alpha^{[\mu} T^{\nu]}, \quad (2.50)$$

and $t_{\mu\nu}$ is a symmetric tensor given by

$$t_{\mu\nu} = \frac{\partial \mathcal{T}}{\partial g^{\mu\nu}} = \frac{1}{2} S_{(\mu}{}^{\alpha\beta} T_{\nu)\alpha\beta} - T^{\alpha\beta}{}_{(\mu} S_{\alpha\beta|\nu)}. \quad (2.51)$$

For simplicity, the field equations (2.49) assume that the connection does not influence the matter sector of the action. As a result, the variation of the matter action with respect to the connection – known as hypermomentum – does not appear. Therefore, the energy-momentum tensor will be the sole source of the gravitational field equations³.

Two important consequences arise from the field equations [82]. First, the metric field equations remain equivalent to Einstein's equations in GR. Second, the connection equations are automatically satisfied as long as the connection obeys (2.44) and (2.45). This implies that the connection equations do not influence the dynamics of the theory; only the metric equations do. Consequently, TEGR exhibits the same dynamics as GR, as demonstrated in (2.47).

2.4.2 Symmetric Teleparallel Equivalent of GR

This subsection will explore the Symmetric Teleparallel Equivalent of GR (STTEGR). As in TEGR, in STTEGR, the Levi-Civita Ricci scalar can be replaced by a different scalar, specifically the non-metricity scalar (2.38). In contrast to TEGR, where the connection is metric-compatible, STTEGR requires a symmetric connection. This condition is expressed as:

$$(\Lambda^{-1})^\alpha{}_\beta \partial_{[\mu} \Lambda^\beta{}_{\nu]} = 0. \quad (2.52)$$

The equation above holds for any transformation of the form $\Lambda^\mu{}_\nu = \partial_\nu \xi^\mu$ where ξ^μ is a collection of four scalar functions. This simple fact has a significant consequence: we can make the connection vanish in a specific frame by choosing $\xi^\mu = M^\mu{}_\nu x^\nu + \xi_0^\mu$. In this expression, ξ_0^μ has constant entries, and $M^\mu{}_\nu$ is a constant, non-degenerate matrix. This gauge choice is known as the *coincident gauge* [35].

³Although hypermomentum plays a significant role in gravitational dynamics and is well-known for its associations with matter and connection – such as the coupling of fermions to torsion in specific gravity theories (see references [83–85]) – we will not address it in this discussion.

Additionally, the expression for the Ricci scalar, given in (2.42), simplifies to:

$$\mathring{R}(g) = -\mathcal{Q} - \mathring{\nabla}_\mu \left(Q^\mu - \tilde{Q}^\mu \right), \quad (2.53)$$

where \mathcal{Q} is defined by (2.38). The additional term on the right-hand side is a boundary term and does not influence the field equations. Therefore, GR and STEGR are dynamically equivalent. In this formulation, the metric is complemented by the field ξ , which introduces four extra dynamical equations alongside the ten equations with respect to the metric.

The action for STEGR takes the form [36]:

$$\mathcal{S}_{\text{STEGR}} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} \mathcal{Q} + \mathcal{S}_m. \quad (2.54)$$

Once again, we adopt the Palatini formalism, treating the metric and connection as independent variables. Varying the action with respect to each of the independent fields gives the following field equations [82]:

$$\frac{2}{\sqrt{-g}} \nabla_\alpha \left[\sqrt{-g} P^\alpha{}_{\mu\nu} \right] + q_{\mu\nu} - \frac{1}{2} \mathcal{Q} g_{\mu\nu} = \kappa \Theta_{\mu\nu}, \quad (2.55)$$

$$\nabla_\mu \nabla_\nu \left(\sqrt{-g} P^{\mu\nu}{}_\alpha \right) = 0. \quad (2.56)$$

Here, $\Theta_{\mu\nu}$ denotes the energy-momentum, $P^{\mu\nu}{}_\alpha$ is the non-metricity conjugate, which is defined by

$$P^\alpha{}_{\mu\nu} = \frac{1}{2} \frac{\partial \mathcal{Q}}{\partial Q^\alpha{}_{\mu\nu}} = -\frac{1}{4} Q^\alpha{}_{\mu\nu} + \frac{1}{2} Q_{(\mu}{}^\alpha{}_{\nu)} + \frac{1}{4} g_{\mu\nu} Q^\alpha - \frac{1}{4} \left(g_{\mu\nu} \tilde{Q}^\alpha + \delta^\alpha{}_{(\mu} Q_{\nu)} \right), \quad (2.57)$$

and $q_{\mu\nu}$ is a symmetric tensor given by

$$q_{\mu\nu} = \frac{\partial \mathcal{Q}}{\partial g^{\mu\nu}} = P_{(\mu|\alpha\beta} Q_{\nu)}{}^{\alpha\beta} - 2P^{\alpha\beta}{}_{(\nu} Q_{\alpha\beta|\mu)}. \quad (2.58)$$

As in TEGR, the connection equations are automatically satisfied for any flat symmetric connection. Consequently, the connection does not contribute dynamically to the theory.

2.4.3 General Teleparallel Equivalent of GR

In this subsection, we extend the teleparallel framework to introduce the General Teleparallel Equivalent of GR (GTEGR), a formulation that includes both TEGR and STEGR.

GTEGR incorporates both torsion and non-metricity while maintaining the flatness condition: $R^\alpha{}_{\mu\nu\rho}(\Gamma) = 0$. In addition to the torsion scalar (as provided in (2.37)) and the non-metricity scalar (as described in (2.38)), GTEGR permits

additional terms that couple torsion and non-metricity. These mixed quadratic terms are expressed as follows:

$$T_{\alpha\mu\nu}Q^{\mu\nu\alpha}, \quad T_\alpha Q^\alpha, \quad T_\alpha \tilde{Q}^\alpha. \quad (2.59)$$

Consequently, the Ricci scalar expression (2.40) takes the following form:

$$\overset{\circ}{R}(g) = -\mathcal{G} - \overset{\circ}{\nabla}_\mu \left(2T^\mu + Q^\mu - \tilde{Q}^\mu \right), \quad (2.60)$$

where

$$\mathcal{G} := \mathcal{T} + \mathcal{Q} + T_{\alpha\mu\nu}Q^{\mu\nu\alpha} - T_\alpha Q^\alpha + T_\alpha \tilde{Q}^\alpha. \quad (2.61)$$

In particular, setting torsion to zero yields $\mathcal{G} = \mathcal{Q}$, while eliminating non-metricity gives $\mathcal{G} = \mathcal{T}$. However, setting either torsion or non-metricity to zero imposes additional constraints on the connection, leading to more specific teleparallel models, as discussed in subsections 2.4.1 and 2.4.2.

The GTEGR action is expressed as [38]:

$$\mathcal{S}_{\text{GTEGR}} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} \mathcal{G} + \mathcal{S}_m. \quad (2.62)$$

From (2.60), it is evident that the GTEGR action differs from the Einstein-Hilbert action only by a boundary term. Therefore, while their formulations differ, both GR and GTEGR describe the same underlying theory. Although we will not present the field equations for the GTEGR action here, we will focus on discussing some implications of the equivalence between GR and GTEGR.

A fundamental distinction between the two formulations lies in the number of fundamental fields: GR relies solely on the metric, whereas GTEGR depends on both the metric and the connection. This apparent inconsistency is resolved by the fact that the connection equations are satisfied off-shell [38]. This indicates that the connection is not a dynamic field of the theory; instead, it appears in the action (2.62) as a boundary term. Consequently, the metric accounts for all degrees of freedom in both theories.

2.4.4 Modified Teleparallel Theories

We now discuss some MG theories in teleparallel geometries. Teleparallel theories can be modified in basically two ways. The first approach is to construct a more general quadratic scalar from torsion and non-metricity. The second consists of introducing non-linear extensions to metric and symmetric teleparallel gravities.

We begin this subsection by examining General Teleparallel Theory, which incorporates both torsion and connection. Unlike GTEGR, which was discussed earlier, this theory is not necessarily equivalent to GR, making it more general.

Within the General Teleparallel gravity framework, the most general scalar takes the form:

$$G := T + Q + b_1 T_{\alpha\mu\nu} Q^{\mu\nu\alpha} + b_2 T_\alpha Q^\alpha + b_3 T_\alpha \tilde{Q}^\alpha, \quad (2.63)$$

where T and Q are the scalars defined by

$$T = a_1 T_{\alpha\mu\nu} T^{\alpha\mu\nu} + a_2 T_{\alpha\mu\nu} T^{\mu\alpha\nu} + a_3 T_\alpha T^\alpha, \quad (2.64)$$

and

$$Q = c_1 Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + c_2 Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} + c_3 Q_\alpha Q^\alpha + c_4 \tilde{Q}_\alpha \tilde{Q}^\alpha + c_5 Q_\alpha \tilde{Q}^\alpha. \quad (2.65)$$

Here, a_i , b_i , and c_i are arbitrary real constants. The quadratic terms in (2.63), (2.64), and (2.65) represent all possible parity-even quadratic combinations of torsion and non-metricity [86].

It is important to clarify the notation used for the scalars. The symbols \mathcal{T} and \mathcal{Q} represent specific choices of torsion and non-metricity scalars, as defined in equations (2.37) and (2.38). The scalar \mathcal{G} , defined in equation (2.61), contains \mathcal{T} and \mathcal{Q} , plus mixed combinations of torsion and non-metricity. Furthermore, the scalars \mathcal{T} , \mathcal{Q} and \mathcal{G} are related to the Levi-Civita curvature as shown in equations (2.47), (2.53) and (2.60), respectively. As a result, these scalars can act as substitutes for $\overset{\circ}{R}$, leading to theories that are dynamically equivalent to GR. In contrast, the scalars G , T and Q recover GR only under the specific parameter choice corresponding to GTEGR ($a_1 = 1/8$, $a_2 = 1/4$, $a_3 = -1/2$, $c_1 = -c_3 = 1/4$, $c_2 = -c_5 = -1/2$, $c_4 = 0$ and $b_1 = -b_2 = -b_3 = 1$).

The scalars G , T and Q , along with the constants a_i , b_i , and c_i , constitute the parameters of the General Teleparallel theory. We have seen above the choice of parameters that lead to GTEGR. Two important subcases of General Teleparallel gravity are:

- **(Metric) Teleparallel Gravity:** For this scenario, we set $b_i = c_i = 0$, and the action is derived directly from T . The 3-parameter theory (with arbitrary a_i) is called New GR [35, 87, 88], and fixing the parameters a_i to $a_1 = 1/8$, $a_2 = 1/4$, $a_3 = -1/2$ results in TEGR.
- **Symmetric Teleparallel Gravity:** Here, we set $a_i = b_i = 0$. The scalar in the action simplifies to Q . If c_i are arbitrary, we have the theory known as Newer GR [35], while STEGR corresponds to a specific parameter choice: $c_1 = -c_3 = 1/4$, $c_2 = -c_5 = -1/2$, $c_4 = 0$.

A key distinction between New GR and TEGR lies in their physical degrees of freedom, and the same applies to Newer GR versus STEGR. Since TEGR and STEGR are equivalent to GR, they both have two degrees of freedom, as expected. In contrast, the number of degrees of freedom in New GR and Newer GR varies

based on the chosen parameters, a_i and c_i , respectively [89]. The same applies to general teleparallel quadratic gravity, where the number of degrees of freedom is determined by the choice of parameters a_i , b_i , and c_i [90].

The Lagrangian of the General Teleparallel theory can be expressed as

$$L = \frac{1}{2} \left(Q_\alpha{}^{\mu\nu} P^\alpha{}_{\mu\nu} + T^\alpha{}_{\mu\nu} S_\alpha{}^{\mu\nu} \right). \quad (2.66)$$

The corresponding field equations for the metric $g_{\mu\nu}$ and the connection $\Lambda^\mu{}_\nu$ are given by [38]:

$$2 \left(\nabla_\alpha + T_\alpha + \frac{1}{2} Q_\alpha \right) P^\alpha{}_{\mu\nu} = g_{\mu\nu} L - 2 \frac{\partial L}{\partial g^{\mu\nu}} + \kappa \Theta_{\mu\nu}, \quad (2.67)$$

and

$$\left(\nabla_\mu + T_\mu + \frac{1}{2} Q_\mu \right) (S_\alpha{}^{\mu\nu} - P^{\mu\nu}{}_\alpha) = 0. \quad (2.68)$$

Notably, the non-metricity conjugate now incorporates torsion-dependent terms and vice versa. Their explicit form is given by:

$$\begin{aligned} P^\alpha{}_{\mu\nu} &= 2c_1 Q^\alpha{}_{\mu\nu} + 2c_2 Q_{(\mu\nu)}{}^\alpha + 2c_3 Q^\alpha g_{\mu\nu} + 2c_4 \delta_{(\mu}^\alpha \tilde{Q}_{\nu)} \\ &+ c_5 \left(\tilde{Q}^\alpha g_{\mu\nu} + \delta_{(\mu}^\alpha Q_{\nu)} \right) - b_1 T_{(\mu\nu)}{}^\alpha + b_2 T^\alpha g_{\mu\nu} + b_3 \delta_{(\mu}^\alpha T_{\nu)}, \end{aligned} \quad (2.69)$$

$$\begin{aligned} S_\alpha{}^{\mu\nu} &= 2a_1 T_\alpha{}^{\mu\nu} - 2a_2 T^{[\mu\nu]}{}_\alpha - 2a_3 \delta_\alpha^{[\mu} T^{\nu]} + b_1 Q^{[\mu\nu]}{}_\alpha \\ &- b_2 \delta_\alpha^{[\mu} Q^{\nu]} - b_3 \delta_\alpha^{[\mu} \tilde{Q}^{\nu]}. \end{aligned} \quad (2.70)$$

Moreover, we have

$$\begin{aligned} \frac{\partial L}{\partial g^{\mu\nu}} &= c_1 \left(Q_{\mu\alpha\beta} Q_\nu{}^{\alpha\beta} - 2Q_{\alpha\beta\mu} Q^{\alpha\beta}{}_\nu \right) - c_2 Q_{\alpha\beta\mu} Q^{\beta\alpha}{}_\nu \\ &+ c_3 (Q_\mu Q_\nu - 2Q^\alpha Q_{\alpha\mu\nu}) - c_4 \tilde{Q}_\mu \tilde{Q}_\nu - c_5 \tilde{Q}^\alpha Q_{\alpha\mu\nu} \\ &+ a_1 \left(2T_{\alpha\beta\mu} T^{\alpha\beta}{}_\nu - T_{\mu\alpha\beta} T_\nu{}^{\alpha\beta} \right) + a_2 T_{\alpha\beta\mu} T^{\beta\alpha}{}_\nu + a_3 T_\mu T_\nu \\ &- b_1 \left(T_{\alpha\beta(\mu} Q_{\nu)}{}^{\alpha\beta} + T_{(\mu|\alpha\beta|} Q^{\alpha\beta}{}_{\nu)} \right) + b_2 (T_{(\mu} Q_{\nu)} - T^\alpha Q_{\alpha\mu\nu}) \\ &= \frac{1}{2} Q_{(\nu}{}^{\alpha\beta} P_{\mu)\alpha\beta} - Q_{\alpha\beta(\mu} P^{\alpha\beta}{}_{\nu)} - \frac{1}{2} T_{(\mu|\alpha\beta} S_{\nu)}{}^{\alpha\beta} + T_{\alpha\beta(\nu} S^{\alpha\beta}{}_{\mu)}. \end{aligned} \quad (2.71)$$

We observe that selecting the coefficients for STEGR recovers the field equations presented in (2.55) and (2.56). Similarly, when we choose the TEGR coefficients, we obtain the equations in (2.49).

We now explore non-linear extensions of TEGR and STEGR, which involve substituting the torsion and non-metricity scalars with functions, $f(\mathcal{T})$ or $f(\mathcal{Q})$,

respectively. The theory that arises from using non-linear functions of the torsion scalar \mathcal{T} is commonly referred to as $f(\mathcal{T})$ theory. Its action is given by [79–81]:

$$\mathcal{S}_{f(\mathcal{T})} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\mathcal{T}) + \mathcal{S}_m. \quad (2.72)$$

The corresponding field equations for the metric and connection are as follows:

$$f_{\mathcal{T}} G_{\mu\nu} - \frac{1}{2} (f(\mathcal{T}) - f_{\mathcal{T}} \mathcal{T}) g_{\mu\nu} + f_{\mathcal{T}\mathcal{T}} S_{(\mu\nu)}{}^\alpha \partial_\alpha \mathcal{T} = \kappa \Theta_{\mu\nu} \quad (2.73)$$

$$(\nabla_\mu + T_\mu) [f_{\mathcal{T}} S_{[\alpha}{}^\mu{}_{\beta]}] = 0. \quad (2.74)$$

In these equations, $f_{\mathcal{T}}$ and $f_{\mathcal{T}\mathcal{T}}$ stand for the first and second derivative of $f(\mathcal{T})$ with respect to \mathcal{T} , respectively. Choosing $f(\mathcal{T}) = \mathcal{T} + 2\Lambda$, where Λ is the cosmological constant, restores Einstein's equations. As a result, deviations from GR arise only when $f_{\mathcal{T}\mathcal{T}} \neq 0$.

In a similar manner, $f(\mathcal{Q})$ gravity is based on the following action [35]:

$$\mathcal{S}_{f(\mathcal{Q})} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\mathcal{Q}) + \mathcal{S}_m. \quad (2.75)$$

The metric and connection field equations are:

$$f_{\mathcal{Q}} G_{\mu\nu} - \frac{1}{2} (f(\mathcal{Q}) - f_{\mathcal{Q}} \mathcal{Q}) g_{\mu\nu} + 2f_{\mathcal{Q}\mathcal{Q}} P^\alpha{}_{\mu\nu} \partial_\alpha \mathcal{Q} = \kappa \Theta_{\mu\nu} \quad (2.76)$$

$$\nabla_\mu \nabla_\nu (\sqrt{-g} f_{\mathcal{Q}} P^{\mu\nu}{}_\alpha) = 0, \quad (2.77)$$

where $f_{\mathcal{Q}}$ and $f_{\mathcal{Q}\mathcal{Q}}$ denote the first and second derivative of $f(\mathcal{Q})$ with respect to \mathcal{Q} , respectively. Equation (2.76) reduces to the Einstein equations with a cosmological constant when $f(\mathcal{Q}) = \mathcal{Q} + 2\Lambda$. Thus, just as in $f(\mathcal{T})$ gravity, the condition $f_{\mathcal{Q}\mathcal{Q}} \neq 0$ is necessary for truly extending GR within the framework of $f(\mathcal{Q})$ theory.

To conclude our discussion, we would like to emphasise that Teleparallel Theories and Scalar-Tensor Theories are special cases of more general theories. For instance, symmetric teleparallel theories can be extended to scalar-tensor theories based on non-metricity [91]. The same applies to Metric Teleparallel [92–94] and General Teleparallel theories [76].

Additionally, Metric and Symmetric Teleparallel theories can be extended to scalar-tensor formulations analogous to Horndeski gravity. Teleparallel Horndeski Gravity [95] replaces curvature with torsion, combining both Horndeski Gravity and Teleparallel Gravity and having $f(\mathcal{T})$ models as special cases. Similarly, Symmetric Teleparallel Horndeski Gravity [96] reformulates Horndeski Gravity for spacetimes where non-metricity is the primary gravitational quantity, including Symmetric Teleparallel Gravity and $f(\mathcal{Q})$ theories.

Chapter 3

Formalism

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This chapter presents the mathematical formalism used in the appended papers. We start with a discussion on the necessity of a gravitational energy-momentum pseudo-tensor, presenting some classical examples. We then extend this discussion to energy tensors in Teleparallel Gravity theories. Then, we review scalar-vector-tensor decomposition and instabilities. In both cases, we present the application of these topics to $f(Q)$ gravity.

3.1 The Gravitational Energy-Momentum Pseudo-Tensor

The conservation of energy and momentum are among the most essential physical principles. These principles can be successfully introduced in Special Relativity, but we find some obstacles when trying to establish energy-momentum conservation in GR.

In flat spacetimes, we express the conservation of energy and momentum using the equation $\partial_\mu \Theta^{\mu\nu} = 0$. In curved spacetimes, the partial derivative should be replaced by the covariant derivative, leading us to the equation $\overset{\circ}{\nabla}_\mu \Theta^{\mu\nu} = 0$. As shown in Section 2.1, this equation is indeed satisfied. However, it is essential to note that this does not imply a conservation law.

To understand this more clearly, let us consider the expression for the divergence of a symmetric rank-two tensor $A^{\mu\nu}$ [97]:

$$\overset{\circ}{\nabla}_\mu A^\mu{}_\nu = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} A^\mu{}_\nu)}{\partial x^\mu} - \frac{1}{2} g_{\mu\eta,\nu} A^{\mu\eta}. \quad (3.1)$$

Consequently, the equation $\overset{\circ}{\nabla}_\mu \Theta^\mu{}_\nu = 0$ becomes:

$$\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} \Theta^\mu{}_\nu)}{\partial x^\mu} - \frac{1}{2} g_{\mu\eta,\nu} \Theta^{\mu\eta} = 0. \quad (3.2)$$

According to Gauss's theorem, for the integral $\int \Theta^\mu{}_\nu \sqrt{-g} dS_\mu^1$ to vanish, we require:

$$\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} \Theta^\mu{}_\nu)}{\partial x^\mu} = 0. \quad (3.3)$$

However, the second term in equation (3.2) prevents this condition from being satisfied.

The above discussion demonstrates that in the presence of gravity, the conservation of the energy-momentum tensor requires some adjustments. These are achieved by realising that, alongside the matter energy-momentum tensor $\Theta^{\mu\nu}$, there also exists a gravitational energy-momentum (pseudo-)tensor, which we will

¹The four-vector dS_μ is directed perpendicularly to the hypersurface and its norm corresponds to the hypersurface volume element.

denote as $t^{\mu\nu}$. The conservation of energy-momentum arises from the combined contributions of matter and gravity as follows:

$$\partial_\mu[\sqrt{-g}(\Theta^{\mu\nu} + t^{\mu\nu})] = 0 \quad (3.4)$$

The gravitational energy-momentum pseudo-tensor is expected to vanish in the absence of gravity. However, due to the presence of an ordinary derivative on the right side of the equation above, it is evident that $t^{\mu\nu}$ is not a proper tensor. Consequently, depending on the coordinate choice, $t^{\mu\nu}$ may not vanish even when $g_{\mu\nu} = \eta_{\mu\nu}$. Conversely, it might vanish at every point, even in the presence of gravity.

The fact that the gravitational contributions to energy and momentum appear in non-tensorial form is rather undesirable since it challenges the physical relevance of these quantities [98]. Moreover, the attempts to solve this problem resulted in an overabundance of pseudo-tensors, which, in general, do not agree with one another. Among the most notable energy-momentum pseudo-tensors are those by Landau-Lifshitz, Einstein, Papapetrou and Weinberg, which are listed below.

Einstein pseudo-tensor

The pseudo-tensor proposed by Einstein is given by [99]:

$$\sqrt{-g}t_{\mu E}^\nu = \frac{1}{16\pi} \left(\delta_\mu^\nu \mathcal{L}_E - \frac{\partial \mathcal{L}_E}{\partial g^{\alpha\beta, \nu}} g^{\alpha\beta, \mu} \right), \quad (3.5)$$

where the Lagrangian density \mathcal{L}_E is given by $\sqrt{-g}g^{\mu\nu}(\dot{\Gamma}^\alpha_{\mu\nu}\dot{\Gamma}^\beta_{\alpha\beta} - \dot{\Gamma}^\alpha_{\mu\beta}\dot{\Gamma}^\beta_{\nu\alpha})$. Since this expression does not contain any second derivatives of the metric, the Einstein energy-momentum pseudo-tensor depends only on the metric and its first derivatives. It is important to note that \mathcal{L}_E is not a scalar, and $t_{\mu E}^\nu$ is not a proper tensor. Additionally, the Einstein pseudo-tensor is generally not symmetric. The lack of symmetry and tensorial properties in Einstein's complex has raised concerns regarding its physical interpretation.

The conserved quantity $\sqrt{-g}(\Theta^\mu{}_\nu + t_{\mu E}^\nu)$ can be expressed as the ordinary derivative of a quantity $\mathfrak{h}_{\nu E}^{\mu\alpha}$, in the following way [100]:

$$\sqrt{-g}(\Theta^\mu{}_\nu + t_{\mu E}^\nu) = \frac{1}{16\pi} \mathfrak{h}_{\nu E}^{\mu\alpha, \alpha}. \quad (3.6)$$

The quantity $\mathfrak{h}_{\nu E}^{\mu\alpha}$ is commonly referred to as a superpotential and, in this context, it is expressed as $\mathfrak{h}_{\nu E}^{\mu\alpha} = (-g)^{-1/2} g_{\nu\beta} [-g(g^{\mu\beta}g^{\alpha\sigma} - g^{\alpha\beta}g^{\mu\sigma})]_{,\sigma}^2$.

Since the superpotential is antisymmetric, $\mathfrak{h}_{\nu E}^{\mu\alpha} = -\mathfrak{h}_{\nu E}^{\alpha\mu}$, equation (3.6) ensures that the conservation equation (3.4) is satisfied. Throughout this section,

²For historical context, we would like to mention that this superpotential was introduced in 1939 by von Freud [101], and Einstein did not use it.

we will adopt the notation $\mathfrak{U}^{\mu\dots\nu\dots}$ to represent (pseudo-)tensor densities, such as $\mathfrak{h}_{\nu E}^{\mu\alpha}$.

Landau-Lifshitz pseudo-tensor

The pseudo-tensor proposed by Landau and Lifshitz is given by [97]:

$$\begin{aligned} 16\pi(-g)t_{\text{LL}}^{\mu\nu} &= \mathfrak{g}^{\mu\nu}{}_{,\alpha}\mathfrak{g}^{\alpha\beta}{}_{,\beta} - \mathfrak{g}^{\mu\alpha}{}_{,\alpha}\mathfrak{g}^{\nu\beta}{}_{,\beta} + \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\mathfrak{g}^{\alpha\rho}{}_{,\sigma}\mathfrak{g}^{\sigma\beta}{}_{,\rho} \\ &- g^{\mu\rho}g_{\sigma\alpha}\mathfrak{g}^{\nu\alpha}{}_{,\beta}\mathfrak{g}^{\sigma\beta}{}_{,\rho} - g^{\nu\rho}g_{\sigma\alpha}\mathfrak{g}^{\mu\alpha}{}_{,\beta}\mathfrak{g}^{\sigma\beta}{}_{,\rho} + g_{\rho\sigma}g^{\alpha\beta}\mathfrak{g}^{\mu\rho}{}_{,\alpha}\mathfrak{g}^{\nu\sigma}{}_{,\beta} \\ &+ \frac{1}{8}(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma})(2g_{\alpha\beta}g_{\gamma\lambda} - g_{\beta\gamma}g_{\alpha\lambda})\mathfrak{g}^{\alpha\lambda}{}_{,\rho}\mathfrak{g}^{\beta\gamma}{}_{,\sigma}, \end{aligned} \quad (3.7)$$

where $\mathfrak{g}^{\beta\gamma} = \sqrt{-g}g^{\beta\gamma}$. The equation above can be rewritten in terms of the Levi-Civita connection $\overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu}$. Although the Levi-Civita connection is not a tensor, it transforms like one under linear transformations of the coordinates because the second term of equation (2.43) is zero. Consequently, the Landau-Lifshitz pseudo-tensor $t_{\text{LL}}^{\mu\nu}$ should exhibit this property as well. Additionally, the Landau-Lifshitz pseudo-tensor has the advantage of being symmetric.

The Landau and Lifshitz pseudo-tensor can be set to zero by selecting an inertial frame, which is a coordinate system where the connection components $\overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu}$ vanish. However, in specific coordinate systems – such as spherical coordinates – the pseudo-tensor may not vanish even in flat spacetimes.

Additionally, the Landau-Lifshitz superpotential $\mathfrak{h}_{\text{LL}}^{\mu\alpha\beta}$ can be expressed as $\frac{1}{16\pi}[(-g)(g^{\mu\alpha}g^{\beta\gamma} - g^{\mu\beta}g^{\alpha\gamma})]_{,\gamma}$, which is clearly antisymmetric in the last two indices. In this context, the conservation equation takes the form $\partial_{\alpha}[-g(\Theta^{\mu\alpha} + t^{\mu\alpha})] = 0$, since $-g(\Theta^{\mu\alpha} + t^{\mu\alpha}) = \mathfrak{h}_{\text{LL}}^{\mu\alpha\beta}{}_{,\beta}$.

Papapetrou pseudo-tensor

Papapetrou introduced the following pseudo-tensor [102]:

$$\Omega^{\mu\nu} = \frac{1}{16\pi} \frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} \left[\sqrt{-g} \left(g^{\mu\nu}\eta^{\alpha\beta} - g^{\mu\alpha}\eta^{\nu\beta} + g^{\alpha\beta}\eta^{\mu\nu} - g^{\nu\beta}\eta^{\mu\alpha} \right) \right], \quad (3.8)$$

where $\eta^{\alpha\beta}$ is the Minkowski metric. Notably, the Papapetrou pseudo-tensor is symmetric, satisfying $\Omega^{\mu\nu} = \Omega^{\nu\mu}$ and obeys the equation $16\pi\Omega^{\mu\nu} = \mathfrak{N}^{\mu\nu\alpha\beta}{}_{,\alpha\beta}$, where

$$\mathfrak{N}^{\mu\nu\alpha\beta} = \sqrt{-g} \left(g^{\mu\nu}\eta^{\alpha\beta} - g^{\mu\alpha}\eta^{\nu\beta} + g^{\alpha\beta}\eta^{\mu\nu} - g^{\nu\beta}\eta^{\mu\alpha} \right). \quad (3.9)$$

The Papapetrou superpotential can be identified as $\mathfrak{h}_{\text{P}}^{\mu\nu\alpha} = \mathfrak{N}^{\mu\nu\alpha\beta}{}_{,\beta}$.

Papapetrou derived his pseudo-tensor using Belifante's method, which assumes that the energy-momentum pseudo-tensor differs from the total energy-momentum density $\sqrt{-g}(\Theta^\nu{}_\alpha + t^\nu{}_\alpha)$ only by a total divergence:

$$\Omega^{\mu\nu} = \eta^{\mu\alpha}[\sqrt{-g}(\Theta^\nu{}_\alpha + t^\nu{}_\alpha)] + \frac{\partial}{\partial x^\alpha}(\mathfrak{B}^{\mu\nu\alpha}). \quad (3.10)$$

For the divergence of $\Omega^{\mu\nu}$ to vanish, i.e., $\Omega^{\mu\nu}{}_{,\nu} = 0$, the quantity $\mathfrak{B}^{\mu\nu\alpha}$ must be antisymmetric in its last two indices: $\mathfrak{B}^{\mu\nu\alpha} = -\mathfrak{B}^{\mu\alpha\nu}$. The quantity $\mathfrak{B}^{\mu\nu\alpha}$ can be expressed in terms of the field spin density $\mathfrak{f}^{\mu\nu\alpha}$ as $\mathfrak{B}^{\mu\nu\alpha} = -\frac{1}{2}(\mathfrak{f}^{\mu\nu\alpha} + \mathfrak{f}^{\alpha\nu\mu} + \mathfrak{f}^{\alpha\mu\nu})$. The spin density is given by:

$$\mathfrak{f}^{\mu\nu\lambda} = \frac{1}{8\pi} \frac{\partial L}{\partial \mathfrak{g}^{\alpha\beta}{}_{,\lambda}} (\mathfrak{g}^{\mu\alpha} \eta^{\nu\beta} - \mathfrak{g}^{\nu\alpha} \eta^{\mu\beta}), \quad (3.11)$$

where $\mathfrak{g}^{\alpha\beta}{}_{,\gamma} = \sqrt{-g} g^{\alpha\beta}{}_{,\gamma}$.

Weinberg pseudo-tensor

Weinberg's energy-momentum pseudo-tensor is based on the premise that the metric asymptotically approaches the Minkowski metric $\eta_{\mu\nu}$ at infinity. This idea is mathematically expressed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ is the spacetime metric and $h_{\mu\nu}$ is a perturbation that vanishes at large distances. By expanding the Einstein equations to second order in $h_{\mu\nu}$, Weinberg derived the following expression for the energy-momentum pseudo-tensor [40]:

$$t_{\mu\nu}^W = \frac{1}{8\pi} \left(-\frac{1}{2} h_{\mu\nu} R^{(1)\lambda}{}_{\lambda} + \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} R^{(1)}{}_{\alpha\beta} + R^{(2)}{}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} R^{(2)}{}_{\alpha\beta} \right) + \mathcal{O}(h^3). \quad (3.12)$$

In the expression above, $R^{(1)}{}_{\mu\nu}$ and $R^{(2)}{}_{\mu\nu}$ represent the first and second-order parts of the Ricci tensor $R_{\mu\nu}$, respectively. Their explicit forms are:

$$R^{(1)}{}_{\mu\nu} = \frac{1}{2} \left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda} \right), \quad (3.13)$$

and

$$\begin{aligned} R^{(2)}{}_{\mu\nu} &= -\frac{1}{2} h^{\alpha\beta} \left(\frac{\partial^2 h_{\alpha\beta}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 h_{\mu\beta}}{\partial x^\nu \partial x^\alpha} - \frac{\partial^2 h_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\beta \partial x^\alpha} \right) \\ &+ \frac{1}{4} \left(2 \frac{\partial h^\alpha{}_\beta}{\partial x^\alpha} - \frac{\partial h^\alpha{}_\alpha}{\partial x^\beta} \right) \left(\frac{\partial h^\beta{}_\mu}{\partial x^\nu} + \frac{\partial h^\beta{}_\nu}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x_\beta} \right) \\ &- \frac{1}{4} \left(\frac{\partial h_{\beta\nu}}{\partial x^\alpha} + \frac{\partial h_{\beta\alpha}}{\partial x^\nu} - \frac{\partial h_{\alpha\nu}}{\partial x^\beta} \right) \left(\frac{\partial h^\beta{}_\mu}{\partial x^\alpha} + \frac{\partial h^{\beta\alpha}}{\partial x^\mu} - \frac{\partial h^\alpha{}_\mu}{\partial x_\beta} \right). \end{aligned} \quad (3.14)$$

To interpret $t_{\mu\nu}^W$ in this context, we note that the linearised Einstein equations take the following form:

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_{\lambda} = 8\pi(\Theta_{\mu\nu} + t_{\mu\nu}^W). \quad (3.15)$$

Consequently, both $t_{\mu\nu}^W$ and $\Theta_{\mu\nu}$ act as sources of the linearized curvature. Therefore, the total energy-momentum of matter and gravity fields, which is expressed as $\tau_W^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}(\Theta_{\mu\nu} + t_{\mu\nu}^W)$, is symmetric and locally conserved: $\tau_W^{\alpha\beta} = \tau_W^{\beta\alpha}$ and $\frac{\partial\tau_W^{\alpha\beta}}{\partial x^\alpha} = 0$. Additionally, although $\tau_W^{\alpha\beta}$ and $t_{\mu\nu}^W$ are Lorentz covariant, they are not generally covariant [40].

The total energy-momentum $\tau_W^{\alpha\beta}$ obeys the equation $\tau_W^{\alpha\beta} = h_W^{\rho\alpha\beta}_{,\rho}$, where $H_W^{\rho\alpha\beta}$ is the Weinberg superpotential, given by [40]:

$$h_W^{\rho\alpha\beta} = \frac{1}{2} \left[\frac{\partial h^\lambda_{\lambda}}{\partial x_\alpha} \eta^{\rho\beta} - \frac{\partial h^\lambda_{\lambda}}{\partial x_\rho} \eta^{\alpha\beta} - \frac{\partial h^{\lambda\alpha}}{\partial x_\lambda} \eta^{\rho\beta} + \frac{\partial h^{\lambda\rho}}{\partial x_\lambda} \eta^{\alpha\beta} + \frac{\partial h^{\alpha\beta}}{\partial x_\rho} - \frac{\partial h^{\rho\beta}}{\partial x_\alpha} \right].$$

In the above equation, $h^\lambda_{\lambda} = \eta^{\lambda\sigma}h_{\lambda\sigma}$. Additionally, We can note that the Weinberg superpotential is antisymmetric in the first two indices: $h_W^{\rho\alpha\beta} = -h_W^{\alpha\rho\beta}$.

3.1.1 Covariantization

As we have seen, there are many energy-momentum pseudo-tensor choices from which we may question whether there is a framework where they would coincide. In addition to that, the pseudo-tensors presented above lack covariance, which can be illustrated by the presence of partial derivatives and/or the Minkowski metric. As previously discussed, this lack of covariance produces some inconvenient unphysical effects, which we would like to avoid.

To ensure that our expressions for the pseudo-tensors are covariant, we must apply the minimal coupling prescription. This procedure involves replacing any partial derivative with the corresponding covariant derivative, i.e., $\partial_\mu \rightarrow \nabla_\mu$. Additionally, we replace the Minkowski metric, $\eta_{\mu\nu}$, with the spacetime metric, $g_{\mu\nu}$.

As a first example, we apply the minimal coupling to the Einstein energy complex $\mathfrak{h}_E^{\mu\nu}{}_\alpha$. For this purpose, we will use a slightly different convention for the superpotential, given by:

$$\mathfrak{h}_E^{\mu\nu}{}_\alpha = \frac{1}{2}(-g)^{-1/2}g_{\alpha\gamma}[-g(g^{\sigma\mu}g^{\nu\gamma} - g^{\gamma\mu}g^{\nu\sigma})]_{,\sigma}. \quad (3.16)$$

Note that, in this convention, the superpotential is antisymmetric with respect to its first two indices. We also introduced the factor 1/2 for convenience. As a result, the relation between superpotential and total energy-momentum should be adapted

accordingly, taking the following form: $\sqrt{-g}(\Theta^\mu{}_\alpha + t^\mu_{\text{E}\alpha}) = \frac{1}{8\pi}\mathfrak{h}^{\mu\nu}{}_{\alpha,\nu}$. However, the conservation equation (3.4) still holds, since $\mathfrak{h}^{\mu\nu}{}_{\alpha,\mu\nu} = 0$.

After the minimal coupling prescription is applied, the Einstein complex takes the following form:

$$\begin{aligned}\mathfrak{h}^{\mu\nu}{}_\alpha &= \frac{1}{2}(-g)^{-1/2}g_{\alpha\gamma}[-g(g^{\sigma\mu}g^{\nu\gamma} - g^{\gamma\mu}g^{\nu\sigma})]_{;\sigma} \\ &= \frac{1}{2}(-g)^{1/2}g_{\alpha\gamma}(g^{\sigma\mu}{}_{;\sigma}g^{\nu\gamma} + g^{\sigma\mu}g^{\nu\gamma}{}_{;\sigma} - g^{\gamma\mu}{}_{;\sigma}g^{\nu\sigma} - g^{\gamma\mu}g^{\nu\sigma}{}_{;\sigma}) \\ &+ \frac{1}{2}(-g)^{-1/2}g_{\alpha\gamma}(g^{\sigma\mu}g^{\nu\gamma} - g^{\gamma\mu}g^{\nu\sigma})(-g)_{;\sigma}\end{aligned}\quad (3.17)$$

In the following steps, we will use the relations given below:

$$g^{\nu\gamma}{}_{;\sigma} = -Q_\sigma{}^{\nu\gamma}, \quad (3.18)$$

$$g^{\sigma\mu}{}_{;\sigma} = -Q_\sigma{}^{\sigma\mu} = -\tilde{Q}^\mu, \quad (3.19)$$

$$(-g)_{;\sigma} = (-g)Q_\sigma, \quad (3.20)$$

Then, the covariant version of the Einstein tensor becomes:

$$\mathfrak{h}^{\mu\nu\alpha} = \frac{1}{2}(-g)^{1/2}[-Q^{\mu\nu\alpha} - g^{\alpha\nu}\tilde{Q}^\mu + Q^{\nu\alpha\mu} + g^{\alpha\mu}\tilde{Q}^\nu + g^{\nu\alpha}Q^\mu - g^{\alpha\mu}Q^\nu],$$

where we have raised the last index. We can rewrite the above equation in a more condensed form, which evidences the superpotential antisymmetry:

$$h^{\mu\nu\alpha} \equiv (-g)^{-1/2}\mathfrak{h}^{\mu\nu\alpha} = -Q^{[\mu\nu]\alpha} + g^{\alpha[\mu}\tilde{Q}^{\nu]} - g^{\alpha[\mu}Q^{\nu]}. \quad (3.21)$$

Similarly, we will use a more convenient convention for the Landau-Lifshitz superpotential, given by:

$$\mathfrak{h}^{\mu\nu\alpha}_{\text{LL}} = \frac{1}{2}(-\eta)^{-1/2}[-g(g^{\sigma\mu}g^{\nu\alpha} - g^{\alpha\mu}g^{\nu\sigma})]_{;\sigma}, \quad (3.22)$$

where η is the Minkowski metric determinant, which was added to adjust the tensorial weight of the pseudo-tensor. Since the Minkowski determinant is a constant, it does not modify the relation between the total energy-momentum and the superpotential.

Although this superpotential is also antisymmetric in the first two indices, it can be shown that the conservation equation (3.4) remains valid, just as we did for the Einstein case. By following the same steps taken with the Einstein superpotential, we obtain the covariant version of the Landau-Lifshitz superpotential:

$$h^{\mu\nu\alpha}_{\text{LL}} \equiv (-g)^{-1/2}\mathfrak{h}^{\mu\nu\alpha}_{\text{LL}} = -Q^{[\mu\nu]\alpha} + g^{\alpha[\mu}\tilde{Q}^{\nu]} - g^{\alpha[\mu}Q^{\nu]}. \quad (3.23)$$

For the remaining superpotentials, we will need slightly different approaches. For the Papapetrou superpotential, for instance, we will promote the partial derivatives to covariant derivatives while maintaining the Minkowski metric. We will substitute $\eta_{\mu\nu}$ with $g_{\mu\nu}$ only after applying the covariant derivative to all terms separately. As a result, we obtain the following covariant Papapetrou energy complex:

$$h_{\text{P}}^{\mu\nu\alpha} \equiv (-g)^{-1/2} \mathfrak{h}_{\text{P}}^{\mu\nu\alpha} = -Q^{[\mu\nu]\alpha} + g^{\alpha[\mu} \tilde{Q}^{\nu]} + g^{\alpha[\mu} Q^{\nu]} \quad (3.24)$$

To arrive at this result, we rearranged the indices in such a way that the superpotential becomes antisymmetric in its first two indices. Specifically, we defined the superpotential as $\mathfrak{h}_{\text{P}}^{\alpha\nu\mu} = \frac{1}{2} \mathfrak{N}^{\mu\nu\alpha\beta}{}_{,\beta}$. However, such a choice modifies neither the relation $16\pi\Omega^{\mu\nu} = \mathfrak{N}^{\mu\nu\alpha\beta}{}_{,\alpha\beta}$ nor the conservation of $\Omega^{\mu\nu}$.

Now, let us turn our attention to the Weinberg superpotential, which, upon covariantisation, assumes the form:

$$h_{\text{W}}^{\mu\nu\alpha} \equiv (-g)^{-1/2} \mathfrak{h}_{\text{W}}^{\mu\nu\alpha} = -Q^{[\mu\nu]\alpha} + g^{\alpha[\mu} \tilde{Q}^{\nu]} - g^{\alpha[\mu} Q^{\nu]} \quad (3.25)$$

In the above, we first adjusted tensor weight of the superpotential by doing $h_{\text{W}}^{\mu\nu\alpha} \rightarrow \mathfrak{h}_{\text{W}}^{\mu\nu\alpha} = \sqrt{-\eta} h_{\text{W}}^{\mu\nu\alpha}$. Next, we made the substitution $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, using the fact that the covariant derivative of the Minkowski metric vanishes. We then replaced $\eta_{\mu\nu}$ with $g_{\mu\nu}$.

We can easily recognise that, upon covariantisation, all superpotentials acquire the same form [103]:

$$h_{\text{E}}^{\mu\nu\alpha} = h_{\text{LL}}^{\mu\nu\alpha} = h_{\text{P}}^{\mu\nu\alpha} = h_{\text{W}}^{\mu\nu\alpha}. \quad (3.26)$$

This approach unifies various earlier superpotentials in a background-independent framework, suggesting a way to solve the long-standing energy problem in GR.

3.1.2 Energy Tensors in Teleparallel Theories

In this subsection, we examine energy-momentum tensors in the context of specific teleparallel theories. We begin with STEGR and then extend the discussion to General Teleparallel Theory.

The STEGR case:

To analyse the gravitational energy-momentum pseudo-tensor, we first revisit the metric and connection field equations from Subsection (2.4.2) of Chapter 2. These equations are shown again below for reference [82]:

$$\frac{2}{\sqrt{-g}} \nabla_{\alpha} [\sqrt{-g} P^{\alpha}{}_{\mu\nu}] + q_{\mu\nu} - \frac{1}{2} \mathcal{Q} g_{\mu\nu} = \kappa \Theta_{\mu\nu} \quad (3.27)$$

$$\nabla_\mu \nabla_\nu (\sqrt{-g} P^{\mu\nu}_\alpha) = 0. \quad (3.28)$$

Here, $\Theta_{\mu\nu}$ denotes the matter energy-momentum tensor, $P^{\mu\nu}_\alpha$ is the non-metricity conjugate defined in equation (2.57), and $q_{\mu\nu}$ is a symmetric tensor given by equation (2.58).

The metric equation can be rewritten as follows [104]:

$$\tau^\mu_\nu - t^\mu_\nu = \Theta^\mu_\nu, \quad (3.29)$$

where

$$\kappa t^\mu_\nu = \frac{1}{2} Q \delta^\mu_\nu + P^{\mu\alpha\beta} Q_{\nu\alpha\beta} \quad (3.30)$$

is identified as the metrical energy-momentum tensor and

$$\kappa \tau^\mu_\nu = \frac{2}{\sqrt{-g}} \nabla_\alpha [\sqrt{-g} P^{\alpha\mu}_\nu] \quad (3.31)$$

is identified as the inertial energy-momentum tensor [82]. When the coincident gauge is applied, meaning $\nabla_\alpha \rightarrow \partial_\alpha$, the metrical energy-momentum takes the form of the Einstein pseudo-tensor ($t_{\mu\nu} \rightarrow t^E_{\mu\nu}$). Additionally, in this gauge, the inertial energy-momentum tensor $\tau_{\mu\nu}$ becomes the Einstein energy-momentum complex, a quantity defined as $\Theta^\mu_\nu + t^E_{\mu\nu}$ [104]. From the connection equation, we can conclude the inertial energy-momentum tensor τ^μ_ν is conserved:

$$\nabla_\mu (\sqrt{-g} \tau^\mu_\nu) = 0. \quad (3.32)$$

We now introduce the concept of the *Canonical Frame*, characterised by the vanishing of the metrical energy-momentum tensor, i.e., $t_{\mu\nu} = 0$ [104]. This condition is generally coordinate-independent, but it becomes coordinate-dependent in the coincident gauge. This behaviour is expected since the metrical energy-momentum tensor coincides with the Einstein pseudo-tensor in this particular case. The canonical frame is the reference frame where physical quantities, such as energy and momentum, should be calculated [103] since it aligns with the principles of Noether's theorems [105]. In other words, the observables coincide with the conserved Noether charges in this frame.

The General Teleparallel case:

We will now turn our attention to the General Teleparallel theory, previously introduced in Subsection 2.4.4 of the preceding chapter. Let us first rewrite the field equations below [38]:

$$2 \left(\nabla_\alpha + T_\alpha + \frac{1}{2} Q_\alpha \right) P^\alpha_{\mu\nu} = g_{\mu\nu} L - 2 \frac{\partial L}{\partial g^{\mu\nu}} + \Theta_{\mu\nu}, \quad (3.33)$$

and

$$\left(\nabla_\mu + T_\mu + \frac{1}{2} Q_\mu \right) (S_\alpha^{\mu\nu} - P^{\mu\nu}{}_\alpha) = 0. \quad (3.34)$$

In the equation above: T_α and Q_α are torsion and nonmetricity traces, respectively; L is the Lagrangian density; $\Theta_{\mu\nu}$ is the energy-momentum tensor; $S_\alpha^{\mu\nu}$ and $P^\alpha{}_{\mu\nu}$ are the torsion and non-metricity conjugates, defined by the equations (2.69) and (2.70), respectively.

By contracting the metric equation with $g^{\beta\mu}$ and adding the term $-2Q_\alpha{}^{\beta\mu}P^\alpha{}_{\beta\nu} = 2(\nabla_\alpha g^{\mu\nu})P^\alpha{}_{\beta\nu}$ to both sides, we arrive at the following expression:

$$2(\nabla_\alpha + T_\alpha)(\sqrt{-g}P^{\alpha\mu}{}_\nu) = \sqrt{-g}(G^\mu{}_\nu + \Theta^\mu{}_\nu), \quad (3.35)$$

where we have introduced the metrical energy-momentum tensor $G_{\mu\nu}$ which is given by [103]:

$$G_{\mu\nu} = g_{\mu\nu}L - 2\frac{\partial L}{\partial g^{\mu\nu}} - 2Q_{\alpha\beta\mu}P^{\alpha\beta}{}_\nu. \quad (3.36)$$

We now substitute equation 3.35 into the connection equation and add the term $-\sqrt{-g}T^\mu{}_{\alpha\beta}S_\nu{}^{\alpha\beta}$ to both sides. After some manipulation, we arrive at the following result [106]:

$$2(\nabla_\alpha + T_\alpha)(\sqrt{-g}S_\nu{}^{\alpha\mu}) - \sqrt{-g}T^\mu{}_{\alpha\beta}S_\nu{}^{\alpha\beta} = \sqrt{-g}(\bar{t}^\mu{}_\nu + \Theta^\mu{}_\nu), \quad (3.37)$$

where we introduced the canonical energy-momentum tensor $\bar{t}_{\mu\nu}$, defined as [103]:

$$\bar{t}_{\mu\nu} = G_{\mu\nu} - T_{\mu\alpha\beta}S_\nu{}^{\alpha\beta}. \quad (3.38)$$

These expressions for the metrical and canonical energy-momentum tensors are quite general and applicable to any teleparallel theory, including the quadratic General Teleparallel gravity as defined by the Lagrangian presented in equation 2.66. In the special case where torsion vanishes, the metrical energy-momentum tensor $G_{\mu\nu}$ reduces to $\bar{t}_{\mu\nu}$.

3.2 Scalar-Vector-Tensor Decomposition

This section presents a brief overview of the scalar-vector-tensor (SVT) decomposition, a fundamental tool in perturbation theory. In perturbation theory, two spacetimes are studied: an unperturbed background spacetime represented by the background metric $\bar{g}_{\mu\nu}$ and a perturbed spacetime denoted by $g_{\mu\nu}$, which deviates slightly from the background. Since the perturbations $\delta g_{\mu\nu}$ are assumed to be small, the perturbed metric can be expressed as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$.

In Cosmology ³, for instance, the background metric $\bar{g}_{\mu\nu}$ depends only on time, while the perturbations $\delta g_{\mu\nu}$ depend on both space and time. More specifically, the background corresponds to the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The metric perturbation $\delta g_{\mu\nu}$ can be decomposed into scalar, vector and tensor modes. The primary advantage of the SVT decomposition in this context is that, at first order, the Einstein equations for scalar, vector, and tensor perturbations decouple [107]. This allows us to study the evolution of each type of perturbation independently from the others.

Before exploring the SVT decomposition of tensors, let us first analyse how a vector can be separated into its scalar and vector components. In three dimensions, a vector \mathbf{V} can be decomposed as a sum of the gradient of a scalar ϕ and the curl of a divergence-free vector \mathbf{A} as follows [108]:

$$\mathbf{V} = \vec{\nabla}\phi + \vec{\nabla}\mathbf{A}, \quad (3.39)$$

It can be shown that the Laplacian of ϕ is equal to the divergence of the vector field \vec{V} , i.e., $\nabla^2\phi = \vec{\nabla} \cdot \vec{V}$. Additionally, the Laplacian of the vector field \vec{A} relates to the curl of \vec{V} as $\nabla^2\vec{A} = -\vec{\nabla} \times \vec{V}$. These relationships arise from the condition that the vector field \vec{A} is divergence-free, which is stated as $\vec{\nabla} \cdot \vec{A} = 0$ and from the following fundamental properties of vector calculus: the curl of the gradient of a scalar field equals zero, which is expressed as $\vec{\nabla} \times \vec{\nabla}\phi = 0$; the divergence of a curl, given by $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})$, is also zero.

According to Helmholtz's theorem, a vector field is fully determined by its divergence and curl [108]. As a result, the decomposition (3.39) is complete because all information about the vector \vec{V} is included. It is important to note that although the vector \vec{V} is fully determined by ϕ and \vec{A} , it is not uniquely specified by them. In fact, we can add a constant to ϕ (i.e., $\phi \rightarrow \phi + \text{constant}$) and the gradient of a scalar function χ to \vec{A} (i.e., $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\chi$) without altering the vector \vec{V} . This fact indicates that we have some freedom in choosing ϕ and \vec{A} , a concept known as *gauge freedom*.

In general, any vector can be decomposed into its scalar and vector components, expressed as $V_i = V_i^S + V_i^V$. This concept can be extended to tensors — particularly rank-2 tensors such as the metric — which can be decomposed into scalar, vector, and tensor components. This approach is referred to as scalar-vector-tensor decomposition.

As an example of the SVT decomposition, we decompose the metric perturbations $\delta g_{\mu\nu}$ around the Minkowski metric as follows [109]:

$$\begin{aligned} \delta g_{00} &= -2\phi, \\ \delta g_{0i} &= B_i + \partial_i B, \\ \delta g_{ij} &= -2\Psi\delta_{ij} + \partial_i\partial_j E + \frac{1}{2}\partial_i E_j + \frac{1}{2}\partial_j E_i + h_{ij}. \end{aligned} \quad (3.40)$$

³For a brief review of Cosmology, we refer the reader to section 4.2.

In the equation above, ϕ , Ψ , E and B are scalars, B_i and E_i are divergence-free vectors ($\delta^{ij}\partial_i B_j = \delta^{ij}\partial_i E_j = 0$), and h_{ij} is a symmetric, traceless and divergence-free tensor ($h_{ij} = h_{ji}$, $\delta^{ij}h_{ij} = 0$ and $\delta^{jk}\partial_k h_{ij} = 0$).

Since the metric is a symmetric tensor, it has, in principle, ten independent components. The four scalars — ϕ , Ψ , E , and B — each contribute one component. The vector fields B_i and E_i each consist of three components, but the divergence-free condition constrains one component in each. Therefore, we have $2 \times 3 - 2 = 4$ independent vector components. The tensor h_{ij} originally has nine components. However, accounting for its symmetry reduces the number of independent components by three. Additionally, the trace-free condition fixes one more component, and the divergence-free condition imposes constraints on three additional components. As a result, the tensor has only $9 - 3 - 1 - 3 = 2$ independent components remaining.

Equation (3.40) decomposes the metric into four scalar modes, four vector modes, and two tensor modes, resulting in a total of ten independent components. Gauge freedom allows us to eliminate four of these modes by selecting a suitable coordinate system. Of these four eliminated modes, two correspond to scalars and two to vectors [110]. As a result, only six modes are considered physical: two scalar modes, two vector modes, and two tensor modes.

3.2.1 SVT decomposition in $f(Q)$ theories

We can now apply the SVT decomposition to the $f(Q)$ theory action and, for this purpose, we will focus on spatially flat cosmological configurations. Before proceeding, let us review some key concepts.

First, it is essential to keep in mind that we are working in a flat and torsionless spacetime. A flat connection is characterized by the equation $\Gamma^\alpha_{\mu\nu} = (\Lambda^{-1})^\alpha_{\beta} \partial_\mu \Lambda^\beta_{\nu}$. Furthermore, the connection satisfies the symmetry condition when $\Lambda^\mu_{\nu} = \partial_\nu \xi^\mu$. In this scenario, the connection takes the form $\Gamma^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\beta} \partial_\mu \partial_\nu \xi^\beta$. Given the symmetries in the cosmological configurations being considered, ξ^μ admits three possible branches of solutions [106], summarized by the following expression [111]:

$$\xi^0 = \xi(t) - \frac{\alpha_I}{2} \sigma_0 \lambda |\vec{x}|^2, \quad \xi^i = \left(\alpha_{II} \lambda \xi(t) + \sigma_0 \right) x^i. \quad (3.41)$$

In the expression above, we have:

- $\alpha_I = \alpha_{II} = 0$: Trivial branch;
- $(\alpha_I = 1, \alpha_{II} = 0)$: Non-trivial branch I;
- $(\alpha_I = 0, \alpha_{II} = 1)$: Non-trivial branch II.

The action for the $f(\mathcal{Q})$ theory will have the following form:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[-\frac{1}{2}f(\mathcal{Q}) - \frac{1}{2}(\partial\chi)^2 - V(\chi) \right]. \quad (3.42)$$

Here, $V(\chi)$ represents the potential of the matter scalar field χ . The connection field equations depend on the selected branch. In the trivial branch, they are trivially satisfied. However, in the non-trivial branches, they lead to:

$$\text{Branch I:} \quad \frac{d}{dt} \left(\frac{\lambda \sigma_0 a n \dot{f}_{\mathcal{Q}}}{\dot{\xi}^2} \right) = 0, \quad (3.43)$$

$$\text{Branch II:} \quad \frac{\lambda}{\lambda \xi + \sigma_0} \frac{d}{dt} \left(\frac{a^3 \dot{f}_{\mathcal{Q}}}{n} \right) = 0. \quad (3.44)$$

Integrating these equations yields two conserved charges, J_{I} and J_{II} , given by:

$$J_{\text{I}} = -\frac{2\lambda \sigma_0 a n \dot{f}_{\mathcal{Q}}}{\dot{\xi}^2} \quad \text{and} \quad J_{\text{II}} = \frac{\lambda a^3 \dot{f}_{\mathcal{Q}}}{n}, \quad (3.45)$$

where numerical factors are included for convenience.

We can now perform the SVT decomposition of the action by perturbing the matter field as $\chi = \bar{\chi}(\eta) + \pi(\eta, \vec{x})$, where η is the conformal time. While we operate in Fourier space, we will use the same symbols for perturbations. The final expressions, obtained after applying the solutions for the non-dynamical fields and performing integration by parts, are given by:

- Scalar sector:

$$\mathcal{S}_{\text{scal}}^{(2)} = \frac{1}{2} \int d\eta d^3k a^2 \left[\Phi' \hat{\mathcal{K}}_s \Phi'^{\dagger} + \Phi' \hat{\mathcal{N}}_s \Phi^{\dagger} - \Phi \hat{\mathcal{V}}_s \Phi^{\dagger} + \text{c.c.} \right]. \quad (3.46)$$

Here, a is the scale factor⁴, Φ denotes the scalar fields ($\Phi = (\phi, \psi, E, \pi)$), the primes denote derivation with respect to the conformal time η , the hats indicate matrices and c.c. represents the complex conjugates of the preceding terms.

- Vector sector:

$$\mathcal{S}_{\text{vec}}^{(2)} = \frac{1}{2} \int d\eta d^3k \alpha_{\text{I}} J_{\text{I}} \xi' \left[\frac{|\vec{E}'|^2}{1 + \frac{J_{\text{I}} \xi'}{a^2 f_{\mathcal{Q}} k^2}} - k^2 |\vec{E}|^2 \right]. \quad (3.47)$$

⁴The conformal time η and the scale factor a will be defined on section 4.2

- Tensor sector:

$$\mathcal{S}_{\text{ten}}^{(2)} = \frac{1}{2} \sum_{\alpha=+, \times} \int d\eta d^3k a^2 f_{\mathcal{Q}} \left[|h'_{\alpha}|^2 - \left(k^2 + \frac{\alpha_I J_I \xi'}{a^2 f_{\mathcal{Q}}} \right) |h_{\alpha}|^2 \right]. \quad (3.48)$$

In the following section, we will assess the viability of $f(\mathcal{Q})$ theories using these results.

3.3 Instabilities

In certain contexts, it can be useful to analyse small perturbations around a given solution, known as the background. A notable example is the Weinberg pseudo-tensor, which is obtained by introducing perturbations around the Minkowski metric. In this case, the metric is expressed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ denotes the perturbation. Similarly, in the SVT decomposition, a perturbed spacetime is described by $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, with $\bar{g}_{\mu\nu}$ as the background metric and $\delta g_{\mu\nu}$ as the perturbation.

As a result, we can observe deviations from the original unperturbed solution. To prevent the perturbed solution from resulting in a significantly different physical scenario, we require that these deviations remain small enough throughout their evolution, so they do not exceed the magnitude of the unperturbed solution. In this context, a solution is considered stable if perturbations remain small during their evolution; otherwise, it is classified as unstable. This section examines common types of instabilities, and, for this purpose, we will closely follow reference [112].

3.3.1 Instabilities in single-field theories.

To illustrate different types of instabilities, we consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2} [a\dot{\varphi}^2 - b(\partial_i \varphi)(\partial^i \varphi) - \mu\varphi^2], \quad (3.49)$$

Here, a , b and μ are coefficients that contain information about the unperturbed solution, while φ represents the perturbation. The corresponding field equations follow from the Euler-Lagrange equation for a scalar field $\varphi = \varphi(x^\mu)$:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) = 0. \quad (3.50)$$

By substituting the Lagrangian (3.49) into the Euler-Lagrange equation, we obtain the field equation for the perturbation:

$$\ddot{\varphi} - \frac{b}{a} \partial_i \partial^i \varphi + \frac{\mu}{a} \varphi = 0. \quad (3.51)$$

Provided the period and wavelength of the perturbations remain sufficiently small throughout their evolution, the coefficients a , b , and μ can be approximated as constants to first order. Thus, an appropriate tentative solution for φ is given by:

$$\varphi_k = A_k e^{-i\left(\sqrt{\frac{b}{a}}\vec{k}\cdot\vec{x}-\omega t\right)} + A_k^* e^{+i\left(\sqrt{\frac{b}{a}}\vec{k}\cdot\vec{x}-\omega t\right)}, \quad (3.52)$$

where $\omega = \sqrt{\frac{b}{a}|\vec{k}|^2 + \frac{\mu}{a}}$.

Different combinations of the signs and magnitudes of a , b , and μ lead to distinct instability scenarios, as outlined below:

- **Stable case** ($a > 0, b > 0, \mu \geq 0$):

In this scenario, the frequency $\omega = \sqrt{\frac{b}{a}|\vec{k}|^2 + \frac{\mu}{a}}$ is real. Consequently, the exponential terms in the ansatz (3.52) exhibit oscillatory behaviour. As a result, the perturbations will not experience exponential growth, ensuring stability at all times.

- **Tachyonic instability** ($a > 0, b > 0, \mu < 0$):

In this case, modes with momentum exceeding $k_{\text{low}} \equiv \sqrt{|\mu|/b}$ have a real frequency ω . As a result, high-frequency modes are oscillatory and, thus, stable. For modes where $|\vec{k}| < k_{\text{low}}$, the frequency ω becomes imaginary ($\omega = i|\omega|$). Consequently, the solution becomes

$$\varphi_k = A_k e^{-i\sqrt{\frac{b}{a}}\vec{k}\cdot\vec{x}-|\omega|t} + A_k^* e^{+i\sqrt{\frac{b}{a}}\vec{k}\cdot\vec{x}+|\omega|t}. \quad (3.53)$$

The first term in the solution decays exponentially, indicating stability. However, the second term includes $e^{|\omega|t}$, which increases exponentially and becomes dominant at later times when $|\omega|t \gg 1$. Such behaviour characterises tachyonic instabilities, which arise whenever the mass term μ is negative.

Let T represent the background time scale and $t_c = |\omega|^{-1}$ denote the characteristic time after which the exponential growth becomes dominant. If $t_c \ll T$, the exponential growth of the perturbation will dominate well before the background undergoes any evolution. In this case, the background is unstable. However, if $t_c \gg T$, the exponential growth will not pose a problem during the background's evolution time.

- **Gradient/Laplacian instability** ($a > 0, b < 0$ or $a < 0, b > 0$):

For modes where $|\vec{k}| > k_{\text{low}}$, the frequency ω will be imaginary and the perturbation presents exponential growth. Unlike tachyonic instabilities, the background will always be unstable, as perturbations will increase rapidly [112].

Tachyonic instability is associated with low-momentum modes, commonly known as *IR modes*. In contrast, gradient instability affects high-momentum modes, referred to as *UV modes*.

- **Ghostly instabilities** ($a < 0, b < 0$):

For $|\vec{k}| > k_{\text{low}}$, the frequency ω remains real, indicating stability in the background, regardless of the sign of μ . However, for $|\vec{k}| < k_{\text{low}}$ two possibilities arise: when $\mu < 0$, ω is always real, and the background remains stable; conversely, when $\mu > 0$, the background becomes unstable. In summary, for $\mu < 0$, the background is consistently stable, while for $\mu > 0$, instabilities occur in the IR modes.

In both cases, whether $\mu > 0$ or $\mu < 0$, the Hamiltonian is unbounded from below. To understand how this occurs, let us calculate the Hamiltonian. The conjugate momentum corresponding to φ is given by

$$\pi_\varphi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = a\dot{\varphi}. \quad (3.54)$$

Therefore, the Hamiltonian density can be expressed as follows:

$$\mathcal{H} = \dot{\varphi}\pi_\varphi - \mathcal{L} = \frac{1}{2} [a\dot{\varphi}^2 + b(\partial_i\varphi)(\partial^i\varphi) + \mu\varphi^2], \quad (3.55)$$

and the Hamiltonian is given by the integral $H = \int d^3x \mathcal{H}$.

For the coefficient combination $a < 0, b < 0, \mu < 0$, although the background is stable, the Hamiltonian is clearly unbounded from below. This means the system can reach arbitrarily negative energy values. In this case, φ is called a *ghost field*. On the other hand, for $a < 0, b < 0, \mu > 0$, we encounter instabilities in the IR modes and the field φ is referred to as a *tachyonic ghost* [113].

3.3.2 Instabilities in multi-field theories

The approach discussed in the previous subsection can be extended to additional fields by rewriting the Lagrangian as follows:

$$\mathcal{L} = \frac{1}{2} [a_{IJ}\dot{\varphi}^I\dot{\varphi}^J - b_{IJ}(\partial_i\varphi^I)(\partial^i\varphi^J) - \mu_{IJ}\varphi^I\varphi^J], \quad (3.56)$$

where φ^I , with $I = 1, \dots, N$ is a N -tuple containing all fields involved. In this context, the coefficients a, b, μ are replaced by matrices: a_{IJ} is the kinetic matrix, b_{IJ} is the gradient matrix, and μ_{IJ} is the mass matrix.

We have a set of field equations derived from the Euler-Lagrange equation for each field φ^I :

$$\frac{\partial \mathcal{L}}{\partial \varphi^I} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^I)} \right) = 0. \quad (3.57)$$

After plugging the Lagrangian given in the equation (3.56), we obtain

$$a_{IJ}\ddot{\varphi}^J - b_{IJ}\partial_i\partial^i\varphi^J + \mu_{IJ}\varphi^J = 0. \quad (3.58)$$

This equation obeys the form

$$\mathcal{K}_{IJ}\ddot{\varphi}^J - \{\text{terms independent of } \ddot{\varphi}^J\}_I = 0, \quad (3.59)$$

where $\mathcal{K}_{IJ} = \frac{\partial^2\mathcal{L}}{\partial\ddot{\varphi}^I\partial\ddot{\varphi}^J}$ represents the Hessian matrix. In this context, the Hessian matrix coincides with the kinetic matrix a_{IJ} . If the Hessian matrix is non-degenerate, meaning that its determinant is non-zero ($\det(\mathcal{K}_{IJ}) \neq 0$), we can solve the corresponding equations for $\ddot{\varphi}^J$. With the appropriate initial conditions, we can then obtain the values of φ^J .

The Lagrangians considered thus far contain only first-order derivatives of the fields. As a result, the field equations produced are, at most, of second order, which means they require two initial conditions each. This pair of initial conditions accounts for a single degree of freedom. As a result, no unwanted degrees of freedom are propagated. However, in gravity theories, we often encounter Lagrangians which include higher-order derivative terms. These higher-order terms require additional initial conditions to fully determine the fields, which can lead to the presence of extra degrees of freedom. Consequently, there may be “ghostly” degrees of freedom that are being propagated, often referred to as *Ostrogradski ghosts*.

In single-field theories, ghost-like instabilities are typically linked to a negative kinetic energy term. Therefore, it is crucial to focus on the sign of the coefficient a in the time-derivative term, which must be positive to ensure stability. For multi-field theories, especially those involving higher derivative terms, we need to pay close attention to the kinetic matrix — or, more specifically, the Hessian matrix — which should remain degenerate. If the Hessian is degenerate, it allows us to eliminate some of the higher derivative terms from the equations and reduce them to second-order equations, indicating that there are no Ostrogradski ghosts in the theory.

Let us come back to the first-order Lagrangian (3.56). To obtain φ^I from $\ddot{\varphi}^I$, we require two initial conditions. Each pair of initial conditions corresponds to one degree of freedom. Therefore, in this example, we have one degree of freedom associated with each field in our Lagrangian. Consequently, there are no additional degrees of freedom that could potentially correspond to Ostrogradski ghosts, as expected. However, this does not guarantee that the theory is entirely free of ghosts, as one or more of the fields φ^I may correspond to a ghost.

To gain a better understanding of this matter, let us return to the single field Lagrangian as represented in equation (3.49). In this straightforward case, the Hessian matrix takes its simplest form, which is simply a constant denoted as a . As noted in the previous subsection, $a < 0$ (and provided that $b < 0$ as well) indicates

the presence of ghosts. In cases where multiple fields are involved, it becomes necessary to examine the Hessian matrix in greater detail.

For that purpose, we assume we have diagonalised the Hessian matrix K_{IJ} , denoting its diagonalised form as \bar{K}_{IJ} . This diagonalisation allows us to rewrite the equations defined by equation (3.59) in a form that removes any coupling between them. Additionally, the determinant of the Hessian is determined by the product of its diagonal elements, which can be expressed as follows:

$$\det(\bar{K}_{IJ}) = \lambda_1 \cdot \lambda_2 \cdots \lambda_N, \quad (3.60)$$

where λ_I represents the eigenvalues of the Hessian matrix, with $I = 1, \dots, N$. If the determinant of \bar{K}_{IJ} is less than zero, it implies that at least one of the eigenvalues must be negative. The presence of one or more negative eigenvalues indicates that the kinetic term of at least one field has the wrong sign, suggesting the presence of a ghost. In conclusion, a negative Hessian determinant is a sufficient condition for classifying a theory as pathological.

Lastly, we will briefly address another form of instability that may arise in perturbation theory: *strong coupling*. This type of instability occurs when the kinetic term of some fields becomes zero at the perturbation level, leading to the non-propagation of those fields. As a result, we generally observe fewer degrees of freedom being propagated at the perturbation level compared to the number of degrees of freedom in the background theory. In some cases, we can identify a decrease in the number of degrees of freedom at first order in perturbation theory. However, it is often necessary to examine higher-order perturbations to get a complete understanding of the situation. Each case should be analysed individually.

3.3.3 Instabilities in $f(Q)$ theories

To study more general Lagrangians involving scalar, vector, and tensor fields, we must treat scalar, vector, and tensor perturbations separately, as discussed in the previous section. The next step is to distinguish between dynamical and auxiliary fields. Auxiliary fields satisfy algebraic equations of motion that provide unique solutions. Once identified, the solutions of auxiliary fields are substituted into the equations for the dynamical fields to identify potential unstable modes.

We will revisit the SVT decomposition from subsection 3.2.1 to examine the presence of instabilities in $f(Q)$ theories in detail. To facilitate this discussion, we will present findings from the publication [111]. We begin with the scalar component of the action, which is expressed as follows:

$$\mathcal{S}_{\text{scal}}^{(2)} = \frac{1}{2} \int d\eta d^3k a^2 \left[\Phi' \hat{\mathcal{K}}_s \Phi'^{\dagger} + \Phi' \hat{\mathcal{N}}_s \Phi^{\dagger} - \Phi \hat{\mathcal{V}}_s \Phi^{\dagger} + \text{c.c.} \right]. \quad (3.61)$$

As previously discussed, a negative Hessian determinant indicates an unhealthy theory. From the equation above, we can see that $\hat{\mathcal{K}}_s$ is the Hessian matrix. Its

determinant is given by

$$\det \hat{\mathcal{K}}_s = \alpha_I \frac{9\lambda^2 \sigma_0^2 f_Q^2 f_{QQ}}{2a^2 \xi'^2 \left[1 - k^2 \frac{(\xi'' - 2\mathcal{H}\xi')^2}{J_I \xi'^3} f_{QQ} \right]} k^4, \quad (3.62)$$

where $\mathcal{H} \equiv \frac{a'}{a}$. This determinant vanishes when $\alpha_I = 0$, which corresponds to the trivial and second non-trivial branches. This leads to a reduction in the number of scalar degrees of freedom, indicating strong coupling. For the first non-trivial branch, all scalar degrees of freedom are present. However, we still need to examine the sign of the Hessian determinant. Let us first examine the other sectors.

In the vector sector, B_i is an auxiliary field, while E_i is the only dynamical field. The vector part of the action is provided by:

$$\mathcal{S}_{\text{vec}}^{(2)} = \frac{1}{2} \int d\eta d^3k \alpha_I J_I \xi' \left[\frac{|\vec{E}'|^2}{1 + \frac{J_I \xi'}{a^2 f_Q k^2}} - k^2 |\vec{E}|^2 \right]. \quad (3.63)$$

This sector vanishes for both the trivial and the second non-trivial branches, which indicates strong coupling. To understand the behaviour of the first non-trivial branch, we examine the UV limit by expanding the expression for $k^2 \gg \frac{J_I \xi'}{a^2 f_Q}$:

$$\mathcal{S}_{\text{vec,UV}}^{(2)} = \frac{1}{2} \int d\eta d^3k \alpha_I J_I \xi' \left[|\vec{E}'|^2 - k^2 |\vec{E}|^2 \right]. \quad (3.64)$$

It is important to note that the condition $J_I \xi' > 0$ is sufficient to prevent ghost instabilities in this branch.

Continuing to the tensor section, we now have:

$$\mathcal{S}_{\text{ten}}^{(2)} = \frac{1}{2} \sum_{\alpha=+, \times} \int d\eta d^3k a^2 f_Q \left[|h'_\alpha|^2 - \left(k^2 + \frac{\alpha_I J_I \xi'}{a^2 f_Q} \right) |h_\alpha|^2 \right]. \quad (3.65)$$

From the expression above, we can see that the condition $f_Q > 0$ is sufficient to prevent the presence of ghosts in all branches. In summary, both the trivial branch and the non-trivial branch II are strongly coupled. Meanwhile, the non-trivial branch I has thus far avoided strong coupling and is considered stable, provided that the conditions $f_Q > 0$ and $J_I \xi' > 0$ hold.

Let us revisit the scalar sector and examine the UV limit of the Hessian determinant, which is given by:

$$(\det \mathcal{K}_s)_{\text{UV}} \simeq -\frac{9J_I \xi' \lambda^2 \sigma_0^2 f_Q^2}{2a^2 (\xi'' - 2\mathcal{H}\xi')^2} k^2. \quad (3.66)$$

It is now evident that while the conditions $f_Q > 0$ and $J_I \xi' > 0$ ensure stability, they also lead to a negative Hessian determinant. As a result, we can conclude that at least one ghost mode is present in the non-trivial branch I.

3.3.4 Key results

The results presented in subsections 3.2.1 and 3.3.3 were published in the paper titled “Pathological Character of Modifications to Coincident General Relativity: Cosmological Strong Coupling and Ghosts in $f(\mathbb{Q})$ Theories” [111]. This paper investigates the viability of $f(\mathbb{Q})$ theories by analysing their cosmological perturbations.

The cosmological perturbations were applied to three distinct branches of spatially flat cosmological solutions. The results are summarised as follows:

- **Trivial and second non-trivial branches:** These branches exhibit reduced linear spectra, indicating an infinite strong coupling problem.
- **First non-trivial branch:** This branch reveals seven degrees of freedom, with at least one behaving as a ghost, implying instability.

These findings suggest that $f(\mathbb{Q})$ cosmologies face significant theoretical challenges, potentially disqualifying them as viable models.

Chapter 4

Applications

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This chapter discusses the main applications of this thesis: Astrophysics and Cosmology. We start with an overview of Stellar Physics, followed by a discussion on how MG theories influence stellar properties. We then review key aspects of Cosmology, briefly discussing their modifications in $f(\mathcal{Q})$ and $f(\mathcal{T})$ gravities. We conclude by presenting a Cosmological Canonical Frame in the context of GTEGR.

4.1 Stellar Physics

This section explores the stellar model used to determine stellar structure and evolution. We begin by outlining the features that define a star [114]:

- It should be held in place by its own gravity;
- It should emit its own light, meaning that it radiates energy, which is supplied by the star.

The energy emitted by a star is predominantly generated from its internal nuclear reactions. This fact suggests that the star's chemical composition changes over time, indicating that the star experiences an evolutionary process. Additionally, much of the information about a star is obtained from its light, which is primarily emitted by the photosphere, one of the star's outermost layers. Key stellar properties, including apparent brightness and surface temperature, are derived from observations of the photosphere.

The brightness B of a star represents its emitted energy flux. Since light spreads spherically, its intensity decreases with the distance d from the star, following the inverse square law. The luminosity L , however, is an intrinsic characteristic of the star and does not depend on the distance. It is defined as the total energy emitted per second [115]. Hence, the relationship between brightness and luminosity can be expressed as $B = L/4\pi d^2$. As we will see, luminosity is intricately connected to several fundamental stellar properties.

A star's energy flux is also connected to its surface temperature. Because a star's spectrum closely resembles that of a blackbody, we can approximate its effective temperature, T_{eff} , to that of a blackbody that emits the same amount of energy flux [115]. The effective temperature provides a reliable estimative of a star's surface temperature, and is related to luminosity via the Stefan-Boltzmann law, expressed as $L = 4\pi R^2 T_{\text{eff}}^4$.

Chemical composition changes within a star originate in the core and gradually propagate outward to its outer layers. Hence, to model stellar evolution, it is necessary to determine the properties of the core and understand how they evolve. This purpose can be achieved by using measurable properties from the star's surface to infer its central counterparts. However, determining a star's effective temperature alone is not sufficient to calculate its core temperature, and the same limitation applies to other stellar properties as well.

To overcome this challenge, we need to develop a theoretical framework that facilitates these calculations. For this purpose, we require the following equations: mass continuity, hydrostatic equilibrium, and the equation of state¹.

Mass Continuity Equation

To simplify our analysis, we will consider non-rotating stars that do not possess strong magnetic fields. In such cases, the star's structure can be well approximated as spherically symmetric. This assumption is valid because the energy contributions from rotation or magnetic fields are typically small compared to the star's gravitational energy [114]. Due to the spherical symmetry, we find that these stars must comply with the mass continuity equation, which can be expressed as:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r). \quad (4.1)$$

In the equation above, r is the radial coordinate, $m(r)$ represents the mass of the star enclosed within the radius r , and $\rho(r)$ denotes the density at a distance r from the centre of the star. The mass continuity equation describes the variation of the mass with respect to r and connects this change to the star's density ρ . Moreover, for a spherically symmetric object, the gravitational force at a distance r from its centre will depend on $m(r)$, which is given by

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'. \quad (4.2)$$

We can notice that $m(0) = 0$. Additionally, $M = m(R)$ represents the total mass of the star, where R is its radius.

Hydrostatic Equilibrium Equation

As a star evolves, nuclear reactions consume its lighter elements, altering the star's chemical composition. This transformation occurs gradually, allowing it to be described as quasi-static. As a result, a star can maintain hydrostatic equilibrium throughout its evolution, meaning that the pressure gradient at a given layer of the fluid balances the gravitational force acting on it.

The balance between gravity and pressure is described by the hydrostatic equilibrium equation, which is given by:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2} \rho(r) = -\rho(r) \frac{d\Phi}{dr}. \quad (4.3)$$

¹To conduct a comprehensive analysis, we require both an equation for radiative transfer and an equation for thermal equilibrium. For simplicity, we will assume that pressure depends solely on density rather than temperature. This assumption allows us to solve these equations independently of the mass continuity and hydrostatic equilibrium equations. Additionally, we will not discuss the heat flow equations in this context.

Here, Φ denotes the gravitational potential, and $\frac{dP}{dr}$ represents the rate of change of pressure inside a star at a distance r from its centre. From the above equation, we can notice that the higher a star's density, the higher the mass within the radius r . This results in a stronger gravitational pull exerted by this mass at radius r . Consequently, a greater pressure gradient is needed to sustain a mass shell dM at this radius. Moreover, the right-hand side of (4.3) is always negative, indicating that pressure decreases as we move away from the star's centre. The central pressure is defined as $P_C \equiv P(0)$, and we assume that the pressure decreases to zero at the surface of the star, i.e., $P(R) = 0$.

Equation of State

We also need to establish a relationship between density and pressure, which can be described by an equation of state (EoS). The form of the EoS depends on the composition of the stellar gas. A standard approximation for the EoS characterising the properties of stellar matter is a power law that relates pressure to energy density. This EoS is expressed by the equation $P = K\rho^\gamma$, known as the *polytropic equation*. This equation can also be applied to ideal gases and less dense regions of a star, such as its atmosphere².

In the polytropic equation, the factor K can represent either a constant or a function, depending on the characteristics of the star and the internal processes being considered, such as the evolution of electron degeneracy. Additionally, the parameter γ is defined as $\gamma = 1 + 1/n$, where n denotes the polytropic index.

The polytropic EoS is a valid approximation, particularly for stars predominantly composed of degenerate electron gas. It is also applicable in cases where the star is fully convective, as the temperature gradient in such stars is typically nearly adiabatic [114]. In this context, the polytropic index n is equal to $3/2$ for a non-relativistic gas and 3 for a highly relativistic gas. Overall, polytropic equations are convenient to use due to their simplicity and broad applicability.

In the following discussion, we will focus on low-mass stars, specifically those with masses below $0.5M_\odot$, where M_\odot denotes the Sun's mass. These stars can be characterised as fully convective, which simplifies their study.

Lane-Emden Equation

To model a star's internal structure more accurately, it is useful to divide a star into concentric spherical shells, each characterised by distinct densities, opacities, and nuclear reaction rates. Given our focus on fully convective stars, we can model them as having a single convective core surrounded by an atmosphere that follows either an ideal EoS or a modified form accounting for ionisation. Therefore, we are

²In this case, if we do not account for ionisation, the analysis reduces to that of an ideal gas.

seeking solutions for either the density ρ or the mass M that satisfy the equations we have discussed so far.

If we multiply equation (4.3) by r^2/ρ , differentiate it with respect to r and use equations (4.1), we will arrive at:

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho. \quad (4.4)$$

If we use $\frac{dP}{dr} = -\rho(r) \frac{d\Phi}{dr}$, we can see that the above equation is equivalent to the Poisson equation, given by

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho. \quad (4.5)$$

By applying the polytropic equation of state, $P = K \rho^{\frac{n+1}{n}}$, we derive a differential equation for the density:

$$\frac{(n+1)K}{4\pi nG} \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho^{\frac{n-1}{n}}} \frac{d\rho}{dr} \right) = -\rho. \quad (4.6)$$

The solutions $\rho(r)$ obtained from the equation above are known as polytropes. Since we are dealing with a second-order differential equation, we require two boundary conditions. The first condition arises from the fact that $P(R) = 0$, which implies that $\rho(R) = 0$ according to the polytropic equation. The second condition is based on the behaviour of the hydrostatic equilibrium equation at the centre. In this case, the right-hand side of the equation (4.3) becomes zero, leading to the conclusion that $dP(0)/dr = 0$. Consequently, it follows that $d\rho(0)/dr = 0$ as well.

To further simplify our equations and facilitate their analysis, we introduce a dimensionless variable, θ , which is related to the energy density as $\rho = \rho_c \theta^n$. Here, ρ_c represents the density at the centre of the star, while θ varies between 0 and 1. This variation allows us to express the corresponding density at the surface and the centre of the star, respectively. The equation (4.6) will become:

$$\left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}} \right] \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) = -\theta^n. \quad (4.7)$$

The terms inside the square brackets have dimensions of square length. Therefore, similar to our approach with variables ρ and θ , we can substitute the coordinate r with a dimensionless coordinate ξ . This relationship will have the form $r = r_c \xi$, where $r_c = \sqrt{\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}}$.

The equation (4.7) is now effectively dimensionless:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n. \quad (4.8)$$

This equation, known as the *Lane-Emden equation*, represents a dimensionless form of the Poisson equation [116, 117]. As discussed previously, $\theta = 1$ at the centre of the star (now given by $\xi = 0$). Additionally, the boundary condition $d\rho(0)/dr = 0$ now translates to $d\theta(\xi = 0)/d\xi = 0$. The Lane-Emden equation can be solved for $\theta(\xi)$, which has its first zero at $\xi = \xi_R$. This point, ξ_R , corresponds to the star's radius, given by the relation $R = r_c \xi_R$.

We are now able to write the star's mass, radius, central density and temperature in terms of solutions θ_n of the Lane-Emden equation [40]:

$$M = 4\pi r_c^3 \rho_c \omega_n, \quad (4.9)$$

$$R = \gamma_n \left(\frac{K}{G} \right)^{\frac{n}{3-n}} M^{\frac{n-1}{n-3}}, \quad (4.10)$$

$$\rho_c = \delta_n \left(\frac{3M}{4\pi R^3} \right), \quad (4.11)$$

$$T_c = \frac{K\mu}{k_B} \rho_c^{\frac{1}{n}} \theta_n. \quad (4.12)$$

In this context, k_B represents the Boltzmann constant, while μ refers to the mean molecular weight. The constants ω_n , γ_n , and δ_n appearing in the equations above have different values for each value of n and are defined as follows:

$$\omega_n = -\xi_R^2 \left. \frac{d\theta}{d\xi} \right|_{\xi=\xi_R}, \quad (4.13)$$

$$\gamma_n = (4\pi)^{\frac{1}{n-3}} (n+1)^{\frac{n}{n-3}} \omega_n^{\frac{n-1}{n-3}} \xi_R, \quad (4.14)$$

$$\delta_n = -\xi_R \left(3 \left. \frac{d\theta}{d\xi} \right|_{\xi=\xi_R} \right)^{-1}. \quad (4.15)$$

It is interesting to note that for values of n such that $1 < n < 3$, the expression $(n-1)/(n-3) < 0$. Thus, the relation $R \propto M^{\frac{n-1}{n-3}}$ indicates that more massive stars will have a smaller radius and consequently will be denser.

HR diagram and Hayashi tracks

Having established a theoretical framework for stellar structure, we now turn our attention to the Hertzsprung-Russell (HR) diagram, named after astronomers Ejnar Hertzsprung and Henry Norris Russell [118, 119]. The HR diagram is a graph in which the horizontal axis represents a star's temperature, while the vertical

axis represents its luminosity. Each point on this graph corresponds to a single star. A notable feature of the diagram is that the majority of stars are found along a thin, diagonal band known as the *Main Sequence* (MS). Stars that lie within this band are called main-sequence stars.

We can note from the Stefan-Boltzmann law ($L = 4\pi R^2 T_{\text{eff}}^4$) that, for a given fixed temperature, the luminosity only depends on the size of the star. Since the MS is approximately a strip, stars that have similar temperatures will exhibit nearly identical luminosities and, consequently, comparable sizes. Moreover, MS stars are found to obey a power law between luminosity and mass: $L/L_{\odot} = (M/M_{\odot})^{\nu}$, where ν ranges from ≈ 3 to ≈ 5 [114].

Before reaching the main sequence, a protostar undergoes a crucial phase in which its luminosity decreases significantly while its temperature rises only slightly. This evolutionary phase, known as the *Hayashi track*, is named after the Japanese astrophysicist Chushiro Hayashi. He was the first to recognise that during this stage of evolution, convection is the primary method of energy transport within the protostar [120]. The Hayashi track serves as the boundary of specific regions in the HR diagram, known as *Hayashi forbidden zones*, which are areas of instability where no stars can be found. The protostar remains in this pre-main-sequence stage until it reaches the luminosity required to enter the main sequence.

When analysing an MS star with a known mass in the HR diagram, we can gather important information about its luminosity, size, and temperature. Furthermore, we can also determine the star’s age. The age of an MS star is particularly influenced by the speed at which it consumes its fuel supply, which is related to its luminosity.

4.1.1 Stars in Modified Gravity

This subsection explores the effects of MG theories on stellar physics. A star’s internal structure is sensitive to deviations from GR, as indicated by modifications to the hydrostatic equilibrium equation, which we will explore shortly. Consequently, several characteristics of a star – such as luminosity [70, 121], age [122, 123], and allowed mass values [123–130] – are influenced by modifications to GR. As a result, these celestial objects can serve as effective probes for testing MG theories.

For instance, the minimum mass required for hydrogen burning in red dwarfs varies across different MG models [124, 131]. Additionally, the mass-radius relationship of white dwarfs and their maximum rotational frequency can also impose constraints on MG theories [125]. Furthermore, light element abundances are affected by the underlying theory [122]. This last point can be used to constrain gravity theories and help explain the “lithium problem” [132–134].

In modified theories of gravity, the hydrostatic equilibrium equation is changed, with the specific form of modifications depending on the theory. As a result, the Poisson equation is also altered. For spherically symmetric, non-rotating objects, it takes the following form [135]:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = -4\pi G\rho + \text{modifications}, \quad (4.16)$$

where Φ is the gravitational potential, ρ is the energy density, and G is Newton's gravitational constant. The additional terms arising from MG corrections generally depend only on r (or its functions), given the spherical symmetry, and on theory-specific parameters.

From now on, we will use the *Lane-Emden equation*, which incorporates both the hydrostatic equilibrium equation and the polytropic equation. In MG theories, modifications to the Poisson and hydrostatic equations lead to additional terms in the Lane-Emden equation [135]:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n + \frac{1}{4\pi G\rho_c} (\text{modifications}), \quad (4.17)$$

where ρ_c is the central density and θ and ξ are defined through $\rho = \rho_c \theta^n$, $P = P_c \theta^{n+1}$ and $r = r_c \xi$. Here, P_c denotes the central pressure while $r_c^2 = \frac{K(n+1)\rho_c^{(n-1)/n}}{4\pi G}$. The usual boundary conditions require $\theta(0) = 1$ and $\theta'(0) = 0$ at the centre and $\theta(\xi_R) = 0$ at the surface, where ξ_R is the first value of ξ_R for which the surface condition is valid.

Lane-Emden solutions determine stellar properties such as mass, radius, and temperature, which depend on the underlying theory. They can also be used to derive the abundances of light elements. In MG theories, the lithium depletion rate takes the following form [122, 135, 136]³:

$$\begin{aligned} \mathcal{F} &= \ln \frac{f_0}{f} = 5.6 \times 10^{14} T_{eff}^{-4} \left(\frac{X}{0.7} \right) \left(\frac{0.1 M_\odot}{M} \right)^3 \left(\frac{0.6}{\mu_{eff}} \right)^6 S f_{scr} a^{16} \\ &\times g(u) \frac{\xi_R^7 (-\theta'(\xi_R))^2}{\delta^2} (1 + \text{extra MG term}), \end{aligned} \quad (4.18)$$

where f_0 is the initial abundance, f_{scr} is the screening correction factor, T_{eff} is the effective temperature, μ_{eff} is the mean molecular weight, $S = 7.2 \times 10^{10}$ and $a = 84.72$ are dimensionless parameters, $\delta = -\xi_R / (3 \frac{d\theta}{d\xi} |_{\xi=\xi_R})$ and $g(u) = u^{-37/2} e^{-u} - 29\Gamma(-37/2, u)$ with $u \equiv aT_{c6}^{-1/3}$ and $\Gamma(-37/2, u)$ being the upper incomplete gamma function.

We should emphasise that the modifications to the theory arise not only from the additional MG terms but also from the values of ξ , θ' and δ , which are related to the solutions of the modified Lane-Emden equation (4.17). One significant effect of this modification is that the age of a star depends on the theoretical framework used to determine it.

³The analysis of lithium depletion within standard Newtonian gravity is discussed in the reference [137].

4.1.2 Key results

Let us discuss some results concerning MG and stellar physics, specifically in the context of DHOST theories of gravity, which were outlined in section 2.2. The findings are further elaborated in the research paper titled “Early evolution of fully convective stars in scalar-tensor gravity” [136].

The paper examines three phases: contraction towards the main sequence, lithium burning, and entry into the main sequence. In DHOST theories, modifications to the hydrostatic equilibrium equation result in shifts in the Hayashi track curves. These shifts alter the Hayashi forbidden zones, which can be tested against astrophysical observations to help constrain theories of gravity.

DHOST modifications significantly affect the age of the star during the lithium-burning evolutionary stage. Our findings indicate that various characteristics of stars – such as photospheric and central temperatures, lithium abundance, and heat transfer processes – are sensitive to DHOST parameters. This provides potential observational tests for these alternative gravity models.

4.2 Cosmology

This section provides an overview of cosmology and explores its application within MG theories. Our starting point is *Copernican principle*, which states that the Universe is uniform on sufficiently large scales, and Earth does not occupy a privileged observational position. Mathematically, this principle can be articulated through the concepts of isotropy and homogeneity. Isotropy expresses the idea that the Universe appears the same in every direction, while homogeneity indicates that it looks the same at every point.

Together, these properties form the *cosmological principle*, which applies to spatial dimensions but does not extend to spacetime as a whole. Consequently, while the Universe exhibits spatial isotropy and homogeneity, it evolves over time. The evolution of the Universe over time is supported by its accelerated expansion [14]. As the Universe expands, the physical distance between any two coordinate points increases over time. This expansion is quantified using the scale factor, denoted as a . Hence, to understand the expansion of the Universe, we first need to understand the evolution of the scale factor. The Friedmann equations describe this evolution, and they are grounded in the assumption that GR accurately describes gravity on large scales.

In addition to the physical distance, which increases as the Universe expands, there exists a conformal distance that remains constant during this expansion. The conformal distance travelled by light during a time interval t is given by $c\eta$, where η is the conformal time defined as

$$\eta \equiv \int_0^t \frac{dt'}{a(t')}. \quad (4.19)$$

To describe an expanding universe, we use the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]. \quad (4.20)$$

The parameter k determines the geometry of the Universe, which can be open, closed or flat. This parameter can assume the following values [138]:

- $k = 0$: In this case, the line element of the hypersurface Σ corresponds to the spatial part of the Minkowski metric in spherical coordinates. It can be expressed as follows:

$$ds_{\Sigma}^2 = a^2 [dr^2 + r^2 d\Omega^2]. \quad (4.21)$$

This geometry corresponds to a flat Universe with zero curvature.

- $k = -1$: In this case, we can consider the hypersurface Σ defined by constant t . By using the transformation $r = \sinh \Psi$, we arrive at the following expression for the line element:

$$ds_{\Sigma}^2 = a^2 [d\Psi^2 + \sinh^2 \Psi d\Omega^2]. \quad (4.22)$$

This geometry corresponds to a generalisation of a hyperboloid, indicating an open Universe with constant negative curvature [39].

- $k = +1$: We now implement the transformation $r = \sin \Psi$. The line element for the $t = \text{constant}$ hypersurface is given by:

$$ds_{\Sigma}^2 = a^2 [d\Psi^2 + \sin^2 \Psi d\Omega^2]. \quad (4.23)$$

This geometry represents a three-dimensional sphere, corresponding to a closed Universe with positive curvature.

From the FLRW metric (4.20), we can straightforwardly derive the Levi-Civita connection components, as well as the components of the Ricci tensor and the Ricci scalar⁴. These derivations lead to differential equations describing the scale factor evolution, which depends on the energy density of the Universe. Therefore, we need to define the form of the energy-momentum tensor, which is typically assumed to be a perfect fluid expressed as follows:

$$\Theta_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu}. \quad (4.24)$$

Here, ρ is the energy density, p is the pressure and u_{μ} is the observer four-velocity.

⁴We will not list all the components here, but we will refer the reader to [39].

The scale factor evolution will be described by the Friedmann equations, derived from the Einstein equations (2.16). The Friedmann equations are given by [39]:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \quad (4.25)$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (4.26)$$

The dot above a variable denotes a derivative with respect to the cosmic time t . In the equations presented, we defined the Hubble parameter as $H = \frac{\dot{a}}{a}$, which quantifies the rate of change of the scale factor. In other words, the Hubble parameter measures the rate of expansion of the Universe. Its present value is represented as H_0 and is known as the Hubble constant. Additionally, we can define the conformal Hubble factor as $\mathcal{H} = \frac{a'}{a}$, where the prime indicates derivatives taken with respect to the conformal time η .

Additionally, the temporal component of the equation $\overset{\circ}{\nabla}_\mu \Theta^{\mu\nu} = 0$ leads to the following relationship [39]:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}, \quad (4.27)$$

where it is assumed that a perfect cosmological fluid satisfies $p = w\rho$. If we assume w is constant, we find that $\rho \propto a^{-3(1+w)}$. The specific value of w depends on the type of cosmological fluid being considered. The two predominant forms of energy in the Universe are radiation, which was the dominant form of energy in the early Universe, and matter, often referred to as dust, which becomes more significant at later times [107].

In a matter fluid, the pressure is significantly smaller than the energy density and can typically be neglected. In matter-dominated universes, the energy density decreases with the scale factor a as follows: $\rho \propto a^{-3}$. In the case of radiation, where the parameter w equals $1/3$, the energy density decreases as $\rho \propto a^{-4}$ [39]. These results illustrate how the Universe's matter content influences the evolution of the scale factor.

4.2.1 Cosmology in Modified Teleparallel Theories

Due to its symmetries, Cosmology provides a straightforward framework for applying MG theories, such as $f(\mathcal{T})$ and $f(\mathcal{Q})$ theories. In $f(\mathcal{T})$ gravity, the Friedmann equations take a modified form [79]:

$$12H^2 f_{\mathcal{T}} + f(\mathcal{T}) = 16\pi G\rho, \quad (4.28)$$

$$48H^2 f_{\mathcal{T}\mathcal{T}}\dot{H} - f_{\mathcal{T}}[12H^2 + 4\dot{H}] - f(\mathcal{T}) = 16\pi Gp. \quad (4.29)$$

When $f(\mathcal{T}) = \mathcal{T}$, the modified equations reduce to the standard Friedmann equations (4.25), (4.26) for a flat Universe, since $\mathcal{T} = -6H^2$ under a proper tetrad choice [79].

On the other hand, for $f(\mathcal{Q})$ gravity, the corresponding Friedmann equations are given by [78]

$$6f_{\mathcal{Q}}H^2 - \frac{1}{2}f(\mathcal{Q}) = 8\pi G\rho, \quad (4.30)$$

$$(12H^2f_{\mathcal{Q}\mathcal{Q}} + f_{\mathcal{Q}})\dot{H} = -4\pi G(\rho + p). \quad (4.31)$$

The equations above can be rewritten to resemble those found in $f(\mathcal{T})$ cosmology. Additionally, in the coincident gauge, the nonmetricity scalar takes the form $\mathcal{Q} = 6H^2$ [78]. Therefore, the standard Friedmann equations for a flat universe are restored when $f(\mathcal{Q}) = \mathcal{Q}$. For both $f(\mathcal{T})$ and $f(\mathcal{Q})$ gravities, the continuity equation maintains its standard form $\dot{\rho} = -3H(\rho + p)$ [78, 79].

4.2.2 General Cosmological Frame

In subsection 3.1.2 of the previous chapter, we presented the metrical and canonical energy tensors in General Parallel Theory. In this subsection, we focus on the GTEGR framework and derive a cosmological canonical frame. For this purpose, we will adopt the following renormalised parameters (originally defined in subsection 2.4.4):

$$c_1 = -\frac{c_2}{2} = -c_3 = \frac{c_5}{2} = -\frac{M_{\text{Pl}}^2}{8}, \quad c_4 = 0, \quad a_1 = \frac{a_2}{2} = -\frac{a_3}{4} = -\frac{M_{\text{Pl}}^2}{8},$$

$$b_1 = -b_2 = b_3 = -\frac{M_{\text{Pl}}^2}{2}. \quad (4.32)$$

Before focusing on GTEGR, we will present the general teleparallel cosmological geometry, which is independent of the theory. Assuming homogeneity and isotropy, we adopt the metric

$$ds^2 = -n^2(t)dt^2 + a^2(t)\gamma_{ij}dx^i dx^j, \quad (4.33)$$

where $\gamma_{ij} = \frac{kx^i x^j}{1-kx^2}$ is a purely spatial metric. We can note that the FLRW metric is parametrised by two time-dependent functions, $n(t)$ and $a(t)$. Similarly, a teleparallel connection is also parametrised by two time-dependent functions, $\mu(t)$ and $\sigma(t)$. This property is in agreement with the fact that both metric and connection should exhibit cosmological symmetries.

To illustrate this point, we should recall that in General Teleparallel theories, the connection is generated by a matrix $\Lambda^\mu{}_\nu \in GL(4, \mathbb{R})$. By imposing the cosmological symmetries on Λ , we find that there exist two classes of solutions, which are called the regular and the exceptional configurations. The regular

configuration further reduces to three distinct branches in the flat limit $k \rightarrow 0$, which can be summarised as [106]:

$$\hat{\Lambda} = \left(\begin{array}{c|c} \mu(t) & -\alpha_{\text{I}} \sigma(t) \lambda x_j \\ \hline \alpha_{\text{II}} \mu(t) \lambda x^i & \sigma(t) \delta_j^i \end{array} \right), \quad (4.34)$$

where λ is a constant parameter denoting the deviation from the trivial configuration. Additionally, the parameters α represent:

- $\alpha_{\text{I}} = \alpha_{\text{II}} = 0$: Trivial branch;
- $(\alpha_{\text{I}} = 1, \alpha_{\text{II}} = 0)$: Non-trivial branch I;
- $(\alpha_{\text{I}} = 0, \alpha_{\text{II}} = 1)$: Non-trivial branch II.

From equation (4.34), we conclude that Λ – and, consequently, the connection Γ – is parametrised by the functions $\mu(t)$ and $\sigma(t)$. In case we limit our analysis to symmetric teleparallelism, the matrix Λ must also satisfy $\Lambda^\mu{}_\nu = \partial_\nu \xi^\mu$, where ξ^μ is defined in equation (3.41), with $\mu(t) = \partial_i \xi(t)$ and $\sigma(t) = \sigma_0$. Hence, in symmetric teleparallelism, the connection depends on a single function of time.

In the following discussion, we will perform a 3+1 decomposition of the metric and other fields. In the 3+1 decomposition, the four-velocity u_μ of the comoving observer – which is normalised to -1 ($u_\mu u^\mu = -1$) – provides a time-like direction. We can then define a spatial projector as:

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad (4.35)$$

where $g_{\mu\nu}$ is the spacetime metric.

A generic tensor (or connection) which is compatible with cosmological symmetries can be decomposed into u^μ and $h_{\mu\nu}$. If parity invariance is not required, the object $\epsilon_{\alpha\beta\gamma} = u^\mu \epsilon_{\mu\alpha\beta\gamma}$ also appears in the decomposition. For example, a generic connection which is compatible with cosmological symmetries can always be parametrised with five time-dependent functions K_i as follows [139]:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{K_1}{n} u^\alpha u_\mu u_\nu + \frac{nK_2}{a^2} u^\alpha h_{\mu\nu} - \frac{K_3}{n} h^\alpha{}_\mu u_\nu - \frac{K_4}{n} u_\mu h^\alpha{}_\nu + \frac{K_5}{a} \epsilon^\alpha{}_{\mu\nu}. \quad (4.36)$$

When teleparallelism is imposed, the coefficients K_i are constrained by several equations, leading to five distinct branches of solutions [140, 141].

However, our solution for the cosmological frame (4.34) corresponds to the teleparallel connection given by⁵:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\mu'}{\mu} u^\alpha u_\mu u_\nu + \alpha_{\text{I}} \frac{n\sigma\lambda}{a^2\mu} u^\alpha h_{\mu\nu} - \frac{\sigma'}{\sigma} h^\alpha{}_\mu u_\nu - \alpha_{\text{II}} \frac{\mu\lambda}{n\sigma} u_\mu h^\alpha{}_\nu, \quad (4.37)$$

⁵The K_5 term only appears in the exceptional configuration.

which contains only the two free functions $\mu(t)$ and $\sigma(t)$. Here, the prime ($'$) denotes derivatives along the direction of the four-velocity ($f' = u^\mu \partial_\mu f$).

By substituting the connection decomposition (4.36) into the torsion definition, we obtain the following expression:

$$T^\alpha{}_{\mu\nu} = 2T_1 h^\alpha{}_{[\mu} u_{\nu]} + 2T_2 \epsilon^\alpha{}_{\mu\nu}. \quad (4.38)$$

Similarly, by incorporating the connection and metric decomposition, we obtain the non-metricity decomposition:

$$Q_\alpha{}^{\mu\nu} = 2Q_1 u_\alpha u^\mu u^\nu + 2Q_2 u_\alpha h^{\mu\nu} + 2Q_3 h_\alpha{}^{(\mu} u^{\nu)}. \quad (4.39)$$

The coefficients T_i and Q_i are related to K_i in the following manner:

$$T_1 = \frac{K_4 - K_3}{n}, \quad T_2 = \frac{K_5}{a}, \quad (4.40)$$

$$Q_1 = N - \frac{K_1}{n}, \quad Q_2 = \frac{K_4}{n} - H, \quad Q_3 = \frac{K_3}{n} - \frac{nK_2}{a^2}, \quad (4.41)$$

where $N \equiv u^\mu \partial_\mu \log n = n'/n$ and $H \equiv u^\mu \partial_\mu \log a = a'/a$ are the associated expansion rates.

At this point, we return to the specific theory specified by equation (4.32). The metrical and inertial energy tensors become:

$$\begin{aligned} \bar{t}^\mu{}_\nu = & - 3M_{\text{Pl}}^2 \left[(Q_2 - T_1)^2 + T_2^2 \right] u^\mu u_\nu \\ & - M_{\text{Pl}}^2 \left[Q_1 Q_3 + (Q_2 - T_1) (3Q_2 - Q_3 - T_1) + T_2^2 \right] h^\mu{}_\nu, \end{aligned} \quad (4.42)$$

$$G^\mu{}_\nu = t^\mu{}_\nu + M_{\text{Pl}}^2 \left(2Q_2 T_1 - Q_3 T_1 - 2T_1^2 + 2T_2^2 \right) h^\mu{}_\nu. \quad (4.43)$$

We can see that if the second term in $G^\mu{}_\nu$ is non-zero, the metrical and inertial energy tensors do not coincide. We can treat the difference between these energy tensors as an effective pressure.

We can observe from the expression of the metrical energy tensor that it vanishes under the following conditions:

$$Q_2 = T_1, \quad T_2 = 0, \quad Q_1 Q_3 = 0. \quad (4.44)$$

Therefore, the torsion will only depend on the term containing T_1 . The condition $T_2 = 0$ also results from parity invariance [106]. Under these assumptions, if we also require metric parallelism – meaning that $Q_a = 0$ – it follows that $T_1 = 0$. This leads to the conclusion that metric parallelism is incompatible with the canonical frame. Furthermore, if we impose symmetric teleparallelism, we conclude that the scale factor must remain constant, indicating that the Universe does not evolve [106]. Consequently, we must also rule out this possibility. In summary, the canonical frame requires that both torsion and non-metricity are non-zero.

Under the canonical conditions, the inertial energy tensors become:

$$G^\mu{}_\nu = t^\mu{}_\nu - M_{\text{Pl}}^2 Q_2 Q_3 h^\mu{}_\nu. \quad (4.45)$$

It is clear that the difference between the metrical and inertial energy tensors can only arise when both Q_2 and Q_3 are non-zero. The condition $Q_2 \neq 0$ is already satisfied, as $T_1 \neq 0$ is necessary for having a canonical frame. The second condition, $Q_3 \neq 0$, is required not to have a curvature-dominated Universe [106]. Moreover, it requires that Q_1 must equal zero in order for $Q_1 Q_3$ also to vanish. Hence, the difference between metrical and canonical energy tensors will be a non-vanishing effective pressure.

For a spatially flat cosmology, there are three branches of solutions, as specified in the subsection 3.2.1. Let us summarise the status of the canonical frame in each of these branches:

- **Trivial and first non-trivial branches:** In both scenarios, the canonical frame results in a trivial configuration where the scale factor remains constant. Therefore, the canonical frame is not viable in these branches.
- **Second non-trivial branch:** The canonical frame can be implemented in this branch.

4.2.3 Key results

The discussion presented in the previous subsection was published in the paper titled “General Teleparallel Cosmology” [106]. This paper investigates a general teleparallel gravity framework, focusing on a quadratic theory that incorporates both torsion and non-metricity while remaining equivalent to GR.

A key finding of the paper was the establishment of a general cosmological frame, established by setting the canonical energy-momentum tensor to zero ($\bar{t}_{\mu\nu} = 0$). The general expressions for the metrical and canonical energy-momentum tensors were presented in subsection 3.1.2. We found that the canonical frame can be consistently applied in one branch of flat solutions, specifically the non-trivial branch II. Furthermore, in this frame, the metrical energy tensor corresponds to an effective pressure.

Summary

Throughout this thesis, we explored different MG theories and their application to Cosmology and Astrophysics. In spite of the diversity of theories studied, they share some key features. For instance, DHOST and $f(Q)$ theories present extra degrees of freedom, while $f(Q)$ and GTEGR are based on teleparallel geometries, where the spacetime is flat. Moreover, they collectively belong to more extensive theories such as Metric and Symmetric Teleparallel Horndeski Gravities. The thesis's main findings are summarised below.

In paper I, we investigated the pre-main sequence evolutionary stages of fully convective, low-mass stars. Our investigation is based on the modification of the hydrostatic equilibrium equation in the context of DHOST theories. We find that these modifications affect many stellar characteristics, such as the lithium-to-hydrogen ratio, Hayashi tracks and the maximal mass of a fully convective star upon entering the Main Sequence. These results suggest possible approaches to constraining gravity theories. However, a more realistic description of the stellar structure would be needed.

Paper II focuses on investigating cosmological symmetries in the context of GTEGR. We constructed the generic frames, which have cosmological symmetries. These frames provide all possible teleparallel connections in cosmology, although the implications of this result go beyond teleparallelism. Out of the three distinct flat connections found, one (non-trivial branch II) is physical because it admits the canonical frame.

Finally, paper III explores the inherent instabilities found in $f(Q)$ theories. We take advantage of the three spatially flat cosmological solutions found in paper II and perturb them around the FLRW metric. Two of these branches are strongly coupled, while the other one contains at least one ghostly degree of freedom. These results illustrate the inevitable pathological issues in this theory.

Acknowledgments

This PhD thesis is the result not only of my effort but also of the collaboration and support of many individuals to whom I would like to express my deepest gratitude. Firstly, I would like to thank my supervisors, Aneta Wojnar and Tomi Koivisto, for their guidance and support during my PhD studies. I am especially grateful to Aneta for her constant encouragement and for being a role model.

I would also like to thank my group colleagues, who provided constructive scientific discussions and moral support. This includes my fellow PhD students Ilaria, Laxmipriya, Luxi, Priidik, Sofia, and Vasiliki, who shared this learning journey with me, and postdoc fellows Sotirios, Damianos, Alejandro, and Pepe. I would also like to extend my gratitude to Professors María José, Margus, Manuel, and the group leader, Laur.

I also had the opportunity to collaborate with incredible people from whom I learned a lot. I want to thank Drs. Agnieszka Janiuk, Fatemeh Hossein Nouri, José Beltrán Jiménez, and Jackson Levi Said, for receiving me during scientific visits and for helping me deepen my understanding of scientific topics. My gratitude also extends to research collaborators Adrián Casado-Turrión, Aleksander Kozak, and Rebecca Briffa.

I am forever grateful to my family and friends, especially Anderson Lemos, Sarah Rosas and Beatriz Graça and to my husband, Ewerton, for their continual support not only during this PhD but also throughout my academic career.

This work would not have been possible without the financial support provided through the following grants: Gauge gravity: unification, extensions and phenomenology (PRG356), The Dark Side of the Universe (TK133) and Foundations of the Universe (TK202). Additional financial funding for visits and conferences was provided by Cost Actions CA18108 and CA21136, as well as the Erasmus Plus Traineeship Program.

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Kokkuvõte (in Estonian)

Laiendatud üldrelatiivsusteooria teoreetilised ja astrofüüsikalised aspektid

Kogu selle lõputöö jooksul uurisime erinevaid MG teooriaid ja nende rakendamist kosmoloogias ja astrofüüsikas. Vaatamata uuritud teooriate mitmekesisusele on neil mõned põhijooned. Näiteks DHOST ja $f(Q)$ teooriad pakuvad täiendavaid vabadusastmeid, samas kui $f(Q)$ ja GTEGR põhinevad teleparalleelsel geomeetrial, kus aegruum on tasane. Lisaks kuuluvad nad kollektiivselt ulatuslikumatesse teooriatesse, nagu Metric ja Symmetric Teleparallel Horndeski Gravities. Lõputöö peamised järeldused on kokku võetud allpool.

Paberis I uurisime täielikult konvektiivsete väikese massiga tähtede põhijärjestuse-eelseid evolutsioonietappe. Meie uurimine põhineb hüdrostaatilise tasakaalu võrrandi muutmisel DHOST-i teooriate kontekstis. Leiame, et need modifikatsioonid mõjutavad paljusid täheomadusi, nagu liitiumi ja vesiniku suhe, Hayashi jäljed ja täielikult konvektiivse tähe maksimaalne mass põhijadasse sisenemisel. Need tulemused viitavad võimalikele lähenemisviisidele gravitatsiooniteooriate piiramiseks. Siiski oleks vaja tähestruktuuri realistlikumat kirjeldust.

Paber II keskendub kosmoloogiliste sümmeetriate uurimisele GTEGRi kontekstis. Ehitasime üldised raamid, millel on kosmoloogiline sümmeetria. Need kaadrid pakuvad kõiki võimalikke teleparalleelseid ühendusi kosmoloogias, kuigi selle tulemuse tagajärjed ulatuvad teleparalleelsusest kaugemale. Kolmest leitud lameühendusest üks (mittetriviaalne haru II) on füüsiline, kuna 1) see lubab kanoonilist raami; 2) see lubab modifitseeritud $f(Q)$ teoorias kõiki füüsikalisi vabadusastmeid.

Lõpuks uurib paber III $f(Q)$ teooriate loomupäraseid ebastabiilsusi. Kasutame ära kolme ruumiliselt tasast kosmoloogilist lahendust, mis on leitud paberist II ja segame neid FLRW mõõdiku ümber. Kaks neist harudest on tugevalt seotud, samas kui teises on vähemalt üks kummituslik vabadusaste. Need tulemused illustreerivad selle teooria vältimatuid patoloogilisi probleeme.

Attached publications

I Early evolution of fully convective stars in scalar-tensor gravity	89
II General Parallel Cosmology	107
III Pathological Character of Modifications to Coincident General Relativity: Cosmological Strong Coupling and Ghosts in $f(Q)$ Theories	149

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Publications:

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Uurimistoetused ja stipendiumid:

2023 COST Action CA18108: Rahastamine osalemiseks 59. Winter School of Theoretical Physics ja kolmandas COST Action Action CA18108 koolituskoolis “Gravitatsioon – klassikaline, kvant- ja fenomenoloogia” (12.–21. veebruar 2023, Pałac Wojanów, Poola)

2023 COST Action CA21136: Rahastamine ühenädalaseks visiidiks Dr Jackson Levi Saidi juurde Malta ülikoolis lühiajalise teadusliku missiooni (STSM) programmi raames.

2023 COST Action CA18108: COST CA18108 neljandal aastakonverentsil (11.–14. juuli 2023, Rijeka, Horvaatia) osalemise rahastamine

2023 Erasmus+: Kahenädalane teaduspraktika Salamanca ülikoolis dr José Beltrán Jiménez juhendamisel.

2023 Erasmus+: Kahenädalane teaduspraktika Poola Teaduste Akadeemia Teoreetilise Füüsika Keskuses, juhendaja dr Agnieszka Janiuk.

2022 Kümnekuuline teaduspraktika Tartu Ülikoolis, juhendajateks dr Tomi Koivisto ja dr Aneta Wojnar.

Publikatsioonid:

1. A. Wojnar, D. A. Gomes. “Bose and Fermi gases in metric-affine gravity and linear Generalized Uncertainty Principle.” *Universe* 10.5 (2024): 217.
2. D. A. Gomes, et al. “Pathological Character of Modifications to Coincident General Relativity: Cosmological Strong Coupling and Ghosts in $f(Q)$ Theories.” *Physical Review Letters* 132.14 (2024): 141401.
3. D. A. Gomes, et al. “Cosmological constraints of Palatini $f(\mathcal{R})$ gravity.” *Journal of Cosmology and Astroparticle Physics* 2024.01 (2024): 011.
4. D. A. Gomes, J. B. Jiménez, T. S. Koivisto. “General parallel cosmology”. *JCAP* 12 (2023) 010.
5. D. A. Gomes, A. Wojnar. “Early evolution of fully convective stars in scalar-tensor gravity”. *Eur.Phys.J.C* 83 (2023) 492.
6. D. A. Gomes, J. B. Jiménez, T. S. Koivisto. “Energy and entropy in the Geometrical Trinity of gravity”. *Phys. Rev. D* 107, 024044 2023.
7. D. A. Gomes, F. C. E. Lima, C. A. S Almeida. “Correlations between emission events in Rainbow Gravity”. *ANNALS OF PHYSICS*, v. 428, p. 168436, 2021.
8. F. C. E. Lima, D. A. Gomes, C. A. S Almeida. “Transforming kinks into compactons in the $O(3)$ -sigma model”. *ANNALS OF PHYSICS*, v. 422, p. 168315, 2020.
9. D. A. Gomes, R. V. Maluf, C. A. S Almeida. “Thermodynamics of Schwarzschild-like black holes in modified gravity models”. *ANNALS OF PHYSICS*, v. 418, p. 168198, 2020.

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