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OPTIMIZATION OF PLASTIC STRUCTURES

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PREFACE

Structural design appears to have been an activity early man already. The modern fields of structural design are related to the aircraft industry, space investigations, ship building, nuclear industry, off-shore industry, chemical and mechanical engineering etc.

Optimization in structural design should be assessed from several points of view. Firstly, for instance, the minimum weight of an optimized aircraft structure is smaller than that of the original sample. This, in its turn improves the flying characteristics of the structure. On the other hand, saving of the structural material gives direct profit in the financial sense.

Optimization in the structural design has developed into a really multi-disciplinary field of science, which requires skillful combining of mechanics and engineering with mathematics. For solving the optimization problems the mathematical programming theory, the calculus of variations, the optimal control theory, as well as direct numerical procedures including the finite element method are used.

In the present work the variational methods of the optimal control theory serve as the optimization tools. In order to shed some light on the behaviour of non-elastic structures optimized according to certain criteria, elastic deformations are disregarded. However, geometrical non-linearity is taken into account. The geometrical non-linearity is meant to be interpreted the same way as in the Von Karman plate theory.

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Tartu, May 1991

J. Lellep

SURVEY OF THE LITERATURE

§1. Traditional plastic design

1.1. The role of analytical and numerical methods

The more readily available literature on structural optimal design will be reviewed herein and suggestions for further work will be offered.

Research work in the structural optimization typically follows one of the two broad directions. One way leads to the use of the methods of non-classical variation, dynamic programming or the principle of maximum of Pontryagin in order to extremize the perfomance index subject to the prescribed constraints.

Another approach, though perhaps less interesting from the mathematical point of view, is to treat it as a problem of mathematical programming and to search for such a set of discrete variables which will extremize the objective function subjected to the constraints. Similarly to that the discrete finite element methods could be used.

Obviously, both kind of studies are important. Perhaps due to the rapid progress in the computer techniques research into numerical methods has been comparatively more intense in comparison with the various analytical methods. However, the analytical methods are able to shed more light onto the mechanical and physical aspects of the designs of structures. Prager, 1981 has emphasized that: "Because they use available resources in a most efficient manner, optimal designs are likely to have unexpected properties requiring special care in the formulation of problems of structural optimization". These words are confirmed by a number of examples. Thus, the role of analytical methods could not be neglected because as a rule, they lead directly to exact results.

In the present work an attempt is made to review the most important analytical methods applied in the optimal design of plastic structures. The direct variational methods, the Drucker-Shield theory and the cost-gradient method are distinguished in the present work, although there are no strict outlines between these.

Linear and non-linear mathematical programming methods, the dynamic programming technique as well as the finite element method and other numerical approaches are outside the scope of the paper.

1.2. Formulation of an optimal design problem

The statement of a problem of optimal design of structures usually involves

a) picking a criterion of merit that can be used for choice of the optimal design from the set of acceptable alternative designs,

b) specification of the material model (elastic, plastic, elastic-plastic, geometrically non-linear rigid-plastic one, for instance), which prescribes the material behaviour in the fixed loading conditions with sufficient adequacy,

c) identification of the loading conditions (quasistatic loading, impulsive or dynamic loading with fixed or unifixed distribution),

d) specification of the limitations imposed on the stress-strain state of the structure,

e) stipulation of limitations on the range of geometrical dimensions and other design variables,

f) selection of appropriate methods of structural analysis and optimization.

The problem of optimal plastic design usually consists in finding a structure of prescribed configuration which will carry given loads and which will be optimal for a given criterion (minimum volume or weight, for instance). Here the design is pursued regardless of the cost of its manufacture.

A number of different approaches to the optimal plastic design problems are developed. A direct design procedure was first given by Michell, 1904 for framed structures composed of a material which has limited strength. Framed steel structures constructed of beams of constant cross-section between joints were studied by Heyman, 1953, Foulkes, 1953; 1954 and Prager, 1956 a. Necessary and sufficient conditions of optimality of such designs were derived with the aid of the linear programming methods.

In the present work the usual structural elements, such as thinwalled shells, plates and beams will be considered.

It is assumed, that the middle surface S as well as the external loading P distributed over the surface S and its boundary are prescribed. At the boundary points either the displacements vanish or the corresponding components of general stresses are given.

The aim of the analysis is to compile the design of the shell, which carries the prescribed external loading and for which the functional

$$J = \int_S F \, dS + G \tag{1.1}$$

takes the minimum value. Here $F = F(P,h)$ stands for a given continuously differentiable function and h is either the thickness or a parameter depending on the area of the cross--section of the shell. The function G depends on constant parameters, only. For instance, in the case of a sandwich type structure h is the thickness of the working sheets carrying bending moments and force resultants by membrane stresses. The core of the prescribed thickness H carries shear forces only.

The function F for the minimum volume design is determined as $F = h$ and for the minimum weight problem as $F = \varrho h$ where ϱ denotes the density of the material. In the present section the material is assumed to be rigid plastic (without strainhardening).

1.3. Drucker-Shield criterion

The start of the broad theory of plastic optimal design was made by Drucker and Shield, 1957 a. A criterion for absolute minimum weight design was established for the structures, which are subjected to direct or membrane stresses. In the case of sandwich beams and plates in transverse bending the condition (which now gives the relative minimum weight) would be written as

$$\frac{D}{h} = \text{const} \tag{1.2}$$

where D denotes the rate of the specific internal energy dissipation.

The previous paper by Drucker and Shield, 1957 a is extended to provide upper and lower bounds to the minimum weight by Drucker and Shield, 1957 b. The generalization also includes the influence of the body forces. The design procedure which yields the result

$$\frac{\partial\Delta/\partial h}{\partial F/\partial h} = \text{const} \tag{1.3}$$

is developed by Shield, 1960. Here Δ stands for the modified dissipation rate per unit area of the middle surface:

$$\Delta = D - F_i u_i h \tag{1.4}$$

where F_i denotes the body forces per unit volume and u_i - - the velocity components.

The general Drucker-Shield condition (1.3) is obtained through the use of the theory of limit analysis and not by direct application of the calculus of variations. The use of the limit analysis theorems established by Drucker, Prager, Greenberg, 1952, also by Gvozdev, 1949 and extended by Prager, 1956 b to the structures characterized by generalized stresses greatly simplifies the derivation of the optimality conditions. Moreover, in the case of a sandwich shell it was shown by Shield, 1960 a,b; 1973 that the result provides the absolute minimum of the optimality criterion. This is a better result than can be obtained by means of the calculus of variations.

In the case of sandwich shells and $F = hf(X)$ where $f(X)$ is a non-negative function of position over the surface S the condition (1.3) takes the form

$$\frac{\Delta}{hf(X)} = \text{const} . \tag{1.5}$$

If the minimum volume design is sought ($f=1$) and no body force is present, the criterion (1.5) evidently coincides with (1.2). The condition (1.5) applies to sandwich structures only; for solid shells it has to be replaced by

$$\frac{1}{f(X)} \frac{\partial\Delta}{\partial h} = \text{const} . \tag{1.6}$$

The design procedure has been extended to the design of multi-purpose structures which are to support different systems of loads at different times. It was shown by Shield, 1963 that the sandwich structure subjected to the multiple loading which consists of two independent sets of loades, has the minimum volume, if

$$\frac{\Delta^1 + \Delta^2}{hf(X)} = \text{const} . \qquad (1.7)$$

Here Δ^1 and Δ^2 stand for the dissipation rate corresponding to different rate fields associated with the different sets of loads.

Save and Shield, 1966 extended the result of Drucker and Shield to sandwich shells subjected to fixed and moveable loads and established a superposition theorem resulting, under certain conditions, in the minimum weight design for special combination of fixed and moving loads as the addition of the minimum weight designs for the separate loads. Following the results by Save, 1975;1977 and Save and Shield, 1966, if a sandwich structure is subjected to an infinite set of alternative loads, e.g. moveable loads, and the location of each set is specified by a parameter $\lambda \in \Lambda$ then the minimum volume design ($f \equiv 1$) is associated with

$$\int_{\Lambda} \frac{\Delta}{h} \, d\lambda = \text{const} . \qquad (1.8)$$

1.3.1. Special problems.

<u>Plates and shells under prescribed loads</u>. Optimality conditions (1.2) - (1.8) impose certain restrictions on the rates of curvatures of the middle surface. In the case of a sandwich structure the thickness h does not enter into these conditions at all because of the linear dependence of D on h. Thus, the optimal thickness distribution could be determined from the equilibrium equations using the relations associated with the preliminarily chosen flow regime. This regards the linear yield conditions, first of all. The Drucker-Shield condition serves for a criterion of practical applicability of the yield regime, which in its turn is influenced by the geometry of the structure and its loading.

Minimum weight design of beams and frames was considered by Heyman, 1953. Using the rationally determined flow regime Prager, 1955 a,b has found minimum volume designs for solid circular and noncircular convex plates whose material obeys the Tresca yield condition. Circular and rectangular solid plates were considered by Craemer, 1955 and plates of infinite length and finite width by Sububi, 1961.

The circular sandwich plates obeying the Tresca yield condition were studied by Onat, Schumann, Shield, 1957, Prager and Shield, 1959 and Shamiev, 1957 a,b. The optimal design of circular plates in the case of the von Mises material was investigated by Eason, 1960. Both built-in and simply supported plates were considered by Eason, 1960; Onat, Schumann and Shield, 1957; Prager and Shield, 1959. A sandwich plate of arbitrary shape from Tresca material was investigated by Shield, 1960. The applicability of different flow regimes has been studied and the minimum weight design for an elliptic plate has been obtained. Plates of arbitrary shape are studied by Nemirovsky and Nebogatov, 1985; Mrôz, 1961.

The minimum weight design of solid plates composed of a material, which obeys the non-linear yield condition has been studied by Sacci, 1980; Zavelani-Rossi, 1969 a,b, Kirakosian, 1977 and Kirakosian, Sarkisian, Minasian, 1982. Circular and annular plates of Tresca material resting on an incompressible liquid were considered by Gasanova and Shamiev, 1977; 1979 and body forces have been taken into account in the minimum weight design of plates and discs by Drucker, Drucker and Shield, 1957, Massonnet, Save, 1977; Save and Massonnet, 1972, 1982.

The problem of the minimum weight design of symmetrically loaded cylindrical shells has been studied by Shield, 1960; Ibragimov, 1968 a,b; Shamiev and Ibragimov, 1966; Shamiev 1963; 1965 and Cinquini, 1983 in the cases of piece-wise linear yield conditions. Kirakosian, 1978 studied a non-linear material. The methods for minimum weight for sandwich shells obeying the von Mises yield criterion were developed by Zavelani-Rossi, 1969 a,b and Shulgin, 1984.

Multiple loading. The minimum weight design of a sandwich Tresca plate for multiple loading was obtained by Shield, 1963. It was assumed that a simply supported circular plate was subjected to the set of lateral loads consisting of a uniformly distributed loading and of the concentrated load which acted at the centre of the plate. Circular plates loaded by two sets of transverse loads were considered later by Save, 1977, whereas beams and frames were studied by Prager, 1967; 1971 and Mayeda and Prager, 1967.

Nagtegaal, 1973 employed the superposition principle to obtain minimum weight designs of beams and frames and Polizzotto, 1974 developed a technique on the basis of the mathematical programming.

Movable loads. The minimum volume plastic design of beams subjected to one single movable load was first studied by Gross and Prager, 1962 starting from a linear programming approach. Solutions of some problems involving both fixed and moving loads were then given by Save and Prager, 1963. After extending the condition of Drucker-Shield to the case under consideration by Save and Shield, 1966, various problems were investigated and discussed by Lamblin and Save, 1971; Lamblin, 1972; Save and Massonnet, 1972 and Save, 1977.

Non-homogeneous plates and shells. The Drucker-Shield condition has been extended to non-homogeneous and composite materials by Mróz, 1970. The particular problems of optimal reinforcement of plates and shells were considered by Mróz, 1970; Mróz and Shamiev, 1970, Love and Melchers, 1972; 1973 and Melchers, 1975.

1.3.2. Sufficiency of Drucker-Shield condition. Uniqueness problems.

It was pointed out above that in the case of a sandwich structure and a convex yield criterion the Drucker-Shield conditions represent the necessary and sufficient conditions of optimality.

However, in general case, the lack of sufficiency has emerged. Using the direct variational methods Mróz, 1961 showed that the minimum weight designs are associated with the corners of the Tresca yield locus only. Thus, the maximum weight could be attained if the stress state corresponds to a side of the yield hexagon.

The uniqueness of the optimal design obtained by the Druckel-Shield procedure was investigated by Hu and Shield, 1961. Following the results by Hu and Shield, 1962 all optimal designs admit a common collapse mode. This conclusion was used to prove the uniqueness of minimum weight designs obtained in the previous work by Shield, 1960. The former result was confirmed by Nemirovsky, 1968; 1977 who showed that the Drucker-Shield conditions could lead to an infinite number of designs. All these designs have the same volume (weight) and correspond to a common collapse field.

1.4. Cost gradient method

1.4.1. Specific cost

Employing the concept of Drucker-Shield as a rule one obtains, structural designs with singular cross-sections of zero area. A theory of optimal plastic design free from this disadvantage was introduced by Prager and Shield, 1967 who generalized a notion by Marcal and Prager, 1964.

The cost of the structure per unit length, area or volume of the structural domain is termed specific cost Ψ. The specific cost Ψ usually depends on the generalized stress vector Q only and thus the total cost subjected to minimization can be expressed as

$$I = \int_S \Psi(Q)\,dS \,. \tag{1.9}$$

The latter may represent the total weight or volume of the structure or of certain specified components, or alternatively, some idealized form of cost in monetary sense.

Denoting the strain rate vector associated with the stress vector Q by q, the internal energy dissipation may be written as $D = Q\cdot q$, if body forces are neglected. Thus, the new notations admit to convert the Drucker-Shield condition (2) into

$$\frac{Q\cdot q}{\Psi} = \text{const.} \tag{1.10}$$

1.4.2. Marcal-Prager-Shield condition

Introducing a fictitious strain field for an "associated" non-linear elastic structure and using the minimum principle of the complementary energy Marcal and Prager, 1964; Prager and Shield, 1967, derived optimality conditions for the total cost (1.9). Following Rozvany, 1973; 1976 the general condition may be represented in the form

$$q_k = G\,\Psi(Q_s) \,. \tag{1.11}$$

Here q_k and Q_s, respectively, are kinematically and statically admissible strain rates and stresses, whereas G stands for generalized gradient operator.

The generalized gradient operator will be determined by the equation $G\Psi = \partial G/\partial\Psi$, if the specific cost function is differentiable. However, if Ψ has slope discontinuities, then $G\Psi$ contains steps. Finally, the impulses corresponding to the form of Dirac's delta function in the cost gradient are associated with the discontinuities of the specific cost function.

In its original form the Marcal-Prager-Shield theory is valid for continuous convex specific cost functions and linear equilibrium equations as a necessary and sufficient condition of optimality. However, it was stated later by Rozvany, 1973 that the criterion (1.11) is useful as a necessary condition for non-convex specific cost functions and for discontinuous cost functions by Rozvany, 1974. Nevertheless, it leads to global minimum in special cases as it was shown by Rozvany and Adidam, 1973.

Optimality conditions for multiple loading were derived by Charrett and Rozvany, 1972, also by Rozvany and Adidam, 1972 by means of the variational methods. A number of special problems regarding unspecified as well as assigned or partially preassigned cost distributions were studied by Marcal, 1967; Marcal and Prager, 1964; Charrett and Rozvany, 1972; Adidam, Lowe and Melchers, 1972; 1973; Melchers, 1975; 1981; Melchers and Rozvany, 1970; Rozvany, 1973 a,b, c,d; 1974 a,b,c; 1975; 1976; Rozvany and Adidam, 1972 a,b; 1973; Rozvany and Charrett, 1971; Rozvany and Hill 1976; 1978; Rozvany and Mróz, 1975; 1977. Systematic reviews of these problems in the context of the Marcal-Prager-Shield theory are represented in the book by Rozvany, 1976 and in the survey by Rozvany and Mróz, 1977, which also includes optimization problems of elastic structures.

Much light has been shed by Rozvany, 1976 to optimal flexure fields, which have two important practical applications: (i) design of fibre-reinforced plates of minimum fibre-volume and (ii) design of minimum weight grillages. Proofs of all properties of optimal flexure fields are given in the study of Rozvany and Hill, 1976, where the general theory of optimal load transmission is outlined.

Further extensions of the cost-gradient method have been made by Rozvany, Olhoff, Cheng and Taylor, 1981, Wang, Rozvany and Olhoff, 1983 and Rozvany and Wang, 1983.

1.5. Direct variational methods

1.5.1. Preliminaries

The plastic optimization problems prescribed above could be considered as variational problems consisting in minimization of the functional (1.1). Depending on the particular statement of the problem one has to take into account the functional as well as special additional constraints and physical and geometrical requirements (equilibrium equations, associated flow law, yield condition, compatibility equations etc.). The problems reduce thus to the constrained non-classical variational problems, mainly. Nevertheless, efficient solutions have been found in many cases.

1.5.2. Minimum weight design of plates and shells

Circular plates obeying the von Mises yield condition were considered by Freiberger and Tekinalp, 1956. By the use of the calculus of variations they derived the necessary optimality conditions which yield the Drucker-Shield criterion and found the optimal thickness distributions for sandwich and solid plates.

In a further study by Freiberger, 1957 the same approach was employed in the case of circular cylindrical shells. The material of working sheets was assumed to obey the von Mises yield condition, which was satisfied in the average (see Robinson, 1971).

Megarefs, 1966; 1967; 1968 developed a static technique of stress variation resorting to the linearity of the volume functional in the case of sandwich plates and the Tresca yield criterion. This approach proved efficient enough for determining of the minimum weight designs of annular plates for any support conditions and one - directional loadings. The static stress variation method was extended by Reiss and Megarefs, 1969; 1971 to sandwich axisymmetric plates and cylindrical shells obeying the von Mises yield condition. These studies encompass various edge conditions and a wide range of axisymmetric loading.

With symmetry conditions removed, the plates of arbitrary shape were considered by Reiss, 1974. Introducing the

Lagrangian multiplier to account for the equilibrium equation the self-adjointness of the problem was observed. By self-adjointness, it is meant that in this particular case the Lagrangian multiplier is proportional to the deflection rate. Probably due to this phenomen the deflection rate appears to be independent of the pressure distribution.

Limit analysis and minimum-weight design of circular and annular plates was investigated also by Mróz, 1958; 1959; 1961; 1963. Both materials obeying the Tresca and Von Mises yield condition were considered. Mróz's, 1961 discussion of the influence of the second order terms neglected in the previous analysis reveals that the minimum weight of a structure will be ensured, if the stress regime over the whole plate corresponds to a corner of the Tresca yield hexagon. Otherwise the nature of the extremum must be investigated.

Kozlovski and Mróz, 1969; 1970 showed that early minimum weight solutions for solid plates and shells represent only local minimum, since the absolute minimum weight converges to zero when the solution reduces to an infinite number of ribs of infinite depth. In the former paper the authors obtained a solution within the constraints that the maximum thickness is prescribed and the plate has a finite number of circumferential ribs. The fact that the need for the formation of flanges or ribs is a natural consequence of the minimum requirements has been emerged in the works by Megarefs, 1966; 1967; 1968 Reiss and Megarefs, 1969; 1971. According to these papers, if the admissible stresses are to be bounded, a minimum stress need not exist at all. But, if the condition of boundedness is removed, the minimum weight design consists of ribs.

Shablii and Zhuk, 1981 investigated the optimization problem in the case of a non-linear approximation of the exact yield surface taking shear forces into account.

Reiss, 1974 has determined minimum weight designs for conical shells subjected to single loading. Circular Tresca plates for multiple loading have been studied by Reiss, 1976. An approximation of the yield surface consisting of two hexagons was employed by Reiss, 1974.

A general variational formulation of the optimal plastic design problems without referring to any particular type of structure is presented by Sacchi, 1971 a; 1975; 1980. The stationarity conditions derived by using the Lagrangian multipliers technique by Sacchi, 1975 coincide with the conditions obtained earlier by Save, 1972.

1.5.3. Optimal design of reinforced structures

Optimal reinforcement of rigid-plastic plates and shells has been studied by Mróz, 1959; 1964; 1967; 1970; 1974, Rozvany, 1976; Lowe and Melchers, 1972; 1973 and others making use of different approaches. Mróz's, 1959 idea of incorporating a lower bound approach with variational techniques has turned out to be quite a fruitful method in this field. Later Mróz, 1967 and Morley, 1966 derived independently the static-kinematic optimality conditions for variable reinforcement in arbitrary directions and presented examples for axisymmetric slabs and simply supported square plates.

Kaliszky, 1965 a,b,c showed that in the case of a curvilinear reinforcement the optimal solution is associated with the elastic moment field for a uniform plate.

A comprehensive set of solutions of optimal reinforcement problems obtained by Charrett, Adidam, Lowe, Melchers and Rozvany is presented in the book by Rozvany, 1976. The latter contains a review of the literature on optimal fibre-reinforced plates, shells and grillage-like continuums.

1.5.4. Parametrical optimization

A broad circle of optimal design problems is formulated by assuming that the design is defined within a set of constant parameters that should be determined from the optimality conditions. A problem of this type is, for instance,the reinforcement problem consisting in optimal orientation of identical fibers.The optimal design of structures with piece-wise constant cross-sections investigated in particular cases by Foulkes, 1953; 1954; Hopkins and Prager, 1955; Sheu and Prager, 1969; Rozvany, 1976; Prager, 1974; Mazalov,1973; Lamblin and Guerlement, 1971; Save, 1985; Lamblin, Guerlement and Save, 1985, as well as optimal location of additional supports studied by Mróz and Rozvany, 1975; Prager and Rozvany, 1975; Rozvany, 1976 also could be conceived as parametrical problems.

Mróz, 1972 has derived the optimality conditions for multiparameter plates and shells

$$\frac{D_1}{V_1} = \dots = \frac{D_n}{V_n} \tag{1.12}$$

assuming that the middle surface is divided into n subregions and the design variable over each subregion is specified within a single parameter. Here D_i, V_i denote the total power of dissipation and volume of each subregion. In particular, (1.12) can be reduced to the conditions obtained by Foulkes, 1954; Sheu and Prager, 1969 and Rozvany, 1976.

Mróz and Garstecki, 1976 studied the problem of optimal distribution and location of loads in order to achieve maximum total load at limit state. The characterization for a particular distribution having the greatest possible total load at the yield point had been found earlier by Collins , 1968. However, the results by Mróz and Garstecki, 1976 are more general.

1.5.5. Existence and uniqueness

It was pointed out above, that the absolute minimum weight design of plastic solid plates appears to be with ribs of infinite thickness and infinitesimal width. To make a minimum weight design practicable, its thickness must be finite everywhere. Therefore, the specification of an upper bound on the plate thickness is necessary for avoiding non--useful designs. But it was established by Rozvany, Olhoff, Cheng and Taylor, 1981, that the maximum thickness constraint alone did not ensure smooth global minimum weight solutions. This turned out to be furnished, in the limit, by a grillage-like continuum consisting of a dense system of ribs of infinitesimal spacing and uniform depth. Wang, Rozvany and Olhoff, 1983 extended the previous paper, introducing the general cost gradient method for deriving minimum weight solutions in the case of solid plates subjected to maximum thickness constraint. Similar elastic problems regularized with reference to the concept of G-closures by Lurie Cherkaev and Fedorov, 1982.

In order to discuss the existence of the optimal solution, one has to specify, first of all, the class of admissible functions. Actually, these may be: (i) continuous smooth functions, (ii) piece-wise continuous and (iii) rib-density functions with unlimited number of discontinuities.

The solid plate paradox discussed by Rozvany, Olhoff, Cheng and Taylor, 1981 showed that the absolute minimum weight solution may not exist in sets (i) and (ii).

Megarefs and Hodge, 1963 demonstrated that in some simple cases the minimum values of the optimality criteria may not be a point of zero variation and that points of zero variation may be relative maximum as well as minimum.

The existence and uniqueness of the optimal solution of variational optimization problems was studied by Cinquini and Mercier, 1976, and by Cinquini and Sacchi, 1980. The results of convex analysis in the abstract Hilbert spaces were invoked, which admitted to establish the existence and uniqueness conditions. These are certain constraints (first of all, the functional must be strictly convex), which have to be met by special sets, defined by the variational problems.

1.6. Pontryagin's maximum principle

Side by side with the direct variational methods also the Pontryagin's maximum principle (see the books by Pontryagin and Boltyanskii, 1971; 1976, Bryson and Yu-Chi-Ho, 1969) may be used for optimal design of structures. The first paper in this field, as for as the authors know was that by Lurie, 1965. He solved some optimization problems regarding elastic structures. Optimal design of reinforced concrete circular plates and cylindrical shells was discussed by Reitman and Shapiro, 1976; 1978; such a lay-out of reinforcement is to be found, for which the general amount of the reinforcement is minimal.

Some problems of optimal design for axisymmetric sandwich plates and shells were examined by Pungar, 1972; 1973; 1974. The material of the structure is rigid-plastic, the yield condition of Von Mises is used. Following Odishvili, 1971 the optimality critirion is taken in the form

$$I = \int_S h^2 dS . \qquad (1.13)$$

Integration of the state and adjoint equations is greatly simplified, if instead of the Mises' yield condition we shall make use of piecewise linear yield conditions. Such an approach was applied by Lepik, 1972; 1973; 1974; 1975, where the Prager's yield condition was used (in this case the yield stresses for tension and compression will be different). This approach was utilized by Vainshtein, Rudis and Polyakov, 1980, also. The latter papers and the one by Lepik, 1972 considered homogeneous structures. However, Lepik, 1973; 1974 studied the sandwich type structures. For the design variable the thickness of the structure (or the thickness of the working sheets) is chosen, this quantity is bound from below and above. The weight of the structure is to be minimized (in the case of a sandwich structure the weight of the core will be neglected). Optimal designs for circular plates were obtained by Lepik, 1972; 1973, whereas axisymmetric cylindrical shells were considered by Lepik, 1974 and Vainshtein et.al., 1980.

A study on optimal design of circular sandwich Tresca plates subjected to concentrated loads was presented by Lellep, 1977. In this note the load carrying capacity is maximized for given weight.

Optimization of rigid-plastic axisymmetric shells by taking into account shear forces was considered by Lepik, 1975.

The papers by Lellep, 1977 and Lepik, 1978 b were devoted to the application of the optimal control theory and to the optimal design of non-linear elastic and viscous beams.

Lepik, 1987 b has studied the problem of optimal location of an additional support in the case of non-elastic beam. The perfomance index and the constraints are given in a quite general form. The aim of the optimization is to reduce the beam's complience.

1.7. Other methods and approaches

1.7.1. Uniform strength

With reference to the concept of uniform strength by an optimal design of a beam such a beam is to be understood for which the yielding takes place in all cross-sections simultaneously. In the case of rotationally symmetric plates the classical concept of uniform strength stipulates a stress state in which both the radial and circumferential moments are equal to the limit moment.

The intuitive approach based on the competing yield mechanisms was used by Prager, 1955; Hopkins and Prager, 1955 in order to obtain minimum weight designs of plates. Onat and Prager, 1955 developed a method for cylindrical shells loaded by a transverse pressure. This technique was extended by Freiberger, 1956 to account for the axial dead--load. However, as it was noted by Hodge, 1957 it is not entirely evident that this technique will result in a shell of minimum weight. The suggested procedure leads to a smaller weight (compared with the shell of a constant thickness) in the case of short shells only.

The classical concept of a rotationally symmetric cupola of uniform strength which foresees the realization of a membrane state of stress was used by Milanković, 1908. This concept was extended by Ziegler, 1958 who introduced a locus of admissible stress points in a stress plane. Issler, 1959; 1964 treated spherical shells under constant vertical dead load per unit area of the middle surface and rotationally symmetric shells of given meridian under their structural weight. The condition of rotational symmetry was abandoned by Schumann and Wüthrich,1972 who discussed a shell of quadratic plan form as an example. Sayir and Schumann, 1972 investigated anisotropic shells obeying the von Mises yield condition. As it was pointed out by Prager and Rozvany, 1980 uniform strength design of a cupola for structural weight alone leads to an unexpted result: the weight of the shell can be made arbitrarily small by choosing the thickness at the apex to be sufficiently small. To avoid this paradox, the combined action of structural and dead weights was considered in the paper by Prager and Rozvany, 1980.

The effect of shear forces on optimal design of plastic beams and circular plates was studied by Nemirovsky, 1975. He revealed that considering the transversal shear prevents obtaining designs with zero thicknesses which emerged by the use of the Kirchoff-Love hypothesis.

General theorems of elastic uniform strength design are established by Save, 1968. Following these results, any of von Mises' plates of elastic restricted uniform strength corresponds to a minimum weight plastic design amplified by a shape factor.

Uniform strength designs of orthotropic shells obeying an approximate yield condition were found by Mikeladze,1959, 1966.

1.7.2. Probabilistic design

The discussed deterministic approaches to optimal plastic design are not applicable if the necessary strict information about the mechanical characteristics, geometry or loading is not available. The structures having elements with random distribution of yield stresses were treated by Sacchi, 1971 a,b making use of probabilistic analysis. Each structural element was considered as an individual drawn by lot from a population of elements whose yield stress is distributed according to the established frequency.

A number of theorems concerning limit analysis as well as the minimum volume design of structures composed of elements with average resistance is formulated and proved by Sacchi, 1971 a. Upper and lower bounds on the average minimum volume are established in the contribution of Sacchi, 1971 b. The bounds are evaluated on the basis of the deterministic limit design.

Multi-criteria probabilistic design of structures was discussed by Parimi and Cohn, 1975.

1.7.3. Superposition principles

Investigating the minimum weight design problems of plastic structures subjected to a set of alternative loads the superposition principles have been established and used by Save and Shield, 1966; Hemp, 1973; Nagtegaal, 1973; Nagtegaal and Prager, 1973. Hemp's approach is developed on the basis of the linear programming, whereas Nagtegaal and Prager used the energy methods of the limit analysis. Nagtegaal, 1973 has shown that the optimal design of a beam-type structure which has to carry the alternative loads P_1 and P_2 can be obtained by the way of determining the optimal solutions for the loads $P^1 = (P_1 + P_2)/2$ and $P^2 = (P_1 - P_2)/2$ separately. Adding then the moments associated with these designs the moment distribution of the optimal design could be obtained.

1.7.4. Other approaches

A number of various methods and approaches have been used in the field of optimal plastic design. The variety of statements of the optimization problems is discussed by Cyras, 1975; 1980; 1982 in the light of the mathematical programming. The two methods based on the mathematical programming and the use of the optimality conditions were discussed by Fleury, 1979. It was emphasized by Fleury, 1979 that "far from being ineluctably opposed,the two approaches have in fact converged to the same method that consists of transforming the original problem to a sequence of simple approximate problems".

Some early works by Prager, 1959; 1970; 1974 are concerned with the optimization of Michell type structures.

Dynamic programming methods have been successfully applied in optimal design by Distefano, 1974; Palmer, 1968; Szefer, 1971; Pochtman and Baranenko, 1975.

Some attempts have been made to develop unified approaches to optimal design of elastic and plastic structures. The most efficient one was probably that by Prager and Taylor, 1968.

There exists a great number of papers devoted to the application of the finite element method in optimal plastic design but this topic is outside the scope of the current review. The authors recommed to refer to the survey by Vanderplaats, 1982, also the papers by Pape and Thierauf, 1980; Maier, Zavelani-Rossi and Beneditti, 1972.

The rheology problems are considered in the context with the minimum weight design by Wojdanowska and Życzkowski, 1980; Życzkowski and Świsterski, 1980; Życzkowski,1971; 1974. A review of recent advances in this field is represented by Życzkowski, 1983; Życzkowski and Krużelecki,1985. Earlier Prager, 1968 has extended the Drucker-Shield condition to the case of stationary creep. Nemirovsky, 1970 established some properties of optimal designs in rheology. Beams and plates of uniform strength are investigated by Nemirovsky and Reznikov, 1969. Minimum weight design of beams and annular plates in stationary creep is studied by Lellep, 1977; 1979 taking into account the difference of the materials' behaviour under tension and compression.

Comprehensive reviews of statements and methods of solution of optimization problems regarding plastic as well as elastic structures are given by Barnett, 1966; Haftka and Prasad, 1981; Lellep and Lepik, 1984; Niordson and Pedersen, 1973; Prager, 1970; Reitman and Shapiro, 1976; 1978; Rozvany and Mróz, 1977; Sheu and Prager, 1968; Vasilyev, 1970; Wasiutyński and Brandt, 1963; Życzkowski, 1974; Życzkowski and Krużelecki, 1985 and others. More detailed information is available in books by Banichuk, 1980; 1986; Brandt(ed.), 1977; Cohn, 1972; Cox, 1965; Cyras, 1982;Hemp,1973; Kirsch, 1981; Narusberg and Teters, 1988; Rozvany, 1976; Save and Massonnet, 1965; 1972; Shanley, 1960; Spunt, 1971 and Wood, 1961.

§2. Optimal design of plastic structures subjected to dynamic loading

2.1. General remarks

The problems of optimal design of non-elastic plates and shells subjected to dynamic loading received the attention of research workers comparatively recently. The first paper in this area was published by Rabinovitch, 1965.

The dynamics problems are complex; remarkable simplification could be obtained by the use of the mode form method suggested by Martin and Symonds, 1966; also by Symonds, 1980. The alternative possibility for simplification is the use of general theorems of dynamic plasticity. This approach was followed by Reitman, 1972; Kaliszky, 1981; Erkhov, 1979.

Comprehensive reviews of the works devoted to the dynamics problems as well as to optimization in the case of non-elastic materials are presented in the monograph book by Lepik, 1982; review articles by Lepik, 1981; Jones, 1989 and Lellep and Lepik, 1984.

2.2. Structures with segmentwise constant thickness

The beams with piece-wise constant thickness are studied by Lepik, 1981; 1982 b; Lepik and Mróz, 1977; 1978 making use of the mode form method. It appeared that the possible mode form is not unique. Exact solutions within the limits of the concept of a rigid-plastic body have been also found by Lepik, 1980; 1981; 1983. Soonets, 1981; 1982 has studied two-stepped beams.

Aunin, Lellep and Sakkov, 1986 as well as Lellep and Sakkov, 1985 have studied the problem of optimization of reinforced beams, subjected to the impulsive loading.

Annular and circular plates of minimum mass have been considered by Lepik, 1982 c; Lepik and Mróz, 1977. Circular plates consisting of two concentric parts with different materials and different thicknesses are discussed by Mazalov, 1973; Mazalov and Nemirovski, 1973.

Optimal design of two-stepped conical and cylindrical shells are examined by Kirs, 1975;1979 a,b using the method of limited interaction between forces and moments. The yield surface was picked in the form of two diamonds in the different planes, suggested by Jones, 1970.

The use of the higher modes and quasi-mode method in the optimal design of structures subjected to the dynamic loads was discussed by Lepik, 1979; 1980 a. However the higher modes appear to be unstable and the motion of the structure steadily goes over to the fundamental mode form.

2.3. Structures with additional supports

An attractive way to diminish the structural compliance and increase the stiffness is to furnish the structures with additional supports. The location of the additional supports is reasonable to select so that the stiffness attains the maximal value.

For non-linear elastic beams under static loading the problem was examined by Mróz and Rozvany, 1975. Making use of the methods of variation the authors derived the optimality condition

$$C[M(s+)] - C[M(s-)] + Rw'(s) = 0 , \qquad (2.1)$$

where C is the complementary energy per unit length of the beam, s is the coordinate of the location of the additional support, R stands for the reaction of the support.

Prager and Rozvany, 1975 obtained the optimality condition for rigid-plastic beams in the form

$$M'(s+)w'(s+) - M'(s-)w'(s-) = 0 . \qquad (2.2)$$

4*

Here M' stands for the shear force and w' is the slope of the deflection.

The applicability of the conditions (2.1), (2.2) was examined by Lepik, 1980; 1981 b; 1982 a with the aid of the methods of the optimal control theory in the case of beams loaded dynamically. It appeared that (2.2) holds good if the method of mode-forms is employed.

In the cases when for optimality criterion are picked (i)-the residual mean deflection, (ii)-the maximal deflection or (iii)-the volume of the beam, a direct technique of determination of the positions of additional supports for plastic beams subjected to the impulsive loading was developed by Lellep, 1978; 1979. It was somewhat amazing that these three criteria led to a common result.

The optimality conditions in the integral form are derived by Lellep, 1981 c for plastic beams loaded impulsively. This approach was extended by Lellep, 1983 a; 1984 for cylindrical shells using the mode form method. In the first paper the case of the rectangular impulse was studied whereas in the second work the shell subjected to the uniform initial impulse was examined. In the both latter papers the problem was converted into a self-adjoint problem of the optimal control theory with distributed parameters.

The optimal location of rigid ring supports for cylindrical shells was examined by Olenev, 1982; 1983; 1985; 1987 in the cases of dynamic pressure loading and impulsive loading. The paper by Olenev, 1988 is devoted to the plastic beams, subjected to the pressure loading.

§3. Large deflections of rigid-plastic structures

3.1. The concept of a geometrically non-linear structure

Within the concept of a rigid-plastic body the structure remains rigid until the external loads attain certain values which correspond to the yield stresses. This concerns the one-dimensional tension or compression. In the two-or three-dimensional case the body is rigid if the stress-state corresponds to an internal point of the yield surface. However, if the corresponding point lies on the yield surface plastic flow will occur.

Introducing the geometrical non-linearity (non-linear terms in the governing equations) one can examine the post-yield behaviour of the structure. In the present work we will presume that the deflections of the thin-walled structures do not exceed the order of the wall thickness.

The number of investigations devoted to the determination of the stress-strain state of geometrically non-linear structures is limited. The present review is not a complete one. Comprehensive reviews of the studies of this kind are presented by Duszek, 1975; Sawczuk, 1980; 1982; Jones, 1969; 1970.

3.2. Large deflections of rigid-plastic beams, plates and shells

Plastic beams and arcs which operate in the post-yield stage are examined by Belenkii, 1973; Dikovitch, 1967; Gill, 1976; Gürkok and Hopkins, 1981; Kondo and Pian, 1981 a. Kondo and Pian, 1981 a suggested a simple method on the basis of the assumption that the beam deforms into a number of rigid regions which are separated by plastic hinges. Kondo and Pian, 1981 b,c,d extended this approach to plastic circular and polygonal plates as well as to shallow spherical shells.

Circular and annular plates, the material of which obeys the Tresca yield condition were investigated by Lepik, 1960. Alternative methods based on the concept of the limited interaction between membrane forces and moments are developed by Jones, 1969; Erkhov and Kislova, 1981; Erkhov and Starov, 1986; 1987. The latter approach was extended to shallow shells by Erkhov and Starov, 1987 b.

Simplified methods of calculation of the stress-strain state of rigid-plastic circular plates and slabs are due to Onat and Haythornthwaite, 1956; Calladine, 1968.

Moderately large deflections of cylindrical shells of the Tresca material are studied by Duszek, 1966; 1967; Duszek and Sawczuk, 1970; Lance and Soechting, 1970 using the yield surface in the three-dimensional space of generalized stresses. Another approach was developed by Lepik, 1966 a,b utilizing the Tresca yield hexagon in the plane of the principal stresses.

The method based on the two yield hexagons on the planes of the forces and moments, respectively, was explored by Lellep and Hein, 1988 when studying the large deflections of plastic shallow spherical shells of the Tresca material.

The circular plates and cylindrical shells of Von Mises material are investigated by Lellep and Majak, 1987; Lellep and Hannus 1983; 1987. The exact yield surface was replaced by a non-linear approximation which corresponds to the satisfaction of the yield condition on an average with respect to the thickness of the shell. For the geometrically linear case this problem was solved by Shulgin, 1985.

§4. Optimal design of geometrically non-linear structures

4.1. Optimality criteria and additional restrictions

Plastic optimization problems discussed above have been stated as minimum weight problems for a given collapse load, thus, under the requirement of the incipient flow. Of course, there were considered the problems consisting in maximization of the limit load, and others, but for all these designs the configuration variations in the post-yield range were neglected. As it was demonstrated by Mróz and Gawecki, 1975; Gawecki and Garstecki, 1978; 1979 such designs appear to be sensitive to geometrical changes which the structures undergo during plastic flow. It was pointed out by Mróz and Gawecki, 1975 that the post-yield stiffness of optimal structures (for geometrically linear approach) is generally smaller than that of a reference structure for which no optimization procedure was carried out. Moreover, it appeared that the load deformation response of an optimal structure could be unstable even when the load deformation curve of a uniform structure was stable. Thus, a necessity arises to consider for geometrical non-linearity in the plastic design. This matter was discussed by Save, Guerlement and Lamblin, 1989.

This involves the question of an optimality criterion as well as of additional restrictions which have to be taken into account by solving a minimum weight problem. The incipient collapse load is no longer suitable as a measure of the strength since the load changes when the structure deforms.

If the post-yield behaviour were accounted for, the optimality criterion would be presented in the form

$$I = \int_S \max_{p \in R} F_0 \, dS + \int_S F \, dS + G \quad . \tag{4.1}$$

Here F_0, F, G stand for continuous differentiable functions depending on displacements U and W, generalized stresses Q and certain parameters. It is assumed that p is a scalar parameter, whereas the set of its admissible values is a closed set. Different particular forms of the functional will be discussed later.

The optimal design of a structure which minimizes the criterion (4.1) has to satisfy the basic equations of the moderately large deflection theory and special additional restrictions imposed on the deflections and stresses as well.

4.2. Optimization for prescribed deflected shape

In the case of piece-wise linear yield surfaces, it appears to be reasonable to state the optimization problem for a given deflected shape associated with the structure of prescribed thickness distribution and subjected to the same loads. In the case of a minimum weight problem now we have $F_0 = G = 0$, $F = h$ in (4.1). For an additional requirement can serve the restriction $W \leq W_*(X,P)$, where W_* denotes the deflection of the prescribed structure. For the simpler problems (concerning with beams or shells obeying piece-wise linear yield conditions for instance) the inequality in the latter relation could be changed by the equality.

Such an approach was developed by Lellep, 1981; 1983 b; Sawczuk and Lellep, 1980;1987 using a variant of the deformation theory of plasticity according to which the strain vector itself is orthogonal to the yield surface. The paper by Lellep, 1981 is devoted to minimum weight design of rigid-plastic beams subjected to an arbitrarily distributed transverse loading and the axial dead-load. The deflection is required to be the one of the beam of a constant area of cross section. With the aid of the variational methods of the optimal control theory necessary optimality conditions are derived. These yield the result

$$\Phi(M,N,h) = 0 \qquad (4.2)$$

which has to be satisfied over the whole beam. In the latter formulae M and N denote, respectively, the bending moment and the axial force, whereas $\Phi \leqslant 0$ represents the yield condition. As an example, a simply supported beam of homogeneous rectangular profile and loaded by the uniformly distributed transverse pressure is studied in greater detail.

A similar problem is investigated in the case of a sandwich cylindrical shell by Lellep, 1983 b; Lellep and Sawczuk, 1980. Lellep and Sawczuk, 1980 studied a structure consisting of a cylindrical shell and of two end plates and subjected to internal pressure is considered and the optimal wall thickness variation is sought for under the requirement of minimum material consumption. The solution procedure regarding optimization of plastic shells obeying a piece-wise linear yield condition has been developed. The optimal design of a shell assuming a required shape beyound the incipient collapse load was found employing the optimal control theory.

The shells of a plastic fiber-reinforced material and rib-reinforced shells are studied by Lellep and Hein, 1987; 1989. Lellep and Mandri, 1987 developed a method for optimization of plastic cylindrical shells with limited thickness.

4.3. Mini-max approach

The optimization technique for prescribed deflected shape in the post-yield range which was discussed above could lead to practically nonuseful designs in more insidious cases. In fact, such a design appears to be the minimum weight design for a given value of the external load P_* and associated with its deflection $W_*(X,P_*)$ only. But it is not clear what happenes if $P < P_*$. As it was noted by Lellep, 1982 a; 1984 a these designs even may not have resistance to all loads $P \in [0,P_*]$ (here the load intensity is assumed to be representable in the form $P \cdot R(X)$, where $R(X)$ is a given function).

To avoid this paradox an optimal design procedure has to be used accounting for a set of loads. Elastic beams optimal for a given class of loads were studied by Banichuk, 1975 ; 1976 making use of the game approach to problems with inadequate information. Some particular elastic and plastic optimal design problems considered in the light of the mathematical game theory are discussed by Aptukov and Pozdeyev, 1982.

The mini-max approach to the optimal design of rigid--plastic structures taking into account the post-yield behaviour is as follows. The optimality criterion subjected to minimization could be represented as

$$I = \int_S \max_P h \, dS \qquad (4.3)$$

where the maximum is attained for $P \in [P_0, P_*]$. Thus, the special case of (4.1) associated with $F = G = 0$; $F_0 = h$, $p = P$ will be considered. Here P_0 stands for the limit load for the structure of specified shape. As before the additional constraint $W = W_*$ and the constitutive equations are assumed to be satisfied.

The papers by Lellep, 1982 a, 1984 a are devoted to the minimum weight design of clamped plastic beams loaded by a distributed transverse pressure and an axial force. As the "associated" structure with the specified shape the beam of a constant thickness was used. With reference to the principle of maximum for non-smooth problems of optimal control theory Boltyanskii, 1971; Demyanov and Malozemov, 1972; Alsevitch, 1976 the necessary optimality conditions were derived for sandwich beams and beams with arbitrary cross-section. Following the notations of the present work the optimal thickness distribution corresponds to

$$h = \max_P \Phi(|M|, N). \qquad (4.4)$$

In (4.4) the piece-wise differentiable function Φ is defined by the yield condition, which now is represented as $h \geqslant \Phi$.

4.4. Parametrical optimization

A set of parametrical optimization problems discussed above admit proper extension to geometrically non-linear structures. The preliminary unknown constant parameters subjected to variation could specify the external load distribution, the cross-sectional area of the structure, layount of the reinforcement or non-homogeneity of the material, support conditions or other factors, which influence the post-yield behaviour of the structure.

For a problem of this kind the optimality criterion may be represented as

$$I = G(p, s, h) + \int_S F(P, H, W, U)\,dS \qquad (4.5)$$

where W,U stand for the displacements, P and H are functions but p,h and s are certain parameters which prescribe, respectively, the load distribution, the thickness or cross-sectional area and the location of additional supports. The parameters may be scalars as well as vectors depending on the formulation of the problem under consideration. Note that the functional (4.5) is a particular case of (4.1)(now $F_0 = 0$).

By minimizing the functional (4.5) one has to take into account some additional requirements

$$\int_S f_i(P, H, W, U)\,dS = A_i \qquad (4.6)$$

and

$$g_j(p, h, s, W(x_k), U(x_k) \leq B_j\,, \qquad (4.7)$$

where A_i and B_j are given constants. A number of constraints (4.6) and (4.7) may be given in the form of equalities. To distiguish the equalities and inequalities is certainly essential from the mathematical point of view, but in the present paper the details of derivation of optimality conditions will be omitted. Thus, this refinement is not necessary herein.

The parametrical approach to optimal design of plastic beams in the post-yield range was developed by Lellep, 1981 a, 1982 b,c. Necessary optimality conditions for the problem of types (4.5) - (4.7) were derived by means of the variational methods of the optimal control theory (see the books by Bryson and Ho, 1969 and Troickii, 1976, Gabasov and Kirillova, 1974). It appeared that the optimal trajectory in the state space which corresponds to the optimal solution of the problem comprises singular as well as usual subarcs (see Gabasov and Kirillova, 1973; Bell and Jacobson, 1975).

An example of the paper by Lellep, 1982 c refers to optimal location of an additional support to the beam clamped at the left-hand end and simply supported at the right end. The sandwich beam of constant cross-section is treated which carries the uniform transverse pressure P and an axial dead-load N. The mean deflection is minimized under the condition that the transverse loading is large enough to generate plastic deformations in both parts of the beam (the latter is divided into two regions by the additional support). This criterion is a particular case of (4.5) associated with $G = 0$, $F = W$. A simple expression was obtained for the optimal layout of the additional support

$$s = \frac{L}{2} + \frac{M_0(N_0 - N)}{PLN_0} \qquad (4.8)$$

where L denotes the length of the beam and M_0, N_0 are respectively, the yield moment and yield load. It should be noted that (4.8) holds good if the load is considerably larger than the limit load.

The optimization technique discussed above was extended to plastic cylindrical shells by Lellep, 1983 b; 1985 a, b. The general theory of optimal design of plastic sandwich shells is developed by Lellep, 1985 b,c assuming that the material obeys the Tresca yield condition and taking into account moderately large deflections. For the sake of simplicity the deformation-type theory of plasticity was employed and the attention is restricted to the short shells. The shells made of a fiber reinforced material were considered by Lellep, 1989 and shells of Von Mises material by Lellep and Hannus, 1988. Two approaches concerning the optimization for prescribed deflected shape and parametrical optimization, respectively, are concerned from the common point of view by Lellep and Sawczuk, 1984; 1987.

Optimal designs for non-homogeneous plastic beams and the beams of piece-wise constant thickness are established by Lellep, 1989 a; Lellep and Majak, 1985; 1988 b. Plastic cylindrical shells of piece-wise constant wall thickness are studied by Lellep and Hannus, 1989 in the case of Von Mises material.

4.5. Minimum weight design in the case of smooth yield surfaces

Minimum weight design of the circular plates for the given collapse load is studied by Freibergar and Tekinalp, 1956; Eason, 1960; Reiss and Megarefs, 1971; Pungar, 1972. Geometrically non-linear annular and circular plates, the material of which obeys the Von Mises yield condition are concerned by Lellep and Majak, 1989. A numerical method is developed which enables to define the minimum weight designs for given maximum deflection.

Cylindrical shells subjected to the transverse pressure were studied in the geometrically linear form, i.e. for given collapse load by Freiberger, 1957; Reiss and Megarefs, 1969; Shulgin, 1984. Geometrical non-linearity has been taken into account by Lellep and Majak, 1988; 1990. In the first paper the shells subjected to the transverse pressure and to the axial tension were considered whereas the latter is devoted to the case when the axial force is generated as the reaction of the supports. In this case the axial displacement vanishes at the edge points of the shell.

CHAPTER I

THEORY OF OPTIMIZATION OF PLASTIC BEAMS

§1.1. Problem formulation

1.1.1. Preliminary remarks and notation

Let us consider a rigid-plastic beam of length L which is subjected to the action of distributed transverse loading of intensity P and to an axial dead-load N (Fig.1.1.1).

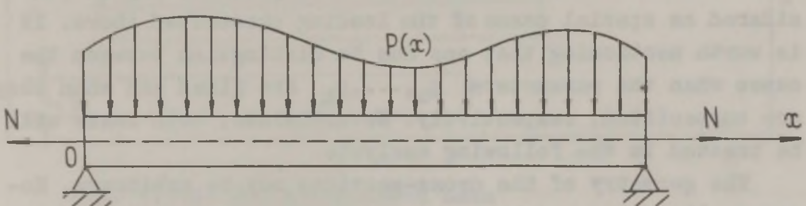

Fig. 1.1.1. Beam subjected to the lateral pressure and axial tension

The ends of the beam may be fixed in different manner; the support conditions will be specified by solving particular problems.

The loading is assumed to be quasi-statical, whereas the axial tension is smaller than the corresponding limit load N_0. If $N = N_0$, then the stress-state of the beam undergoes to the membrane state and shape optimization is no more possible.

The displacements of the beam are assumed to be finite (not exceeding the order of the thickness of the beam), whereas the deformations are small. Thus, the deformation components and equilibrium equations are non-linear, they correspond to the non-linear shell theory of Von Karman. Since the concept of an ideally rigid-plastic body will be used, the load intensity P must exceed the load-carrying capacity

P_o. The quantity P_o will be determined in special cases. It depends on the yield stress of the material (or yield stresses when different materials are utilized),on the geometry of cross-sections,on the load distribution,as well as on the support conditions of the ends the beam. The limit load may be evaluated by using the corresponding lower and higher bound theorems of the limit analysis.

The load distribution is not necessarily a uniform one. Let $P = P(x, p_o,\ldots,p_K)$, where $p_o,\ldots,p_K$ stand for preliminarily unknown constant parameters. The coordinate axis Ox coincides with the axis of the beam in the non-deformed state with the origin coinciding with the left-hand end of the beam. The function $P(x,p_o,\ldots,p_K)$ is assumed to be a piece-wise continuous function. Thus, the piece-wise constant load distribution and the uniform pressure may be considered as special cases of the loading prescribed above. It is worth mentioning that one has to distinguish between the cases when the parameters $p_o,\ldots,p_K$ are fixed and when they are unspecified, respectively. Nevertheless, both cases will be treated in the following analysis.

The geometry of the cross-sections may be arbitrary. However, it is assumed that the cross-sections are of a common configuration and the geometrical sizes are either constants or continuously variable quantities in each interval $D_j = (a_j, a_{j+1})$, where $j = 0,\ldots,K$ and $a_o = 0, a_{k+1} = L$. The geometrical dimensions of the configurations of the cross-sections of the beam are specified by the coordinate x and parameters $h_1,\ldots,h_m$, which may be previously fixed or unfixed quantities.

The area of the cross-section of the beam may be presen-

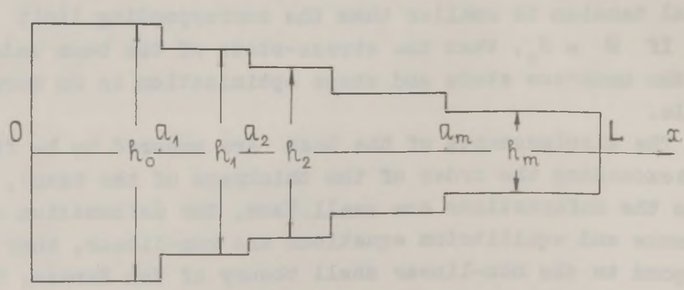

Fig. 1.1.2. Beam of piece-wise constant thickness

ted as $S = S_j(x, H_j, h_1, \ldots, h_m)$ for $x \in D_j$. Here $H_j = H_j(x)$ stands for an unknown differentiable function, whereas S_j is the given function. Thus, the function S is piece-wise differentiable with respect to its arguments.

The parameters $h_1, \ldots, h_m$ may be interpreted as different thicknesses in the case of a beam of piece-wise constant thickness (Fig. 1.1.2). But $h_1, \ldots, h_m$ may stand for the dimensions of different layers in the cases of non-homogeneous reinforced or layered beams (Fig. 1.1.3).

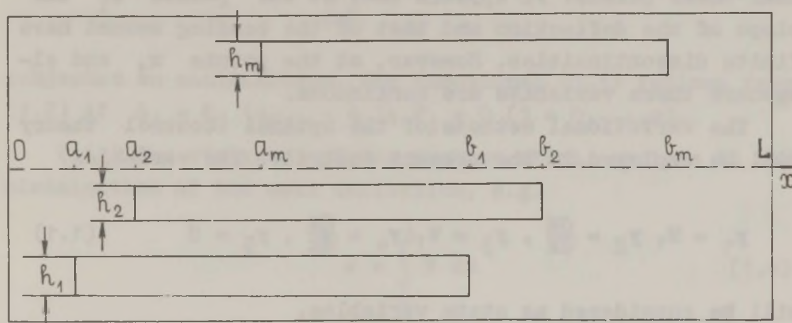

Fig. 1.1.3. Non-homogeneous beam

Generally, these are the parameters denoting arbitrary geometrical dimensions of physical constants of the structure under consideration.

Let us assume that at the points $x = s_1, \ldots, x = s_n$ additional rigid supports are located (Fig. 1.1.4).

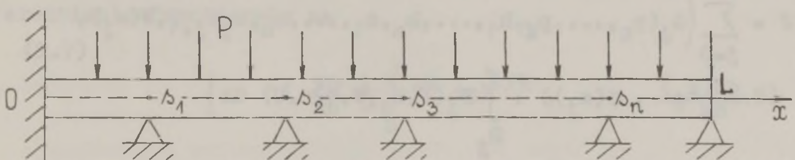

Fig. 1.1.4. Beam with additional supports

The parameters s_j $(j = 1, \ldots, n)$ may be preliminarily fixed or unfixed parameters. In the latter case the problem consists in the determination of the quantities s_j so that the optimality criterion obtains the minimal value.

Let $x_j \in D_j$ $(j = 0,\ldots,K)$ denote the coordinates of the cross-sections of the beam where certain restrictions are imposed on the stress-strain state of the beam. At $x = x_j$ the bending moment M, the transverse and axial displacements W, U as well as the slopes of these variables may be restricted. The numbers K, m, n are fixed integers.

The need for different notation of the corresponding points at s_i $(i = 1,\ldots,n)$ and x_j $(j = 0,\ldots,K)$ springs from the different behavior of stresses and displacements near these points. It appears that at the points s_i the slope of the deflection and that of the bending moment have finite discontinuities. However, at the points x_j and elsewhere these variables are continuous.

The variational methods of the optimal control theory will be employed in the present analysis. The variables

$$y_1 = M,\ y_2 = \frac{dM}{dx},\ y_3 = W,\ y_4 = \frac{dM}{dx},\ y_5 = U \qquad (1.1)$$

will be considered as state variables.

1.1.2. Optimality criteria

In the present work an optimization theory will be developed for plastic structures which takes into account the post-yield point behavior. It is assumed that the optimization criterion is a differentiable functional.Thus, the total cost of the design is expressed as

$$J = \sum_{j=0}^{K} \Big\{ G_j(p_0,\ldots,p_K,h_1,\ldots,h_m,s_1,\ldots,s_n,a_j a_{j+1},W(x_j), \frac{dW(x_j)}{dx}, U(x_j)) + \int_{D_j} F_j(P,S_j,W,\frac{dW}{dx},U)\, dx \Big\} \qquad (1.2)$$

where the functions F_j and G_j are given differentiable functions.

Let us consider some particular cases of the functional (1.1).

(i) The cost functional corresponding to the minimization of the maximal deflection at the center of the beam may be expressed as

$$J = W\left(\frac{L}{2}\right)$$

provided both ends are fixed in the same manner. Evidently, this is a particular case of (1.2), associated with $G_0 = W(x_0)$, $G_j = 0$, $j \neq 0$, $x_0 = L/2$, $F_j = 0 (j = 1,\dots,m)$.

(ii) In the case of weight minimization of the beam of piece-wise constant thickness one has cost function

$$J = \sum_{j=0}^{m} h_j(a_{j+1} - a_j) \tag{1.3}$$

subjected to minimization. The functional (1.3) follows from (1.2) if $G_j = h_j(a_{j+1} - a_j)$; $F_j = 0 \ (j = 0,\dots,m)$.

(iii) The statement of the problem which consists in the minimization of the mean deflection, e.g.

$$J = \int_0^L W \, dx \tag{1.4}$$

is a particular case of (1.2) associated with $G_j = 0 (j = 0, \dots,m)$; $F_j = W$. It is worth mentioning that (1.4) may be considered as a linear approximation of the non-differentiable functional

$$J = \max_x W \tag{1.5}$$

where the maximum is calculated with respect to $x \in [0, L]$.

It was indicated by Banichuk (1980, 1986) that the functional of type (1.5) may be approximated in the class of differentiable functionals as

$$J = \left(\frac{1}{L} \int_0^L W^k \, dx \right)^{1/k}, \tag{1.6}$$

where k is a positive number. If $k \to \infty$ then (1.6) tends to (1.5). Evidently, (1.6) coincides with (1.4) if $k = 1$.

1.1.3. Additional restrictions

The minimum of the cost criterion is sought for under the condition that the optimal solution satisfies the basic equations of the geometrically non-linear theory of plastic beams as well as the additional constraints. The additional constraints may be divided into two groups. These are the state constraints which are imposed on the stress-strain state of the beam at each point of the optimal trajectory and local constraints which apply at singular points, respectively.

The state constraints are assumed to be expressed as

$$R_i \left(P, S, W, \frac{dW}{dx}, U\right) \leq 0; \; i = 1,\ldots,r \tag{1.7}$$

and

$$\sum_{j=0}^{K} \int_{D_j} S_{io}\left(P, S_j, W, \frac{dW}{dx}, U\right) dx = A_i; \; i=1,\ldots,s \tag{1.8}$$

where the functions R_i $(i = 1,\ldots,r)$ are given continuous and differentiable functions. The functions $S_{io}(i = 1,\ldots,s)$ are assumed to be piece-wise differentiable functions whereas A_i stand for given constants. The numbers r and s are fixed integers.

Particular cases of (1.7) are, for instance, the constraints

$$W - W_o \leq 0 \tag{1.9}$$

and

$$H - H_o \leq 0 \tag{1.10}$$

which impose the upper bounds to the deflection and to the thickness, respectively. The quantities W_o and H_o in (1.9) and (1.10) are given constants.

Relations (1.8) present these constraints, which restrict the quantities of integral type. For instance, in the case of problems with a given volume (mass) of the beam one has

$$\int_0^L S(H(x))\,dx = A_1 . \tag{1.11}$$

Similarly, the assumption that the mean deflection is given leads to the constraint

$$\int_0^L W dx = A_2 . \tag{1.12}$$

Evidently, the integral constraints (1.11) and (1.12) are the particular cases of (1.8), associated with $S_{10} = S$ and $S_{20} = W$, respectively.

The local constraints may be presented as inequalities

$$f_{ij}(p_0,\dots,p_k,h_1,\dots,h_m,\ x_j,\ M(x_j)\ \frac{dM(x_j)}{dx},\ W(x_j),$$
$$\frac{dW(x_j)}{dx},\ U(x_j)) \leq 0;\ i = 1,\dots,f_{ji}; j \in K_f \tag{1.13}$$

and equalities

$$g_{ij}\,(p_0,\dots,p_k,h_1,\dots,h_m,\ s_1,\dots,s_n,\ M(s_j),\ \frac{dM(s_j^{\pm})}{dx},$$
$$\frac{dW(s_j^{\pm})}{dx},\ U(s_j^{\pm})) = 0;\ i = 1,\dots,g_j;\ j = 1,\dots,n . \tag{1.14}$$

Here the set K_f is a subset of integers $0,\dots,K$. It may also be an empty set. In (1.13) and (1.14) f_{ij} and g_{ij} stand for given differentiable functions; numbers f_j, g_j, K, n are specified. As a rule, these numbers are arbitrary. The only limitation is that $g_j \leq K + m + n + 8$ for each $j = 1,\dots,n$, provided the functions g_{ij} depend upon the variables indicated in (1.14). If this inequality is violated, then the number of unknown quantities in (1.14) is smaller than the number of equations, and thus the equations may be contradictory ones.

It is assumed that (1.13) and (1.14) express intermediate conditions, only. The boundary conditions at $x = 0$ and $x = L$ will be specified later.

If, for instance, at $x = s_j$ ($j = 1,\dots,n$) the additional rigid supports are located, one has

$$W(s_j) = 0, \quad M(s_j) = M_* \; ; \quad j = 1,\dots,n \tag{1.15}$$

where the quantity M_* is to be determined according to the yield condition. Thus, in the present case the requirements (1.14) take the form $q_{1j} = W(s_j)$ and $g_{2j} = M_*$.

§1.2. Basic equations

1.2.1. Equilibrium equations and geometrical relations

In the present study the theory of moderately large deflections will be used. Within this concept, the post-yield behavior of structures can be properly treated when the changes in geometry are not large and the transverse displacements do not exceed the order of the thickness.

The equilibrium equations of the beam element have the form

$$\frac{d^2M}{dx^2} + N_1 \frac{d^2W}{dx^2} + P = 0, \quad \frac{dN_1}{dx} = 0 \tag{2.1}$$

where N_1 stands for the axial force. According to the second equation in (2.1), the axial force is constant along the span of the beam. Thus, $N_1 = N$, where N stands for the value of the axial force at the end of the beam. The quantity N may be a fixed as well as an unfixed parameter, depending on the statement of the problem under consideration.

The deformation components may be expressed as

$$\varepsilon = \frac{dU}{dx} + \frac{1}{2}\left(\frac{dW}{dx}\right)^2, \quad \varkappa = -\frac{d^2W}{dx^2} \tag{2.2}$$

where ε and $\varkappa$ stand for the elongation and curvature of the neutral axis of the beam, respectively.

1.2.2. Yield condition and associated deformation law

The material of the beam is assumed to be ideally rigid-plastic (without strain-hardening). Within the limits of the concept of a rigid-plastic body the experimental sample remains rigid in one-dimensional stress state until the stress is less than the yield stress σ_0 . Thus, in the one-dimensional case the yield condition is $|\sigma| - \sigma_0 \leq 0$.

The yield condition may be expressed in terms of generalized stresses, e.g. via axial force and bending moment. The form of the yield curve depends upon the yield stress of the material as well as on the geometry of the cross-sections of the beam. Generally, the yield curve of a beam is a piecewise smooth closed curve. The area surrounded by the curve comprises the origin of coordinates.

Let us denote the equation of the yield curve for $x \in D_j$ by $\Phi_j = 0$ in the plane of the membrane force and bending moment. Thus, the yield (plasticity) condition may be presented as

$$\Phi_j \, (M, N_1 H_j, h_1, \ldots, h_m) \leq 0 \qquad (2.3)$$

for $x \in D_j$; $j = 0, \ldots, K$. In (2.3) Φ_j stands for a piece-wise differentiable function.

For instance, in the case of a cross-section of sandwich type one has (Fig. 1.1.5)

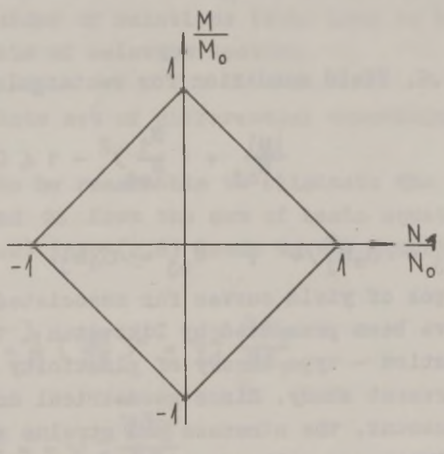

Fig. 1.1.5. Yield condition for sandwich cross-section

$$\frac{|M|}{M_{oj}} + \frac{|N_1|}{N_{oj}} - 1 \leqslant 0 \qquad (2.4)$$

where $M_{oj} = \sigma_o H_o h_j$, $N_{oj} = 2\,\sigma_o h_j$, σ_o being the yield stress. Here h_j stands for the thickness of carrying layers in the region D_j, whereas H_o is the total thickness of the beam. Similarly, in the case of a beam of a homogeneous rectangular cross-section, the yield curve may be presented as (Fig. 1.1.6)

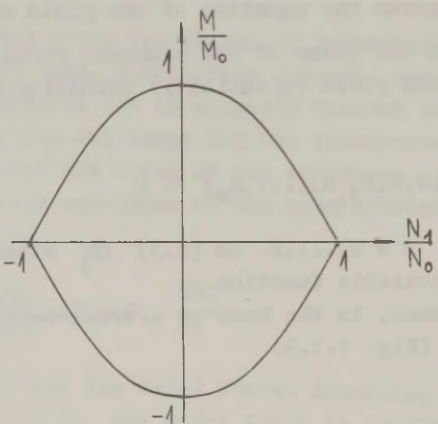

Fig. 1.1.6. Yield condition for rectangular cross-section

$$\frac{|M|}{M_{oj}} + \left(\frac{N_1}{N_{oj}}\right)^2 - 1 \leqslant 0 . \qquad (2.5)$$

In (2.5) $M_{oj} = \sigma_o h_j^2/4$, $N_{oj} = \sigma_o h_j$.

Different types of yield curves for associated types of cross-sections have been presented by Dikovitch [1967].

A deformation – type theory of plasticity will be utilized in the present study. Since geometrical non-linearity is taken into account, the stresses and strains are coupled. The relations between generalized stresses and strain components are furnished by the associated deformation law, which states that the vector with strain components (2.2) is directed along the outward normal to the yield curve (2.3). Inside the region determined by (2.3) the strain vector must vanish (corresponding to these points the zone of the beam remains rigid).

The normality rule yields the relations

$$\varepsilon = \nu_j^2 \frac{\partial \Phi_j}{\partial N_1}, \quad \varkappa = \nu_j^2 \frac{\partial \Phi_j}{\partial M_1} \tag{2.6}$$

for $x \in D_j$ provided $\Phi_j = 0$. If $\Phi_j < 0$ then $\varepsilon = \varkappa = 0$. In (2.6) ν_j^2 stands for an unspecified scalar multiplier. Stipulating $\nu_j = 0$ for $\Phi_j < 0$ e.g. assuming that

$$\nu_j \Phi_j = 0 \tag{2.7}$$

one can employ the equations (2.6) in the plastic regions (associated with $\Phi_j = 0$) as well as in rigid domains ($\Phi_j < 0$).

It is worth noting that the gradientality rule in its original form (2.6) is applied at the regular points of the yield curve. At the points of intersection of smooth curves the strain vector is to be specified as an arbitrary positive linear combination of the normal vectors for the adjacent parts of the curve at the non-regular point. In this case the right-hand sides of relations (2.6) have to be handled as scalar products of relevant vectors.

1.2.3. Complete set of differential constraints

It appears to be reasonable to eliminate the deformation components ε and $\varkappa$ from the set of basic equations. Substitution of (2.2) into (2.6) leads to the equations

$$\frac{dU}{dx} + \frac{1}{2}\left(\frac{dW}{dx}\right)^2 = \nu_j^2 \frac{\partial \Phi_j}{\partial N_1}$$
$$\frac{d^2W}{dx^2} = -\nu_j^2 \frac{\partial \Phi_j}{\partial M} \tag{2.8}$$

for $x \in D_j$ $(j = 0,\dots,K)$, provided $\nu_j = 0$ if $\Phi_j < 0$.

Making use of the notation (1.1) one can present (2.1) and (2.8) as

$$
\begin{aligned}
y_1' &= y_2 , \\
y_2' &= \vartheta_j^2 N_1 \frac{\partial \Phi_j}{\partial y_1} - P, \\
y_3' &= y_4 , \qquad\qquad\qquad\qquad\qquad\qquad (2.9) \\
y_4' &= - \vartheta_j^2 \frac{\partial \Phi_j}{\partial y_1} , \\
y_5' &= - \frac{1}{2} y_4^2 + \vartheta_j^2 \frac{\partial \Phi_j}{\partial N_1}
\end{aligned}
$$

for $x \in D_j$. Here primes denote the differentiation with respect to x. The quantity N_1 is to be considered as a constant (with respect to x) parameter.

Evidently, $y_1,\ldots,y_5$ in (2.9) must be referred to as the state variables but ϑ_j as a control function (see Bryson and Ho, 1969, Pontryagin et al., 1962).

The state variables have to satisfy certain boundary conditions imposed at the points $x = 0$ and $x = L$. The boundary requirements depend on the type of supports. Since the strict boundary values of the state variables may be unspecified at this stage of the solution, we assume that

$$y_i(0) = y_{oi}, \quad i \in I_o \qquad (2.10)$$

and

$$y_j(L) = y_{Lj} , \quad j \in I_L . \qquad (2.11)$$

In (2.10), (2.11) I_o and I_L stand for certain subsets of the set of integers 1, 2, 3, 4, 5; whereas y_{oi} and y_{Lj} denote the given constants. If, for instance, the left end of the beam is simply supported, one has $y_1(0) = y_3(0) = 0$. Thus, $I_o = \{1,3\}$ in this case.

Alternatively, in the case of the built in right-hand end of the beam, the corresponding boundary conditions are $y_1(L) = M_*$, $y_3(L) = 0$ where M_* is the value of the bending moment which corresponds to plastic hinge and is to be determined from (2.3). Consequently, $I_L = \{1,3\}$ similarly to the case of the hinged end. However, in the case of the absolutely free end $I_L = \{1,2\}$, since now $y_1(L) = y_2(L) = 0$.

§1.3. Necessary optimality conditions

1.3.1. Order of the state constraints

The posed problem is a control problem with state variable inequality constraints. It appears that the form of the extended functional depends upon the order of the state constraints (1.7) and (2.3) (see Berkovitz and Dreyfus, 1965; Troickii, 1976; Speyer and Bryson, 1968; Jacobson, Lele and Speyer, 1971; Kreindler, 1982.

We say that a state constraint $F(x,y_1,\dots,y_5) \leq 0$ is the constraint of order k if the derivatives $F', F'', \dots, F^{(k-1)}$ do not depend explicitly on control functions but $F^{(k)}$ does depend. The differentiation must be performed by the sample

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \sum_{i=1}^{5} \frac{\partial F}{\partial y_i} y_i' \tag{3.1}$$

where the derivatives y_i' have to be substituted with the help of (2.9). If, for instance, F depends only on x and y_1 then according to (3.1) and (2.9) $F' = F'(x, y_1, y_2)$. Therefore, the order of this constraint is not equal to one. Evidently, it equals two in this case.

Let us consider the state constraints (1.7) in greater detail. Let us assume that the equality sign applies in (1.7) and consequently we have

$$R_i(P, S, y_3, y_4, y_5) = 0; \; i = 1,\dots,r \tag{3.2}$$

in the regions $(\beta_{ij}, \beta_{ij}^1)$, where $j = 0,\dots,K$.

Differentiation of (3.2) with respect to x making use of (2.9) leads to

$$\frac{\partial R_i}{\partial P}\frac{dP}{dx} + \frac{\partial R_i}{\partial S}\frac{dS}{dx} + \frac{\partial R_i}{\partial y_3} y_4 + \frac{\partial R_i}{\partial y_5}\Big(-\frac{1}{2}y_4^2 + \\ + \nu_j^2 \frac{\partial \Phi_j}{\partial N_1}\Big) - \frac{\partial R_i}{\partial y_4}\nu_j^2 \frac{\partial \Phi_j}{\partial y_1} = 0 \tag{3.3}$$

for $x \in (\beta_{ij}, \beta_{ij}^1)$. It yields from (3.3) that the const-

7

raints $R_i \leqslant 0$ generally, are of order one, since the first derivative of R_i comprises the controls ν_j expicitly. If, however

$$\frac{\partial R_i}{\partial y_5}\ \frac{\partial \Phi_j}{\partial N_1} - \frac{\partial R_i}{\partial y_4}\ \frac{\partial \Phi_j}{\partial y_1} = 0 \qquad (3.4)$$

then (3.3) takes the form

$$\frac{\partial R_i}{\partial P} P' + \frac{\partial R_i}{\partial S} S' + \frac{\partial R_i}{\partial y_3} y_4 - \frac{1}{2} \frac{\partial R_i}{\partial y_5} y_4^2 = 0 \qquad (3.5)$$

and thus, R_i' does not depend explicitly on controls. Differentiation of (3.5) with the help of (2.9) shows that R''_i does depend on the controls. Thus, the constraints $R_i \leqslant 0$ are the second order constraints in this case.

It is worth noting that (3.4) is satisfied if $\partial R_i / \partial y_5 = \partial R_i / \partial y_4 = 0$. Therefore, the order of (1.7) is equal to two when R_i does not depend on W' and U.

Now let us study the constraints (2.3). We assume that the equality sign in (2.3) applies in a subdomain of D_j. Let

$$\Phi_j(y_1, N_1, H_j, h_1, \ldots, h_m) = 0 \qquad (3.6)$$

for $x \in (a_{oj}, a_{oj}^1)$, where $a_{oj} \in D_j$ and $a_{oj}^1 \in D_j$.

Differentiation of (3.6) with respect to x gives

$$\frac{\partial \Phi_j}{\partial y_1} y_2 = 0 \qquad (3.7)$$

for $x \in (a_{oj}, a_{oj}^1)$. Since Φ_j depends on the bending moment, generally, $\partial \Phi / \partial y_1 \neq 0$. Therefore, (3.7) yields

$$y_2 = 0 \qquad (3.8)$$

for $x \in (a_{oj}, a_{oj}^1)$.

Differentiating (3.8) and making use of (2.9) one obtains

$$\nu_j^2 N_1 \frac{\partial \Phi_j}{\partial y_1} - P = 0 \qquad (3.9)$$

for $x \in (a_{oj}, a_{oj}^1)$. It follows from (3.7) - (3.9) that the state constraints (2.3) are of the second order.

1.3.2. Extended functional

In order to derive the necessary optimality conditions, one has to compile an augmented (extended) functional introducing the corresponding Lagrangian multipliers. This functional has to take into account the cost criterion (1.2)and additional constraints (1.7), (1.8), (1.13), (1.14),as well as the state constraints (2.3), differential constraints (2.9) and boundary requirements (2.10), (2.11). Assuming that (1.7) or (3.2) are the first order constraints,the following functional will be used

$$J_* = \sum_{j=0}^{K} \Big\{ G_j + \int_{D_j} \Big(\sum_{i=1}^{5} \psi_i y_i' - \mathcal{L}_j \Big) dx + \varrho_{oj} \Phi_{oj} +$$

$$+ \varrho_{2j} y_2(a_{oj}) + \sum_{i=1}^{r} \nu_{ij} R_{ij}^{o} \Big\} + \sum_{i=1}^{g_j} \sum_{j=1}^{n} \lambda_{ij} g_{ij} + \tag{3.10}$$

$$+ \sum_{i\in I_o} \eta_i (y_i(0) - y_{oi}) + \sum_{i\in I} \varrho_i (y_i(L) - y_{Li}) +$$

$$+ \sum_{j\in K_f} \sum_{i=1}^{f_j} \mu_{ij}(f_{ij} + r_{ij}^2).$$

In (3.10) ϱ_{oj}, μ_{ij}, ϱ_{2j}, ν_{ij}, λ_{ij}, η_i, ϱ_i stand for unknown Lagrangian multipliers which are assumed to be constants whereas $\psi_i (i = 1,\ldots,5)$ are the so-called adjoint variables. The quantities r_{ij} are constant parameters which meet the requirements

$$f_{ij} + r_{ij}^2 = 0; \quad i = 1,\ldots,f_j; \quad j \in K_f . \tag{3.11}$$

Here Φ_{oj} and R_{ij}^{o} stand for the left sides of (3.6) and (3.2) evaluated at a_{oj} and b_{ij}, respectively, e.g.

$$\Phi_{oj} = \Phi_j(y_1(a_{oj}), N_1, H_j, h_1,\ldots,h_m) \; ; \; j = 0,\ldots,K \tag{3.12}$$

$$R_{ij}^{o} = R_i(P(b_{ij}), S(b_{ij}), y_3(b_{ij}), y_4(b_{ij}), y_5(b_{ij})); \quad i = 1,\ldots,r.$$

The Lagrangian function $\mathcal{L}_j$ in (3.10) is defined as

$$\mathcal{L}_j = H_j + \sum_{i=1}^{r} \varphi_{ij}^1 R_{ij}^1 + \varphi_{ij}(\nu_j^2 N_1 \frac{\partial \Phi_j}{\partial y_1} - P) \qquad (3.13)$$

where $j = 0,\dots,K$ and

$$H_j = - F_j + \psi_1 y_2 + \psi_2(\nu_j^2 N_1 \frac{\partial \Phi_j}{\partial y_1} - P) +$$

$$+ \psi_3 y_4 - \psi_4 \nu_j^2 \frac{\partial \Phi_j}{\partial y_1} + \qquad (3.14)$$

$$+ \psi_5(\nu_j^2 \frac{\partial \Phi_j}{\partial N_1} - \frac{1}{2} y_4^2) + \sum_{i=1}^{s} \psi_{oi} S_{io}.$$

In (3.13) and (3.14) R_{ij}^1 stand for the left-hand sides of equations (3.3), whereas φ_{ij}^1 and φ_{ij} denote preliminarily unknown multipliers. These should be certain functions of coordinate x, but ψ_{oi} are presumably constants. The latter is due to the isoperimetric nature of the constraints (1.8) (see Pontryagin et al. 1962; Troickii, 1976).

1.3.3. Total variation of the functional

For optimality of the solution, it is necessary that the total variation of the functional (3.10) should be equal to zero. The variations of the state variables at unspecified positions will be determined by the following sample (see Troickii 1976, Sage and White, 1977)

$$\Delta y\ (s \pm 0) = \delta y(s \pm 0) + \frac{dy}{dx}\Big|_{x = s\pm} \cdot \Delta s. \qquad (3.15)$$

Here δy stands for the variation of the state variable y which is due to the variation of the trajectory in the state space, whereas $\Delta y(s+)$ and $\Delta y(s-)$ denote the total variations of y, Δs being the increment of the parameter s. If the variable y must be continuous at $x = s$ then, of course, $\Delta y(s-) = \Delta y(s+)$. While

$$y(s \pm) = \lim_{x \to s\pm} y(x) .$$

Employing the rule (3.15) in the case of points whose coordinates are assumed to be fixed, one easily obtains

$$\begin{aligned} \delta y_i(0) &= \Delta y_i(0) \ ; \quad i = 1,\ldots,5; \\ \delta y_i(L) &= \Delta y_i(L) \end{aligned} \tag{3.16}$$

However, at $x = a_j$, $x = a_{oj}$, $x = b_{ij}$, $x = x_j$ one has

$$\begin{aligned} &\Delta y(a_j) = \delta y(a_j) + y'(a_j)\ \Delta a_j \ ; \quad j = 0,\ldots,K \\ &\Delta y(a_{oj}) = \delta y(a_{oj}) + y'(a_{oj})\ \Delta a_{oj} \\ &\Delta y(b_{ij}) = \delta y(b_{ij}) + y'(b_{ij})\ \Delta b_{ij}; \ i=1,\ldots,r; \ j \in K \\ &\Delta y\ (x_j) = \delta y(x_j) + y'(x_j)\ \ \Delta x_j \ ; \quad j \in K_f \end{aligned} \tag{3.17}$$

where the variable y may be replaced by any of the state variables $y_1,\ldots,y_5$. At these intermediate points of the optimal trajectory all state variables are assumed to be continuous.

Since at $x = s_j$ $(j = 1,\ldots,n)$ the state variables y_2, y_4, y_5 have discontinuities, one has to take into account that

$$\begin{aligned} &\Delta y_i(s_j \pm) = \delta y_i(s_j \pm) + y_i'(s_j \pm)\ \Delta s_j \\ &\qquad\qquad i = 2,\ 4,\ 5; \quad j = 1,\ldots,n\ . \end{aligned} \tag{3.18}$$

On the other hand, y_1 and y_3 are fixed at $x = s_j$. Therefore, $\Delta y_i(s_j+) = \Delta y_i(s_j-) = 0$ for $i = 1,\ 3$ and according to (3.15)

$$\begin{aligned} &\delta y_i(s_j \pm) = -\ y_i'\ (s_j \pm)\ \Delta s_j; \quad i = 1,3; \\ &\qquad\qquad\qquad\qquad j = 1,\ldots,n. \end{aligned} \tag{3.19}$$

It is worth noting that despite y_1 and y_3 have not any discontinuities at $x = s_j$, δy_1 and δy_3 are not continuous at these points. Moreover, certain adjoint coordinates may have finite jumps at the points $s_j (j = 1,\ldots,n)$ as well as a_{oj}, a_j, x_j $(j = 0,\ldots,K)$ and b_{ij} . Therefore, for each ψ_k and y_k

$$\int_0^L \delta(\psi y')dx = -\sum_{j=0}^{K}\Big\{\int_{D_j}\psi'\delta y\,dx + [\psi(a_{oj})]\,\delta y(a_{oj}) +$$

$$+\sum_{i=1}^{r}[\psi(b_{ij})]\,\delta y(b_{ij})\Big\} - \sum_{j\in K_f}[\psi(x_j)]\,\delta y(x_j) -$$

$$-\sum_{j=1}^{K}[\psi(a_j)]\,\delta y(a_j) + \sum_{j=1}^{n}(\psi(s_j-)\,\delta y(s_j-) - \tag{3.20}$$

$$-\psi(s_j+)\,\delta y(s_j+)) + \psi(L)\,\delta y(L) - \psi(0)\,\delta y(0).$$

In (3.20) square brackets denote the finite discontinuities, e.g.

$$[\psi(s)] = \psi(s+) - \psi(s-). \tag{3.21}$$

Performing the variation of (3.10), making use of (3.12)-(3.14) and (3.16) - (3.20) one obtains the total variation of the cost criterion which may be presented as

$$\Delta J_* = \Delta G + \Delta Y + \Delta J_o + \sum_{j\in K_f}\sum_{i=1}^{f_j} 2\mu_{ij}r_{ij}\,\Delta r_{ij} = 0\,. \tag{3.22}$$

Here

$$J_o = \sum_{j=0}^{K}\int_{D_j}\Big(\sum_{i=1}^{5}\psi_i\,y_i' - \mathcal{L}_j\Big)\,dx \tag{3.23}$$

and

$$G = \sum_{j=0}^{K}G_j + \sum_{j\in K_f}\sum_{i=1}^{f_j}(\mu_{ij}\,f_{ij}) + \sum_{j=1}^{n}\sum_{i=1}^{g_j}\lambda_{ij}g_{ij}$$

$$Y = +\sum_{i\in I_o}\eta_i(y_i(0) - y_{oi}) + \sum_{i\in I_L}\varrho_i(y_i(L) - y_{Li}) + \tag{3.24}$$

$$+\sum_{j=0}^{K}(\varrho_{oj}\,\Phi_{oj} + \varrho_{2j}y_2(a_{oj}) + \sum_{i=1}^{r}\nu_{ij}\,R_{ij}^{o}).$$

The first term in (3.22) becomes

$$\Delta G = \sum_{i=1}^{m} \frac{\partial G}{\partial h_i} \Delta h_i + \sum_{i=1}^{n} \frac{\partial G}{\partial s_i} \Delta s_i +$$

$$+ \sum_{i=0}^{K} \frac{\partial G}{\partial p_i} \Delta p_i + \sum_{j=1}^{K} \frac{\partial G}{\partial a_j} \Delta a_j + \tag{3.25}$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{g_j} \lambda_{ij} \sum_{k=1}^{5} \Big(\frac{\partial g_{ij}}{\partial y_k(s_j-)} \Delta y_k(s_j-) +$$

$$+ \frac{\partial g_{ij}}{\partial y_k(s_j+)} \Delta y_k(s_j+)\Big) + \sum_{j\in K_f} \frac{\partial G}{\partial x_j} \Delta x_j +$$

$$+ \sum_{j\in K_f} \sum_{i=1}^{f_j} \mu_{ij} \sum_{k=1}^{5} y_k(x_j) .$$

Similarly, the second term in (3.22) yields

$$\Delta Y = \sum_{i=1}^{m} \frac{\partial Y}{\partial h_i} \Delta h_i + \sum_{i=1}^{n} \frac{\partial Y}{\partial s_i} \Delta s_i +$$

$$+ \sum_{i=0}^{K} \frac{\partial Y}{\partial p_i} \Delta p_i + \sum_{i\in I_0} \eta_i \Delta y_i(0) + \tag{3.26}$$

$$+ \sum_{i\in I_L} \varrho_i \Delta y_i(L) + \sum_{j=0}^{K} \Big(\varrho_{0j} \frac{\partial \Phi_{0j}}{\partial y_1(a_{0j})} \Delta y_1(a_{0j}) +$$

$$+ \varrho_{2j} \Delta y_2(a_{0j}) + \sum_{i=1}^{r} \sum_{k=3}^{5} \nu_{ij} \frac{\partial R^{o}_{ij}}{\partial y_k(b_{ij})} \Delta y_k(b_{ij})\Big)$$

and the third term gives

$$\Delta J_0 = \sum_{j=0}^{K} \Big\{ \int_{D_j} \Big(\sum_{i=1}^{5} \Big(- \frac{\partial \mathcal{L}_j}{\partial y_i} - \psi_i'\Big) \delta y_i -$$

$$- \frac{\partial \mathcal{L}_j}{\partial v_j} \partial v_j + \frac{\partial \mathcal{L}_j}{\partial H_j} \delta H_j - \sum_{i=0}^{K} \frac{\partial \mathcal{L}_j}{\partial p_i} \Delta p_i - \tag{3.27}$$

$$- \sum_{i=1}^{m} \frac{\partial \mathcal{L}_j}{\partial h_i} \Delta h_i\Big) dx - \sum_{i=1}^{r} \sum_{k=1}^{5} [\psi_k(b_{ij})] \delta y_k -$$

$$- \sum_{i=1}^{5} [\psi_i(a_{oj})\,\delta y_i(a_{oj})] - \sum_{j \in K_r} \sum_{i=1}^{5} [\psi_i(x_j)]\delta y_i(x_j) -$$

$$- \sum_{j=1}^{K} \sum_{i=1}^{5} [\psi_i(a_j)]\,\delta y_i(a_j) + \sum_{i=1}^{5} \Big\{ \psi_i(L)\,\Delta y_i(L) -$$

$$- \psi_i(0)\,\Delta y_i(0) + \sum_{j=1}^{n} (\psi_i(s_j-)\delta y_i(s_j-) - \psi_i(s_j+)\delta y_i(s_j+)) \Big\}.$$

1.3.4. Adjoint set and stationarity conditions

Equalizing ΔJ_* with zero and taking into account that $\delta y_i(x)$ and $\delta v_j(x)$ are the independent variations of the corresponding quantities, one obtains from equations (3.22), (3.23), (3.27)

$$\psi_i' = - \frac{\partial \mathcal{L}_j}{\partial y_i} \; ; \quad i = 1,\ldots,5 \tag{3.28}$$

and

$$\frac{\partial \mathcal{L}_j}{\partial v_j} = 0 \quad ; \qquad j = 0,\ldots,K \tag{3.29}$$

$$\frac{\partial \mathcal{L}_j}{\partial H_j} = 0 \; ,$$

for $x \in D_j$. Equations (3.28) are the so-called adjoint equations which will be used for determination of the variables $\psi_i(x)$ $(i = 1,\ldots,5)$, whilst (3.29) may be interpreted as the stationarity conditions of the Lagrange function.

Making use of (3.22), (3.26), (3.27) and taking into account that $\Delta y_i(0)$, $\Delta y_i(L)$ may be handled as independent variations one obtains from (3.27) the transversality conditions, e.g. boundary conditions for adjoint variables at $x = 0$ as

$$\psi_i(0) = 0, \quad i \bar{\in} I_o$$

$$\psi_j(0) = \eta_j, \; j \in I_o \tag{3.30}$$

and at $x = L$ as

$$\psi_i(L) = 0, \quad i \bar{\in} I_L$$
$$\psi_j(L) = -\ \varrho_j, \ j \in I_L. \tag{3.31}$$

Since Δr_{ij} in (3.22) are arbitrary from (3.22) follow the equations

$$\mu_{ij}\, r_{ij} = 0 \ ; \quad i = 1,\ldots,f_j; \ j \in K_f \tag{3.32}$$

which may be used for determination of parameters r_{ij}.

Bearing in mind that the quantities Δp_i and Δh_k have to be considered as constant parameters, the relations (3.22), (3.25) - (3.27) permit to get the equations

$$\frac{\partial}{\partial p_i}(G + Y) - \sum_{j=0}^{K} \int_{D_j} \frac{\partial \mathcal{L}_j}{\partial p_i}\, dx = 0; \ i = 0,\ldots,K \tag{3.33}$$

and

$$\frac{\partial}{\partial h_k}(G + Y) - \sum_{j=0}^{K} \int_{D_j} \frac{\partial \mathcal{L}_j}{\partial h_k}\, dx = 0; \ k = 1,\ldots,m\,. \tag{3.34}$$

1.3.5. Intermediate conditions

Considering $\Delta y_i(a_j)$ for $j = 1,\ldots,K$ and $\Delta y_i(x_j)$ for $j \in K_f$ as arbitrary variations in (3.25) - (3.27) and making use of (3.17), one obtains for $i = 1, 2, 3, 4, 5$

$$[\psi_i(a_j)] = 0, \quad j = 1,\ldots,K \tag{3.35}$$

and

$$[\psi_i(x_j)] = \sum_{k=1}^{f_j} \sum_{m \in K_f} \mu_{km} \frac{\partial f_{km}}{\partial y_i(x_j)} \ ; \quad j \in K_f\,. \tag{3.36}$$

Note that (3.35) holds good, provided $x_j \neq a_j$. If, however, $x_j = a_j$, then (3.36) must be utilized for $x_j = a_j$.

Similarly to the previous cases it follows from (3.25) - (3.27) that for $j = 0,\dots,K$

$$[\psi_1(a_{oj})] = \varrho_{oj} \frac{\partial \Phi_{oj}}{\partial y_1(a_{oj})} ,$$

$$[\psi_2(a_{oj})] = \varrho_{2j} , \tag{3.37}$$

$$[\psi_i(a_{oj})] = 0 ; \quad i = 3,4,5$$

and

$$[\psi_1(b_{ij})] = [\psi_2(b_{ij})] = 0 ,$$

$$[\psi_k(b_{ij})] = \nu_{ij} \frac{\partial R_{ij}}{\partial y_k(b_{ij})} ; \quad k = 3,4,5 \tag{3.38}$$

where $i = 1,\dots,r$.

Taking (3.28) - (3.38) into account (3.22) may now be converted into

$$\sum_{j=1}^{K} \Big(\frac{\partial G}{\partial a_j} + \sum_{i=1}^{5} [\psi_i(a_j)]\, y_i'(a_j) \Big) \Delta a_j + \sum_{j \in K_f} \Big(\frac{\partial G}{\partial x_j} +$$

$$+ \sum_{i=1}^{5} [\psi_i(x_j)]\, y_i'(x_j) \Big) \Delta x_j + \sum_{j=0}^{K} \sum_{k=1}^{5} \Big\{ [\psi_k(a_{oj})]\, y_k'(a_{oj})\, \Delta a_{oj} +$$

$$+ \sum_{i=1}^{r} [\psi_k(b_{ij})]\, y_k'(b_{ij})\, \Delta b_{ij} \Big\} + \sum_{j=1}^{n} \sum_{k=1}^{5} \Big\{ (\psi_k(s_j-) \delta y_k(s_j-) -$$

$$- \psi_k(s_j+) \delta y_k(s_j+)) + \sum_{i=1}^{g_j} \lambda_{ij} \Big(\frac{\partial g_{ij}}{\partial y_k(s_j-)} \Delta y_k(s_j-) +$$

$$+ \frac{\partial g_{ij}}{\partial y_k(s_j+)} \Delta y_k(s_j+) \Big\} + \sum_{j=1}^{n} \frac{\partial G}{\partial s_j} \Delta s_j = 0 .$$

Due to the independence of variations Δa_j, Δx_j, Δa_{oj} and Δb_{ij} and to the continuity of the functions F_j and S_{io} in (3.14) it immediately follows from the previous equation that

$$[\mathcal{H}_j^*(a_j)] + \frac{\partial G}{\partial a_j} = 0; \quad j = 1,\dots,K;$$

$$[\mathcal{H}_j(x_j)] + \frac{\partial G}{\partial x_j} = 0; \quad j = 0,\dots,K; \tag{3.39}$$

and

$$[\mathcal{H}_j(a_{oj})] = [\mathcal{H}_j(b_{ij})] = 0 \ ; \ i = 1,\ldots,r; \ j = 0,\ldots,K \quad (3.40)$$

provided $\mathcal{H}_j^*(a_j+) = \mathcal{H}_j(a_j+)$ and $\mathcal{H}_j^*(a_j-) = \mathcal{H}_{j-1}(a_j-)$.
Substitution of the variations $\delta y_k(s_j^{\pm})$ according to (3.18), (3.19) into (3.39) leads to the equation

$$\sum_{j=1}^{n}\Big\{\sum_{k=2,4,5}\Big\{\Big(\psi_k(s_j-) + \sum_{i=1}^{g_j}\lambda_{ij}\frac{\partial g_{ij}}{\partial y_k(s_j-)}\Big)\Delta y_k(s_j-) +$$

$$+\Big(\sum_{i=1}^{g_j}\lambda_{ij}\frac{\partial g_{ij}}{\partial y_k(s_j+)} - \psi_k(s_j+)\Big)\Delta y_k(s_j+) +$$

$$+\sum_{k=1,3}\Big(-[\psi_k(s_j)] + \sum_{i=1}^{g_j}\lambda_{ij}\frac{\partial g_{ij}}{\partial y_k(s_j)}\Big)\Delta y_k(s_j) +$$

$$+\Big(\frac{\partial G}{\partial s_j} + \sum_{k=1}^{5}[\psi_k(s_j)y_i'(s_j)]\Big)\Delta s_j = 0$$

from where it directly yields that for $j = 1,\ldots,n$

$$\psi_k(s_j^{\pm}) = \pm\sum_{i=1}^{g_j}\lambda_{ij}\frac{\partial g_{ij}}{\partial y_k(s_j^{\pm})} \ ; \quad k = 2,4,5 \qquad (3.41)$$

$$[\psi_k(s_j)] = \sum_{i=1}^{g_j}\lambda_{ij}\frac{\partial g_{ij}}{\partial y_k(s_j)} \ ; \qquad k = 1,3$$

and

$$[\mathcal{H}_j^*(s_j)] + \frac{\partial G}{\partial s_j} = 0 \ ; \quad j = 1,\ldots,n \ ; \qquad (3.42)$$

provided $H_j^* = H_j$, if $s_j \in D_j$.

On the basis of (3.13), (3.14) in (3.39), (3.40), (3.42) $\mathcal{H}_j$ may be substituted by $\mathcal{L}_j$. Thus, in the case of the present approach, the Lagrangian function may have finite discontinuities at $x = a_j$, $x = x_j$ ($j = 0,\ldots,K$), $x = s_j$ ($j = 1,\ldots,n$), whereas at $x = a_{oj}$, $x = b_{ij}$ as well as elsewhere it is continuous. Note that the adjoint variables are continuous at a_{oj}^1 and b_{ij}^1, being discontinuous at a_{oj} and b_{ij}.

1.3.6. The second order inequality constraints

In the previous section it was assumed that the inequalities (1.7) present the first order state constraints. In this case equations (3.3) explicitly depend on controls ν_j. Let us assume now that it is not the case and (3.3) converts into (3.5). Differentiation of (3.5) with respect to x leads to the equations

$$\left(\frac{\partial R_1}{\partial p} P' + \frac{\partial R_1}{\partial s} s_j'\right)' + \left(\frac{\partial R_1}{\partial y_3}\right)' y_4 + \\ + \frac{\partial R_1}{\partial y_3}(-\nu_j^2)\frac{\partial \Phi_1}{\partial y_1} - \frac{1}{2}\left(\frac{\partial R_1}{\partial y_5}\right)' y_4^2 + \\ + \frac{\partial R_1}{\partial y_5} y_4 \nu_j^2 \frac{\partial \Phi_1}{\partial y_1} = 0 . \tag{3.43}$$

Assuming that the controls do not disappear from (3.43), the constraints (1.7) turn out to be the second order constraints imposed on the state variables. In this case the optimization technique utilized above must be slightly modified. Namely, the sums with terms $\nu_{ij} R_{ij}^{o}$ and $\varphi_{ij}^{1} R_{ij}^{1}$ in (3.10), (3.13) have to be replased by $\nu_{ij} R_{ij}^{o} + \nu_{ij1}^{o} R_{ij}^{10}$ and $\varphi_{ij2} R_{ij}^{2}$, respectively. Here $R_{ij}^{10} = R_{ij}^{1}(b_{ij})$ and R_{ij}^{2} stand for the left-hand side of (3.43). As a result of this substitution the corresponding intermediate conditions (3.38) have to be modified.

However, in the present work another approach which was used by the author in 1982 will be employed . This consists in the following. Introducing new control variables v_j ($j = 1,\dots r$) and θ_j ($j = 0,\dots,K$) one can present the inequalities (1.7) and (2.3) as

$$R_j(P, S, y_3, y_4, y_5) + v_j^2 = 0; \quad j = 1,\dots,r \tag{3.44}$$

and

$$\Phi_j(y_1, N_1, H_j, h_1,\dots,h_m) + \theta_j^2 = 0 ; \quad j = 0,\dots,K ; \tag{3.45}$$

respectively. Now the extended functional will be presented as previously in the form

$$J_* = G + Y + J_o + \sum_{j \in K_f} \sum_{i=1}^{f_j} \mu_{ij} r_{ij}^2 ,$$

whereas J_o and G are specified by (3.23) and (3.27), respectively, whilst

$$Y = \sum_{i \in I_o} \eta_i (y_i(0) - y_{oi}) + \sum_{i \in I_\ell} \varrho_i (y_i(L) - y_{\ell i}) +$$
$$+ \sum_{j=0}^{K} \left\{ \varrho_{oj} \, y_2(a_{oj}) + \varrho_{1j} \, y_2(a_{oj}^1) \right\} + \tag{3.46}$$
$$+ \sum_{j=0}^{K} \sum_{i=1}^{r} \left\{ \nu_{ij}^0 \, R_{ij}^1 \, (b_{ij}) + \nu_{ij}^1 \, R_{ij}^1 \, (b_{ij}^1) \right\}$$

and

$$\mathcal{L}_j = \mathcal{H}_j + \varphi_j (\Phi_j + \theta_j^2) + \sum_{i=1}^{r} \varphi_{io} (R_i + v_i^2) . \tag{3.47}$$

The derivation of the necessary conditions is similar to that presented above. Therefore, only final results will be presented herein. For determination of the controls θ_j and v_i, one obtains the equations

$$\varphi_j \, \theta_j = 0; \; j = 0, \ldots, K \tag{3.48}$$

and

$$\varphi_{io} v_i = 0; \; i = 1, \ldots, r. \tag{3.49}$$

After their application , equations (3.29), (3.47) and (3.14) give

$$\nu_j \left\{ (\psi_2 N_1 - \psi_4) \frac{\partial \Phi_j}{\partial y_1} + \psi_5 \frac{\partial \Phi_j}{\partial N_1} \right\} = 0; \; j = 0, \ldots, K . \tag{3.50}$$

Bearing in mind that the control variables θ_j and v_i have to specify the parameters a_{oj}, a_{oj}^1 , and b_{ij}, b_{ij}^1 , the variations of the latter are equal to zero. Thus, the intermediate conditions obtained in the previous section have to be suitably changed.

It appears that instead of (3.37) one has now for $j = 0,\dots,K$

$$[\psi_i(a_{oj})] = [\psi_i(a_{oj}^1)] = 0; \quad i = 1, 3, 4, 5$$

$$[\psi_2(a_{oj})] = \varrho_{oj}; \tag{3.51}$$

$$[\psi_2(a_{oj}^1)] = \varrho_{1j}.$$

Similarly to that the requirements (3.38) have to be replaced by the following conditions

$$[\psi_k(b_{ij})] = [\psi_k(b_{ij}^1)] = 0; \quad k = 1,2;$$

$$[\psi_k(b_{ij})] = \nu_{ij}^{o} \frac{\partial R_{ij}^1}{\partial y_k(b_{ij})}; \quad k = 3,4,5 \tag{3.52}$$

$$[\psi_k(b_{ij}^1)] = \nu_{ij}^{1} \frac{\partial R_{ij}^1}{\partial y_k(b_{ij}^1)}; \quad k = 3,4,5.$$

It is worth noting that the requirements (3.35), (3.36), (3.39), (3.41), (3.42) hold good, whereas (3.40) must be omitted, in the present case. Thus, the Lagrangian function is not continuous at the points a_{oj} and b_{ij}, now. Evidently, (3.28) - (3.34) are also valid.

The adjoint set (3.28) takes the form for $x \in D_j$ (here $j = 0,\dots,K$)

$$\psi_1' = -\varphi_j \frac{\partial \Phi_j}{\partial y_1},$$

$$\psi_2' = -\psi_1,$$

$$\psi_3' = \frac{\partial F_j^*}{\partial y_3}, \tag{3.53}$$

$$\psi_4' = -\psi_3 + y_4\psi_5 + \frac{\partial F_j^*}{\partial y_4},$$

$$\psi_5' = \frac{\partial F_j^*}{\partial y_5}$$

where the following notation is introduced

$$F_j^* = F_j - \sum_{i=1}^{s} \psi_{oi}\, S_{io} - \sum_{i=1}^{r} \varphi_{io} R_i - \varphi_j \Phi_j . \qquad (3.54)$$

By solving an optimization problem one has to distinguish between the cases when the axial force N_1 is specified and unspecified, respectively. In the first case, the quantity $N_1(0) = N$ is to be considered as a given parameter. In the second case, the axial force is generated as a reaction of supports which must not permit the axial displacements. Now the parameter N_1 is subjected to the variation with the variation of the extended functional. This yields

$$\sum_{j=0}^{K} \int_{D_j} \left\{ \varphi_j \frac{\partial \Phi_j}{\partial N_1} + \nu_j^2 \frac{\partial}{\partial N_1} \left((\psi_2 N_1 - \psi_4) \frac{\partial \Phi_j}{\partial y_1} + \right.\right.$$
$$\left.\left. + \psi_5 \frac{\partial \Phi_j}{\partial N_1} \right) \right\} dx = 0 . \qquad (3.55)$$

Equation (3.55) may be considered as the equation for determination of the quantity N_1.

Now let us consider the particular case of the posed problem when neither the optimality criterion (1.2) nor the requirements (1.7), (1.8) depend on y_4 and y_5. We assume that $F_j = F_j(P, S_j, y_3)$; $R_i = R_i(P, S, y_3)$; $S_{io} = S_{io}(P, S, y_3)$. Now, according to (3.54)

$$\frac{\partial F_*^j}{\partial y_4} = \frac{\partial F_*^j}{\partial y_5} = 0 . \qquad (3.56)$$

Consequently, the last equation in the set (3.53) gives $\psi_5' = 0$. Thus, in the case when the axial force is specified and the axial displacement does not vanish at one end of the beam one has for $x \in (0, L)$

$$\psi_5 = 0 . \qquad (3.57)$$

In fact, now $\psi_5 = \text{const}$ and according to the transversality conditions (3.30), (3.31) $\psi_5(0) = 0$ or $\psi_5(L) = 0$.

Substitution of (3.56), (3.57) into (3.53) results in

$$\psi_1' = -\varphi_j \frac{\partial \Phi_j}{\partial y_1}$$

$$\psi_2' = -\psi_1 ,$$

$$\psi_3' = \frac{\partial F_j^*}{\partial y_3} \tag{3.58}$$

$$\psi_4' = -\psi_3 .$$

Considerable simplification may also be achieved in (3.50). Taking (3.56) - (3.58) into account one can easily establish that

$$\psi_4 = N_1 \psi_2, \quad \psi_3 = N_1 \psi_1, \tag{3.59}$$

and

$$N_1 \varphi_j \frac{\partial \Phi_j}{\partial y_1} = -\frac{\partial F_j^*}{\partial y_3} \tag{3.60}$$

if $\nu_j \neq 0$. In the regions where $\nu_j = 0$ (these correspond to the rigid zones of the beam) relations (3.59), (3.60) do not hold good. In the plastic zones $\Phi_j = 0$, whereas the rigid ones are associated with $\Phi_j < 0$ and $\nu_j = 0$.

§1.4. Optimal location of additional supports

1.4.1. Statement of the problem

As the first illustration of the previous analysis can serve a problem consisting in the determination of the optimal positions of additional supports. It consists in the determination of the coordinates $s_1, \ldots, s_n$ which specify the locations of additional absolutely rigid supports (Fig. 1.1.4).

The left end of the beam is clamped, whereas the right end is simply supported. Assume that the beam is loaded by the uniformly distributed lateral pressure of intensity P and by the axial tension N .

Such a layout of rigid supports is sought for which the criterion

$$J = \int_0^L y_3 dx \qquad (4.1)$$

attains the minimal value.

The deformations of the beam are expected to take place in the range of finite deflections. Thus, the load intensity has to exceed the load carrying capacity for each section of the beam, which yields the requirements

$$P - P_j \geqslant 0; \quad j = 0,\dots,n . \qquad (4.2)$$

In (4.2) $P_j = P_j(s_j, s_{j+1}, N)$ stands for the limit load for the section of the beam which is located between the supports s_j and s_{j+1} . Here $s_o = 0$ and $s_{n+1} = L$, while $N_1 = N$ for $x \in (0,L)$.

The problem posed herein could be considered as a particular case of (1.2), (1.7), (1.8), (1.13), associated with $G_j = 0$, $F_j = y_3$, $S_{io} = A_{io} = 0$, $R_i = 0$, $f_{ij} = P_j - P$ ($j = 0,\dots,n$).

It is reasonable to assume that at $x = s_j (j = 0,\dots,n)$ plastic hinges are located. Thus, the state variables have to meet the following boundary conditions

$$\begin{aligned} y_1(0) &= - M_* , \quad y_1(L) = 0, \\ y_3(0) &= y_3(L) = y_5(0) = 0 . \end{aligned} \qquad (4.3)$$

Similarily, at $x = s_j$ one has

$$y_1(s_j) = - M_*, \quad y_3(s_j) = 0; \; j = 1,\dots,n ; \qquad (4.4)$$

where M_* stands for the value of the bending moment which corresponds to a plastic hinge. For instance, in the case of a sandwich beam the yield curve is specified by (2.4) or Fig. 1.1.5. Now

$$M_* = M_0(1 - \frac{N}{N_0}) \tag{4.5}$$

and

$$\Phi = |y_1| - M_* \tag{4.6}$$

where $M_0 = \sigma_0 Hh$; $N_0 = 2\sigma_0 h$.

Evidently, the boundary requirements (4.3) conform to (2.10), (2.11), if $I_0 = \{1, 3, 5\}$; $I_L = \{1, 3\}$ and (4.4) may be conceived as a particular case of (1.14) .

1.4.2. Optimality conditions and the adjoint set

It appears that the equations for determination of the parameters $s_1, \dots, s_n$ are given by (3,42) which may be presented as

$$[\mathcal{L}(s_j)] + \frac{\partial G}{\partial s_j} = 0 \; ; \quad j = 1, \dots, n \tag{4.7}$$

where $\mathcal{L}$ is specified by (3.47) and (3.14). According to (3.24), (4.2) - (4.4) the function G in (4.7) changes

$$G = \sum_{j=0}^{n} \mu_j(P_j - P) + \sum_{j=1}^{n} (\lambda_{1j}(y_1(s_j) + M_*) + \\ + \lambda_{2j} y_3(s_j)) . \tag{4.8}$$

On the other hand, in the present case (3.54) may be converted into

$$F^* = Y_3 - \varphi\Phi . \tag{4.9}$$

Since according to (4.6) $\Phi = \Phi(y_1, N_1, h, H)$ the validity of (3.56) immediately follows from (4.9) . Moreover, now $y_5(L)$ is not fixed and therefore $\psi_5 = 0$, as shown by (3.57). Making use of (4.9), one can present the adjoint set as

$$\psi_1' = \begin{cases} \frac{1}{N_1}, & \Phi = 0 \\ 0, & \Phi < 0 \end{cases} \tag{4.10}$$

$$\psi_2' = -\psi_1 ,$$

$$\psi_3' = 1 ,$$

$$\psi_4' = -\psi_3$$

where (3.60) is taken into account.

According to (4.3) the transversality conditions (3.30), (3.31) take the form

$$\psi_2(0) = \psi_4(0) = \psi_2(L) = \psi_4(L) = 0. \tag{4.11}$$

From (3.41) and (4.8) it follows that

$$\psi_2(s_j) = \psi_4(s_j) = 0 ; \quad j = 1,\dots,n \tag{4.12}$$

and

$$[\psi_1(s_j)] = \lambda_{1j} ; [\psi_3(s_j)] = \lambda_{2j} ; \quad j = 1,\dots,n. \tag{4.13}$$

Therefore, adjoint variables ψ_1, ψ_3 are discontinuous at $x = s_j$ in the present case. However, the Lagrangian function is not necessarily discontinuous.

Proceeding from (4.8) one easily obtains

$$\frac{\partial G}{\partial s_j} = \mu_j \frac{\partial P_j}{\partial s_j} + \mu_{j-1} \frac{\partial P_{j-1}}{\partial s_j} \tag{4.14}$$

for $j = 1,\dots,n$. Substitution of (4.14) into (4.7) leads to the jump condition

$$[\mathscr{L}(s_j)] = -\frac{\partial}{\partial s_j}(\mu_j P_j + \mu_{j-1} P_{j-1}); \quad j = 1,\dots,n \tag{4.15}$$

where the Lagrangian function is given by (3.14), (3.47). Making use of (3.14) and taking into account (4.12), (3.57), one can establish that on the other hand

$$[\mathscr{L}(s_j)] = [\psi_1(s_j)y_2(s_j) + \psi_3(s_j)y_4(s_j)]; \; j=1,\dots,n. \tag{4.16}$$

The relations (4.15), (4.16) furnish the optimality conditions

$$\left[\psi_1(s_j)y_2(s_j) + \psi_3(s_j)\, y_4(s_j)\right] = \\ = -\frac{\partial}{\partial s_j}\left(\mu_j P_j + \mu_{j-1}P_{j-1}\right) \quad j = 1,\ldots n \tag{4.17}$$

where $\mu_j = 0$, if $P - P_j > 0$.

1.4.3. Adjoint variables

In order to employ the requirements (4.17), one has to find the solutions of the basic equations (2.9) and of the adjoint set (4.10). Since these are not coupled in the present case, there exists a possibility of solving the adjoint set separately.

In the rigid-plastic analysis one has to distinguish between the rigid and plastic zones of a beam under consideration. In the plastic zones, at each point the stress-strain state corresponds to a point lying on the yield curve $\Phi = 0$, whereas rigid zones are associated with the inequality $\Phi < 0$.

Suppose that the plastic zones are (a_j, b_j) for each $j = 0,\ldots,n$; where $s_{j-1} < a_j < b_j < s_j$. Note that in comparison to the notation used above $b_j = a_j^1$ herein. Thus, according to (4.6)

$$\Phi = y_1 - M_* = 0 \tag{4.18}$$

for each $x \in (a_j, b_j)$; $j = 0,\ldots,n$; provided the bending moment is positive in these regions.

The adjoint coordinates have to meet not only (4.10) - (4.13) but (3.51), (3.52) as well. The requirements of (3.52) are satisfied spontaneously, while (3.51) establishes the continuity of variables ψ_1, ψ_3, ψ_4 at $x = a_j$ and $x = b_j$. However, ψ_2 may have finite jumps at these points. The jump conditions in (3.51) should be used for determination of corresponding Lagrangian multipliers.

Therefore, in each region (s_j, s_{j+1}) variables ψ_3, ψ_4 are continuous whereas according to (4.13), ψ_3 has finite jumps at the boundaries of these regions. Integrating the latter equations in (4.10) and making use of (4.12), one obtains

$$\psi_3 = \frac{1}{2}(2x - s_{j+1} - s_j)$$
$$\psi_4 = \frac{1}{2}(x - s_j)(x - s_{j+1}) \tag{4.19}$$

for $x \in (s_j, s_{j+1})$, where $j = 0,\ldots,n$.

In the plastic regions (a_j, b_j) the relations (3.59) hold good. It is easy to check that the solution of the adjoint set (4.10) which satisfies (3.59), (4.11)-(4.13) has the form

$$\psi_1 = \begin{cases} (2a_j - s_j - s_{j+1})/2N, & x \in (s_j, a_j), \\ (2x - s_j - s_{j+1})/2N, & x \in (a_j, b_j), \\ (2b_j - s_j - s_{j+1})/2N, & x \in (b_j, s_{j+1}), \end{cases}$$
$$\psi_2 = \begin{cases} (x - s_j)(s_j + s_{j+1} - 2a_j)/2N, & x \in (s_j, a_j), \\ (x - s_j)(s_{j+1} - x)/2N, & x \in (a_j, b_j), \\ (s_{j+1} - x)(2b_j - s_j - s_{j+1})/2N, & x \in (b_j, s_{j+1}) \end{cases} \tag{4.20}$$

for the region (s_j, s_{j+1}) where $j = 0,\ldots,n$.

Due to (4.19) and (4.20) one has

$$\psi_1(s_j-) = (2b_{j-1} - s_j - s_{j-1})/2N, \quad \psi_1(s_j+) =$$
$$= (2a_j - s_j - s_{j+1})/2N, \psi_3(s_j-) = \frac{1}{2}(s_j - s_{j-1}); \tag{4.21}$$
$$\psi_3(s_j+) = \frac{1}{2}(s_j - s_{j+1}).$$

1.4.4. State variables

Let us study the integral curves of the basic set (2.9) now. In the present case $K = 0$, evidently. Thus, in (2.9), (3.6) - (3.9) the subscripts may be omitted and (3.9) leads to

$$\nu^2 = \frac{P}{N} \tag{4.22}$$

which holds good in plastic zones (a_j, b_j), where $\Phi = 0$. The most essential attribute of rigid zones (where $\Phi < 0$) is the vanishing of the deformation components. Thus

$$\nu = 0 \tag{4.23}$$

for $x \bar{\in} (a_j, b_j)$.

These considerations accompanied with (4.18) make possible the integration of (2.9). The first two equations in (2.9) after integration lead to the relations

$$y_1 = \begin{cases} -\frac{P}{2}(x - a_j)^2 + M_*, & x \in (s_j, a_j), \\ M_*, & x \in (a_j, b_j), \\ -\frac{P}{2}(x - b_j)^2 + M_*, & x \in (b_j, s_{j+1}) \end{cases}$$

$$y_2 = \begin{cases} -P(x - a_j), & x \in (s_j, a_j) \\ 0, & x \in (a_j, b_j), \\ -P(x - b_j), & x \in (b_j, s_{j+1}), \end{cases} \tag{4.24}$$

for $x \in (s_j, s_{j+1})$, where the continuity requirements for the state variables are taken into account.

The bending moment and its slope given by (4.24) have to meet the boundary and intermediate conditions (4.3), (4.4). For that reason

$$a_j = s_j + a_o \; ; \quad j = 0,\dots,n \; ;$$

$$b_i = s_{i+1} - a_o \; ; \quad i = 0,\dots,n-1 \; ; \tag{4.25}$$

$$b_n = L - \sqrt{\frac{2M_*}{P}},$$

where

$$a_o = \sqrt{\frac{4M_*}{P}}. \tag{4.26}$$

Making use of (4.22), (4.23) as well as (4.5), (4.6) the state variables are obtained from (2.9)

$$y_3 = \begin{cases} (-\frac{P}{N} a_j + A_j)(x - s_j), & x \in (s_j, a_j), \\ -\frac{P}{2N} x^2 + A_j x + B_j, & x \in (a_j, b_j), \\ (-\frac{P}{N} b_j + A_j)(x - s_{j+1}), & x \in (b_j, s_{j+1}), \end{cases}$$

$$y_4 = \begin{cases} -\frac{P}{N} a_j + A_j, & x \in (s_j, a_j), \\ -\frac{P}{N} x + A_j, & x \in (a_j, b_j), \\ -\frac{P}{N} b_j + A_j, & x \in (b_j, s_{j+1}), \end{cases} \quad (4.27)$$

$$y_5 = \begin{cases} -\frac{1}{2} (\frac{P}{N} a_j + A_j)^2 x + A_j x + C_j, & x \in (s_j, a_j), \\ \frac{1}{6} (A_j - \frac{P}{N} x)^3 + \frac{PM_0}{NN_0} x + D_j, & x \in (a_j, b_j), \\ -\frac{1}{2} (A_j - \frac{P}{N} b_j)^2 x + E_j & x \in (b_j, s_{j+1}) \end{cases}$$

where A_j, B_j, C_j, D_j, E_j $(j = 0,\dots,n)$ stand for arbitrary constants of integration.

For the determination of the constants A_j, B_j the conditions of continuity of y_3 at $x = a_j$ and $x = b_j$ may be utilized. This yields

$$A_j = \frac{P}{2N(s_j - s_{j+1})} (b_j^2 - a_j^2 + 2a_j s_j - 2b_j s_{j+1}); \quad j = 0,\dots,n; \quad (4.28)$$

$$B_j = \frac{P}{2N(s_j - s_{j+1})} (a_j s_{j+1}(a_j - 2s_j) + b_j s_j(2s_{j+1} - b_j)).$$

It is easy to draw the conclusion from (4.24), (4.27), (4.28) that, in particular

$$y_2(s_j - 0) = P(b_{j-1} - s_j), \quad y_4(s_j-) =$$

$$= \frac{P(a_{j-1}-b_{j-1})(a_{j-1}+b_{j-1}-2s_{j-1})}{2N(s_j-s_{j-1})} \tag{4.29}$$

$$y_2(s_j+) = P(a_j - s_j), \quad y_4(s_j+) =$$

$$= \frac{P(a_j-b_j)(a_j+b_j-2s_{j+1})}{2N(s_{j+1}-s_j)} \ .$$

When deriving (4.22) - (4.29) it is tacitly assumed that the measure of a plastic region (a_j, b_j) is non-negative, e.g. $a_j \leqslant b_j$. The case $a_j = b_j$ is associated with the limit state of the section of beam which lies between the supports s_j and s_{j+1}. Thus, the corresponding limit loads may be obtained from (4.25), (4.26) as

$$P_j = \frac{16M_*}{(s_{j+1}-s_j)^2} \ ; \ j = 0,\dots,n-1;$$

$$P_n = 2(1 + \sqrt{2})^2 \frac{M_*}{(L-s_n)^2} \ . \tag{4.30}$$

1.4.5. Optimal layout of additional supports

The parameters $s_1,\dots,s_n$ will be determined with the aid of the equations (4.17). Substituting the jumps of the state and adjoint variables from (4.21) and (4.29) into (4.17), one obtains the equations

$$P\Big\{(2a_j - s_j - s_{j+1})(a_j - s_j) - (2b_{j-1} - s_j -$$

$$- s_{j-1})(b_{j-1} - s_j) - \tfrac{1}{2}(a_j - b_j)(a_j + b_j -$$

$$- 2s_{j+1}) - \tfrac{1}{2}(a_{j-1} - b_{j-1})(a_{j-1} + b_{j-1}2s_{j-1})\Big\} = \tag{4.31}$$

$$= -2N\Big(\mu_j \frac{\partial P_j}{\partial s_j} + \mu_{j-1} \frac{\partial P_{j-1}}{\partial s_j}\Big) \ ; \ j = 1,\dots,n \ .$$

Making use of (4.30) the equations (4.31) may be presented as

$$\frac{P}{4N}\left\{(s_j - s_{j-1})^2 - (s_{j+1} - s_j)^2\right\} =$$

$$= -\frac{32 M_* \mu_j}{(s_{j+1}-s_j)^3} + \frac{32 M_* \mu_{j-1}}{(s_j - s_{j-1})^3} ;$$

$$j = 1,\ldots,n-1 \tag{4.32}$$

$$-\frac{P}{4N}\, 2s_n (L - s_n) + s_{n-1}^2 - L^2 - \frac{a_0^2}{2} =$$

$$= \frac{32 M_* \mu_{n-1}}{(s_n - s_{n-1})^3} - \frac{4(1+\sqrt{2})^2 M_* \mu_n}{(L - s_n)^3} .$$

In order to solve the set of algebraic equations (4.32) with respect to the coordinates s_j, one has to consider the two principally different cases. Firstly, if $\mu_j = 0$, then according to (4.2) $P > P_j$. Secondly, $\mu_j \neq 0$ when $P = P_j$. It means that the Lagrangian multipliers μ_j vanish in each section of the beam where the plastic deformations take place.

Analysis of the possible versions leads to the solution of (4.32)

$$\frac{s_j}{L} = \begin{cases} 2j\sqrt{\dfrac{M_*}{PL^2}}, & P_0 \leqslant P \leqslant P_1, \\ \dfrac{j}{n^2-1}\left(n - \sqrt{1 - \dfrac{a_0^2}{2L^2}(n^2 - 1)}\right), & P_1 \leqslant P \end{cases} \tag{4.33}$$

where P_0 and P_1 are the following values of the load intensity

$$\begin{aligned} P_0 &= \frac{2}{L^2} M_* \left(1 + (1+2n)\sqrt{2}\right)^2, \\ P_1 &= \frac{2}{L^2} M_* \left(\sqrt{8}n + \sqrt{7}\right)^2 . \end{aligned} \tag{4.34}$$

It is easy to recheck that

$$s_j(P_o) = \frac{2\sqrt{2}\, jL}{1+(1+2n)\sqrt{2}} \qquad (4.35)$$

Thus, for $P = P_o$ the layout of the additional supports corresponds to the maximum load carrying capacity, P_o being the common limit load for each section of the beam.

It should be noted that the optimal location of supports (4.35) corresponds not only to the optimal solution of a geometrically linear problem which consists in the maximization of the load carrying capacity but it was established in the cases of dynamic loading of beams and cylindrical shells as well by the author, 1978, 1981, 1983, 1984.

Economy of the design established could be assessed by the ratio

$$e = \frac{J(s_1,\dots,s_n)}{J(s_1^1,\dots,s_n^1)}\,, \qquad (4.36)$$

where $J(s_1,\dots,s_n)$ stands for the value of the cost criterion (4.1) which corresponds to the optimal layout of additional supports given by (4.33). $J(s_1^1,\dots,s_n^1)$ in (4.36) is the value of (4.1) calculated for the uniform distribution of additional supports specified by

$$s_j^1 = \frac{jL}{n+1}\,; \quad j = 1,\dots,n\,. \qquad (4.37)$$

Numerical results are presented in Tables 1.4.1 and 1.4.2 for the case $n = 1$. Table 1.4.1 corresponds to the axial force $N = 0{,}2\ N_o$, Table 1.4.2 to the case $N = 0{,}5\ N_o$.

Table 1.4.1. Optimal location of the additional support in the case $N = 0{,}2\ N_o$

p	2,769	2,775	2,80	3,00	4,00	5,00	6,00
s_{10}	0,539	0,518	0,517	0,516	0,513	0,510	0,508
e	0	0,302	0,685	0,946	0,992	0,997	0,998

Here the following notation is introduced:

$$p = \frac{PL^2}{16M_*}\ \ p_o = \frac{P_oL^2}{16M_*}\ ,\ \ s_{10} = \frac{s_1}{L},\ \ s_{11} = \frac{s_1^1}{L}\ . \qquad (4.38)$$

Table 1.4.2. Optimal location of the additional support for $N = 0,5\ N_o$.

p	1,718	1,75	2,0	2,5	3,0	3,5	4,0
s_{10}	0,539	0,518	0,516	0,512	0,510	0,509	0,508
ϵ	0	0,685	0,973	0,993	0,996	0,998	0,999

In Tables 1.4.1, 1.4.2 the values of the economy coefficient are presented, too. It appears that the quantity e vanishes when the lateral load equals the load carrying capacity of the beam. This phenomenon is consistent with (4.36) because $J(s_1) = 0$, but $J(s_1^1) \neq 0$ for $p = p_o$.

Table 1.4.3. Optimal coordinates for different numbers of additional supports

n	1	2	3	4	5	6
p_o	1,718	4,071	7,425	11,78	17,13	23,49
p_1	1,873	4,308	7,744	12,18	17,61	24,05
s_{10}	0,539	0,350	0,260	0,206	0,171	0,146
s_{11}	0,500	0,333	0,250	0,200	0,167	0,143

In Table 1.4.3 the results are presented for several numbers of additional supports. The data accommodated herein correspond to the limit state of beam and $N = 0,5\ N_o$.

1.4.6. Non-self-adjointness of the problem

The problems of optimal design of beams and axisymmetric plates and shells posed in the geometrically linear form appear to be self-adjoint problems as a rule (see Lepik, 1982; Grinjev and Filippov, 1979; Lellep and Lepik, 1984).The problems are called self-adjoint ones if there exists a linear relation between the solutions of the basic set and adjoint set, respectively. However, if the configuration changes are taken into account the problem seems to be a non-self--adjoint one.

Indeed, making use of (4.19), (4.20) and (4.24), (4.27), (4.28) it could be checked that

$$
\begin{aligned}
\psi_1 &= -\frac{y_4}{P},\\
\psi_2 &= \frac{y_3}{P} + \begin{cases} 0, & x \in (s_j, a_j),\\ \frac{2M_*}{PN}, & x \in (a_j, b_j),\\ 0, & x \in (b_j, s_{j+1})\end{cases} \qquad (4.38)\\
\psi_3 &= -\frac{1}{P}(y_2 + Ny_4),\\
\psi_4 &= \frac{1}{P}(y_1 + Ny_3 + M_*).
\end{aligned}
$$

Note that the relations (4.38) hold good in the regions (s_j, s_{j+1}) where $j = 0,\ldots,n-1$. For (s_n, L) similar relations may be derived.

According to (4.19), (4.20) and (4.24) - (4.28) one has for $x \in (s_n, L)$

$$
\begin{aligned}
\psi_1 &= \frac{1}{P}\left(-y_4 + \frac{M_*}{NP(L-s_n)}\right),\\
\psi_2 &= \frac{y_3}{P} - \frac{M_* x}{N\,P(L-s_n)} - \frac{M_*}{NP(L-s_n)} \cdot \begin{cases} s_n, & x \in (s_n, a_n),\\ (2-s_n), & x \in (a_n, b_n),\\ 1, & x \in (b_n, L).\end{cases} \qquad (4.39)\\
\psi_3 &= -\frac{1}{P}(y_2 + Ny_4) + \frac{M_*}{P(L-s_n)}\\
\psi_4 &= \frac{1}{P}(y_1 + Ny_3) + \frac{M_*(L-x)}{P(L-s_n)}.
\end{aligned}
$$

Thus, it follows from (4.38), (4.39) that the relations between state and adjoint variables are not linear.

§1.5. Piece-wise homogeneous beams

1.5.1. Concept of a quasi-homogeneous beam

Let us study the behaviour of a piece-wise homogeneous beam (Fig. 1.1.3) in the range of finite deflections. The structure consists of a matrix with layers of finite dimensions embedded reinforcement. The thicknesses of the layers are marked with $h_1,\dots,h_m$, whereas their materials have yield stresses $\sigma_1,\dots,\sigma_m$, respectively, the yield stress of the matrix being σ_0.

Let a_j and b_j $(j = 1,\dots,m)$ be the coordinates of the beginning and end points of the layers.

We call a section of the beam lying in an interval D a quasi-homogeneous section if for each $x \in D$ the geometry and dimensions of the cross-sections remain the same. Thus, the quasi-homogeneous sections of the beam do not comprise the end points of reinforcements.

The points $x = a_j$ and $x = b_j$ subdivide the interval $(0,L)$ into $2m + 1$ regions D_j (the length of the beam is assumed to be $2L$). Within each region D_j $(j = 0,\dots,2m)$ the beam is quasi-homogeneous. Let D_* be a quasi-homogeneous region (a,b) of finite length where the plastic deformations take place.

It seems to be convenient to use the notation $a_{j+m} = b_j$ for $j = 1,\dots,m$.

Employing the methods of plastic analysis of structures one could derive a yield curve for each quasi-homogeneous section of the beam. The equations of the yield curves expressed via a bending moment and membrane force depend not only on the thicknesses of the layers $h_1,\dots,h_m$, but on the thicknesses of those layers which are located between the layers of the reinforcement, as well. Thus, the number of thickness parameters is equal to $2m + 1$ and it appears to be reasonable to assume that for $x \in D_j$ the generalized yield condition may be presented as

$$\Phi_j(y_1, \varkappa, h_1,\dots,h_{2m+1}) \leq 0; \quad j = 0,\dots,2m, \qquad (5.1)$$

provided the yield stresses of different materials are fixed. Here the parameters $h_{m+1},\dots,h_{2m+1}$ stand for the distances between adjacent layers.

It should be noted that the functions Φ_j might be obtained using the hypothesis of Kirchoff accompanied with the linear distribution of strains along the normal to the curved axis of the beam. Particular cases of piece-wise quasi-homogeneous beams subjected to static and dynamic loads are studied by the author and Majak, 1988 and Sakkov, 1984, 1985.

The functions Φ_j in (5.1) are continuous and piece-wise differentiable with respect to their arguments.

1.5.2. Optimization problem

In the present study the attention will be confined to the problems in the case of which the optimality criterion may be presented as

$$I = \sum_{j=1}^{m} k_j\, h_j(a_{j+m} - a_j) \qquad (5.2)$$

where k_j will be handled as given constants. If the problem consists in the minimization of the amount of the reinforcement then k_j coincides with the density of the corresponding material.

When the minimum of (5.2) is looked for, it is required that the deflection of the beam does not exceed deflection of the associated quasi-homogeneous beam. The latter means that a state constraint

$$y_3 - W_* \leq 0 \qquad (5.3)$$

is imposed on the optimal trajectory at each $x \in (0, L)$. Here W_* stands for the deflection of the associated beam.

Evidently, the posed problem is a particular case of the problem expressed by (1.2), (1.7), (1.8). In the present case one has according to (5.2), (5.3)

$$G_j = k_j\, h_j(a_{j+m} - a_j), \; F_j = 0, \; R_1 = y_3 - W_*, \qquad (5.4)$$
$$S_{io} = A_i = 0$$

where $K = 2m + 1$, $a_0 = 0$, $a_{2m+1} = L$, $r = 1$, $s = 0$.

1.5.3. Necessary optimality conditions

Let us adjust the set of the necessary conditions to the present problem.

It was assumed above that the only plastic region of finite length is D_*. Thus,

$$\Phi_*(y_1, N, h_1, \dots, h_{2m+1}) = 0 \qquad (5.5)$$

for each $x \in D_*$. The region D_* is a subset of one of the regions $D_j (j = 0, \dots, 2m)$.

Besides the zone D_* there may exist single points which correspond to the plastic hinges. The minimum material requirement (5.2) implies that the utilized material is to be extremely stressed. However, the extremal bending moment distribution must not violate the flow condition (5.1) in each region D_j. Obviously, the most dangerous cross-sections correspond to the ends of the reinforcing layers. Thus, it is reasonable to assume that

$$\Phi_j(y_1(a_j), \; N, h_1, \dots, h_{2m+1}) = 0; \; j = 1, \dots, 2m \quad (5.6)$$

or $\Phi_j(a_{j+1}) = 0$. Since the points a_j may be rearranged, attention will be paid to the case (5.6), only.

It appears that the assumption (5.6) leads to a jump condition for adjoint variables. This should be applied for a suitable endpoint of the region D_j.

Making use of (5.5), it follows from the state equations (2.9) that $y_2 = 0$ in the plastic zone D_*. Thus, in particular case

$$y_2(a) = y_2(b) = 0 . \qquad (5.7)$$

Introducing a new control variable v_1 one can (5.3) put into the form

$$y_3 - W_* + v_1^2 = 0 . \qquad (5.8)$$

Due to (5.3), (5.8) the posed problem is a problem with the second-order state constraints. Thus, (3.47) - (3.55) hold good in the present case.

Assuming that the beam is subjected to the transverse pressure P and axial load N, which implies that the axial displacement y_5 does not vanish at $x = L$, for instance. Consequently, $\psi_5 = 0$ and (3.54) takes the form

$$F_j^* = -\varphi_{10}(y_3 - W_*) - \varphi_j \Phi_j , \qquad (5.9)$$

Φ_j being introduced by (5.1). In the plastic region D_* one has according to (3.59), (3.60),

$$N\varphi_* \frac{\partial \Phi_*}{\partial y_1} = \varphi_{10} , \quad \psi_3 = N\psi_1 , \quad \psi_4 = N\psi_2 . \qquad (5.10)$$

It is assumed herein that $v_1 = 0$ in (3.49) and $\varphi_{10} \neq 0$ for $x \in D_*$ whereas in the rigid zones $\nu_j = 0$, respectively.

Since, in the rigid zones $\Phi_j < 0$ for $x \in D_j$ (except the boundary points of D_j) one has the non-trivial controls $\theta_j \neq 0$ and (3.48) lead to

$$\varphi_j = 0 \qquad (5.11)$$

for $x \bar{\in} D_*$. Thus, it might be expected that $\varphi_j = c\,\nu_j^2$, which implies that φ_j = const. Evidently, this is not the only case.

Taking (5.8)-(5.11) into account one can put the adjoint set (3.58) into the form

$$\psi_1' = \begin{cases} \dfrac{-\varphi_{10}}{N} , & x \in D_* , \\ 0 , & x \bar{\in} D_* , \end{cases}$$
$$\psi_2' = -\psi_1 , \qquad (5.12)$$
$$\psi_3' = -\varphi_{10} ,$$
$$\psi_4' = -\psi_3 .$$

The adjoint variables have to meet the transversality conditions (3.30), (3.31) as well as the jump conditions (3.39), (3.51), (3.52). The requirements (3.39) may be presented as

$$\mathcal{L}_j^*(a_j) + \frac{\partial I}{\partial a_j} = 0 \;; \quad j = 1,\dots,2m+1 \tag{5.13}$$

where (5.2) and (3.24) are taken into account.

The equations (3.34) take the form

$$\frac{\partial G}{\partial h_j} - \sum_{i=0}^{2m} \int_{D_i} \frac{\partial \mathcal{L}_i}{\partial h_i} dx = 0 \;, \quad i = 1,\dots,m, \tag{5.14}$$

where $\mathcal{L}_j$ is given by (3.47) and according to (3.24),(5.2), (5.6)

$$G = I + \sum_{j=1}^{2m} \mu_j \Phi_j(y_1(a_j), N, h_1,\dots,h_{2m+1}) \,. \tag{5.15}$$

Due to the requirement (5.6) one can assert that in comparison with the general analysis $x_j = a_j$, in the present case. Thus, the jump condition (3.35) must be replaced by (3.36), which leads to

$$[\psi_1(a_j)] = \mu_j \frac{\partial \Phi_j}{\partial y_1(a_j)} \;, \quad j = 1,\dots,2m \tag{5.16}$$

whereas

$$[\psi_i(a_j)] = 0 \;, \quad i = 2,3,4,5 \,.$$

Bearing in mind that at $x = a_j$ the only discontinuous variable (among the state and adjoint variables) is ψ_2 on the grounds of (3.47), (3.14), one obtains the jump of the Lagrangian function as

$$[\mathcal{L}_j^*(a_j)] = [\psi_1(a_j)]\, y_2(a_j) \;; \quad j = 1,\dots,2m \,. \tag{5.18}$$

The relations (5.18) with (5.13) lead to

$$[\psi_1(a_j)] = \frac{-1}{y_2(a_j)} \cdot \frac{\partial I}{\partial a_j} \;; \quad j = 1,\dots,2m \;; \tag{5.19}$$

which, in turn, with (5.16) give the Lagrangian multipliers

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$$\mu_j = -\frac{\partial I}{\partial a_j}\left(y_2(a_j)\frac{\partial \Phi_j}{\partial y_1(a_j)}\right)^{-1}; \; j = 1,\dots,2m. \tag{5.20}$$

Let us study the equations (5.14) in a greater detail, now. Substituting (5.15), (3.47), (3.14) into (5.14) may be obtained

$$\frac{\partial I}{\partial h_j} + \sum_{i=1}^{2m} \mu_i \frac{\partial \Phi_i(a_i)}{\partial h_j} - \sum_{i=0}^{2m} \int_{D_i} \Big(\varphi_i \frac{\partial \Phi_i}{\partial h_j} + \nu_i^2(\psi_2 N - \psi_4)\frac{\partial}{\partial h_j}\frac{\partial \Phi_i}{\partial y_1}\Big)\, dx = 0, \; j = 1,\dots,m \tag{5.21}$$

provided $\psi_5 = 0$ and $\partial W_* / \partial h_j = 0$.

It is worth emphasizing that the thicknesses which are subjected to the variation are $h_1,\dots,h_m$. Evidently, if $h_{m+1},\dots,h_{2m+1}$ are previously unknown the equations (5.21) hold good for $j = 1,\dots,2m+1$.

Substituting (5.20) into (5.21) and taking into account that $\nu_i = 0$ (except of $x \in D_*$) where in turn $\psi_2 N = \psi_4$ and that $\varphi_i = 0$ (since $\theta_i \neq 0$) one obtains

$$\frac{\partial I}{\partial h_i} - \sum_{j=1}^{2m} \frac{\partial I}{\partial a_j}\left(y_2(a_j)\frac{\partial \Phi_j}{\partial y_1(a_j)}\right)^{-1}\frac{\partial \Phi_j(a_j)}{\partial h_i} - \int_{D_*} \varphi_* \frac{\partial \Phi_*}{\partial h_i}\, dx = 0, \; i = 1,\dots,m . \tag{5.22}$$

The latter equations serve for determihation of unknown parameters $h_1,\dots,h_m$, whereas the coordinates a_j may be specified according to (5.6) after solving the basic and adjoint equations. In a particular case when the section of the beam for $x \in D_*$ is either homogeneous or quasi- homogeneous with fixed geometrical parameters so that $\partial \Phi_* / \partial h_i = 0$, (5.22) takes the form

$$\sum_{j=1}^{2m} \frac{\partial I}{\partial a_j}\left(y_2(a_j)\frac{\partial \Phi_j}{\partial y_1(a_j)}\right)^{-1}\frac{\partial \Phi_j(a_j)}{\partial h_i} = \frac{\partial I}{\partial h_i}; \; i = 1,\dots,m. \tag{5.23}$$

§1.6. Beams of piece-wise constant thickness

1.6.1. Preliminaries

Let us consider a beam of piece-wise constant thickness hinged at both ends. The beam is subjected to the uniform lateral pressure and to the axial dead load N. It is assumed that the structure is symmetric with respect to the central cross-section and (Fig. 1.1.2)

$$h = h_j \tag{6.1}$$

for $x \in D_j$ $(j = 0,\ldots,m)$. Here $a_0 = 0$ and $a_{m+1} = L$.

The material of the beam is assumed to be a homogeneous rigid-plastic one and the cross-sections of the beam are rectangles with dimensions h_j and B. The latter stands for the width of the beam.

The yield condition for the present case is given by (2.5) and Fig. 1.1.6. Assuming that $M \geqslant 0$ one has now

$$\frac{y_1}{M_{oj}} + \left(\frac{N}{N_{oj}}\right)^2 - 1 \leqslant 0 \tag{6.2}$$

for $x \in D_j$. In (6.2) $M_{oj} = \sigma_0 B\, h_j^2/4$, $N_{oj} = \sigma_0 B\, h_j$. Multiplying (6.2) by h_j^2 one obtains the equation of the yield curve as

$$\Phi_j = \frac{4y_1}{N_*} + \left(\frac{N}{N_*}\right)^2 - h_j \tag{6.3}$$

where $N_* = \sigma_0 B$.

The minimum volume problem with the cost criterion

$$I = \sum_{j=0}^{m} h_j(a_{j+1} - a_j) \tag{6.4}$$

will be investigated in this section. The parameters $a_j (j = 1,\ldots,m)$ and $h_j (j = 0,\ldots,m)$ are handled as previously unknown constants for which (6.4) attains the minimal value.

The minimum of (6.4) is sought for under the requirement that the deflection of the beam with optimal parameters does

not exceed that of an associated beam of constant thickness h_0. Thus, the problem could be considered as a particular case of the one studied in (1.6).
According to (6.3) in this case

$$\frac{\partial \Phi_j}{\partial y_1} = \frac{4}{N_*} , \quad \frac{\partial \Phi_j}{\partial h_j} = - 2h_j . \tag{6.5}$$

Let us study the boundary conditions (2.10), (2.11),now. Due to summetry at the center of the beam

$$y_2(0) = y_4(0) = y_5(0) = 0 . \tag{6.6}$$

The end of the beam at $x = L$ is hinged, thus

$$y_1(L) = y_3(L) = 0 . \tag{6.7}$$

Making use of (6.6), (6.7) one can compile the sets $I_0 = \{2, 4, 5\}$ and $I_L = \{1, 3\}$.

1.6.2. Optimality conditions and state variables

Evidently, the optimality requirements established in the previous section are applicable in the present case after slight modifications. First of all, the number of unknown coordinates a_j is equal to m, now. Thus the index j in (5.6), (5.13) - (5.23) changes up to m.

Making use of (6.4), the relations (5.23) may be converted into

$$\sum_{j=1}^{m} (h_{j-1} - h_j) \frac{\partial \Phi_j(a_j)}{\partial h_j} \left(y_2(a_j) \frac{\partial \Phi_j(a_j)}{\partial y_1(a_j)} \right)^{-1} - \tag{6.8}$$

$$- (a_{i+1} - a_i) = 0 , \quad i = 1,\ldots,m .$$

Since $\partial \Phi_j / \partial h_i = 0$, if $i \neq j$ it follows from (6.8) that

$$(h_{j-1} - h_j) \frac{\partial \Phi_j(a_j)}{\partial h_j} - (a_{j+1} - a_j) y_2(a_j) \frac{\partial \Phi_j(a_j)}{\partial y_1(a_j)} = 0 ,$$

$$j = 1,\ldots,m . \tag{6.9}$$

Finally, substituting the partial derivatives from (6.5) into (6.9), one obtains

$$h_j(h_{j-1} - h_j) + \frac{2}{N_*} y_2(a_j)(a_{j+1} - a_j) = 0; \; j=1,\dots,m \,. \tag{6.10}$$

The second group of equations may be obtained on the grounds of (5.6). Thus, according to (6.3) one has

$$\frac{4}{N_*} y_1(a_j) + \left(\frac{N}{N_*}\right)^2 - h_j^2 = 0 \; ; \quad j = 1,\dots,m \,. \tag{6.11}$$

The set (6.10), (6.11) enables to calculate the values of parameters a_j and h_j. A characteristic feature of this set is that it does not depend upon the values of adjoint variables.

The solution of the state equations (2.9) could be found assuming that a half of the beam is divided into the plastic zone $(0,b) = D_*$ and the rigid zone (b,L). In the plastic zone, according to (6.3) $y_1 = M_*$, whereas in the rigid zone $\nu_j = 0$. Here

$$M_* = \frac{N_*}{4}\left(h_o^2 - \left(\frac{N}{N_*}\right)^2\right) \,. \tag{6.12}$$

Substituting (6.3) into (2.9), one can integrate these equations by turn in different regions. After satisfying the boundary conditions (6.6), (6.7) as well as the continuity requirements of the state variables,one obtains for $x \in (0,b)$

$$\begin{aligned} y_1 &= M_* \,, \\ y_2 &= 0 \,, \\ y_3 &= \frac{P}{2N}(2bL - b^2 - x^2) \,, \\ y_4 &= -\frac{P}{N} x \,, \\ y_5 &= -\frac{1}{6}\left(\frac{P}{N}\right)^2 x^3 + \frac{Px}{2N_*} \,, \end{aligned} \tag{6.13}$$

provided $b < a_1$. For $x \in (b,L)$ the state variables turn to be as follows :

$$y_1 = M_* - \frac{P}{2}(x - b)^2 \quad ,$$

$$y_2 = P(b - x) \quad ,$$

$$y_3 = \frac{Pb}{N}(L - x) \quad , \tag{6.14}$$

$$y_4 = - \frac{P}{N} b \quad ,$$

$$y_5 = - \frac{1}{2}\left(\frac{Pb}{N}\right)^2 x + \frac{1}{3}\left(\frac{P}{N}\right)^2 b^3 + \frac{Pb}{N_*} \quad .$$

The length of the plastic zone in (6.13), (6.14) may be evaluated as

$$b = L - \sqrt{\frac{2M_*}{P}} \quad . \tag{6.15}$$

Calculating $y_1(a_j)$ and $y_2(a_j)$ according to (6.14) and substituting into (6.10) and (6.11), one could obtain the equations

$$N_* h_j(h_{j-1} - h_j) + 2P(b - a_j)(a_{j+1} - a_j) = 0 \; ;$$

$$j = 1,\ldots,m \tag{6.16}$$

and

$$\frac{4}{N_*}\left(M_* - \frac{P}{2}(a_j - b)^2\right) + \left(\frac{N}{N_*}\right)^2 - h_j^2 = 0 \; ;$$

$$j = 1,\ldots,m \; , \tag{6.17}$$

where b stands for the length on the plastic zone, M_* is given by (6.12), whilst P and N stand for loading parameters.

1.6.3. Optimal design of a beam of piece-wise constant thickness

Optimal values of parameters a_j and h_j may be obtained solving the non-linear set of equations (6.16),(6.17). Introducing non-dimensional quantities

$$p = \frac{PL^2}{N_* h_o^2} \quad , \qquad n = \frac{N}{N_* h_o} \tag{6.18}$$

this set may be converted into

$$\begin{aligned} &2(h_o^2 - h_j^2)L^2 - p(a_j - b)^2 h_o^2 = 0 \ , \quad j = 1,\ldots,m \\ &2h_j(h_{j-1} - h_j)L^2 - p(a_j - b)(a_{j+1} - a_j)h_o^2 = 0 \ , \end{aligned} \tag{6.19}$$

where according to (6.15) and (6.12)

$$b = L\left(1 - \sqrt{\frac{2(1 - n^2)}{p}}\right) . \tag{6.20}$$

For numerical solution of the system (6.19) the method of Newton was used. The results of the calculations are presented in Tables 1.6.1 - 1.6.5. In Table 1.6.1 - 1.6.4 optimal values of parameters a_j, h_j are accommodated for different values of loading parameters (here $m = 7$).

Economy of the design established is assessed by the coefficient

$$e = \frac{a_1}{L} + \frac{1}{h_o L}\sum_{i=1}^{m} h_j(a_{j+1} - a_j) \quad , \tag{6.21}$$

which is equal to the ratio of the optimal volume and of the beam of constant thickness, respectively.

Tables 1.6.1, 1.6.2 correspond to the limit state, i.e. $y_3(0) = 0$. The same results may be obtained when using the methods of the limit analysis. It was observed that the data given in Table 1.6.2 also provide the optimal solution in the post-yield range. However, the coordinates of the steps depend on the load intensity.

In Tables 1.6.3, 1.6.4 the optimal values of the coordinates of the cross-sections where the jumps of the thickness occur are accommodated for different values of the load intensity. The first lines in the both tables correspond to the limit state. They coincide with the second and the third rows in Table 1.6.1 and 1.6.2, respectively. The calculations carried out reveal that the coordinates of the steps which are located near the supports are much less sensitive to the changes of the loading intensity (in comparison with

Table 1.6.1. Optimal non-dimensional coordinates of the steps

n	a_1/L	a_2/L	a_3/L	a_4/L	a_5/L	a_6/L	a_7/L	e
0,0	0,320	0,475	0,600	0,707	0,800	0,880	0,948	0,825
0,2	0,315	0,469	0,593	0,700	0,792	0,873	0,943	0,835
0,4	0,307	0,458	0,580	0,685	0,778	0,862	0,935	0,861
0,6	0,299	0,446	0,566	0,671	0,765	0,850	0,929	0,899
0,8	0,290	0,434	0,553	0,658	0,753	0,841	0,923	0,946

Table 1.6.2. Optimal non-dimensional thicknesses

n	h_1/h_o	h_2/h_o	h_3/h_o	h_4/h_o	h_5/h_o	h_6/h_o	h_7/h_o	e
0,0	0,948	0,880	0,800	0,707	0,600	0,475	0,320	0,825
0,2	0,951	0,888	0,814	0,728	0,630	0,518	0,383	0,835
0,4	0,960	0,908	0,847	0,778	0,701	0,617	0,512	0,861
0,6	0,971	0,934	0,892	0,844	0,791	0,733	0,669	0,899
0,8	0,985	0,965	0,943	0,919	0,892	0,863	0,833	0,946

Table 1.6.3. Optimal coordinates of the steps for n=0,2

p	a_1/L	a_2/L	a_3/L	a_4/L	a_5/L	a_6/L	a_7/L	e
1,92	0,315	0,469	0,593	0,700	0,792	0,873	0,943	0,835
2,22	0,363	0,506	0,622	0,721	0,807	0,882	0,947	0,846
2,52	0,402	0,536	0,645	0,738	0,819	0,889	0,950	0,858

Table 1.6.4. Optimal coordinates of the steps for n=0,4

p	a_1/L	a_2/L	a_3/L	a_4/L	a_5/L	a_6/L	a_7/L	e
1,68	0,307	0,458	0,580	0,685	0,778	0,862	0,935	0,861
1,88	0,345	0,487	0,603	0,702	0,791	0,869	0,939	0,869
2,08	0,377	0,513	0,622	0,717	0,801	0,876	0,942	0,875
2,28	0,405	0,534	0,639	0,730	0,810	0,881	0,945	0,881

Table 1.6.5. Volume ratios for different numbers of steps

m	1	2	3	4	5	10	50	100
e	0,914	0,878	0,858	0,846	0,837	0,815	0,792	0,789

those located near center of the beam).

The values of the economy coefficient (6.21) are presented in Table 1.6.5. for different numbers of jumps of the thickness. The given values correspond to load carrying capacity. Here $N = 0$.

The earlier investigations have predicted the value of the economy coefficient e = 0,785 for a beam of continuously variable thickness.The volumes of the designs of piece-wise constant thickness are quite close to this value in the cases of large numbers of steps, as shown in Table 1.6.5.

In order to be convinced in the non-singularity of the problem let us examine the solution of the adjoint system (5.12). In keeping with the boundary conditions for the state variables (6.6), (6.7) the transversality conditions become now as

$$\psi_1(0) = \psi_3(0) = \psi_2(L) = \psi_4(L) = \psi_5(L) = 0. \quad (6.22)$$

In addition to (6.22), one has to take into account the intermediate conditions at $x = a_j$ and $x = b$. According to (5.19) and (6.4) one has

$$[\psi_1(a_j)] = \frac{h_j - h_{j-1}}{y_2(a_j)} \; ; \quad j = 1, \ldots, m \quad (6.23)$$

$$[\psi_i(a_j)] = 0 \; ; \; i = 2, 3, 4, 5.$$

The jump conditions at the boundary point of the plastic zone might be obtained from (3.51) as

$$[\psi_2(b)] = \varphi_{10} \; ; \quad [\psi_j(b)] = 0 \; ; \quad j = 1, 3, 4, 5. \quad (6.24)$$

When integrating the set (5.12) considering φ_{10} as a constant and accounting for (6.22) - (6.24), one can state that

$$\psi_1 = \begin{cases} - dx, & x \in (0, b), \\ - db, & x \in (b, a_1), \\ - db + d_{1j}, & x \in D_j, \end{cases}$$

$$(6.25)$$

$$\psi_2 = \begin{cases} \frac{d}{2}(x^2 - a_2), & x \in (0, b) \; , \\ db(x - a_1) + d_{21}, & x \in (b, a_1) \\ (db - d_j)(x - a_j) + d_{2j}, & x \in D_j, \end{cases}$$

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$$\psi_3 = - dNx, \qquad x \in (0, L),$$

$$\psi_4 = \frac{1}{2} dN(x^2 - L^2), \qquad x \in (0, L).$$

In (6.25) the following notation is used:

$$d = \frac{\varphi_{10}}{N}, \quad d_{1j} = [\psi_1(a_j)], \qquad (6.26)$$

$$d_{2j} = -\sum_{i=j+1}^{m} (bd - d_{1j})(a_{i+1} - a_i); \; j = 1,\dots,m.$$

Note finally that d is an unknown constant in (6.25), (6.26). If, however, the thickness h_0 is not fixed previously (5.22) furnishes the relation

$$\varphi_{10} = - \frac{a_1}{2bh_0}$$

which enables us to determine the quantity d.

§1.7. Reinforced beams

1.7.1. Statement of the problem

Let us consider a non-homogeneous beam which consists of a rigid-plastic matrix with two layers of the reinforcement (Fig. 1.7.1). It is assumed that the structure is symmetric

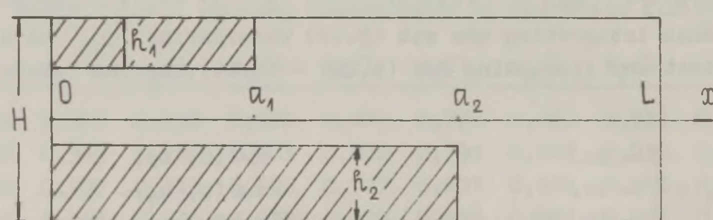

Fig. 1.7.1. Non-homogeneous beam with two layers of reinforcement

with respect to the central cross-section. Thus, the upper and lower layers of the reinforcement are of the lengths $2a_1$ and $2a_2$, respectively. The matrix material as well as that of reinforcement are assumed to be rigid-plastic ones, σ_0

and σ_1 being the yield stresses. It is reasonable to suppose that $\sigma_1 > \sigma_0$.

Let the structure be subjected to the axial tension N and uniformly distributed trahsverse pressure of intensity P. The ends of the beam are simply supported. Therefore, (6.6) and (6.7) might be applied in the present section as well.

The total thickness (heigth of the cross-section) of the beam H is fixed, whereas the dimensions of the layers of the reinforcement a_1, a_2, h_1, h_2 are preliminarily unknown constants. The optimal values of these parameters are sought for under the condition that the volume of layers

$$I = B\,(h_1 a_1 + h_2 a_2) \tag{7.1}$$

attains the minimal value. We look for the minimum of (7.1) taking into account that the deflection of the beam must not exceed that of the associated quasi-homogeneous beam. In the quasi-homogeneous case $a_1 = a_2 = L$.

1.7.2. Yield conditions for non-homogeneous beams

The problem set up above was examined by the author and Majak, 1988; 1989. In these studies a method for construction of the yield curve was suggested. This technique will be used in the present study. The parametrical equations of the yield curve will be derived for each region $D_j = (a_j, a_{j+1})$, where $j = 0, 1, 2$ and $a_0 = 0$, $a_3 = L$.

The yield condition (5.1) could be expressed as

$$\Phi_j = y_1 - M_j^0\,(N, h_1, h_2)\ ; \quad j = 0, 1, 2\,. \tag{7.2}$$

The generalized stresses M and N are related to the normal stress as follows

$$M = B \int_{-H/2}^{H/2} z\sigma\,dz\ ; \quad N = B \int_{-H/2}^{H/2} \sigma\,dz \tag{7.3}$$

where B is width of the cross-section of the beam and z stands for the local coordinate axis. Since $N_1 = N$ the subscripts relevant to N are omitted.

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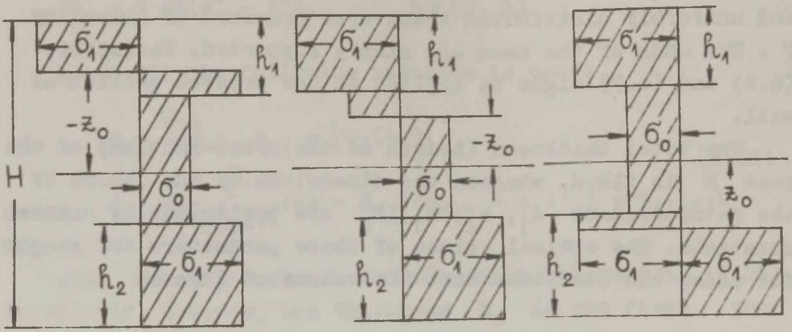
The statically admissible stress distributions along the thickness of the beam are presented in Fig.1.7.2 a,b,c.



a b c

Fig. 1.7.2. Admissible distributions of stresses

Here z_0 stands for the coordinate of neutral axis. As shown in Fig. 1.7.2, three different cases of the stress distribution must be examined. If

$$-\frac{H}{2} + h_1 \leq z_0 \leq \frac{H}{2} - h_2 \tag{7.4}$$

then according to Fig. 1.7.2 and (7.3) one has

$$\begin{aligned} M &= B\,б_0\left\{-z_0^2 + \frac{1}{8}(1-k)((H-2h_1)^2 + (H-2h_2)^2)\right\}, \\ N &= B\,б_0\left\{-z_0 + (k-1)(h_2-h_1)\right\}, \end{aligned} \tag{7.5}$$

where $k = б_1/б_0$.

If, however,

$$-\frac{H}{2} \leqslant z_0 \leqslant -\frac{H}{2} + h_1 \tag{7.6}$$

or

$$\frac{H}{2} - h_2 \leqslant z_0 \leqslant \frac{H}{2} \tag{7.7}$$

similary to the previous case one obtains

$$M = B\sigma_0\left\{-kz_0^2 + \frac{k}{4}H^2 + \frac{1}{8}(k-1)((H-2h_1)^2-(H-2h_2)^2)\right\},$$
$$N = B\sigma_0\left\{-2kz_0 \pm (k-1)(H-h_1-h_2)\right\}, \tag{7.8}$$

where the sign plus applies when (7.6) is satisfied and minus is associated with the case (7.7).

Taking into account that the ends of the beam are simply supported, it seems reasonable to assume that $M \geqslant 0$; $N \geqslant 0$. Elimination of z_0 from (7.5) and (7.8) gives the equations of the yield curves. The yield curves have different equations for different ranges of axial tension. Using the notation (7.2), one has for $x \in D_0$

$$M_0^0 = M_0\left\{k - \frac{1}{k}\left(\frac{N}{N_0} - \frac{(k-1)}{H}(H-h_1-h_2)\right)^2 + \right.$$
$$\left. + \frac{(k-1)}{2H^2}((H-2h_2)^2 - (H-2h_1)^2)\right\}, \tag{7.9}$$

if $0 \leqslant N \leqslant N_1^0$,

$$M_0^0 = M_0\left\{k - \left(\frac{N}{N_0} - \frac{(k-1)}{H}(h_2-h_1)\right)^2 - \right.$$
$$\left. - \frac{(k-1)}{2H^2}((H-2h_2)^2 + (H-2h_1)^2)\right\}, \tag{7.10}$$

if $N_1^0 \leqslant N \leqslant N_2^0$ and

$$M_0^0 = M_0\left\{k - \frac{1}{k}\left(\frac{N}{N_0} + \frac{(k-1)}{H}(H-h_1-h_2)\right)^2 + \right.$$
$$\left. + \frac{(k-1)}{2H^2}((H-2h_1)^2 - (H-2h_2)^2)\right\}, \tag{7.11}$$

if $N_2^0 \leqslant N \leqslant N_2^0 + 2\sigma_1 Bh_1$.

In (7.9) - (7.11) the following notation is used

$$N_0 = \sigma_0 BH, \quad M_0 = \frac{\sigma_0}{4}BH^2, \quad k = \frac{\sigma_1}{\sigma_0},$$
$$N_1^0 = N_0\left(-1 + 2\frac{h_2}{H} + \frac{(k-1)}{H}(h - h)\right), \tag{7.12}$$
$$N_2^0 = N_0\left(1 - 2\frac{h_1}{H} + \frac{(k-1)}{H}(h_2 - h_1)\right).$$

It was assumed above that $N \geqslant 0$. Thus, according to (7.12) N_1^0 is non-negative, if

$$k \geqslant \frac{H - h_1 - h_2}{h_2 - h_1} \quad . \tag{7.13}$$

If, however, (7.13) is not satisfied then (7.9) may be disregarded and (7.10) applies for $0 \leqslant N \leqslant N_2^o$. Furthermore, if N_2^o is not positive the relation (7.11) holds good for each value of N .

In the region D_1, making use of (7.3) one obtains

$$\begin{aligned} M &= B\sigma_o(-z_o^2 + \tfrac{1}{8}(1 - k)(H^2 + (H - 2h_2)^2)) \ , \\ N &= B\sigma_o(-2z_o + (k - 1)h_2) \ , \end{aligned} \tag{7.14}$$

if

$$-\frac{H}{2} \leqslant z_o \leqslant \frac{H}{2} - h_2 \tag{7.15}$$

and

$$\begin{aligned} M &= B\sigma_o(-kz_o^2 + \tfrac{k}{4}H^2 + \tfrac{1}{8}(k - 1)(H^2 - (H - 2h_2)^2)) \ , \\ N &= B\sigma_o(-2kz_o - (k - 1)(H - h_2)) \ , \end{aligned} \tag{7.16}$$

if

$$\frac{H}{2} - h_2 \leqslant z_o \leqslant \frac{H}{2} \quad . \tag{7.17}$$

The relations (7.14) - (7.17) lead to the yield curve associated with

$$\begin{aligned} M_1^2 = M_o(&\frac{(1+k)}{2} - \frac{1}{k}(\frac{N}{N_o} - \frac{(k-1)}{H}(H - h_2))^2 + \\ &+ \frac{(k-1)}{2H^2}(H - 2h_2)^2) \ , \end{aligned} \tag{7.18}$$

if

$$0 \leqslant N \leqslant N_o(-1 + \frac{(1+k)}{H}h_2) \tag{7.19}$$

and

$$M_1^o = M_o(k - \frac{1}{k}(\frac{N}{N_o} + \frac{(1-k)}{H}h_2)^2 +$$

$$+ \frac{(1-k)}{2H^2}((H - 2h_2)^2 - H^2)) \;, \qquad (7.20)$$

if

$$N_o(-1 + \frac{(1+k)}{H}h_2) \leqslant N \leqslant N_o(1 + \frac{(k-1)}{H}h_2) \;. \qquad (7.21)$$

Finally, for $x \in D_2$ one obtains

$$M_2^o = M_o(1 - (\frac{N}{N_o})^2) \qquad (7.22)$$

as might be expected, since in this section the material of the beam is uniform. The relation (7.22) holds good for each N from the interval $0 \leqslant N \leqslant N_o$.

Thus, the equation of the yield curve for the reinforced beam is specified by (7.2), where M_j^o for $x \in D_j$ is given by (7.9) - (7.11), (7.18) - (7.21) and (7.22), respectively.

1.7.3. Necessary optimality conditions

The problem posed herein may be considered as a particular case of the problem set up in the section 1.5.1. The unknown parameters are a_1, a_2, h_1, h_2, now.

In the present case according to (3.24), (7.1)

$$G = B(h_1a_1 + h_2a_2) + \mu_1\Phi_1(a_1) + \mu_2\Phi_2(a_2) \;,$$

$$f_j = \Phi_j(y_1(a_j), N, h_1, h_2) \;; \quad j = 1,2 \qquad (7.23)$$

$$\Phi_* = \Phi_o \;.$$

Therefore, (5.5) yields

$$y_1 - M_o^o(N, h_1, h_2) = 0 \qquad (7.24)$$

for $x \in (0,b)$. From (5.13) and (7.1) one obtains

$$[\mathcal{L}^*(a_j)] + Bh_j = 0 \;; \quad j = 1,2 \qquad (7.25)$$

where the jumps of the Lagrangian function may be determined by (5.18). Combining (5.18) and (7.25) gives

$$[\psi_1(a_j)] = -\frac{Bh_j}{y_2(a_j)}\ ; \quad j = 1,2\ . \qquad (7.26)$$

On the other hand, it appears that (5.16) and (5.17) hold good for $j = 1,2$, too. Thus,

$$[\psi_i(a_j)] = \mu_j\frac{\partial\Phi_j(a_j)}{\partial y_i(a_j)}\ ; \quad i = 1,\ldots,5; \quad j = 1,2\ . \qquad (7.27)$$

In the case $i = 1$ the latter may be used for determination of the Lagrangian multipliers $\mu_j (j = 1,2)$.

According to (7.23), (7.24), $\partial\Phi_*/\partial h_j$ does not vanish, which implies that (5.23) may not be applied. The equations (5.22) with (7.1), (7.23), (7.24) lead to

$$Ba_i - \sum_{j=1}^{2} Bh_j \frac{1}{y_2(a_j)}\frac{\partial\Phi_j(a_j)}{\partial h_i} - \int_0^b \varphi_o \frac{\partial\Phi_o}{\partial h_i}\, dx = 0,\ i=1,2. \qquad (7.28)$$

It follows from (7.24) that

$$\frac{\partial\Phi_o}{\partial h_i} = -\frac{\partial M_o^o(N,h_1,h_2)}{\partial h_i} = \text{const}\ . \qquad (7.29)$$

Furthermore, $\varphi_o' = 0$ for $x \bar{\in} (0,b)$. Thus, it appears that $\varphi_o = c\nu_o^2$, where ν_o is constant and consequently, $\varphi_o = \text{const}$.

Integrating (7.28) making use of (7.24), (7.29) gives

$$\frac{Bh_1}{y_2(a_1)}\frac{\partial\Phi_1}{\partial h_j} + \frac{Bh_2}{y_2(a_2)}\frac{\partial\Phi_2}{\partial h_j} + \varphi_o b\frac{\partial\Phi_o}{\partial h_j} - Ba_j = 0, \quad j=1,2. \qquad (7.30)$$

It is easy to check using (7.2), (7.18) - (7.22) that

$$\frac{\partial\Phi_1}{\partial h_1} = \frac{\partial\Phi_2}{\partial h_1} = 0 \qquad (7.31)$$

which enables us to determine from (7.30)

$$\varphi_o = -\frac{Ba_1}{b}\left(\frac{\partial M_o^o}{\partial h_1}\right)^{-1}\ . \qquad (7.32)$$

Substituting (7.32) into (7.30) gives the equation

$$\frac{h_1}{y_2(a_1)}\frac{\partial M_1^o}{\partial h_2} - a_1\frac{\partial M_o^o}{\partial h_2}\left(\frac{\partial M_o^o}{\partial h_1}\right)^{-1} + a_2 = 0 \qquad (7.33)$$

which, accompanied with (5.3) and (5.6) leads to the optimal values of unknown parameters.

1.7.4. Optimal design of the reinforced beam

It appears that the state variables are specified by (6.13), (6.14) in the present case too, where M_* is to be substituted by M_o^o. Thus, (6.15) leads to

$$b = L - \sqrt{\frac{2M_o^o}{P}} \quad . \tag{7.34}$$

It is supposed that $y_3(0) = W_o$, where W_o is to be considered as a given constant. This constant is equal to the maximal deflexion of the reference beam. Making use of (6.13), one obtains

$$W_o = \frac{P}{2N}(2bL - b^2) \ , \tag{7.35}$$

which enables to determine

$$b = L - \sqrt{L^2 - \frac{2}{P} NW_o} \quad . \tag{7.36}$$

Combining (7.34) and (7.36) gives

$$M_o^o = \frac{1}{2} PL^2 - NW_o \tag{7.37}$$

which, in turn, with (5.6) and (6.14) leads to the following system

$$\begin{aligned} M_1^o &= M_o^o - \frac{P}{2}(a_1 - b)^2 \ , \\ M_2^o &= M_o^o - \frac{P}{2}(a_2 - b)^2 \ . \end{aligned} \tag{7.38}$$

The set (7.38) may be converted into

$$\begin{aligned} a_1 &= b + \sqrt{L^2 - \frac{2}{P}(NW_o + M_1^o)} \ , \\ a_2 &= b + \sqrt{L^2 - \frac{2}{P}(NW_o + M_2^o)} \ . \end{aligned} \tag{7.39}$$

Equations (7.39) with (7.33) and (7.37) are solved numerically after substituting into (7.33) partial derivatives calculated making use of (7.9) - (7.11), (7.18) - (7.21).

The adjoint set (5.12) might be integrated similarly to the previous case. The solution which satisfies (6.22), (6.24), (7.26), (7.27) may be presented as

$$\psi_1 = \begin{cases} -\varphi_o x , & x \in (0,b) , \\ -\varphi_o b , & x \in (b,a_1) , \\ -\varphi_o b + d_1 , & x \in (a_1,a_2) , \\ -\varphi_o b + d_1 + d_2 , & x \in (a_2,L) , \end{cases}$$

$$\psi_2 = \begin{cases} \frac{\varphi_o}{2}(x^2 - L^2) , & x \in (0,b), \\ \varphi_o b(x-a_1)+(\varphi_o b-d_1)(a_1-a_2)+(\varphi_o b-d_1-d_2)(a_2-L) , & x \in (b,a_1), \\ (\varphi_o b-d_1)(x-a_2) + (\varphi_o b-d_1-d_2)(a_2-L) , & x \in (a_1,a_2), \\ (\varphi_o b-d_1-d_2)(x-L) , & x \in (a_2,L), \end{cases} \qquad (7.40)$$

$$\psi_3 = -\varphi_o N x ,$$

$$\psi_4 = \frac{\varphi_o}{2} N(x^2 - L^2) ,$$

where

$$d_j = [\psi_1(a_j)] \; ; \quad j = 1,2 \; ; \qquad (7.41)$$

and φ_o is given by (7.32). Evidently, the constants d_j could be found from (7.26).

The economy of the design established could be assessed by the ratio

$$e = \frac{IL}{V_o} \qquad (7.42)$$

where I stands for the minimal value of (7.1), whereas $V_o = B(h_1 + h_2)L$. The quantity V_o may be interpreted as the amount of the reinforcement in the case when both layers stretch up to the end of the beam. Naturally, both, the optimal and reference beams have a common maximal deflection.

Numerical results (the solutions of the set (7.33), (7.37) with (7.39)) are presented in Tables 1.7.1 - 1.7.4. Here the following notation is used

$$w_o = \frac{N_o W_o}{M_o} , \qquad p = \frac{PL^2}{M_o} , \qquad n_1 = \frac{N}{N_o} . \qquad (7.43)$$

Tables 1.7.1 and 1.7.2 correspond to $k = 2$ and $n_1 = 0,5$. In Table 1.7.1 $p = 3$, whereas Table 1.7.2 is associated with fixed maximal deflection $w_o = 1$. It follows from Table 1.7.1 that for $w_o \geqslant 1,2$ the upper layer of reinforcement is unnecessary, provided the lateral loading and axial tension are fixed.

Table 1.7.1. Optimal dimensions of the layers for different deflections

w_o	h_1/H	h_2/H	a_1/L	a_2/L	b/L	e
0	0,116	0,283	0,307	0,707	0	0,591
0,4	0,057	0,200	0,285	0,675	0,069	0,588
0,8	0,017	0,126	0,261	0,627	0,144	0,583
1,2	0	0,053	0,225	0,542	0,225	0,542
1,4	0	0,017	0,270	0,452	0,270	0,452
1,5	0	0	0,293	0,293	0,293	0

Table 1.7.2. Optimal dimensions of the layers for different values of the load intensity

p	h_1/H	h_2/H	a_1/L	a_2/L	b/L	e
2,5	0	0	0,225	0,225	0,225	0
2,6	0	0,017	0,216	0,412	0,216	0,412
2,8	0	0,053	0,210	0,526	0,198	0,525
3,0	0,005	0,089	0,245	0,592	0,184	0,574
3,8	0,075	0,245	0,365	0,727	0,142	0,642
4,6	0,256	0,474	0,478	0,791	0,115	0,681

According to Table 1.7.2 for $p \leqslant 2.8$ the reinforcement must be utilized in the lower layer only. In the case $p = 2,5$ the beam carries given loads without reinforcement.

Table 1.7.3 corresponds to the case when $p = 4,2$; $n_1 = 0,5$ and $w_o = 1$. It appears that the length of the lower layer of the reinforcement does not depend on the ratio of the yield stresses of the reinforcement and matrix, respectively. The same may be asserted with respect to the length of the plastic region. In Table 1.7.3 might be accommodated

$b = 0{,}127$ and $a_2 = 0{,}763$ for each value of the ratio k.

Table 1.7.3. Influence of the yield stresses on the optimal values of parameters

k	h_1/H	h_2/H	a_1/L	e
2,0	0,144	0,338	0,419	0,661
2,2	0,102	0,268	0,404	0,664
2,4	0,079	0,223	0,394	0,667
2,6	0,064	0,192	0,387	0,669
2,8	0,054	0,168	0,381	0,671
3,0	0,046	0,150	0,376	0,673

Table 1.7.4. Optimal values of parameters for different values of the axial tension

n_1	h_1/H	h_2/H	a_1/L	a_2/L	b	e
0,2	0,062	0,116	0,329	0,545	0,069	0,470
0,4	0,016	0,095	0,263	0,560	0,144	0,517
0,6	0,000	0,08989	0,232	0,642	0,225	0,642
0,8	0	0,103	0,318	0,793	0,317	0,793

Table 1.7.4 is associated with $p = 3$; $w_0 = 1$ and $k = 2$. It follows from Table 1.7.4 that in the case of large values of the axial tension (for $n_1 = 0{,}6$) the upper layer of the reinforcement is unnecessary ($h_1 = 0$).

Let us consider the same optimization problem under the condition that the dimensions of the upper and lower layer are equal to each other (Fig. 1.7.3).

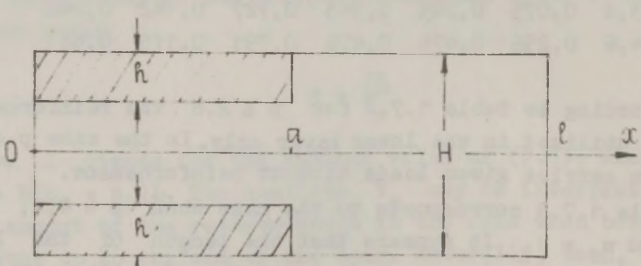


Fig. 1.7.3. Beam with two identical layers of reinforcement

The yield curves for the beam presented in Fig. 1.7.3 may be found according to (7.2) - (7.22) putting $h_1 = h_2 = h$. The derivation of the optimality conditions is similar to the previous case and therefore it will be omitted herein.

The results of the calculations are presented in Tables 1.7.5. and 1.7.6. The results accommodated herein correspond to the case $k = 2$, $n_1 = 0,5$. Now $h = h_1 = h_2$ and $a = a_1 = a_2$.

Table 1.7.5. Optimal values of parameters for the case associated with two identical layers of the reinforcement

p	h/H	a/L	b/L	V
2,0	0,032	0,418	0,065	0,027
2,5	0,105	0,599	0,051	0,125
3,0	0,194	0,688	0,043	0,267
3,5	0,317	0,744	0,036	0,471

The data presented in Table 1.7.5 are found for the maximal deflection $w_0 = 0,25$. The economy coefficient $e = a/L$ in the present case. Thus, the corresponding values of the ratio e are presented in the third column of Table 1.7.5. In the last column the non-dimensional volumes of the reinforcement layers are given, i.e. $V = 2ah/BHL$.

Table 1.7.6. Influence of the optimal dimensions of layers on the maximal deflection

w_0	h/H	a/L	b/L	V
0	0,250	0,707	0	0,354
0,25	0,194	0,688	0,043	0,267
0,50	0,146	0,665	0,087	0,195
0,75	0,105	0,634	0,134	0,133
1,00	0,067	0,592	0,184	0,079
1,25	0,032	0,525	0,236	0,034
1,50	0	0,293	0,293	0

Table 1.7.6 corresponds to $p = 3$. It appears that the greater deflection, the smaller the amount of the reinforcement. For $w_0 \geq 1,5$ the reinforcement is unnecessary ($V = 0$).

Note finally that the two types of optimization problems should be considered as independent problems since the loading parameters as well as material constants and the deflection of the beam are assumed to be fixed previously. Thus, the comparison of the respective solutions is quite problematic.

1.8. Optimal design of plastic beams for given deflected shape

1.8.1. Problem formulation

Let us consider a beam of length $2L$ simply supported at the ends. The beam is made of a uniform rigid-plastic material and has a variable thickness $H = H(x)$. The beam is subjected to a distributed transverse loading of intensity $P(x)$ and to the axial force N. Let the origin of the coordinate system be located at the median cross-section of the beam. Due to the symmetry the attention will be confined to the right-hand side of the beam, only.

We are looking for the design of the beam with variable cross-section whose volume

$$J = \int_0^L H(x)dx \tag{8.1}$$

attains the minimum value. Minimum of (8.1) is to be determined under the condition that the deflection y_3 does not exceed the deflection W_* of a beam with given thickness H_0. Here H_0 may stand for a constant as well as a given continuous function. Thus, one has to take into account that

$$y_3 - W_* \leq 0 \tag{8.2}$$

when minimizing the cost criterion (8.1).

This problem was studied by the author in 1981 assuming that in (8.2) applies the equality sign for each x.

1.8.2. Necessary optimality conditions

The posed problem may be considered as a particular case of the problem set up in §1.1. Now, according to (1.2),(1.13), (1.14), one has

$$K = 0 \ , \quad G_o = 0 \ , \quad F_o = H(x) \ , \quad R_1 = y_3 - W_* \ ,$$
$$r = 1 \ , \quad S_{io} = A_i = 0 \ , \quad f_{ij} = g_{ij} = 0. \tag{8.3}$$

Since (8.2) presents the second-order constraint imposed on the state variables, the optimality conditions derived in section 1.3.6 might be used in the present case. Making use of a control variable v_1, inequality (8.2) may be converted into the equality (5.8) which leads to the optimality condition (3.49). The latter gives

$$\varphi_{10} v_1 = 0 \ . \tag{8.4}$$

The beams with homogeneous rectangular cross-sections as well as the sandwich-type beams with variable face-sheet thickness will be studied. The yield condition is assumed to be expressed as

$$\Phi(y_1, N, H(x)) + \theta^2 = 0 \tag{8.5}$$

where H stands for the variable thickness and according to (3.48) the control function θ must meet the requirement

$$\varphi\theta = 0 \ . \tag{8.6}$$

The Lagrangian function is obtained from (3.14), (3.47), (8.1) - (8.3) as

$$\mathscr{L} = -H + \psi_1 y_2 + \psi_2\left(\nu^2 N \frac{\partial\Phi}{\partial y_1} - P\right) + \psi_3 y_4 - \psi_4 \nu^2 \frac{\partial\Phi}{\partial y_1} +$$
$$+ \psi_5\left(\nu^2 \frac{\partial\Phi}{\partial N} - \frac{1}{2} y_4^2\right) + \varphi(\Phi + \theta^2) + \varphi_{10}(y_3 - W_* + v_1^2) \ , \tag{8.7}$$

where the unnecessary subscripts are omitted.

Making use of (8.7) one can put the optimality conditions (3.29) into the form

$$-1 + \varphi \frac{\partial\Phi}{\partial H} + \nu^2 \frac{\partial}{\partial H}\left(\psi_2 N \frac{\partial\Phi}{\partial y_1} + \psi_4 \frac{\partial\Phi}{\partial y_1} + \psi_5 \frac{\partial\Phi}{\partial N}\right) = 0 \tag{8.8}$$

and

$$\nu\left((\psi_2 N - \psi_4) \frac{\partial\Phi}{\partial y_1} + \psi_5 \frac{\partial\Phi}{\partial N}\right) = 0 \ . \tag{8.9}$$

It appears that the boundary conditions for the state variables may be given by (6.6) and (6.7) for the present case. Thus, from the transversality conditions one obtains $\psi_5 = 0$ which implies that (5.10) - (5.12) are also applicable.

Bearing in mind that $\nu = 0$ in a rigid zone and (5.10) applies in a plastic region, equation (8.8) may be presented as

$$- 1 + \varphi\frac{\partial\Phi}{\partial H} = 0 \, . \tag{8.10}$$

Evidently, (8.9) coincides with (5.10) in the plastic zones where $\nu \neq 0$. When substituting the multiplier φ from (8.10) into (5.10) and (5.12) leads to the adjoint set

$$\psi_1' = \begin{cases} -\frac{\partial\Phi}{\partial y_1}\left(\frac{\partial\Phi}{\partial H}\right)^{-1} , & \Phi = 0 \, ; \\ 0 , & \Phi < 0 \, ; \end{cases} \tag{8.11}$$

$$\psi_2' = -\psi_1 \; ; \quad \psi_3' = -\varphi_{10} \; ; \quad \psi_4' = -\psi_3$$

which could be integrated accounting for corresponding transversality and jump conditions.

Necessary optimality conditions are presented by (8.4), (8.6) and (8.9), whereas (8.10)may be considered as an equation for determination of the multiplier φ. Equation (8.10) gives a solution $\varphi \neq 0$. Consequently, in (8.6) $\theta = 0$ and according to (8.5) one has

$$\Phi(y_1, N, H(x)) = 0 \, , \tag{8.12}$$

from where the unknown function $H = H(x)$ may be determined.

Equation (8.4) results in

$$\varphi_{10} = 0 \, , \quad v_1 \neq 0 \tag{8.13}$$

or

$$\varphi_{10} \neq 0 \, , \quad v_1 = 0 \, . \tag{8.14}$$

The latter case corresponds to the equality $y_3 = W_*$. Bearing in mind (8.13), (8.14) one has to compile the optimal solution, so that in certain regions the deflections of the optimal beam and the reference beam coincide, but out of these regions the respective deflections might be different. However, if the reference beam has a constant thickness, the equality applies at each point $x \in (0,L)$.

1.8.3. Large deflections of a beam of given shape

The solution of the state equations (2.9) corresponding to a beam of constant thickness and rectangular cross-section was presented by (6.13) - (6.15). Evidently, this solution holds good in the case of sandwich beams when M_* and N_* in (6.13) - (6.15) are modified suitably. In the present section this solution will be extended to the case of beams with variable thickness.

The attention will be confined to the class of beams for which the stress-strain state corresponds to a plastic state near the center of the beam for $x \in (0,b)$ and to the rigid state out of this region, Thus, it may be assumed that

$$y_1 = M_*(N,x) \tag{8.15}$$

for $x \in (0,b)$ and

$$\vartheta = \frac{d^2 y_3}{dx^2} = 0 \tag{8.16}$$

for $x \in (b,L)$. Here $M_*(N,x)$ stands for the solution of (8.12) with respect to y_1.

Integrating of (2.9) and making use of (8.15), (8.16) leads to (it is assumed that $P = \text{const}$)

$$y_3 = \begin{cases} -\frac{P}{2N}x^2 - \frac{1}{N}(M_*(N,x) - M_*(N,0)) + W_0, & x \in (0,b), \\ \frac{1}{N}(L - x)(Pb - Q(b)), & x \in (0,L), \end{cases} \tag{8.17}$$

and

$$y_1 = \begin{cases} M_*(N,x), & x \in (0,b), \\ -\frac{P}{2}(x - b)^2 + Q(b)(x - b) + M_*(N,b), & x \in (b,L) \end{cases} \tag{8.18}$$

where the boundary conditions (6.6), (6.7) as well as the continuity requirements for y_1, y_2, y_3, y_4 $x = b$ are taken into account. Here

$$Q(x) = \frac{dM_*(N,x)}{dx} \quad . \tag{8.19}$$

The maximal deflection W_0 may be evaluated as

$$W_0 = \frac{1}{N}(PLb - \frac{P}{2}b^2 + Q(b)(L - b) + M_*(N,b) - M_*(N,0)) \tag{8.20}$$

whereas the length of the plastic zone has to be determined according to the equation

$$\frac{P}{2}(b - L)^2 + Q(b)(L - b) + M_*(N,b) = 0 \ . \tag{8.21}$$

In the case when the thickness is a constant (8.20) gives

$$W_0 = \frac{P}{N}(Lb - \frac{1}{2}b^2) \tag{8.22}$$

and (8.21) coincides with (6.15), provided M_* is in consistent with (8.15).

1.8.4. Optimal design of plastic beams in the post--yield range

Let us consider the beams of uniform rectangular cross--section with width B and heigth H. The yield curve for this case is presented in Fig. 1.1.6, also by (6.2), (6.3). It should be presented as

$$\Phi = \frac{4|y_1|}{N_*} + (\frac{N}{N_*})^2 - H^2 = 0 \ . \tag{8.23}$$

For the reference beam let us employ the beam of constant thickness H_0. In this case $M_* = \text{const}$ and $Q = 0$ in (8.17) - (8.21). Besides, the validity of (8.14) might be expected. The latter implies that the deflection y_3 coincides with the corresponding deflection of the reference beam.

Within the limits of the non-linear beam theory the bending moment y_1 and the deflection y_3 are coupled due to the equilibrium equations and the associated deformation law (2.9). Thus, for determination of the optimal design one can use the bending moment distribution (8.18) associated with the beam of constant thickness.

The optimality criterion for the general case is expressed by (8.12), where y_1 is the distribution of the bending moment. In the present case it means the use of (8.23) and (8.18) under the above assumptions. Thus, one obtains easily the optimal thickness (evidently $y_1 \geqslant 0$)

$$v = \begin{cases} 1 & x \in (0,b), \\ \sqrt{1 - (1 - n^2)\left(\frac{x - b}{L - b}\right)^2}, & x \in (b,L), \end{cases} \tag{8.24}$$

where

$$b = L\left(1 - \sqrt{\frac{2(1 - n^2)}{p}}\right) \tag{8.25}$$

and

$$v = \frac{H}{H_0}, \quad n = \frac{N}{N_0^o}, \quad p = \frac{PL^2}{M_0^o}. \tag{8.26}$$

Here M_0^o, N_0^o stand for the limit moment and limit load for the beam of constant thickness, i.e. $M_0^o = \sigma_0 BH_0^2/4$, $N_0^o = \sigma_0 BH_0$.

In the case of the beams of sandwich-type (8.23) must be replaced by

$$\Phi = \frac{y_1}{hN_*} + \frac{N}{2N_*} - H = 0. \tag{8.27}$$

H being the thickness of carrying layers and h is the total thickness of the beam. Evidently (8.18) holds good in the present case if $Q = 0$ and

$$M_* = h(H_0 N_* - N/2). \tag{8.28}$$

Combining (8.18) and (8.27), (8.28) leads to the distributinn of the material in carrying layers as

$$v = \begin{cases} 1 & x \in (0,b), \\ 1 - \frac{p}{2L^2}(x - b)^2, & x \in (b,L), \end{cases} \tag{8.29}$$

where (8.26) might be used for suitable N_0^o, M_0^o and

$$b = L\left(1 - \sqrt{\frac{2(1 - n)}{p}}\right). \tag{8.30}$$

Economy of the designs established herein could be assessed by the coefficient

$$e = \frac{1}{H_0 L} \int_0^L H dx \; . \tag{8.31}$$

When integrating in (8.31) making use of (8.24), (8.25) and (8.29), (8.30), respectively leads to

$$e = \frac{1}{L}(b - \frac{p}{2}(L - b) + \frac{L - b}{2\sqrt{1 - n^2}} \arcsin \sqrt{1 - n^2}) \tag{8.32}$$

and

$$e = 1 - \frac{p}{6L^3}(L - b)^3 \; . \tag{8.33}$$

The values of the economy coefficient calculated by (8.23) and (8.33) are accommodated in Tables 1.8.1 and 1.8.2, respectively, for certain values of b and p. Both Tables correspond to the axial tension $n = 0,2$.

It follows from Tables 1.8.1 and 1.8.2 that maximal economy for given axial load may be obtained for the limit load (load carrying capacity). If $n = 0$, $p = 2$ one obtains $e = 0,785$ for sandwich beams and $e = 0,667$ in the case of uniform rectangular cross-section.

Table 1.8.1. Economy of the design for a beam with homogeneous rectangular cross-section

b/L	0	0,1	0,2	0,4	0,6	0,8
p	1,920	2,370	3,000	5,334	12	48
e	0,799	0,819	0,839	0,879	0,919	0,960

Table 1.8.2. Economy of the design for sandwich beam

b/L	0	0,1	0,2	0,4	0,6	0,8
p	1,600	1,975	2,5	4,444	10	40
e	0,733	0,760	0,787	0,840	0,893	0,947

The optimal thickness distribution is presented in Fig. 1.8.1 for several values of the load intensity. Here $n = 0,2$. It may be observed that the optimal non-dimensional thickness tends to unit when the load intensity increases.

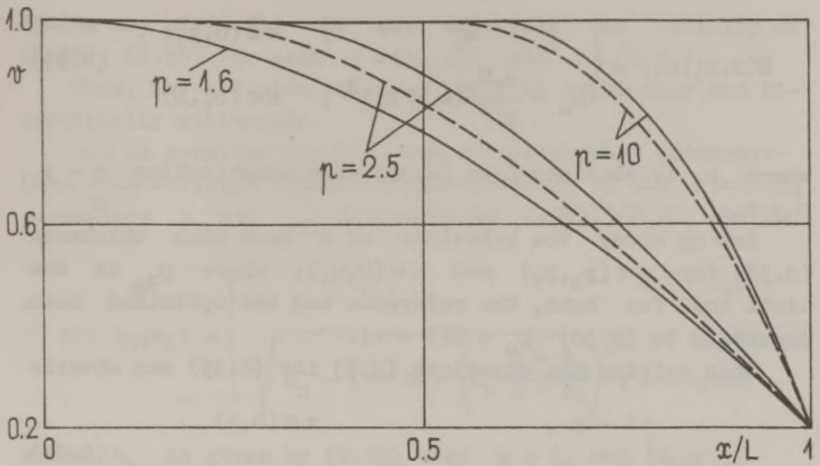


Fig. 1.8.1. Optimal thicknesses for the simply supported beams subjected to the transverse pressure and axial tension

The continuous lines in Fig.1.8.1 correspond to the sandwich beams, whereas the dotted curves are associated with the uniform beams.

Note that adjoint set (8.11) could be integrated accounting for the transversality conditions (6.22). For instance, in the case of sandwich beams the solution procedure results in

$$\psi_1 = \frac{x}{hN_*} \, , \quad \psi_2 = \frac{-1}{2hN_*}(x^2 - L^2) \qquad (8.34)$$
$$\psi_3 = N\psi_1 \, , \quad \psi_4 = N\psi_2 \, .$$

The solution (8.34) may be non-unique.

Note finally that the optimal thicknesses (8.24), (8.25) and (8.29), (8.30) were found under the assumption that the external loading parameters p and n are given constants. Therefore, it is not clear whether the designs obtained above for the values p and n are applicable for lower values of the loading parameters.

In order to ansver this question let us consider the design of the beam for $p = p_1$ and $n = n_1$. In the case of a sandwich beam according to (8.29)

$$H(x,p_1,n_1) = \begin{cases} H_0 , & x \in (0,b_1) , \\ H_0 - \frac{p_1 H_0}{2L^2}(x - b_1)^2 , & x \in (b_1,L) \end{cases} \tag{8.35}$$

where b_1 is obtained from (8.30) when substituting $p = p_1$, $n = n_1$.

Let us study the behaviour of a beam with thickness (8.35) for $p \in (p_0,p_1)$ and $n \in (0,n_1)$, where p_0 is the limit load for both, the reference and the optimized beam. According to (8.30) $p_0 = 2(1 - n)$.

When solving the equations (2.9) for (8.35) one obtains

$$m = \begin{cases} 1 - n , & x \in (0,b) \\ 1 - n - \frac{p}{2L^2}(x - b)^2 , & x \in (b,L) \end{cases} \tag{8.36}$$

where $m = y_1/N_* hH_0$ and b is determined by (8.30). Now one has to check the statical admissibility of the stress distribution (8.36). Thus, according to (8.5) and (8.27)the following inequality

$$m + n \leq \frac{H}{H_0} \tag{8.37}$$

must be satisfied for each $n \in (0,n_1)$ and $p \in (p_0,p_1)$.

Evidently, for the above assumptions $b \leq b_1$. Thus, the inequality (8.37) must be checked in the regions $(0,b)$;(b,b_1) and (b_1,L), respectively. For $x \in (0,b)$ (8.37) applies as equality. In fact, substituting (8.36) into (8.37) gives for $x \in (0,b)$

$$1 \leq \frac{H}{H_0}$$

which is satisfied spontaneously due to (8.35).

For $x \in (b,b_1)$ combining (8.35) - (8.37) leads to the inequality

$$1 - n - \frac{p}{2L^2}(x - b)^2 \leq \frac{H_0}{H_0} , \tag{8.38}$$

whereas for (b_1,L) one has

$$- \frac{p}{2L^2}(x - b)^2 \leq - \frac{p_1}{2L^2}(x - b_1)^2 . \tag{8.39}$$

Making use of (8.30) one can establish the validity of (8.38), (8.39) for each $p \in (p_o, p_1)$ and $n \in (0, n_1)$.

Thus, the solution obtained above is statically and kinematically admissible.

Let us consider now the beam of rectangular cross-section with variable thickness corresponding to the loading parameters p_1 and n_1. According to (8.24), (8.26) one has

$$H(x,p_1,n_1) = \begin{cases} H_0 , & x \in (0,b_1) \\ \sqrt{H_o - (1-n_1^2)\left(H_o \cdot \dfrac{x-b_1}{L-b_1}\right)^2} , & x \in (b_1,L) \end{cases} \tag{8.40}$$

where b_1 is given by (8.25) when $p = p_1$ and $n = n_1$.

Integrating the state equations (2.9) one obtains

$$m = \begin{cases} 1 - n^2 , & x \in (0,b) \\ 1 - n^2 - \dfrac{p}{2L^2}(x-b)^2 , & x \in (b,L) . \end{cases} \tag{8.41}$$

Using (8.41) it is easy to recheck that there exists statically and kinematically admissible stress-strain state for the beam of thickness (8.40) for each $p \in (p_o, p_1)$ and $n \in (0, n_1)$.

CHAPTER II

OPTIMIZATION OF PLASTIC CYLINDRICAL SHELLS FOR PRESCRIBED DEFLECTED SHAPE

§2.1. Problem formulation and the basic equations

2.1.1. Optimality criterion and additional restrictions

Let us consider the moderately large deflections of a rigid-plastic circular cylindrical shell of length 2L and radii A . The shell is subjected to the axial dead load N and to the internal pressure loading of intensity P(x). The coordinate system with its x-axis coinciding with the undeformed generator of the shell has its origin at the central cross-section of the shell (Fig. 2.1.1).

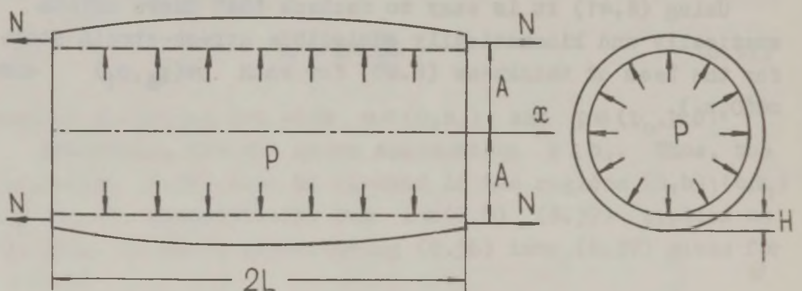


Fig. 2.1.1. Shell geometry

The intensity of the pressure loading as well as the deformations are expected to be symmetric with respect to the central cross-section of the shell. Due to symmetry, the right-hand side of the shell will be considered ($0 \leqslant x \leqslant L$).

The stress-strain state of the shell is specified by the axial and transverse displacemants U and W as well as the moment M , the axial force N_1 , the hoop force N_2 and the tangential force Q . The positive directions of generalized stresses and strains are shown in Fig. 2.1.2.

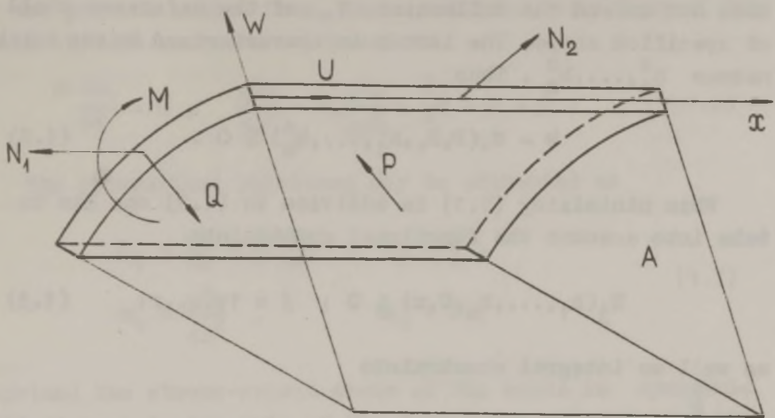


Fig. 2.1.2. Sign convention

It should be noted that the internal forces contributing to the energy dissipation are N_1, N_2 and M.

It is assumed that the yield moment and yield force depend upon the functions $h_1(x),\ldots,h_m(x)$. These quantities could be termed thicknesses but in particular cases they could stand for arbitrary dimensions or parameters of the cross-section of the shell. The shells with non-homogeneous cross-sections including layered and rib-reinforced tubes will be considered.

The functions $h_j(x)(j = 1,\ldots,m)$ have to be determined so that the optimality criterion

$$J = \int_0^L F(h_1,\ldots,h_m)dx \qquad (1.1)$$

attains the minimum value. Here F stands for a given differentiable function. For instance, if the shell wall is homogeneous and the problem consists in the weight minimization of the shell, one can state $F = h$, h being the wall thickness. For a similar problem in the case of a layered shell one has $F = \varrho_1 h_1 + \ldots + \varrho_m h_m$, where $\varrho_i (i = 1,\ldots,m)$ stand for the densities of different layers.

Minimum of (1.1) will be looked for under the condition that the deflection corresponding to the optimal solution does not exceed the deflection W_* of the reference shell of specified shape. The latter is characterized by the thicknesses $h_1^0,\ldots,h_m^0$. Thus

$$W - W_*(P,N_1,h_1^0,\ldots,h_m^0) \leq 0 \,. \tag{1.2}$$

When minimizing (1.1) in addition to (1.2) one has to take into account the functional constraints

$$R_j(h_1,\ldots,h_m,U,x) \leq 0 \;; \quad j = 1,\ldots,r; \tag{1.3}$$

as well as integral constraints

$$\int_0^L S_j(h_1,\ldots,h_m,W,U,x)dx = A_j \;; \quad j = 1,\ldots,s \tag{1.4}$$

where $R_j (j = 1,\ldots,r)$ and $S_j (j = 1,\ldots,s)$ are given differentiable functions.

The geometrical and physical meanings of the constraints (1.3) and (1.4) are similar to those of the corresponding restrictions in the case of plastic beams (chapter I). If, for instance, the statement of a particular problem is such that the thicknesses of the layers h_j must not exceed the corresponding thicknesses h_j^0 of these layers for the reference shell, the constraints (1.3) could be expressed as $R_j = h_j - h_j^0$ $(j = 1,\ldots,m)$. However, if $S_j = h_j$, then the weight (volume) of different layers is fixed previously.

In the present section the shells with simply supported ends will be studied.Although, the attention will be focused to the case when the axial tension applied to the edges of the shell is fixed the analysis should also be carried out for the shells with fully fixed ends. Thus, the boundary conditions may be expressed as

$$\begin{aligned} M'(0) = W'(0) = U(0) = 0, \\ M(L) = W(L) = 0 \,. \end{aligned} \tag{1.5}$$

primes denoting differentiation with respect to x.

2.1.2. Basic equations

Moderately large deflections will be taken into account in the present study. Thus, the equilibrium equations of the shell element could be expressed as

$$\frac{dN_1}{dx} = 0\ , \quad \frac{d^2M}{dx^2} - N_1\frac{d^2W}{dx^2} + \frac{N_2}{A} - P = 0\ . \tag{1.6}$$

The geometrical relations may be presented as

$$\begin{aligned} \varepsilon_1 &= \frac{dU}{dx} + \frac{1}{2}\left(\frac{dW}{dx}\right)^2\ , \quad & \varepsilon_2 &= \frac{W}{A}\ , \\ \varkappa_1 &= \frac{d^2W}{dx^2}\ , & \varkappa_2 &= 0\ , \end{aligned} \tag{1.7}$$

provided the stress-strain state of the shell is symmetric with respect to the axis of the tube. Due to symmetry, the hoop moment is eliminated from the set of relations (1.6), (1.7). The consistence of (1.6) an (1.7), respectively was studied by Duszek, 1975; Jones, 1971; Sawczuk, 1982.

The material of the shell is assumed to be a rigid-plastic one obeying the Tresca yield condition or generally, a piece-wise linear yield condition. Although the yield surface in the space of stress resultants associated with a piece-wise linear yield condition in the plane of principal stresses may consist of linear and non-linear parts, it is assumed that the stress profile lies on the flat

$$N_2 = N_0(h_1, \ldots, h_m) \tag{1.8}$$

only, where N_0 is the yield force. In the case of the uniform shell wall $N_0 = \sigma_0 h$, $M_0 = \sigma_0 h^2/4$, whereas in the case of a sandwich shell one has $N_0 = 2\sigma_0 h$, $M_0 = \sigma_0 hH$, h and H being the thickness of carrying layers and the total thickness, respectively.

The stress points lying on the flat (1.8) of a yield surface must meet the restriction

$$\Phi(M, N_1, h_1, \ldots, h_m) \leq 0 \tag{1.9}$$

where Φ is a given continuous and piece-wise differentiable function.

Note that the hypothesis about the stress profile (1.8) was successfully utilized in the plastic analysis by Duszek, 1975 and Sawczuk, 1982 when accounting for moderately large deflections of rigid-plastic cylindrical shells. It is useful tool in the optimal design of plastic cylindrical shells in the post-yield range as shown by the author and Sawczuk 1980, 1987.

Evidently, foregoing analysis remains valid for the approximations of the yield surfaces for which the stress regime is specified by the flat of type (1.8), but the quantity N_0 must not be interpreted as the yield force. Such a situation occurs when studying the post-yield behaviour of cylindrical shells manufactured of a fiber-reinforced anisotropic material.

A type of deformation theory of plasticity will be used which states that the strain vector with components given by (1.7) is directed along the outward normal to the yield surface. Thus, according to the associated deformation law and the hypothesis about the stress profile (1.8), (1.9)

$$\varepsilon_1 = \lambda^2 \frac{\partial \Phi}{\partial N_1} \; , \quad \varkappa_1 = \lambda^2 \frac{\partial \Phi}{\partial M} \tag{1.10}$$

whereas $\varepsilon_2 = \lambda_1^2$. Here λ^2 stands for a non-negative scalar multiplier, which vanishes, if $\Phi < 0$. The equation regarding to ε_2 will be omitted in the further analysis , since it may be conceived as an equation for determination of the quantity λ_1.

At the non-regular points of the yield surface e.g., at the intersections of the smooth pieces of the yield surface, the strain vector may be specified as an arbitrary positive linear combination of normal vectors to the adjacent flats at this point. Thus, the relations of the type given by (1.10) remain valid at non-regular points of the curve specified by (1.9) if the products in (1.10) are interpreted as the scalar products of appropriate vectors.

Elimination from (1.7), (1.10) the components of deformation gives

$$\frac{dU}{dx} + \frac{1}{2}\left(\frac{dW}{dx}\right)^2 = \lambda^2 \frac{\partial \Phi}{\partial N} \; , \quad \frac{d^2 W}{dx^2} = \lambda^2 \frac{\partial \Phi}{\partial M} \; , \tag{1.11}$$

also

$$W = A \lambda_1^2 \; .$$

2.1.3. Yield surfaces for shells of a Tresca material

Let us consider a cylindrical shell made of a material which obeys the Tresca yield condition. In its original form the Tresca condition represents the hexagon in principal stresses (Fig. 2.1.3).

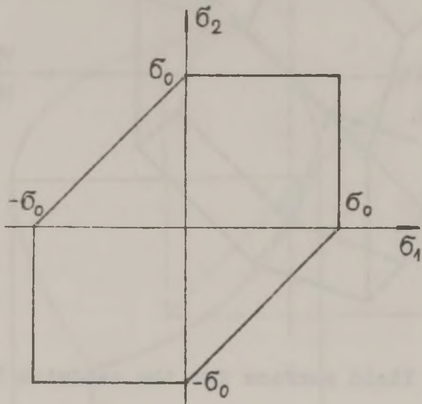


Fig. 2.1.3. Tresca yield hexagon

Equations of the exact yield surface in the space of the stress resultants were first derived by Onat and Prager, 1954 using the assumption of straight normals.

In the case of a circular cylindrical shell with shell wall of sandwich type, the yield surface is a polyhedron whose faces are

$$
\begin{aligned}
\frac{N_2}{N_0} &= \pm 1 , \\
\frac{N_2}{N_0} - \frac{N_1}{N_0} &= \pm 1 , \\
-\frac{N_1}{N_0} \pm \frac{M}{M_0} &= \pm 1 , \\
2\frac{N_2}{N_0} - \frac{N_1}{N_0} \pm \frac{M}{M_0} &= \pm 2 .
\end{aligned}
\qquad (1.12)
$$

The polyhedron defined by (1.12) is presented in Fig. 2.1.4.

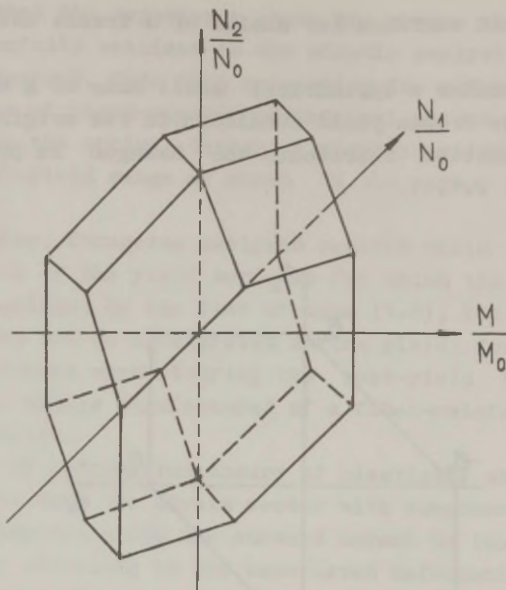


Fig. 2.1.4. Yield surface for the sandwich Tresca shell

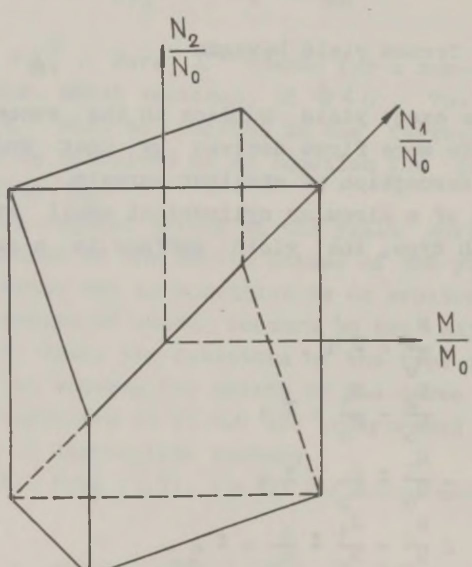


Fig. 2.1.5. An approximation of the yield surface for sandwich shells

In Fig. 2.1.5 there is shown a linear approximation of this surface for which the face $N_2 = N_0$ is bounded by the lines $N_1/N_0 \pm M/M_0 = 1$.

For the cylindrical shells with uniform shell walls the yield surface becomes non-linear (Fig. 2.1.6) which may be

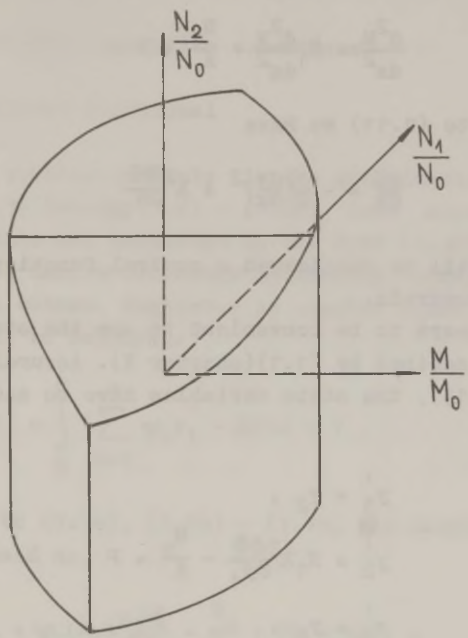


Fig. 2.1.6. Yield surface for homogeneous shells of Tresca material

presented as

$$\begin{aligned} \frac{N_2}{N_0} &= \pm 1 , \\ \pm \frac{M}{M_0} + \left(\frac{N_1}{N_0}\right)^2 &= 1 , \qquad (1.13) \\ \frac{N_1}{N_0} - \frac{N_2}{N_0} &= \pm 1 . \end{aligned}$$

Making use of the strain-mapping method suggested by Onat and Prager 1954, one can derive the equations of the yield surface corresponding to an arbitrary piece-wise linear yield condition.

2.1.4. Normal form of state equations

The differential constraints are presented by (1.6) and (1.11). Evidently, the first equation in (1.6) may be omitted, stipulating that N_1 is a constant. The second equation in (1.6) after using (1.8) leads to the equation

$$\frac{d^2M}{dx^2} = N_1\frac{d^2W}{dx^2} - \frac{N_o}{A} + P . \tag{1.14}$$

According to (1.11) we have

$$\frac{dU}{dx} = -\frac{1}{2}\left(\frac{dW}{dx}\right)^2 + \lambda^2\frac{\partial\Phi}{\partial N_1} \tag{1.15}$$

where λ will be considered a control function. $h_1,\ldots,h_m$ are also controls.

It appears to be convenient to use the state variables $y_1,\ldots,y_5$ defined by (1.1)(chapter I). According to (1.11), (1.14),(1.15), the state variables have to satisfy the state equations

$$\begin{aligned}
y_1' &= y_2 ,\\
y_2' &= N_1\lambda^2\frac{\partial\Phi}{\partial y_1} - \frac{N_o}{A} + P ,\\
y_3' &= y_4 ,\\
y_4' &= \lambda^2\frac{\partial\Phi}{\partial y_1} ,\\
y_5' &= -\frac{1}{2}\,y_4^2 + \lambda^2\frac{\partial\Phi}{\partial N_1} .
\end{aligned} \tag{1.16}$$

The quantity N_1 will be regarded as a given constant, if the axial tension of the shell is specified. However, if N_1 is generated as a reaction of supports, N_1 will be treated as a parameter subjected to the variation.

Introducing new control variables v_j, one can put the inequalities (1.2) and (1.3) into the form of equalities

$$y_3 - W_* + v_o^2 = 0 \tag{1.17}$$

and

$$R_j(h_1,\dots,h_m,y_5,x) + v_j^2 = 0 \; ; \quad j = 1,\dots,r \; . \qquad (1.18)$$

Similarly to the previous case (1.9) may be converted into

$$\Phi(y_1,N_1,h_1,\dots,h_m) + \theta^2 = 0 \; . \qquad (1.19)$$

§ 2.2. Necessary optimality conditions

2.2.1. Extended functional

The posed problem consists in the minimization of the functional (1.1) taking (1.2) - (1.10) into account. The state constraints are presented in the form (1.16) - (1.20).

In order to derive necessary optimality conditions resorting to the method suggested in chapter I, the following functional will be employed:

$$J_* = \int_0^L \Big(\sum_{i=1}^5 \psi_i y_i' - \mathcal{L}\Big)dx + Y \; . \qquad (2.1)$$

In (2.21) due to (1.14), (1.16) - (1.19) the Lagrangian function is expressed as

$$\mathcal{L} = -F + \psi_1 y_2 + \psi_2\Big(N_1\lambda^2\frac{\partial\Phi}{\partial y_1} - \frac{N_o}{A} + P\Big) + \psi_3 y_4 + \\ + \psi_4\lambda^2\frac{\partial\Phi}{\partial y_1} + \psi_5\Big(-\frac{1}{2}y_4^2 + \lambda^2\frac{\partial\Phi}{\partial N_1}\Big) + \sum_{i=1}^{s}\psi_{oi}S_i + \qquad (2.2) \\ + \varphi_o(y_3 - W_* + v_o^2) + \sum_{j=1}^{r}\varphi_j(R_j + v_j^2) + \varphi(\Phi + \theta^2) \; ,$$

φ, φ_j being the Lagrangian multipliers. The quantities $\psi_1,\dots,\psi_5$ are termed adjoint variables.

The term Y in (2.1) may be picked according to (1.5) as

$$Y = \varrho_1 y_1(L) + \varrho_2 y_2(0) + \varrho_3 y_3(L) + \varrho_4 y_4(0) + \varrho_5 y_5(0) . \qquad (2.3)$$

Note that the terms of type $R_j(b_{ij})$ are omitted in (2.3) for the sake of simplicity. Evidently, the jump conditions (3.37), (3.38) or (3.51), (3.52) from chapter I remain valid for the present case.

2.2.2. Variation of the extended functional

Variation of the functional (2.1) results in

$$\delta J_* = \int_0^L \Big\{ \sum_{i=1}^{5} (\psi_i \delta y_i' - \frac{\partial \mathscr{L}}{\partial y_i}\delta y_i) - \frac{\partial \mathscr{L}}{\partial \lambda}\delta\lambda - \sum_{j=1}^{m} \frac{\partial \mathscr{L}}{\partial h_j}\delta h_j -$$

$$- \sum_{j=1}^{r} 2\varphi_j v_j \delta v_j - 2\varphi_o v_o \delta v_o - 2\varphi\theta\delta\theta \Big\} dx + \varrho_1 \delta y_1(L) + \tag{2.4}$$

$$+ \varrho_2 \delta y_2(0) + \varrho_3 \delta y_3(L) + \varrho_4 \delta y_4(0) + \varrho_5 \delta y_5(0) .$$

Since $\delta\lambda$ and δh_j are arbitrary in (2.4), one has

$$\frac{\partial \mathscr{L}}{\partial \lambda} = 0 \tag{2.5}$$

and

$$\frac{\partial \mathscr{L}}{\partial h_j} = 0 , \quad j = 1,\dots,m . \tag{2.6}$$

Similarly to that

$$\varphi_j v_j = 0 ; \quad j = 0,\dots,r \tag{2.7}$$

and

$$\varphi\theta = 0 . \tag{2.8}$$

Integrating by parts the terms $\psi_i \delta y_i' dx$ in (2.4) and bearing in mind (2.5) - (2.8) leads to the equation

$$- \int_0^L \sum_{i=1}^{5} (\psi_i' \delta y_i + \frac{\partial \mathscr{L}}{\partial y_i}\delta y_i) dx + \sum_{i=1}^{5} (\psi_i(L)\delta y_i(L) - \psi_i(0)\delta y(0)) + \tag{2.9}$$

$$+ \varrho_1 \delta y_1(L) + \varrho_2 \delta y_2(0) + \varrho_3 \delta y_3(L) + \varrho_4 \delta y_4(0) + \varrho_5 \delta y_5(0) = 0.$$

One readily obtains from (2.9) the adjoint set

$$\psi_i' = -\frac{\partial \mathcal{L}}{\partial y_i} \; ; \quad i = 1,\dots,5 \tag{2.10}$$

as well as the transversality conditions

$$\psi_1(0) = \psi_2(L) = \psi_3(0) = \psi_4(L) = \psi_5(L) = 0 \tag{2.11}$$

and

$$\begin{gathered} \psi_1(L) = -\varrho_1 \,, \quad \psi_3(L) = -\varrho_3 \,, \\ \psi_2(0) = \varrho_2 \,, \quad \psi_4(0) = \varrho_4 \,, \quad \psi_5(0) = \varrho_5 \,. \end{gathered} \tag{2.12}$$

The relations (2.12) could be regarded as the equations for determination of the corresponding Lagrangian multi - pliers $\varrho_1,\dots,\varrho_5$. Therefore, they will be out of the attention of the further analysis.

2.2.3. Optimality conditions

Let us study the relations (2.5) - (2.11) in a greater detail. Equation (2.5) may be put into the following form when using (2.2)

$$\lambda\left\{(N_1\psi_2 + \psi_4)\frac{\partial \Phi}{\partial y_1} + \psi_5 \frac{\partial \Phi}{\partial N_1}\right\} = 0 \,. \tag{2.13}$$

Evidently, it follows from (2.13) that either

$$\lambda = 0 \tag{2.13}$$

or

$$(N_1\psi_2 + \psi_4)\frac{\partial \Phi}{\partial y_1} + \psi_5 \frac{\partial \Phi}{\partial N_1} = 0 \,. \tag{2.15}$$

Inserting (2.2) in (2.6) leads to the set of equations

$$\begin{gathered} -\frac{\partial F_*}{\partial h_j} - \frac{\psi_2}{A}\frac{\partial N_o}{\partial h_j} + \lambda^2\left\{(N_1\psi_2 + \psi_4)\frac{\partial}{\partial h_j}\frac{\partial \Phi}{\partial y_1} + \psi_5 \frac{\partial}{\partial h_j}\frac{\partial \Phi}{\partial N_1}\right\} + \\ + \varphi\frac{\partial \Phi}{\partial h_j} = 0 \; ; \quad j = 1,\dots,m \,, \end{gathered} \tag{2.16}$$

where for the conciseness sake the following notation is introduced

$$F_* = F - \sum_{i=1}^{r} \varphi_i R_i - \sum_{i=1}^{s} \psi_{oi} S_i \; . \qquad (2.17)$$

The adjoint system (2.10) could be expressed as

$$\begin{aligned}
\psi_1' &= -\varphi \frac{\partial \Phi}{\partial y_1} - \lambda^2 \left\{ (N_1 \psi_2 + \psi_4) \frac{\partial^2 \Phi}{\partial y_1^2} + \psi_5 \frac{\partial^2 \Phi}{\partial y_1 \partial N_1} \right\} , \\
\psi_2' &= -\psi_1 \, , \\
\psi_3' &= -\varphi_o - \sum_{i=1}^{s} \psi_{oi} \frac{\partial S_i}{\partial y_3} \, , \\
\psi_4' &= -\psi_3 + \psi_5 y_4 \, , \\
\psi_5' &= -\sum_{i=1}^{s} \psi_{oi} \frac{\partial S_i}{\partial y_5} - \sum_{i=1}^{r} \varphi_i \frac{\partial R_i}{\partial y_5} \; .
\end{aligned} \qquad (2.18)$$

In a particular case when the constraints (1.3) and (1.4) do not depend upon the axial displacement, i.e.

$$\frac{\partial R_j}{\partial y_5} = \frac{\partial S_i}{\partial y_5} = 0 \qquad j = 1,\dots,r \; ; \quad i = 1,\dots,s \qquad (2.19)$$

it follows from (2.18) and (2.11) that

$$\psi_5 = 0 \; . \qquad (2.20)$$

Note that (2.20) takes into account the case when $J(L) \neq 0$.

2.2.4. Optimality conditions for the minimum weight design associated with the given deflected shape

Consider now the optimization problem which consists in the minimization of the material volume of the shell

$$J = \int_0^L h(x)dx \qquad (2.21)$$

under the conditions that the thickness is constrained below by

$$h - h_1 \geqslant 0 \qquad (2.22)$$

and above by

$$-h + h_2 \geqslant 0 \; . \qquad (2.23)$$

Evidently, the problem (2.21) - (2.23) is a particular case of that stated by (1.1) - (1.4). In the present case

$$F = h\ ,\quad R_1 = -h + h_1\ ,\quad R_2 = h - h_2\ ,\quad S_j = 0 \qquad (2.24)$$

which according to (2.19), (2.20) implies that $\psi_5 = 0$. Therefore, (2.13), (2.15) simplify considerably taking the form

$$\lambda(N_1\psi_2 + \psi_4) = 0\ . \qquad (2.25)$$

Substituting (2.25) into (2.15) leads to the equation

$$-\frac{\partial F_*}{\partial h} - \frac{\psi_2}{A}\frac{\partial N_o}{\partial h} + \varphi\frac{\partial\Phi}{\partial h} = 0 \qquad (2.26)$$

where according to (2.17) and (2.21) - (2.24)

$$F_* = (-h_1 + h)\varphi_1 + \varphi_2(h_2 - h) + h\ . \qquad (2.27)$$

According to the relation (2.25) we put the adjoint equations (2.18) into the form

$$\begin{aligned} \psi_1' &= -\varphi\frac{\partial\Phi}{\partial y_1}\ , \\ \psi_2' &= -\psi_1\ , \\ \psi_3' &= -\varphi_o\ , \\ \psi_4' &= -\psi_3 \end{aligned} \qquad (2.28)$$

which, in turn, combined with (2.25) yields

$$\psi_4 = -N_1\psi_2\ ,\quad \psi_3 = -N_1\psi_1 \qquad (2.29)$$

and

$$\varphi_o = -\varphi N_1\frac{\partial\Phi}{\partial y_1}\ . \qquad (2.30)$$

Note that (2.29) and (2.30) is applied in the regions where the stress state of the shell corresponds to the regime $\Phi = 0$.

In the further analysis the minimum weight problems will be studied for which W_* is the deflection of the reference shell of constant thickness. It is assumed that $\varphi_0 \neq 0$ in (2.7) and thus, $v_0 = 0$. This means that the deflections of the corresponding shells coincide at each point.

From (2.26) making use of (2.27) one readily obtains

$$\varphi = (1 + \varphi_1 - \varphi_2 + \frac{\psi_2}{A}\frac{\partial N_0}{\partial h})(\frac{\partial \Phi}{\partial h})^{-1} . \tag{2.31}$$

On the grounds of the latter relation one can draw the conclusion that φ does not vanish simultaneously, at least if the constraints (2.22) and (2.23) are passive, i.e. $\varphi_1 = \varphi_2 = 0$. Therefore, in (2.8) $\theta = 0$ and (1.19) yields

$$\Phi(M, N_1, h) = 0 . \tag{2.32}$$

The equation (2.32) could be interpreted as the optimality condition for the shell with the specified deflected shape, provided no additional restrictions are imposed. If, however, the thickness of the shell is constrained by (2.22) and (2.23), one can start with the solution procedure from (2.32) and construct the admissible solution when suitably combining the requirements $\varphi_j = 0$ or $v_j = 0$ in (2.7). Solving the so-called problem of synthesis certain regions could occur with $\theta \neq 0$ for the problem with additional restrictions.

Note finally that we cannot provide a guarantee that there really exists the optimal solution for arbitrary function W_*. The problems of the existence and uniqueness of the optimal solution exceed the scope of the present work. Valuable achievements in this field have been made by Cinquini and Sacchi, 1980.

§2.3. Reference solutions for shells of constant thickness

2.3.1. Shells of sandwich type

Large deflections of rigid-plastic cylindrical shells the material of which obeys the Tresca yield condition have been studied by Duszek, 1966, 1967; Duszek and Sawczuk, 1970; Lance and Soechting, 1970 and Sawczuk, 1982 using the yield surface in the space of the stress resultants. Another approach to these problems was developed by Lepik, 1966; 1967 on the basis of the original Tresca yield condition in the space of principal stresses.

Consider first the shell if ideal sandwich type. The yield surface for shells of sandwich type is presented in Fig. 2.1.4, the equations of the faces are given by (1.12). The stress state of the shell entirely belongs to the face $N_2 = N_0$ of the yield surface (Fig. 2.1.4). The sketch of this face of the yield polyhedron is presented in Fig.2.3.1.

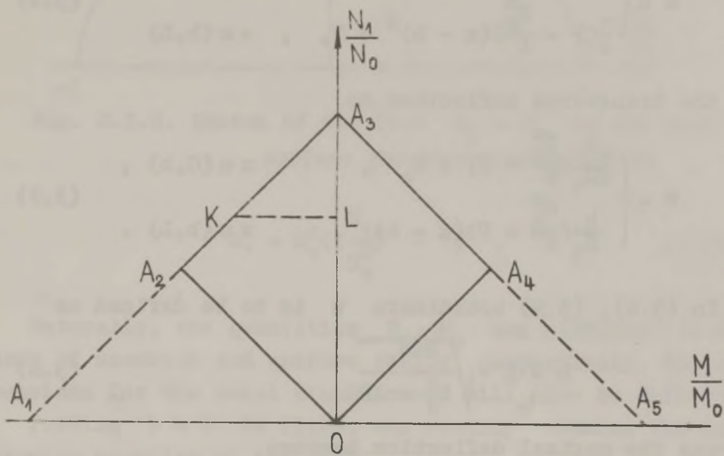


Fig. 2.3.1. Sketch of the face $N_2 = N_0$ of the yield polyhedron

Assuming that the stress state of the shell corresponds to the point K in Fig. 2.3.1 in the central zone for $x \in (0,b)$ and to the line KL for $x \in (b,L)$ one states that

$$M = M_* \tag{3.1}$$

for $x \in (0,b)$ and

$$\varepsilon_1 = \varkappa_1 \tag{3.2}$$

for $x \in (b,L)$. In (3.1) according to (1.12)

$$M_* = M_0^0\left(\frac{N_1}{N_0^0} - 1\right) \tag{3.3}$$

where $M_o^o = 6_o H h_o$, $N_o^o = 26_o h_o$.

Making use of (3.1) and (3.2) one can integrate the equilibrium equation (1.14) and the relations (1.11) separately in the regions (0,b) and (b,L), respectively. Satisfying the boundary conditions (1.20) as well as the continuity requirements of the quantities M, M', W and W' at x = b one obtains the bending moment as

$$M = \begin{cases} M_* , & x \in (0,b) , \\ \frac{1}{2}(P - \frac{N_o^o}{A})(x - b)^2 + M_* , & x \in (b,L) \end{cases} \tag{3.4}$$

and the transverse deflection as

$$W = \begin{cases} \frac{x^2}{2N_1}(\frac{N_o^o}{A} - P) + W_o , & x \in (0,b) , \\ \frac{b}{N_1}(\frac{N_o^o}{A} - P)(x - L) , & x \in (b,L) . \end{cases} \tag{3.5}$$

In (3.4), (3.5) coordinate b is to be defined as

$$b = L - \sqrt{\frac{2AM_*}{N_o^o - AP}} \tag{3.6}$$

whereas the maximal deflection becomes

$$W_o = \frac{1}{2N_1}(\frac{N_o^o}{A} - P)(b^2 - 2bL) . \tag{3.7}$$

Note that the solution (3.4) - (3.7) remains valid until N_1 is not less than value $N_o^o/2$. For $N_1 < N_o^o/2$ the results (3.4) - (3.7) remain valid if M_* in (3.3) is replaced with $-M_o N_1/N_o$. However, the obtained results hold good for the approximation of the exact yield surface (Fig. 2.1.5), for which the face $N_2 = N_o$ is presented by $A_1A_3A_5$ in Fig. 2.3.1.

2.3.2. Homogeneous shells

It appears that (3.4) - (3.7) remain valid for shells with solid shell walls when using the plastic regime $N_2 = N_o$ on the yield surface (Fig. 2.1.6). The sketch of this face of the surface is presented in Fig. 2.3.2. Now (3.3) must be replaced by

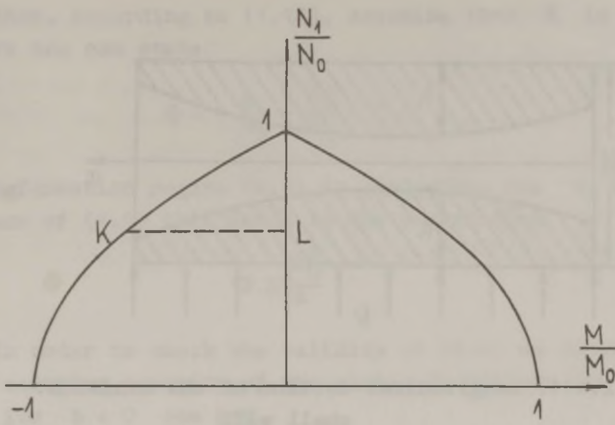


Fig. 2.3.2. Sketch of the face $N_2 = N_0$ of the yield surface for homogeneous shells

$$M_* = M_0^o\left(\left(\frac{N_1}{N_0^o}\right)^2 - 1\right) \quad . \tag{3.8}$$

Naturally, the quantities N_0, M_0 are different in the cases of sandwich and uniform shells, respectively. The expressions for the axial displacement will also be different.

Putting $b = 0$ in (3.6), one readily obtains the load carrying capacity of the shell

$$P_0 = \frac{N_0^o}{A} - \frac{2M_*}{L^2} \tag{3.9}$$

which holds good in both cases, if M_* is interpreted suitably. In the further analysis it is assumed that $P \geqslant P_0$.

§ 2.4. Minimum weight design of the closed sandwich shell of the Tresca material

2.4.1. Problem formulation

Consider a closed sandwich cylindrical shell hinged at the end sections and allowed to displace in the axial direction. The shell wall consists of a core layer carrying shear forces and of two layers of thickness $h(x)$ carrying membrane forces and moments (Fig. 2.4.1.).

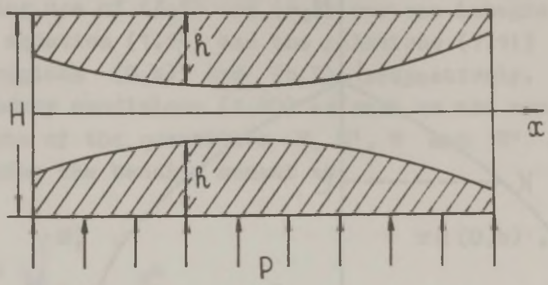


Fig. 2.4.1. Longitudinal section of the sandwich shell wall

The structure consisting of a cylindrical shell and of two end plates is subjected to the internal pressure of intensity P considered as a dead load at small configuration changes. For the considered shell and loading the axial force is generated by the internal pressure. The equilibrium of the end plates yields

$$N_1 = \frac{1}{2}AP \ . \tag{4.1}$$

In the present section the optimum design of a short closed cylindrical shell will be established for which the volume of carrying layers

$$J = \int_0^L h(x)dx \tag{4.2}$$

attains the minimum value under the requirement that the shell deflections are as given by (3.5) - (3.7). In (3.5) - - (3.7) the axial force is specified by (4.1). Besides,when minimizing (4.2) there have to be satisfied the basic equations (1.5) with boundary conditions (1.20) as well as the yield condition (1.9) or (1.19).

We are looking for the optimal thickness distribution among continuous and smooth functions.

Material of the shell is assumed to be rigid-plastic obeying the Tresca yield condition. Besides, the stress state of the shell corresponds to the plane $N_2 = N_0$ (Fig. 2.3.1) and thus, according to (1.12), assuming that M is non-positive one can state

$$\Phi = -\frac{M}{M_o} + \frac{N_1}{N_o} - 1 . \tag{4.3}$$

The deformation regime (4.3) is admissible for $N_1 \geqslant N_o^o/2$. Because of (4.1) this leads to the restriction

$$P \geqslant \frac{N_o^o}{A} . \tag{4.4}$$

In order to check the validity of (4.4) we determine the load carrying capacity of the shell. It follows from (3.6) that for $b = 0$ one has

$$P_o = \frac{N_o^o}{A} - \frac{2}{L^2} M_* . \tag{4.5}$$

Substituting (3.3) and (4.1) in (4.5) gives

$$P_o = \frac{N_o^o}{A}\left(1 + \frac{AM_o^o}{AM_o^o + L^2 N_o^o}\right) \tag{4.6}$$

which shows that for $P \geqslant P_o$ (4.4) is satisfied spontaneously. Thus, the deformation regime (4.3) is associated with the exact yield surface, with the quadrate $OA_2A_3A_4$ in Fig. 2.3.1.

2.4.2. Optimal solution

For the intended optimization the necessary optimality condition is expressed by (2.32). Employing (2.32) with (4.3) and (4.1) one obtains the relation

$$M = -M_o\left(1 - \frac{AP}{2N_o}\right) \tag{4.7}$$

which should be satisfied at each $x \in (0,L)$. It is worth emphasizing that M_o and N_o depend on the actual face sheet thickness and thus (4.7) states the relation between the bending moment M and the thickness h. This permits to integrate the basic equations (1.16), or (1.14) and (1.15).

It appears to be convenient to use the following non-dimensional quantities

$$p = \frac{PA}{N_o^o}\ , \quad a = \frac{N_o^o L^2}{AM_o^o}\ , \quad v = \frac{h}{h_o}\ , \quad n = \frac{N_1}{N_o^o}\ ,$$

$$m = \frac{M}{M_o^o}\ , \quad u = \frac{L(N_o^o)^2 U}{4(M_o^o)^2}\ , \quad w = \frac{N_o^o W}{2M_o^o}\ . \tag{4.8}$$

Using (4.8) one could put (4.7) into the form

$$m = \tfrac{1}{2}p - v \tag{4.9}$$

whereas the load carrying capacity (4.6) may be expressed as

$$p_o = 1 + \frac{1}{1 + a}\ . \tag{4.10}$$

Calculating the curvature with the aid of (3.5) gives

$$\frac{d^2W}{dx^2} = \begin{cases} \frac{2}{Ap}(1 - p)\ , & x \in (0,b)\ , \\ 0\ , & x \in (b,L)\ , \end{cases} \tag{4.11}$$

where

$$b = L\left(1 - \sqrt{\frac{2 - p}{a(p - 1)}}\right)\ . \tag{4.12}$$

Substituting (4.11) in (1.14) - (1.16) leads to the following set of equations

$$m' = q\ ,$$

$$q' = \begin{cases} \frac{a}{L^2}(1 - v)\ , & x \in (0,b)\ , \\ \frac{a}{L^2}(p - v)\ , & x \in (b,L)\ , \end{cases} \tag{4.13}$$

$$u' = \begin{cases} -\frac{a^2(p - 1)^2x^2}{2L^3p^2} + \frac{a}{2Lp}(p - 1)\ , & x \in (0,b)\ , \\ -\frac{a^2(p - 1)^2b^2}{2L^3p^2}\ , & x \in (b,L)\ , \end{cases}$$

where q may be considered as an auxiliary variable.

The equations (4.13) could be integrated with the aid of

(4.9) separately in the regions $(0,b)$ and (b,L). For determination of the constants of integration one can utilize the boundary conditions (1.5) as well as the continuity requirements imposed on variables m, q and u at $x = b$.

Doing so one eventually obtains the solution of (4.13) in the form

$$
\begin{aligned}
m &= -1 + \frac{p}{2} + \left(\frac{p}{2} - (p-1)\,\mathrm{ch}\, d(L-b)\right)\frac{\mathrm{ch}\, dx}{\mathrm{ch}\, dL}\,, \\
q &= \frac{d}{\mathrm{ch}\, dL}\left(\frac{p}{2} - (p-1)\,\mathrm{ch}\, d(L-b)\right)\mathrm{sh}\, dx\,, \qquad (4.14) \\
u &= \frac{a^2x^3}{6p^2L^3}(p-1)^2 + \frac{ax}{2pL}(p-1)
\end{aligned}
$$

for $x \in (0,b)$ and

$$
\begin{aligned}
m &= -\frac{p}{2} + \frac{1}{\mathrm{ch}\, dL}\left(\frac{p}{2}\mathrm{ch}\, dx + (p-1)\,\mathrm{sh}\, db \cdot \mathrm{sh}\, d(L-x)\right)\,, \\
q &= \frac{d}{\mathrm{ch}\, dL}\left(\frac{p}{2}\mathrm{sh}\, dx - (p-1)\,\mathrm{sh}\, db \cdot \mathrm{ch}\, d(L-x)\right)\,, \qquad (4.15) \\
u &= -\frac{a^2b^2}{2p^2L^3}(p-1)^2\left(x - \frac{2}{3}b\right) + \frac{ab}{2pL}(p-1)
\end{aligned}
$$

for $x \in (b,L)$. Here the notation $d = \sqrt{a}/L$ is used.

Inserting the quantity m according to (4.14) and (4.15) into (4.9) leads to the optimal thickness distribution

$$
v = \begin{cases} 1 - \left(\frac{p}{2} - (p-1)\,\mathrm{ch}\, d(L-b)\right)\frac{\mathrm{ch}\, dx}{\mathrm{ch}\, dL}\,, & x \in (0,b)\,, \\ p - \frac{1}{\mathrm{ch}\, dL}\left(\frac{p}{2}\mathrm{ch}\, dx + (p-1)\,\mathrm{sh}\, db \ \mathrm{sh}\, d(L-x)\right)\,, & x \in (b,L)\,. \end{cases} \qquad (4.16)
$$

In the case when $p = p_0$ according to (4.12) $b = 0$ and optimal thickness will be

$$
v = p - \frac{p}{2}\frac{\mathrm{ch}\, dx}{\mathrm{ch}\, dL}\,. \qquad (4.17)
$$

However, for $p = 2$ one has $b = L$ and the optimal thickness coincides with the constant thickness $h = h_0$.

2.4.3. Adjoint variables

The optimality condition (2.32) holds under the condition that the problem is not a singular one. This implies that the adjoint variables must not be equal to zero simultaneously.

As there are not any constraints of type (2.21), (2.22) imposed on the optimal thickness one can put $\varphi_1 = \varphi_2 = 0$ into (2.31) which using (4.3) yields

$$\varphi = -h(1 + \frac{2\sigma_0}{A}\psi_2) \, . \tag{4.18}$$

Due to the validity of (2.29) only the first two equations in the set (2.28) have to be integrated. Substitution of (4.18) as well as (4.3) in (2.28) gives the set

$$\begin{aligned} \psi_1' &= -\frac{2}{AH}(\psi_2 + A_*) \, , \\ \psi_2' &= -\psi_1 \, , \end{aligned} \tag{4.19}$$

which has the solution

$$\begin{aligned} \psi_1 &= -(C_1 \mathrm{sh}\, dx + C_2 \mathrm{ch}\, dx) d \, , \\ \psi_2 &= C_1 \mathrm{ch}\, dx + C_2 \mathrm{sh}\, dx - A_* \, . \end{aligned} \tag{4.20}$$

Here $A_* = A/2\sigma_0$ and C_1, C_2 are arbitrary constants. The constants of integration may be determined when satisfying the transversality conditions. This results in

$$\begin{aligned} \psi_1 &= -\frac{A_* d}{\mathrm{ch}\, dL} \mathrm{sh}\, dx \, , \\ \psi_2 &= A_*(-1 + \frac{\mathrm{ch}\, dx}{\mathrm{ch}\, dL}) \, . \end{aligned} \tag{4.21}$$

Consequently, the solution of the adjoint system (4.19) is non-trivial and thus the posed problem is non-singular.

2.4.4. Applicability of the optimal design

The optimal solution (4.14), (4.15) and thicknesses (4.16), (4.17) were obtained under the assumption that the intensity of the pressure loading was fixed. It means that the optimal thickness is associated with the given value of the internal pressure. This involves the question about the applicability of the design established herein for the pressures which are lower than those corresponding to the constructed solution.

In order to answer this question consider design (4.16) for $p = p_1$

$$v(p_1,x) = \begin{cases} 1 - \left(\frac{p_1}{2} - (p_1 - 1)\operatorname{ch} d(L - b_1)\right)\frac{\operatorname{ch} dx}{\operatorname{ch} dL}, & x \in (0,b_1), \\ p_1 - \frac{1}{\operatorname{ch} dL}\left(\frac{p_1}{2}\operatorname{ch} dx + (p_1 - 1)\operatorname{ch} db_1 \operatorname{sh} d(L - x)\right), & x \in (b_1,L), \end{cases} \tag{4.22}$$

where

$$b_1 = L\left(1 - \sqrt{\frac{2 - p_1}{a(p_1 - 1)}}\right). \tag{4.23}$$

It will be shown that the load carrying capacity of the shell with thickness (4.22) coincides with (4.10) and that an admissible solution to the problem of determination of the stress-strain state of the design (4.22) exists for $p \in (p_0,p_1)$. It appears to be reasonable to assume that the stress state of the shell corresponding to (4.22), (4.23) is associated with point K in Fig. 2.3.1, i.e.

$$m(p,x) = \tfrac{1}{2}p - v(p_1,x) \tag{4.24}$$

for $x \in (0,b)$ and with the profile KL for $x \in (b,L)$, where b is given by (4.12).

For the statical admissibility of this solution it is necessary that the stress point should not exceed the limits of the interval KL in Fig. 2.3.1, i.e.

$$m(p,x) \geqslant \tfrac{1}{2}p - v(p_1,x) \tag{4.25}$$

for $x \in (b,L)$ and each $p \in (p_0,p_1)$.

Note, that for $b = 0$ (4.23) gives the limit load p_0 which coincides with (4.10) as might be expected. Making use of (4.12) and (4.23) it is easy to recheck that $b \leq b_1$ if $p \leq p_1$. Hence, the validity of (4.26) must be studied in the regions (b,b_1) and (b_1,L).

For $x \in (0,b)$ by means of (4.22), (4.24), one obtains

$$m = -1 + \frac{p}{2} + \left(\frac{p_1}{2} - (p_1 - 1)\text{ch } d(L - b)\right)\frac{\text{ch } dx}{\text{ch } dL} ,$$
$$q = \frac{d}{\text{ch } dL}\left(\frac{p_1}{2} - (p_1 - 1)\text{ch } d(L - b)\right)\text{sh } dx . \tag{4.26}$$

Since the equations (4.13) have to be satisfied, the variable u given by (4.14) remains valid in the present case as well. Naturally, (4.26) is in consistence with (4.13), too.

Now one has to integrate (4.13) inserting first (4.22) and making use of (4.12) and (4.23). Performing the integration and determining the integration constants according to the continuity conditions at $x = b$ and $x = b_1$ leads to the solution

$$m = \frac{p}{2} - v(p_1,x) + \frac{d^2}{2}(p - 1)(x - b)^2 \tag{4.27}$$

for $x \in (b,b_1)$ and

$$m = \frac{p}{2} - v(p_1,x) + \frac{d^2}{2}(p - p_1)(x - b_1)^2 + $$
$$+ \frac{d^2}{2}(p - 1)(b_1 - b)(2x - b - b_1) \tag{4.28}$$

for $x \in (b_1,L)$. Evidently, the boundary condition $m(L) = 0$ is satisfied spontaneously, if b and b_1 are determined by (4.12) and (4.23), respectively.

Employing (4.26) - (4.28), it is easy to recheck that (4.25) is satisfied for each $x \in (b,L)$.

2.4.5. Discussion

Economy of the optimal design established could be assessed by the coefficient

$$e = \frac{1}{h_0 L}\int_0^L h(x)dx , \tag{4.29}$$

where h is the optimal thickness and h_0 stands for the constant thickness of the reference shell.

Inserting (4.16) in (4.29) leads to the relation

$$e = \frac{1}{L}(b + p(L - b)) + \frac{1}{dL\, ch\, dL}((p - 1)sh\, db - \frac{p}{2}sh\, dL). \quad (4.30)$$

Certain values of the economy coefficient are placed in Table 2.4.1 for the given values of the load intensity. Table 2.4.1 corresponds to the case when $a = 4$ and $dL = 2$.

Table 2.4.1. Economy of the design of a closed sandwich shell

b/L	p	e
0	1,200	0,911
0,1	1,230	0,921
0,2	1,281	0,931
0,3	1,338	0,943
0,4	1,410	0,954
0,5	1,500	0,967
0,6	1,610	0,978
0,7	1,735	0,988
0,8	1,862	0,996
0,9	1,962	0,999
1,0	2,000	1,000

In the present work the attention is focused on comparatively short shells. Now the bending moment is a monotonic function. Naturally, it is not easy to establish the strict boundaries between short and long shells. This depends not only on the geometrical parameter a but on the material utilized by the manufacturing of the shell as well as on the loading and on the support conditions. Therefore the numerical investigations are carried out for shells with $a \in (1,8)$.

Table 2.4.1 shows that the amount of the material saving is not large (maximally less than 9% in the case when $a = 4$). The first row in Table 2.4.1 corresponds to the load carrying capacity of the shell, $p = 1,2$ being the limit load. The last row in Table 2.4.1 illustrates the fact that for $p = 2$ the axial force $n = 1$ and the only admissible project of the shell is the design of a constant thickness. Evidently, when $n = 1$ then the shell operates in the membrane state, i.e. $m = 0$. However, for $p = 1{,}2$ one has $n = 0{,}6$. Thus, due to comparatively large values of the axial force (membrane stress) generated by the internal pressure the economy of the material need may not be large.

Fig. 2.4.2 presents the optimal thickness of the face sheet

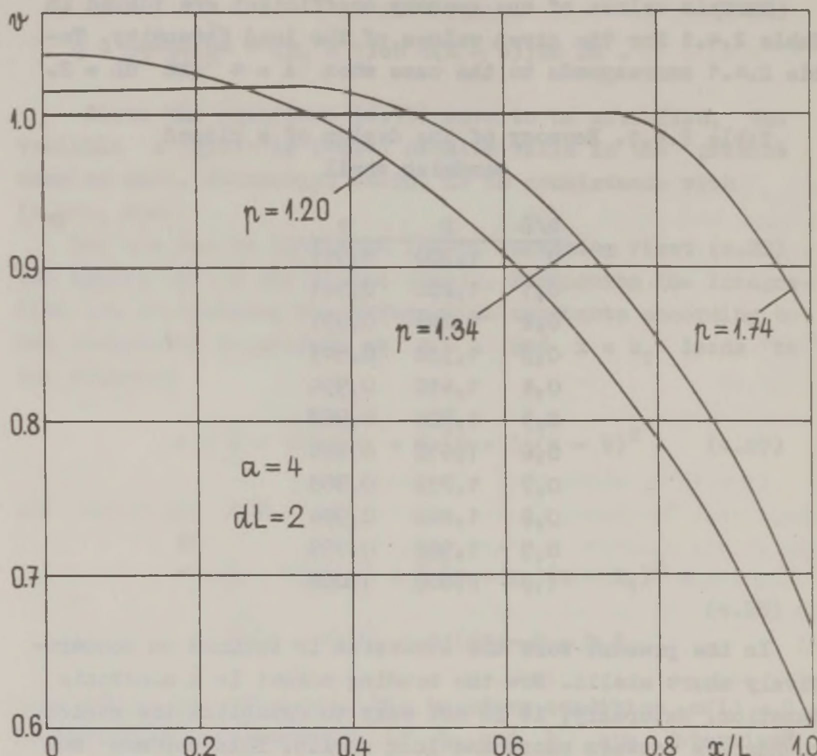


Fig. 2.4.2. Optimal thickness of a closed sandwich shell

and the bending moment is shown in Fig. 2.4.3. The curves presented in Fig. 2.4.2 and 2.4.3 correspond to the shell with $a = 4$. Here $b = 0$; $b = 0{,}3$ and $b = 0{,}7$, respectively. The optimal thickness in Fig. 2.4.2 seems to be constant near the center of the shell. However, the calculations carried out reveal the dependence of the thickness on the coordinate. Specific values of the thickness v at different extents of the central zone corresponding to the stress regime K on the ridge of the yield surface are given in Table 2.4.2.

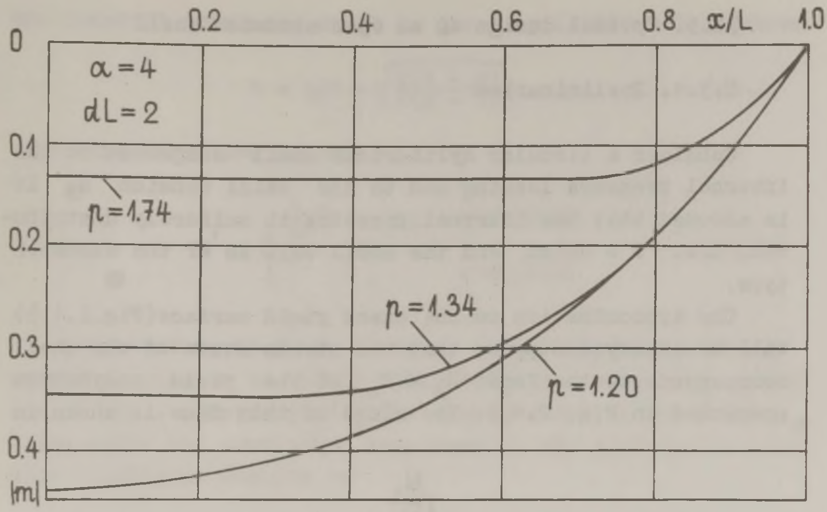


Fig. 2.4.3. Bending moment of a closed sandwich shell

Table 2.4.2. Optimal face sheet thickness of a closed shell

x/L	b/L=0	0,3	0,5	0,7
0,0	1,0405	1,0154	1,0057	1,0011
0,2	1,0276	1,0166	1,0062	1,0012
0,4	0,9867	1,0138	1,0077	1,0014
0,6	0,9112	0,9651	1,0003	1,0019
0,8	0,7889	0,8561	0,9220	0,9880
1,0	0,6000	0,6700	0,7500	0,8700

It follows from Table 2.4.2 that in the central part of the shell the optimal thickness exceeds that of the reference shell. However, if b tends to unity, the function h(x) uniformly tends to h_0.

§ 2.5. Optimal design of an open sandwich shell

2.5.1. Preliminaries

Consider a circular cylindrical shell subjected to the internal pressure loading and to the axial tension N. It is assumed that the internal pressure is uniformly distributed, i.e. $P = \text{const}$ and the shell wall is of the sandwich type.

The approximation on the exact yield surface(Fig.2.1.5) will be employed. Suppose that the stress state of the shell corresponds to the face $N_2 = N_0$ of the yield polyhedron presented in Fig. 2.1.5. The ridge of this face is shown in

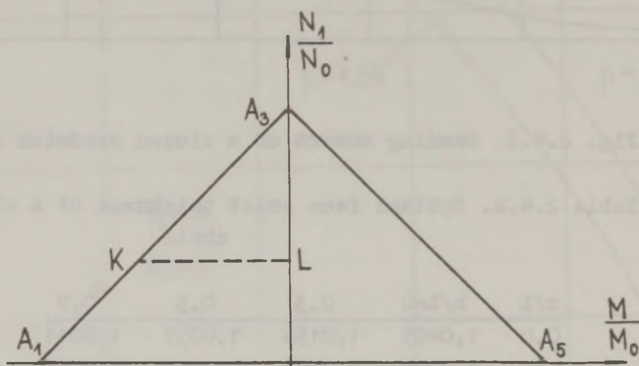


Fig.2.5.1. Sketch of the face $N_2=N_0$ of the yield polyhedron

Fig. 2.5.1. Here the line KL represents the stress regime. Thus, the relation (4.3) holds good for each N which does not exceed the value N_0, in the present case.

The non-dimensional quantities (4.8) will be utilized in the present section, as well. Since the optimality condition (2.32) remains valid the relation (4.3) gives

$$m = n - v \tag{5.1}$$

which should be used at each point $x \in (0,L)$.

2.5.2. Optimal solution

For an open shell the reference solution associated with the constant thickness shell is given by (3.4) - (3.7), where

$$b = L\left(1 - \sqrt{\frac{2(1 - n)}{a(p - 1)}}\right) . \tag{5.2}$$

Thus, the curvature becomes

$$W' = \begin{cases} \frac{1}{An}(1 - p), & x \in (0,b) , \\ 0 , & x \in (b,L) . \end{cases} \tag{5.3}$$

Substituting (5.3) in (1.14) - (1.16) leads to a set of equations similar to (4.13) which could be integrated making use of the condition (5.1). The integration of basic equations under the continuity requirements and appropriate boundary conditions results in

$$\begin{aligned} m &= n - 1 - (n - p + (p - 1)\operatorname{ch} d(L - b))\frac{\operatorname{ch} dx}{\operatorname{ch} dL} , \\ q &= -\frac{d}{\operatorname{ch} dL}(n - p + (p - 1)\operatorname{ch} d(L - b))\operatorname{sh} dx , \\ u &= -\frac{a^2x^3}{24n^2L^3}(p - 1)^2 + \frac{ax}{4nL}(p - 1) \end{aligned} \tag{5.4}$$

for $x \in (0,b)$ and

$$\begin{aligned} m &= n - p - \frac{1}{\operatorname{ch} dL}((n - p)\operatorname{ch} dx + (1 - p)\operatorname{sh} db\cdot\operatorname{sh} d(L - x)), \\ q &= \frac{-d}{\operatorname{ch} dL}((n - p)\operatorname{sh} dx - (1 - p)\operatorname{sh} db\cdot\operatorname{ch} d(L - x)), \\ u &= -\frac{a^2b^2}{8n^2L^3}(p - 1)^2\left(x - \frac{2}{3}b\right) + \frac{ab}{4Ln}(p - 1) \end{aligned} \tag{5.5}$$

for $x \in (b,L)$.

Combining (5.1) and (5.4), (5.5) leads to the optimal non-dimensional face-sheet thickness

$$v = \begin{cases} 1 + (n - p + (p - 1)\operatorname{ch} d(L - b))\frac{\operatorname{ch} dx}{\operatorname{ch} dL} , \\ \qquad\qquad x \in (0,b) , \\ p + \frac{1}{\operatorname{ch} dL}((n - p)\operatorname{ch} dx + (1 - p)\operatorname{sh} db\cdot\operatorname{sh} d(L-x)), \\ \qquad\qquad x \in (b,L) . \end{cases} \tag{5.6}$$

For the limit load $p_0 = 2(1 - n)/a$ the formula (5.6) gives

$$v = p - (p - n)\frac{\operatorname{ch} dx}{\operatorname{ch} dL} \quad . \tag{5.7}$$

However, in the case when $n = 1$ according to (5.2) $b = L$ and (5.6) yields $v = 1$.

Note that the adjoint variables could be determined the same way as in the case of a closed shell. It may be rechecked that (4.18) - (4.21) remain valid in the present case, as well.

2.5.3. Applicability of the optimal design

Let us study the applicability of the shell with optimal thickness $v(n_1,p_1,x)$ when loaded by $n \in (0,n_1)$ and $p \in (p_0,p_1)$. Here $v(n_1,p_1,x)$ stands for the thickness (5.6) corresponding to the axial load n_1 and the pressure loading of intensity p_1.

Assume that the stress state of the shell with variable thickness corresponds to the point K (Fig. 2.5.1) for $x \in (0,b)$ and to the line KL for $x \in (b,L)$. Therefore, the verification of the statical admissibility of the solution reduces to the checking of the inequality

$$m(n,p,x) \geqslant n - v(n_1,p_1,x) \tag{5.8}$$

for $x \in (b,b_1)$ and $x \in (b_1,L)$. The coordinate b_1 is to be specified by (5.2) for $n = n_1$ and $p = p_1$. Evidently, $b \leqslant b_1$.

Integrating the set of basic equations for $v=v(n_1,p_1,x)$ one eventually obtains

$$m = \begin{cases} n - v(n_1,p_1,x) \ , & x \in (0,b), \\ \frac{d^2}{2}(p - 1)(x - b)^2 + n - v(n_1 p_1,x) & , \ x \in (b,b_1), \\ n - v(n_1,p_1,x) + y \ , & x \in (b,L) \end{cases} \tag{5.9}$$

where

$$\begin{aligned} y = {} & \frac{d^2}{2}((p - p_1)(x - b_1)^2 + (p - 1)(b_1 - b)^2) + \\ & + d^2(p - 1)(b_1 - b)(x - b_1) \ . \end{aligned} \tag{5.10}$$

Making use of (5.9), (5.10) one can establish that (5.8) is satisfied for each $x \in (0,L)$. Thus, the solution of the direct problem is statically and kinematically admissible.

2.5.4. Discussion of the results

Economy coefficient (4.29) could be expressed as

$$e = \frac{1}{L}(b + p(L - b)) + \frac{1}{dL\ ch\ dL}((n - p)sh\ dL + (p - 1)sh\ db).$$

Different values of (5.11) corresponding to different values of b and n_1 are presented in Table 2.5.1.

Table 2.5.1. Economy of the optimal design of an open shell

b/L	n = 0	0,2	0,4	0,6	0,8
0	0,777	0,822	0,866	0,911	0,955
0,2	0,809	0,847	0,885	0,924	0,962
0,4	0,846	0,877	0,908	0,938	0,969
0,6	0,889	0,911	0,933	0,955	0,978
0,8	0,939	0,951	0,964	0,976	0,988

Table 2.5.2. Load intensities for fixed extent of the central zone

b/L	n = 0	0,2	0,4	0,6	0,8
0	1,500	1,400	1,300	1,200	1,100
0,2	1,781	1,625	1,469	1,313	1,156
0,4	2,389	2,111	1,833	1,556	1,278
0,6	4,125	3,500	2,875	2,250	1,625
0,8	13,500	11,000	8,500	6,000	3,500

Optimal thickness distribution is presented in Fig.2.5.2 for different values of the transverse pressure. Fig. 2.5.2 corresponds to the value of the geometrical parameter $a = 4$ and the axial force $n = 0,2$. Evidently, $v(1) = n$. If the axial force tends to unity or the load intensity increases the thickness tends to unity as shown in Table 2.5.3.

It should be noted that when n is approaching unity the limiting process is uniformly convergent. However, for a fixed value of n near the edges of the shell there occur the regions where $v < 1$.

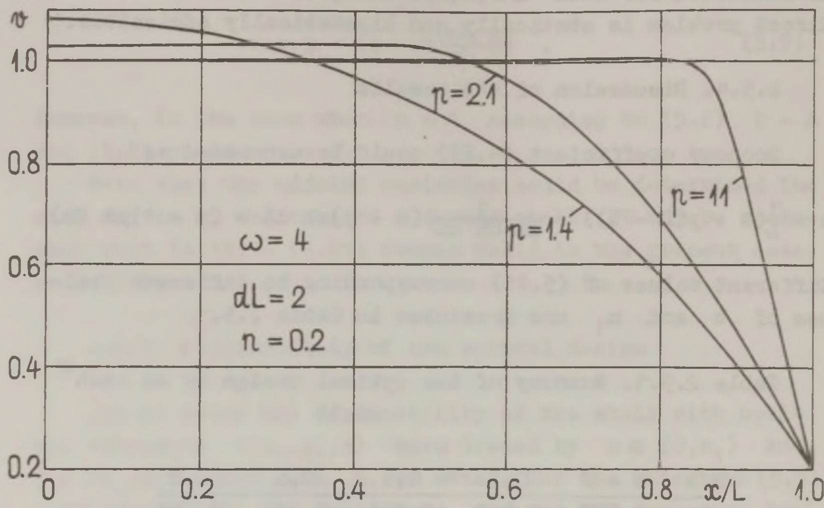


Fig. 2.5.2. Optimal thickness of an open sandwich shell

Table 2.5.3. Optimal shape of an open sandwich shell

p	x/L = 0	0,2	0,4	0,6	0,8	1,0
1,400	1,081	1,055	0,973	0,822	0,578	0,200
3,500	1,012	1,013	1,015	1,021	0,827	0,200
11,000	1,003	1,003	1,004	1,005	1,007	0,200

The maximal economy which can be achieved corresponds to $n = 0$ and $p = p_0$ for each value of the parameter a. For $a = 4$ the maximal material saving equals 22,3%. If the axial force or load intensity increases the eventual economy decreases.

§2.6. Weight minimization of an open homogeneous shell

2.6.1. Preceding remarks

The optimal design problem posed in the previous paragraph will be investigated herein assuming the shell wall is homogeneous. Let the thickness of the shell wall be denoted by $h(x)$. The shell is subjected to the internal pressure and to the axial tension, edges of the shell being hinged.

Material of the shell is assumed to obey the Tresca yield surface shown in Fig. 2.1.6. The face $N_2 = N_0$ of this surface is outlined in Fig. 2.3.2. The stress regime KL (Fig. 2.3.2.) will be used, provided the point K corresponds to the region (0,b) and KL to the interval (b,L). Thus, the function Φ in (4.3) could be expressed as

$$\Phi = -\frac{M}{M_0} + \left(\frac{N_1}{N_0}\right)^2 - 1 \,, \tag{6.1}$$

where $M_0 = \sigma_0 h^2/4$, $N_0 = \sigma_0 h$.

Note that the non-dimensional quantities defined by(4.8) are also applicable in the present case.Naturally,the yield force N_0^0 and yield moment M_0^0 have to be replaced by appropriate expressions.

Evidently, the posed problem may be considered as a particular case of the one studied in section 2.2.4. Thus, the optimality condition (2.32) holds good in the present case. Combining (2.32) and (6.1) one obtains

$$v = \sqrt{n^2 - m} \,, \tag{6.2}$$

where the notation (4.8) is used.

2.6.2. Minimum weight design of the shell

Substituting the curvature of a generator of the shell (5.3) into the system (1.14) - (1.16) leads to the equations

$$m'' = \begin{cases} \frac{a}{L^2}(1 - v), & x \in (0,b) \,, \\ \frac{a}{L^2}(p - v), & x \in (b,L) \,, \end{cases} \tag{6.3}$$

and

$$u' = \begin{cases} -\frac{a^2}{8L^3 n^2}(p - 1)^2 x^2 + \frac{a}{L}(p - 1), & x \in (0,b) \,, \\ -\frac{a^2}{8L^3 n^2}(p - 1)^2 x^2 \,, & x \in (b,L) \,, \end{cases} \tag{6.4}$$

where according to (3.6), (3.8) and (4.8)

$$b = L\left(1 - \sqrt{\frac{2(1 - n^2)}{a(p - 1)}}\right). \tag{6.5}$$

From (6.5) one could obtain the limit load

$$p_o = 1 + \frac{2}{a}(1 - n^2) . \tag{6.6}$$

Evidently (6.4) could be integrated analytically. This results in

$$u = \begin{cases} -\dfrac{a^2}{24n^2L^3}(p - 1)^2x^3 + \dfrac{a}{2L}(p - 1)x , & x\in(0,b) , \\ -\dfrac{ab^2}{8n^2L^3}(p - 1)^2(x - \frac{1}{3}b) + \dfrac{a}{2L}(p - 1)b , & x\in(b,L) . \end{cases} \tag{6.7}$$

Taking the optimality condition (6.2) into account the equation (6.3) may be converted into the set

$$\frac{dm}{dt} = q ,$$

$$\frac{dq}{dt} = \begin{cases} a(1 - \sqrt{n^2 - m}) , & t\in(0,b/L) , \\ a(p - \sqrt{n^2 - m}) , & t\in(b/L,1) , \end{cases} \tag{6.8}$$

where $t = x/L$. The set of equations (6.8) is integrated numerically under the following boundary conditions

$$q(0) = m(1) = 0 . \tag{6.9}$$

2.6.3. Numerical results

The boundary value problem (6.8),(6.9) was solved by the use of the Runge-Kutta method of the fourth order. The results are presented in Table 2.6.1 and Fig. 2.6.1. Two curves in Fig. 2.6.1 correspond to the case when $a = 4$ and $n = 0,5$. Here $p = 1,5$ and $1,7$; respectively. Calculations carried out reveal the matter that the optimal thickness is comparatively weakly sensitive to the changes in the intensity of the transverse pressure loading for fixed axial tension. This was observed in the case of sandwich shells, too. The third curve in Fig. 2.6.1 is associated with the axial tension $n = 0,8$.

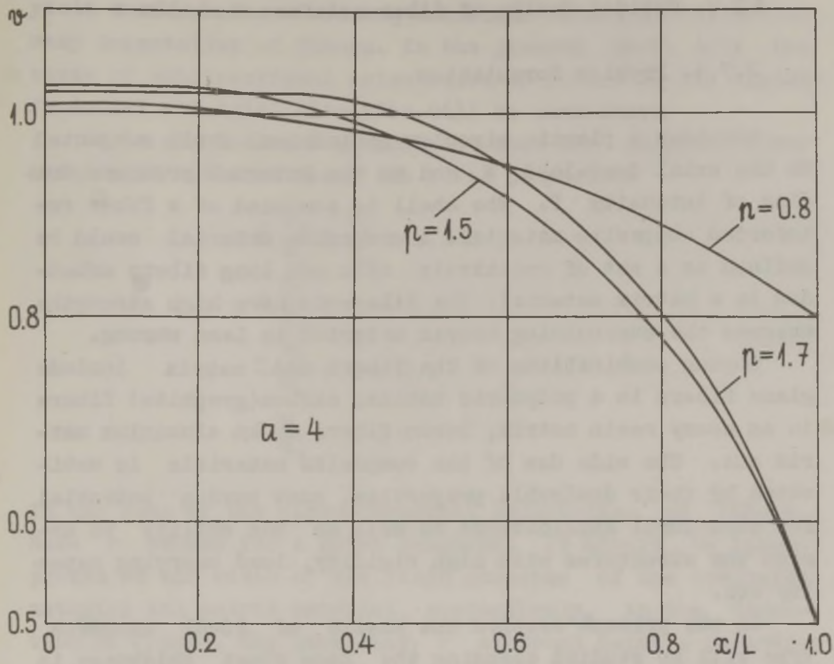


Fig. 2.6.1. Optimal thickness of an homogeneous Tresca shell

The economy of the design established is assessed by the economy coefficient defined by (4.29). Different values of the economy coefficient are presented in Table 2.6.1 for different values of the transverse pressure. Here a = 4 and n = 0,5.

Table 2.6.1. Economy of the design for an homogeneous Tresca shell

p	1,5	1,6	1,7	1,8	1,9
e	0,902	0,911	0,917	0,920	0,924

§ 2.7. Optimal design of fiber-reinforced shells

2.7.1. Problem formulation

Consider a plastic circular cylindrical shell subjected to the axial dead-load N and to the internal pressure loading of intensity P. The shell is composed of a fiber reinforced composite material. A composite material could be defined as a set of relatively thin and long fibers embedded in a matrix material. The filaments have high strengths, whereas the surrounding matrix material is less strong.

Common combinations of the fibers and matrix include glass fibers in a polymeric matrix, carbon(graphite) fibers in an epoxy resin matrix, boron fibers in an aluminium matrix etc. The wide use of the composite materials is motivated by their desirable properties, cost saving potential for structural applications as well as the ability to design the structures with high rigidity, load carrying capacity etc.

In the present section the shells of ideal sandwich type will be studied assuming the face sheet thickness is variable. The optimal thickness distribution is being sought that corresponds to the minimum weight and for which the deflections coincide with that of the reference shell of constant thickness. It is reasonable to demand that

$$N - 2\sigma_* h(x) \leq 0 \,, \tag{7.1}$$

σ_* being the yield stress of the material in the axial direction.

2.7.2. Yield surfaces for fiber reinforced shells

Anticipating the need for the ductile failure theory, a simple approximation of the yield surface for fiber reinforced materials was suggested by Lance and Robinson; 1971; 1972. This surface was used in evaluating the load carrying capacity of cylindrical shells and plates by Lance and Robinson, 1972; 1973.

Because of the strong dependence of the yield behaviour on the angle between the direction of fibers and principal stresses it appears to be impossible to draw one single yield surface characterizing material's behaviour for arbitrary orientation of fibers. In the present work, only the cases of unidirectional orientation of fibers in the circumferential and axial direction will be considered.

Lance and Robinson established that the plastic fiber reinforced material obeyd the yield polyhedron

$$\begin{aligned} &\pm \frac{N_1}{N_o} \mp \frac{N_2}{N_o} - r = 0 , \\ &\pm \frac{N_1}{N_o} - 1 = 0 , \\ &\pm \frac{N_2}{N_o} - r = 0 , \\ &\pm \frac{M}{M_o} - 1 = 0 \end{aligned} \tag{7.2}$$

in the case of the circumferential orientation of fibers. Here r stands for a given constant. It should be interpreted as the ratio of the yield stresses of the composite material and matrix material, respectively, in the direction of fibers. The parameter r although being a parameter of the reinforced material is not a physical constant.

However,

$$\begin{aligned} &\pm \frac{N_2}{N_o} + 1 = 0 , \\ &\pm \frac{N_1}{N_o} \mp \frac{N_2}{N_o} - r = 0 , \\ &\pm \frac{N_1}{N_o} - r = 0 , \\ &- \frac{M}{M_o} - r = 0 \end{aligned} \tag{7.3}$$

in the case of axial orientation of fibers.

It appears to be reasonable to introduce a formal parameter (a distinctive mark) k so that $k = 1$ corresponds to the axial orientation and $k = r$ to the circumferential orientation of fibers. The use of this attribute allows to present the yield surfaces (7.2) and (7.3) as a single closed polyhedron formed by the intersection of the faces

$$\begin{aligned} &\pm n_1 - \frac{r}{k} v = 0 , \qquad &&\pm n_2 - kv = 0 , \\ &\pm m - \frac{r}{k} v = 0 , \qquad &&\pm n_1 \mp n_2 - rv = 0 \end{aligned} \tag{7.4}$$

where notation (4.8) is employed.

It will be assumed that $r \geqslant 1$ as the fibers should be stronger than the matrix material, as a rule.

2.7.3. Shell of constant thickness

The solution obtained for the sandwich shell of a Tresca material in the section 2.3.1 must be modified slightly when using the shells manufactured of a reinforced composite. Substituting $v = 1$ and assuming that the stress profile lies on the face $n_2 = k$ of the yield surface (7.4) leads to the restriction imposed on the bending moment

$$|m| \leqslant \frac{r}{k}. \tag{7.5}$$

Assuming $m = -r/k$ for $x \in (0,b)$ and $m \geqslant -r/k$ for $x \in (b,L)$ yields the curvature distribution

$$w'' = \begin{cases} \frac{a}{2nL^2}(k - p), & x \in (0,b), \\ 0, & x \in (b,L), \end{cases} \tag{7.6}$$

which could be used when integrating the equilibrium equation (1.14) where N_o is to be replaced by kN_o. Thus, (7.6) with (7.4) and (1.14) lead to the displacement field

$$w = \begin{cases} \frac{a}{2nL^2}(k - p)x^2 + w_o, & x \in (0,b), \\ \frac{ab}{2nL^2}(k - p)(x - L), & x \in (b,L). \end{cases} \tag{7.7}$$

In (7.7) L stands for half of the length of the shell, whereas

$$b = L - \sqrt{\frac{2rL^2}{ak(p - k)}} \tag{7.8}$$

and

$$w_o = \frac{1}{4n}\left(a(p - k) - 2\frac{r}{k}\right). \tag{7.9}$$

The bending moment corresponding to (7.5) - (7.9), (1.14) may be expressed as

$$m = \begin{cases} -\frac{r}{k}, & x \in (0,b), \\ \frac{a(p-k)}{2L^2}(x-b)^2 - \frac{r}{k}, & x \in (b,L). \end{cases} \tag{7.10}$$

By the use of (7.10) it is easy to recheck that (7.5) is met.

From (7.8) and (7.9) follows the load carrying capacity in the form

$$p_o = k + \frac{2r}{ak} . \tag{7.11}$$

2.7.4. Minimum weight design for a fiber reinforced shell

According to (7.2) - (7.4) the function Φ could be picked as

$$\Phi = -\frac{M}{M_o} - \frac{r}{k} \tag{7.12}$$

whereas (7.1) could be expressed as

$$n - \frac{r}{k}v + \theta_1^2 = 0 \tag{7.13}$$

θ_1 being an additional control function.

It should be noted that the present problem differs slightly from the particular problems studied in the previous sections. It is caused by restriction (7.13). Therefore, the optimality condition (2.32) holds good in a section of the optimal trajectory, only. In some other parts of the interval (0,L) this requirement must be substituted by the condition $\theta_1 = 0$. Thus, according to (7.12) and (7.13)

$$m = -\frac{r}{k}v \tag{7.14}$$

or

$$v = \frac{k}{r}n . \tag{7.15}$$

It appears that (7.14) holds good in the region $x \in (0,b_1)$, whereas (7.15) applies for $x \in (b,L)$.

Assuming that the stress state of the shell of a variable face sheet thickness corresponds to the plane $n_2 = kv$ of the yield polyhedron (7.4) one readily obtains from(1.14)-(1.16) the equations

$$m'' = \begin{cases} \frac{ak}{L^2}(1 - v) , & x \in (0,b) , \\ \frac{a}{L^2}(p - kv) , & x \in (b,L) , \end{cases} \tag{7.16}$$

$$u' = \frac{-a^2}{8n^2L^3}(k - p)^2 x^2 , \quad x \in (0,L).$$

The first equation in (7.16) when substituting (7.14) and (7.15) leads to

$$m'' = \begin{cases} \frac{ak}{L^2}(1 + \frac{k}{r}m) , & x \in (0,b) , \\ \frac{a}{L^2}(p + \frac{k^2}{r}m), & x \in (b,b_1) , \\ \frac{a}{L^2}(p - \frac{k^2}{r}n), & x \in (b_1,L) \end{cases} \tag{7.17}$$

whereas the last equation in (7.16) gives

$$u = - \frac{a^2}{24n^2L^3}(k - p)^2 x^3 . \tag{7.18}$$

The solution of (7.17) can be presented as

$$m = - \frac{r}{k} + A_1 \mathrm{ch}\, dx + A_2 \mathrm{sh}\, dx \tag{7.19}$$

for $x \in (0,b)$,

$$m = - \frac{r}{k^2} pB_1 \mathrm{ch}\, dx + B_2 \mathrm{sh}\, dx \tag{7.20}$$

for $x \in (b,b_1)$ and

$$m = \frac{a}{2L^2}(p - \frac{n}{r}k^2)x^2 + C_1 x + C_2 \tag{7.21}$$

for $x \in (b_1,L)$, where $d = k\sqrt{a/rL^2}$.

For determination of the arbitrary constants A_1, A_2, B_1, B_2, C_1, C_2 one can use the boundary conditions

$$m(1) = m'(0) = 0 , \tag{7.22}$$

the requirement

$$m(b_1) = - n \tag{7.23}$$

which follows from (7.14), (7.15) as well as the continuity conditions for $m(x)$ and $m'(x)$ at $x = b$ and $x = b_1$.

These result in

$$m = \frac{\text{ch } dx}{\text{sh } db}\left(\frac{r}{k^2}(p - \frac{n}{r}k^2)\text{sh } d(b - b_1) + \right.$$

$$\left. + \frac{q_*}{d}\text{ch } d(b - b_1) - \frac{r}{k}\right) \tag{7.24}$$

for $x \in (0,b)$,

$$m = \frac{-r}{k^2}p + \frac{r}{k^2}(p - \frac{n}{r}k^2)\text{ch } d(x - b_1) +$$

$$+ \frac{q_*}{d}\text{sh } d(x - b_1) \tag{7.25}$$

for $x \in (b,b_1)$ and

$$m = (\frac{a}{2L}(p - \frac{n}{r}k^2)(x - b_1) - \frac{n}{b_1 - L})(x - L) \tag{7.26}$$

for $x \in (b_1,L)$.

In (7.24) - (7.26) the following notation is introduced:

$$q_* = \frac{a}{2L}(p - \frac{n}{r}k^2)(b_1 - L) - \frac{nL}{b_1 - L} \text{ .} \tag{7.27}$$

The boundary conditions (7.22), (7,23) and continuity conditions are satisfied if b_1 is the root of the equation

$$\frac{Ln\text{ch } db_1}{d(L - b_1)} - \frac{a}{2dL}(p - \frac{n}{r}k^2)(L - b)\text{ch } db_1 -$$

$$- (\frac{r}{k^2}p - n)\text{sh } db_1 + \frac{r}{k^2}(p - k)\text{sh } db = 0 \text{ .} \tag{7.28}$$

The optimal thickness distribution can be defined according to (7.14), (7.15), (7.24) - (7.26) as

$$v = \begin{cases} 1 - \frac{\text{ch } dx}{\text{sh } db}((\frac{p}{k} - \frac{n}{r}k)\text{sh } d(b - b_1) + \frac{kq_*}{rd}\text{ch } d(b - b_1)), & x \in (0,b), \\ \frac{p}{k} - (\frac{p}{k} - \frac{n}{r}k)\text{ch } d(x - b_1) - \frac{kq_*}{rd}\text{sh } d(x - b_1), & x \in (b,b_1), \\ \frac{k}{r}n \text{ ,} & x \in (b_1,L). \end{cases} \tag{7.29}$$

The adjoint variables could be determined according to

(2.28) - (2.31). As φ_1 is not zero over the region (0,L) the relation (4.18) is not applicable. However, $\varphi_1 = 0$ for $x \in (0,b_1)$ and $\varphi = 0$ for $x \in (b_1,L)$.

Integrating (2.28), making use of (2.29) - (2.31) and satisfying the transversality conditions and continuity requirements imposed on the adjoint variables at $x = b_1$ leads to the solution of the adjoint set

$$\psi_1 = \begin{cases} - C \text{ sh } dx , & x \in (0,b_1) , \\ - C \text{ sh } db_1 , & x \in (b_1,L) , \end{cases}$$
$$\psi_2 = \begin{cases} - \frac{1}{ak} + \frac{C}{dL}\text{ch } dx , & x \in (0,b_1) \\ C \text{ sh } db_1(x - L) , & x \in (b_1,L) \end{cases} \tag{7.30}$$

where

$$C = \frac{1}{ak}(\text{ch } db_1 + d(L - b_1)\text{sh } db_1)^{-1} .$$

Consequently, the adjoint variables are not spontaneously equal to zero which implies that the problem is not a singular one.

2.7.5. Applicability of the design

In order to study the behaviour of the minimum weight design established herein let us consider the shell of thickness

$$v = \begin{cases} 1 - \frac{\text{ch } dx}{\text{sh } db_1^1}((\frac{p_1}{k} - \frac{n_1 k}{r})\text{sh } d(b_1^1 - b_2^1) + \frac{kq_1}{rd}\text{ch } d(b_1^1 - b_2^1)), & x \in (0,b_1^1), \\ \frac{p_1}{k} - (\frac{p_1}{k} - \frac{n_1 k}{r})\text{ch } d(x - b_2^1) - \frac{kq_1}{rd}\text{sh } d(x - b_2^1), & x \in (b_1^1,b_2^1) \\ \frac{kn_1}{r} , & x \in (b_2^1,L) \end{cases} \tag{7.31}$$

which is obtained from (7.29) for $p = p_1$ and $n = n_1$.

In (7.31)

$$q_1 = \frac{a}{2L}(p_1 - \frac{n_1}{r}k^2)(b_2^1 - L) - \frac{n_1 L}{b_2^1 - L} \tag{7.32}$$

and b_2^1 stands for a root of the equation

$$\frac{Ln_1 \operatorname{ch} db_2^1}{d(L - b_2^1)} - \frac{a}{2dL}(p_1 - \frac{n_1}{r}k^2)(L - b_1^1)\operatorname{ch} db_2^1 - \tag{7.33}$$
$$- (\frac{r}{k^2}p_1 - n_1)\operatorname{sh} db_2^1 + \frac{r}{k^2}(p - k)\operatorname{sh} db_1^1 = 0 .$$

The quantity $b_{1_1}^1$ is defined by (7.8) for $p = p_1$. It should be noted that b_1^1 does not depend on the axial force n_1.

Let us study the post-yield behaviour of the shell of variable thickness defined by (7.31). Assume that $p \in (p_0, p_1)$ and $n \in (0, n_1)$, where p_0 is the load carrying capacity of the shell.

The stress distribution is assumed to be such that

$$m = -\frac{r}{k}v \tag{7.34}$$

for $x \in (0, b_0)$ and

$$m \geqslant -\frac{r}{k}v , \quad w'' = 0 \tag{7.35}$$

for $x \in (b_0, L)$, where v is given by (7.31) - (7.33).

Integrating the equilibrium equation (1.14) taking (7.34), (7.35) into account leads to the relations

$$m = -\frac{r}{k}v + \frac{a}{2L^2}(p - k)(x - b_0)^2 \tag{7.36}$$

for $x \in (b_0, b_1^1)$,

$$m = -\frac{r}{k}v + \frac{a}{2L^2}(p - k)(b_1^1 - b_0)(2x - b_0 - b_1^1) \tag{7.37}$$

for $x \in (b_1^1, b_2^1)$ and

$$m = -\frac{r}{k}v + \frac{a}{2L^2}(b_2^1 - x)^2(p - \frac{n_1}{r}k^2) + \frac{q_1}{L}(x - b_0^2) +$$
$$+ \frac{a}{2L^2}(p - p_1)(b_2^1 - b_1^1)(2x - b_1^1 - b_2^1) +$$
$$+ \frac{a}{2L^2}(p - k)(b_1^1 - b_0)(2x - b_1^1 - b_0) \tag{7.38}$$

for $x \in (b_2^1, L)$.

The unknown quantity b_0 in (7.36) - (7.38) is to be calculated as a root of equation

$$\frac{a}{2L^2}(p - \frac{n_1}{r}k^2)(L - b_2^1)^2 + \frac{q_1}{L}(L - b_2^1) +$$
$$+ \frac{a}{2L^2}(p - p_1)(b_2^1 - b_1^1)(2L - b_1^1 - b_2^1) - n_1 + \qquad (7.39)$$
$$+ \frac{a}{2L^2}(p - k)(b_1^1 - b_o)(2L - b_1^1 - b_o) = 0 .$$

Making use of (7.31) - (7.33) and (7.36) - (7.39), one can recheck that the inequality (7.35) holds good for each $x \in (b_o, L)$. Thus, the solution is admissible for each $p \in (p_o, p_1)$ and $n \in (0, n_1)$. Evidently, the load carrying capacity of the shell of variable thickness coincides with the limit load for the reference shell of constant thickness.

2.7.6. Discussion and conclusions

The results of calculations are presented in Tables 2.7.1, 2.7.2 and in Fig. 2.7.1, 2.7.2, 2.7.3. The dashed

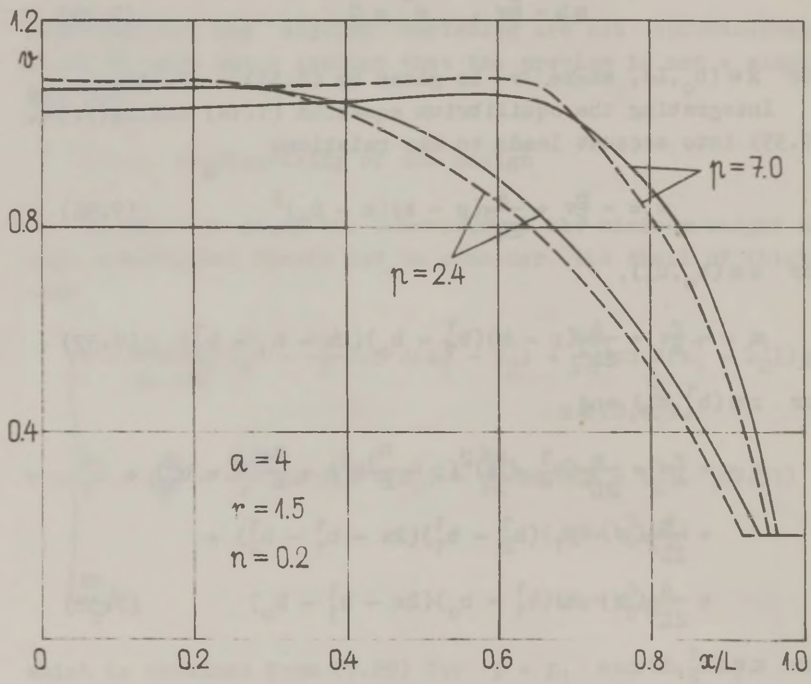


Fig. 2.7.1. Optimal thickness for a fiber-reinforced shell (n = 0,2)

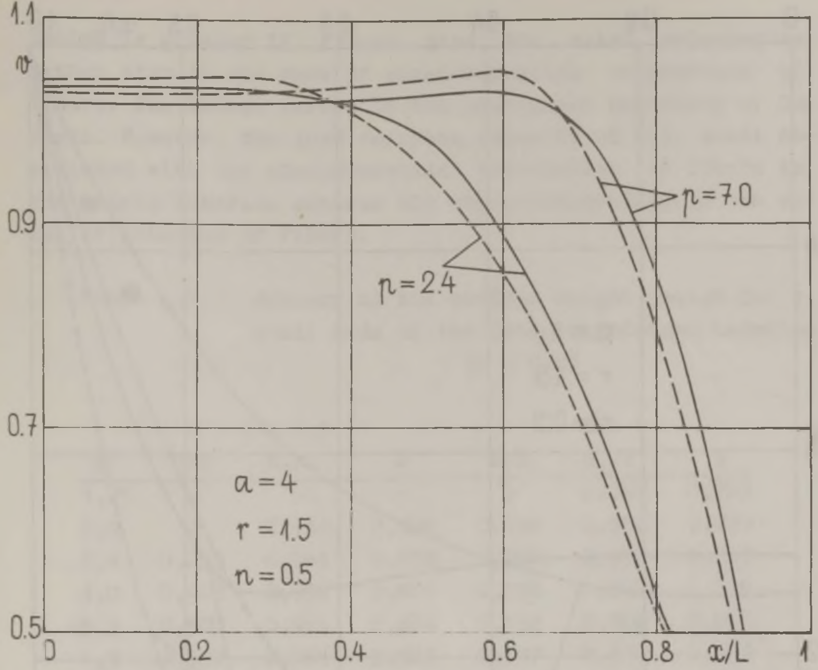


Fig. 2.7.2. Optimal thickness of a fiber-reinforced shell (n = 0,5)

lines in Fig. 2.7.1 - 2.7.3 correspond to the shells with axial orientation of fibers and the continuous ones to the circumferential orientation of fibers.

It is somewhat surprising that the optimal thickness is comparatively unsensitive to the orientation of fibers, while the stress-strain state of the shell strongly depends on that. However, this discrepancy springs from the statement of the problem - the minimum material consumption is looked for under the requirement that the deflections of the shell of variable thickness and that of the reference one, respectively, coincide. Naturally, the deflections corresponding to the axial and circumferential orientations of fibers, essentially differ from each other.

Fig. 2.7.3 presents the bending moment distribution for a relatively long shell. In the case of greater values of the transverse pressure the crests of these curves lie off

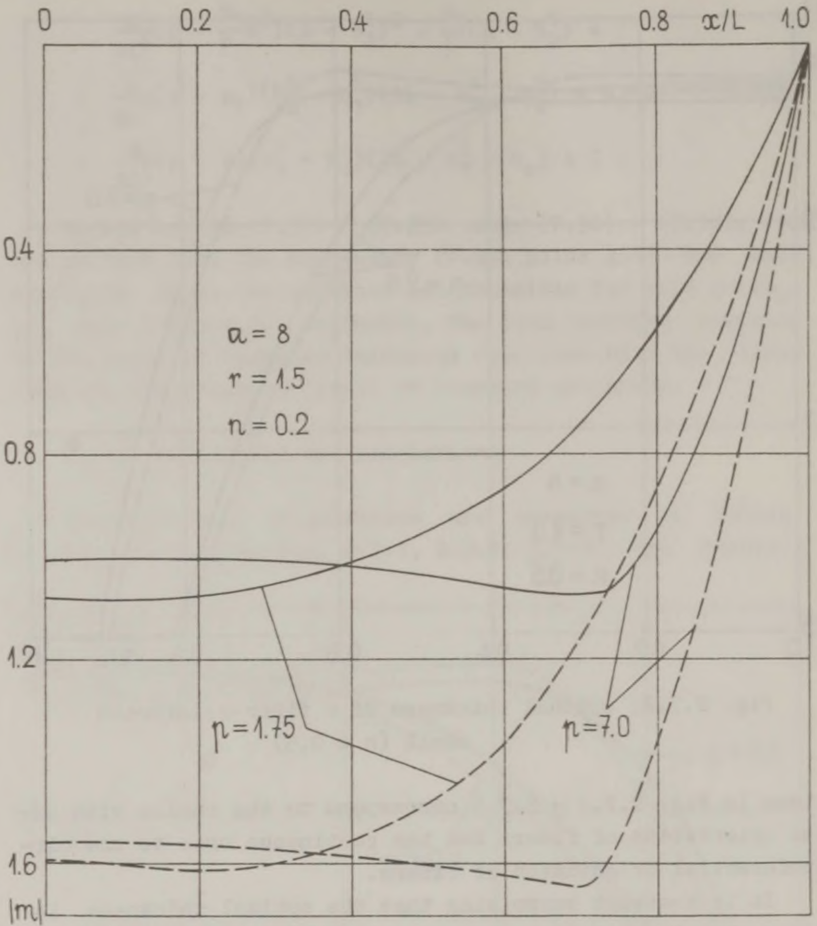


Fig. 2.7.3. Moment distribution for a cylindrical shell made of a fiber-reinforced composite

the center of the shell as shown in Fig. 2.7.3.

Specific values of the economy coefficient e which may be expressed by (4.29) are accommodated in Tables 2.7.1 and 2.7.2. The Tables correspond to the axial tension $n = 0,2$ and $n = 0,5$, respectively. The load carrying capacity of the shell with $a = 4$, $r = 1,5$ is equal to 1,75 in the case of the axial orientation of filaments and to 2,0 in the case of circumferential arrangement of fibers in a composite material. The exposed data reveal the fact that the material saving is greater if fibers have the axial orientation rather than in the case of circumferential orientation of fibers. The latter refers to the post-yield behaviour of the shell. However, the load carrying capacity of the shell associated with the circumferential orientation of fibers in the matrix material exceeds the one corresponding to the axial orientation of fibers.

Table 2.7.1. Economy of the minimum weight design for a shell made of the fiber-reinforced material (n = 0,2)

	k = r = 1,5			k = 1		
p	b/L	b_1/L	e	b/L	b_1/L	e
1,75	-	-	-	0	0,947	0,763
2,0	0	0,932	0,804	0,134	0,952	0,789
2,4	0,255	0,941	0,838	0,268	0,957	0,817
3,0	0,423	0,950	0,867	0,388	0,962	0,846
3,5	0,500	0,955	0,882	0,452	0,966	0,863
4,5	0,592	0,961	0,903	0,537	0,970	0,889
5,0	0,622	0,964	0,911	0,567	0,972	0,900
7,0	0,698	0,970	0,936	0,646	0,976	0,934

Table 2.7.2. Economy of the design for n = 0,5

	k = r = 1,5			k = 1		
p	b/L	b_1/L	e	b/L	b_1/L	e
1,75	-	-	-	0	0,856	0,781
2,0	0	0,797	0,838	0,134	0,868	0,806
2,4	0,255	0,828	0,868	0,268	0,883	0,832
3,0	0,423	0,856	0,892	0,388	0,898	0,859
3,5	0,500	0,870	0,905	0,452	0,907	0,875
4,5	0,592	0,890	0,923	0,537	0,920	0,900
5,0	0,622	0,897	0,930	0,567	0,924	0,910
7,0	0,698	0,915	0,952	0,646	0,936	0,943

§2.8. Optimal design of rib-reinforced cylindrical shells

2.8.1. Statement of problem

The shells strengthened by means of longitudinal and circumferential rib-reinforcements have quite a broad range of applications. In the last decades, in addition to the traditional fields of applications of rib-reinforced shells, the attention has been paid to the structures of the off-shore industry.

The load carrying capacity of rib-reinforced rigid-plastic cylindrical shells has been studied by Biron, 1970; Nemirovsky, 1969; Nemirovsky and Rabotnov, 1963; 1964; Biron and Sawczuk, 1967; Cinquini and Kouam, 1983.The authors have studied three types of shells:

(i) the structure consists of two cylindrical layers, the ribs being between them (Fig. 2.8.1),

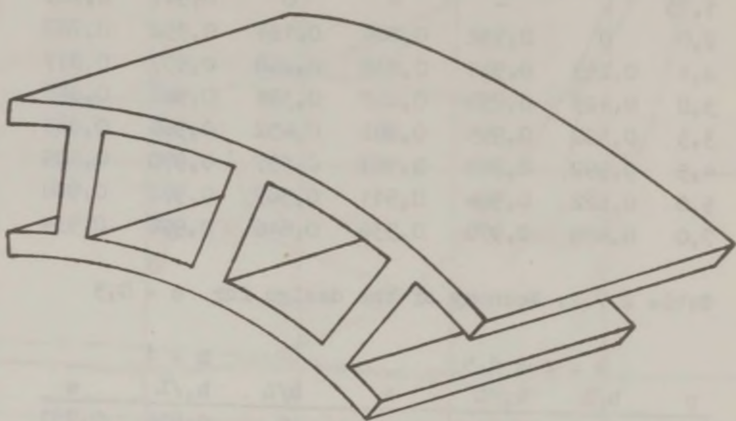


Fig. 2.8.1. An element of a rib-reinforced shell (the stringers lying between the cylindrical layers)

(ii) the ribs are located symmetrically at both sides of the rim (Fig. 2.8.2) and

(iii) the cylindrical layer is strengthened with the ribs at one side of the shell (Fig. 2.8.3).

In the present work, for the conciseness' sake the attention is focused to the first type of the shell (Fig. 2.8.1). Let $H(x)$ and h be the variable thickness of the ribs and the constant thickness of the layers, respectively. The distances between the ribs are denoted by d_1 (Fig. 2.8.4).

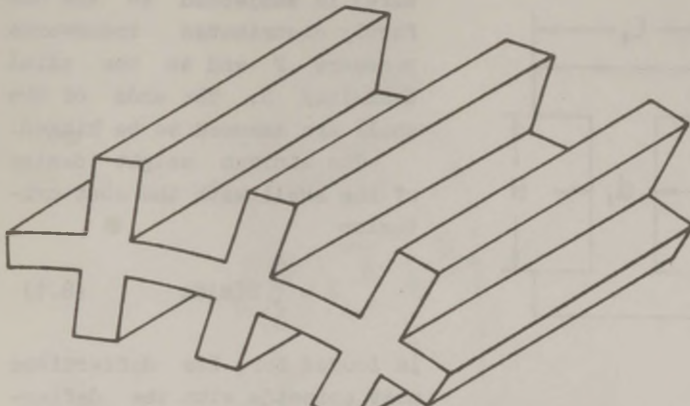


Fig. 2.8.2. A shell element with ribs lying in both sides of the rim

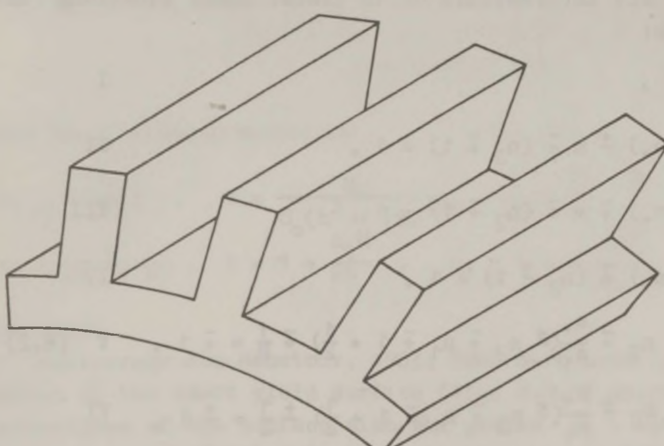


Fig. 2.8.3. A shell element with ribs lying in one side of the shell

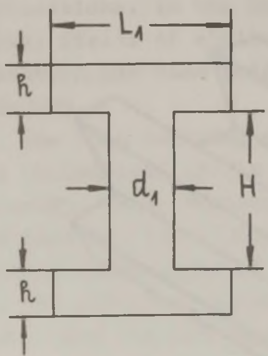


Fig. 2.8.4. The dimensions of the shell element

The cylindrical shell(structure) is subjected to the uniformly distributed transverse pressure P and to the axial dead-load N. The ends of the shell are assumed to be hinged.

The minimum weight design of the shell with the cost criterion

$$J = \int_0^L H(x)dx \qquad (8.1)$$

is looked for. Its deflections must coincide with the deflections of the shell of constant dimensions.

2.8.2. Yield surfaces for rib-reinforced shells

Nemirovsky and Rabotnov, 1963; 1964 have derived the yield surfaces for the rib-reinforced cylindrical shells. For the case presented in Fig. 2.8.1 the yield surface is formed by the intersection on 18 flats. These equations are as follows:

$$\pm n_2 = 1 \,, \qquad \text{I}$$

$$\pm (n_1 - n_*) \pm m \mp (n_2 \mp 1) = 1 \,, \qquad \text{II}$$

$$\pm (n_1 \pm n_*) \mp m \mp (n_2 \mp 1) = 1 \,, \qquad \text{III}$$

$$\pm (n_1 \pm n_*) \mp (n_2 \pm 1) = 1 \,, \qquad \text{IV}$$

$$m \pm m_* - n_2 \mp \frac{a_*}{a_1}(\pm n_2 \mp n_1 \mp 1 + \tfrac{1}{2}) \mp \tfrac{1}{2} = \mp 1 \,, \qquad \text{V} \quad (8.2)$$

$$m \mp m_* + n_2 \pm \frac{a_*}{a_1}(\pm n_2 \mp n_1 \mp 1 + \tfrac{1}{2}) \pm \tfrac{1}{2} = \pm 1 \,, \qquad \text{VI}$$

$$m \mp m_* \pm \frac{a_*}{a_1} n_1^2 = 0 \,, \qquad \text{VII}$$

$$m \pm (n_1 - n_*) = 0 \,, \qquad \text{VIII}$$

$$m \mp (n_1 + n_*) = 0 \,. \qquad \text{IX}$$

The yield surface with flats (8.2) is presented in Fig. 2.8.5.

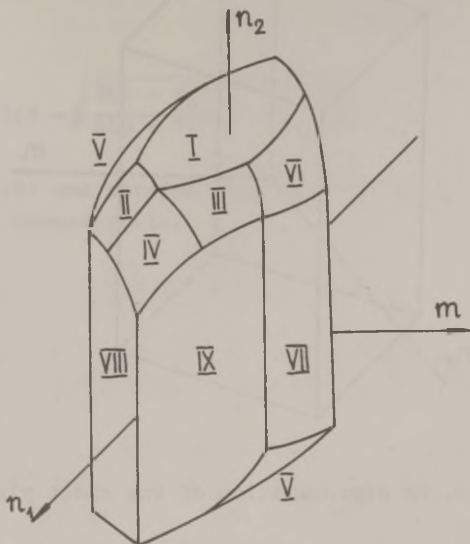


Fig. 2.8.5. Sketch of the exact yield surface for a cylindrical shell reinforced by the longitudinal ribs

Here the following notations

$$n_{1,2} = \frac{N_{1,2}}{2\sigma_0 h}\ , \quad m = \frac{M}{\sigma_0(h^2 + hH)}\ , \quad a_1 = \frac{d_1}{L_1}\ ,$$
$$a_* = \frac{h^2}{h(h + H)}\ , \quad n = 1 + \frac{a_1 H}{2h}\ , \quad m_* = 1 + a_* \frac{a_1 H^2}{4h^2} \tag{8.3}$$

are used.

Nemirovsky and Rabotnov, 1963; 1964 have used an approximation of the exact yield surface (Fig. 2.8.5) where the intersections of the surface with the planes n_1 = const have the rectangular form. This approximation was achieved when omitting faces II - VI (Fig. 2.8.5) and elongating planes I and VII - IX.

The approximation of the exact yield surface (Fig.2.8.6) used in the present work has been formed by the intersection

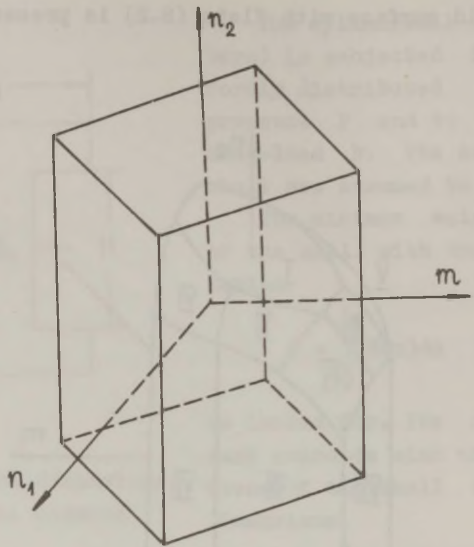


Fig. 2.8.6. An approximation of the exact yield surface

of the faces I, VIII and IX. Thus, in the plane

$$N_2 = 2\sigma_0 h \tag{8.4}$$

the bending moment is restricted by the inequality

$$\frac{M}{\sigma_0(H+h)h} \leq 1 + \frac{a_1 H}{2h} - \frac{N}{2\sigma_0 h} \; . \tag{8.5}$$

According to (8.4), (8.5) the function Φ could be put into the form

$$\Phi = -M + \sigma_0 h(H+h)\left(\frac{H}{2\sigma_0 h} - \frac{a_1 H}{2h} - 1\right) \; . \tag{8.6}$$

2.8.3. Minimum weight design of the rib-reinforced shell

Consider at first the reference shell of constant thickness H_0. Let us denote $v = H/h$ and $v_0 = H_0/h$. Similarly to the results of the previous sections, we get the bending moment for a shell with thickness v_0 as

$$m = \begin{cases} (1 + v_o)(n - 1 - \frac{1}{2} a_1 v_o) , & x \in (0,b) , \\ (1 + v_o)(n - 1 - \frac{1}{2} a_1 v_o) + \frac{a}{2L^2}(p - 1)(x - b)^2 , & \\ & x \in (b,L) . \end{cases} \qquad (8.7)$$

Here

$$b = L(1 - \sqrt{\frac{2(1 + v_o)}{a(p - 1)}(1 - n - \frac{a_1}{2} v_o)}) . \qquad (8.8)$$

From (8.8) one obtains the load carrying capacity for the rib-reinforced shell:

$$p_o = 1 + \frac{2}{a}(1 + v_o)(1 - n - \frac{a_1}{2} v_o) . \qquad (8.9)$$

Evidently, for $v_o = 0$, we can derive from (8.9) the limit load for the sandwich shell without ribs, familiar from section 2.3.

In order to get the optimal solution for the shell of variable thickness v, let us assume that according to (2.7), (2.8) and (2.32) $v = v_o$ for $x \in (0,b)$ and $v = v(x)$, $\Phi = 0$ for $x \in (b,L)$. Therefore, according to (8.6) - (8.8) one has the optimal thickness in the form

$$v = \begin{cases} v_o , & x \in (0,b) , \\ (\frac{1}{2} + \frac{n - 1}{a_1})^2 - \frac{a}{a_1 L^2}(p - 1)(x - L)(L + x - 2b) + & \\ + \frac{n - 1}{a_1} - \frac{1}{2} , & x \in (b,L) . \end{cases} \qquad (8.10)$$

The economy of the design (8.10) could be assessed as

$$e = \frac{1}{v_o L} \int_0^L v(x) dx . \qquad (8.11)$$

Substituting (8.10) into (8.11) gives

$$e = \frac{1}{v_o L} (L - b)(\frac{n - 1}{a} - \frac{1}{2}) +$$

$$+ \frac{1}{v_o} \int_b^L \sqrt{(\frac{1}{2} + \frac{n-1}{a_1})^2 - \frac{a}{a_1 L^2}(p-1)(x-L)(L + x - 2b)} \, dx \qquad (8.12)$$

The results of calculations are presented in Fig. 2.8.7 and in Tables 2.8.1 and 2.8.2.

In Fig. 2.8.7 the optimal thickness distribution is shown for several values of the transverse pressure. Here $v_o = 4,5$; $a = 4$; $a_1 = 0,1$; $n = 1,1$. It is shown that the

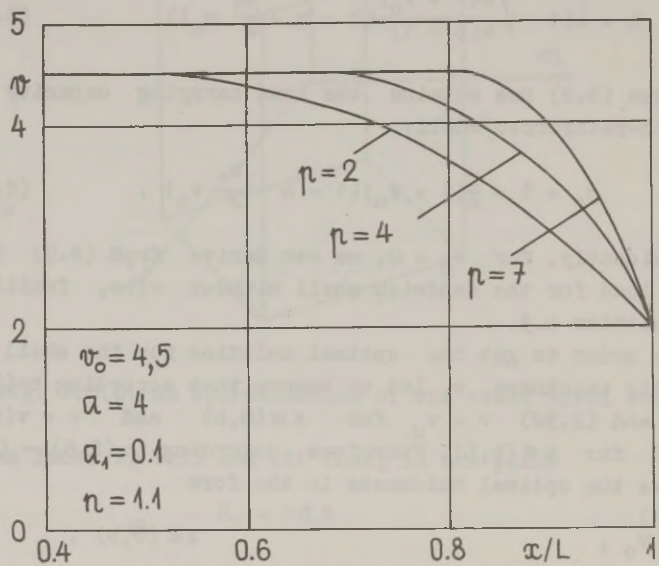


Fig. 2.8.7. Optimal thickness of a rib

optimal thickness tends to the constant value v_o when the transverse pressure increases.

Tables 2.8.1 and 2.8.2 present the corresponding values of the economy coefficient (8.12) as well as those of the coordinates b/L. Table 2.8.1 corresponds to the axial tension $n = 1,1$ whereas Table 2.8.2 is associated with $n = 1,2$.

Table 2.8.1. Economy of the design of a rib-reinforced shell for $n = 1,1$

p	2,0	2,5	3,0	3,5	4,0	4,5	5,0	5,5	6,0
b/L	0,414	0,521	0,585	0,629	0,661	0,687	0,707	0,724	0,738
e	0,910	0,926	0,936	0,943	0,948	0,952	0,955	0,958	9,60

Table 2.8.2. Economy of the design of a rib-reinforced shell for $n = 1,2$

p	2,0	2,5	3,0	3,5	4,0	4,5	5,0	5,5	6,0
b/L	0,738	0,786	0,815	0,834	0,849	0,860	0,870	0,876	0,883
e	0,991	0,992	0,993	0,994	0,995	0,995	0,995	0,996	0,996

In order to be convinced in the validity of the design of type (8.10) for an interval of load intensities let us consider the shell with thickness (8.10) for $p = p_1$ and $n = n_1$. For the statical admissibility of the solution for $p \leq p_1$ and $n \leq n_1$ there must exist such a stress distribution which can satisfy the requirement

$$m \geq (1 + v)(n - 1 - \frac{1}{2} a_1 v) \qquad (8.13)$$

where v is the variable thickness corresponding to the loading parameters p_1 and n_1. The inequality (8.13) is checked numerically.

The absolute values of the bending moment are presented in Table 2.8.3 for different values of the transverse pressure. The shell studied in Table 2.8.3 corresponds to the thickness (8.10) with $p = 7$; $n = 1,1$ and $a = 4$; $v_0 = 4,5$; $a_1 = 0,1$. The last row in Table 2.8.3 is associated with $p = 7$, thus it is equal to the right hand side of the inequality (8.13).

Table 2.8.3. Moment distributions for a shell of variable thickness

p \ x/L	0,471	0,588	0,647	0,706	0,765	0,824	0,882	0,941
2,0	0,681	0,627	0,579	0,517	0,441	0,352	0,248	0,131
4,0	0,688	0,688	0,688	0,676	0,624	0,530	0,395	0,218
5,0	0,688	0,688	0,688	0,688	0,661	0,579	0,441	0,248
6,0	0,688	0,688	0,688	0,688	0,680	0,614	0,479	0,274
7,0	0,688	0,688	0,688	0,688	0,688	0,640	0,510	0,296

Table 2.8.3 demonstrates the fact that (8.13) is satisfied at each point of the shell.

2.8.4. Minimum weight design of the shell in the case of the rims of variable thickness.

Consider now such a case of a rib-reinforced shell where the thickness of ribs is constant but the cylindrical layers have variable thicknesses. Thus, $H = \text{const}$, $h = h(x)$. As above, we are looking for the design which corresponds to the minimum weight.

Using the non-dimensional quantities

$$v = \frac{h}{h_o} , \quad H_1 = \frac{H}{h_o} \tag{8.14}$$

the relation (8.6) may be substituted by

$$\Phi = - m + (v + H_1)(n - \tfrac{1}{2}a_1H_1 - v) . \tag{8.15}$$

Combining (2.32) and (8.15) leads to the relation

$$v = \sqrt{\tfrac{1}{4}(n - H_1(1 + \tfrac{1}{2}a_1))^2 + nH_1 - \tfrac{1}{2}a_1H_1^2 - m} + \tfrac{1}{2}(n - H_1(1 + \tfrac{1}{2}a_1)) . \tag{8.16}$$

However, the bending moment m in (8.16) is defined as a solution of the equation

$$m'' = \begin{cases} a\sqrt{\tfrac{1}{4}(n - H_1(1 + \tfrac{1}{2}a_1))^2 + hH_1 - \tfrac{1}{2}a_1H_1^2 - m} + \\ + a(1 - \tfrac{n}{2} + \tfrac{1}{2}H_1(1 + \tfrac{1}{2}a_1)) ; \qquad x \in (0,b) , \\ a\sqrt{\tfrac{1}{4}(n - H_1(1 + \tfrac{1}{2}a_1))^2 + nH_1 - \tfrac{1}{2}a_1H_1^2 - m} + \\ + a(p - \tfrac{n}{2} + \tfrac{1}{2}H_1(1 + \tfrac{1}{2}a_1)) , \qquad x \in (b,L) . \end{cases} \tag{8.17}$$

The equation (8.17) must be integrated under the boundary conditions $m(1) = m'(0) = 0$. The integration has been accomplished numerically. Several values of the economy coefficient and coordinate b are presented in Table 2.8.4 . Here $a = 4$; $n = 1,1$; $a_1 = 0,1$; $H_1 = 4,5$.

Table 2.8.4. Economy of the design of variable thickness

p	1,5	2,5	3,0
b/L	0,171	0,521	0,585
e	0,943	0,957	0,958

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