DISSERTATIONES
MATHEMATICAE
UNIVERSITATIS
TARTUENSIS
126

ANNIKA KRUTTO

Empirical Cumulant Function Based Parameter Estimation in Stable Distributions

TARTU ÜLIKOOL · UNIVERSITAS TARTUENSIS · 1632

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

126

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

126

ANNIKA KRUTTO

Empirical Cumulant Function
Based Parameter Estimation
in Stable Distributions

UNIVERSITY OF TARTU
Press

Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Estonia.

Dissertation has been accepted for the commencement of the degree of Doctor of Philosophy (PhD) in Mathematical Statistics on December 31, 2018, by the Council of the Institute of Mathematics and Statistics, University of Tartu.

Supervisor	Professor Tõnu KOLLO, PhD Institute of Mathematics and Statistics Faculty of Science and Technology University of Tartu
Opponents	Professor Anatoliy MALYARENKO, PhD Division of Applied Mathematics of the School of Education, Culture, and Communication, Mälardalen University, Västerås, Sweden
	Professor Olga JANUŠKEVIČIENE, PhD Institute of Mathematics and Informatics, Vilnius University

Commencement will take place on February 07, 2019 at 14:15 in J. Liivi 2-111 Tartu, Estonia.

The publication of this dissertation is financed by the Institute of Mathematics and Statistics, University of Tartu.

This research was funded by Estonian Research Council Grants ETF9127 and IUT34-5, Estonian Doctoral School in Mathematics and Statistics, Kristjan Jaak Scholarships, Dora Plus Doctoral Studies and Internationalisation Programme and Robust Analysis Inc.

This thesis was composed using the free cross-platform TEX/LATEX editor `Texmaker` and the `MiKTeX 2.9` system, an open-source free-distribution of the TEX/LATEX typesetting system developed as a joint effort by the TEXUsers Group.

ISSN 1024-4212
ISBN 978-9949-77-964-2 (print)
ISBN 978-9949-77-965-9 (PDF)

University of Tartu Press
http://www.tyk.ee/

Contents

List of original publications

This thesis is based on the following publications:

1. Krutto, A. (2016). Parameter Estimation in Stable Law. *Risks* **4**(4) 1–15.

2. Krutto, A. (2018). Empirical cumulant function based parameter estimation in stable laws. *Acta et Commentationes Universitatis Tartuensis de Mathematica* **22**(2) 311–338.

Publications not included in this thesis:

1. Ramsay, C., Oguledo, V., Krutto, A. (2018). Exploring the Optimal Design of an Employer-Sponsored Sickness-Disability Compensation Insurance Plan When Sickness Presenteeism Is Penalized. *North American Actuarial Journal* **22**(3) 426–457.

Acknowledgements

I am sincerely thankful to my supervisor, Professor Tõnu Kollo, for introducing me to stable laws and for his long term supportive attitude, optimism, and guidance. I appreciate the many times he carefully and cheerfully read and edited my thesis. Deserving special mention is Dr. Jüri Lember for his probing questions and insightful suggestions that helped to improve my thesis. I wish to express my very great appreciation to the two external peer-reviewers for providing many heedful remarks and comments. I wish to express my heartfelt thanks to all of the staff of our Institute of Mathematics and Statistics for their continuous caring and support and for making my experience such a pleasant and rewarding one. I especially would like to thank Professor Ene–Margit Tiit, Dr. Imbi Traat, Dr. Ene Käärik, Mare Vähi, Dr. Kristi Kuljus, Anne Selart, Professor Kalev Pärna, Dr. Raul Kangro and Dr. Meelis Käärik. I am thankful to the Head of Institute, Professor Viktor Abramov, the Vice Director, Dr. Märt Põldvere, and our secretary, Svetlana Saprõkova and Kelli Sander, for helping me navigate the procedures required for the official submission and defence processes. I also would like to thank Kelly Sander for her guidance during my studies, the library's Liina Jürimaa for her help in processing many orders for books and Kersti Taurus for her technical help. I must also thank the University of Tartu Press, especially editor Aive Maamets, for their fast service and compliance. I wish to thank the members and the Chair of my defense committee. I am particularly grateful for Professor Anatoliy Malyarenko and Professor Olga Januškevičiene for agreeing to be the opponents of my thesis and for all the time and effort that it brings along. I am genuinely appreciative of Professor John Nolan, whom I met a few years ago, for his interest in my research and for his kind encouragement over the years. With warm feelings and joyful memories, I remember all my fellow Ph.D. students, especially Liivika, Ants, Joonas and Roel. I thank Liivika for her cheerful helpfulness in many significant moments while attending conferences. My deepest gratitude is for my beloved family and close friends for their endless love and encouragement. This thesis would not have succeeded without their help. I thank Kalev and Riina for their unconditional support and always having faith in me. I thank Laur for all his contribution and understanding. I thank Rosalind for her endearing seriousness, Mildred for her serene cheerfulness and Konrad for his undeniable wisdom. I thank Kairi for her aspiring attitude and for always believing in me. I thank Zenni and Elmar for their kind heartedness and inspirational industriousness. I thank Hindrik and Mall for their long term kind support. I thank Reet for her broad-minded attitude and for her help in proofreading my thesis. I thank Anne S. for always being helpful and for her prudence. I thank Anne H., Liana, Merike for their patience and lightness of spirits. Last but not least I wish to express my unfeigned thanks to Wolfgang A. M. for all the inspiration, joy, and conciliation.

Annika Krutto,
Tartu, January 2019

Introduction

This thesis aims to provide a method of estimating the parameters of stable distributions. Stable distributions, also known as the Lévy-stable, α-stable, sum-stable, or Pareto-stable distributions, form a sub-class of infinitely divisible distributions that are the only possible limiting distributions for normalized sums of independent identically distributed random variables. The first appearance of stable laws was observed within the results of the theory of two classic limit theorems – the law of large numbers and the central limit theorem. However, the family of stable laws was first described by Lévy (1925). The intriguing theoretical properties of stable laws have engaged the interest of many mathematicians and statisticians ever since. All stable laws possess a stability property: the sum of independent stable random variables is distributed as a stable law. However, the normal law is the only stable law with finite variance and light tails. All other non-degenerate stable laws have infinite variance and heavier tails. Univariate stable laws account for location, scale and, in some appropriate senses, skewness and tail weight parameters. The flexible 4–parameter stable laws can capture the fuzzy dynamics and large fluctuations that result from stochastic processes occurring in diverse fields of business, science, and engineering. Starting with a series of publications in the 1960's (e.g., Mandelbrot (1960a,b), Good (1961)) stable laws have been extensively used in various areas of applications. For example, radiophysics (e.g., Nikias and Shao (1995), Cek (2015), Wang et al. (2017), and Pad et al. (2017)); biostatistics (e.g., Hougaard (1986)); hydrology (e.g., Menabde and Sivapalan (2000) and Kohlbecker et al. (2006)); finance (e.g., Leitch and Paulson (1975), McCulloch (1996), Nolan (2003), Curto et al. (2009), Kring et al. (2009), Xu et al. (2011), and Kateregga et al. (2017)); and actuarial science (e.g., Goovaerts et al. (2003), Brahimi and Abdelli (2016), and Luong (2016)). For more on the fields of applications, see, e.g., Uchaikin and Zolotarev (1999), Nolan (2018a).

One can list few alternatives to stable distributions: Johnson distributions (Johnson et al. (1994)); generalised hyperbolic distributions (Barndorff-Nielsen (1978)), the sinh-arcsinh distributions (Jones and Pewsey (2009)), skew-elliptical families (e.g., Azzalini and Capitanio (2014)), extreme value distributions (e.g., Kotz and Nadarajah (2000)), geometric stable distributions (e.g., Kozubowski (1999)), and tempered stable laws (e.g., Rosiński (2007)). However, when solving problems related to the limit distributions of normalized sums of independent identically distributed random variables then stable laws have the most appropriate structure (Uchaikin and Zolotarev, 1999, p. 66). When compared to most of the aforementioned alternative distributions, stable distributions provide heavier tails. In addition, stable distributions are intimately related to the stable Lévy processes, a significant subclass of Lévy processes.

Motivation and Research Task

A challenging problem in applying stable distributions to practical problems is estimating their parameters because many stable distributions have infinite moments and, with a few exceptions, the densities cannot be explicitly expressed in the terms of elementary functions. In spite of these limitations, various methods of constructing estimators of the parameters of stable laws have been proposed. Unfortunately, many of these methods have restrictions in the parameter space while the most popular methods involve algorithmic procedures rather than closed-form estimators. Press (1972) proposed a procedure that is based on the logarithm of the characteristic function of stable laws at two different pairs of arbitrary non-zero arguments along the real line, (u_1, u_2) and (u_3, u_4), and estimators are obtained by a so-called version of the method of moments. Unfortunately, Press (1972) provided no guidance on how to choose these two pairs of arguments. Furthermore, he added (Press, 1972, Footnote 1, p. 843):

> It may be that the same pair of values (u_1, u_2), which is used to estimate tail index and scale parameter, will also serve well to estimate location and skewness parameters. However, this question requires further study.

Those problems have remained unsolved ever since thus making the method not very useful in practice. Paulson et al. (1975, p. 168) found the version of the method of moments to be ineffective and very often yielding impossible (in the sense of the parameter space) results. Borak et al. (2005, Section 1.4.3) comment that Press (1972) estimates turn out to be of poor quality and do not recommend the method for more than preliminary estimation.

The aim of this thesis is to provide new insights into the Press (1972) parameter estimation procedure. More precisely, to revise its formulation, to study the statistical inference and, most importantly, with the aid of substantiative simulations give suggestions on the selection of the arguments.

Contribution of the Thesis

The main results of this thesis are a class of closed-form estimators, called the empirical cumulant function[1] (ECuF) based estimators, and a novel sample based solution, called the Argument–Selection–Rule, to the problem of ECuF estimators depending on two arbitrary different positive

[1] For clarity, the term cumulant function refers to the logarithm of characteristic function.

arguments. Another significant contribution of this thesis is providing the asymptotic normality of the ECuF estimators. In more detail, the results provided in this thesis are as follows,

(i) It is proven that the parameters of general stable laws can be expressed through the real and imaginary parts of the cumulant function at two arbitrary $u_1 > 0, u_2 > 0,$, $u_1 \neq u_2$.

(ii) The Press (1972) estimation procedure is reformulated to use only two (rather than four) different positive real numbers, called the ECuF estimators.

(iii) Based on the study of the convergence of the real part of the empirical cumulant function a sample based Argument–Selection–Rule for selecting the arguments $u_1 > 0, u_2 > 0$, $u_1 \neq u_2$, of ECuF estimators is proposed.

(iv) Via exhaustive Monte–Carlo simulations it is shown that the closed-form ECuF estimators make an considerable alternative to the well-known algorithmic methods.

(v) Based on the applications it is shown that the ECuF estimators can be successfully used in practice, and that stable laws can be suggested for modelling non-life insurance claim sizes distributions.

(vi) The asymptotic normality for the real and imaginary parts of the empirical cumulant function in general is proven.

(vii) The asymptotic normality for the ECuF estimators (throughout the interior of the parameter space) is proven.

The thesis is formulated as a monograph based on Krutto (2016) and Krutto (2018). This thesis also includes results that have not been published. For convenience of those interested in the code developed for this thesis, files with the raw working code are available at `https://github.com/akrutto/StableECuF`.

Thesis Outline

The thesis is organized as follows. Chapter 1 and Chapter 2 are preliminary and lay out the mathematical foundations for the following chapters. Short theory of the characteristic and cumulant functions is given in Chapter 1. An overview of stable laws is given in Chapter 2. In Chapter 3 the parameters of stable laws are expressed via the real and imaginary parts of their cumulant function. In Chapter 4 the empirical cumulant function (ECuF) based estimators are formulated and asymptotic normality is provided. In Chapter 5 simulations for ECuF estimators at various selections of arguments are carried out. In Chapter 6 a Argument–Selection–Rule for the selection of u_1, u_2 is proposed. In Chapter 7 the effectiveness of ECuF estimators is assessed and comparison to other estimation methods is provided. In Chapter 8 two applications to non-life insurance claim sizes data are presented. Six appendices (A–F) are included in the thesis. All citations given in this thesis are alphabetically listed in the references section.

Chapter 1

Characteristic and Cumulant Functions

Let X be a random variable on $\mathbb{R}$ with some distribution $P = P_{\boldsymbol{\theta}}$ where $\boldsymbol{\theta}$ denotes the vector of parameters. The distribution function of X, $F_X(x) = F_X(x|\boldsymbol{\theta}) = \mathbf{P}(X \leq x)$, completely describes the distribution of random variable X (and vice versa). The characteristic function, a kind of Fourier transform, also contains the complete information about the random variable under consideration. Characteristic functions are especially useful in the problems of summation of independent random variables because they transform convolutions (of distribution functions) into products (of characteristic functions).

Characteristic functions are often written in a simpler form via the (natural) logarithm. The logarithm of characteristic function is called the cumulant generating function (e.g., Lukacs (1970), Knight and Satchell (1997), Grabchak (2016)), the second characteristic function (e.g., Uchaikin and Zolotarev (1999)), the log-characteristic function (Meerschaert and Scheffler (2001)) or shortly the cumulant function (Kollo and von Rosen (2005)). In this thesis the latter is used.

In what follows, let $\Re$ and $\Im$ denote the real and imaginary part operators, respectively: given a complex number $z = x + iy$ then $\Re z = x$, $\Im z = y$, $\Re^2 z = x^2$, $\Im^2 z = y^2$, and $i^2 = -1$.

1.1 Characteristic Functions

The theory of characteristic functions on real line is amply described[1] in Lukacs (1970), Feller (1971, Chapter XV), Ushakov (1999), Uchaikin and Zolotarev (1999, Chapter 3). The following definition is based on Feller (1971, p. 499).

[1]For multivariate extension, see, e.g., Ushakov (1999, Section 1.8), Sato (1999, Section 1.2), Meerschaert and Scheffler (2001, Section 1.3).

Definition 1.1. *Let X be a real valued random variable with the distribution function $F_X(x)$. The characteristic function is a complex-valued function $\varphi : \mathbb{R} \to \mathbb{C}$,*

$$\varphi_X(u) = \int_{\mathbb{R}} \exp\{iux\}\,\mathrm{d}F_X(x), \qquad u \in \mathbb{R} \tag{1.1}$$

with

$$\Re\varphi_X(u) = \int_{\mathbb{R}} \cos(ux)\,\mathrm{d}F_X(x) \tag{1.2}$$

and

$$\Im\varphi_X(u) = \int_{\mathbb{R}} \sin(ux)\,\mathrm{d}F_X(x). \tag{1.3}$$

For an absolutely continuous X with density $f_X(x)$ the characteristic function $\varphi_X(u|\boldsymbol{\theta})$ is the ordinary Fourier transform of $f_X(x)$,

$$\varphi_X(u) = \int_{\mathbb{R}} \exp\{iux\} f_X(x)\,\mathrm{d}x, \qquad u \in \mathbb{R}. \tag{1.4}$$

Note that

$$\varphi_X(u) = \mathbf{E}\exp\{iuX\} = \mathbf{E}\cos(uX) + i\,\mathbf{E}\sin(uX),$$

where $\mathbf{E}$ is the expectation operator. The inverse Fourier transform allows to reconstruct the density from a known characteristic function (e.g., Ushakov (1999, Theorem 1.2.6, p. 6), Meerschaert and Scheffler (2001, Theorem 1.3.7, p. 15)).

Theorem 1.1 (The Fourier Inversion Theorem)**.** *Suppose a random variable X has an absolutely integrable (with respect to the Lebesgue measure) characteristic function $\varphi_X(u)$. Then the corresponding distribution function $F_X(x)$ is absolutely continuous, the density $f_X(x)$ is bounded and continuous, and*

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\{-iux\}\varphi_X(u)\,\mathrm{d}u, \qquad x \in \mathbb{R} \tag{1.5}$$

with respect to the Lebesgue measure on $\mathbb{R}$.

The following properties of characteristic functions immediately follow from their definition (e.g., Feller (1971, Section XV.I), Ushakov (1999, Section 1.1), Uchaikin and Zolotarev (1999, Section 3.1)).

Proposition 1.1. *Let $\varphi_X(u)$ be the characteristic function of an absolutely continuous distribution given by (1.1). Then,*

(a) $\varphi_X(0) = 1$; $\Re\varphi_X(0) = 1, \Im\varphi_X(0) = 0$;

(b) $|\varphi_X(u)| \leq 1$ *for all* $u \in \mathbb{R}$*;*

(c) $|\varphi_X(u)| \to 1$ *as* $|u| \to 0$*;*

(d) $|\varphi_X(u)| \to 0$ *as* $|u| \to \infty$;

(e) $\varphi_X(u)$ *is a uniformly continuous function on* $\mathbb{R}$*;*

(f) $\varphi_X(-u) = \varphi_{-X}(u) = \overline{\varphi_X(u)}$ *(the horizontal bar denotes the complex conjugate);*

(g) A random variable X *is symmetric,* $X \stackrel{d}{=} -X$*, if and only if its characteristics function is real,* $\varphi_X(u) = \Re\varphi_X(u)$, $u \in \mathbb{R}$*;*

(h) $\varphi_{a+bX}(u) = \exp\{iua\}\varphi_X(bu)$*;*

(i) If $S_n = X_1 + \cdots + X_n$ *is the sum of independent random variables then*

$$\varphi_{S_n}(u) = \varphi_{X_1}(u) \ldots \varphi_{X_n}(u).$$

As $|\varphi(u)| \leq 1$ then by the mean value theorem $|\mathbf{E}\,\varphi(u)| \leq 1$. Furthermore, $|\mathbf{E}\,\varphi(u)|^2 \leq 1$ (see, Feller (1971, p. 498)).

A sequence of distribution functions $F_{X_1}(x), F_{X_2}(x), \ldots, x \in \mathbb{R}$ converges in distribution to a distribution function $F(x)$ if $\lim_{k\to\infty} F_{X_k}(x) = F(x)$ for all continuity points x of $F(x)$ (e.g., Ushakov (1999, p. 4)). Let $\stackrel{\mathscr{D}}{\to}$ denote the convergence in distribution (or weak convergence).

The Lèvy continuity theorem (see, e.g., Uchaikin and Zolotarev (1999, Theorem 3.1.2, p.72), Ushakov (1999, Theorems 1.2.1, 1.2.2, p. 4–5), or Meerschaert and Scheffler (2001, Theorem 1.3.6, p. 15)) is of special importance regarding the problems of limit distributions.

Theorem 1.2 (The Lèvy Continuity Theorem)**.** *Let, for every* $x \in \mathbb{R}$, $F_{X_1}(x), F_{X_2}(x), \ldots$ *be a sequence of distribution functions, and, for every* $u \in \mathbb{R}$, $\varphi_{X_1}(u), \varphi_{X_2}(u), \ldots$ *be the corresponding sequence of characteristic functions.*

(a) The sequence $F_{X_1}, F_{X_2}, \ldots$ *converges in distribution to some distribution function* $F(x)$ *if and only if the sequence* $\varphi_{X_1}(u), \varphi_{X_2}(u), \ldots$ *converges at all points to some function* $\varphi(u)$ *which is continuous at zero. In this case,* $\varphi(u)$ *is the characteristic function*[2] *corresponding to* $F(x)$*.*

(b) The sequence $F_{X_1}, F_{X_2}, \ldots$ *converges in distribution to some distribution function* $F(x)$ *if and only if the sequence* $\varphi_{X_1}(u), \varphi_{X_2}(u), \ldots$ *converges uniformly on each bounded interval to some function* $\varphi(u)$*. In this case,* $\varphi(u)$ *is the characteristic function corresponding to* $F(x)$*.*

[2]Note that a sequence of characteristic functions may converge at all points to a function which is not a characteristic function (see, e.g., Ushakov (1999), Example 28 of Appendix A).

There is one-to-one correspondence between the characteristic and distribution functions (e.g., Ushakov (1999, Theorem 1.1.2), Meerschaert and Scheffler (2001, Proposition 1.3.5)).

Theorem 1.3 (Uniqueness Theorem)**.** *Two distribution functions are identical if and only if their characteristic functions are identical.*

1.2 Cumulant Functions

Cumulant function is defined as the logarithm of characteristic function.

Definition 1.2. *For a real valued random variable X on $\mathbb{R}$ the cumulant function is a complex-valued function $\psi : \mathbb{R} \to \mathbb{C}$,*

$$\psi_X(u) = \ln \varphi_X(u), \qquad u \in \mathbb{R}, \tag{1.6}$$

where $\varphi_X(u)$ is given by (1.1) and $\varphi_X(u) \neq 0$.

From elementary complex analysis (e.g., Brown and Churchill (1996), p. 75-76)

$$\psi_X(u) = \ln|\varphi_X(u)| + i \arg \varphi_X(u), \tag{1.7}$$

where $\arg \varphi_X(u)$ is the argument of $\varphi_X(u)$. Recall that given a complex number $z = x + iy$, the argument of z, $\arg z$, has an infinite number of possible values, including negative ones, that differ by integer multiples of 2π. More precisely,

$$\arg z = \operatorname{Arg} z + 2k\pi \qquad k = 0, \pm 1, \pm 2, \dots, \tag{1.8}$$

where $\operatorname{Arg} z$, called the principal value of $\arg z$, is a unique value such that $\operatorname{Arg} \varphi_X(u) \in (-\pi, \pi]$. The principal value $\operatorname{Arg} z$ can be evaluated in terms of standard arctan function (see, e.g., Kasana (2005), p. 14), and in a variety of computer programming languages (e.g., `R` (R Core Team (2018)), it is provided under the function name of atan2, i.e., $\operatorname{Arg} z \equiv \operatorname{atan2}(\Im z, \Re z)$,

$$\operatorname{Arg} z = \operatorname{atan2}(\Im z, \Re z) = \begin{cases} \arctan(\Im z/\Re z) & \text{if } \Re z > 0, \\ \arctan(\Im z/\Re z) + \pi & \text{if } \Re z < 0 \text{ and } \Im z \geq 0, \\ \arctan(\Im z/\Re z) - \pi & \text{if } \Re z < 0 \text{ and } \Im z < 0, \\ +\pi/2 & \text{if } \Re z = 0 \text{ and } \Im z > 0, \\ -\pi/2 & \text{if } \Re z = 0 \text{ and } \Im z < 0, \\ \text{undefined} & \text{if } \Re z = 0 \text{ and } \Im z = 0. \end{cases} \tag{1.9}$$

Hereby, the logarithm of z can be written

$$\ln z = \ln|z| + i\,(\operatorname{atan2}(\Im z, \Re z) + 2k\pi) \qquad k = 0, \pm 1, \pm 2, \dots \tag{1.10}$$

and the principal value[3] of z is the value obtained from (1.10) when $k = 0$.

In what follows, the cumulant function $\psi_X(u)$ is assumed its principal value (with the same notation),

$$\psi_X(u) \equiv \ln|\varphi_X(u)| + i\operatorname{atan2}(\Im\varphi_X(u), \Re\varphi_X(u)), \tag{1.11}$$

with $\varphi_X(u)$ given by (1.1) and $\varphi_X(u) \neq 0$. The following properties of cumulant functions immediately follow from their definition and Proposition 1.1.

Proposition 1.2. *Let $\psi_X(u) = \psi_X(u)$ be a cumulant function of an absolutely continuous distribution given by (1.11). Then,*

(a) $\psi_X(0) = 0$;

(b) $\Re\psi_X(u) \leq 0$ *for all* u;

(c) $|\psi_X(u)| \to \infty$ *as* $|u| \to \infty$;

(d) $\psi_{a+bX}(u) = \{iua\} + \psi_X(bu)$ *for any* $b \neq 0$, $a \in \mathbb{R}$;

(e) If $S_n = X_1 + \cdots + X_n$ *is the sum of independent random variables then*

$$\psi_{S_n}(u) = \psi_{X_1}(u) + \cdots + \psi_{X_n}(u).$$

Characteristic function can be expressed through real and imaginary parts of cumulant function,

$$\varphi_X(u) = \exp\{\Re\psi_X(u)\}\exp\{i\Im\psi_X(u)\},$$

and then

$$\Re\varphi_X(u) = \exp\{\Re\psi_X(u)\}\cos\Im\psi_X(u), \tag{1.12}$$

$$\Im\varphi_X(u) = \exp\{\Re\psi_X(u)\}\sin\Im\psi_X(u). \tag{1.13}$$

Clearly, $Re\psi_X(u) = \ln|\varphi_X(u)|$ and $\Im\psi_X(u) = \operatorname{atan2}(\Im\varphi_X(u), \Re\varphi_X(u))$.

[3] In a variety of computer programming languages (e.g., R (R Core Team (2018)), the logarithm of complex numbers is by default its principal value.

1.3 Empirical Characteristic Function

For a random sample formed by independent and identically distributed (i.i.d.) random variables on $\mathbb{R}$ the idea of empirical or sample characteristic function was originally initiated by Parzen (1962) (see, also Press (1972)).

Definition 1.3. *Let $Y_1, \ldots, Y_n$ be i.i.d. random variables on $\mathbb{R}$. The empirical characteristic function associated with the random sample $\{Y_1, \ldots, Y_n\}$, denoted by $\varphi_n(u) = \varphi_{\{Y_1,\ldots,Y_n\}}(u)$, is a complex valued function, $\varphi_n : \mathbb{R} \to \mathbb{C}$,*

$$\varphi_n(u) = \int_{\mathbb{R}} \exp\{iuy\} \, \mathrm{d}F_n(y) = \frac{1}{n} \sum_{j=1}^{n} \exp\{iuY_j\} \qquad u \in \mathbb{R}, \tag{1.14}$$

with

$$\Re\varphi_n(u) = \frac{1}{n} \sum_{j=1}^{n} \cos(uY_j), \tag{1.15}$$

and

$$\Im\varphi_n(u) = \frac{1}{n} \sum_{j=1}^{n} \sin(uY_j), \tag{1.16}$$

where $F_n(x)$ is the empirical distribution function,

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{Y_i \leq y\}}, \qquad x \in \mathbb{R}. \tag{1.17}$$

Ushakov (1999, Section 3.1, p. 160) explains, that empirical characteristic function is a random function whose all realizations are characteristic functions of discrete (concentrated at most n points) distributions. A realization of empirical characteristic function at the counterpart $y_1, \ldots, y_n$ of the random sample $Y_1, \ldots, Y_n$ is denoted by $\hat{\varphi}_n(u)$,

$$\hat{\varphi}_n(u) = \frac{1}{n} \sum_{j=1}^{n} \exp\{iuy_j\}. \tag{1.18}$$

The following properties of empirical characteristic function immediately follow from its definition (see, e.g., Ushakov (1999, Section 3.1)).

Proposition 1.3. *Let $\varphi_n(u)$ be the empirical characteristic function given by (1.14). Then,*

- $\varphi_n(0) = 1$*;*
- $|\varphi_n(u)| \leq 1$*;*

- $\varphi_n(-u) = \overline{\varphi_n(u)}$ *(the horizontal bar denotes the complex conjugate);*
- $\limsup_{|u|\to\infty} |\varphi_n(u)| = 1$*;*
- $\varphi_n(u)$ *has derivatives of all orders.*

Let $Y_1, \ldots, Y_n$ be the i.i.d. copies of a random variable X. Then,

$$\mathbf{E}\,\varphi_n(u) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\,e^{iuY_j} = \mathbf{E}\exp\{iuX\} = \varphi_X(u), \tag{1.19}$$

which implies that empirical characteristic function is an unbiased estimator of the corresponding characteristic function. Moreover (e.g., Ushakov (1999, p. 161)), for every fixed $u_1, u_2 \in \mathbb{R}$,

$$\begin{aligned} \mathrm{Cov}(\varphi_n(u_1), \varphi_n(u_2)) &= \mathbf{E}[(\varphi_n(u_1) - \varphi_X(u_1|))\overline{(\varphi_n(u_2) - \varphi_X(u_2))}] \\ &= \frac{1}{n}[\varphi_X(u_1 - u_2) - \varphi_X(u_1)\overline{\varphi_X(u_2)}], \end{aligned} \tag{1.20}$$

and in particular,

$$\mathbf{E}\,|\varphi_n(u) - \varphi_X(u)|^2 = \frac{1}{n}[1 - |\varphi_X(u)|^2],$$

which implies that the empirical characteristic function $\varphi_n(u)$ converges in mean square to $\varphi_X(u)$ at every fixed point $u \in \mathbb{R}$ as $n \to \infty$,

$$\lim_{n\to\infty} \mathbf{E}\,|\varphi_n(u) - \varphi_X(u)|^2 = 0. \tag{1.21}$$

For any fixed $u \in \mathbb{R}$, $\varphi_n(u)$ is an average of i.i.d. random variables with $\mathbf{E}\,\varphi_n(u) = \varphi(u)$ and finite variance. Therefore it follows by the strong law of large numbers (e.g., DasGupta (2008, Theorem 3.1, p. 35)) that at every fixed $u \in \mathbb{R}$ the empirical characteristic function $\varphi_n(u)$ converges almost surely to $\varphi_X(u)$,

$$\mathbf{P}\left(\lim_{n\to\infty} \varphi_n(u) = \varphi_X(u)\right) = 1, \tag{1.22}$$

and $\varphi_n(u)$ is almost surely consistent estimator of $\varphi_X(u)$. The Glivenko-Cantelli theorem implies that $\varphi_n(u)$ is almost surely consistent uniformly on each bounded interval of $\mathbb{R}$ (e.g., Feuerverger and Mureika (1977, Theorem 2.1)),

$$\mathbf{P}\left(\lim_{n\to\infty} \sup_{|u|\le U} |\varphi_n(u) - \varphi_X(u)| = 0\right) = 1. \tag{1.23}$$

for any fixed positive $U < \infty$. For $U \to \infty$ Csörgo and Totik (1983) pointed out that (1.23) is true if and only if the distribution of X is discrete. When the distribution of X is continuous then empirical characteristic function is not almost surely consistent uniformly on the whole real line. In fact (e.g., Ushakov (1999), p. 16), for every n,

$$\mathbf{P}\left(\limsup_{|u|\to\infty}|\varphi_n(u)-\varphi_X(u)|>0\right)=1.$$

Csörgo (1981, Theorem 1) showed that for some sequence $\{U_n\}$ of finite positive numbers such that

$$\lim_{n\to\infty}U_n\sqrt{\frac{\ln\ln n}{n}}=0$$

then

$$\mathbf{P}\left(\lim_{n\to\infty}\sup_{|u|\le U_n}|\varphi_n(u)-\varphi_X(u)|=0\right)=1 \tag{1.24}$$

for any characteristic function $\varphi_X(u)$. Csörgo and Totik (1983, Theorem 1) showed that for some sequence $\{U_n\}$ of finite positive numbers converging to infinity such that

$$\lim_{n\to\infty}\frac{\ln U_n}{n}=0$$

then

$$\mathbf{P}\left(\lim_{n\to\infty}\sup_{|u|\le U_n}|\varphi_n(u)-\varphi_X(u)|=0\right)=1 \tag{1.25}$$

for any characteristic function $\varphi_X(u)$. The problem of deriving the rates of uniform convergence for the empirical characteristic function $\varphi_n(u)$ is addressed in Csörgo (1985).

For every fixed $u_1,u_2\in\mathbb{R}$ the product–to–sum identities of cosine and sine functions yield (e.g., Heathcote (1977), Ushakov (1999, p. 162))

$$\begin{aligned}
2n\,\mathrm{Cov}(\Re\varphi_n(u_1),\Re\varphi_n(u_2)) &= \Re\varphi_X(u_1-u_2)\\
&\quad+\Re\varphi_X(u_1+u_2)-2\Re\varphi_X(u_1)\Re\varphi_X(u_2), && (1.26)\\
2n\,\mathrm{Cov}(\Im\varphi_n(u_1),\Im\varphi_n(u_2)) &= \Re\varphi_X(u_1-u_2)\\
&\quad-\Re\varphi_X(u_1+u_2)-2\Im\varphi_X(u_1)\Im\varphi_X(u_2), && (1.27)\\
2n\,\mathrm{Cov}(\Re\varphi_n(u_1),\Im\varphi_n(u_2)) &= \Im\varphi_X(u_1+u_2)\\
&\quad-\Im\varphi_X(u_1-u_2)-2\Re\varphi_X(u_1)\Im\varphi_X(u_2). && (1.28)
\end{aligned}$$

Let

$$Z_n(u)=\sqrt{n}(\varphi_n(u)-\varphi_X(u)) \tag{1.29}$$

be a random complex process in $u\in\mathbb{R}$. It is easy to see, that $\mathbf{E}Z_n(u)=0$, $\mathbf{E}[Z_n(u_1)\overline{Z_n(u_2)}]=$

$\varphi_X(u_1-u_2)-\varphi_X(u_1)\overline{\varphi(u_2)}$ and $Z_n(u)$ has a covariance matrix with the structure following from (1.26) – (1.28). Define $Z(u)$ a zero mean complex normal process satisfying $Z(u)=Z(-u)$ and having the same covariance structure as $Z_n(u)$. By means of multivariate central limit theorem (e.g., DasGupta (2008, Theorem 1.17, p. 9)) it follows that for every finite collection of points the process $Z_n(u)$ converges in distribution (weakly) to $Z(u)$ as $n\to\infty$.

Theorem 1.4. *Let $Z_n(u)$, $Z(u)$ be as defined above. Then the process $Z_n(u)$ converges in distribution to $Z(u)$ for every finite collection of points, $u_1,u_2,\dots,u_K\in\mathbb{R}$.*

The weak convergence in finite interval is studied in, e.g., Kent (1975), Feuerverger and Mureika (1977, Theorem 3.1), Csörgo (1981, Theorem 2), Marcus (1981, Theorem 1). The efficiency of empirical characteristic function based parameter estimation procedures is discussed, e.g., in Feuerverger and McDunnough (1981) and Yu (2004).

1.4 Empirical Cumulant Function

For the logarithm of empirical characteristic function the term empirical cumulant (generating) function is used (e.g., Knight and Satchell (1997), Krutto (2016)).

Definition 1.4. *Let $Y_1,\dots,Y_n$ be i.i.d. random variables on $\mathbb{R}$. The empirical cumulant function associated with the random sample $\{Y_1,\dots,Y_n\}$, denoted by $\psi_n(u)=\varphi_{\{Y_1,\dots,Y_n\}}(u)$, is a complex valued function, $\psi_n:\mathbb{R}\to\mathbb{C}$,*

$$\psi_n(u)=\ln\varphi_n(u)=\ln|\varphi_n(u)|+i\arg\varphi_n(u)\qquad u\in\mathbb{R}, \tag{1.30}$$

where $\varphi_n(u)$ is given by (1.14).

In what follows, the cumulant function $\psi_n(u)$ is assumed as its principal value (with the same notation),

$$\psi_n(u)\equiv\ln|\varphi_n(u)|+i\,\mathrm{Arg}(\varphi_n(u)), \tag{1.31}$$

where

$$\Re\psi_n(u)=\ln|\varphi_n(u)|=\ln\sqrt{\Re^2\varphi_n(u)+\Im^2\varphi_n(u)} \tag{1.32}$$

$$\Im\psi_n(u)\equiv\mathrm{Arg}\,\varphi_n(u)=\mathrm{atan2}\,(\Im\varphi_n(u),\Re\varphi_n(u)) \tag{1.33}$$

The realization of empirical cumulant function at the counterpart $y_1,\dots,y_n$ of the random sample $Y_1,\dots,Y_n$ is denoted by $\hat{\psi}_n(u)$,

$$\hat{\psi}_n(u)=\ln\frac{1}{n}\sum_{j=1}^{n}\exp\{iuy_j\}. \tag{1.34}$$

Based on (1.19) empirical characteristic function $\varphi_n(u)$ is an unbiased estimator of $\varphi_X(u)$. However, for $\psi_n(u) = \ln \varphi_n(u)$ the relation between $\mathbf{E}\,\psi_n(u)$ and $\mathbf{E}\,\psi_X(u)$ is not as elementary yielding that $\psi_n(u)$ may be a biased estimator of $\psi_X(u)$. On the asymptotic properties of empirical cumulant function not much has been published. To the best of our knowledge, the only study is by Knight and Satchell (1997) in the framework of the generalized nonlinear least square parameter estimation procedure. In Section 4.2 the asymptotic normality of the real and imaginary parts of the empirical cumulant function at every fixed $u \in \mathbb{R}$ is provided.

Chapter 2

Overview of Stable Laws

Stable distributions form a sub-class of infinitely divisible distributions that are the only possible limiting distributions for normalized sums of independent identically distributed (i.i.d.) random variables.

On the univariate limit theorems for sums we refer, for example, to Gnedenko and Kolmogorov (1954), Feller (1971, Chapter VIII), Rao and Swift (2006, Chapter 5) and on the multivariate case to Meerschaert and Scheffler (2001).

The class of univariate infinitely divisible distributions is treated in detail by Feller (1971, Chapter VI, Chapter XVII), Steutel and van Harn (2004), and the multivariate case by Meerschaert and Scheffler (2001, Chapter 3), Sato (1999).

The class of stable laws is amply described in Lukacs (1970, Chapter 5), Gnedenko and Kolmogorov (1954, Chapter 6), Feller (1971, Chapter VI.1, XVII.5), Zolotarev (1986) (i.e., the translation of Zolotarev (1983)), Samorodnitsky and Taqqu (1994), Uchaikin and Zolotarev (1999), Meerschaert and Scheffler (2001, Section 7), Rao and Swift (2006, Section 8.4) and Nolan (2018c).

A comprehensive bibliography on stable distributions, processes and related topics is given by Nolan (2018a).

2.1 Formulation and Some Properties

The class of stable laws is formulated by Lévy (1925) as an asymptotic statement on the limiting distribution for sums of random variables.

Theorem 2.1 (Lévy via Uchaikin and Zolotarev (1999), Theorem 1.12.4, p. 33)**.** *Assume $X_1, X_2, \dots, X_n$ are i.i.d. random variables on $\mathbb{R}$ and denote $S_n = X_1 + \cdots + X_n$. Let for every $n \geq 2$ there exist $b_n > 0$*

and $a_n \in \mathbb{R}$ such that

$$\mathbf{P}\left(\frac{S_n - a_n}{b_n} \le x\right) \xrightarrow{\mathscr{D}} G(x),\ n \to \infty,$$

for some function $G(x)$ which is not degenerate. Then $G(x)$ is stable law.

Feller (1971, Definition VI.1.2, p. 172) restates the fact that all stable distributions and no others occur as limits of sums of i.i.d. random variables: the common distribution of the independent random variables $X_1, X_2, \ldots, X_n$ belongs to the domain of attractions of a stable distribution G if there exist constants $b_n > 0$, $a_n \in \mathbb{R}$ such that the distribution of $(S_n - a_n)/b_n$ tends to G. In other words, a distribution G possesses a domain of attraction if and only if it is stable.

Let $\stackrel{\mathrm{d}}{=}$ denote equality in distribution.

Another formulation of stable laws, also called the definition of stability, is given by Feller (1971, Definition VI.1.1, p. 170).

Definition 2.1. *Let $X, X_1, \ldots$ denote independent random variables on $\mathbb{R}$ with a common distribution G, and $S_n = X_1 + \cdots + X_n$, $n \ge 2$. The random variable X (or distribution G) is stable if for each n there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ such that*

$$S_n \stackrel{\mathrm{d}}{=} c_n X + d_n, \tag{2.1}$$

and X is not concentrated[1] at one point. The distribution G (or the random variable X) is strictly stable if

$$S_n \stackrel{\mathrm{d}}{=} c_n X. \tag{2.2}$$

The scaling constants are found to be of the form $c_n = n^{1/\alpha}$ with $0 < \alpha \le 2$ (Feller (1971), Theorem VI.1.1, p. 170) and α is called the characteristic exponent. The special case of $\alpha = 2$ corresponds to the normal distribution. Like the normal law, all stable distributions remain stable under linear transformations.

Another equivalent definition of stable laws (or stability) can be found in Feller (1971, Problem VI.13.1 p. 215), Samorodnitsky and Taqqu (1994, Definition 1.1.1, p. 2).

Definition 2.2. *A random variable X on $\mathbb{R}$ is said to have a stable distribution if for any positive numbers A and B, there is a positive number C and a real number D such that*

$$AX_1 + BX_2 \stackrel{\mathrm{d}}{=} CX + D, \tag{2.3}$$

where X_1 and X_2 are independent copies of X, and X is not concentrated at one point.

[1] A random variable X concentrated at one point is always stable but this degenerate case is of no interest. In the following it is assumed that X is non-degenerate.

For any stable random variable X there is a number $\alpha \in (0,2]$ such that (Samorodnitsky and Taqqu (1994), Theorem 1.1.2, p. 3)

$$C^{\alpha} = A^{\alpha} + B^{\alpha}. \tag{2.4}$$

For example, for independent normal random variables $X, X_1, X_2 \sim N(\mu, \sigma^2)$ it holds $AX_1 + BX_2 \sim N((A+B)\mu, (A^2+B^2)\sigma^2)$. From the equalities of means $D = (A+B-C)\mu$ and of variances $C^2 = A^2 + B^2$ it implies that (2.3) and (2.4) hold.

Every stable distribution is a continuous distribution (for proof, see Feller (1971, Problem VI.13.2, p. 215)) and completely described by a closed form characteristic function. The characteristic function can be derived from the general canonical form of that of the infinitely divisible distributions (see Section 2.2) or as the limit of normalized sums of i.i.d. variables as the number of variables $n \to \infty$ (see Section 2.4).

Parameters

A univariate stable distribution is completely described by four real-valued parameters:

- The characteristic exponent $\alpha \in (0,2]$, also called the index of stability or tail index, describes the rate of decay of the tails of stable distribution. The smaller the α, the slower is the decay and the heavier are the tails. The case of $\alpha = 2$ corresponds to the light-tailed normal distribution.
- The skewness parameter $\beta \in [-1,1]$ illustrates the degree of asymmetry[2] of a stable distribution. For $\beta = 0$ the stable random variable is symmetric and for $\beta = \pm 1$ it is maximally asymmetric.
- The scale parameter $\gamma > 0$ and the location parameter $\delta \in \mathbb{R}$.

The formulation of scale and location parameters may differ due to various representations of stable laws (see, Section 2.3). For any admissible parameter quadruple, $(\alpha, \beta, \gamma, \delta)$, it holds (e.g., Uchaikin and Zolotarev (1999, Property 3.7.(2), p. 99), Nolan (2018b, Proposition 1.11, p. 12))

$$X(\alpha, -\beta, \gamma, -\delta) \stackrel{\mathrm{d}}{=} -X(\alpha, \beta, \gamma, \delta), \tag{2.5}$$

stating, that all stable laws have the reflection property.

A stable random vector on $\mathbb{R}^p$, $p \geq 2$ requires a more sophisticated approach: it is described by the characteristic index $\alpha \in (0,2]$, the shift vector $\boldsymbol{\delta} \in \mathbb{R}^p$ and some distribution concentrated on the unit sphere, called the spectral measure (see, e.g., Samorodnitsky and Taqqu (1994, Theorem 2.3.1, p.

[2]The value of β means little when $\alpha \to 2$ as for $\alpha \to 2$ stable distributions tend to symmetry regardless of the value of β.

65), Sato (1999, Theorem 14.3(ii), p. 77, Theorem 14.10, p. 84), Meerschaert and Scheffler (2001, Theorem 7.3.16, p. 272)).

Moments

A feature of stable laws is that not all moments exist. For example, taking the variance from both sides of (2.1), where $c_n = n^{1/\alpha}$, gives

$$n\,\mathrm{Var}X = n^{2/\alpha}\,\mathrm{Var}X. \tag{2.6}$$

For non-degenerate distributions with finite variance the index α in (2.6) must be equal to 2. If $\alpha \neq 2$, then (2.6) can be formally satisfied only for $\mathrm{Var}(X) = \infty$. Indeed, all stable distributions with $0 < \alpha < 2$ have infinite variance. More general, if $\alpha \in (1,2)$ then second and higher order moments are infinite while if $\alpha \in (0,1]$ then the first and higher order moments are infinite (e.g., Meerschaert and Scheffler (2001, Remark 7.3.21, p. 276), Nolan (2018b)). The fractional lower order moments (e.g., Nikias and Shao (1995, Theorem 3 p. 22), Nolan (2018c)) and negative order moments (e.g., Nikias and Shao (1995, Proposition 2 p. 34)) exist. On logarithmic moments see, e.g., Zolotarev (1986, Section 3.6.).

Densities

All stable distributions have a continuous infinitely differentiable density function (see, e.g., Zolotarev (1986, Chapter 2)). However, it is complicated to calculate the densities by the direct application of the Fourier inversion theorem and in general, there is no closed analytic form for the distribution and density functions of stable laws. Nevertheless, the densities of stable laws are well studied (see, e.g., Zolotarev (1986), Zolotarev (1995), Nolan (1997, 1999), Uchaikin and Zolotarev (1999, Chapter 4), and references therein). Numerous graphs and tables of densities can be found in Samorodnitsky and Taqqu (1994), Uchaikin and Zolotarev (1999)), Nolan (2018b). Three members of the family of stable distributions, the normal, Cauchy and Lévy distributions (and the reflection of the Lévy distribution), have densities expressed by elementary functions.

2.2 Stable Laws as Infinitely Divisible Distributions

A distribution F is infinitely divisible if for every n there exists a distribution F_n such that F is the n-fold convolution of F_n (e.g., Feller (1971, Definition VI.3.1, p. 176), Meerschaert and Scheffler (2001, Definition 3.1.1, p. 37), Sato (1999, Definition 7.1, p. 31)). A characteristic function is infinitely divisible if and only if, for each positive integer n, a nth root of the characteristic function can be chosen in such way that it is the characteristic function of some probability distribution (e.g., Feller (1971, p. 554), Sato (1999, p. 31)). Stable distributions are infinitely divisible distributions and distinguished by the fact that F_n differs from F only by location (shift) parameter: an infinitely divisible characteristic function $\varphi(u)$, $u \in \mathbb{R}$ is stable, if, for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}$, such that $[\varphi(u)]^a = \varphi(bu)\exp\{icu\}$ (e.g., Sato (1999, Definition 13.1, p. 69)).

The fundamental contributions of infinitely divisible distributions were developed[3] by Kolmogorov, Lévy, and Khintchine in the 1930's (e.g., Khintchine and Lèvy (1936), Khintchine (1937)). The representation of the general canonical form of the characteristic function of all infinitely divisible distributions on $\mathbb{R}^p$ can be found in Sato (1999, Theorem 8.1, p. 37). It is pointed out (Sato (1999, Remark 8.4, p. 38)), that there are many ways of formulating the canonical form of the characteristic function of an infinitely divisible distribution (see also Feller (1971, p. 564-565), DasGupta (2008, p. 73-74), Pitman and Pitman (2016)). However, the Lévy–Khintchine form (e.g., Uchaikin and Zolotarev (1999, Equation (3.5.14), p. 89), DasGupta (2008, Theorem 5.13)) has become a standard.
Theorem 2.2 (Lévy–Khintchine form)**.** *The Lévy–Khintchine representation of the characteristic function of infinitely divisible distribution is of form*

$$\varphi(u) = \exp\left\{ iua - bu^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dH(x) \right\} \tag{2.7}$$

where $a \in \mathbb{R}$, $b \geq 0$ *are real numbers and* $H(x)$ *is called the spectral function (measure).*

Based on Lévy–Khintchine representation given by (2.7) an explicit form of the characteristic function of stable laws, given by (2.3), has been derived in Gnedenko and Kolmogorov (1954, Equation (7.34.1), p. 164)[4], Uchaikin and Zolotarev (1999, Theorem 3.5.1, p. 89-92), Sato (1999, Theorem 14.15, p. 86), Meerschaert and Scheffler (2001, Theorem 7.3.16., p. 272).

2.3 Various Representations of Stable Laws

When considering stable laws one should be careful as many different definitions of (the characteristic function of) stable distributions can be found in books and papers. For example, in the parameter estimation procedures various forms of stable laws have been used: Press (1972), Paulson et al. (1975) estimated the parameters of stable laws in form (2.12), Koutrouvelis (1980, 1981), Krutto (2016) in form (2.14), Kogon and Williams (1998) an adaptation of form (2.18) and Krutto (2018) used stable laws in form (2.19). Nevertheless, all representations of stable laws are uniquely related.

[3]For an overview on history we refer to Mainardi and Rogosin (2006) with English translation of the Khintchine (1937) paper.

[4]In Gnedenko and Kolmogorov (1954, Equation (7.34.1), p. 164) there is a misprint concerning the sign of the parameter β when $\alpha = 1$. However, it has been corrected by various authors later.

Lévy–Khintchine Representations

The explicit form of the characteristic function of univariate stable laws following Lévy–Khintchine representation of canonic form of characteristic function of infinitely divisible distributions, given by (2.7), is restated by the following theorem (see, e.g., Uchaikin and Zolotarev (1999, Theorem 3.5.1, p. 89-92), Sato (1999, Theorem 14.15, p. 86)).

Theorem 2.3. *[Lévy–Khintchine form] Let X be a stable random variable on $\mathbb{R}$ represented in Lévy–Khintchine form given by (2.7). Then its characteristic function $\varphi_X(u) = \varphi_X(u|\alpha,\beta,\gamma,\delta)$ can equivalently be represented as*

$$\varphi_X(u) = \exp\{\Re\psi_X(u) + i\Im\psi_X(u)\} \tag{2.8}$$

where

$$\Re\psi_X(u) = -\gamma|u|^{\alpha}, \tag{2.9}$$

$$\Im\psi_X(u) = \begin{cases} -\gamma|u|^{\alpha}[\beta(\operatorname{sign} u)\tan\frac{\pi\alpha}{2}] + \delta u & \textit{for } \alpha \neq 1 \\ -\gamma|u|[\beta\frac{2}{\pi}(\operatorname{sign} u)\ln|u|] + \delta u & \textit{for } \alpha = 1 \end{cases} \tag{2.10}$$

with $u \in \mathbb{R}$, $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma > 0$ and $\delta \in \mathbb{R}$.

In this thesis stable distributions in Lévy–Khintchine representation are denoted by $S(\alpha,\beta,\gamma,\delta;KL)$. Normal distribution $N(\mu,\sigma^2)$ is $S(2,\cdot,\sigma^2/2,\mu;\mathrm{KL})$ while the value of β has no influence (as for $\alpha = 2$ it follows $\tan\frac{\pi\alpha}{2} = 0$).

Definition 2.3. *Let in Theorem 2.3 the scale parameter be fixed as $\gamma = 1$ and the location parameter as $\delta = 0$. The corresponding stable distributions are called standard stable distributions of Lévy–Khintchine representation and denoted by $S(\alpha,\beta;KL)$,*

$$S(\alpha,\beta;KL) \equiv S(\alpha,\beta,\gamma = 1,\delta = 0;KL).$$

From Theorem 2.3 the characteristic function of a standard stable random variable Z in Lévy–Khintchine representation, $Z \sim S(\alpha,\beta;\mathrm{KL})$, is of form

$$\varphi_Z(u) = \begin{cases} \exp\{-|u|^{\alpha}[1 - i\beta(\operatorname{sign} u)\tan\frac{\pi\alpha}{2}]\} & \text{for } \alpha \neq 1 \\ \exp\{-|u|[1 + i\beta\frac{2}{\pi}(\operatorname{sign} u)\ln|u|]\} & \text{for } \alpha = 1 \end{cases} \tag{2.11}$$

with $u \in \mathbb{R}$, $\alpha \in (0,2]$, $\beta \in [-1,1]$.

A relation between general stable random variable $X \sim S(\alpha,\beta,\gamma,\delta;\mathrm{KL})$ and standard stable random variable $Z \sim S(\alpha,\beta;\mathrm{KL})$ is given as follows.

Proposition 2.1. *Let $Z \sim S(\alpha,\beta;KL)$ be a standard stable distribution in Lévy–Khintchine form with the characteristic function given by (2.11). For $\gamma > 0$ and $\delta \in \mathbb{R}$ a random variable*

$$X \overset{d}{=} \begin{cases} \gamma^{1/\alpha} Z + \delta & \text{for } \alpha \neq 1 \\ \gamma(Z + \beta \frac{2}{\pi} \ln \gamma) + \delta & \text{for } \alpha = 1 \end{cases} \tag{2.12}$$

is a general stable random variable in Lévy–Khintchine form, $X \sim S(\alpha,\beta,\gamma,\delta;KL)$, with characteristic function given by (2.8).

Proof. Note that $\operatorname{sign}(\gamma^{1/\alpha} u) = \operatorname{sign} u$. Based on Proposition 1.1(h), and the characteristic function of $Z = Z(\alpha,\beta;\mathrm{KL})$, given by (2.11), it holds

$$\begin{aligned}
\varphi_X(u) &= \begin{cases} \varphi_{\gamma^{1/\alpha} Z + \delta}(u) & \text{for } \alpha \neq 1 \\ \varphi_{\gamma Z + \delta + \gamma\beta \frac{2}{\pi} \ln \lambda}(u) & \text{for } \alpha = 1 \end{cases} \\
&= \begin{cases} \exp\{i\delta u\} \varphi_Z(\gamma^{1/\alpha} u) & \text{for } \alpha \neq 1 \\ \exp\{i(\delta + \gamma\beta \frac{2}{\pi} \ln \lambda) u\} \varphi_Z(\gamma u) & \text{for } \alpha = 1 \end{cases} \\
&= \begin{cases} \exp\{-\gamma |u|^\alpha [1 - i\beta (\operatorname{sign} u) \tan \frac{\pi\alpha}{2}] + i\delta u\} & \text{for } \alpha \neq 1 \\ \exp\{-\gamma |u| [1 + i\beta \frac{2}{\pi} (\operatorname{sign} u) \ln |u|] + i\delta u\} & \text{for } \alpha = 1 \end{cases}
\end{aligned}$$

with $u \in \mathbb{R}$, $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma > 0$ and $\delta \in \mathbb{R}$, which is the same as the characteristic function of $X \sim S(\alpha,\beta,\gamma,\delta;\mathrm{KL})$ given by (2.8). □

A slightly different representation of stable laws has been introduced in Zolotarev (1986, Equation (A), p. 9), called the form A, and here restated through $Z \sim S(\alpha,\beta;\mathrm{KL})$.

Corollary 2.1. *Let $Z \sim S(\alpha,\beta;KL)$ be a standard stable distribution in Lévy–Khintchine form with characteristic function given by (2.11). Then, for $\gamma > 0$ and $\delta \in \mathbb{R}$ a random variable*

$$X \overset{d}{=} \begin{cases} \gamma^{1/\alpha} Z + \gamma\delta & \text{for } \alpha \neq 1 \\ \gamma(Z + \beta \frac{2}{\pi} \ln \gamma) + \gamma\delta & \text{for } \alpha = 1 \end{cases} \tag{2.13}$$

is a stable random variable in Zolotarev form A.

In (2.12) and (2.13) the parameter γ is not straightforwardly the scale (and in (2.13) the parameter δ is not straightforwardly the shift) of $Z \sim S(\alpha,\beta;\mathrm{KL})$. Therefore, in literature, e.g., in Samorodnitsky and Taqqu (1994, Definition 1.1.6, p. 5), Nikias and Shao (1995, Definition 1, p. 13), Kotz and Nadarajah (2000, p. 55), Meerschaert and Scheffler (2001, Theorem 7.3.16, p. 272), Sato (1999, Theorem 14.15, p. 86), Embrechts et al. (2013, Theorem 2.2.3, p. 71), a modification is introduced, called a 1–parametrization in Nolan (2018b, Definition 1.8).

Corollary 2.2. *Let $Z \sim S(\alpha,\beta;KL)$ be a standard stable distribution in Lévy–Khintchine form with characteristic function given by (2.11). Then, for $\gamma > 0$ and $\delta \in \mathbb{R}$ a random variable*

$$X \stackrel{d}{=} \begin{cases} \gamma Z + \delta & \text{for } \alpha \neq 1 \\ \gamma(Z + \beta \frac{2}{\pi} \ln \gamma) + \delta & \text{for } \alpha = 1 \end{cases} \tag{2.14}$$

is a stable random variable in 1–parametrization.

In the following definition the characteristic function of form (2.14), see, e.g., Samorodnitsky and Taqqu (1994, Definition 1.1.6, p. 5)), is restated in terms of real and imaginary parts of the cumulant function.

Definition 2.4. *Stable random variable given by (2.14) has characteristic function*

$$\varphi_X(u) = \varphi_X(u|\alpha,\beta,\gamma,\delta)$$

of the form

$$\varphi_X(u) = \exp\{\Re\psi_X(u) + i\Im\psi_X(u)\} \tag{2.15}$$

where

$$\Re\psi_X(u) = -\gamma^\alpha |u|^\alpha, \tag{2.16}$$

$$\Im\psi_X(u) = \begin{cases} -\gamma^\alpha |u|^\alpha [\beta(\operatorname{sign} u)\tan\frac{\pi\alpha}{2}] + \delta u, & \text{for } \alpha \neq 1 \\ -\gamma|u|[\beta\frac{2}{\pi}(\operatorname{sign} u)\ln|u|] + \delta u, & \text{for } \alpha = 1 \end{cases} \tag{2.17}$$

with $u \in \mathbb{R}$, $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma > 0$, $\delta \in \mathbb{R}$.

Stable distributions in 1–parametrization, given by (2.14), are denoted by $S(\alpha,\beta,\gamma,\delta;1)$.

Let in equation (2.14), or equivalently in (2.15), the scale parameter be one, $\gamma = 1$, and the location parameter zero, $\delta = 0$. Then the corresponding stable laws are called *standard stable laws in 1–parametrization*, denoted by $S(\alpha,\beta;1)$,

$$S(\alpha,\beta;1) \equiv S(\alpha,\beta,\gamma=1,\delta=0;1).$$

For various values of α the graphs of the characteristic function of $S(\alpha,\beta=1;\mathrm{KL}) = S(\alpha,1;1)$, given by (2.11), are presented in Figure A.1 of Appendix A, its absolute value, real and imaginary parts in Figure B.1 of Appendix B, while the corresponding absolute value, real and imaginary parts of cumulant function in Figure C.1 of Appendix C.

Other Representations

In addition to the Lévy–Khintchine representation based forms many others have been introduced in Zolotarev (1986) (see, also Uchaikin and Zolotarev (1999, Section 3.6)) and Nolan (1998b), Nolan (2018b, Section 1.3).

Zolotarev (1983, p. 21-22) points out that since in (2.13) $\tan \pi\alpha/2 \to \infty$ as $\alpha \to 1$ and $\beta \neq 0$ then in forms (2.12), (2.13) and (2.14) the characteristic functions of stable laws (as functions of parameters) are not continuous. This discontinuity is of special importance in the asymptotic parameter convergence problems in the parameter estimation procedures. A continuous form is proposed by Zolotarev (1983, Equation (M), p. 22), called form M, and here restated through $Z \sim S(\alpha, \beta; \mathrm{KL})$.

Corollary 2.3. *Let $Z \sim S(\alpha, \beta; KL)$ be a standard stable distribution in Lévy–Khintchine form with characteristic function given by (2.11). Then, for $\gamma > 0$ and $\delta \in \mathbb{R}$ a random variable*

$$X \stackrel{d}{=} \begin{cases} \gamma^{1/\alpha} Z + \gamma\delta - \gamma\beta \tan(\pi\alpha/2) & \textit{for } \alpha \neq 1 \\ \gamma(Z + \beta \frac{2}{\pi} \ln \gamma) + \gamma\delta & \textit{for } \alpha = 1 \end{cases} \tag{2.18}$$

is a stable random variable in Zolotarev form M.

Pitman and Pitman (2016, Corollary 4.1, p. 269) show that representation (2.18) admits the alternative canonic measure for infinitely divisible distributions advocated by Feller (1971, Chapter XVII)[5].

Nolan (2018b, Definition 1.7) introduces another representation, also continuous in all parameters, called 0–parametrization.

Corollary 2.4. *Let $Z \sim S(\alpha, \beta; KL)$ be a standard stable distribution in Lévy–Khintchine form with characteristic function given by (2.11). Then, for $\gamma > 0$ and $\delta \in \mathbb{R}$, a random variable*

$$X \stackrel{d}{=} \begin{cases} \gamma Z + \delta - \gamma\beta \tan(\pi\alpha/2) & \textit{for } \alpha \neq 1 \\ \gamma Z + \delta & \textit{for } \alpha = 1 \end{cases} \tag{2.19}$$

is a stable random variable in Nolan 0–parametrization.

The characteristic function of stable random variable in (2.19) is given by Nolan (2018b, Definition 1.7) and here restated in terms of real and imaginary parts of the cumulant function.

Definition 2.5. *Stable random variable given by (2.19) has characteristic function $\varphi_X(u) = \varphi_X(u|\alpha, \beta, \gamma, \delta)$ of the form*

$$\varphi_X(u) = \exp\{\Re\psi_X(u) + i\Im\psi_X(u)\} \tag{2.20}$$

[5]Feller (1971, p. 565) says that although the measure used in (2.7) avoids the unboundedness then it at the same time complicates many arguments unnecessarily, especially when considering stable distributions.

where

$$\Re\psi_X(u) = -\gamma^\alpha |u|^\alpha, \tag{2.21}$$

$$\Im\psi_X(u) = \begin{cases} u[\beta\gamma\tan\frac{\pi\alpha}{2}(|\gamma u|^{\alpha-1}-1)+\delta] & \text{for } \alpha \neq 1 \\ u[-\beta\gamma\frac{2}{\pi}\ln(\gamma|u|)+\delta] & \text{for } \alpha = 1 \end{cases} \tag{2.22}$$

with $u \in \mathbb{R}$, $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma > 0$, $\delta \in \mathbb{R}$.

Stable distributions in 0–parametrization, given by (2.19), are denoted by $S(\alpha,\beta,\gamma,\delta;0)$.

Let in (2.19) the scale parameter be 1 ($\gamma = 1$), and the location parameter 0 ($\delta = 0$). The corresponding stable laws are called *standard stable laws in 0–parametrization*, denoted by $S(\alpha,\beta;0)$,

$$S(\alpha,\beta;0) \equiv S(\alpha,\beta,\gamma=1,\delta=0;0).$$

Remark 1. *Let* γ_1, δ_1 *denote the scale and shift parameters of stable distributions in 1–parametrization, given by (2.14), and* γ, δ *those of 0–parametrization, given by (2.19). Then* $\gamma_1 = \gamma$ *while for* $\alpha \neq 1$ $\delta_1 = \delta - \beta\gamma\tan\frac{\pi\alpha}{2}$ *and for* $\alpha = 1$, $\delta_1 = \delta - \frac{2}{\pi}\beta\gamma\ln\gamma$.

Last remark easily follows from the definitions of 1– and 0–parametrization (see also Nolan (2018b, Equation (1.7), p. 11)).

For $\alpha = 2, 1.8, 1.5, 1.2, 1.1, 1, 0.9, 0.8, 0.5, 0.2$, the graphs of corresponding characteristic functions of $S(\alpha,1;0)$, given by (2.20), are presented in Appendix A Figure A.2, the corresponding absolute values, real and imaginary parts in Appendix B Figure B.2, and the absolute values, real and imaginary parts of corresponding cumulant functions in Appendix C Figure C.2.

2.4 Stable Laws as Limiting Distributions for Sums

For normalized sums of i.i.d. random variables the only possible limiting distributions are stable distributions. There are several other classes of distributions which involve the term "stable laws". The max-stable laws, also called extreme value distributions (e.g., Kotz and Nadarajah (2000)), arise as the limits of normalized maximum of i.i.d. random variables (e.g., Embrechts et al. (2013, Section 3.3), Beranger and Padoan (2016), Dey et al. (2016)). Geometric stable distributions arise as a limiting class in the random summation scheme where the number of summands is a geometrically distributed random variable (e.g., Kozubowski and Rachev (1999b,a)). An overview of various stable distributions, called alternative stable laws, is given by Mittnik and Rachev (1991). To distinguish stable laws from alternative stable distributions the term sum-stable is sometimes used.

Following Uchaikin and Zolotarev (1999, p. 62) the generalized central limit theorem is reformulated (the multivariate case can be found in Meerschaert and Scheffler (2001, Section 7.3)).

Theorem 2.4. *For i.i.d. random variables $X_1,\dots,X_n$ with the common distribution function $F_X(x)$ satisfying the conditions*

$$1-F_X(x) \sim \quad cx^{-\mu}, \quad x\to\infty, \tag{2.23}$$

$$F_X(x) \sim \quad d|x|^{-\mu}, \quad x\to-\infty, \tag{2.24}$$

with $c,d\geq 0$, $c+d>0$, and $\mu>0$, there exist sequences $a_n\in\mathbb{R}$ and $b_n>0$ such that for $n\to\infty$ the centred and normalized sum

$$Z_n = \frac{X_1+\cdots+X_n-a_n}{b_n}$$

converges in distribution to a standard stable random variable Z in Lévy–Khintchine form, given by (2.11),

$$Z_n \xrightarrow{\mathscr{D}} Z, \quad Z\sim S(\alpha,\beta;LK), \text{ as } n\to\infty, \tag{2.25}$$

with parameters

$$\alpha = \begin{cases} \mu & \mu\leq 2, \\ 2 & \mu>2 \end{cases} \text{ and } \beta = \frac{c-d}{c+d}. \tag{2.26}$$

The expressions for a_n, b_n and the sketch of the proof of Theorem 2.4 can be found in Uchaikin and Zolotarev (1999, p. 62–64). By Theorem 1.2 the convergence in (2.25) is equivalently expressed in the terms of the characteristic functions,

$$\varphi_Z(u;\alpha,\beta) = \lim_{n\to\infty}\varphi_{Z_n}(u).$$

By replacing the conditions (2.23) and (2.24) with a simple Zipf–Pareto form,

$$\mathrm{d}F_X(x) = \begin{cases} \alpha c x^{-\alpha-1} & x>\varepsilon \\ 0 & -\varepsilon<x<\varepsilon \\ \alpha d|x|^{-\alpha-1} & x<-\varepsilon, \end{cases} \tag{2.27}$$

where $0<\alpha\leq 2$, and c, $d\geq 0$, $c+d>0$, $\varepsilon>0$ are determined from $\int_{-\infty}^{\infty}\mathrm{d}F_X(x)=1$, the explicit form of characteristic function of Z is derived in Uchaikin and Zolotarev (1999, Sections 3.2–3.4, p. 72–85). It follows, that $\varphi_Z(u;\alpha,\beta)=\lim_{n\to\infty}\varphi_{Z_n}(u)$ is of the form of the characteristic function of the standard stable random variable in Lévy–Khintchine form, given by (2.11).

2.5 Some Special Cases of Stable Laws

For $\beta = 0$ stable laws are symmetric and for $\beta = \pm 1$ maximally skewed, being often called totally skewed stable laws. Totally right-skewed ($\beta = 1$) stable laws with $0 < \alpha < 1$ yield one-sided stable distributions concentrated on the positive semi-axes only (e.g., Uchaikin and Zolotarev (1999, p. 53)) that are often called positive[6] stable laws (e.g., Hougaard (2001, p. 503-504), Simon (2014), Grabchak (2018)). Note that positive stable random variables form a special sub-class of (strictly) stable laws (see, e.g., Feller (1971, Theorem XIII.7.1, p. 448), Uchaikin and Zolotarev (1999, p. 145-150)).

The scheme of the summation of i.i.d. random variables has been studied by numerous scholars. Among those results several members of the class of stable laws were explored as two-parameter (location and scale) distributions (see, e.g., Feller (1971, p. 173-176)), Uchaikin and Zolotarev (1999, p. xviii-xix)). In Table 1, some of them are listed (the origins are referred via Uchaikin and Zolotarev (1999) except Fréchet distribution which is referred via Kotz and Nadarajah (2000)).

TABLE 2.1: Some members of stable laws.

Distribution	α	β
Normal	2	·
Cauchy	1	0
Lévy	1/2	1
Lorentz (1906), von Holtsmark (1919)*	1/2	1
von Holtsmark (1919)*	3/2	0
von Holtsmark (1919)*	3/4	0
Fréchet (1927)	2/3	1
Landau (1944)	1	1
Monin (1955)*	2/3	0
Lifshits (1956)	5/3	1

* – originally obtained for the three-dimensional case

Note, that Frechet distribution is considered as a stable (e.g, Simon (2014)) as well as a max-stable distribution (e.g., Embrechts et al. (2013, p. 121)). This can be reasoned (see, e.g., Feller (1971, p. 172, p. 277, p. 465)) by the understanding that for strictly stable (i.e., $0 < \alpha < 1$) distributions the maximum $M_n = \max\{X_1, \ldots, X_n\}$ of i.i.d $X_1, \ldots, X_n$ gives a primary contribution to the sum $S_n = X_1 + \cdots + X_n$ and $F_{S_n} \sim F_{M_n}$ (e.g., Uchaikin and Zolotarev (1999, p. 54, p. 150)).

[6] Mandelbrot (1960b) refers to positive stable laws as an abbreviation for all stable laws that are maximally skewed in the positive direction ($0 < \alpha < 2$, $\beta = 1$), and in p. 87 defines a special sub-class for $1 < \alpha < 2$, called *Pareto-Lévy distributions*.

2.6 Estimation in Stable Laws

The process of parameter estimation of stable distributions is often complicated due to the lack of availability of the density function in explicit form and due to the fact that not all moments exist. In spite of these limitations, various methods have been proposed for estimating the parameters. Then again, many of these methods have restrictions to the parameter space or focus on estimating the characteristic exponent (tail index) α only.

For the methods restricted to symmetric ($\beta = 0$) stable distributions see, e.g., Fama and Roll (1971), Arad (1980), Ma and Nikias (1995), Nikias and Shao (1995), Tsihrintzis and Nikias (1996), Bodnar and Gupta (2011), Brouste and Masuda (2018). A comparison of the estimation methods in symmetric stable laws is given by Höpfner and Rüschendorf (1999). For procedures focusing on estimating the characteristic exponent α see, e.g., DuMouchel (1983), Höpfner (1998), Mittnik and Paolella (1999), Fan (2006), Yanushkevichiene and Saenko (2017).

For estimating the four parameters of general stable distribution, two broad classes of methods have been proposed: algorithmic procedures and closed-form estimators. The primary algorithmic procedures, with no restrictions to the parameter space, include: the quantile based look-up method by McCulloch (1986); the empirical characteristic function based methods by Koutrouvelis (1980, 1981) and Kogon and Williams (1998); and the (numerical) maximum likelihood estimation methods (Mittnik et al. (1999), and Nolan (2001)). The primary closed-form estimators, with no restrictions to the parameter space, include: empirical characteristic function based estimators (Press (1972), Krutto (2016, 2018)), the logarithmic and fractional lower order moments based and the extreme value theory based estimators (Kuruoglu (2001)).

More discussion on the methods of estimating the parameters of stable laws can be found in Nolan (2001), Borak et al. (2005), for example.

Of the aforementioned methods, the maximum likelihood method has high accuracy but high computational complexity. On the other hand, the quantile method is a simple technique that is based on look-up tables of quantiles, which naturally affects the accuracy of this method. The empirical characteristic function based estimation is discussed in the next section: shortly, the method by Koutrouvelis (1980) requires a look-up table and that of Koutrouvelis (1981) numerous iterations; the procedures in Koutrouvelis (1980, 1981), Kogon and Williams (1998) require pre-estimates of scale and location parameters. A disadvantage of the closed-form estimators in Kuruoglu (2001) is the lack of an estimator for the location parameter and, as showed in Kuruoglu (2001), his estimators did not show as high performance when estimating the tail index and scale parameter compared to the Kogon and Williams (1998) procedure.

On simulation-based comparison of some of the aforementioned methods, see, for example, Akgiray and Lamoureux (1989), Kogon and Williams (1998), Kuruoglu (2001), Kateregga et al. (2017), Krutto (2018). On simulating stable random variables, see, e.g., Chambers et al. (1976), Weron (1996), Nolan (1998a) and Devroye and James (2014).

In addition, the Bayesian approach has been discussed in Buckle (1995), Efthymios (2000), Lemke et al. (2015), Achcar and Lopes (2016), and a special focus on estimating mixtures of stable laws is given in Salas-Gonzalez et al. (2009), Peng et al. (2013), Teimouri et al. (2018).

2.7 Characteristic Function Based Estimation in Stable Laws

Press (1972) introduced a procedure which is based on the transformation of the logarithm of characteristic function (i.e., cumulant function) at four arbitrary different non-zero arguments $u_k, k = 1, \ldots, 4$ along the real line, called a version of the method of moments[7]. He provided no suggestions on how to select these four arguments and the method has been considered as not very useful in practice. To get around the Press (1972) difficulties, several algorithmic modifications relying on a number of arguments have been proposed.

Paulson et al. (1975) provided a method that minimizes (by a gradient projection algorithm) the integrated squared error $\int |\varphi(u) - \varphi_n(u)|^2 \mathrm{d}u$ along the real line (see, also Heathcote (1977)) by a Hermitian quadrature at specified points u_k, $k = 1, \ldots, 20$. The asymptotic distribution of Paulson et al. (1975) approach is discussed in Thornton and Paulson (1977).

Koutrouvelis (1980) followed Press (1972) closed-form estimators but reformulated them in a 2-step regression analysis: first estimating α and γ by regressing $\ln(-\ln(\psi_n(u))$ onto $\ln(-\ln(\psi(u))$ and then β and δ by regressing $\Im\,\psi_n(u)$ onto $\Im\,\psi(u)$ at the points $u_k = \pi k/25$ for $k = 1, 2, \ldots, K$ with K having values from $K = 10$ to $K = 134$, depending on sample size n and parameter α (with $u_1 \approx 0.13$ up to $u_{K=134} \approx 16.84$). For implementation one needs to look up the number of points to be used for the regression from Table 1 in Koutrouvelis (1980).

A procedure proposed by Kogon and Williams (1998) provides a modification of Koutrouvelis (1980) by an ordinary least squares regression at $u_k = k/10$ for $k = 1, 2, \ldots, 10$ Kogon and Williams (1998) found empirically that using more than $K = 10$ values from interval $[0.1, 1]$ does not remarkably improve the estimates of their procedure. Note that Knight and Satchell (1997) provided a six-step procedure for estimating cumulant function (in general, not of stable distribution) at a set of q points: how many points and which set of q points should be used remained an open question.

All the aforementioned empirical characteristic function based methods assume standardized data, i.e., they require pre-estimates for γ, δ. Paulson et al. (1975) proposed an algorithmic procedure for the scaling parameter, and median (in symmetric case) or minimum-maximum (in asymmetric case)

[7] Kozubowski (1999) adapted Press (1972) method for geometric stable laws. Similarly to Press (1972) his approach requires four arbitrary different non-zero arguments $u_k, k = 1, \ldots, 4$ along the real line.

for shift parameter; Koutrouvelis (1980) applied the fractile based method (Fama and Roll (1971))[8] for the scale parameter, and truncated mean for shift parameter; Kogon and Williams (1998) used quantile based estimators (McCulloch (1986))[9].

Krutto (2016) revised Press (1972) approach and provided closed-form estimators that require only two distinct positive real points u_1 and u_2, called cumulant estimators. Instead of standardizing the data it was proposed to divide the data elements by the absolute value of their median, called reduced values' estimators. In order to provide some guidance on how to choose these two arguments, Krutto (2016) performed an empirical search over various pairs of positive real arguments. In Krutto (2018) the asymptotic normality for the estimators was proved and a sample based selection for u_1 and u_2 was proposed, called Argument–Selection–Rule.

[8]Fama and Roll (1971) estimators are based on the peculiar fact that for $1 \leq \alpha \leq 2$ the 0.72 quantile of a standard symmetric stable distribution is in the interval 0.827 ± 0.003.

[9]McCulloch (1986) method works best for $0.5 \leq \alpha \leq 2$; to estimate the parameters γ and δ the estimates of α and β are calculated first.

Chapter 3

Parameters of Stable Laws via Cumulant Function

Note that unlike the case of the characteristic function, the parameters β and δ have no influence on the real part of the cumulant function of stable laws. This forms the basis of Krutto (2016, Theorem 1) which states that the parameters α, γ, β, δ of stable laws can be expressed via real and imaginary part of the corresponding cumulant function at two arbitrary different arguments u_1, u_2 on the positive real line.

Theorem 3.1. *Let $u_1 > 0, u_2 > 0, u_1 \neq u_2$. Let $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ with the real and imaginary parts of the cumulant function $\psi_X(u)$ given by (2.16) and (2.17), respectively. Then*

$$\alpha = \frac{\ln(-\Re\psi_X(u_1)) - \ln(-\Re\psi_X(u_2))}{\ln u_1 - \ln u_2}, \tag{3.1}$$

$$\gamma = \exp\left\{\frac{\ln u_1 \ln(-\Re\psi_X(u_2)) - \ln u_2 \ln(-\Re\psi_X(u_1))}{\ln(-\Re\psi_X(u_1)) - \ln(-\Re\psi_X(u_2))}\right\}, \tag{3.2}$$

and in the case of $\alpha \neq 1$,

$$\beta = \frac{u_2 \Im\psi_X(u_1) - u_1 \Im\psi_X(u_2)}{\gamma^\alpha \left(u_2 u_1^\alpha - u_1 u_2^\alpha\right) \tan\frac{\pi\alpha}{2}}, \tag{3.3}$$

$$\delta = \frac{u_1^\alpha \Im\psi_X(u_2) - u_2^\alpha \Im\psi_X(u_1)}{u_2 u_1^\alpha - u_1 u_2^\alpha}, \tag{3.4}$$

where α is given by (3.1) and γ by (3.2), and, in the case of $\alpha = 1$,

$$\beta = \pi\frac{u_2 \Im(\psi_X(u_1)) - u_1 \Im\psi_X(u_2)}{2\gamma u_1 u_2(\ln u_2 - \ln u_1)}, \tag{3.5}$$

$$\delta = \frac{u_2 \Im\psi_X(u_1) \ln u_2 - u_1 \Im\psi_X(u_2) \ln u_1}{u_1 u_2(\ln u_2 - \ln u_1)}, \tag{3.6}$$

where γ is given by (3.2).

Proof. Let us choose constants $u_1, u_2 \in \mathbb{R}$ so that $u_1 > 0, u_2 > 0, u_1 \neq u_2$. Assuming that the parameters of a stable random variable $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ are fixed, we can write the following system of equations:

$$\begin{cases} \psi_X(u_1) = -\gamma^\alpha u_1^\alpha + i \begin{cases} \left(\beta\gamma^\alpha u_1^\alpha \tan\frac{\pi\alpha}{2} + \delta u_1\right) & \text{for } \alpha \neq 1 \\ \left(-\frac{2}{\pi}\beta\gamma u_1 \ln u_1 + \delta u_1\right) & \text{for } \alpha = 1 \end{cases} \\ \psi_X(u_2) = -\gamma^\alpha u_2^\alpha + i \begin{cases} \left(\beta\gamma^\alpha u_2^\alpha \tan\frac{\pi\alpha}{2} + \delta u_2\right) & \text{for } \alpha \neq 1 \\ \left(-\frac{2}{\pi}\beta\gamma u_2 \ln u_2 + \delta u_2\right) & \text{for } \alpha = 1. \end{cases} \end{cases} \tag{3.7}$$

Because (3.7) is a system of complex valued functions, it must simultaneously hold for real and imaginary parts of $\psi_X(u_1)$ and $\psi_X(u_2)$. The real parts of $\psi_X(u_1)$ and $\psi_X(u_2)$ in (3.7) give

$$\begin{cases} \Re\psi_X(u_1) = -\gamma^\alpha u_1^\alpha, \\ \Re\psi_X(u_2) = -\gamma^\alpha u_2^\alpha. \end{cases} \tag{3.8}$$

Solving the system (3.8) for α and γ gives (3.1) and (3.2). The imaginary parts of $\psi_X(u_1)$ and $\psi_X(u_2)$ in (3.7) form two systems. First, in the case of $\alpha \neq 1$ the imaginary parts in system (3.7) give the system

$$\begin{cases} \Im\psi_X(u_1) = \beta\gamma^\alpha u_1^\alpha \tan\frac{\pi\alpha}{2} + \delta u_1, \\ \Im\psi_X(u_2) = \beta\gamma^\alpha u_2^\alpha \tan\frac{\pi\alpha}{2} + \delta u_2. \end{cases} \tag{3.9}$$

Solving system (3.9) for δ and β gives (3.3) and (3.4), respectively, with α and γ from (3.1) and (3.2). In the case of $\alpha = 1$ the imaginary parts in (3.7) give the system

$$\begin{cases} \Im\psi_X(u_1) = -\frac{2}{\pi}\beta\gamma u_1 \ln u_1 + \delta u_1, \\ \Im\psi_X(u_2) = -\frac{2}{\pi}\beta\gamma u_2 \ln u_2 + \delta u_2. \end{cases} \tag{3.10}$$

Solving system (3.10) for δ and β gives (3.5) and (3.6), respectively, with γ given by (3.2). □

Theorem 3.1 can be extended to any parametrization introduced in Section 2.3. For asymptotic parameter convergence problems the characteristic functions of stable laws in continuous form, given by (2.18) and (2.19), are more suitable because the characteristic function is continuous in all parameters. For stable laws in 0–parametrization, given by (2.19), a modification of Theorem 3.1 is formulated in Krutto (2018, Theorem 1).

Corollary 3.1. *Let $X \sim S(\alpha, \beta, \gamma, \delta; 0)$ be a stable random variable with the real and imaginary parts of cumulant function $\psi_X(u)$ given by (2.21) and (2.22), respectively, and let, for every fixed* *real* $u_1 > 0, u_2 > 0, u_1 \neq u_2$,

$$\boldsymbol{b} = \boldsymbol{b}(u_1,u_2,X) = \begin{pmatrix} b_1(u_1,X) \\ b_2(u_2,X) \\ b_3(u_1,X) \\ b_4(u_2,X) \end{pmatrix} = \begin{pmatrix} \Re\varphi_X(u_1) \\ \Re\varphi_X(u_2) \\ \Im\varphi_X(u_1) \\ \Im\varphi_X(u_2) \end{pmatrix} \tag{3.11}$$

be a 4*-dimensional real valued vector. The parameters of stable random variable* $X \sim S(\alpha,\beta,\gamma,\delta;0)$ *can be expressed as* $\alpha = g_1(\boldsymbol{b})$, $\gamma = \exp\{g_2(\boldsymbol{b})\}$, $\beta = g_3(\boldsymbol{b})$, $\delta = g_4(\boldsymbol{b})$, *where*

$$g_1(\boldsymbol{b}) = \frac{\ln\big(-\ln\sqrt{b_1^2+b_3^2}\big) - \ln\big(-\ln\sqrt{b_2^2+b_4^2}\big)}{\ln u_1 - \ln u_2}, \tag{3.12}$$

$$g_2(\boldsymbol{b}) = \frac{\ln u_1 \ln\big(-\ln\sqrt{b_2^2+b_4^2}\big) - \ln u_2 \ln\big(-\ln\sqrt{b_1^2+b_3^2}\big)}{\ln\big(-\ln\sqrt{b_1^2+b_3^2}\big) - \ln\big(-\ln\sqrt{b_2^2+b_4^2}\big)}, \tag{3.13}$$

if $\alpha \neq 1$ *then*

$$g_3(\boldsymbol{b}) = \frac{u_2 \operatorname{atan2}(b_3,b_1) - u_1 \operatorname{atan2}(b_4,b_2)}{\exp\{g_1(\boldsymbol{b})g_2(\boldsymbol{b})\}\big(u_2 u_1^{g_1(\boldsymbol{b})} - u_1 u_2^{g_1(\boldsymbol{b})}\big)\tan(\pi g_1(\boldsymbol{b})/2)} \tag{3.14a}$$

while if $\alpha = 1$ *then*

$$g_3(\boldsymbol{b}) = \frac{\pi}{2}\frac{u_2 \operatorname{atan2}(b_3,b_1) - u_1 \operatorname{atan2}(b_4,b_2)}{\exp\{g_2(\boldsymbol{b})\}u_1u_2(\ln u_2 - \ln u_1)}, \tag{3.14b}$$

if $\alpha \neq 1$ *then*

$$g_4(\boldsymbol{b}) = \frac{u_2^{g_1(\boldsymbol{b})}\operatorname{atan2}(b_3,b_1)\Big[(\exp\{g_2(\boldsymbol{b})\}u_1)^{1-g_1(\boldsymbol{b})} - 1\Big]}{u_2 u_1^{g_1(\boldsymbol{b})} - u_1 u_2^{g_1(\boldsymbol{b})}} - \frac{u_1^{g_1(\boldsymbol{b})}\operatorname{atan2}(b_4,b_2)\Big[(\exp\{g_2(\boldsymbol{b})\}u_2)^{1-g_1(\boldsymbol{b})} - 1\Big]}{u_2 u_1^{g_1(\boldsymbol{b})} - u_1 u_2^{g_1(\boldsymbol{b})}} \tag{3.15a}$$

while if $\alpha = 1$ *then*

$$g_4(\boldsymbol{b}) = \frac{u_2 \operatorname{atan2}(b_3,b_1)\, g_2(\boldsymbol{b}) \ln u_2 - u_1 \operatorname{atan2}(b_4,b_2)\, g_2(\boldsymbol{b}) \ln u_1}{u_1u_2(\ln u_2 - \ln u_1)}. \tag{3.15b}$$

Proof. From (1.31)

$$\Re\psi_X(u) = \ln|\varphi_X(u)| = \ln\sqrt{\Re^2\varphi_X(u) + \Im^2\varphi_X(u)}, \tag{3.16}$$

$$\Im\psi_X(u) = \operatorname{atan2}(\Im\varphi_X(u), \Re\varphi_X(u)). \tag{3.17}$$

Then (3.12) through (3.15b) immediately follow from Theorem 3.1 applied to the real and imaginary parts of the cumulant function of stable laws in 0–parametrization, given by (2.21) and (2.22), respectively. □

Note that from (1.12) and (1.13)

$$\boldsymbol{b} = \begin{pmatrix} \Re\varphi_X(u_1) \\ \Re\varphi_X(u_2) \\ \Im\varphi_X(u_1) \\ \Im\varphi_X(u_2) \end{pmatrix} = \begin{pmatrix} \exp\{\Re\psi_X(u_1)\}\cos\Im\psi_X(u_1) \\ \exp\{\Re\psi_X(u_2)\}\cos\Im\psi_X(u_2) \\ \exp\{\Re\psi_X(u_1)\}\sin\Im\psi_X(u_1) \\ \exp\{\Re\psi_X(u_2)\}\sin\Im\psi_X(u_2) \end{pmatrix}.$$

For the same parametrization the expressions of the parameters in Theorem 3.1 and Corollary 3.1 are equivalent. For 1- and 0–parametrizations there is a difference in the expression of the location parameter δ.

For the real part of cumulant function of stable laws the following property holds.

Proposition 3.1. *Let $X \sim S(\alpha,\beta,\gamma,\delta;k)$ with $k = 1,0$. Then $\Re\psi_X(0) = 0$,*

$$\Re\psi_X(\pm\frac{1}{\gamma}) = -1,$$

and $\Re\psi_X(u) \in [0,-1]$ for every $u \in [-1/\gamma, 1/\gamma]$.

Proof. Proof immediately follows from the expressions of the real part of cumulant functions of $X \sim S(\alpha,\beta,\gamma,\delta;k)$, $k = 1,0$, given by (2.16) and (2.21),

$$\Re\psi_X(u) = -\gamma^\alpha |u|^\alpha,$$

and then $\Re\psi_X(0) = 0$, $\Re\psi_X(\pm\frac{1}{\gamma}) = -1$ while $\Re\psi_X(u) \in [0,-1]$ for any $u \in [-\frac{1}{\gamma}, \frac{1}{\gamma}]$. □

For the imaginary part of cumulant function of stable laws in Nolan 0–parametrization the following property holds.

Proposition 3.2. *Let $X \sim S(\alpha,\beta,\gamma,\delta;0)$ with the imaginary part of its cumulant function given by (2.22). For $u = 1/\gamma$ it holds*

$$\Im\psi_X(1/\gamma) = \delta/\gamma$$

and for $u = -1/\gamma$ it holds

$$\Im\psi_X(-1/\gamma) = -\delta/\gamma.$$

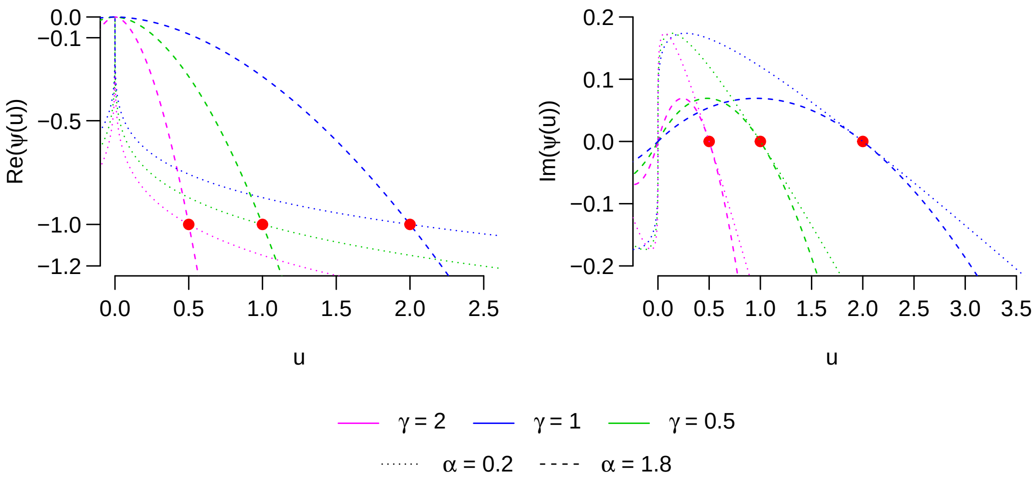

FIGURE 3.1: The real (on left) and imaginary (on right) parts of the cumulant functions of $S(\alpha,\beta=1,\gamma,\delta=0;0)$.

Proof. Immediately follows from (2.22) at $u=\pm 1/\gamma$,

$$\Im\psi_X(\pm\frac{1}{\gamma}) = \begin{cases} \pm 1/\gamma[\beta\gamma \tan\frac{\pi\alpha}{2}(|1|^{\alpha-1}-1)+\delta] & \alpha\neq 1 \\ \pm 1/\gamma[-\beta\gamma\frac{2}{\pi}\ln(1)+\delta] & \alpha=1, \end{cases} \tag{3.18}$$

$$= \begin{cases} \pm\delta/\gamma & \alpha\neq 1 \\ \pm\delta/\gamma & \alpha=1. \end{cases} \tag{3.19}$$

□

For $\alpha=0.2,1.8$ and $\gamma=2,1,0.5$ Propositions 3.1 and 3.2 are illustrated by Figure 3.1.

Chapter 4

Empirical Cumulant Function (ECuF) Based Estimators

The key to the empirical cumulant function (ECuF) based estimators is the substituting principle (e.g., Knight (1999, Section 4.5, p. 190)): in Theorem 3.1, and Corollary 3.1, the real and imaginary parts of the cumulant function $\psi_X(u)$ are replaced by those of the empirical cumulant function $\psi_n(u)$. In this chapter the estimators are formulated, their statistical inference is discussed, an empirical search and sample based rule for the selection of u_1, u_2 is provided.

4.1 Formulation of ECuF Estimators

The ECuF estimators for the parameters of $X \sim S(\alpha,\ \beta,\ \gamma, \delta; 1)$ are formulated by Krutto (2016, Definition 2).

Definition 4.1 (ECuF estimators)**.** *Assume $Y_1, Y_2, \dots, Y_n$ form a sample of i.i.d. random variables from $S(\alpha,\ \beta,\ \gamma, \delta; 1)$, and $u_1 > 0, u_2 > 0, u_1 \neq u_2$. The ECuF estimators*

$$\alpha_n = \alpha_n(u_1, u_2, Y_1, \dots, Y_n), \tag{4.1}$$

$$\beta_n = \beta_n(u_1, u_2, Y_1, \dots, Y_n), \tag{4.2}$$

$$\gamma_n = \gamma_n(u_1, u_2, Y_1, \dots, Y_n), \tag{4.3}$$

$$\delta_n = \delta_n(u_1, u_2, Y_1, \dots, Y_n), \tag{4.4}$$

for the parameters of $S(\alpha,\ \beta,\ \gamma, \delta; 1)$ are defined to satisfy (3.1)–(3.6), where the real and imaginary parts of cumulant function, given by (2.16) and (2.17), respectively, are replaced by those of the empirical cumulant function, given by (1.32) and 1.33, respectively.

From Definition 4.1 it follows, that ECuF estimators are found step-by-step: first the tail index α and the scale γ are estimated and based on those results the estimates of the asymmetry β and location δ are obtained.

In general, especially for smaller samples, ECuF estimators may give non-admissible values, that is, one or more estimates may turn out of the parameter space: $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma > 0$, $\delta \in \mathbb{R}$ (see Section 5.2 on estimating tail index α). It can be dealt by truncating the values of estimators: replace α_n with $\min(\max(\alpha_n, 0.01), 2)$, β_n with $\beta_n = \min(\max(\beta_n, -1), 1)$ and γ with $\gamma_n = \max(0, \gamma_n)$. However, it complicates statistical properties and in this thesis the non-truncated estimators are studied. In fact, using non-truncated estimators is interesting as it provides insights to the range and occurrence of invalid values (see Section 5.2 and Table 8.2). However, in simulations in Chapter 7 and in applications in Section 8 the truncated ECuF estimators are applied.

The ECuF estimators can equivalently be formulated based on Corollary 3.1.

Definition 4.2 (ECuF Estimators)**.** *Let $Y_1, Y_2, \ldots, Y_n$ form a sample of i.i.d. random variables from $S(\alpha, \beta, \gamma, \delta; 1)$, given by (2.19), and let, for every fixed $u_1 > 0, u_2 > 0, u_1 \neq u_2$,*

$$\boldsymbol{b}_n = \boldsymbol{b}_n(u_1, u_2, Y_1, \ldots, Y_n) = \begin{pmatrix} \Re\varphi_n(u_1) \\ \Re\varphi_n(u_2) \\ \Im\varphi_n(u_1) \\ \Im\varphi_n(u_2) \end{pmatrix} \tag{4.5}$$

be a $4-$dimensional real vector, where $\varphi_n(u)$ is the empirical characteristic function given by (1.14). The ECuF estimators of the parameters of $S(\alpha, \beta, \gamma, \delta; 0)$ are defined to satisfy (3.12)–(3.15b), where the elements of $\boldsymbol{b}$, given by (3.11), are replaced with those of $\boldsymbol{b}_n$, given by (4.5),

$$\alpha_n = \alpha_n(u_1, u_2, Y_1, \ldots, Y_n) \equiv g_1(\boldsymbol{b}_n), \tag{4.6}$$

$$\gamma_n = \gamma_n(u_1, u_2, Y_1, \ldots, Y_n) \equiv \exp\{g_2(\boldsymbol{b}_n)\}, \tag{4.7}$$

$$\beta_n = \beta_n(u_1, u_2, Y_1, \ldots, Y_n) \equiv g_3(\boldsymbol{b}_n), \tag{4.8}$$

$$\delta_n = \delta_n(u_1, u_2, Y_1, \ldots, Y_n) \equiv g_4(\boldsymbol{b}_n). \tag{4.9}$$

It has been discussed (see, e.g., Paulson et al. (1975), Koutrouvelis (1980), Koutrouvelis (1981), Kogon and Williams (1998)) that $\gamma >> 1$ influences the empirical characteristic function based estimation in stable laws and because of that the pre–standardizing of data has been suggested. In this thesis the approach byKrutto (2003, 2016) is followed: sample elements are divided by the absolute value of the sample median (assuming that it is greater than 1). Let $m = q_{0.5}$ denote the sample median, where q_p is the k-th percentile, $p = k/100$.

Definition 4.3. *Let $Y_1, Y_2, \ldots, Y_n$ form a sample of i.i.d random variables from stable law in 1–parametrization, $S(\alpha,\ \beta,\ \gamma, \delta; 1)$ with the sample median m. For $|m| > 1$ the reduced values' estimators $\alpha_n, \beta_n, \gamma_n, \delta_n$ for the parameters of $S(\alpha,\ \beta,\ \gamma, \delta; 1)$ are*

$$\alpha_n = \alpha_n', \tag{4.10}$$

$$\beta_n = \beta_n', \tag{4.11}$$

$$\gamma_n = |m|\gamma_n', \tag{4.12}$$

$$\delta_n = \begin{cases} |m|\delta_n' & \text{for } \alpha \neq 1, \\ |m|\delta_n' - \frac{2}{\pi}\beta_n'\gamma_n'|m|\ln|m| & \text{for } \alpha = 1. \end{cases} \tag{4.13}$$

where $\alpha_n', \beta_n', \gamma_n', \delta_n'$ are ECuF estimators, given by Definition 4.1, evaluated from the sample reduced by $Y_i' = Y_i/|m|$, $i = 1, \ldots, n$.

Similarly, the reduced values' estimators can be defined for the 0–parametrization.

Definition 4.4. *Let $Y_1, Y_2, \ldots, Y_n$ form a sample of i.i.d random variables from $S(\alpha,\ \beta,\ \gamma, \delta; 0)$ with the sample median m. For $|m| > 1$ the reduced values' estimators $\alpha_n, \beta_n, \gamma_n, \delta_n$ for the parameters of $S(\alpha,\ \beta,\ \gamma, \delta; 0)$ are*

$$\alpha_n = \alpha_n', \tag{4.14}$$

$$\beta_n = \beta_n', \tag{4.15}$$

$$\gamma_n = |m|\gamma_n', \tag{4.16}$$

$$\delta_n = |m|\delta_n' \tag{4.17}$$

where $\alpha_n', \beta_n', \gamma_n', \delta_n'$ are ECuF estimators, given by Definition 4.2, evaluated from the sample reduced by $Y_i' = Y_i/|m|$, $i = 1, \ldots, n$.

The expressions of reduced values' estimators in Definitions 4.3 and 4.4 immediately follow from Proposition 1.2(d) of the cumulant function (see also Nolan (2018b, Propositions 1.16 and 1.17)), i.e., for any $m \neq 0$,

$$\psi_{mX}(u) = \ln\left(\mathbf{E}[e^{imuX}]\right) = \psi_X(mu).$$

Remark 2. *An improvement of reduced values' ECuF estimators that would be applicable to the large shift case would be standardize Y by*

$$Y' = \frac{Y - m}{\mathrm{IQR}},$$

where $m = q_{0.5}$ is the median of Y and $\mathrm{IQR} = q_{0.75} - q_{0.25}$ the interquartile range of Y, and q_p is the k-th percentile, $p = k/100$.

The realizations of the ECuF estimators at the counterpart $y_1, \ldots, y_n$ of the random sample $Y_1, \ldots, Y_n$ are denoted by $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\delta}_n$, respectively.

4.2 Asymptotic Normality for the Real and Imaginary Parts of Empirical Cumulant Function

Before deriving the asymptotic normality of ECuF estimators the asymptotic normality of the real and imaginary parts of the empirical cumulant function is established.

Theorem 4.1. *Let X be a real valued random variable with the cumulant function $\psi_X(u)$, $u \in \mathbb{R}$, given by (1.11). Let $\psi_n(u)$, $u \in \mathbb{R}$, be the empirical cumulant function, given by (1.31). For every fixed $u \in \mathbb{R}$,*

(i) $\Re\psi_n(u)$ is strongly consistent estimator for $\Re\psi_X(u)$ and $\Im\psi_n(u)$ is strongly consistent estimator for $\Im\psi_X(u)$;

(ii) $\Re\psi_n(u)$, $\Im\psi_n(u)$ are asymptotically normal,

$$\sqrt{n}\big(\Re\psi_n(u) - \Re\psi_X(u)\big) \xrightarrow{\mathscr{D}} N_1\big(0, \kappa_{\Re}(u)\big), \text{ as } n \to \infty, \tag{4.18}$$

$$\sqrt{n}\big(\Im\psi_n(u) - \Im\psi_X(u)\big) \xrightarrow{\mathscr{D}} N_1\big(0, \kappa_{\Im}(u)\big), \text{ as } n \to \infty, \tag{4.19}$$

with

$$\kappa_{\Re}(u) = \frac{1}{2\Re^2\psi_X(u)}\left(1 + \exp\{\Re\psi_X(2u)\} - 2\exp\{2\Re\psi_X(u)\}\right), \tag{4.20}$$

$$\kappa_{\Im}(u) = \frac{1}{2\Re^2\psi_X(u)}\left(1 - \exp\{\Re\psi_X(2u)\}\right). \tag{4.21}$$

Proof. For a real vector $\boldsymbol{x} = (x_1, x_2)' \neq \mathbf{0}$ define $h_1(\boldsymbol{x}) = \ln\sqrt{x_1^2 + x_2^2}$ and $h_2(\boldsymbol{x}) = \operatorname{atan2}(x_2, x_1)$. Let

$$\boldsymbol{a} = (\Re\varphi_X(u), \Im\varphi_X(u))'$$

with $\Re\varphi_X(u), \Im\varphi_X(u)$ given by (1.2) and (1.3), respectively, and

$$\boldsymbol{a}_n = (\Re\varphi_n(u), \Im\varphi_n(u))'$$

with $\Re\varphi_n(u), \Im\varphi_n(u)$ given by (1.15) and (1.16), respectively. The real and imaginary parts of cumulant function in (1.11) are the functions of the elements of $\boldsymbol{a}$,

$$h_1(\boldsymbol{a}) = \Re\psi_X(u),\, h_2(\boldsymbol{a}) = \Im\psi_X(u),$$

and the real and imaginary parts of the empirical cumulant function in (1.31) are the functions of the elements of $\boldsymbol{a}_n$,

$$h_1(\boldsymbol{a}_n) = \Re\psi_n(u),\, h_2(\boldsymbol{a}_n) = \Im\psi_n(u).$$

Clearly, h_1 is a continuous function. Based on (1.9), the function $h_2(\boldsymbol{a})$ has a discontinuity at $\Re\varphi_X(u)=0$ and $\Im\varphi_X(u)=0$, i.e., at $|\varphi_X(u)|=0$. Based on Proposition 1.1(d) the term $|\varphi_X(u)|$ tends to 0 as $|u|\to\infty$. However, in the assumption it is said that $u\in\mathbb{R}$ is fixed, $|u|<\infty$, and $h_2(\boldsymbol{a})$ is continuous on any bounded interval.

(i) Based on relation (1.22), for every fixed $u\in\mathbb{R}$

$$\boldsymbol{a}_n \xrightarrow{a.s.} \boldsymbol{a}, \text{ as } n\to\infty. \tag{4.22}$$

By the continuous mapping theorem (e.g., van der Vaart (1998, Theorem 2.3)),

$$h_j(\boldsymbol{a}_n) \xrightarrow{a.s.} h_j(\boldsymbol{a}), \text{ as } n\to\infty, \tag{4.23}$$

and the estimators $h_j(\boldsymbol{a}_n)$ are consistent for $h_j(\boldsymbol{a})$, $j=1,2$.

(ii) The quantities $\Re\varphi_{X_n}(u_j)$, $\Im\varphi_{X_n}(u_j)$, $j=1,2$ are sample means of i.i.d. random variables with $\mathbf{E}\,\boldsymbol{a}_n=\boldsymbol{a}$ and with finite variance. Therefore, as stated in Theorem 1.4, for every fixed $u\in\mathbb{R}$,

$$\sqrt{n}(\boldsymbol{a}_n-\boldsymbol{a}) \xrightarrow{\mathscr{D}} N_2(\mathbf{0},\boldsymbol{\Sigma}(u)), \text{ as } n\to\infty$$

where $\boldsymbol{\Sigma}(u)\equiv\boldsymbol{\Sigma}=(\sigma_{kl})$ is a 2×2 covariance matrix with the structure following from (1.26)–(1.28),

$$2\sigma_{11} = 1+\Re\varphi_X(2u)-2\Re^2\varphi_X(u) \tag{4.24}$$

$$2\sigma_{22} = 1-\Re\varphi_X(2u)-2\Im^2\varphi_X(u) \tag{4.25}$$

$$2\sigma_{12} = 2\sigma_{21} = \Im\varphi_X(2u)-2\Re\varphi_X(u)\Im\varphi_X(u), \tag{4.26}$$

where $\varphi_X(u)=\exp\psi_X(u)$. From Kollo and von Rosen (2005, Theorem 3.1.3), or Anderson (2003, p. 132-133), it immediately follows, that for every fixed $u\in\mathbb{R}$,

$$\sqrt{n}\big(h_1(\boldsymbol{a}_n)-h_1(\boldsymbol{a})\big) \xrightarrow{\mathscr{D}} N_1\big(0,\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu}\big), \text{ as } n\to\infty, \tag{4.27}$$

$$\sqrt{n}\big(h_2(\boldsymbol{a}_n)-h_2(\boldsymbol{a})\big) \xrightarrow{\mathscr{D}} N_1\big(0,\boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta}\big), \text{ as } n\to\infty, \tag{4.28}$$

with $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ as matrix derivatives (Kollo and von Rosen (2005, Definition 1.4.1)),

$$\boldsymbol{\nu}=\left.\frac{\mathrm{d}h_1(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{a}} \text{ and } \boldsymbol{\eta}=\left.\frac{\mathrm{d}h_2(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{a}}.$$

It is easy to see that

$$\frac{\mathrm{d}h_1(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}=\frac{\mathrm{d}\ln\sqrt{x_1^2+x_2^2}}{\mathrm{d}\boldsymbol{x}}=\left(\frac{x_1}{x_1^2+x_2^2},\frac{x_2}{x_1^2+x_2^2}\right)'$$

and

$$\frac{\mathrm{d}h_2(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}} = \frac{\mathrm{d}\,\mathrm{atan2}(x_2, x_1)}{\mathrm{d}\boldsymbol{x}} = \left(-\frac{x_2}{x_1^2+x_2^2}, \frac{x_1}{x_1^2+x_2^2}\right)'.$$

Hereby,

$$\boldsymbol{\nu} = \left.\frac{\mathrm{d}h_1(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{a}} = \left(\frac{\Re\varphi_X(u)}{|\varphi_X(u)|^2}, \frac{\Im\varphi_X(u)}{|\varphi_X(u)|^2}\right)' \neq \mathbf{0},$$

and

$$\boldsymbol{\eta} = \left.\frac{\mathrm{d}h_2(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{a}} = \left(-\frac{\Im\varphi_X(u)}{|\varphi_X(u)|^2}, \frac{\Re\varphi_X(u)}{|\varphi_X(u)|^2}\right)' \neq \mathbf{0}.$$

Obtaining $\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu}$ is straightforward,

$$\begin{aligned}
\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu} &= (\nu_1)^2\sigma_{11} + \nu_1\nu_2(\sigma_{12}+\sigma_{21}) + (\nu_2)^2\sigma_{22} \\
&= \frac{1}{2|\varphi_X(u)|^4}\Big[\Re^2\varphi_X(u)\big(1+\Re\varphi_X(2u) - 2\Re^2\varphi_X(u)\big) \\
&\quad + \Im^2\varphi_X(u)\big(1-\Re\varphi_X(2u) - 2\Im^2\varphi_X(u)\big) \\
&\quad + 2\Re\varphi_X(u)\Im\varphi_X(u)\big(\Im\varphi_X(2u) - 2\Re\varphi_X(u)\Im\varphi_X(u)\big)\Big] \\
&= \frac{1}{2|\varphi_X(u)|^4}\Big[|\varphi_X(u)|^2 - 2|\varphi_X(u)|^4 \\
&\quad + \underbrace{\Re\varphi_X(2u)\big(\Re^2\varphi_X(u) - \Im^2\varphi_X(u)\big) + 2\Im\varphi_X(2u)\Re\varphi_X(u)\Im\varphi_X(u)}_{(*)}\Big],
\end{aligned}$$

with $\Re\varphi_X(u) + \Im\varphi_X(u) = |\varphi_X(u)|^2$. By relations (1.12) and (1.13),

$$\begin{aligned}
(*) &= \Re\varphi_X(2u)2\big(\Re^2\varphi_X(u) - \Im^2\varphi_X(u)\big) + 2\Im\varphi_X(2u)\Re\varphi_X(u)\Im\varphi_X(u) \\
&= \exp\{\Re\psi_X(2u)\}\exp\{2\Re\psi_X(u)\}\big[\cos(2u)\big(\cos^2(u) - \sin^2(u)\big) + 2\sin(u)\cos(u)\sin(2u)\big] \\
&= \exp\{\Re\psi_X(2u)\}\exp\{2\Re\psi_X(u)\}. \qquad (4.29)
\end{aligned}$$

Since $|\varphi_X(u)| = \exp\{\Re\psi(u)\}$, then

$$\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu} = \frac{1}{2\exp\{2\Re\psi(u)\}}\left(1 - 2\exp\{2\Re\psi(u)\} + \exp\{\Re\psi(2u)\}\right),$$

which is the same as (4.20) and $\kappa_{\Re} = \boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu}$. In a similar manner,

$$
\begin{aligned}
\boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta} &= (\eta_1)^2\sigma_{11} + \eta_1\eta_2(\sigma_{12}+\sigma_{21}) + (\eta_2)^2\sigma_{22} \\
&= \frac{1}{2|\varphi_X(u)|^4}\Big[\Im^2\varphi_X(u)\big(1+\Re\varphi_X(2u) - 2\Re^2\varphi_X(u)\big) \\
&\quad + \Re^2\varphi_X(u)\big(1-\Re\varphi_X(2u) - 2\Im^2\varphi_X(u)\big) \\
&\quad - 2\Re\varphi_X(u)\Im\varphi_X(u)\big(\Im\varphi_X(2u) - 2\Re\varphi_X(u)\Im\varphi_X(u)\big)\Big] \\
&= \frac{1}{2|\varphi_X(u)|^4}\Big[|\varphi_X(u)|^2 - \underbrace{\Re\varphi_X(2u)\big(\Re^2\varphi_X(u) - \Im^2\varphi_X(u) + 2\Im\varphi_X(2u)\Re\varphi_X(u)\Im\varphi_X(u)}_{(*)}\Big].
\end{aligned}
$$

Replacing $(*)$ by (4.29) and $|\varphi_X(u)|$ by $\exp\{\Re\psi(u)\}$ yields

$$
\boldsymbol{\nu}'\boldsymbol{\Sigma}\boldsymbol{\nu} = \frac{1}{2\exp\{2\Re\psi(u)\}}\left(1-\exp\{\Re\psi(2u)\}\right),
$$

which is the same as (4.21) and $\kappa_{\Im} = \boldsymbol{\eta}'\boldsymbol{\Sigma}\boldsymbol{\eta}$.

$\square$

From Proposition 1.1(c), $\Re\psi_X(u) = \ln|\varphi_X(u)| \to 0$ for $|u| \to 0$ and the asymptotic variances given by (4.20) and (4.21) tend to 0 as $|u| \to 0$. Hereby, the smaller the argument u, and the greater the sample size n, the better the real and imaginary parts of empirical cumulant function estimate those of the cumulant function of X. However, as $\psi_n(0) = \psi_X(0) = 0$ then at $u = 0$ cumulant function holds no info about the various parameters of the distribution.

the asymptotic covariance matrix of empirical cumulant function in the framework of the least squares estimation has been discussed in Knight and Satchell (1997).

From Theorem 4.1 the asymptotic normality of real and imaginary parts of empirical cumulant function of stable distributions immediately follows.

Corollary 4.1. *Let $X \sim S(\alpha,\beta,\gamma,\delta;0)$ be a stable random variable with real and imaginary parts of its cumulant function given by (2.21) and (2.22), respectively, and ψ_n be as given by (1.31). Then, for every fixed $u \in \mathbb{R}$,*

$$
\sqrt{n}\big(\Re\psi_n(u) - \Re\psi_X(u)\big) \xrightarrow{\mathscr{D}} N_1\big(0,\kappa_{\Re}(u)\big),\ \textit{as } n\to\infty, \tag{4.30}
$$

$$
\sqrt{n}\big(\Im\psi_n(u) - \Im\psi_X(u)\big) \xrightarrow{\mathscr{D}} N_1\big(0,\kappa_{\Im}(u)\big),\ \textit{as } n\to\infty, \tag{4.31}
$$

where

$$
\kappa_{\Re}(u) = \exp\{2(\gamma u)^\alpha\}\left(1+\exp\{-(2\gamma u)^\alpha)\} - 2\exp\{-2(\gamma u)^\alpha\}\right)/2, \tag{4.32}
$$

and

$$\kappa_{\Im}(u) = \exp\{2(\gamma u)^{\alpha}\}\left(1-\exp\{-(2\gamma u)^{\alpha})\}\right)/2. \tag{4.33}$$

By Corollary 4.1 for $u \to 0$ the asymptotic variances tend to zero, $\kappa_{\Re}(u) \to 0$ and $\kappa_{\Im}(u) \to 0$, as $n \to \infty$.

4.3 Asymptotic Normality for ECuF Estimators

This section is based on Krutto (2018). The asymptotic normality of ECuF estimators[1] for the parameters of stable laws in 0–parametrization, given by Definition 4.2, is established. Note that the parameter space of stable laws has a boundary at $\alpha = 2$ and $\beta = \pm 1$ and the asymptotic normality does not make sense there. To obtain asymptotic distribution of estimators the standard assumption is that the true parameter is in the interior of the parameter space (e.g., Andrews (1997)). The distribution in the boundary case is not considered.

Theorem 4.2. *Let $\boldsymbol{b}$ be given by (3.11), $\boldsymbol{b}_n$ by (4.5), and functions $g_j(\boldsymbol{b})$, $j = 1,2,3,4$ represent parameters of $X \sim S(\alpha,\beta,\gamma,\delta;0)$ as in Theorem 4.2, with $\varphi_X(u) = \varphi_X(u|\alpha,\beta,\gamma,\delta)$. Let $g_j(\boldsymbol{b}_n)$, $j = 1,2,3,4$, be the ECuF estimators given by Definition 4.2. Then,*

(i) $g_j(\boldsymbol{b}_n)$ are strongly consistent estimators of $g_j(\boldsymbol{b})$, $j = 1,2,3,4$;

(ii) $g_1(\boldsymbol{b}_n)$ is asymptotically normal for $g_1(\boldsymbol{b}) \in (0,2)$, $g_2(\boldsymbol{b}_n)$ is asymptotically normal for $g_2(\boldsymbol{b})$, $g_3(\boldsymbol{b}_n)$ is asymptotically normal for $g_3(\boldsymbol{b}) \in (-1,1)$, and $g_4(\boldsymbol{b}_n)$ is asymptotically normal for $g_4(\boldsymbol{b})$,

$$\sqrt{n}\left(g_j(\boldsymbol{b}_n) - g_j(\boldsymbol{b})\right) \xrightarrow{\mathscr{D}} N_1(0, \upsilon_j(u_1,u_2)), \quad \text{as } n \to \infty \tag{4.34}$$

with $\upsilon_j(u_1,u_2) \equiv \boldsymbol{\xi}_j' \boldsymbol{\Lambda} \boldsymbol{\xi}_j$ where

$$\boldsymbol{\Lambda} = (\lambda_{ij}) \tag{4.35}$$

[1] For symmetric stable laws centered around zero ($\beta = \delta = 0$ and $\Im\varphi_X(u) = 0$), the asymptotic normality is provided in Press (1972).

is the 4×4 *covariance matrix having elements*

$$2\lambda_{11} = 1 + \Re\varphi(2u_1) - 2\Re^2\varphi(u_1), \tag{4.36}$$
$$2\lambda_{22} = 1 + \Re\varphi(2u_2) - 2\Re^2\varphi(u_2), \tag{4.37}$$
$$2\lambda_{33} = 1 - \Re\varphi(2u_1) - 2\Im^2\varphi(u_1), \tag{4.38}$$
$$2\lambda_{44} = 1 - \Re\varphi(2u_2) - 2\Im^2\varphi(u_2), \tag{4.39}$$
$$2\lambda_{12} = 2\lambda_{21} = \Re\varphi(u_1 - u_2) + \Re\varphi(u_1 + u_2) - 2\Re\varphi(u_1)\Re\varphi(u_2), \tag{4.40}$$
$$2\lambda_{13} = 2\lambda_{31} = \Im\varphi(2u_1) - 2\Re\varphi(u_1)\Im\varphi(u_1), \tag{4.41}$$
$$2\lambda_{14} = 2\lambda_{41} = \Im\varphi(u_1 + u_2) - \Im\varphi(u_1 - u_2) - 2\Re\varphi(u_1)\Im\varphi(u_2), \tag{4.42}$$
$$2\lambda_{23} = 2\lambda_{32} = \Im\varphi(u_1 + u_2) - \Im\varphi(u_2 - u_1) - 2\Re\varphi(u_2)\Im\varphi(u_1), \tag{4.43}$$
$$2\lambda_{24} = 2\lambda_{42} = \Im\varphi(2u_2) - 2\Re\varphi(u_2)\Im\varphi(u_2), \tag{4.44}$$
$$2\lambda_{34} = 2\lambda_{43} = \Re\varphi(u_1 - u_2) - \Re\varphi(u_1 + u_2) - 2\Im\varphi(u_1)\Im\varphi(u_2); \tag{4.45}$$

$\boldsymbol{\xi}_1$ *has elements*

$$\xi_{1i} = \frac{b_i}{|\varphi(u_1)|^2 \Re\psi(u_1)\ln(u_1/u_2)} \qquad \textit{for } i = 1, 3, \tag{4.46}$$
$$\xi_{1i} = \frac{-b_i}{|\varphi(u_2)|^2 \Re\psi(u_2)\ln(u_1/u_2)} \qquad \textit{for } i = 2, 4, \tag{4.47}$$

$\boldsymbol{\xi}_2$ *has elements*

$$\xi_{2i} = \frac{1}{|\varphi(u_1)|^2 \Re\psi(u_1)} \frac{-b_i(\ln u_2 + g_2(\boldsymbol{b}))}{\ln(\Re\psi(u_1)/\Re\psi(u_2))} \qquad \textit{for } i = 1, 3 \tag{4.48}$$
$$\xi_{2i} = \frac{1}{|\varphi(u_2)|^2 \Re\psi(u_2)} \frac{b_i(\ln u_1 + g_2(\boldsymbol{b}))}{\ln(\Re\psi(u_1)/\Re\psi(u_2))} \qquad \textit{for } i = 2, 4. \tag{4.49}$$

If $\alpha \neq 1$ *then* $\boldsymbol{\xi}_3$ *has elements*

$$\xi_{31} = \frac{g_3(\boldsymbol{b})}{|\varphi(u_1)|^2 \Re\psi(u_1)} \left(\frac{-b_3 u_2 \ln|\varphi(u_1)|}{C_1} - \frac{b_1 C_2}{\ln(u_1/u_2)} - \frac{b_1 u_2 u_1^{g_1(\boldsymbol{b})}}{C_3} \right), \tag{4.50}$$
$$\xi_{32} = \frac{g_3(\boldsymbol{b})}{|\varphi(u_2)|^2 \Re\psi(u_2)} \left(\frac{-b_4 u_1 \ln|\varphi(u_2)|}{C_1} + \frac{b_2 C_2}{\ln(u_1/u_2)} + \frac{b_2 u_1 u_2^{g_1(\boldsymbol{b})}}{C_3} \right), \tag{4.51}$$
$$\xi_{33} = \frac{g_3(\boldsymbol{b})}{|\varphi(u_1)|^2 \Re\psi(u_1)} \left(\frac{b_1 u_2 \ln|\varphi(u_1)|}{C_1} - \frac{b_3 C_2}{\ln(u_1/u_2)} - \frac{b_3 u_2 u_1^{g_1(\boldsymbol{b})}}{C_3} \right), \tag{4.52}$$
$$\xi_{34} = \frac{g_3(\boldsymbol{b})}{|\varphi(u_2)|^2 \Re\psi(u_2)} \left(\frac{b_2 u_1 \ln|\varphi(u_2)|}{C_1} + \frac{b_4 C_2}{\ln(u_1/u_2)} + \frac{b_4 u_1 u_2^{g_1(\boldsymbol{b})}}{C_3} \right), \tag{4.53}$$

and $\boldsymbol{\xi}_4$ has elements (with the elements of $\boldsymbol{\xi}_3$ given by (4.50)-(4.53))

$$\xi_{41} = \frac{u_2^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_1)|^2}\left(\quad b_1 u_1^{g_1(\boldsymbol{b})} C_1 - b_3\right) + \frac{C_1(\xi_{31}/g_3(\boldsymbol{b}) + \xi_{21} + \xi_{11}C_2)}{C_3 \exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}}, \tag{4.54}$$

$$\xi_{42} = \frac{u_1^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_2)|^2}\left(-b_2 u_2^{g_1(\boldsymbol{b})} C_1 + b_4\right) + \frac{C_1(\xi_{32}/g_3(\boldsymbol{b}) + \xi_{22} + \xi_{12}C_2)}{C_3 \exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}}, \tag{4.55}$$

$$\xi_{43} = \frac{u_2^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_1)|^2}\left(\quad b_3 u_1^{g_1(\boldsymbol{b})} C_1 + b_1\right) + \frac{C_1(\xi_{33}/g_3(\boldsymbol{b}) + \xi_{23} + \xi_{13}C_2)}{C_3 \exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}}, \tag{4.56}$$

$$\xi_{44} = \frac{u_1^{g_1(\boldsymbol{b})}}{C_3^2|\varphi(u_2)|^2}\left(-b_4 u_2^{g_1(\boldsymbol{b})} C_1 - b_2\right) + \frac{C_1(\xi_{34}/g_3(\boldsymbol{b}) + \xi_{24} + \xi_{14}C_2)}{C_3 \exp\{g_2(\boldsymbol{b})(g_1(\boldsymbol{b})-1)\}}, \tag{4.57}$$

where

$$\begin{aligned}
C_1 &= u_2 \Im\psi(u_1) - u_1 \Im\psi(u_2),\\
C_2 &= \pi/2[\cot(\pi g_1(\boldsymbol{b})/2) + \tan(\pi g_1(\boldsymbol{b})/2)]\\
C_3 &= u_2 \exp\{g_1(\boldsymbol{b}) \ln u_1\} - u_1 \exp\{g_1(\boldsymbol{b}) \ln u_2\}
\end{aligned}$$

If $\alpha = 1$ then $\boldsymbol{\xi}_3$ has elements

$$\xi_{31} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{21} - \frac{b_3}{u_1|\varphi(u_1)|^2 \ln(u_2/u_1)}\right), \tag{4.58}$$

$$\xi_{32} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{22} + \frac{b_4}{u_2|\varphi(u_2)|^2 \ln(u_2/u_1)}\right), \tag{4.59}$$

$$\xi_{33} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{23} + \frac{b_1}{u_1|\varphi(u_1)|^2 \ln(u_2/u_1)}\right), \tag{4.60}$$

$$\xi_{34} = \frac{\pi}{2\exp\{g_2(\boldsymbol{b})\}}\left(\xi_{24} - \frac{b_2}{u_2|\varphi(u_2)|^2 \ln(u_2/u_1)}\right), \tag{4.61}$$

and $\boldsymbol{\xi}_3$ has elements (with the elements of $\boldsymbol{\xi}_3$ given by (4.58)-(4.61))

$$\xi_{41} = \frac{-b_3 \ln u_2}{u_1 \ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{31} g_2(\boldsymbol{b}) \exp\{g_2(\boldsymbol{b})\} + \xi_{31} g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)), \tag{4.62}$$

$$\xi_{42} = \frac{b_4 \ln u_1}{u_2 \ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{32} g_2(\boldsymbol{b}) \exp\{g_2(\boldsymbol{b})\} + \xi_{32} g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)), \tag{4.63}$$

$$\xi_{43} = \frac{b_1 \ln u_2}{u_1 \ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{33} g_2(\boldsymbol{b}) \exp\{g_2(\boldsymbol{b})\} + \xi_{33} g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)), \tag{4.64}$$

$$\xi_{44} = \frac{-b_2 \ln u_1}{u_2 \ln(u_2/u_1)} + \frac{2}{\pi}(\xi_{34} g_2(\boldsymbol{b}) \exp\{g_2(\boldsymbol{b})\} + \xi_{34} g_4(\boldsymbol{b})(g_2(\boldsymbol{b})+1)). \tag{4.65}$$

Proof. Based on (1.9), the function atan2, which is involved in g_j, $j = 3,4$, has a discontinuity at $\Re\varphi_X(u) = 0$ and $\Im\varphi_X(u) = 0$, i.e., $|\varphi_X(u)| = 0$. Based on Proposition 1.1(d) $|\varphi_X(u)| \to 0$ as $|u| \to \infty$. However, in the theorem assumption it is said that $u_1, u_2 \in \mathbb{R}$ are fixed, $u_1, u_2 < \infty$, and it follows that under the theorem assumptions atan2 and g_j, $j = 3,4$ may assumed as continuous. In what follows, functions g_j, $j = 1,2,3,4$ are assumed to be continuous.

(i) *Consistency*. Based on relation (1.22), for every pair of $u_1 > 0, u_2 > 0$, with $u_1 \neq u_2$,

$$\boldsymbol{b}_n \stackrel{a.s.}{\to} \boldsymbol{b}, \text{ as } n \to \infty. \tag{4.66}$$

By the continuous mapping theorem (e.g., van der Vaart (1998, Theorem 2.3)),

$$g_j(\boldsymbol{b}_n) \stackrel{a.s.}{\to} g_j(\boldsymbol{b}), \text{ as } n \to \infty, \tag{4.67}$$

and the estimators $g_j(\boldsymbol{b}_n)$ are consistent for $g_j(\boldsymbol{b})$, $j = 1, \ldots, 4$.

(ii) *Asymptotic normality*. The quantities $\Re\varphi_n(u)$, $\Im\varphi_n(u)$ are sample means of i.i.d. random variables with finite variance and $\mathbf{E}\,\boldsymbol{b}_n = \boldsymbol{b}$. By Theorem 1.4,

$$\sqrt{n}(\boldsymbol{b}_n - \boldsymbol{b}) \stackrel{\mathscr{D}}{\to} N_4(\mathbf{0}, \boldsymbol{\Lambda}), \text{ as } n \to \infty, \tag{4.68}$$

where $\boldsymbol{\Lambda}$ is the 4×4 covariance matrix with elements' structure following from (1.26)–(1.28) and yielding (4.36)–(4.45). Applying Kollo and von Rosen (2005, Theorem 3.1.3) or Anderson (2003, p. 132-133), for $j = 1,2,3,4$,

$$\sqrt{n}\,(g_j(\boldsymbol{b}_n) - g_j(\boldsymbol{b})) \stackrel{\mathscr{D}}{\to} N_1(0, \boldsymbol{\xi}_j' \boldsymbol{\Lambda} \boldsymbol{\xi}_j), \quad \text{as } n \to \infty \tag{4.69}$$

where

$$\boldsymbol{\xi}_j = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} g_j(\boldsymbol{x}) \bigg|_{\boldsymbol{x}=\boldsymbol{b}} \neq 0 \tag{4.70}$$

are matrix derivatives (Kollo and von Rosen (2005, Definition 1.4.1)). The elements of $\boldsymbol{\xi}_j$, $j = 1,2,3,4$ in (4.46) - (4.65) easily follow from the matrix derivatives of $g_j(\boldsymbol{b})$, $j = 1,2,3,4$, given by (3.12) through (3.15b), and elementary complex analysis (in a similar manner to the proof of Theorem 4.1).

□

Due to the complexity of the ECuF estimators it is difficult to determine whether or not they are biased. For example, from Definition 4.1 and Definition 4.2, $\alpha_n = \alpha_n(u_1, u_2) = g_1(b_n)$, and

$$\mathbf{E}\,\alpha_n = \mathbf{E}\,g_1(b_n) = \frac{1}{\ln\frac{u_1}{u_2}}\,\mathbf{E}\left\{\ln\frac{\ln\left[(\frac{1}{n}\sum_{j=1}^n \cos u_1 Y_j)^2 + (\frac{1}{n}\sum_{j=1}^n \sin u_1 Y_j)^2\right]}{\ln\left[(\frac{1}{n}\sum_{j=1}^n \cos u_2 Y_j)^2 + (\frac{1}{n}\sum_{j=1}^n \sin u_2 Y_j)^2\right]}\right\}. \tag{4.71}$$

Corollary 4.2. *The asymptotic normality for the symmetric stable laws centred around zero is a special case of Theorem 4.2. In this case* $\beta = 0 = \delta = 0$, $\Im\varphi_X = 0$ *and* $\varphi_X = \Re\varphi_X$. *Therefore*

$$\boldsymbol{b} = (\varphi_X(u_1), \varphi_X(u_2), 0, 0)',$$

and in the covariance matrix $\boldsymbol{\Lambda} = (\lambda_{kl})$, *given by (4.36)–(4.45), all the terms of* $\Im\varphi_X(u) = 0$ *are omitted. That is, Theorem 4.2 holds with* $\boldsymbol{\Lambda} = (\lambda_{kl})$ *having elements*

$$\begin{aligned}
2\lambda_{11} &= 1 + \Re\varphi(2u_1) - 2\Re^2\varphi(u_1),\\
2\lambda_{22} &= 1 + \Re\varphi(2u_2) - 2\Re^2\varphi(u_2),\\
2\lambda_{33} &= 1 - \Re\varphi(2u_1),\\
2\lambda_{44} &= 1 - \Re\varphi(2u_2),\\
2\lambda_{12} &= 2\lambda_{21} = \Re\varphi(u_1 - u_2) + \Re\varphi(u_1 + u_2) - 2\Re\varphi(u_1)\Re\varphi(u_2),\\
2\lambda_{34} &= 2\lambda_{43} = \Re\varphi(u_1 - u_2) - \Re\varphi(u_1 + u_2)\\
\lambda_{13} &= \lambda_{31} = \lambda_{14} = \lambda_{41} = \lambda_{23} = \lambda_{32} = \lambda_{24} = \lambda_{42} = 0,
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\xi}_1 &= (\xi_{11}, \xi_{12}, 0, 0)'\\
\boldsymbol{\xi}_2 &= (\xi_{21}, \xi_{22}, 0, 0)'\\
\boldsymbol{\xi}_j &= \mathbf{0}, j = 3, 4.
\end{aligned}$$

Chapter 5

Monte–Carlo Simulations for ECuF Estimators

In this chapter the behaviour of ECuF estimators $\alpha_n = \alpha_n(u_1,u_2)$, $\beta_n = \beta_n(u_1,u_2)$, $\gamma_n = \gamma_n(u_1,u_2)$ and $\delta_n = \delta_n(u_1,u_2)$ at various selections of (u_1,u_2) is studied. By Definition 4.1 (and equivalently by Definition 4.2) any pair of arguments $u_1 > 0, u_2 > 0, u_1 \neq u_2$ can be used. Recall that Press (1972) estimators were defined at two pairs of arguments, (u_1,u_2) and (u_3,u_4), while there are not many studies that apply Press (1972) estimators. For example, Fan (2006) proposed a pair of $u_1 = 0.5$ and $u_2 = 1.5$ for estimating the tail index α of standard symmetric stable laws around zero (that is, the pair (u_3,u_4) is not necessary). In Section 5.1 the accuracy of ECuF estimators is assessed through Monte–Carlo simulations at various pairs of $u_1 > 0, u_2 > 0$, $u_1 \neq u_2$ and in addition the accuracy and the asymptotic variance of $\alpha_n = \alpha_n(u_1,u_2)$ at $(u_1,u_2) \in (0,1] \times (0,1]$ is studied. All the computation and graphs are made by `R` (R Core Team (2018)) while simulations are made by packages `stabledist` (Würtz and Mächler (2016)) or `STABLE`® (Robust Analysis Inc. (2017)).

5.1 Estimating $S(\alpha,\beta;1)$ via ECuF Estimators at Selection of Arguments

In this section ECuF and reduced values' ECuF estimates for the parameters of $S(\alpha,\beta,\gamma,\delta;1)$, given by (2.14), are obtained. Without the loss of generality, the standard stable laws with $\delta = 0$ and $\gamma = 1$ are studied. By the reflection property, e.g., Uchaikin and Zolotarev (1999, Property (2), p. 99), Nolan (2018b, Proposition 1.11, p. 12), only non-negative values of β are used, $\beta \in [0,1]$. The realizations of ECuF estimators are denoted by $\hat{\alpha}_n$, $\hat{\beta}_n$, $\hat{\gamma}_n$, $\hat{\delta}_n$, respectively. The quality of estimates is assessed by the mean squared errors (MSE),

$$\mathrm{MSE}(\hat{\theta}_n) = \tfrac{1}{K}\textstyle\sum_{k=1}^{K}\left(\theta - \hat{\theta}_n(k)\right)^2,$$

where K is the number of replications, θ is the true parameter and $\hat{\theta}_n(k)$ is the estimate of the parameter from the k^{th} sample, $k = 1, 2, \ldots, K$. The realizations of mean squared errors are denoted by MSE($\hat{\alpha}_n$), MSE($\hat{\beta}_n$), MSE($\hat{\gamma}_n$), and MSE($\hat{\delta}_n$).

5.1.1 Estimating $S(\alpha, \beta; 1)$ via ECuF Estimators at Various (u_1, u_2)

For the empirical search on the arguments u_1 and u_2 various arbitrary pairs of of $u_1 > 0, u_2 > 0$, $u_1 \neq u_2$ are chosen. The pairs are presented in Table 5.1.

TABLE 5.1: A selection of (u_1, u_2) for ECuF estimators.

(0.03, 0.09)
(0.03, 0.9)
(0.03, 9)
(0.03, 90)
(0.3, 0.09)
(0.3, 0.9)
(0.3, 9)
(0.3, 90)
(3, 0.09)
(3, 0.9)
(3, 9)
(3, 90)

The ECuF estimates and corresponding mean squared errors are obtained at all pairs given by Table 5.1 for $K = 200$ replicates from $S(\alpha, \beta; 1)$, with $\alpha = 0.5, 1.5$, each with size $n = 10^5$. At every pair (u_1, u_2) in Table 5.1 the MSEs of reduced values' ECuF estimates[1] are presented in Appendix D. It follows, that the pair $(u_1 = 0.03, u_2 = 0.09)$ has the smallest values of MSEs of the estimates for parameters of $S(1.5, 0.1; 1)$ as well as it gives small values of MSEs (less than 0.004 at least) of estimates for parameters of other considered stable distributions. Based on simulation results in Appendix D the following remark is given.

Remark 3. *Based on Appendix D the pair $(u_1 = 0.03, u_2 = 0.09)$ is proposed as empirically optimal for obtaining the ECuF estimators.*

5.1.2 Estimating $S(\alpha, \beta; 1)$ via ECuF Estimators at $(0.03, 0.09)$

To assess the quality of ECuF estimators at $u_1 = 0.03, u_2 = 0.09$ more generally the MSEs of the estimates are obtained for $K = 200$ replicates from $S(\alpha, \beta; 1)$ with various values of α, that is, $\alpha \neq 1$, α is in the neighbourhood of 1, and $\alpha = 1$. For all α various values of β were studied. The simulations were carried out for various sample sizes, $n = 10^2, 10^3, 10^4, 10^5$, while in this thesis the

[1]MSEs of ECuF estimates turned out bigger than those of reduced values' ECuF estimates. For the sake of space the MSEs of ECuF estimates are not presented.

MSEs are presented for 10^5 only. In conclusion, the MSEs of ECuF estimators remarkably decrease while sample size increases (see also Section 7.2).

The results for the tail index $\alpha \neq 1$: In Appendix E the MSEs of ECuF and reduced values' ECuF (denoted by RVECuF) estimates at $u_1 = 0.03, u_2 = 0.09$ of $K = 200$ replicates from $S(\alpha,\beta;1)$ with $\alpha = 0.25$, 0.5, 0.75, 1.25, 1.5, 1.75 and $\beta = 0.1, 0.25, 0.5, 0.75, 1$, each with size $n = 10^5$, are given. Based on simulation results, presented in Appendix E, the MSEs of reduced values' ECuF estimates at $u_1 = 0.03, u_2 = 0.09$ for the parameters α,β,γ turned out of the order 10^{-6} while for the parameter δ of the order from 10^{-6} to 10^{-2}. In other words, the pair $u_1 = 0.03, u_2 = 0.09$ gives similar MSEs for all considered stable laws. When comparing the MSEs of the ECuF estimates with the reduced values' ECuF estimates, in Appendix E the reduced values' ECuF estimates have smaller values of MSEs for stable laws which do not tend to symmetry (with median close to 0). Indeed, as stated in Definition 4.2, the reduced values' EcuF estimators should be used only for samples with median more than 1 (for the current empirical study that requirement was omitted). Based on simulation results, presented in Appendix E, the MSEs of reduced values' ECuF estimates for stable laws with $\alpha < 1.75$ and $\beta > 0.1$ turned out to be smaller than those of the ECuF estimates. For the cases with $\alpha \geqslant 1.75$ (i.e., stabe distribution is tending to symmetry for any β) the MSEs of reduced values' ECuF estimates turned out bigger than those of the ECuF estimates. Note that for symmetric cases median is less than 1 and dividing by median yields increasing the scale of data which, based on Section 6.2, causes the increase of MSEs. Note that in Definition 4.3 it is assumed that data median is more than 1. In Remark 2 a modification of reduced values' ECuF estimates was introduced where the assumption of median less than 1 can be omitted.

The results for the tail index α close to 1: In Table 5.2 the MSEs of ECuF estimates at $u_1 = 0.03, u_2 = 0.09$ of $K = 200$ replicates from $S(\alpha,\beta;1)$, each with size $n = 10^5$, in the neighbourhood of $\alpha = 1$ with $\beta = 0.1, 1$ are given. For comparison, the MSEs of estimates for β are obtained both by formulas (3.3) and (3.5), and for δ by formulas (3.4) and (3.6). In the neighbourhood of $\alpha = 1$, based on Table 5.2, the ECuF estimates fail (based on MSEs) for the location parameter δ while the MSEs of estimates for other parameters are quite small. For the asymmetry parameter β, even when α is very close to 1, the MSEs of estimates by formula (3.3) are not smaller than estimates by formula (3.5). In addition to the pair $(u_1 = 0.03, u_2 = 0.09)$ the ECuF estimates were obtained at all pairs (u_1, u_2) in Table 5.1: some of the pairs gave smaller MSEs of estimates for the location parameter δ but muchbigger for the other parameters.

The results for the tail index $\alpha = 1$: In Table 5.3 the MSEs of ECuF estimates at $(u_1 = 0.03, u_2 = 0.09)$ of $K = 200$ replicates from $S(1,\beta;1)$ with $\beta = 0.1, 0.25, 0.5, 0.75, 1$, each with size $n = 10^5$, are given. For comparison, the MSEs of estimates for β are obtained both by formulas (3.3) and (3.5), and estimates for δ both by formulas (3.4) and (3.6).

TABLE 5.2: MSEs of ECuF estimates for the parameters of $S(\alpha, \beta; 1)$.

			by **(3.3)**	by **(3.5)**		by **(3.4)**	by **(3.6)**
α	β	**MSE** $(\hat{\alpha}_n)$	**MSE** $(\hat{\beta}_n)$	**MSE** $(\hat{\beta}_n)$	**MSE** $(\hat{\gamma}_n)$	**MSE** $(\hat{\delta}_n)$	**MSE** $(\hat{\delta}_n)$
0.95	0.1	0.0000	0.0001	0.0003	0.0000	0.0458	1.6567
0.95	1	0.0003	0.0008	0.0688	0.0023	9×10^1	2×10^2
0.96	0.1	0.0000	0.0001	0.0002	0.0001	0.1537	2.6231
0.96	1	0.0005	0.0012	0.0496	0.0044	5×10^4	3×10^2
0.98	0.1	0.0001	0.0002	0.0003	0.0003	7×10^1	1×10^1
0.98	1	0.0009	0.0027	0.0309	0.0135	4×10^5	1×10^3
0.99	0.1	0.0002	0.0005	0.0005	0.0007	1×10^6	4×10^1
0.99	1	0.0017	0.0045	0.0553	0.0333	2×10^4	4×10^3
1.01	0.1	0.0002	0.0004	0.0004	0.0007	2×10^5	4×10^1
1.01	1	0.0016	0.0060	0.0383	0.0278	4×10^6	4×10^3
1.02	0.1	0.0001	0.0002	0.0002	0.0002	1×10^3	1×10^1
1.02	1	0.0009	0.0027	0.0202	0.0108	4×10^4	9×10^2
1.04	0.1	0.0001	0.0001	0.0002	0.0001	0.1101	2.4367
1.04	1	0.0006	0.0015	0.0340	0.0046	2×10^4	2×10^2
1.05	0.1	0.0000	0.0001	0.0001	0.0000	0.0458	1.5814
1.05	1	0.0005	0.0013	0.0392	0.0032	3×10^2	1×10^2

TABLE 5.3: MSEs of ECuF estimates for the parameters of $S(\alpha = 1, \beta; 1)$.

	by **(3.1)**	by **(3.3)**	by **(3.5)**	by **(3.2)**	by **(3.4)**	by **(3.6)**
β	**MSE** $(\hat{\alpha}_n)$	**MSE** $(\hat{\beta}_n)$	**MSE** $(\hat{\beta}_n)$	**MSE** $(\hat{\gamma}_n)$	**MSE** $(\hat{\delta}_n)$	**MSE** $(\hat{\delta}_n)$
0.1	0.00018	1×10^1	0.00021	0.0013	5×10^1	0.0012
0.25	0.00018	2×10^1	0.00032	0.0021	3×10^1	0.0072
0.5	0.00022	2×10^1	0.00049	0.0034	2×10^1	0.0043
0.75	0.00021	2×10^1	0.00051	0.044	1×10^1	0.0093
1	0.00019	2×10^1	0.00053	0.0385	5×10^1	0.0092

In Table 5.3, as expected, the MSEs of the ECuF estimates for β and δ by formula (3.6) are smaller then those by formula (3.4).

The MSEs of ECuF estimates at $u_1 = 0.03, u_2 = 0.09$ of $K = 200$ replicates from $S(\alpha, \beta; 1)$ for $\alpha \downarrow 0$, $\alpha \uparrow 2$ with $\beta = 0.1, 0.25, 0.5, 0.75, 1$, each with size $n = 10^5$, are not presented here while the results are formulated in following remarks.

Remark 4. *In the case of $\alpha \downarrow 0$, stable distributions are very condensed (and scale factor γ has not much influence on the shape of the distribution and hence may be difficult to estimate). For $\alpha < 0.2$ the MSEs of the ECuF estimates of α, δ turned out of good quality (order of 10^{-6} to 10^{-3}) while those of γ and β have lower quality.*

Remark 5. *For the case of $\alpha \uparrow 2$ stable distributions get close to the normal distributions (and the parameter β loses its effect and may be difficult to estimate). For $\alpha > 1.8$ the MSEs of the ECuF estimates for α, γ, δ turned out of good quality (order of 10^{-6} to 10^{-3}) while those of β have lower quality.*

5.2 Estimating α via ECuF Estimator at $(u_1,u_2) \in (0,1] \times (0,1]$.

In this section the ECuF estimator for α of $S(\alpha,\beta;0)$ at the pairs of arguments $u_1 \in (0,1]$ and $u_2 \in (0,1]$, with $u_1 \neq u_2$ and step size 0.01, are studied. The interest is in exploring an empirical evidence of the existence of a good area for the selection of $(u_1,u_2) \in (0,1] \times (0,1]$. Also, the pairs at which the non-admissible estimates occur are aimed to explore. Another interest is to empirically study the possible bias of $\alpha_n = \alpha_n(u_1,u_2)$ by comparing the means of absolute errors (MAE) and the means of the asymptotic standard deviations (MASD) of $\hat{\alpha}_n = \hat{\alpha}_n(u_1,u_2)$. More precisely, in Section 5.2.1 the means of $\hat{\alpha}_n(u_1,u_2)$ are obtained on the bases of $K = 100$ replicates from $S(1.5,0.5;0)$ and $S(0.5,0.5;0)$, each with various sample sizes such as $n = 50$, 200, 500, 3000, and for a single replicate with $n = 1000$, and in Section the MAEs and MASDs of $\hat{\alpha}_n(u_1,u_2)$ are obtained on the bases of $K = 1000$ replicates from $S(1.5,0.5;0)$ and $S(0.5,0.5;0)$, each with a large sample size such as $n = 10000$. ECuF estimators are not defined for $u_1 = u_2$ and in all illustrative figures the diagonal of $(0,1] \times (0,1]$ has no values of $\hat{\alpha}_n(u_1,u_2)$. Also, due to the properties of cumulant function ECuF estimates behave symmetrically with respect to the diagonal of $(0,1] \times (0,1]$.

5.2.1 Estimating α from Various Size Samples

In this section the means of ECuF estimates $\hat{\alpha}_n(u_1,u_2)$ are obtained on the bases of $K = 100$ replicates from $S(\alpha = 1.5, \beta = 0.5;0)$ and $S(\alpha = 0.5, \beta = 0.5;0)$, each with various sample sizes such as $n = 50$, 200, 500, 3000. In Figure 5.1 the means of $\hat{\alpha}_n$ on the bases of $K = 100$ replicates, each with sample size $n = 50$, are presented.

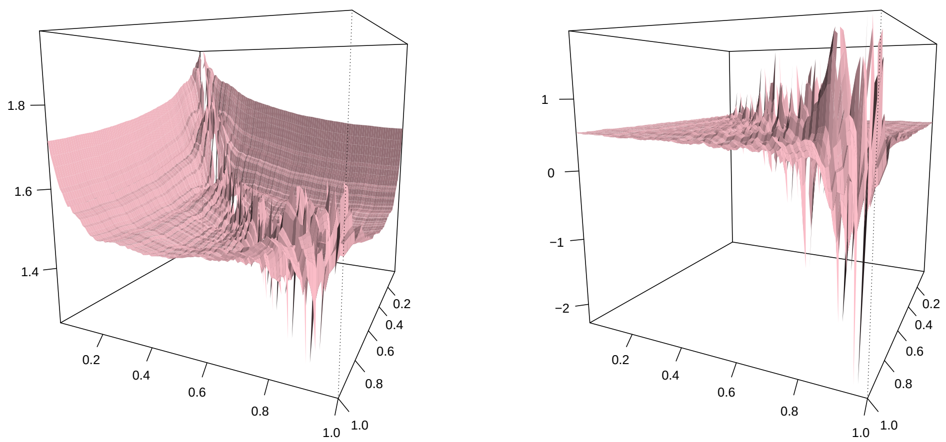

FIGURE 5.1: Means of ECuF estimators $\hat{\alpha}_n$ for $\alpha = 1.5$ (on left) and $\alpha = 0.5$ (on right) of $K = 100$ replicates with $n = 50$.

The means of $\hat{\alpha}_n$ at $(u_1,u_2) \in (0,1] \times (0,1]$ on left and right of Figure 5.1 vary quite a lot from the actual values $\alpha = 1.5$ and $\alpha = 0.5$, respectively. ECuF estimates $\hat{\alpha}_n$, smaller than actual values $\alpha = 1.5$ and $\alpha = 0.5$ relate to the underestimating and the estimates bigger than the actual values to the overestimating. ECuF estimates outside of the parameter interval $\alpha \in (0,2]$ are non-admissible for α. It could be dealt by truncating the values of estimators, see Chapter 7. However, in this chapter non-truncated estimates are studied. For $\alpha = 1.5$ (on left) the means of $\hat{\alpha}_n$ have 1.36 as the minimum, 1.97 as the maximum and 1.5 as the mean and median, and for $\alpha = 0.5$ (on right) the means of $\hat{\alpha}_n$ have -2.29 as the minimum, 1.90 as the maximum and 0.5 as the mean and median.

The pairs (u_1,u_2) at which the MAEs of $\hat{\alpha}_n$ are less than 1% of the true parameter $\alpha = 1.5$ and $\alpha 0.5$ are presented on the left and right of Figure 5.2, respectively. Note that

$$\mathrm{MAE}(\theta_n) = \frac{1}{K}\sum_{k=1}^{K} \left|\hat{\theta}_n(k) - \theta\right| \tag{5.1}$$

where K is the number of replications, θ is the true value and $\hat{\theta}_n(k)$ is the estimate for k^{th} sample, $k = 1,2,\ldots,K$.

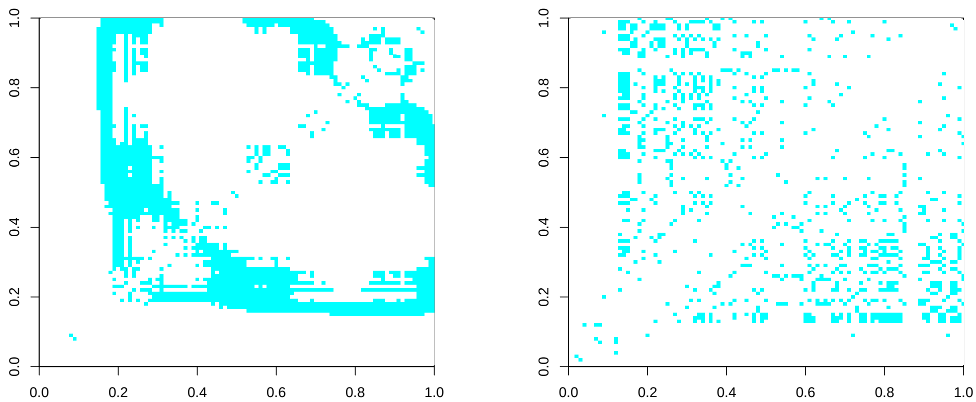

FIGURE 5.2: For $K = 100$ replicates with $n = 50$ the pairs (u_1,u_2) at which the MAEs of $\hat{\alpha}_n$ are less than 1% of $\alpha = 1.5$ (on left) and of $\alpha = 0.5$ (on right).

The locations of the pairs at which ECuF estimates have MAEs less than 1% of the true parameter $\alpha = 1.5$ and $\alpha = 0.5$, respectively, are different for $\alpha = 1.5$ and for $\alpha = 0.5$.

The means of ECuF estimates $\hat{\alpha}_n(u_1,u_2)$ at $(u_1,u_2) \in (0,1] \times (0,1]$ on the basis of $K = 100$ replicates from $S(\alpha = 1.5, \beta = 0.5; 0)$ and $S(\alpha = 0.5, \beta = 0.5; 0)$ with $n = 200$ and $n = 500$ are presented in Figures 5.3 and 5.4, respectively. Comparing Figures 5.3 and 5.4 with Figure 5.1 then for larger samples the range of means of $\hat{\alpha}_n(u_1,u_2)$ turn out to be smaller. For example, in Figure 5.4 the means of ECuF estimates for $\alpha = 1.5$ (on left) have 1.39 as the minimum, 1.72 as the maximum and 1.5 as the mean and median while ECuF estimates for $\alpha = 0.5$ (on right) have -0.02 as the minimum, 1.11 as the maximum and 0.5 as the mean and median, approximately.

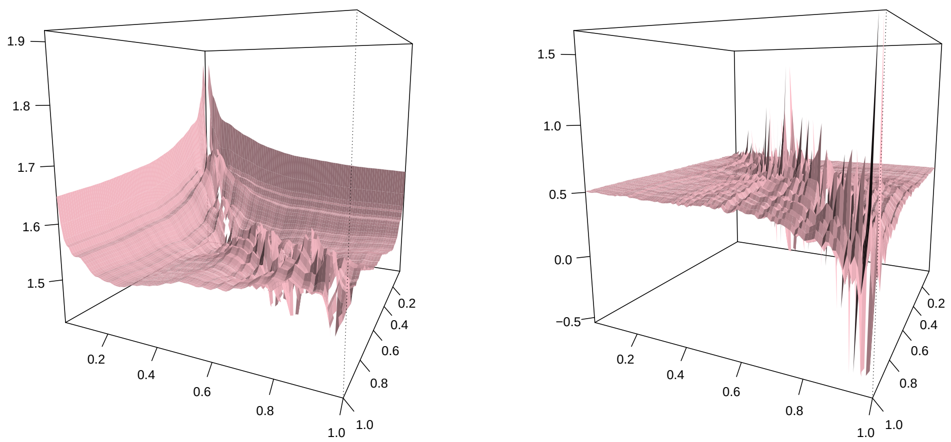

FIGURE 5.3: Means of ECuF estimates of $\alpha = 1.5$ (on left) and $\alpha = 0.5$ (on right) of $K = 100$ replicates with $n = 200$.

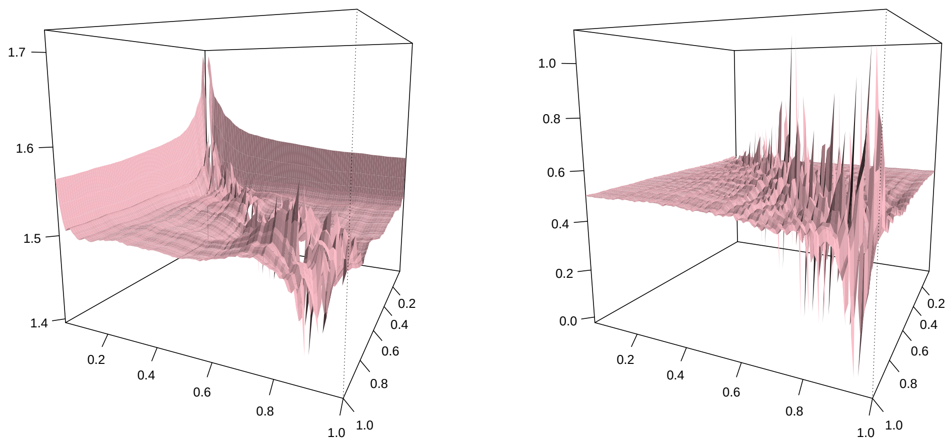

FIGURE 5.4: Means of ECuF estimates of $\alpha = 1.5$ (on left) and $\alpha = 0.5$ (on right) of $K = 100$ replicates with $n = 500$.

The means of ECuF estimates $\hat{\alpha}_n$ at $(u_1, u_2) \in (0,1] \times (0,1]$ on the basis of $K = 100$ replicates with $n = 3000$ are presented in Figure 5.5. As expected, in Figure 5.5 the means of ECuF estimates

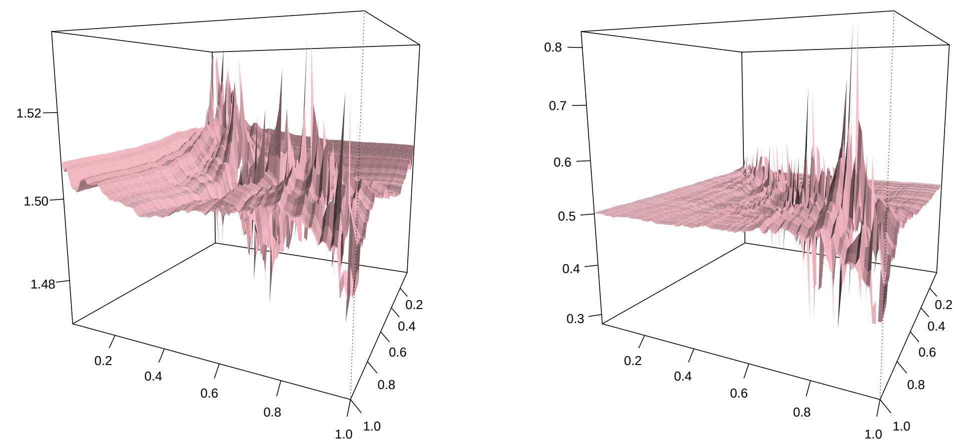

FIGURE 5.5: Means of ECuF estimates for $\alpha = 1.5$ (on left) and for $\alpha = 0.5$ (on right) of $K = 100$ replicates with $n = 3000$.

fluctuate less comparing to Figures 5.1 – 5.4. Indeed, in Figure 5.5 the means of ECuF estimates for $\alpha = 1.5$ (on left) have 1.46 as the minimum, 1.54 as the maximum and 1.5 as the mean and median while ECuF estimates for $\alpha = 0.5$ (on right) have 0.28 as the minimum, 0.83 as the maximum and 0.5 as the mean and median, approximately. The pairs (u_1, u_2) at which the MAEs of $\hat{\alpha}_n$ are less than 1%of the true parameter $\alpha = 1.5$ and $\alpha 0.5$ are presented on the left and right of Figure 5.6, respectively.

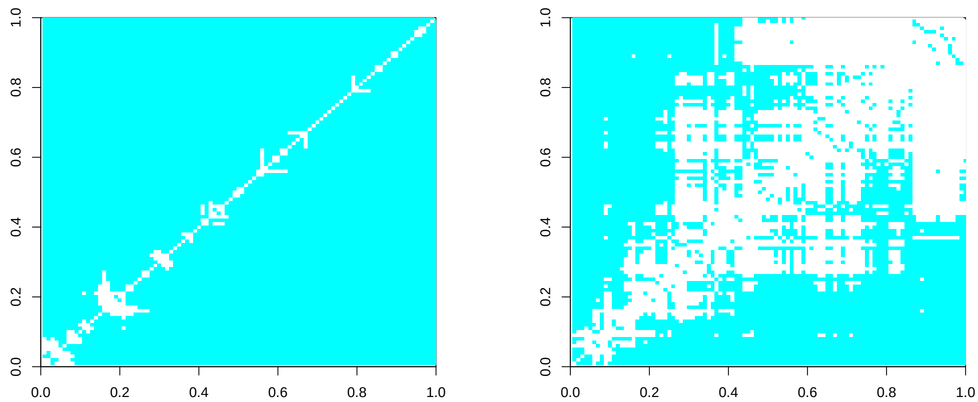

FIGURE 5.6: For $K = 100$ replicates with $n = 3000$ the pairs (u_1, u_2) at which the MAE's of $\hat{\alpha}_n$ are less than 1% of $\alpha = 1.5$ (on left) and of $\alpha = 0.5$ (on right).

As expected, for $\alpha = 1.5$ there are more pairs (u_1, u_2) leading to small MAEs than for $\alpha = 0.5$. Nevertheless, in Figure 5.6 there are many pairs $(u_1, u_2) \in (0,1] \times (0,1]$ at which the MAEs are less than 1% both $\alpha = 1.5$ (on left of Figure 5.6) and $\alpha = 0.5$ (on right of Figure 5.6).

In closing, ECuF estimates $\hat{\alpha}_n(u_1, u_2)$ for a single ($K = 1$) replicate from $S(\alpha = 1.5, \beta = 0.5; 0)$ and $S(\alpha = 0.5, \beta = 0.5; 0)$, both with sample size $n = 1000$, are obtained. It follows that ECuF estimates $\hat{\alpha}_n(u_1, u_2)$ at $(u_1, u_2) \in (0,1] \times (0,1]$ for $\alpha = 1.5$ have 1.33 as the minimum, 1.98 as the maximum and 1.54 as the median and the mean while for $\alpha = 0.5$ have -3.38 as the minimum, 5.29 as the maximum, 0.493 as the median and 0.477 the mean, approximately. As it is expected, comparing to the MAEs of $K = 100$ replicates the absolute erros of $K = 1$ have much bigger variation and the absolute errors less than 1% turn out at considerably less pairs (u_1, u_2). The pairs (u_1, u_2) at which the absolute errors of $\hat{\alpha}_n$ are less than 1% are presented in Figure 5.7 with $\alpha = 1.5$ on left and $\alpha = 0.5$ on right.

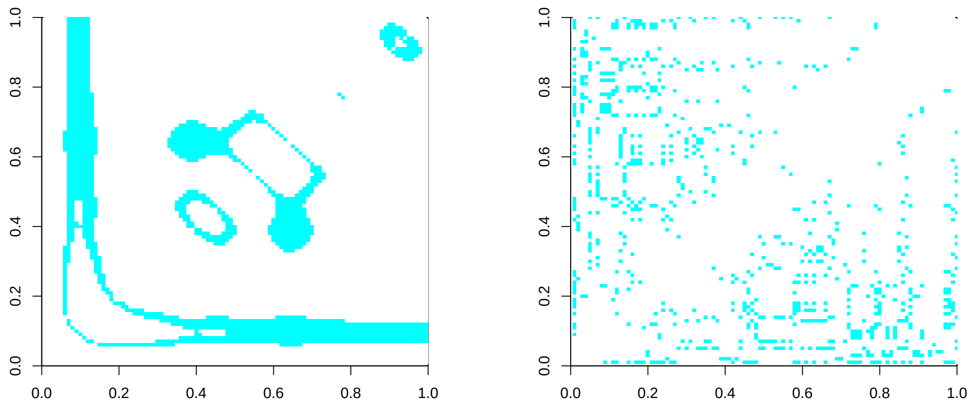

FIGURE 5.7: For $K = 1$ replicates with $n = 1000$ the pairs of (u_1, u_2) at which the MAEs of $\hat{\alpha}_n$ for $\alpha = 1.5$ (on left) and for $\alpha = 0.5$ (on right) are less than 1%.

In Figure 5.7 the absolute errors less than 1% occur at considerably less pairs (u_1, u_2) than in Figure 5.2 and in Figure 5.6. Similarly to the results in Figure 5.2 and in Figure 5.6 the distribution (location) of the pairs (u_1, u_2) at which the absolute errors of ECuF estimates are less than 1% of the true values $\alpha = 1.5$ (on the left of Figure 5.7) and $\alpha = 0.5$ (on the right of Figure 5.7) is not similar.

In conclusion, on the basis of the $K = 100$ and $K = 1$ replicates from stable laws $S(\alpha = 1.5, \beta = 0.5; 0)$ and $S(\alpha = 0.5, \beta = 0.5; 0)$ there is no evidence of the existence of a consistently good area for selecting the arguments (u_1, u_2) in $(0,1] \times (0,1]$ for the ECuF estimator $\alpha_n(u_1, u_2)$.

5.2.2 Estimating α from a Large Sample

Due to the complexity of the ECuF estimators it is difficult to determine whether or not they are biased (see (4.71) for $\mathbf{E}\,\alpha_n$). The aim of this section is to estimate empirically the bias or unbiasedness of the ECuF estimators α_n. For that, the means of absolute errors (MAE) and the means of the asymptotic standard deviation (MASD) of $\hat{\alpha}_n(u_1,u_2)$ on the bases of $K = 1000$ replicates from stable distributions $S(\alpha = 1.5, \beta = 0.5; 0)$ and $S(\alpha = 0.5, \beta = 0.5; 0)$, each with a large sample $n = 10000$, are compared. The MAEs are given by (5.1). The asymptotic standard deviation of ECuF estimator $\alpha_n(u_1,u_2) = g_1(\boldsymbol{b}_n)$ is obtained from its asymptotic variance $\upsilon_1(u_1,u_2) = \boldsymbol{\xi}_1'\boldsymbol{\Lambda}\boldsymbol{\xi}_1$ with $\boldsymbol{\xi}_1$ and $\boldsymbol{\Lambda}$ given by Theorem 4.2. Calculations for $\upsilon_1(u_1,u_2)$ are made by R (R Core Team (2018)), that is, $\upsilon_1(u_1,u_2)$ is not analytically derived.

The results for the tail index $\alpha = 1.5$. In Table 5.4 the summary of the MAEs and MASDs of $\hat{\alpha}_n(u_1,u_2)$ at $(u_1,u_2) \in (0,1] \times (0,1]$, each MAE and MASD on the bases of $K = 1000$ replicates from $S(1.5, 0.5; 0)$ each with a large sample size $n = 10000$, is given.

TABLE 5.4: The summary of MAEs and MASDs of $\hat{\alpha}_n(u_1,u_2)$ for $\alpha = 1.5$.

	$q_{0.00}$	$q_{0.25}$	$q_{0.50}$	$q_{0.75}$	$q_{1.00}$	mean	sd
MAEs	-0.0047	0.0005	0.0009	0.0014	0.0218	0.0011	0.0013
MASDs	1.7496	2.1907	2.7112	3.6339	21.9102	3.1554	1.4565

q_p–the k-th percentile, $p = k/100$; sd–standard deviation

By Table 5.5 at least half of the MAEs turned out to be less than 0.0009 and at all pairs $(u_1,u_2) \in (0,1] \times (0,1]$ the MAEs were less than 0.0218. The empirical distributions of MAEs and the MASDs of $\hat{\alpha}_n(u_1,u_2)$ at $(u_1,u_2) \in (0,1] \times (0,1]$ are presented on the left and the right of Figure 5.8, respectively .

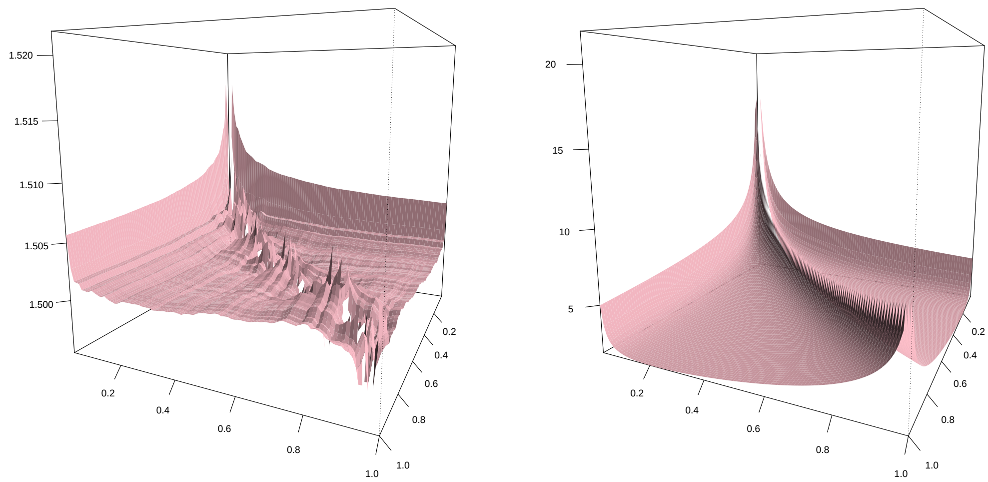

FIGURE 5.8: The MAEs (on left) and the MASDs (on right) of $\hat{\alpha}_n(u_1,u_2)$ on the basis of $K = 1000$ replicates from $S(1.5, 0.5; 0)$ with $n = 10000$.

The biggest values of MAEs occur at pairs where one or two of the arguments are close to 0 while the biggest values of MASDs occur around the diagonal of $(0,1] \times (0,1]$. Hereby, based on Figure

5.8 the pairs (u_1,u_2) at which the MASDs of $\hat{\alpha}_n$ have the smallest values seem not to guarantee the smallest MAEs. A heat-map corresponding to the results in Figure 5.8 is given in Figure 5.9 where the lightest areas indicate the smallest values of MAEs and MASDs while red areas indicate the biggest values. It is expected that if small asymptotic variance would indicate accurate ECuF estimates then the patterns on the left and right half of Figure 5.9 would look similar. The MAEs

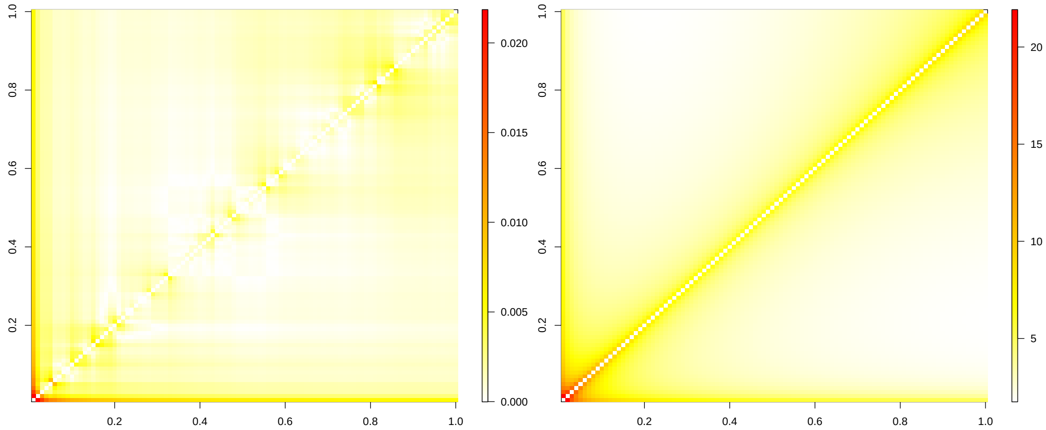

FIGURE 5.9: The MAEs (on left) and the MASDs (on right) of $\hat{\alpha}_n(u_1,u_2)$ on the basis of $K = 1000$ replicates from $S(1.5,0.5;0)$ with $n = 10000$.

on the left of Figure 5.9 are biggest at pairs (u_1,u_2) where at least one of the u_1,u_2 is less than 0.2, approximately. However, in contradiction, the MASDs on left of Figure 5.9 turn out to be the smallest at the at pairs (u_1,u_2) where u_1,u_2 is less than 0.1, approximately. A less detailed version of

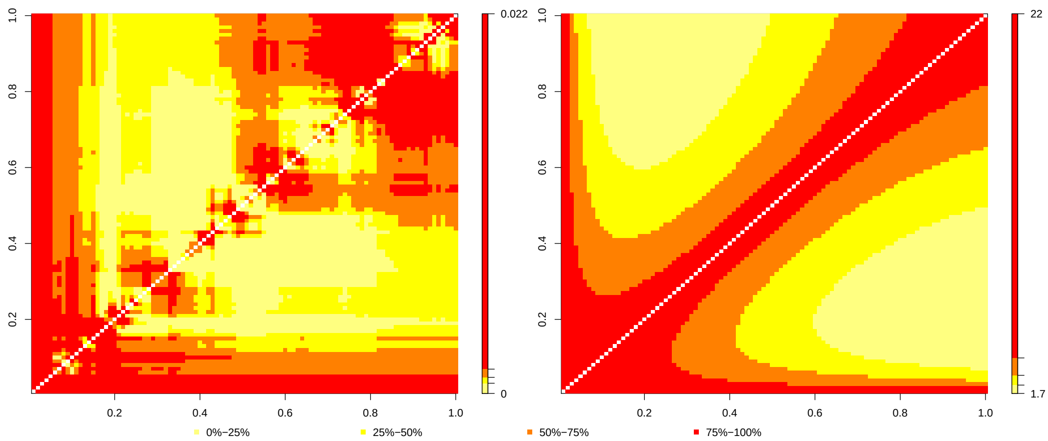

FIGURE 5.10: The quartiles of MAEs (on left) and the MASDs (on right) of $\hat{\alpha}_n(u_1,u_2)$ on the basis of $K = 1000$ replicates from $S(1.5,0.5;0)$ with $n = 10000$.

Figure 5.9 is provided in Figure 5.10 where the MAEs and MASDs are grouped into their quartiles (of all of the MAEs and MASDs at $(u_1,u_2) \in (0,1] \times (0,1]$, each on the bases of $K = 1000$ replicates with size $n = 10000$) with the lightest areas indicate pairs (u_1, u_2) at which the MAEs and MASDs belong into the first quartile while red areas present pairs at which they belong within their fourth quartile. Many of the pairs of u_1 and u_2 that minimize the MASDs (i.e., MASDs within the first quarter) on the right of Figure 5.9 lead to quite poor MAEs while other such pairs lead to rather good MAEs (on the left of Figure 5.9).

The results for the tail index $\alpha = 0.5$. In Table 5.5 the summary of the MAEs and MASDs of $\hat{\alpha}_n(u_1, u_2)$ at $(u_1, u_2) \in (0, 1] \times (0, 1]$, each MAE and MASD on the bases of $K = 1000$ replicates from $S(0.5, 0.5; 0)$ each with a large sample size $n = 10000$, is given. By Table 5.5 at least half of

TABLE 5.5: The summary of MAEs and MASDs of $\hat{\alpha}_n(u_1, u_2)$ for $\alpha = 1.5$.

	$q_{0.00}$	$q_{0.25}$	$q_{0.50}$	$q_{0.75}$	$q_{1.00}$	**mean**	**sd**
MAEs	0.0000	0.0003	0.0006	0.0016	0.0452	0.0015	0.0028
MASDs	0.8567	1.6699	2.8506	5.6891	83.8996	5.6615	8.6857

q_p–the k-th percentile, $p = k/100$; sd–standard deviation

the MAEs turned out to be less than 0.0006 and they all were less than 0.0452. However, when comparing with the results for $\alpha = 1.5$ in Table 5.4 the MAEs as well as the MASDs for $\alpha = 0.5$ in Table 5.5 have wider range and vary more. The empirical distributions of MAEs and the MASDs of $\hat{\alpha}_n(u_1, u_2)$ at $(u_1, u_2) \in (0, 1] \times (0, 1]$ are presented on the left and the right of Figure 5.11, respectively. In Figure 5.11 the biggest values of MAEs and MASDs are around the diagonal

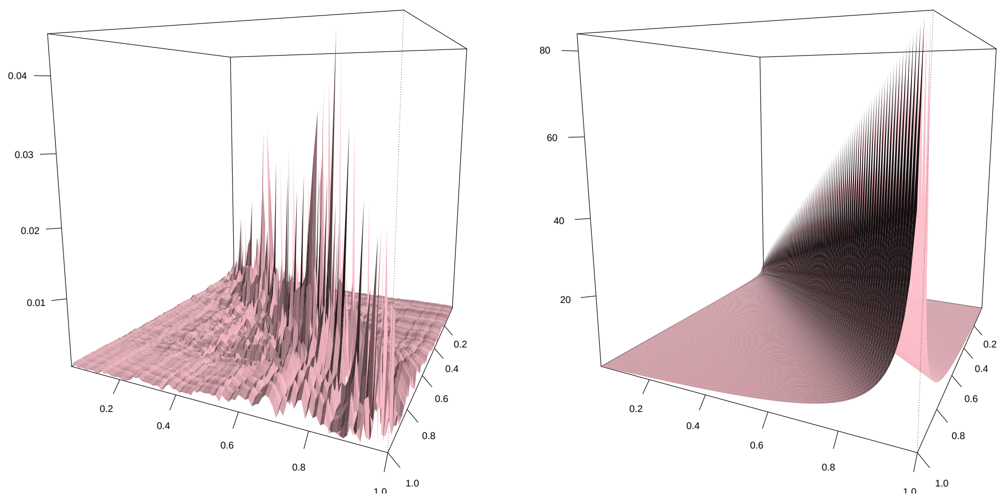

FIGURE 5.11: The MAEs (on left) and the MASDs (on right) of $\hat{\alpha}_n(u_1, u_2)$ on the basis of $K = 1000$ replicates from $S(0.5, 0.5; 0)$ with $n = 10000$.

of $(0, 1] \times (0, 1]$. Based on Figure 5.11 the pairs (u_1, u_2) at which the MASDs of $\hat{\alpha}_n$ have the smallest values seem to lead to the smallest MAEs. A heat-map corresponding to the results in Figure 5.11 is given in Figure 5.12 where the lightest areas correspond to the smallest values of MAEs and MASDs while red areas indicate the biggest values.

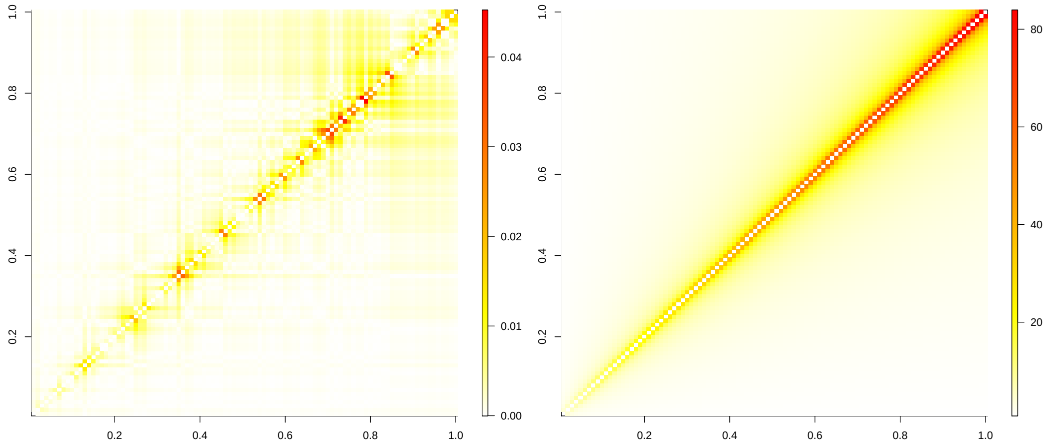

FIGURE 5.12: The MAEs (on left) and the MASDs (on right) of $\hat{\alpha}_n(u_1,u_2)$ on the basis of $K = 1000$ replicates from $S(0.5,0.5;0)$ with $n = 10000$.

The patterns on the left and right in Figure 5.9 look quite similar indicating that small asymptotic variance indeed yields more accurate ECuF estimates (that is, smaller MAEs). A more robust version of Figure 5.12 is provided by Figure 5.13 where the MAEs and MASDs are grouped within their quartiles (oof all of the MAEs and MASDs at $(u_1,u_2) \in (0,1] \times (0,1]$, each on the bases of $K = 1000$ replicates with size $n = 10000$) with the lightest areas indicating the pairs (u_1,u_2) at which the MAEs and MASDs belong to the first quartile (MAEs less than 0.0003, MASDs less than 0) while red areas present pairs at which they belong to their fourth quartile (MAEs more than 0.0452, MASDs more than 5.689). Many of the pairs of u_1,u_2 within the first quartile of MASDs (on right of Figure 5.13)

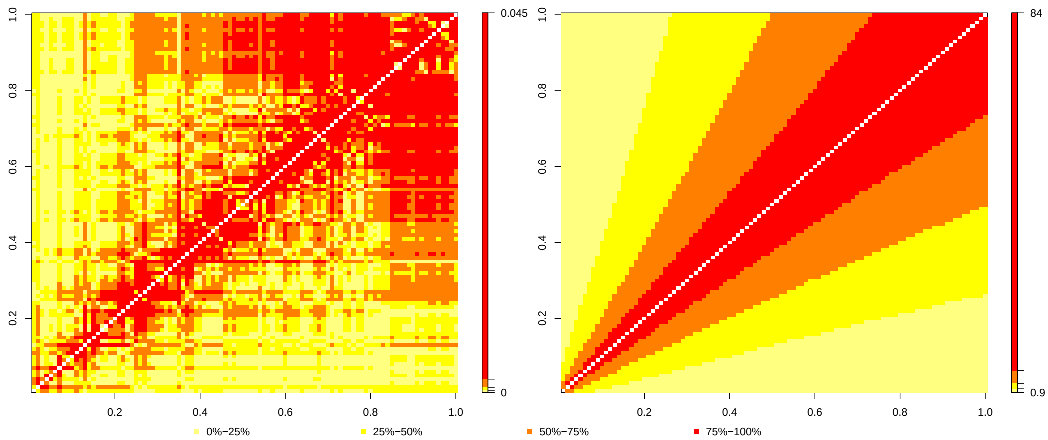

FIGURE 5.13: The quartiles of MAEs (on left) and the MASDs (on right) of $\hat{\alpha}_n(u_1,u_2)$ on the basis of $K = 1000$ replicates from $S(0.5,0.5;0)$ with $n = 10000$.

lead to the first quartile of MAEs (on left Figure 5.13) while other such pairs lead to rather poor MAEs.

On the bases of the MAEs and MASDs of $\hat{\alpha}_n$ of $K = 1000$ replicates from $S(\alpha = 1.5, \beta = 0.5; 0)$ and $S(\alpha = 0.5, \beta = 0.5; 0)$, each with sample size $n = 10000$, it may concluded that many of the pairs of u_1, u_2 at which MASDs were small lead to small MAEs while other such pairs lead to rather poor MAEs, even for the large samples such as $n = 10000$. In practice, for smaller single smaller size samples the corespondance may turn out at less pairs. Krutto (2018) studied single replicates from $S(\alpha = 1.5, \beta = 0.5; 0)$ and $S(\alpha = 0.5, \beta = 0.5; 0)$, each with sample size $n = 10000$, and found that selecting (u_1, u_2) of the ECuF estimator $\alpha_n(u_1, u_2)$ on the bases of minimizing the asymptotic variance of the $\alpha_n(u_1, u_2)$ is not suggested. We illustrate the same buy a single replicate from $S(\alpha = 1.5, \beta = 0.5; 0)$ with $n = 1000$. In Figure the

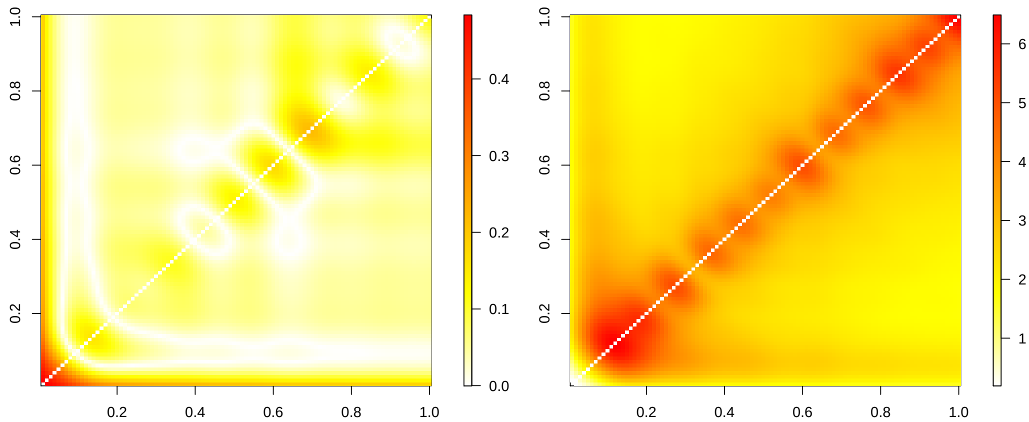

FIGURE 5.14: The MAEs (on left) and the MASDs (on right) of $\hat{\alpha}_n(u_1, u_2)$ on the basis of a single replicate from $S(1.5, 0.5; 0)$ with $n = 1000$.

Chapter 6

Sample Based Selection of the Arguments of ECuF Estimators

Perhaps the most important aspect about the ECuF estimators $\alpha_n(u_1,u_1) = g_1(\boldsymbol{b}_n)$, $\ln\gamma_n(u_1,u_1) = g_2(\boldsymbol{b}_n)$, $\beta_n(u_1,u_1) = g_3(\boldsymbol{b}_n)$, and $\delta_n(u_1,u_1) = g_4(\boldsymbol{b}_n)$ is the determination of the two real arguments $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$ at which they are obtained. The aim of this chapter is to provide suggestions on the selection of the arguments without any modification[1] in the estimation procedure. As seen in Section 5.2, selecting (u_1,u_2) on the bases of minimizing the asymptotic variance of one or more of ECuF estimators is not suggested due to the possible biasedness. In this chapter an approach by Krutto (2018), called the Argument–Selection–Rule, is introduced: $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$ are suggested to to select on the bases of the real part of empirical cumulant function, $\Re\psi_n$. In Section 6.1 the convergence of the real part of empirical cumulant function is studied and in Section 6.2 suggestions for the selection of $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$ are given. In what follows, let $\vee$ and $\wedge$ denote the max and min operators, respectively: given a real numbers a,b, $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

6.1 Convergence of the Real Part of Empirical Cumulant Function

In this section the convergence of the real part of empirical cumulant function on the real line for stable laws $X \sim S(\alpha,\beta,\gamma,\delta;0)$ is considered. More precisely, we consider the quantity

$$|\Re\psi_n(u) - \Re\psi_X(u)| = |\ln|\varphi_n(u)| - \ln|\varphi_X(u)||,$$

[1]Modifications to algorithmic minimizing procedure on a number of arguments have been proposed by Koutrouvelis (1980, 1981), Knight and Satchell (1997), Kogon and Williams (1998), see Section 2.7.

where $\psi_n(u)$ is the empirical cumulant function, given by (1.31), $\Re\psi_X(u)$ is the real part of cumulant function of X, given by (2.21), $\varphi_n(u)$ is the empirical characteristic function, given by (1.14), and $\varphi_X(u)$ is the characteristic function of $X \sim S(\alpha,\beta,\gamma,\delta;0)$, given by (2.20),

$$\varphi_X(u) = \exp\{\Re\psi_X(u) + i\Im\psi_X(u)\}$$

where

$$\Re\psi_X(u) = -\gamma^\alpha|u|^\alpha,$$
$$\Im\psi_X(u) = \begin{cases} u[\beta\gamma\tan\frac{\pi\alpha}{2}(|\gamma u|^{\alpha-1}-1)+\delta] & \text{for } \alpha \neq 1 \\ u[-\beta\gamma\frac{2}{\pi}\ln(\gamma|u|)+\delta] & \text{for } \alpha = 1 \end{cases}$$

with $u \in \mathbb{R}$, $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\gamma > 0$, $\delta \in \mathbb{R}$. Recall that $S(\alpha,\beta,\gamma=1,\delta=0;0)$ is denoted by $S(\alpha,\beta;0)$ Continuous mapping theorem (e.g., van der Vaart (1998, Theorem 2.3)) and (1.22) imply that at every fixed $u \in \mathbb{R}$ the empirical cumulant function $\psi_n(u)$ converges almost surely to $\psi_X(u)$ as $n \to \infty$. Hereby, at every fixed $u \in \mathbb{R}$

$$\Re\psi_n(u) \overset{a.s.}{\to} \Re\psi_X(u) \text{ as } n \to \infty.$$

Following proposition gives for any fixed n the bound for $|\Re\psi_n(u) - \Re\psi_X(u)|$.

Proposition 6.1. *Assume $X \sim S(\alpha,\beta,\gamma,\delta;0)$. Let $0 < \varepsilon < 1$ and $u \in \mathbb{R}$. If*

$$e^{\gamma^\alpha|u|^\alpha}\Delta_n(u) \leq \frac{\varepsilon}{1+\varepsilon} \tag{6.1}$$

where $\Delta_n(u) = |\varphi_n(u) - \varphi_X(u)|$, then

$$|\Re\psi_n(u) - \Re\psi_X(u)| \leq \varepsilon.$$

Proof. We have $|\Re\psi_n(u) - \Re\psi_X(u)| = |\ln|\varphi_n(u)| - \ln|\varphi_X(u)||$. By the mean value theorem and the reverse triangle inequality, for every fixed $u \in \mathbb{R}$,

$$|\Re\psi_n(u) - \Re\psi_X(u)| \leq \frac{\Delta_n(u)}{\min(|\varphi_n(u)|, |\varphi_X(u)|)}$$

where $\Delta_n(u) = |\varphi_n(u) - \varphi_X(u)| \geq 0$. Note that if $0 < |\varphi_n(u)| < |\varphi_X(u)|$ then by the reverse triangle inequality,

$$|\varphi_n(u)| = |\varphi_n(u) + \varphi_X(u) - \varphi_X(u)| \geq |\varphi_X(u)| - |\varphi_n(u) - \varphi_X(u)| = |\varphi_X(u)| - \Delta_n(u).$$

Hereby,

$$|\Re\psi_n(u) - \Re\psi_X(u)| \le \begin{cases} \dfrac{\Delta_n(u)}{|\varphi_X(u)|} & \text{for } 0 < |\varphi_X(u)| < |\varphi_n(u)| \\ \dfrac{\Delta_n(u)}{|\varphi_X(u)| - \Delta_n(u)} & \text{for } 0 < |\varphi_n(u)| < |\varphi_X(u)| \end{cases}$$

$$\le \frac{\Delta_n(u)}{|\varphi_X(u)| - \Delta_n(u)} = \frac{\Delta_n(u)}{\mathrm{e}^{-\gamma^\alpha |u|^\alpha} - \Delta_n(u)} \tag{6.2}$$

Condition

$$\frac{\Delta_n(u)}{\mathrm{e}^{-\gamma^\alpha |u|^\alpha} - \Delta_n(u)} \le \varepsilon$$

is equivalent to the condition (6.1). Hereby, if (6.1) holds, then

$$|\Re\psi_n(u) - \Re\psi_X(u)| = |\ln|\varphi_n(u)| - \ln|\varphi_X(u)|| \le \varepsilon.$$

$\square$

Note, that condition (6.1) is equivalent to

$$|u| \le \frac{1}{\gamma}\left(\ln\frac{\varepsilon}{\varepsilon+1} - \ln\Delta_n(u)\right)^{1/\alpha} \equiv z_n(\gamma,\alpha,\varepsilon,\Delta_n(u)). \tag{6.3}$$

In other words, if (6.3) holds then $|\Re\psi_n(u) - \Re\psi_X(u)| \le \varepsilon$.

In Figures 6.1, 6.2, and 6.3 the means of absolute estimation errors ε, (that is, the MAEs, given by (5.1)) versus $u > 0$ are plotted for $\alpha = 0.2, 1, 1.8$ and $\gamma = 0.5, 1, 2$. More precisely, the MAEs of $K = 200$ replicates from $S(\alpha, 0.5, \gamma, 0; 0)$, each with size $n = 1000$, are obtained.

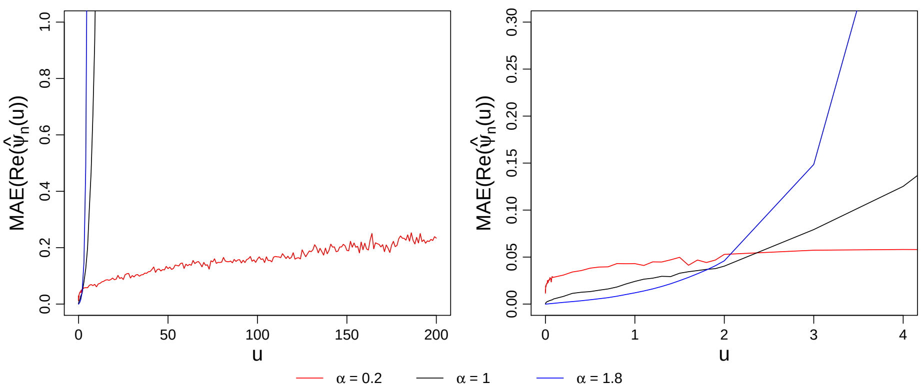

FIGURE 6.1: MAEs of $\Re\hat{\psi}_n(u)$ of $K = 200$ replicates from $S(\alpha, \frac{1}{2}, \frac{1}{2}, 0; 0)$ each with $n = 1000$.

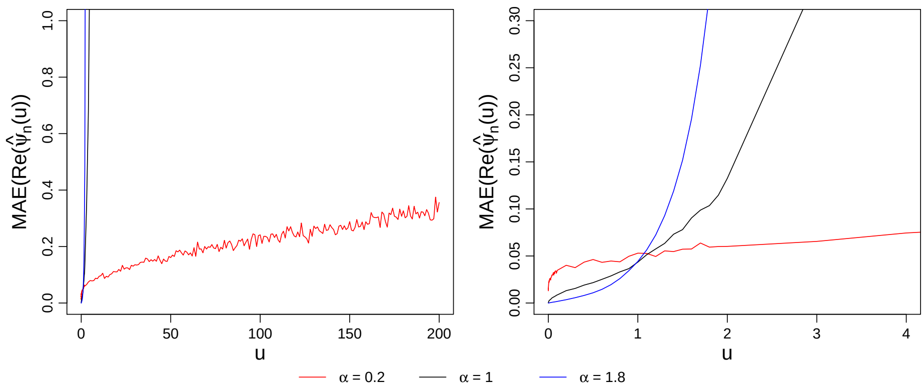

FIGURE 6.2: MAEs of $\Re\hat{\psi}_n(u)$ of $K = 200$ replicates from $S(\alpha, \frac{1}{2}, 1, 0; 0)$ each with $n = 1000$.

As seen in Figures 6.1, 6.2, and 6.3 for all $\alpha = 0.2, 1, 1.8$, and $\gamma = 0.5, 1, 2$, the MAEs notably increase for $u > 1/\gamma$, roughly. Based on Figures 6.1, 6.2 and 6.3 for samples with a size around $n = 1000$ the arguments $|u| \le 1/\gamma$ should yield $|\Re\psi_n(u) - \Re\psi_X(u)| < \varepsilon < 0.1$, in average. Note that by Proposition 3.1 the condition $|u| \le 1/\gamma$ is equivalent to $\Re\psi_X(u) \in [-1, 0]$.

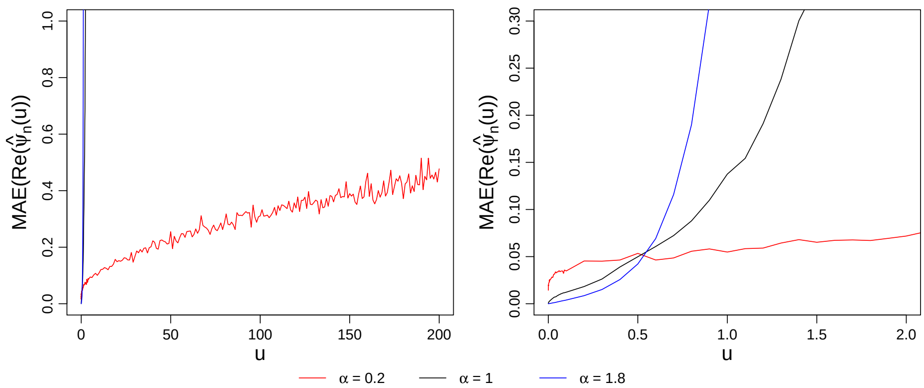

FIGURE 6.3: MAEs of $\Re\hat{\psi}_n(u)$ of $K = 200$ replicates from $S(\alpha, \frac{1}{2}, 2, 0; 0)$ each with $n = 1000$.

Proposition 6.1 indicates that the estimation error (convergence rate) of the real part of empirical cumulant function is always bigger than the estimation error of empirical characteristic function Δ_n. Indeed, if (6.3) holds then $|\Re\psi_n(u) - \Re\psi_X(u)| \le \varepsilon$. However, z_n in (6.3) is defined only if

$$\ln \frac{\varepsilon}{\varepsilon + 1} \ge \ln \Delta_n(u)$$

which for $0 < \Delta_n < 1$ holds if and only if

$$\varepsilon \geq \frac{\Delta_n(u)}{1-\Delta_n(u)}$$

and $\varepsilon \geq \Delta_n(u)$. The following Corollary 6.1 which implies that for any $0 < U < \infty$ the convergence of

$$\sup_{|u|\leq U} |\Re\psi_n(u) - \Re\psi_X(u)| \overset{a.s.}{\to} 0$$

is slower then the convergence of

$$\sup_{|u|\leq U} \Delta_n(u) \overset{a.s.}{\to} 0.$$

Corollary 6.1. *Assume $X \sim S(\alpha, \beta, \gamma, \delta; 0)$ and let $0 < U < \infty$. If*

$$\mathrm{e}^{(\gamma U)^\alpha} \sup_{|u|\leq U} \Delta_n(u) \overset{a.s.}{\to} 0 \text{ as } n \to \infty \tag{6.4}$$

then

$$\sup_{|u|\leq U} |\Re\psi_n(u) - \Re\psi_X(u)| \overset{a.s.}{\to} 0 \text{ as } n \to \infty. \tag{6.5}$$

Proof. The proof follows from the Proposition 6.1 and equation (1.23). □

Note that if $n \to \infty$ then $\mathbf{P}\left(|\Re\psi_n(u) - \Re\psi_X(u)| \leq \varepsilon \text{ eventually }\right) = 1$, while for fixed $n < \infty$ the quantity $|\Re\psi_n(u) - \Re\psi_X(u)|$ is less than $\varepsilon > 0$ with some probability $1 - p$.

For various sample sizes the behaviour of $\Re\psi_n(u)$ is illustrated in Figures 6.4, 6.5 and 6.6.

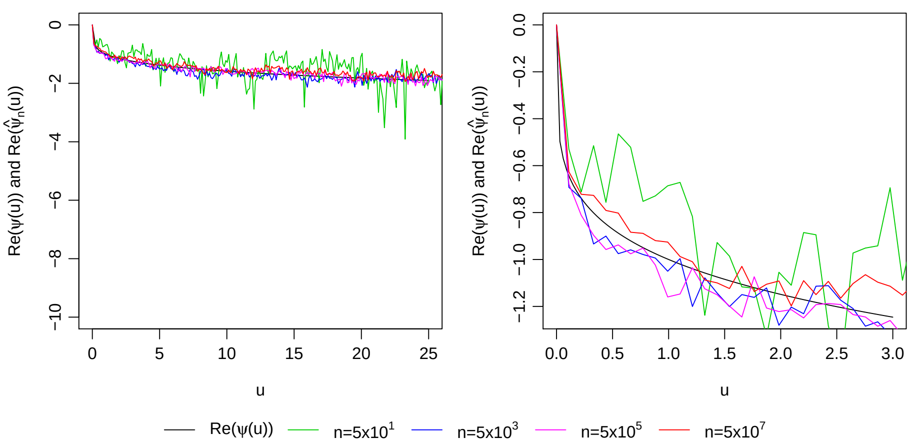

FIGURE 6.4: The values of $\Re\psi_n(u)$ of single replicates from $S(0.2, 0.5; 0)$ and the corresponding values of $\Re\psi_X(u)$.

In Figures 6.4, 6.5 and 6.6 the values of $\Re\psi_n(u)$ of single replicates from $X \sim S(\alpha, 0.5; 0)$ with $\alpha = 0.2, 1, 1.8$ and the corresponding values of $\Re\psi_X(u)$ are presented for $n = 5\times10^1, 5\times10^3, 5\times10^5, 5\times10^7$. Figures 6.4, 6.5 and 6.6 indicate that for smaller values of n, and smaller values of α, the interval

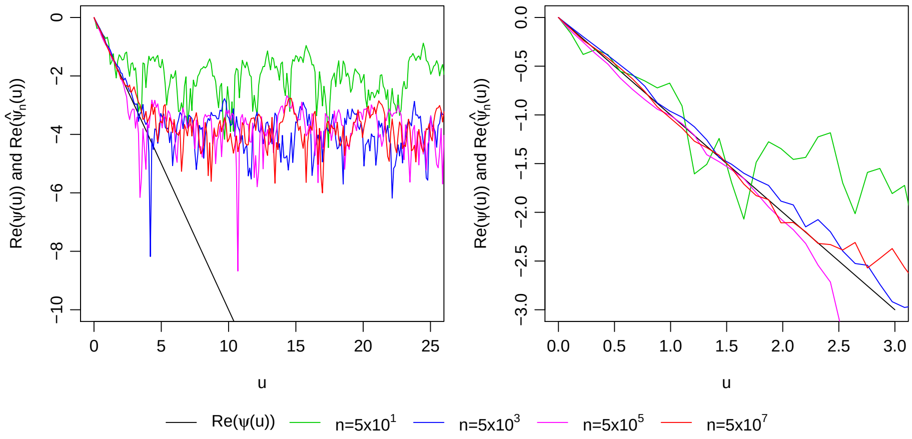

FIGURE 6.5: The values of $\Re\psi_n(u)$ of single replicates from $S(1, 0.5; 0)$ and the corresponding values of $\Re\psi_X(u)$.

of the arguments u at which the estimation error turns out to be good, say $|\Re\psi_n(u) - \Re\psi_X(u)| \le 0.5$, is smaller than $[0, 1/\gamma]$, and for bigger values of n, and bigger values of α, the interval is bigger than $[0, 1/\gamma]$, while in Figures 6.4, 6.5, 6.6 $\gamma = 1$). In conclusion, based on Figures 6.1-6.6 the interval

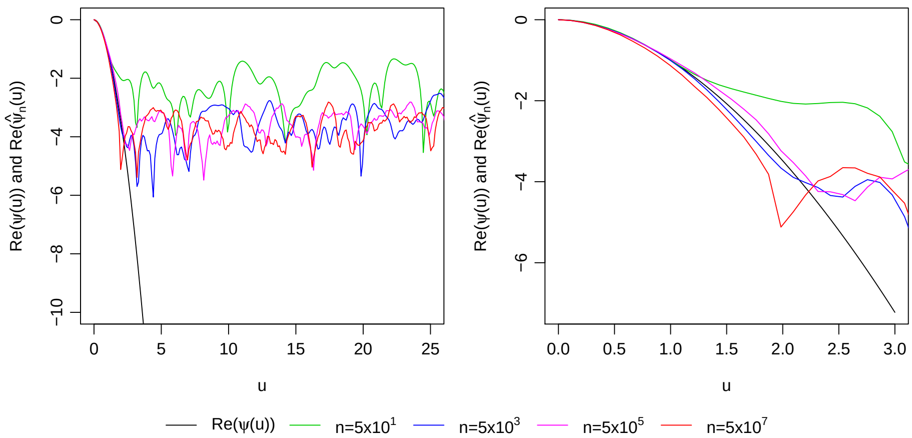

FIGURE 6.6: The values of $\Re\psi_n(u)$ of single replicates from $S(1.8, 0.5; 0)$ and the corresponding values of $\Re\psi_X(u)$.

of u where $|\Re\psi_n(u) - \Re\psi_X(u)| < \varepsilon$, $0 < \varepsilon \le 0.1$, roughly, is quite small, that is, the convergence of the real part of the empirical cumulant function is quite slow.

Recall that if for any $\varepsilon > 0$ and $0 < \kappa < 1$

$$\exp\{(\gamma u)^{\alpha}\}\Delta_n(u) \leq \frac{\varepsilon}{(1+\varepsilon)} = \kappa,$$

then by Proposition 6.1

$$|\Re\psi_n(u) - \Re\psi_X(u)| \leq \varepsilon.$$

The Proposition 6.2 below sets a bound for

$$\mathbf{P}\left(\exp\{(\gamma U)^{\alpha}\} \sup_{|u|\leq U} \Delta_n(u) > \kappa\right).$$

First note that for $X \sim S(\alpha,\beta,\gamma,\delta;0)$ there exists $L = L(\alpha) > 1$ such that (e.g., Nolan (2018b, Theorem 1.12, p. 14))

$$\mathbf{P}(X \leq x) \leq L\gamma^{\alpha}c(\alpha)(1-\beta)|x|^{-\alpha}, \forall x < 0, \quad \mathbf{P}(X > x) \leq L\gamma^{\alpha}c(\alpha)(1+\beta)x^{-\alpha}, \forall x > 0 \tag{6.6}$$

where

$$c(\alpha) = \frac{\Gamma(\alpha)}{\pi}\sin\frac{\pi\alpha}{2} \leq \frac{1}{2}. \tag{6.7}$$

For some arbitrarily small $\upsilon > 0$ let $A_{\upsilon} = A(\upsilon, F)$ be such that $F(-A_{\upsilon}) \leq \frac{\upsilon}{6}$ and $1 - F(A_{\upsilon}) \leq \frac{\upsilon}{6}$ where $F = F_X$ is the distribution function of X. Hereby,

$$A_{\upsilon} = \max\left\{-F^{-1}\left(\frac{\upsilon}{6}\right), (1-F)^{-1}\left(\frac{\upsilon}{6}\right)\right\} \leq K_{\upsilon}$$

where by (6.6)

$$K_{\upsilon} = \gamma\left(\frac{6L}{\upsilon}c(\alpha)\right)^{1/\alpha}[(1-\beta)^{1/\alpha} \vee (1+\beta)^{1/\alpha}]. \tag{6.8}$$

Proposition 6.2. *Assume $X \sim S(\alpha,\beta,\gamma,\delta;0)$. Let $\kappa > 0$, $\delta_n > 0$ be arbitrarily small. Then for any $0 < U < \infty$*

$$\mathbf{P}\left(\exp\{(\gamma U)^{\alpha}\} \sup_{|u|\leq U} \Delta_n(u) > \kappa\right) \leq 2\exp\left\{-n \cdot \frac{\delta_n^2}{18}\right\} + 2\exp\left\{-n \cdot \frac{(\kappa\exp\{-(\gamma U)^{\alpha}\} - \delta_n)^2}{2(K_{\delta_n}U)^2}\right\}. \tag{6.9}$$

where K_{δ_n} is given by (6.8).

Proof. Let $n < \infty$ be fixed. If $\|F_n - F\| = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \le \frac{\delta_n}{6}$, where F_n is given by (1.17), then for any $0 < U < \infty$ (see, e.g., Csörgo (1981, p. 136)),

$$\sup_{|u|\le U} |\varphi_n(u) - \varphi(u)| \le \delta_n + 2UK_{\delta_n}\|F_n - F_X\|,$$

where K_{δ_n} is given by (6.8). Then

$$\exp\{(\gamma U)^\alpha\} \sup_{|u|\le U} |\varphi_n(u) - \varphi(u)| \le \exp\{(\gamma U)^\alpha\}\left(\delta_n + 2UK_{\delta_n}\|F_n - F\|\right).$$

Hereby,

$$\left\{\exp\{(\gamma U)^\alpha\} \sup_{|u|\le U} \Delta_n(u) > \kappa\right\} \subset \left\{\|F_n - F\| > \frac{\delta_n}{6}\right\} \cup \left\{\exp\{(\gamma U)^\alpha\}\left(\delta_n + 2UK_{\delta_n}\|F_n - F\|\right) > \kappa\right\}$$

and

$$P\left(\exp\{(\gamma U)^\alpha\} \sup_{u\le U} \Delta_n(u) > \kappa\right) \le P\left(\|F_n - F_X\| > \frac{\delta_n}{6}\right) + P\left(\|F_n - F_X\| > \frac{\kappa\exp\{-(\gamma U)^\alpha\} - \delta_n}{2UK_{\delta_n}}\right).$$

By the Dvoretzky–Kiefer–Wolfowitz inequality (e.g., van der Vaart (1998, p. 268)), for any $\varepsilon^* > 0$,

$$\mathbf{P}\left(\|F_n - F\| > \varepsilon^*\right) \le 2\exp[-2n(\varepsilon^*)^2].$$

Hereby,

$$\mathbf{P}\left(\exp\{(\gamma U)^\alpha\} \sup_{|u|\le U} \Delta_n(u) > \kappa\right) \le 2\exp\left\{-n\frac{\delta_n^2}{18}\right\} + 2\exp\left\{-n\frac{(\kappa\exp\{-(\gamma U)^\alpha\} - \delta_n)^2}{2(K_{\delta_n}U)^2}\right\}.$$

□

The following proposition 6.3 sets a bound for

$$\mathbf{P}\left(\sup_{|u|\le U} |\Re\psi_n(u) - \Re\psi_X(u)| \le \varepsilon\right)$$

for any $0 < U < \infty$, n and $\varepsilon > 0$.

Proposition 6.3. *Assume $X \sim S(\alpha, \beta, \gamma, \delta; 0)$. Let $n > 1$, $0 < U < \infty$ and $p \in (0,1)$ be such that*

$$\left(\frac{72(\ln 4 - \ln p)}{n}\right)^{\frac{1}{2}} < 1 \tag{6.10}$$

and

$$2\left(\frac{\ln 4 - \ln p}{nk(\alpha, \gamma, \beta, U)}\right)^{1/s} < 1 \tag{6.11}$$

where $s = 2(1 + \frac{1}{\alpha})$ *and*

$$k(\alpha,\gamma,\beta,U) = \frac{\exp\{-2(1+\frac{1}{\alpha})(\gamma U)^{\alpha}\}}{2(\gamma U)^2\left(6L\frac{\Gamma(\alpha)}{\pi}\sin\frac{\pi\alpha}{2}\right)^{2/\alpha}[(1-\beta)^{2/\alpha}\vee(1+\beta)^{2/\alpha}]}. \tag{6.12}$$

Let

$$\varepsilon = \frac{\sqrt{72(\ln 4-\ln p)}}{\sqrt{n}-\sqrt{72(\ln 4-\ln p)}} \vee \frac{2\,(\ln 4-\ln p))^{1/s}}{(nk(\alpha,\gamma,\beta,U))^{1/s}-2\,(\ln(4-\ln p)^{1/s}}, \tag{6.13}$$

then

$$\mathbf{P}\left(\sup_{|u|\le U}|\Re\psi_n(u)-\Re\psi_X(u)|\le\varepsilon\right)\ge 1-p. \tag{6.14}$$

Proof. Proposition 6.1 implies that with $\kappa = \frac{\varepsilon}{\varepsilon+1}$

$$\mathbf{P}\left(\sup_{|u|\le U}|\Re\psi_n(u)-\Re\psi_X(u)|\le\varepsilon\right)\ge\mathbf{P}\left(\exp\{(\gamma U)^{\alpha}\}\sup_{|u|\le U}\Delta_n(u)\le\frac{\varepsilon}{\varepsilon+1}\right),$$

or equivalently

$$\mathbf{P}\left(\sup_{|u|\le U}|\Re\psi_n(u)-\Re\psi_X(u)|>\varepsilon\right)\le\mathbf{P}\left(\exp\{(\gamma U)^{\alpha}\}\sup_{|u|\le U}\Delta_n(u)>\frac{\varepsilon}{\varepsilon+1}=\kappa\right).$$

In Proposition 6.2 choose $\delta_n = \kappa\exp\{(\gamma U)^{\alpha}\}/2$. Then

$$\mathbf{P}\left(\exp\{(\gamma U)^{\alpha}\}\sup_{u\le U}\Delta_n(u)>\kappa\right)\le 4\exp\left\{-n\frac{\kappa^2}{72}\right\}\vee 4\exp\left\{\left(\frac{\kappa}{2}\right)^{2(1+\frac{1}{\alpha})}k(\alpha,\gamma,\beta,U)\right\}.$$

Let $p\in(0,1)$ be fixed. Observe,

$$\kappa\ge\left(\frac{72(\ln 4-\ln p)}{n}\right)^{\frac{1}{2}},\quad\Leftrightarrow\quad 4\exp\left\{-n\frac{\kappa^2}{72}\right\}\le p$$

and

$$\kappa\ge 2\left(\frac{\ln 4-\ln p}{nk(\alpha,\gamma,\beta,U)}\right)^{\frac{1}{2(1+\frac{1}{\alpha})}},\quad\Leftrightarrow\quad 4\exp\left\{-n\left(\frac{\kappa}{2}\right)^{2(1+\frac{1}{\alpha})}k(\alpha,\gamma,\beta,U)\right\}\le p.$$

Also observe that

$$\kappa=\frac{\varepsilon}{1+\varepsilon}\ge\left(\frac{72(\ln 4-\ln p)}{n}\right)^{\frac{1}{2}},\quad\Leftrightarrow\quad\varepsilon\frac{\sqrt{72(\ln 4-\ln p)}}{\sqrt{n}-\sqrt{72(\ln 4-\ln p)}},$$

and

$$\kappa = \frac{\varepsilon}{1+\varepsilon} \geq 2\left(\frac{\ln 4 - \ln p}{nk(\alpha,\gamma,\beta,U)}\right)^{\frac{1}{s}}, \quad \Leftrightarrow \quad \varepsilon \geq \frac{2\left(\ln 4 - \ln p\right)^{1/s}}{\left(nk(\alpha,\gamma,\beta,U)\right)^{1/s} - 2\left(\ln 4 - \ln p\right)^{1/s}}.$$

Therefore, as ε satisfies (6.13), we have

$$\mathbf{P}\left(\sup_{|u|\leq U} |\Re\psi_n(u) - \Re\psi_X(u)| > \varepsilon\right) \leq p,$$

or, equivalently

$$\mathbf{P}\left(\sup_{|u|\leq U} |\Re\psi_n(u) - \Re\psi_X(u)| \leq \varepsilon\right) \geq 1-p.$$

□

In what follows we fix $p \in (0,1)$, $\varepsilon > 0$, n, $\alpha \in (0,2]$, $\gamma > 0$ and $\beta \in [-1,1]$, and aim to find the $0 < U < \infty$ such that (6.14) in Proposition 6.3 holds. By (6.10) and (6.13) the sample size must be so big that

$$\frac{\varepsilon}{\varepsilon+1} \geq \left(\frac{72(\ln 4 - \ln p)}{n}\right)^{\frac{1}{2}}. \tag{6.15}$$

Then Proposition 6.3 holds for all $0 < U < \infty$ such that

$$\frac{\varepsilon}{\varepsilon+1} \geq 2\left(\frac{\ln 4 - \ln p}{nk(\alpha,\gamma,\beta,U)}\right)^{1/s}, \tag{6.16}$$

where $k(\alpha,\gamma,\beta,U)$ is given by (6.12). In the following Corollary 6.2 we aim to find the biggest possible U, that is is the upper bound of the solutions U to (6.16), denoted by $U' = U'(\varepsilon,p,n,\alpha,\gamma,\beta)$.

Corollary 6.2. *Assume $X \sim S(\alpha,\beta,\gamma,\delta;0)$. Fix $\varepsilon > 0$, $p \in (0,1)$, $\alpha \in (0,2]$, $\gamma > 0$ and let n be such that* (6.10) *and* (6.15) *hold. Let U' be such that*

$$\frac{\varepsilon}{\varepsilon+1} = 2\left(\frac{\ln 4 - \ln p}{nk(\alpha,\gamma,\beta,U')}\right)^{1/s}, \tag{6.17}$$

where $k(\alpha,\gamma,\beta,U)$ is given by (6.12)*. Then for all $U < U'$*

$$\mathbf{P}\left(\sup_{|u|\leq U} |\Re\psi_n(u) - \Re\psi_X(u)| \leq \varepsilon\right) \geq 1-p, \tag{6.18}$$

Proof. If (6.15) holds then

$$\varepsilon \geq \frac{\sqrt{72(\ln 4 - \ln p)}}{\sqrt{n} - \sqrt{72(\ln 4 - \ln p)}}.$$

If U' is such that (6.17) holds then

$$\varepsilon = \frac{2\,(\ln 4 - \ln p)^{1/s}}{(nk(\alpha,\gamma,\beta,U))^{1/s} - 2\,(\ln 4 - \ln p)^{1/s}}$$

holds. Hereby the assumption (6.13) in Proposition 6.3 holds and for all $U < U'$

$$\mathbf{P}\left(\sup_{|u|\le U} |\Re\psi_n(u) - \Re\psi_X(u)| \le \varepsilon\right) \ge 1-p.$$

□

For $\beta \in (0,1]$ (6.17) gives

$$\frac{\exp\{-2(1+\frac{1}{\alpha})(\gamma U')^\alpha\}}{2\,[(\gamma U')^\alpha]^{2/\alpha}} = \frac{\ln(4/p)}{n}\left(\frac{2(\varepsilon+1)}{\varepsilon}\right)^{2(1+\frac{1}{\alpha})}\left(6L(1+\beta)\frac{\Gamma(\alpha)}{\pi}\sin\frac{\pi\alpha}{2}\right)^{2/\alpha}, \tag{6.19}$$

and for $\beta \in [-1,0)$ (6.17) gives

$$\frac{\exp\{-2(1+\frac{1}{\alpha})(\gamma U')^\alpha\}}{2\,[(\gamma U')^\alpha]^{2/\alpha}} = \frac{\ln(4/p)}{n}\left(\frac{2(\varepsilon+1)}{\varepsilon}\right)^{2(1+\frac{1}{\alpha})}\left(6L(1-\beta)\frac{\Gamma(\alpha)}{\pi}\sin\frac{\pi\alpha}{2}\right)^{2/\alpha}, \tag{6.20}$$

and for $\beta = 0$ (6.17) gives

$$\frac{\exp\{-2(1+\frac{1}{\alpha})(\gamma U')^\alpha\}}{2\,[(\gamma U')^\alpha]^{2/\alpha}} = \frac{\ln(4/p)}{n}\left(\frac{2(\varepsilon+1)}{\varepsilon}\right)^{2(1+\frac{1}{\alpha})}\left(6L\frac{\Gamma(\alpha)}{\pi}\sin\frac{\pi\alpha}{2}\right)^{2/\alpha}, \tag{6.21}$$

where $L = L(\alpha)$ is given by (6.6). Note that for all β the solution U to equations (6.19)-(6.21) is expressed through the quantity $-(\gamma U)^\alpha = \Re\psi_X(u)$.

We illustrate Proposition 6.3 for the case $\beta = 0$, that is, solve the equation (6.21) for U'. Fix $p = 0.2$ and $\varepsilon = 0.5$. Then by (6.10) and (6.13) it must be choosen $n \ge 250$. By Nolan (2018b, Theorem 1.12, p. 14), for $S(\alpha,\beta,\gamma,\delta;0)$,

$$\lim_{x\to\infty} \frac{\mathbf{P}(X > x)}{\gamma^\alpha c(\alpha)(1+\beta)x^{-\alpha}} = 1, \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{\mathbf{P}(X \le -x)}{\gamma^\alpha c(\alpha)(1-\beta)x^{-\alpha}} = 1,$$

where $c(a)$ is given by (6.7). Hereby, following (6.6) and (6.8), for our illustrative example we take $L = 2$ in (6.21). For $\alpha = 0.1, 0.2, \ldots, 2$ consider the cases $\gamma = 1$ and $\gamma = 10$. The numerical solutions U' to (6.21) are obtained by a search function combining the so-called gold section procedure with the parabolic interpolation (e.g., Forsythe et al. (1977)), available by function `uniroot` in R-package `stats4` (R Core Team (2018)), and presented in Figure 6.7. In Figure 6.7 the solution U' to (6.21), that is, the upper bound for the solutions U to (6.16), increase as the values of α and n increase. However, for $\gamma = 10$ the scale of U' notably decrease for all α and n implying that the upper bound

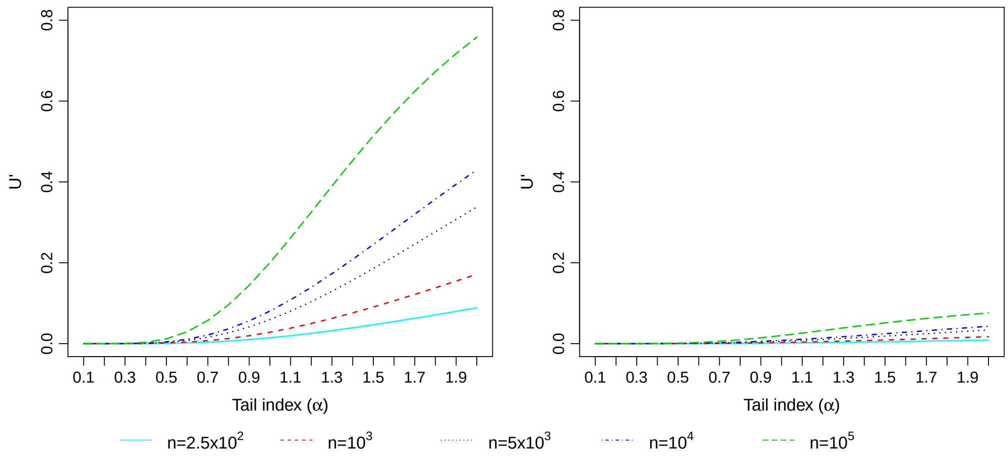

FIGURE 6.7: Solution to (6.21) plotted U' versus α for $\gamma = 1$ (on left) and for $\gamma = 10$ (on right) with $\varepsilon = 0.5, p = 0.2, L = 2$.

U' to (6.16) depends on the scale γ of the underlying stable distribution. Hereby, for fixed ε, p it seems complicated to set a general rule for $U' > 0$ for all values of α, γ, n. Krutto (2018) pointed out that the quantity $\Re\psi_X(U) = -(\gamma U)^\alpha$ should be considered instead. In Figure 6.8 the values of $-(\gamma U')^\alpha$ are presented. Note that $0 < U \leq U' < \infty$ is equivalent to $-(\gamma U)^\alpha \geq -(\gamma U')^\alpha$. In Figure

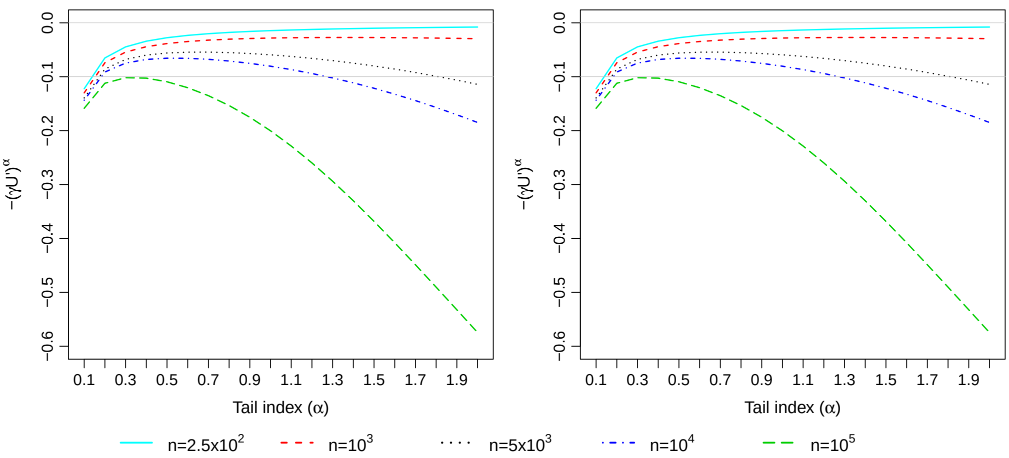

FIGURE 6.8: Solutions U' to (6.21) plotted for $-(\gamma U')^\alpha$ with $\gamma = 1$ (on left) and for $\gamma = 10$ (on right) with $\varepsilon = 0.5$,$p = 0.2$, $L = 2$.

6.8, for small values of n the quantity $-(\gamma U')^\alpha$ is almost constant among the values of $\alpha = 0.1, \ldots, 2$, and the quantity $-(\gamma U')^\alpha$ decreases when n (and α) increase. Also, as seen on the right of Figure 6.8 and (6.21), the change in $\gamma > 0$ has no influence for the quantity $-(\gamma U')^\alpha$. The results in Figure

6.8 imply there is a fixed level $r > 0$ such that

$$U \leq U' = \Re \hat{\psi}_n^{-1}(-r)$$

would satisfy (6.16) for any α, γ, n. Indeed, by Figure 6.8 if $n \geq 10^5$ then all $|u| < U < U'$ with $-(\gamma U')^\alpha = -0.1$ lead to $\varepsilon \leq 0.5$ with probability $1 - p \geq 0.8$ for any α and γ. Similar verity is visible in Figures 6.4, 6.5, 6.6: for each considered n and α the estimation error $\varepsilon \leq 0.5$, roughly, in the interval such that

$$\Re \hat{\psi}_n(u) \in [-0.5, 0],$$

at least, or equivalently such that

$$|u| \leq \Re \hat{\psi}_n^{-1}(-0.5).$$

Figure 6.8 and Figures 6.4, 6.5 and 6.6 suggest choosing $|u| < U < U'$ on the bases of $\Re \psi_X^{-1}(-r)$, where $r > 0$ is fixed, that is,

$$U' = \operatorname{argmin}_{u>0}(\Re \psi_X(u) + r).$$

Comparing to the empirical result in Figures 6.4, 6.5, 6.6 the results in Figure 6.8, that is, the solutions U' to (6.21), lead to more conservative estimate for r. By Figures 6.4, 6.5 for samples $n \geq 50$ it is suggested $r \leq 0.5$ while by Figure 6.8 for samples with $n \geq 10^5$ it is suggested $r \leq 0.1$.

6.2 Sample Based Selection of the Arguments of ECuF Estimators

ECuF estimators for the parameters of $S(\alpha, \beta, \gamma, \delta; 0)$ are obtained step-by-step: the estimators for α and γ are expressed through the real part of the empirical cumulant function $\Re \psi_n(u)$ at the pair (u_1, u_2) and the estimators for β and δ are expressed through $\Re \psi_n(u)$ and $\Im \psi_n(u)$ at the same pair (u_1, u_2), where $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$. Hereby, the interest is in estimating some lower bound $0 < \underline{u} < \infty$ and upper bound $0 < \underline{u} < \overline{u} < \infty$ such that $\underline{u} < u_1 < \overline{u}$, $\underline{u} < u_2 < \overline{u}$, $u_1 \neq u_2$ that would accurately estimate the real part of cumulant function of stable laws, and then also ECuF estimates. Corollary 6.2 gives for fixed $\varepsilon > 0$, $p \in (0,1)$, $\alpha \in (0,2]$, $\gamma > 0$, n the upper bound $U' > 0$ such that $\mathbf{P}\left(\sup_{|u| \leq U} |\Re \psi_n(u) - \Re \psi_X(u)| \leq \varepsilon\right) \geq 1 - p$ for every $0 < U < U'$, while the corresponding lower bound is zero. However, ECuF estimators are defined for non-zero values only, $u_1 > 0$, $u_2 > 0$, $u_1 \neq u_2$, and moreover, as $\psi_n(0) = \psi_X(0) = 0$ then at $u = 0$ cumulant function holds no info about the parameters of the distribution. The lower and upper bounds for the arguments of ECuF estimators are estimated on the basis of $\alpha_n = \alpha_n(u_1, u_2)$,

$$\alpha_n = \frac{\ln(-\Re \psi_n(u_1)) - \ln(-\Re \psi_n(u_2))}{\ln u_1 - \ln u_2} = \frac{\ln(-\Re \psi_n(u_2)) - \ln(-\Re \psi_n(u_1))}{\ln u_2 - \ln u_1}. \tag{6.22}$$

Proposition 6.4. *For every* $0 < \overline{u} < \infty$ *it holds*

$$\sup_{|u|\le\overline{u}} |\Re\psi_n(u) - \Re\psi_X(u)| \overset{a.s.}{\to} 0 \text{ as } n \to \infty,$$

and then for every $0 < \underline{u} < \overline{u}$

$$\sup_{\underline{u}\le|u|\le\overline{u}} |\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| \overset{a.s.}{\to} 0 \text{ as } n \to \infty.$$

Proof.

$$\sup_{\underline{u}\le|u|\le\overline{u}} |\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| = \sup_{\underline{u}\le|u|\le\overline{u}} \ln\frac{\Re\psi_n(u))}{\Re\psi_X(u))} = \ln \sup_{\underline{u}\le|u|\le\overline{u}} \frac{\Re\psi_n(u))}{\Re\psi_X(u))} \overset{a.s.}{\to} 0,$$

as $n \to \infty$. Note that as $\Re\psi_X(0)) = 0$ then in the proof the lower bound $\underline{u} > 0$ is required. □

Similarly to the proof of Proposition 6.1, for every fixed $u \in \mathbb{R}$,

$$|\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| \le \frac{\Delta'_n(u)}{-\Re\psi_X(u) - \Delta'_n(u)} = \frac{\Delta'_n(u)}{(\gamma|u|)^\alpha - \Delta'_n(u)},$$

where $\Delta'_n = |\Re\psi_n(u) - \Re\psi_X(u)|$. Hereby, if for $\nu > 0$

$$\frac{\Delta'_n(u)}{\gamma^\alpha|u|^\alpha - \Delta'_n(u)} \le \nu \Leftrightarrow (\gamma|u|)^{-\alpha}\Delta'_n(u) \le \frac{\nu}{\nu+1} < 1 \tag{6.23}$$

then

$$|\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| \le \nu.$$

Now, if (6.23) holds then $(\gamma|u|)^\alpha > \Delta'_n(u)$ and then for every $0 < \underline{u} < \infty$,

$$\sup_{\underline{u}\le|u|} |\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| \le \sup_{\underline{u}\le|u|} \frac{\Delta'_n(u)}{(\gamma|u|)^\alpha - \Delta'_n(u)} \le \frac{\sup_{\underline{u}\le|u|}\Delta'_n(u)}{(\gamma\underline{u})^\alpha - \sup_{\underline{u}\le|u|}\Delta'_n(u)} \le \nu$$

Hereby,

$$\mathbf{P}\Big(\sup_{\underline{u}\le|u|\le\overline{u}} |\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| \le \nu\Big) \ge \mathbf{P}\Big(\sup_{\underline{u}\le|u|\le\overline{u}} \Delta'_n(u) \le (\gamma\underline{u})^\alpha \frac{\nu}{\nu+1}\Big).$$

Let $R = R(u) = -\Re\psi_X(u) = (\gamma|u|)^\alpha$. Let $0 < \underline{r} < \overline{r} < 1$ denote the values such that $\overline{u} = R^{-1}(\overline{r})$ and $\underline{u} = R^{-1}(\underline{r})$. From now on consider a function $\varepsilon' = \varepsilon'(r,p,n)$ such that

$$\mathbf{P}\left(\sup_{|u|\le\overline{u}} |\Re\psi_n(u) - \Re\psi_X(u)| \le \varepsilon'\right) = \mathbf{P}\left(\sup_{|u|\le R^{-1}(\overline{r})} \Delta'_n(u) \le \varepsilon'\right) \ge 1 - p. \tag{6.24}$$

Proposition 6.5. *Let $\bar{r} > 0$ be such that $\varepsilon' = \varepsilon'(r,p,n) < \bar{r}$. Then for any $\underline{r} > 0$ such that $\varepsilon' < \underline{r} < \bar{r}$ it holds*

$$\mathbf{P}\left(|\alpha_n(\underline{u},\bar{u}) - \alpha| \leq 2\frac{\varepsilon'/(\underline{r}-\varepsilon')}{\ln\bar{u} - \ln\underline{u}}\right) \geq 1-p \tag{6.25}$$

where α_n is given by (6.22) *and* $\alpha \in (0,2]$.

Proof. By (6.23) and 6.24

$$\begin{aligned}
\mathbf{P}\left(\sup_{\underline{u}\leq|u|\leq\bar{u}} |\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| \leq \frac{\varepsilon'/(\gamma\underline{u})^\alpha}{1-\varepsilon'/(\gamma\underline{u})^\alpha}\right) &\geq \mathbf{P}\left(\sup_{\underline{u}\leq|u|\leq\bar{u}} \Delta_n'(u) \leq (\gamma\underline{u})^\alpha \frac{\nu}{\nu+1}\right) \\
&\geq \mathbf{P}\left(\sup_{|u|\leq R^{-1}(\bar{r})} \Delta_n'(u) \leq \varepsilon'\right) \\
&\geq 1-p.
\end{aligned}$$

Note that

$$\frac{\varepsilon'/(\gamma\underline{u})^\alpha}{1-\varepsilon'/(\gamma\underline{u})^\alpha} = \frac{\varepsilon'}{\underline{r}-\varepsilon'}.$$

Therefore

$$\mathbf{P}\left(\sup_{\underline{u}\leq|u|\leq\bar{u}} |\ln(-\Re\psi_n(u)) - \ln(-\Re\psi_X(u))| \leq \frac{\varepsilon'}{\underline{r}-\varepsilon'}\right) \geq 1-p.$$

Hence, with probability $1-p$ we have

$$\begin{aligned}
|\ln(-\Re\psi_n(\bar{u})) - \alpha(\gamma - \ln\bar{u})| &\leq \frac{\varepsilon'}{\underline{r}-\varepsilon'}, \\
|\ln(-\Re\psi_n(\underline{u})) - \alpha(\gamma - \ln\underline{u})| &\leq \frac{\varepsilon'}{\underline{r}-\varepsilon'},
\end{aligned}$$

and therefore

$$|\ln(-\Re\psi_n(\bar{u})) - \ln(-\Re\psi_n(\underline{u})) - \alpha(\ln\bar{u} - \ln\underline{u})| \leq 2\frac{\varepsilon'}{\underline{r}-\varepsilon'}.$$

Hereby,

$$\begin{aligned}
\mathbf{P}\left(\left|\frac{\ln(-\Re\psi_n(\bar{u})) - \ln(-\Re\psi_n(\underline{u}))}{\ln\bar{u} - \ln\underline{u}} - \alpha\right| \leq \frac{2\frac{\varepsilon'}{\underline{r}-\varepsilon'}}{\ln\bar{u} - \ln\underline{u}}\right) &= \mathbf{P}\left(|\alpha_n(\underline{u},\bar{u}) - \alpha| \leq 2\frac{\varepsilon'/(\underline{r}-\varepsilon')}{\ln\bar{u} - \ln\underline{u}}\right) \\
&\geq 1-p
\end{aligned}$$

□

Note that
$$\ln\overline{u} - \ln\underline{u} = \frac{1}{\alpha}(\ln\overline{r} - \ln\underline{r})$$
and then
$$\frac{2\frac{\varepsilon'}{\underline{r}-\varepsilon'}}{\ln\overline{u} - \ln\underline{u}} = \frac{2\alpha\frac{\varepsilon'}{\underline{r}-\varepsilon'}}{\ln\overline{r} - \ln\underline{r}} \leq \frac{4\frac{\varepsilon'}{\underline{r}-\varepsilon'}}{\ln\overline{r}| - \ln\underline{r}}.$$
Therefore,
$$\mathbf{P}\left(|\alpha_n - \alpha| \leq 4\frac{\varepsilon'/(\underline{r}-\varepsilon')}{\ln\overline{r} - \ln\underline{r}}\right) \geq \mathbf{P}\left(|\alpha_n - \alpha| \leq 2\frac{\varepsilon'/(\underline{r}-\varepsilon')}{\ln\overline{u} - \ln\underline{u}}\right) \geq 1-p.$$

By Proposition 6.5, the arguments (u_1, u_2) of ECuf estimators are suggested such that $0 < \underline{u} \leq u_1 < u_2 \leq \overline{u} < \infty$ where
$$\underline{u} \geq \text{argmin}_{u>0}(\Re\psi_X(u) + \underline{r}) \tag{6.26}$$
and
$$\overline{u} \leq \text{argmin}_{u>0}(\Re\psi_X(u) + \overline{r}) \tag{6.27}$$
with $0 < \underline{r} < \overline{r} < 1$ being fixed. However, fixing $0 < \underline{r} < \overline{r} < 1$ is complicated. Proposition 6.3 gave a quite conservative estimate for $0 < \overline{r} < 1$: by Figure 6.8 it is suggested that if $n \geq 10^5$ then $\overline{r} \leq 0.1$. On the other hand, on the bases of empirical results in Figures 6.4, 6.5, 6.5 it is suggested that if $n \geq 50$ then $\overline{r} \leq 0.5$. By (6.25) the estimation error is as smaller as bigger is u_2/u_1, that is, $\underline{r}$ and $\overline{r}$ should not be too close. In Section 5.2 the tail index α was estimated via (6.22) at $(u_1, u_2) \in (0,1] \times (0,1]$. In addition, the influence of the change in u_1 for fixed $u_2 = 1$ is assessed through the estimation error of $\hat{\alpha}_n$. More precisely, for fixed sample size, $n = 1000$, the quantity MAE($\hat{\alpha}_n$) is studied for $K = 200$ replicates from $S(\alpha, \beta = 0.5; 0)$ with $\alpha = 0.2, 1, 1.8$ at $u_1 \in (0, 100]$ and $u_2 = 1$. Results are presented in Figure 6.9. In Figure 6.9, as expected by (6.22), the MAEs are big in the area where u_1 is close to u_2. The ECuF estimates for $\alpha = 1.8$ are smallest at u_1 around 0.03, for $\alpha = 1$ at u_1 around 0.01 and for $\alpha = 0.2$ at u_1 around 0.

In conclusion, the arguments u_1, u_2 should be chosen such that $u_1 \geq \underline{u}$ is not too small, that is, $\underline{r}$ is not too close too 0, and $u_2 \leq \overline{u}$ is not too big, that is, $\overline{r}$ is not too close too big, and by (6.25) u_1, u_2 should not be too close, that is, r_1 and r_2 should not be too close. Hereby, on the basis of Section 5.2, Corollary 6.2, Proposition 6.5, and Figures 6.1-6.9, a following sample based selection is proposed by the substituting principle (e.g., Knight (1999, Section 4.5, p. 190)).

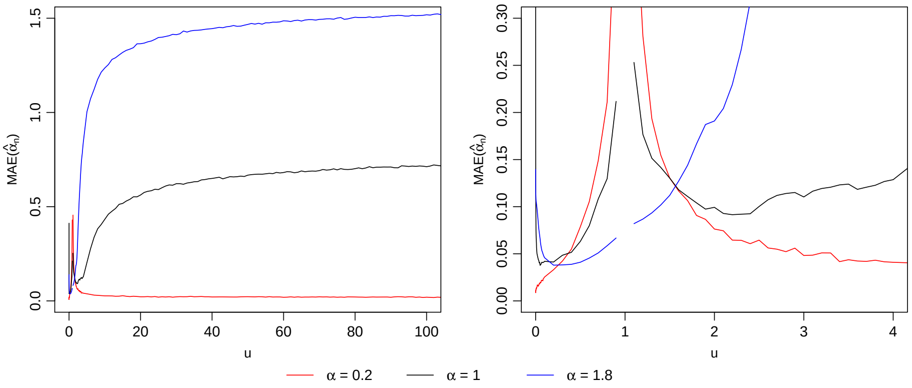

FIGURE 6.9: MAEs of $\hat{\alpha}_n(u_1, u_2)$ at $u_1 \in (0, 100]$ and $u_2 = 1$.

Argument–Selection–Rule 1. *For a general stable law the ECuF-estimators in Definitions 4.2 are obtained at* $u_1 > 0$ *and* $u_2 > 0$ *satisfying*

$$\underline{u} \geq \operatorname{argmin}_{u>0}(\Re\hat{\psi}_n(u) + 0.1) \tag{6.28}$$

and

$$\overline{u} \leq \operatorname{argmin}_{u>0}(\Re\hat{\psi}_n(u) + 0.5) \tag{6.29}$$

where $\Re\hat{\psi}_n(u)$ *is the realization of* $\Re\psi_n(u)$*, given by (1.32).*

Implying Argument–Selection–Rule requires solving noisy equations (6.28) and (6.29). For solving (6.28) and (6.29) with respect to u various numerical methods can be used. For example, a look-up procedure with the mid-range rule or some simple one-dimensional search function (see, e.g., Brent (1973)). For illustration, an example of the selection procedure by the Argument–Selection–Rule 1 is presented. For that, single replicates with $n = 1000$ from $X \sim S(\alpha, \beta = 0.5; 0)$ where

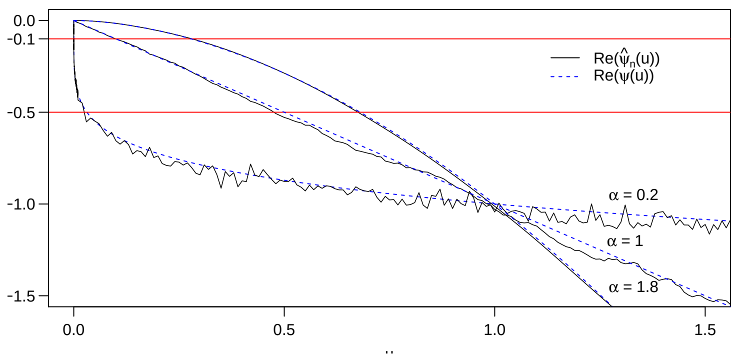

FIGURE 6.10: Argument–Selection–Rule 1 for a sample from $S(\alpha, 0.5; 0)$.

$\alpha = 0.2, 1.0, 1.8$ are simulated. In Figure 6.10 the graphs of corresponding $\Re\psi_X(u)$ (dashed blue lines) and graphs of $\Re\hat{\psi}_n(u)$ of the simulated samples (solid black lines) at $u > 0$ are presented. In Figure 6.10, the solid red lines show the levels where $\Re\psi(u) = -0.1$ and $\Re\psi(u) = -0.5$. The corresponding u_1 and u_2 where obtained through the look-up procedure (`R` (R Core Team (2018)) function `which.min` or `which.max`) and the mid-range rule. Based on Argument–Selection–Rule 1, the ECuF estimators for the parameters of $X \sim S(\alpha, \beta = 0.5; 0)$ for $\alpha = 0.2$ are suggested to evaluate at $u_1 = 0.6 \times 10^{-5}$ and $u_2 = 0.03$; for $\alpha = 1$ are suggested to evaluate at $u_1 = 0.11$ and $u_2 = 0.48$; and for $\alpha = 1.8$ are suggested to evaluate at $u_1 = 0.29$ and $u_2 = 0.68$. The graphs of the

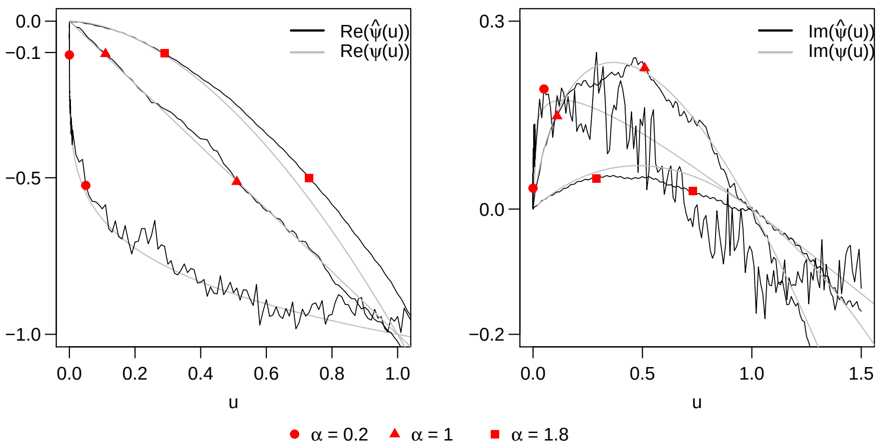

FIGURE 6.11: The Argument–Selection–Rule 1 for a sample from $S(\alpha, 0.5; 0)$.

real and imaginary parts of cumulant functions and the selected u_1, u_2 and the corresponding values of $\Im\hat{\psi}_n(u_1)$ and u_2 from $\Im\hat{\psi}_n(u_2)$ are illustrated by Figure 6.11.

Remark 6. *Höpfner and Rüschendorf (1999, p. 157) proposed a selection of u_1 and u_2 for Press (1972) estimators in symmetric stable law centered around zero ($\beta = \delta = 0$) where the arguments u_3, u_4 are not used. Their selection is based on the real part of empirical characteristic function. Note that for symmetric stable laws around zero $\varphi_X(u) = \Re\varphi_X(u) = \exp\{\Re\psi_X(u)\}$. For the Press (1972) estimators for α and γ^α they stated the following rule: u_1 is based on the 0.3-quantile and u_2 on the 0.7-quantile of the empirical characteristic function. In terms of the real part of empirical cumulant function it can be written $\Re\hat{\psi}_n(u_1) = -0.36$ and $\Re\hat{\psi}_n(u_2) = -1.2$.*

Chapter 7

Monte–Carlo Simulations for ECuF Estimators with the Argument–Selection–Rule

In this section, Monte–Carlo simulations for assessing the quality of ECuF estimators at u_1, u_2 selected by Argument–Selection-Rule 1 are carried out. Without loss of generality, standard stable laws are studied, $\delta = 0$ and $\gamma = 1$, and by reflection property (e.g., Uchaikin and Zolotarev (1999, Property (2), p. 99), Nolan (2018b, Proposition 1.11, p. 12)) only non-negative values of β are used, $\beta \in [0,1]$. In simulation study $K = 100$ replicates $S(\alpha, \beta; 0)$ are generated. The ECuF estimates $\hat{\alpha}_n = g_1(\hat{\boldsymbol{b}}_n)$, $\ln \hat{\gamma}_n = g_2(\hat{\boldsymbol{b}}_n)$, $\hat{\beta}_n = g_3(\hat{\boldsymbol{b}}_n)$, and $\hat{\delta}_n = g_4(\hat{\boldsymbol{b}}_n)$ are obtained from (3.12) through (3.15b) at $\hat{\boldsymbol{b}}_n$. When $|\hat{\alpha}_n - 1| < 0.01$, then it is set $\hat{\alpha}_n = 1$. For $\hat{\alpha}_n = 1$ the estimates of β and δ are calculated by (3.14b) and (3.15b), respectively, and for $\hat{\alpha}_n \neq 1$ by (3.14a) and (3.15a), respectively. The arguments u_1 and u_2 are obtained from (6.28) and (6.29) by a search function combining the so-called gold section procedure with the parabolic interpolation (e.g., Forsythe et al. (1977)), available by function `uniroot` in `R`-package `stats4` (R Core Team (2018)). Note that in ECuF estimation procedure the values $\Re\psi_n(u_1)$ and $\Re\psi_n(u_2)$ are used, i.e., they are not replaced by -0.1 and -0.5. In results the admissible parameter values are used: the values of $\hat{\beta}_n$ are replaced with $\hat{\beta}_n = \min(\max(\hat{\beta}_n, 0), 1)$ and the values of $\hat{\alpha}_n$ are replaced with $\hat{\alpha}_n = \min(\max(\hat{\alpha}_n, 0.001), 2)$. Overview of estimation methods for the parameters of stable laws is given in Sections 2.6 and 2.7. For comparison purposes, the ECuF estimates are compared with the estimates by the following algorithmic methods: the maximum likelihood (ML) based estimators by Nolan (2001), the empirical characteristic function (EChF) based estimators by Kogon and Williams (1998), and the quantile based (QB) estimators by McCulloch (1986). The closed-form logarithmic moments (Log), fractional lower order moments (FLOM) and extreme value theory (EVT) methods by Kuruoglu (2001) were not considered because they do not provide estimators for the location parameter δ and, as mentioned in Kuruoglu (2001), though well-performing (in the sense of the estimation error) in general, did not outperform the EChF methods in estimating parameters α and γ. All statistical computing (including for ECuF estimators) and graphics are done by open-source free-software `R` (R Core Team (2018)) while simulations and the estimates of ML, EChF and QB methods are made by its package `STABLE`® (Robust Analysis Inc. (2017)). Similarly to Kogon and Williams (1998) and Kuruoglu (2001) all results are reported in the terms of the root mean-square error (RMSE) of the parameter estimates,

$$\mathrm{RMSE}(\hat{\theta}_n) = \sqrt{\tfrac{1}{K}\textstyle\sum_{k=1}^{K}\left(\theta - \hat{\theta}_n(k)\right)^2} \tag{7.1}$$

where $K = 100$ is the number of replications, θ is the true parameter value and $\hat{\theta}_n(k)$ is the estimate of the parameter from the k^{th} sample, $k = 1, 2, \ldots, K$.

7.1 Estimating $S(\alpha, \beta; 0)$ from Samples with $n = 5000$ and $n = 50$

From $S(\alpha, \beta = 0.5; 0)$ with $\alpha = 0.2, 0.3, 0.5, 0.8, 1, 1.2, 1.5, 1.8, 2$ the $K = 100$ replicates with size $n = 5000$ are simulated. The RMSE of $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\delta}_n$ versus the values of tail index α for the ECuF, ML, EChF, QB (solid red line, dashed, dotted and dash-dot lines, respectively) estimates are shown in Figure 7.1.

The results for the tail index α: According to RMSEs of α, as shown in Figure 7.1 (upper left), the ECuF and EChF methods remarkably outperform other estimators at the lower values of α while the ECuF method performs slightly better than the EChF method. For $0.5 \leq \alpha \leq 1.8$, the methods give similar results except for the ML method, which performs better. For $\alpha > 1.8$, the ECuF, EChF, and ML methods perform in a similar manner while they all outperform the QB method.

The results for the asymmetry index β: According to RMSEs of β, as shown in Figure 7.1 (upper right), the ECuF and ML methods notably outperform the other estimators at the lower values of α while for $0.5 \leq \alpha \leq 1.5$ the ML and QB methods perform better than ECuF and EChF methods. For $1.5 \leq \alpha \leq 2$ the ECuF, EChF and ML methods outperform the QB method (the fact that RMSE $(\hat{\beta}_n) \to \infty$ as $\alpha \to 2$ is not relevant in practice as β means little when $\alpha \to 2$). In estimating the tail index α and asymmetry index β our ECuF method performs most steadily over the whole space of the values of parameter α.

The results for the scale parameter γ and the shift parameter δ: According to RMSEs of $\hat{\gamma}_n$, as shown in Figure 7.1 (lower left), the ECuF method outperforms other methods at the lower values of α. For $0.5 \leq \alpha \leq 1.2$, all methods perform similarly. Based on RMSEs of the shift parameter δ, as shown in in Figure 7.1 (lower right), the proposed ECuF method does not outperform others.

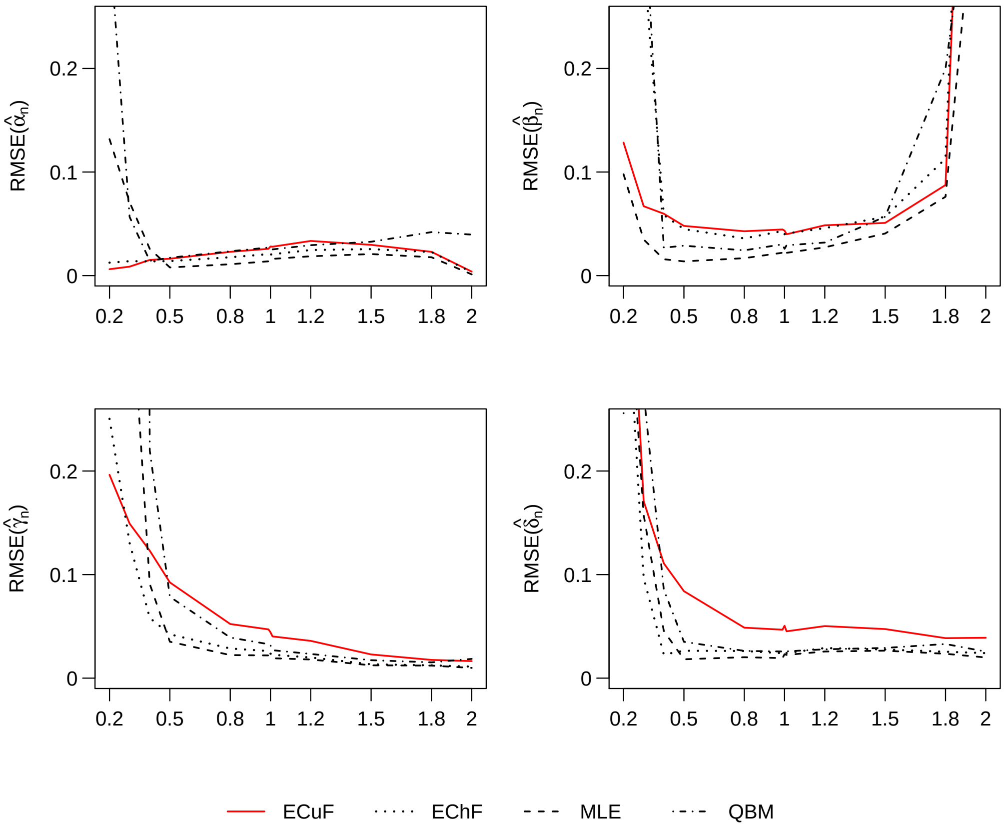

FIGURE 7.1: For $n = 5000$ the performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimators for α (upper left), β (upper right), γ (lower left) and δ (lower right) plotted as RMSE versus tail index α.

The results for all parameters from the samples with $n = 50$: To see, if any of these methods work well when the sample size is small for $K = 100$ replicates from $S(\alpha, 0.5; 0)$, $\alpha = 0.5, 1, 1.5$, each with $n = 50$, are simulated. For the ECuF, ML, EChF, QB (solid red line, dashed, dotted and dash-dot lines, respectively) estimates the RMSEs of $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\delta}_n$ versus the tail index α are shown in Figure 7.2.

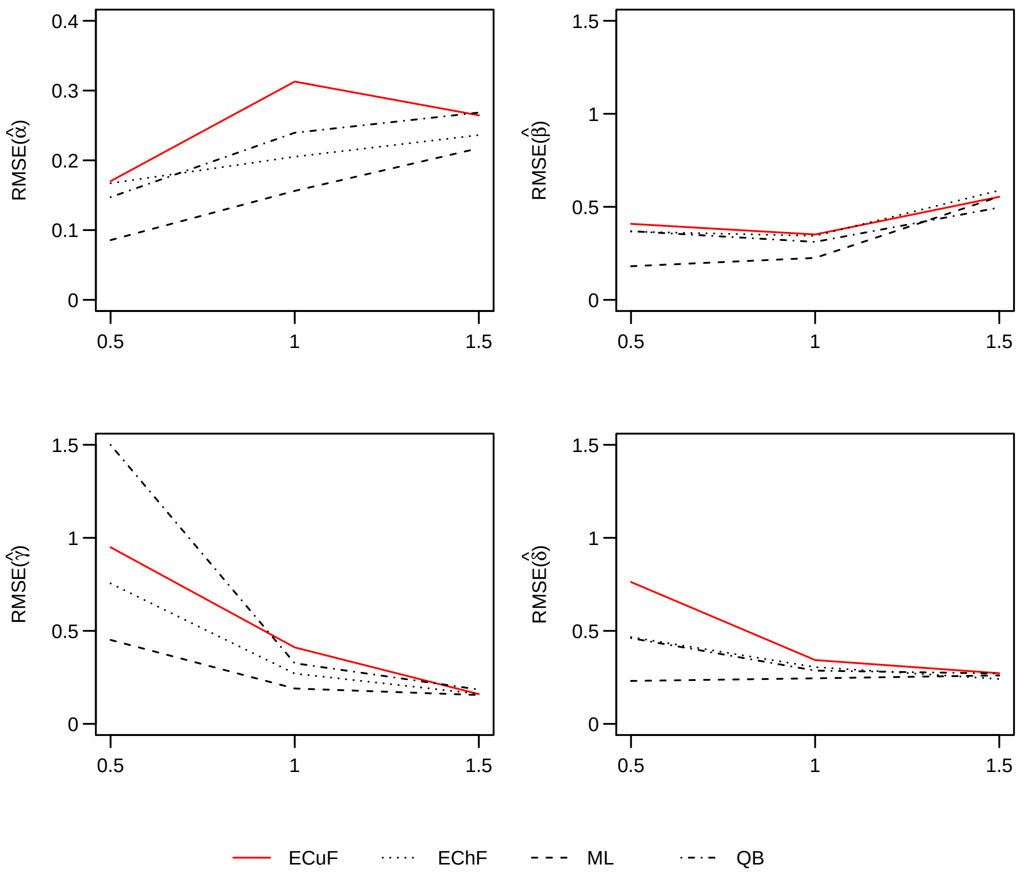

FIGURE 7.2: For $n = 50$ the performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimators for α (upper left), β (upper right), γ (lower left) and δ (lower right) plotted as RMSE versus tail index α.

Comparing the results in Figure 7.2 (i.e., $n = 50$) with the results at $\alpha = 0.5, 1, 1.5$ in Figure 7.1 (i.e., $n = 5000$), the values of RMSEs of ECuF estimators for all parameters are remarkably bigger, as it is natural to expect according to theoretical results in Chapter 6. However, sample size influences the RMSEs of all methods while among the studied estimation methods the RMSEs of MLE estimates turned out smallest for all parameters. In the next Section 7.2 results on more sample sizes versus the RMSEs of estimates are presented.

Comparing the computational speed: The extension time of ECuF estimators compared to other methods is illustrated by the total user's CPU time, measured by the `R` (R Core Team (2018)) function `proc.time`. For comparing the CPU time the sample size $n = 1000$ is fixed, and $K = 20$ replicates from $S(\alpha,\beta;0)$ with $\alpha = 0.2, 1, 1.8$ and $\beta = 0, 0.5, 1$ are estimated. Corresponding user's CPU time in seconds is given in Table 7.1.

TABLE 7.1: Comparison of user's CPU time.

	ECuF estimates		
β	$\alpha = 0.2$	$\alpha = 1$	$\alpha = 1.8$
0.0	0.006708204	0.008366600	0.007745967
0.5	0.008366600	0.005477226	0.006324555
1.0	0.008944272	0.005000000	0.008062258
	ML estimates		
β	$\alpha = 0.2$	$\alpha = 1$	$\alpha = 1.8$
0	0.1082359	0.03646917	0.04842520
0.5	0.1305565	0.03794733	0.05196152
1	0.2952372	0.05554278	0.06024948
	EChF estimates		
β	$\alpha = 0.2$	$\alpha = 1$	$\alpha = 1.8$
0	0.005	0.002236068	0.006324555
0.5	0.000	0.002236068	0.003162278
1	0.000	0.002236068	0.004472136
	QB estimates		
β	$\alpha = 0.2$	$\alpha = 1$	$\alpha = 1.8$
0	0.000000000	0.000000000	0
0.5	0.003162278	0.004472136	0
1	0.000000000	0.000000000	0

7.2 Estimating $S(\alpha,\beta;0)$ from Samples with Various Sizes

Simulations for $K = 100$ replicates from $S(\alpha,\beta;0)$, $\alpha = 0.3, 1, 1.8$, $\beta = 0, 1$, each with sample sizes $n = 200$, 500, 1000, 3000, 5000 are carried out. The RMSEs of $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\delta}_n$ of the ECuF, ML, EChF, QB (solid red line, dashed, dotted and dash-dot lines, respectively) estimates versus the sample size are presented in Appendix F , Figures F.1-F.4. Figure F.1 shows results for α, Figure F.2 for β, Figure F.3 for γ, and Figure F.4 for δ. Not surprisingly, the effectiveness of all methods discussed depends on the sample size (the greater the sample size the better the performance). However, the results in Appendix F show that the ECuF method performs best at the lower values of α and in comparison to other methods, it is more robust to changes in the value of β.

Chapter 8

Applications

Two examples of modelling via stable laws are considered. In Section 8.1 stable laws are fitted on Estonian property and causality (PC) insurance claim sizes, presented in Krutto (2016), and in Section 8.2 on Danish fire insurance claim sizes. In Section 8.1 ECuF and reduced values' ECuF estimates are evaluated at all pairs (u_1, u_2) given by Table 5.1. In Section 8.2 the ECuF estimators are obtained by the Argument–Selection–Rule 1. All statistical computing, evaluation of ECuF estimates and graphics are done by the open-source free-software `R` (R Core Team (2018)).

8.1 Case study I: Estonian PC Insurance Data

A dataset[1] of Estonian property and causality (PC) insurance, containing fire, natural forces and other property insurance claim sizes in euros (EUR) by legal persons in a calendar year, is studied. There are a total of $n = 2802$ observations. The k-th 100-quantile, denoted by q_p, $p = k/100$, mean and standard deviation (sd) of claim sizes are given in Table 8.1 and the histogram in Figure 8.1. Based on Table 8.1 and histogram in Figure 8.1 the data of claim sizes are heavy tailed and right-

TABLE 8.1: Summary of Estonian PC insurance claim sizes (EUR).

$q_{0.00}$	$q_{0.25}$	$q_{0.50}$	$q_{0.75}$	$q_{0.99}$	$q_{1.00}$	**mean**	**sd**
15.3	358.0	955.0	6703.0	102206.1	1166000.0	2781.0	42011.15

skewed. When fitting a stable law it would be expected to get the estimate for β close to 1 and the estimate for α not too close to 2 (i.e., normal law) or 0 (because the interquartile range of data is quite large and values of claim sizes are not that condensed).

[1] Data were communicated by Meelis Käärik, (meelis.kaarik@ut.ee).

The claim sizes distribution is modelled via $S(\alpha,\beta,\gamma,\delta;1)$ by ECuF and reduced values' ECuF estimates at all pairs (u_1,u_2) given by Table 5.1. In the process of estimation a simple non-parametric bootstrap with replacement is applied.

ECuF estimates ECuF estimators for the parameters of $S(\alpha,\beta,\gamma,\delta;1)$ are given by Definition 4.1. In Table 8.2 the means of ECuF estimates for $K=200$ bootstrap replicates are presented. Claim size data has a interquartile range of 1165642 and standard deviation of 42011.1. The ECuF estimates in Table 8.2 illustrate that scale statistic with such large scaling influence the accuracy of ECuF estimates. Indeed, at most of the pairs of u_1,u_2, the estimates for scale γ turned out to be infinite, implying that ECuF estimates may produce meaningless results, as also mentioned in Paulson et al. (1975) about the Press (1972) estimates. In addition, the means of estimates for β are not close to 1 (as expected), and those of α are close to 0 (as not expected).

TABLE 8.2: Means of ECuF estimates (at u_1,u_2 from Table 5.1) for the parameters of $S(\alpha,\beta,\gamma,\delta;1)$ on Estonian PC insurance claim sizes.

u_1	u_2	Mean $(\hat{\alpha}_n)$	Mean $(\hat{\beta}_n)$	Mean $(\hat{\gamma}_n)$	Mean $(\hat{\delta}_n)$
0.03	0.09	0.13	1.62	9×10^{77}	–3.77
0.03	0.9	0.01	6.70	Inf	1.53
0.03	9	0.03	2.98	Inf	–0.10
0.03	90	0.02	–0.76	Inf	–0.02
0.3	0.09	0.04	–0.42	Inf	3.05
0.3	0.9	–0.17	–1.22	Inf	1.93
0.3	9	–0.01	–1.46	Inf	–0.17
0.3	90	–0.01	0.79	Inf	–0.02
3	0.09	0.01	–2.35	Inf	0.12
3	0.9	0.14	2.18	3×10^{125}	–0.87
3	9	–0.04	0.36	5×10^{184}	–0.29
3	90	–0.03	0.59	Inf	–0.02

Inf–infinity

As the median of claim sizes data is greater than 1, $q_{0.50}=955.0$, then the reduced values' ECuF estimators, given by Definition 4.3, are suggested.

Reduced values' ECuF estimates In Table 8.2 the means of reduced values' ECuF estimates[2] and the coefficient of variation (c_v), which is the ratio of the standard deviation to the mean (see, e.g., Everitt (1998)), are presented.

It follows that the reduced values' ECuF estimators yield much better estimates. Compared to Table 8.2, the means of reduced values' ECuF estimates in Table 8.3 are more meaningful and less varying. It confirms that reduced values' ECuF estimators are useful for the cases where data median (in absolute value) is more than 1. Table 8.3 is sorted increasingly by $c_v(\hat{\alpha}_n)$. The mean of reduced

[2]Based on Definition 4.3 the estimates in Table 8.3 are presented for the original (not-reduced) claim sizes.

TABLE 8.3: Reduced values' ECuF estimates for the parameters of $S(\alpha,\beta,\gamma,\delta;1)$ on Estonian PC insurance claim sizes.

u_1	u_2	Mean ($\hat{\alpha}_n$)	c_v ($\hat{\alpha}_n$)	Mean ($\hat{\beta}_n$)	c_v ($\hat{\beta}_n$)	Mean ($\hat{\gamma}_n$)	c_v ($\hat{\gamma}_n$)	Mean ($\hat{\delta}_n$)	c_v ($\hat{\delta}_n$)
0.03	9	0.71	0.030	1.19	0.059	382.46	0.073	–432.25	0.206
3	0.09	0.72	0.030	1.12	0.042	444.48	0.045	–574.15	0.209
0.3	9	0.67	0.032	1.17	0.046	410.33	0.059	–335.48	0.181
0.03	0.9	0.77	0.039	1.05	0.057	568.07	0.057	–1113.55	0.328
0.03	90	0.56	0.039	1.78	0.079	119.88	0.192	–103.29	0.181
0.3	0.9	0.80	0.048	1.06	0.055	578.91	0.059	–1460.33	0.342
0.3	90	0.48	0.058	1.87	0.085	181.45	0.175	–85.05	0.183
3	0.9	0.60	0.064	1.18	0.061	475.25	0.046	–283.09	0.345
0.3	0.09	0.75	0.069	1.00	0.072	523.66	0.145	–819.13	0.793
0.03	0.09	0.78	0.099	1.09	0.101	581.48	0.304	–1989.91	1.272
3	9	0.60	0.107	0.94	0.133	484.41	0.092	–139.00	0.608
3	90	0.33	0.151	2.00	0.157	690.79	0.194	–60.62	0.204

c_v–coefficient of variation

values' ECuF estimates at $u_1 = 3$, $u_2 = 0.09$ has the smallest value of $c_v(\hat{\alpha}_n)$ and $c_v(\hat{\beta}_n)$. Hereby, stable distribution $S(\alpha = 0.72, \beta = 1, \gamma = 444, \delta = -574; 1)$ is proposed as a model for Estonian PC insurance claim sizes.

For comparison the maximum likelihood (ML) based estimators by Nolan (2001), the empirical characteristic function (EChF) based estimators by Kogon and Williams (1998), and the quantile based (QB) estimators by McCulloch (1986) are obtained[3] by a free-software `STABLE`® (Nolan (2005)). Results are presented in Table 8.4.

TABLE 8.4: Modelling Estonian PC insurance claim sizes via $S(\alpha,\beta,\gamma,\delta;1)$.

Estimation Method	**Fitted Stable Law**
Reduced values' ECuF	$S(\alpha = 0.72, \beta = 1, \gamma = 444, \delta = -574; 1)$
Characteristic function based	$S(\alpha = 0.78, \beta = 1, \gamma = 581, \delta = -1117; 1)$
Maximum likelihood based	$S(\alpha = 0.60, \beta = 1, \gamma = 606, \delta = -189; 1)$
Quantile based	$S(\alpha = 0.82, \beta = 1, \gamma = 1213, \delta = -3258; 1)$

The estimates for the parameters of the stable laws $S(\alpha,\beta,\gamma,\delta;1)$ in Table 8.4 are quite similar: all estimation methods propose a maximally skewed stable distribution ($\beta = 1$) with a negative location parameter ($\delta < 0$) and index of stability less than 1 (but not close to 0).

To illustrate the matches of the fitted stable laws the corresponding (numerical) density functions (by `R`-package `stabledist` Würtz and Mächler (2016)) are presented in Figure 8.1 (for the full data) and Figure 8.2 (for the tail area). According to Figures 8.1 and 8.2, the reduced values' ECuF estimates seem to give the best match for the Estonian PC insurance claim size data.

[3] An overview of the estimation methods of the parameters of stable laws is given in Sections 2.6 and 2.7.

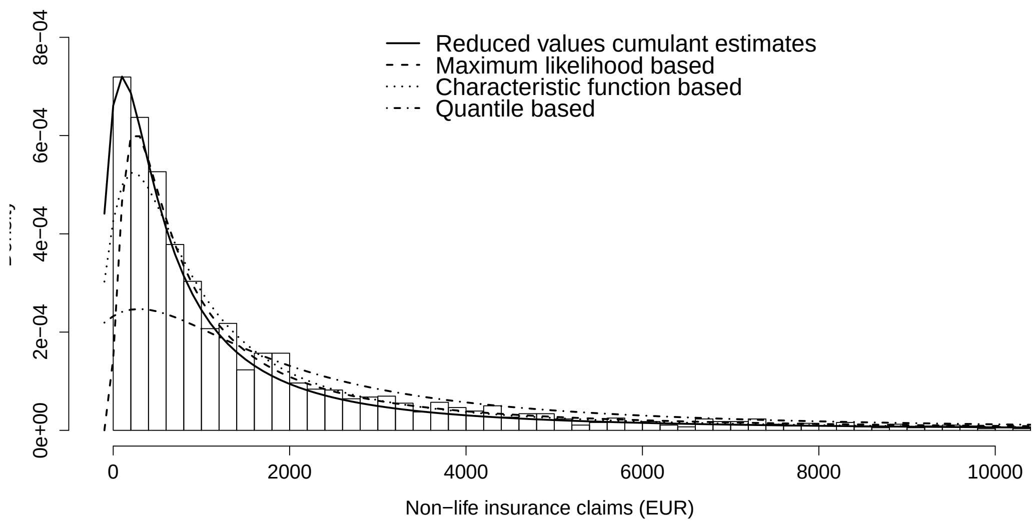

FIGURE 8.1: Histogram of Estonian PC insurance claim sizes and the densities of the fitted stable laws (given by Table 8.4).

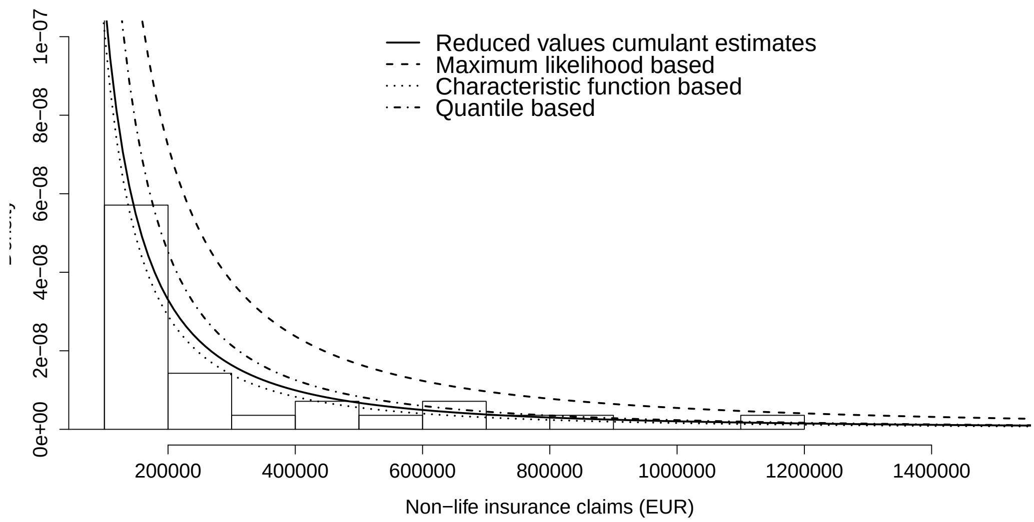

FIGURE 8.2: Histogram of the tail values of Estonian PC insurance claim sizes and the densities of the fitted stable laws (given by Table 8.4).

8.2 Case study II: Danish Fire Insurance Data

A dataset of Danish fire insurance claim sizes (losses), available in R-package `SMPracticals` (Davison (2015)), is studied. Claim sizes are given in millions of Danish krone (DKK) and are not adjusted for inflation over time. For exploratory analysis see Embrechts et al. (2013, Example 6.2.9). The dataset corresponds to the period 1980-1990, inclusive, and there are a total of $n = 2493$ observations. The k-th 100-quantile, denoted by q_p, $p = k/100$, mean and standard deviation of claim sizes are given by Table 8.5 and the histogram in Figure 8.3.

TABLE 8.5: Summary of Danish fire insurance losses (in millions DKK).

$q_{0.00}$	$q_{0.25}$	$q_{0.50}$	$q_{0.75}$	$q_{0.99}$	$q_{1.00}$	**mean**	**sd**
0.31	1.16	1.63	2.65	24.61	263.25	3.06	7.98

Based on Table 8.5 and histogram in Figure 8.3 the distribution of Danish fire insurance claim sizes is heavy-tailed and right-skewed. When fitting stable laws it would be expected the estimates for β be close to 1, and the estimates for α not too close to 2 (i.e., normal law) or 0 (because the interquartile range of data is quite large and values of claim sizes are not that condensed).

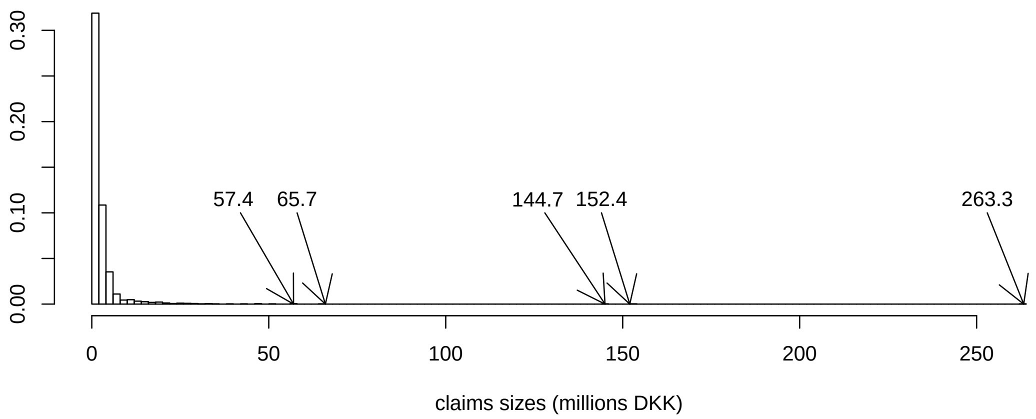

FIGURE 8.3: Danish fire insurance data.

Applying Argument–Selection–Rule 1 on Danish fire insurance claim sizes the values of u_1, u_2 for ECuF estimators are selected as

$$u_1 = 0.186,$$
$$u_2 = 1.036.$$

Note that if the selection would have been done on the original values in millions (DKK) then $u_1 = 1.86 \times 10^{-7}$ and $u_2 = 1.036 \times 10^{-6}$.

For comparison[4], the maximum likelihood (ML) based (Nolan (2001)), the empirical characteristic function (EChF) based (Kogon and Williams (1998)), and the quantile based (QB) estimates (McCulloch (1986)) are found by `STABLE`® Robust Analysis Inc. (2017).

The goodness-of-fit within different estimates is assessed[5] by log-likelihood function (e.g., Knight (1999)) based criteria. Given $\boldsymbol{y}$ is the realization of $\boldsymbol{Y}$, the log-likelihood function is denoted by $l(\boldsymbol{\theta})$, and for stable laws the numerically estimated by the joint density function $f^*(\boldsymbol{y}|\boldsymbol{\theta}) = \prod_{j=1}^{n} f^*(y_j|\boldsymbol{\theta})$,

$$l(\boldsymbol{\theta}) = \ln f^*(\boldsymbol{y}|\boldsymbol{\theta}) = \sum_{j=1}^{n} \ln f^*(y_j|\boldsymbol{\theta}) \tag{8.1}$$

where $f^*(y_j|\boldsymbol{\theta})$ is the numerically estimated density of a stable distribution, depending on $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \delta)'$. Table 8.6 is ordered increasingly by the values of negative log-likelihood (NLL) function, which is defined as $\text{NLL} = -l(\theta)$. In addition, the risk at tail is assessed by the 99th percentile, $q_{0.99}$. Results are presented in Table 8.6.

TABLE 8.6: Modelling of Danish fire insurance claim sizes via stable law $S(\alpha, \beta, \gamma, \delta; 0)$.

Method	**Fitted Stable Law**	NLL	$q_{0.99}$
ML	$S(\alpha = 0.88, \beta = 0.99, \gamma = 0.37, \delta = 1.32; 0)$	3855.5	46.6
ECuF	$S(\alpha = 0.93, \beta = 1, \gamma = 0.45, \delta = 1.32; 0)$	3940.3	43.9
EChF	$S(\alpha = 0.95, \beta = 1, \gamma = 0.46, \delta = 1.31; 0)$	3969.4	40.5
QB	$S(\alpha = 1.01, \beta = 1, \gamma = 0.74, \delta = 1.63; 0)$	4202.4	49.7

In Table 8.6, the ECuF estimates with the Argument–Selection–Rule 1 give lower value of NLL than those of the EChF and QE estimates illustrating that ECuF estimators compare favourably with other estimation methods for stable laws.

In Table 8.7 the comparison[6] of the approximation of stable laws (and its special cases Cauchy and Lévy distributions) with other distribution models[7] proposed for Danish fire insurance claim sizes (see, e.g., Bakar et al. (2015)), Miljkovic and Grün (2016)), is presented. In Table 8.7, in addition to NLL, the Akaike Information Criterion (AIC), given by AIC$= -2l(\boldsymbol{\theta}) + 2p$, where p is the number of parameters of distribution, is provided. Except for stable, Cauchy and Levy distributions, the numerical results given by Table 8.7 can be found in Bakar et al. (2015) and Miljkovic and Grün (2016). However, in this thesis the values of NLL, AIC in Table 8.7 are recalculated (by `R`–function `mle` for maximum likelihood method in `R`-package `stats4` (R Core Team (2018))) based on the

[4]The overview of estimation methods for the parameters of stable laws is given in Sections 2.6 and 2.7.

[5]There are various (characteristic function based) goodness-of-fit tests available, see, e.g, Ushakov (1999, Section 3.10), Matsui and Takemura (2008), Meintanis (2016). However, in modelling Danish loss data the common goodness-of-fit criteria are based on log-likelihood (see, e.g., Bakar et al. (2015), Miljkovic and Grün (2016)), and for comparison purpose the same are used in this thesis.

[6]To the best of our knowledge, there are no publications that fit stable laws on the Danish fire insurance losses data.

[7]For composite models proposed for the Danish fire insurance losses data see, e.g., Bakar et al. (2015), for mixture models, e.g., Miljkovic and Grün (2016), and for a combination of both, see, e.g., Reynkens et al. (2017).

TABLE 8.7: Some distributions fitted on Danish fire insurance claim sizes.

Model	NLL	AIC	$q_{0.99}$
Burr	3835.1	7676.2	31.0
Stable	3855.5	7719.1	46.6
Inv. Burr	3966.8	7941.8	14.0
Gen. Pareto	4102.3	8210.6	13.0
Loglogistic	4280.6	8565.2	10.0
Log-Normal	4433.9	8871.8	10.8
Cauchy	4563.5	9131.0	17.1
Inv. Pareto	4647.2	9299.4	161.7
Levy	5039.8	10083.5	7647.5
Pareto	5051.9	10109.8	17.1
Gamma	5243.0	10490.1	12.6
Weibull	5270.5	10544.9	14.8

closed form densities (available in various `R`-packages, e.g., `actuar` (Dutang et al. (2008)) and `rmutil` (Swihart and Lindsey (2018)). Table 8.7 is ordered by the values of AIC. It follows that only Burr distribution has lower values of NLL and AIC than stable distribution, and comparing to other proposed models (distributions) stable laws give better fit on Danish fire insurance claim sizes. However, compared to Burr distribution, the proposed stable law gives a much higher value for the risk at tail, that means, it is a more conservative model for the risk of extreme losses.

Conclusions

The aim of this thesis was to address the problem of estimating the parameters of general stable laws. This was done by providing new insights to the method of closed-form estimators used by Press (1972). The main results of this thesis are as follows.

(i) It is proven that the parameters of general stable laws can be expressed through the real and imaginary parts of the cumulant function at two arbitrary different arguments u_1, u_2 on the positive real line.

(ii) The Press (1972) estimation procedure is reformulated to use only two (rather than four) arguments, called the ECuF estimators.

(iii) The asymptotic normality for the real and imaginary parts of the empirical cumulant function of arbitrary distribution is proven.

(iv) The asymptotic normality for the ECuF estimators (throughout the interior of the parameter space) is proven.

(v) A sample based Argument–Selection–Rule for choosing the arguments u_1, u_2 of ECuF estimators is proposed.

In conclusion, via exhaustive Monte–Carlo simulations and two applications it is shown that under the proposed Argument–Selection–Rule the performance of the proposed ECuF estimators may be considered effective in general and they make an attractive alternative to the algorithmic methods for estimating the parameters of stable laws. This results contradict (reject) the established view in literature (e.g., (Paulson et al., 1975, p. 168), (Borak et al., 2005, Section 1.4.3)). More precisely, the ECuF estimators applied with the Argument–Selection–Rule compare favourably with the commonly used quantile (QB), empirical characteristic function (EChF), and maximum likelihood (ML) based estimation methods for stable laws. In estimating the tail index, α, the closed-form ECuF estimators outperform other methods in the case when $\alpha < 0.5$, while at the higher values of α the closed-form ECuF estimators perform similarly to the algorithmic methods. In estimating the skewness parameter β, scale parameter γ, and location parameter δ the ECuF method outperforms algorithmic methods in some cases but not always. The main argument in favour of the ECuF estimators (with the Argument–Selection–Rule) is their computational simplicity; there is no need for data standardization, no restrictions have to be put to the parameter space, and they perform steadily across the values of the tail index α and skewness index β. For practitioners concerned about data with the heaviest tails, i.e.,where $\alpha < 0.5$, the closed-form ECuF estimators make an attractive alternative to the algorithmic methods. Based on the applications, the ECuF estimators can be successfully used in practice, and stable laws can be suggested for modelling non-life insurance claim sizes distributions.

An area of further research is developing the optimal implementation of the Argument–Selection–Rule. Another unresolved question is determining the bias of ECuF estimators and the possible selection of u_1, u_2 based on minimizing the asymptotic variance.

Appendix A

Characteristic Functions of Stable Laws

Figure A.1. The graphs of the characteristics function of $S(\alpha,1;1)$ (with characteristic function given by (2.11)) for $\alpha = 2, 1.8, 1.5, 1.2, 1.1, 1, 0.9, 0.8, 0.5, 0.2$.
Figure A.2. The graphs of the characteristics function of $S(\alpha,1;0)$ (with characteristic function given by (2.20)) for $\alpha = 2, 1.8, 1.5, 1.2, 1.1, 1, 0.9, 0.8, 0.5, 0.2$.

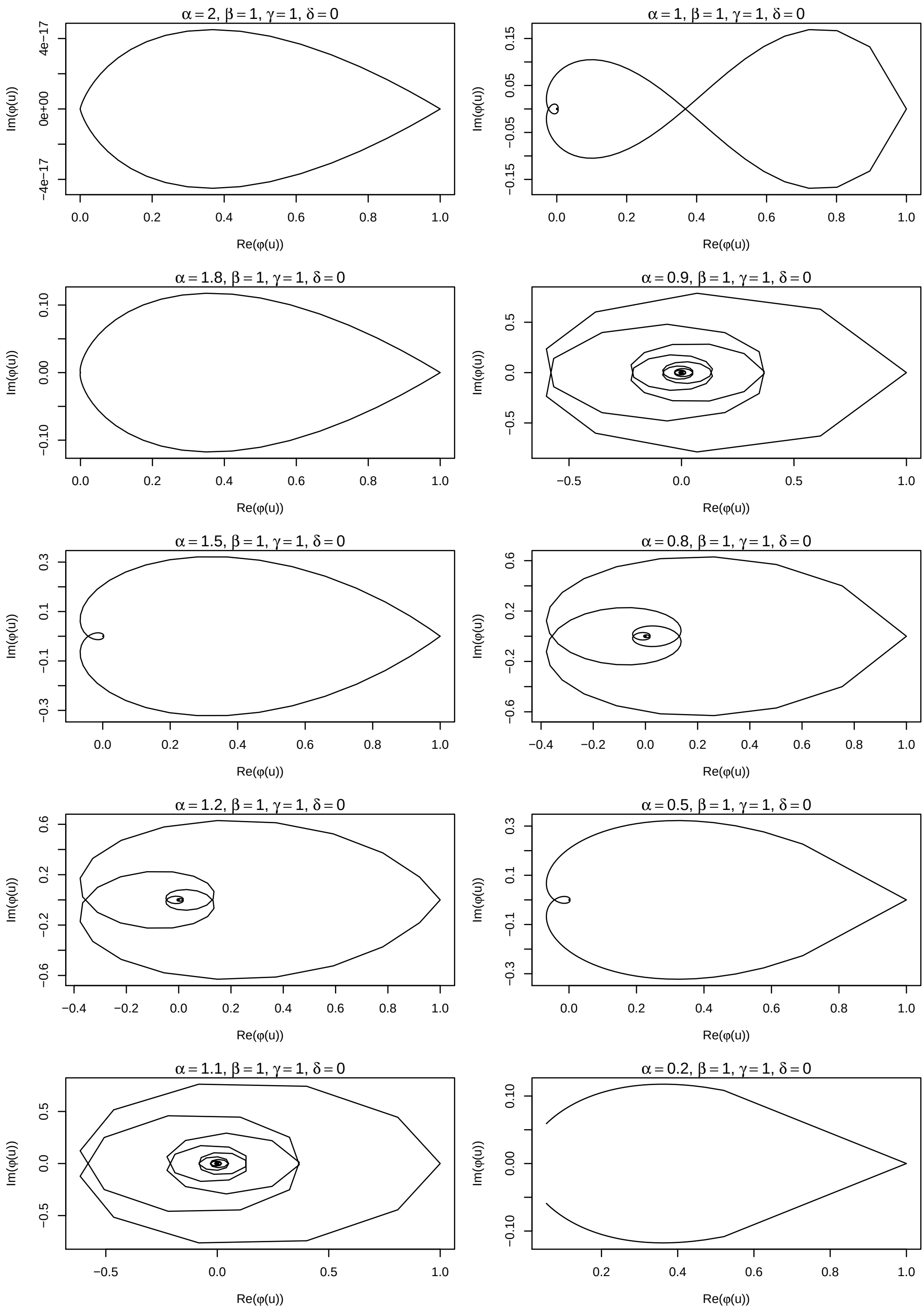

FIGURE A.1: Characteristic functions of $S(\alpha, 1; 1)$.

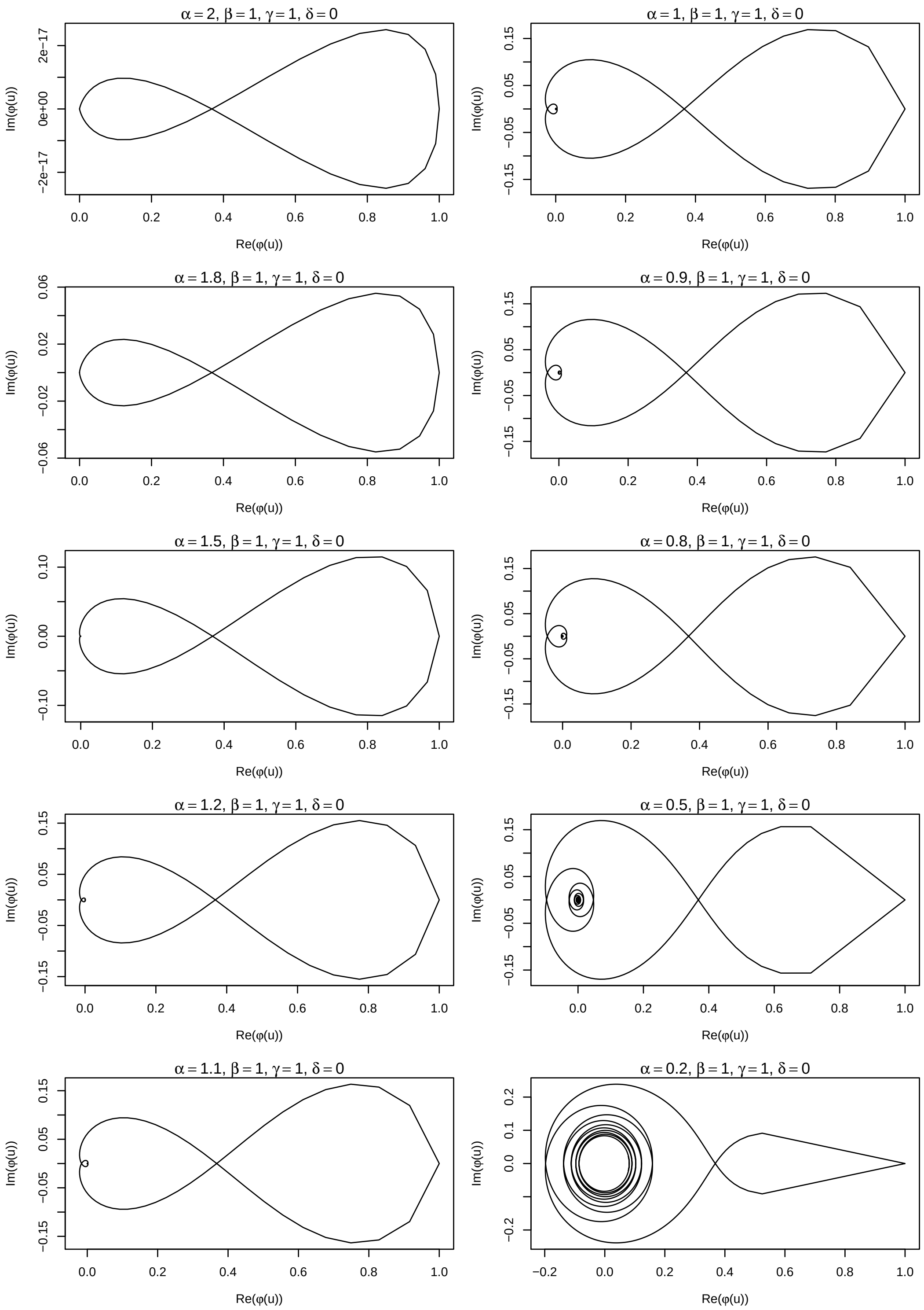

FIGURE A.2: Characteristic functions of $S(\alpha,1;0)$.

Appendix B

Absolute Value, Real and Imaginary Parts of Characteristic Functions

Figure B.1. The graphs of absolute value, real and imaginary parts of the characteristic functions of standard stable distributions $S(\alpha, \beta = 1; 1)$ (with characteristic function given by (2.11)) for $\alpha = 2, 1.8, 1.5, 1.2, 1.1, 1, 0.9, 0.8, 0.5, 0.2$.

Figure B.1. The graphs of absolute value, real and imaginary parts of the characteristic functions of standard stable distributions $S(\alpha, \beta = 1; 0)$ (with characteristic function given by (2.20)) for $\alpha = 2, 1.8, 1.5, 1.2, 1.1, 1, 0.9, 0.8, 0.5, 0.2$.

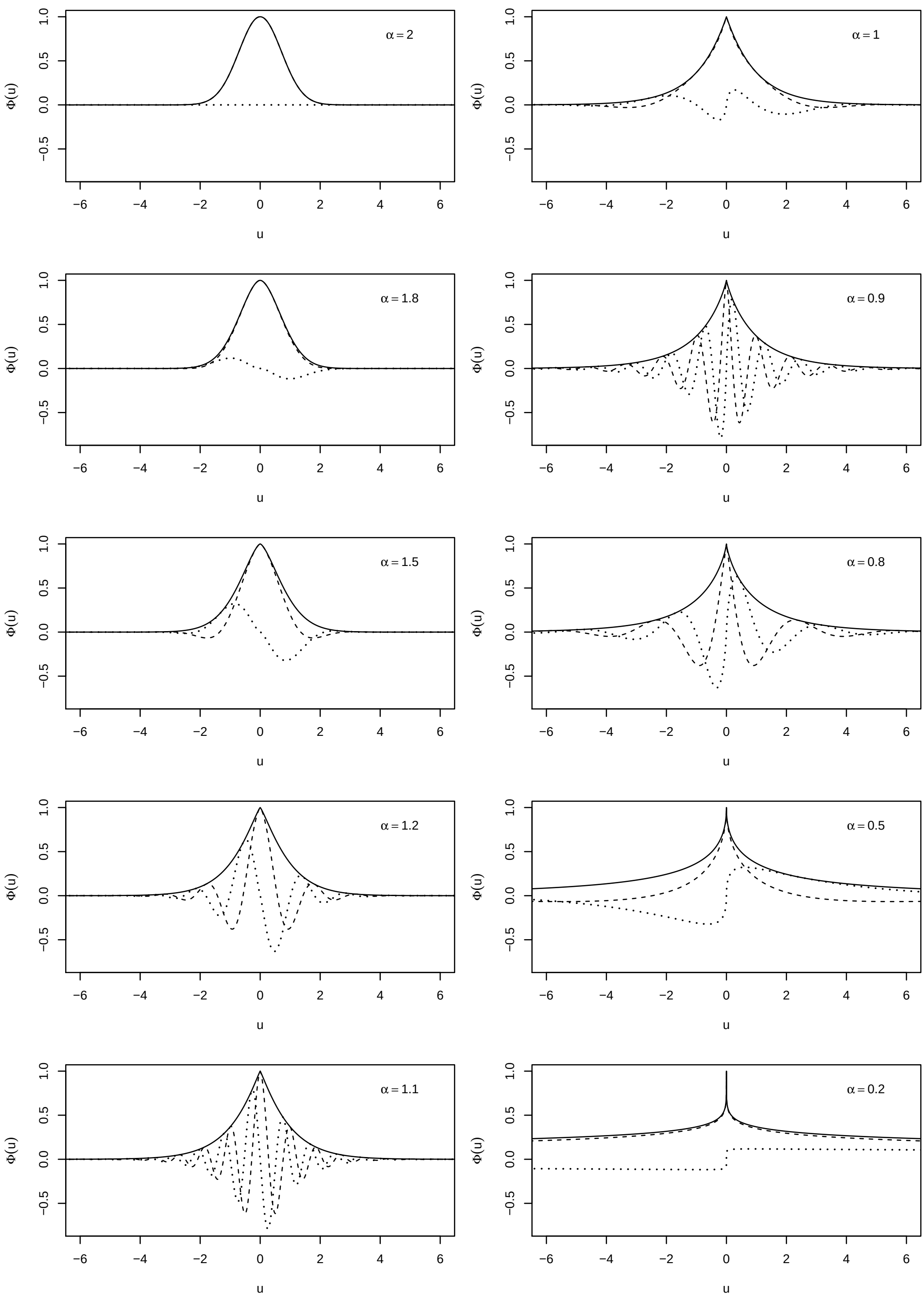

FIGURE B.1: The absolute value (solid line), the real part (dashed line) and imaginary part (dotted line) of the characteristic function of $S(\alpha, 1; 1)$.

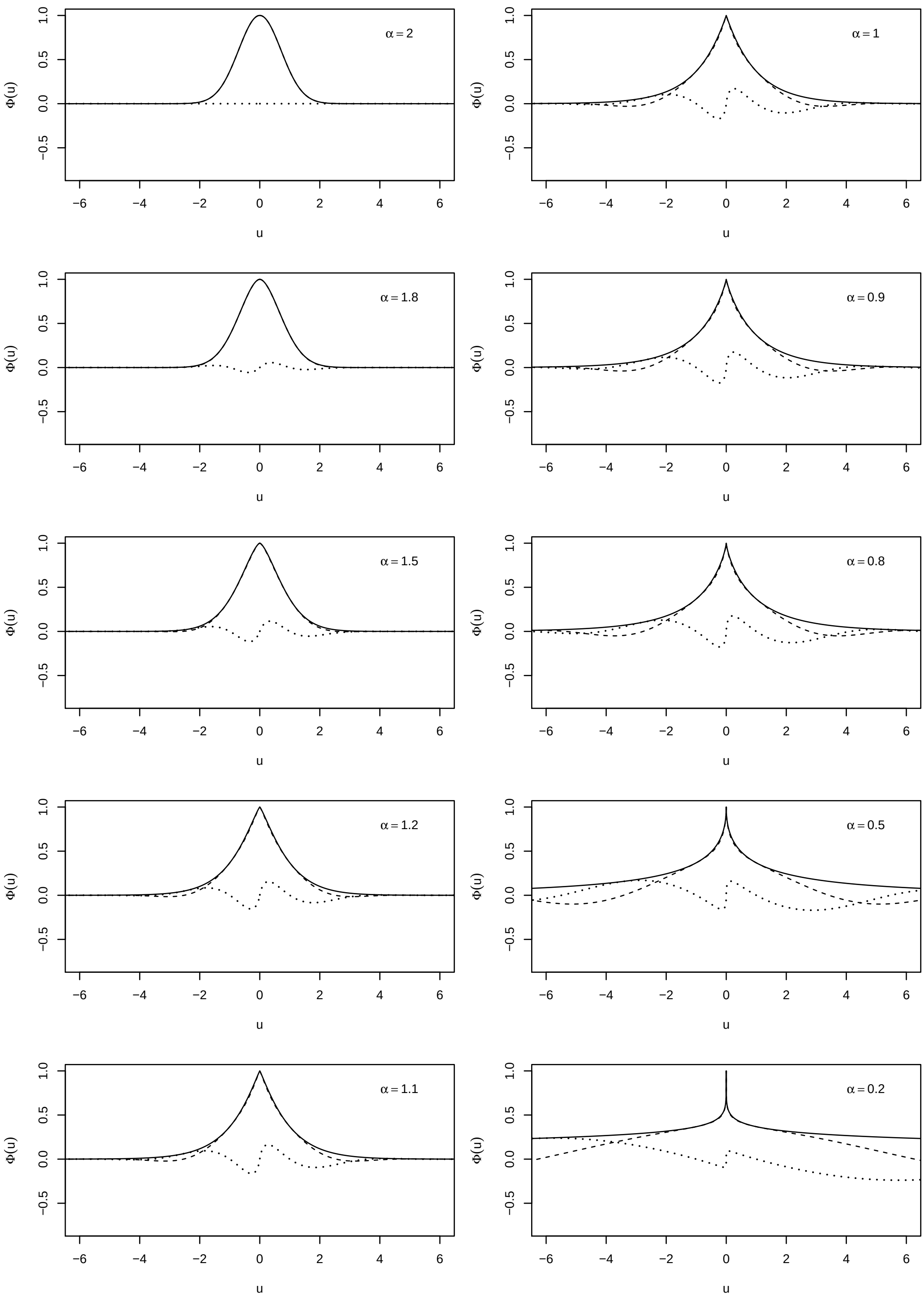

FIGURE B.2: The absolute value (solid line), the real part (dashed line) and imaginary part (dotted line) of the characteristic function of $S(\alpha, 1; 0)$.

Appendix C

Absolute Value, Real and Imaginary Parts of Cumulant Functions

Figure C.1. The graphs of absolute value, real and imaginary parts of the cumulant functions of standard stable distributions $S(\alpha, \beta = 1; 1)$ (with the real part given by (2.9) and imaginary part by (2.10)) for $\alpha = 2, 1.8, 1.5, 1.2, 1.1, 1, 0.9, 0.8, 0.5, 0.2$.
Figure C.2. The graphs of the absolute, real and imaginary parts of the cumulant functions of standard stable distributions $S(\alpha, \beta = 1; 0)$ (with the real part of cumulant function given by (2.21) and imaginary part by (2.22)) for $\alpha = 2, 1.8, 1.5, 1.2, 1.1, 1, 0.9, 0.8, 0.5, 0.2$.

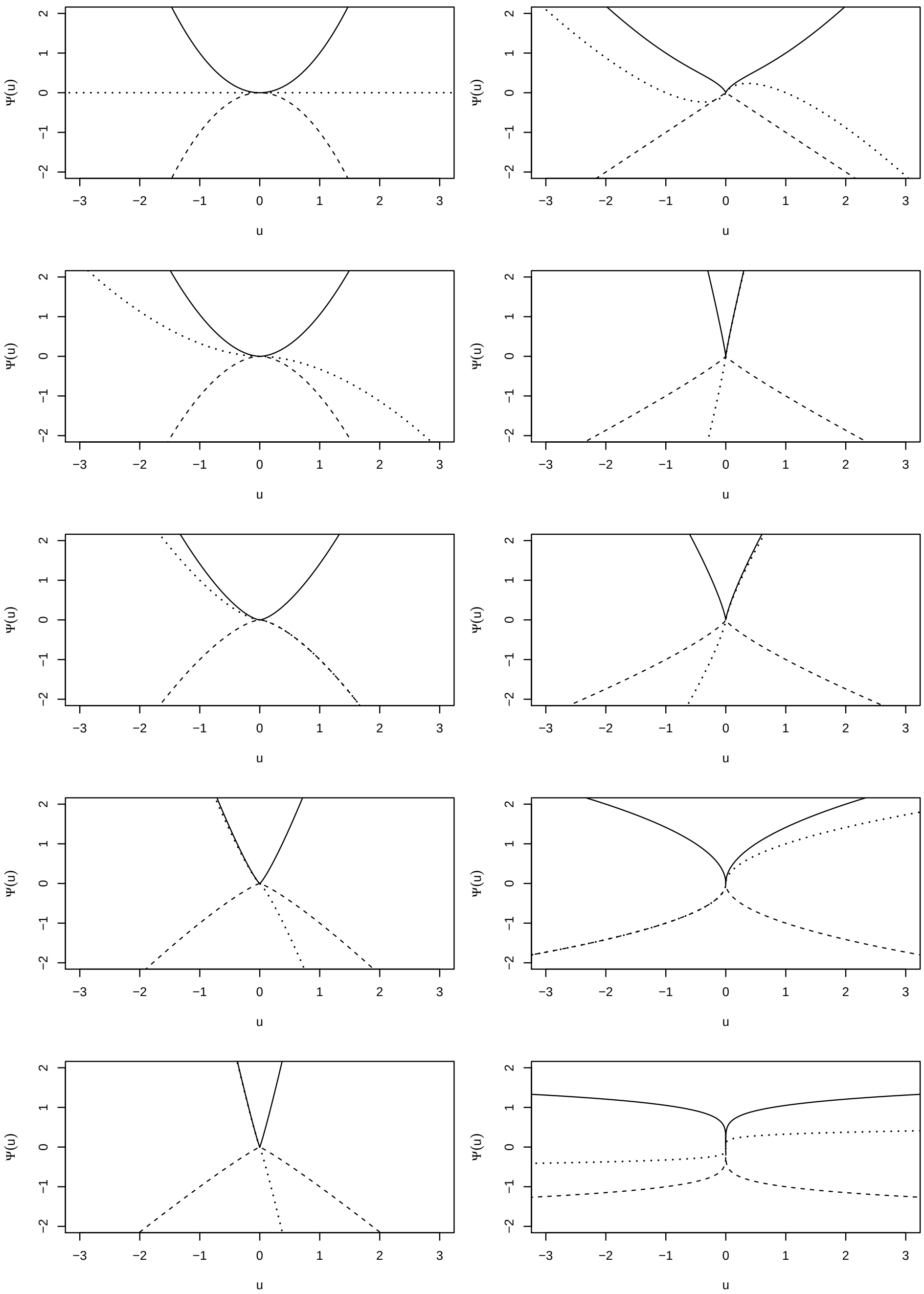

FIGURE C.1: The absolute value (solid line), the real part (dashed line) and imaginary part (dotted line) of the cumulant function of $S(\alpha, 1; 1)$.

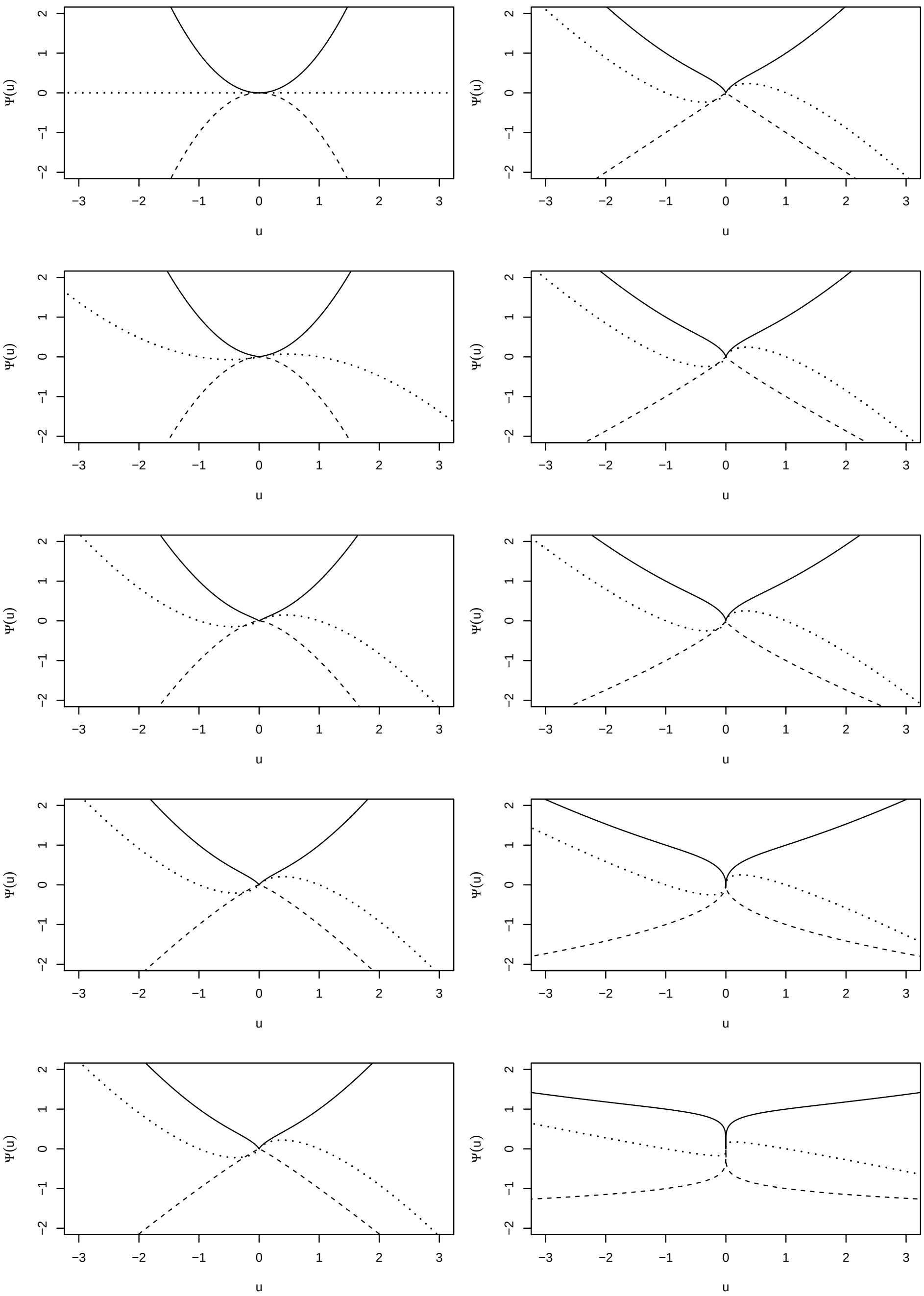

FIGURE C.2: The absolute value (solid line), the real part (dashed line) and imaginary part (dotted line) of the cumulant function of $S(\alpha, 1; 0)$.

Appendix D

Reduced values' ECuF Estimates at Selection of (u_1, u_2)

Table D.1. The MSEs of ECuF estimates for $K = 200$ replicates, each with size of $n = 10^5$, from $S(\alpha, \beta; 1)$ with $\alpha = 0.5$ and $\beta = 0$ at all pairs of u_1, u_2 in Table 5.1.

Table D.2. The MSEs of ECuF estimates for $K = 200$ replicates, each with size of $n = 10^5$, from $S(\alpha, \beta; 1)$ with $\alpha = 0.5$ and $\beta = 1$ at all pairs of u_1, u_2 in Table 5.1.

Table D.3. The MSEs of ECuF estimates for $K = 200$ replicates, each with size of $n = 10^5$, from $S(\alpha, \beta; 1)$ with $\alpha = 1.5$ and $\beta = 0$ at all pairs of u_1, u_2 in Table 5.1.

Table D.4. The MSEs of ECuF estimates for $K = 200$ replicates, each with size of $n = 10^5$, from $S(\alpha, \beta; 1)$ with $\alpha = 1.5$ and $\beta = 1$ at all pairs of u_1, u_2 in Table 5.1.

TABLE D.1: The MSEs of ECuF estimates for $K = 200$ replicates from the stable law $S(\alpha = 0.5, \beta = 0, \gamma = 1, \delta = 0; 1)$ each with size $n = 10^5$.

$(u_1$	$u_2)$	MSE $(\hat{\alpha}_n)$	MSE $(\hat{\beta}_n)$	MSE $(\hat{\gamma}_n)$	MSE $(\hat{\delta}_n)$
0.03	0.0	0.0000	0.0002	0.0001	0.0003
0.03	0.9	0.0000	0.0001	0.0002	0.0000
0.03	9	0.0059	0.0012	0.0425	0.0006
0.03	90	0.0392	0.0091	0.3183	0.0000
0.3	0.09	0.0000	0.0002	0.0001	0.0001
0.3	0.9	0.0002	0.0007	0.0011	0.0002
0.3	9	0.0163	0.0086	0.2077	0.0006
0.3	90	0.0755	0.0311	7.7057	0.0000
3	0.09	0.0007	0.0005	0.0002	0.0008
3	0.9	0.0045	0.0231	0.1267	0.0030
3	9	0.1520	4×10^1	Inf	0.0014
3	90	0.2151	3×10^1	Inf	0.0000

Inf–infinity

TABLE D.2: The MSEs of ECuF estimates for $K = 200$ replicates from the stable law $S(\alpha = 0.5, \beta = 0, \gamma = 1, \delta = 0; 1)$ each with size $n = 10^5$.

$(u_1$	$u_2)$	MSE $(\hat{\alpha}_n)$	MSE $(\hat{\beta}_n)$	MSE $(\hat{\gamma}_n)$	MSE $(\hat{\delta}_n)$
0.03	0.0	0.0000	0.0002	0.0001	0.0003
0.03	0.9	0.0000	0.0001	0.0002	0.0000
0.03	9	0.0059	0.0012	0.0425	0.0006
0.03	90	0.0392	0.0091	0.3183	0.0000
0.3	0.09	0.0000	0.0002	0.0001	0.0001
0.3	0.9	0.0002	0.0007	0.0011	0.0002
0.3	9	0.0163	0.0086	0.2077	0.0006
0.3	90	0.0755	0.0311	7.7057	0.0000
3	0.09	0.0007	0.0005	0.0002	0.0008
3	0.9	0.0045	0.0231	0.1267	0.0030
3	9	0.1520	4×10^1	Inf	0.0014
3	90	0.2151	3×10^1	Inf	0.0000

Inf–infinity

TABLE D.3: The MSEs of ECuF estimates for $K = 200$ replicates from the stable law $S(\alpha = 0.5, \beta = 0, \gamma = 1, \delta = 0; 1)$ each with size $n = 10^5$.

$(u_1$	$u_2)$	MSE ($\hat{\alpha}_n$)	MSE ($\hat{\beta}_n$)	MSE ($\hat{\gamma}_n$)	MSE ($\hat{\delta}_n$)
0.03	0.0	0.0000	0.0002	0.0001	0.0003
0.03	0.9	0.0000	0.0001	0.0002	0.0000
0.03	9	0.0059	0.0012	0.0425	0.0006
0.03	90	0.0392	0.0091	0.3183	0.0000
0.3	0.09	0.0000	0.0002	0.0001	0.0001
0.3	0.9	0.0002	0.0007	0.0011	0.0002
0.3	9	0.0163	0.0086	0.2077	0.0006
0.3	90	0.0755	0.0311	7.7057	0.0000
3	0.09	0.0007	0.0005	0.0002	0.0008
3	0.9	0.0045	0.0231	0.1267	0.0030
3	9	0.1520	4×10^1	Inf	0.0014
3	90	0.2151	3×10^1	Inf	0.0000

Inf–infinity

TABLE D.4: The MSEs of ECuF estimates for $K = 200$ replicates from the stable law $S(\alpha = 0.5, \beta = 0, \gamma = 1, \delta = 0; 1)$ each with size $n = 10^5$.

$(u_1$	$u_2)$	MSE ($\hat{\alpha}_n$)	MSE ($\hat{\beta}_n$)	MSE ($\hat{\gamma}_n$)	MSE ($\hat{\delta}_n$)
0.03	0.0	0.0000	0.0002	0.0001	0.0003
0.03	0.9	0.0000	0.0001	0.0002	0.0000
0.03	9	0.0059	0.0012	0.0425	0.0006
0.03	90	0.0392	0.0091	0.3183	0.0000
0.3	0.09	0.0000	0.0002	0.0001	0.0001
0.3	0.9	0.0002	0.0007	0.0011	0.0002
0.3	9	0.0163	0.0086	0.2077	0.0006
0.3	90	0.0755	0.0311	7.7057	0.0000
3	0.09	0.0007	0.0005	0.0002	0.0008
3	0.9	0.0045	0.0231	0.1267	0.0030
3	9	0.1520	4×10^1	Inf	0.0014
3	90	0.2151	3×10^1	Inf	0.0000

Inf–infinity

Appendix E

Reduced Values' ECuF Estimates at $(0.03, 0.09)$

Table E.1. The MSEs of ECuF estimates (ECuF) and reduced values' ECuF (RVECuF) estimates at $u_1 = 0.03, u_2 = 0.09$ for $K = 200$ replicates from $\boldsymbol{S(\alpha, \beta; 1)}$ with sample size $n = 10^5$.

TABLE E.1: The MSEs of ECuF estimates (ECuF) and reduced values' ECuF (RVECuF) estimates at $u_1 = 0.03, u_2 = 0.09$.

α	β	**Method**	**MSE** ($\hat{\alpha}_n$)	**MSE** ($\hat{\beta}_n$)	**MSE** ($\hat{\gamma}_n$)	**MSE** ($\hat{\delta}_n$)
0.25	0.1	RVECuF	5.7×10^{-6}	6.9×10^{-5}	1.5×10^{-4}	1.7×10^{-6}
0.25	0.1	ECuF	3.8×10^{-5}	6.3×10^{-4}	5.5×10^{-3}	5.5×10^{-3}
0.25	0.25	RVECuF	4.1×10^{-6}	4.9×10^{-5}	6.6×10^{-5}	1.4×10^{-5}
0.25	0.25	ECuF	3.5×10^{-5}	4.6×10^{-4}	4.6×10^{-3}	4.6×10^{-3}
0.25	0.5	RVECuF	4.3×10^{-6}	5.2×10^{-5}	3.8×10^{-4}	1.7×10^{-4}
0.25	0.5	ECuF	3.9×10^{-5}	6.1×10^{-4}	5.8×10^{-3}	5.8×10^{-3}
0.25	0.75	RVECuF	3.4×10^{-6}	5.9×10^{-5}	5.8×10^{-4}	8.9×10^{-4}
0.25	0.75	ECuF	3.6×10^{-5}	6.1×10^{-4}	5.4×10^{-3}	5.4×10^{-3}
0.25	1	RVECuF	4.2×10^{-6}	1.0×10^{-4}	1.1×10^{-3}	4.3×10^{-3}
0.25	1	ECuF	4.3×10^{-5}	7.1×10^{-4}	6.3×10^{-3}	6.3×10^{-3}
0.5	0.1	RVECuF	4.1×10^{-6}	1.8×10^{-5}	1.5×10^{-5}	3.0×10^{-5}
0.5	0.1	ECuF	4.6×10^{-5}	3.2×10^{-4}	4.3×10^{-3}	4.3×10^{-3}
0.5	0.25	RVECuF	3.7×10^{-6}	2.5×10^{-5}	5.5×10^{-5}	1.2×10^{-4}
0.5	0.25	ECuF	6.7×10^{-5}	2.4×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
0.5	0.5	RVECuF	5.5×10^{-6}	2.4×10^{-5}	1.5×10^{-4}	3.2×10^{-4}
0.5	0.5	ECuF	5.8×10^{-5}	2.6×10^{-4}	4.4×10^{-3}	4.4×10^{-3}
0.5	0.75	RVECuF	7.4×10^{-6}	3.6×10^{-5}	2.9×10^{-4}	1.1×10^{-3}
0.5	0.75	ECuF	5.5×10^{-5}	2.8×10^{-4}	6.1×10^{-3}	6.1×10^{-3}
0.5	1	RVECuF	8.5×10^{-6}	3.9×10^{-5}	4.5×10^{-4}	2.5×10^{-3}
0.5	1	ECuF	5.5×10^{-5}	3.0×10^{-4}	8.4×10^{-3}	8.4×10^{-3}
0.75	0.1	RVECuF	4.5×10^{-6}	1.6×10^{-5}	2.0×10^{-5}	1.9×10^{-4}
0.75	0.1	ECuF	9.9×10^{-5}	3.1×10^{-4}	7.3×10^{-3}	7.3×10^{-3}
0.75	0.25	RVECuF	8.3×10^{-6}	2.5×10^{-5}	8.8×10^{-5}	6.7×10^{-4}
0.75	0.25	ECuF	1.2×10^{-4}	3.1×10^{-4}	9.6×10^{-3}	9.6×10^{-3}
0.75	0.5	RVECuF	1.2×10^{-5}	3.9×10^{-5}	2.0×10^{-4}	2.5×10^{-3}
0.75	0.5	ECuF	9.9×10^{-5}	2.5×10^{-4}	1.7×10^{-2}	1.7×10^{-2}
0.75	0.75	RVECuF	2.0×10^{-5}	4.0×10^{-5}	4.2×10^{-4}	8.3×10^{-3}
0.75	0.75	ECuF	9.5×10^{-5}	2.3×10^{-4}	2.8×10^{-2}	2.8×10^{-2}

RVECuF - reduced values' ECuF

TABLE E.1: *Cont.*

α	β	**Method**	**MSE** ($\hat{\alpha}_n$)	**MSE** ($\hat{\beta}_n$)	**MSE** ($\hat{\gamma}_n$)	**MSE** ($\hat{\delta}_n$)
0.75	1	RVECuF	2.1×10^{-5}	5.4×10^{-5}	5.3×10^{-4}	1.6×10^{-2}
0.75	1	ECuF	9.4×10^{-5}	2.3×10^{-4}	4.3×10^{-2}	4.3×10^{-2}
1.25	0.1	RVECuF	5.8×10^{-6}	1.5×10^{-5}	5.7×10^{-6}	5.7×10^{-5}
1.25	0.1	ECuF	3.2×10^{-4}	8.5×10^{-4}	1.4×10^{-3}	1.4×10^{-3}
1.25	0.25	RVECuF	1.8×10^{-5}	3.9×10^{-5}	4.3×10^{-5}	1.3×10^{-4}
1.25	0.25	ECuF	4.1×10^{-4}	1.0×10^{-3}	2.7×10^{-3}	2.7×10^{-3}
1.25	0.5	RVECuF	3.8×10^{-5}	7.8×10^{-5}	1.5×10^{-4}	4.0×10^{-4}
1.25	0.5	ECuF	4.3×10^{-4}	7.7×10^{-4}	4.8×10^{-3}	4.8×10^{-3}
1.25	0.75	RVECuF	5.6×10^{-5}	1.4×10^{-4}	3.0×10^{-4}	8.9×10^{-4}
1.25	0.75	ECuF	4.0×10^{-4}	7.2×10^{-4}	7.9×10^{-3}	7.9×10^{-3}
1.25	1	RVECuF	8.8×10^{-5}	1.3×10^{-4}	5.5×10^{-4}	1.9×10^{-3}
1.25	1	ECuF	3.8×10^{-4}	6.0×10^{-4}	1.2×10^{-2}	1.2×10^{-2}
1.5	0.1	RVECuF	3.8×10^{-6}	1.8×10^{-5}	1.2×10^{-6}	1.2×10^{-5}
1.5	0.1	ECuF	5.9×10^{-4}	1.9×10^{-3}	2.2×10^{-4}	2.2×10^{-4}
1.5	0.25	RVECuF	5.0×10^{-6}	2.2×10^{-5}	2.9×10^{-6}	1.2×10^{-5}
1.5	0.25	ECuF	6.0×10^{-4}	2.0×10^{-3}	2.4×10^{-4}	2.4×10^{-4}
1.5	0.5	RVECuF	1.6×10^{-5}	4.7×10^{-5}	1.7×10^{-5}	1.8×10^{-5}
1.5	0.5	ECuF	6.2×10^{-4}	1.9×10^{-3}	2.6×10^{-4}	2.6×10^{-4}
1.5	0.75	RVECuF	2.4×10^{-5}	7.4×10^{-5}	4.0×10^{-5}	2.3×10^{-5}
1.5	0.75	ECuF	5.6×10^{-4}	1.9×10^{-3}	3.3×10^{-4}	3.3×10^{-4}
1.5	1	RVECuF	3.4×10^{-5}	9.8×10^{-5}	6.9×10^{-5}	3.2×10^{-5}
1.5	1	ECuF	5.9×10^{-4}	1.7×10^{-3}	3.9×10^{-4}	3.9×10^{-4}
1.75	0.1	RVECuF	8.3×10^{-4}	4.7×10^{-3}	4.6×10^{-5}	4.6×10^{-5}
1.75	0.1	ECuF	4.5×10^{-2}	4.6×10^{-1}	1.2×10^{-3}	4.2×10^{-1}
1.75	0.25	RVECuF	4.0×10^{-6}	6.3×10^{-5}	1.2×10^{-6}	6.3×10^{-6}
1.75	0.25	ECuF	8.0×10^{-4}	6.1×10^{-3}	7.1×10^{-5}	7.1×10^{-5}
1.75	0.5	RVECuF	3.0×10^{-6}	3.1×10^{-5}	1.0×10^{-6}	4.2×10^{-6}
1.75	0.5	ECuF	7.8×10^{-4}	7.1×10^{-3}	6.5×10^{-5}	6.5×10^{-5}
1.75	0.75	RVECuF	5.7×10^{-6}	4.4×10^{-5}	2.0×10^{-6}	4.6×10^{-6}
1.75	0.75	ECuF	7.6×10^{-4}	7.4×10^{-3}	5.8×10^{-5}	5.8×10^{-5}
1.75	1	RVECuF	8.1×10^{-6}	9.1×10^{-5}	3.4×10^{-6}	5.2×10^{-6}
1.75	1	ECuF	9.7×10^{-4}	9.4×10^{-3}	7.3×10^{-5}	7.3×10^{-5}

RVECuF - reduced values' ECuF

Appendix F

Estimating $S(\alpha, \beta; 0)$ from Samples with Various Sizes

Figure F.1. The results for the tail index (characteristic exponent) α: The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimates for the parameter α for the $K = 100$ replicates from $S(\alpha, \beta, \gamma = 1, \delta = 0; 0)$ plotted as RMSE vs. sample size $n = 200$, 500, 1000, 3000, 5000.

Figure F.2. The results for the asymmetry parameter β: The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimates β for the $K = 100$ replicates from $S(\alpha, \beta, \gamma = 1, \delta = 0; 0)$ plotted as RMSE vs. sample size $n = 200$, 500, 1000, 3000, 5000.

Figure F.3. The results for the scale parameter γ: The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimates for the parameter γ for the $K = 100$ replicates from $S(\alpha, \beta, \gamma = 1, \delta = 0; 0)$ plotted as RMSE vs. sample size $n = 200$, 500, 1000, 3000, 5000.

Figure F.4. The results for the shift parameter δ: The performance of empirical cumulant function (ECuF), empirical characteristic function (EChF), maximum likelihood (ML) and quantile based (QB) estimates for the parameter γ for the $K = 100$ replicates from $S(\alpha, \beta, \gamma = 1, \delta = 0; 0)$ plotted as RMSE vs. sample size $n = 200$, 500, 1000, 3000, 5000.

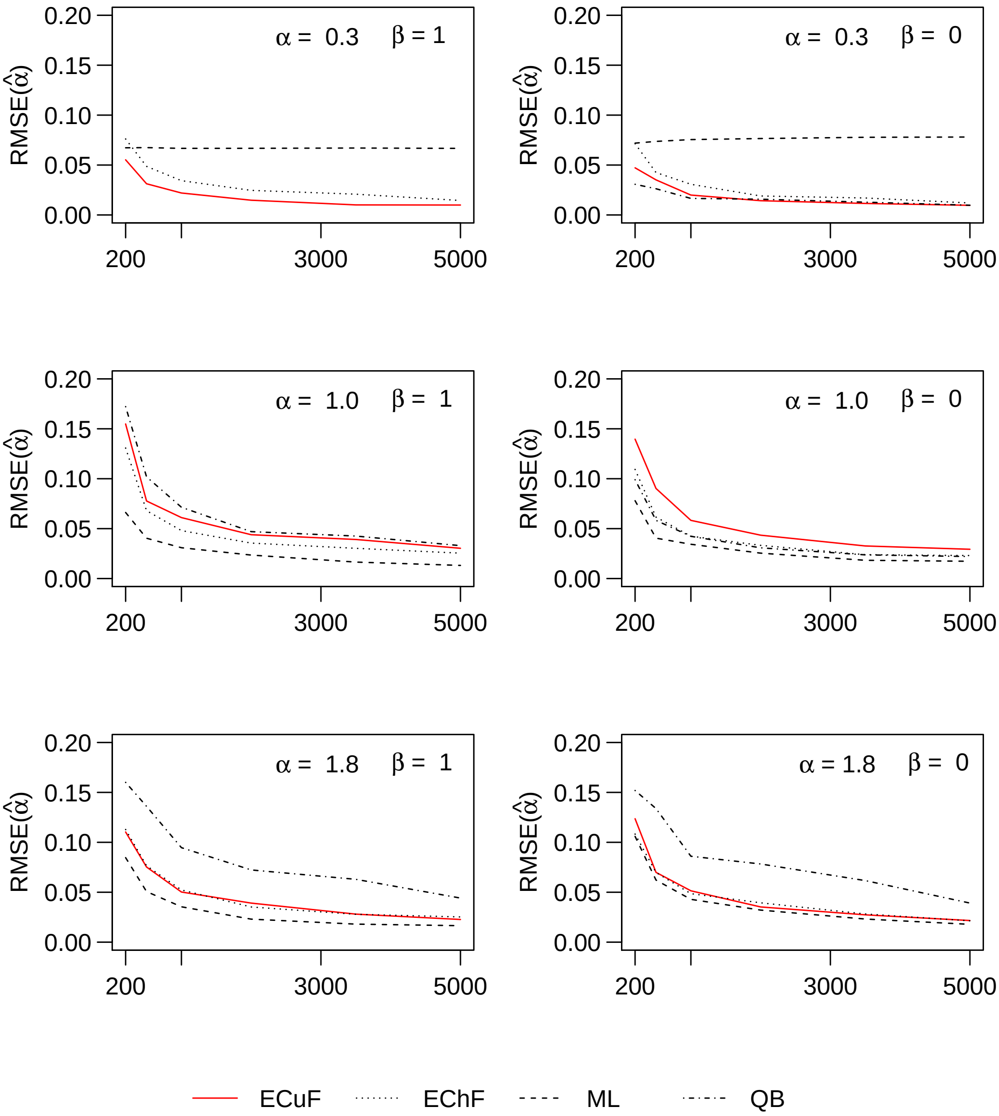

FIGURE F.1: Results for the tail index α plotted as RMSE vs. sample size n.

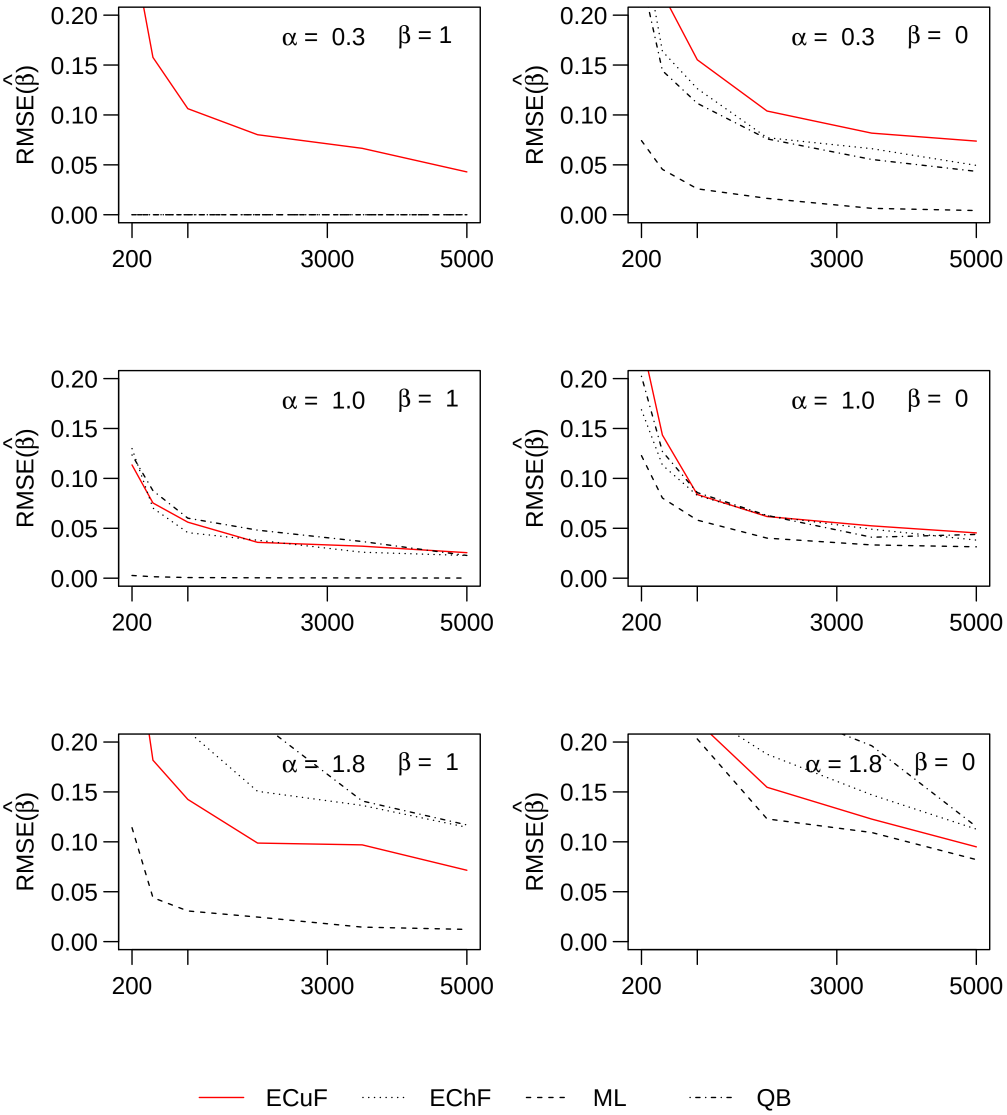

FIGURE F.2: Results for the asymmetry index β plotted as RMSE vs. sample size n.

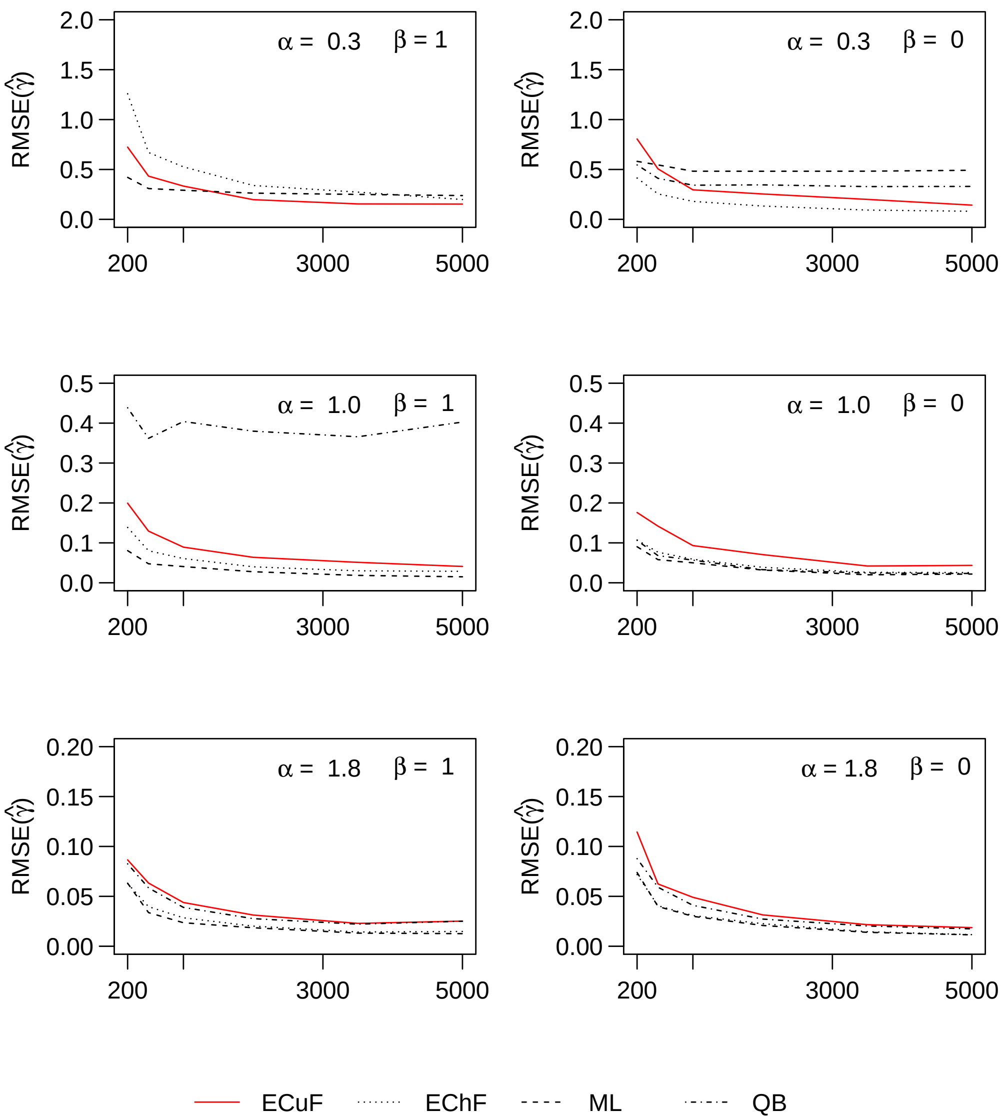

FIGURE F.3: Results for the scale parameter γ plotted as RMSE vs. sample size n.

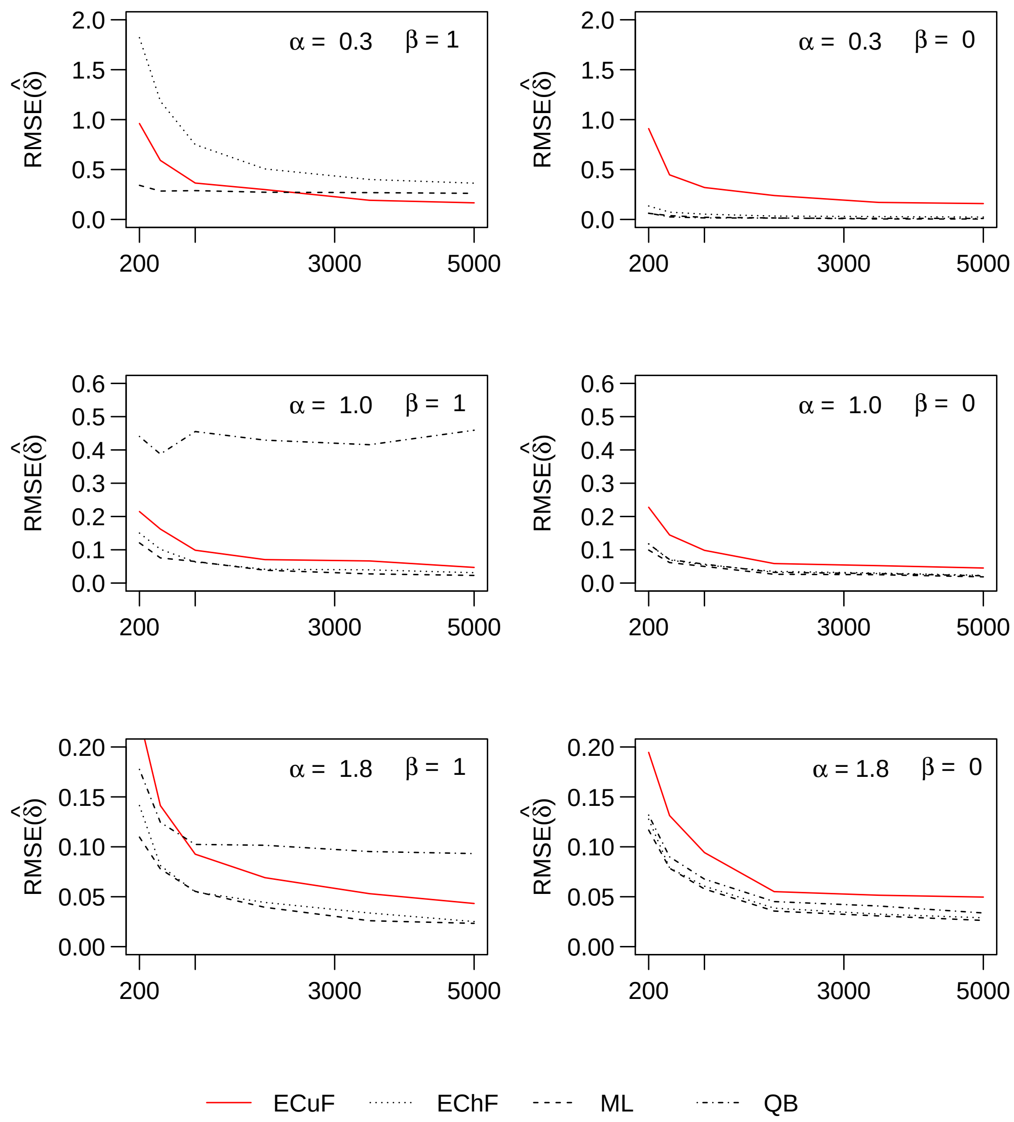

FIGURE F.4: Results for the shift parameter δ plotted as RMSE vs. sample size n.

References

[1] Achcar, J. A. and Lopes, S. R. (2016). Linear and non-linear regression models assuming a stable distribution. *Revista Colombiana de Estadística* **39** 109–128.

[2] Akgiray, V. and Lamoureux, C. G. (1989). Estimation of stable-law parameters: A comparative study. *Journal of Business & Economic Statistics* **7**(1) 85–93.

[3] Anderson, T. (2003). *An Introduction to Multivariate Statistical Analysis* (3rd ed.). New York: Wiley.

[4] Andrews, D. W. (1997, June). Estimation When a Parameter Is on a Boundary: Theory and Applications. Cowles Foundation Discussion Papers 1153, Cowles Foundation for Research in Economics, Yale University.

[5] Arad, R. W. (1980). Parameter estimation for symmetric stable distribution. *International Economic Review* **21**(1) 209–220.

[6] Azzalini, A. and Capitanio, A. (2014). *The Skew-Normal and Related Families*. Cambridge: Cambridge University Press.

[7] Bakar, S. A., Hamzah, N., Maghsoudi, M., and Nadarajah, S. (2015). Modeling loss data using composite models. *Insurance: Mathematics and Economics* **61** 146 – 154.

[8] Barndorff-Nielsen, O. (1978). Hyperbolic distributions and distributions on hyperbolae. *Scandinavian Journal of Statistics* **5**(3) 151–157.

[9] Beranger, B. and Padoan, S. (2016). Extreme dependence models. In Dey, D. and Yan, J. (Eds.), *Extreme Value Modeling and Risk Analysis: Methods and Applications*, pp. 325–352. Boca Raton, FL: CRC Press.

[10] Bodnar, T. and Gupta, A. K. (2011). Estimation of the precision matrix of a multivariate elliptically contoured stable distribution. *Statistics* **45**(2) 131–142.

[11] Borak, S., Härdle, W., and Weron, R. (2005). Stable distributions. In Cizek, P., Härdle, W., and Weron, R. (Eds.), *Statistical Tools for Finance and Insurance*, pp. 21–44. Berlin Heidelberg: Springer.

[12] Brahimi, B. and Abdelli, J. (2016). Estimating the distortion parameter of the proportional hazards premium for heavy-tailed losses under Lèvy-stable regime. *Insurance: Mathematics and Economics* **70** 135–143.

[13] Brent, R. (1973). *Algorithms for Minimization without Derivatives*. Englewood Cliffs N.J.: Prentice-Hall.

[14] Brouste, A. and Masuda, H. (2018). Efficient estimation of stable Lévy process with symmetric jumps. *Statistical Inference for Stochastic Processes* **21**(2) 289–307.

[15] Brown, J. and Churchill, R. (1996). *Complex variables and applications*. New York: McGraw-Hill.

[16] Buckle, D. J. (1995). Bayesian inference for stable distributions. *jasa* **90**(430) 605–613.

[17] Cek, M. E. (2015). Covert communication using skewed α-stable distributions. *Electronics Letters* **51**(1) 116–118.

[18] Chambers, J. M., Mallows, C. L., and Stuck, B. W. (1976). A method for simulating stable random variables. *jasa* **71**(354) 340–344.

[19] Csörgo, S. (1981). Limit behaviour of the empirical characteristic function. *The Annals of Probability* **9**(1) 130–144.

[20] Csörgo, S. (1985). Rates of uniform convergence for the empirical charateristic function. *Acta Scientiarum Mathematicarum* **48** 97–102.

[21] Csörgo, S. and Totik, V. (1983). On how long interval is the empirical characteristic function uniformly consistent? *Acta Scientiarum Mathematicarum* **45** 141–149.

[22] Curto, J., Pinto, J., and Tavares, G. (2009). Modeling stock markets' volatility using GARCH models with normal, Student's t and stable Paretian distributions. *Statistical Papers* **50**(2) 311–321.

[23] DasGupta, A. (2008). *Asymptotic Theory of Statistics and Probability*. New York: Springer.

[24] Davison, A. (2015). *SMPracticals. R package version n 1.4-2*. `https://cran.r-project.org/package=SMPracticals`.

[25] Devroye, L. and James, L. (2014). On simulation and properties of the stable law. *Statistical Methods & Applications* **23**(3) 307–343.

[26] Dey, D., Jiang, Y., and Yan, J. (2016). Multivariate extreme value analysis. In Dey, D. and Yan, J. (Eds.), *Extreme Value Modeling and Risk Analysis: Methods and Applications*, pp. 23–40. Boca Raton, FL: CRC Press.

[27] DuMouchel, W. H. (1983). Estimating the stable index α in order to measure tail thickness: A critique. *The Annals of Statistics* **11**(4) 1019–1031.

[28] Dutang, C., Goulet, V., and Pigeon, M. (2008). actuar: An R package for actuarial science. *Journal of Statistical Software* **25**(7) 38. `https://CRAN.R-project.org/package=actuar`.

[29] Efthymios, G. T. (2000). Efficient posterior integration in stable Paretian models. *Statistical Papers* **41**(3) 305–325.

[30] Embrechts, P., Klüppelberg, C., and Mikosch, T. (2013). *Modelling Extremal Events: for Insurance and Finance*. Berlin Heidelberg: Springer.

[31] Everitt, B. (1998). *Cambridge Dictionary of Statistics*. Cambridge: Cambridge University Press.

[32] Fama, E. F. and Roll, R. (1971). Parameter estimates for symmetric stable distributions. *jasa* **66**(334) 331–338.

[33] Fan, Z. (2006). Parameter estimation of stable distributions. *Communications in Statistics - Theory and Methods* **35**(2) 245–255.

[34] Feller, W. (1971). *An Introduction to Probability Theory and its Applications. Vol. II* (2nd ed.). New York: Wiley.

[35] Feuerverger, A. and McDunnough, P. (1981). On the efficiency of empirical characteristic function procedures. *Journal of the Royal Statistical Society. Series B (Methodological)* **43**(1) 20–27.

[36] Feuerverger, A. and Mureika, R. A. (1977). The empirical characteristic function and its applications. *The Annals of Statistics* **5**(1) 88–97.

[37] Forsythe, G. E., Malcolm, M. A., and Moler, C. B. (1977). *Computer Methods for Mathematical Computations*. Prentice Hall Professional Technical Reference.

[38] Fréchet, M. (1927). Sur la loi de probabilité de l'écart maximum. *Ann. Soc. Polon. Math. Cracovie* **6** 93–116.

[39] Gnedenko, B. and Kolmogorov, A. (1954). *Limit distributions for sums of independent random variables*. Cambridge: Addison-Wesley Pub. Co.

[40] Good, I. J. (1961). The real stable characteristic functions and chaotic acceleration. *Journal of the Royal Statistical Society. Series B (Methodological)* **23**(1) 180–183.

[41] Goovaerts, M. J., Schepper, A. D., Vyncke, D., Dhaene, J., and Kaas, R. (2003). Stable laws and the present value of fixed cash flows. *North American Actuarial Journal* **7**(4) 32–43.

[42] Grabchak, M. (2016). *Tempered Stable Distributions: Stochastic Models for Multiscale Processes*. Springer International Publishing.

[43] Grabchak, M. (2018). Domains of attraction for positive and discrete tempered stable distributions. *Journal of Applied Probability* **55**(1) 30–42.

[44] Heathcote, C. R. (1977). The integrated squared error estimation of parameters. *Biometrika* **64**(2) 255–264.

[45] Hougaard, P. (1986). Survival models for heterogeneous populations derived from stable distributions. *Biometrika* **73**(2) 387–396.

[46] Hougaard, P. (2001). *Analysis of Multivariate Survival Data* (2nd ed.). New York: Springer.

[47] Höpfner, R. (1998). Estimating a stability parameter: Asymptotics and simulations. *Statistics* **30**(4) 291–305.

[48] Höpfner, R. and Rüschendorf, L. (1999). Comparison of estimators in stable models. *Mathematical and Computer Modelling* **29**(10) 145–160.

[49] Johnson, N., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions* (2nd ed.), Volume 1. New York: Wiley.

[50] Jones, M. C. and Pewsey, A. (2009). Sinh-arcsinh distributions. *Biometrika* **96**(4) 761–780.

[51] Kasana, H. S. (2005). *Complex Variables: Theory And Applications* (2nd ed.). New Delhi: Prentice-Hall of India Private Limited.

[52] Kateregga, M., Mataramvura, S., and Taylor, D. (2017). Parameter estimation for stable distributions with application to commodity futures log-returns. *Cogent Economics & Finance* **5**(1) 1318813.

[53] Kent, J. T. (1975). A weak convergence theorem for the empirical characteristic function. *Journal of Applied Probability* **12**(3) 515–523.

[54] Khintchine, A. Y. (1937). A new derivation of a formula by p. Lèvy. *Bulletin of the Moscow State University 1* **1**(1) 1–5. In Russian.

[55] Khintchine, A. Y. and Lèvy, P. (1936). Sur les lois stables. *C. R. Acad. Sci. Paris* **202** 374–376.

[56] Knight, J. L. and Satchell, S. E. (1997). The cumulant generating function estimation method: Implementation and asymptotic efficiency. *Econometric Theory* **13**(2) 170–184.

[57] Knight, K. (1999). *Mathematical Statistics*. Boca Raton, FL: CRC Press.

[58] Kogon, S. M. and Williams, D. B. (1998). Characteristic function based estimation of stable distribution parameters. In Adler, R. J., Feldman, R. E., and Taqqu, M. S. (Eds.), *A Practical Guide to Heavy Tails*, pp. 311–335. Boston: Birkhäuser.

[59] Kohlbecker, M. V., Wheatcraft, S. W., and Meerschaert, M. M. (2006). Heavy-tailed log hydraulic conductivity distributions imply heavy-tailed log velocity distributions. *Water Resources Research* **42**(4) 1944–1973.

[60] Kollo, T. and von Rosen, D. (2005). *Advanced Multivariate Statistics with Matrices*. Dordrecht: Springer.

[61] Kotz, S. and Nadarajah, S. (2000). *Extreme Value Distributions*. London: Imperial College Press.

[62] Koutrouvelis, I. A. (1980). Regression-type estimation of the parameters of stable laws. *jasa* **75**(372) 918–928.

[63] Koutrouvelis, I. A. (1981). An iterative procedure for the estimation of the parameters of stable laws. *Communications in Statistics-Simulation and Computation* **10**(1) 17–28.

[64] Kozubowski, T. (1999). Geometric stable laws: Estimation and applications. *Mathematical and Computer Modelling* **29**(10) 241–253.

[65] Kozubowski, T. and Rachev, T. (1999a). Multivariate geometric stable laws. *Journal of Computational Analysis and Applications* **1** 349–385.

[66] Kozubowski, T. and Rachev, T. (1999b). Univariate geometric stable laws. *Journal of Computational Analysis and Applications* **1** 177–217.

[67] Kring, S., Rachev, S. T., Höchstötter, M., and Fabozzi, F. J. (2009). Estimation of α-stable sub-Gaussian distributions for asset returns. In Bol, G., Rachev, S. T., and Würth, R. (Eds.), *Risk Assessment*, pp. 111–152. Heidelberg: Physica-Verlag HD.

[68] Krutto, A. (2003). Estimating the parameters of stable distribution. Master's thesis, Tartu Ülikool. In Estonian.

[69] Krutto, A. (2016). Parameter estimation in stable law. *Risks* **4**(4) 43.

[70] Krutto, A. (2018). Empirical cumulant function based estimation in stable laws. *Acta et Commentationes Universitatis Tartuensis de Mathematica* **22**(2) 311–338.

[71] Kuruoglu, E. E. (2001). Density parameter estimation of skewed alpha-stable distributions. *IEEE Transactions on Signal Processing* **49**(10) 2192–2201.

[72] Landau, L. (1944). On the energy ionizational losses by fast particles. *Journal of Physics (USSR)* **8** 201–210.

[73] Leitch, R. A. and Paulson, A. S. (1975). Estimation of stable law parameters: stock price behavior application. *jasa* **70**(351a) 690–697.

[74] Lemke, T., Riabiz, M., and Godsill, S. J. (2015). Fully Bayesian inference for α-stable distributions using a Poisson series representation. *Digital Signal Processing* **47** 96 – 115.

[75] Lévy, P. (1925). *Calcul des probabilités*. Gauthier-Villars.

[76] Lifshits, I. (1956). On the temperature flashes in a medium under the action of nuclear radiation. *Proceedings of the USSR Academy of Sciences* **109** 1109–1111. In Russian.

[77] Lorentz, H. (1906). The absorption and emission lines of gaseous bodies. *Proceedings of the Royal Netherlands Academy of Arts and Sciences* **8**(II) 591–611.

[78] Lukacs, E. (1970). *Characteristic Functions* (2nd ed.). London: Griffin.

[79] Luong, A. (2016). Cramer-Von Mises distance estimation for some positive infinitely divisible parametric families with actuarial applications. *Scandinavian Actuarial Journal* **2016**(6) 530–549.

[80] Ma, X. Y. and Nikias, C. L. (1995). Parameter estimation and blind channel identification in impulsive signal environments. *IEEE Transactions on Signal Processing* **43**(12) 2884–2897.

[81] Mainardi, F. and Rogosin, S. (2006). The origin of infinitely divisible distributions: from de Finetti's problem to Lévy-Khintchine formula. *Mathematical Methods in Economics and Finance* **1** 37–55.

[82] Mandelbrot, B. (1960a). Stable Paretian random functions and the multiplicative variation of income. *Econometrica* **29** 517–543.

[83] Mandelbrot, B. (1960b). The Pareto-Lévy law and the distribution of income. *International Economic Review* **1** 79–106.

[84] Marcus, M. B. (1981). Weak convergence of the empirical characteristic function. *The Annals of Probability* **9**(2) 194–201.

[85] Matsui, M. and Takemura, A. (2008). Goodness-of-fit tests for symmetric stable distributions – empirical characteristic function approach. *TEST* **17**(3) 546–566.

[86] McCulloch, J. H. (1986). Simple consistent estimators of stable distribution parameters. *Communications in Statistics-Simulation and Computation* **15**(4) 1109–1136.

[87] McCulloch, J. H. (1996). Financial applications of stable distributions. In Maddala, G. and Rao, C. (Eds.), *Statistical Methods in Finance*, Volume 14 of *Handbook of Statistics*, pp. 393 – 425. Elsevier.

[88] Meerschaert, M. M. and Scheffler, H.-P. (2001). *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. New York: Wiley.

[89] Meintanis, S. G. (2016). A review of testing procedures based on the empirical characteristic function. *South African Statistical Journal* **50**(1) 1–14.

[90] Menabde, M. and Sivapalan, M. (2000). Modeling of rainfall time series and extremes using bounded random cascades and Lévy-stable distributions. *Water Resources Research* **36**(11) 3293–3300.

[91] Miljkovic, T. and Grün, B. (2016). Modeling loss data using mixtures of distributions. *Insurance: Mathematics and Economics* **70** 387 – 396.

[92] Mittnik, S. and Paolella, M. (1999). A simple estimator for the characteristic exponent of the stable paretian distribution. *Mathematical and Computer Modelling* **29**(10) 161 – 176.

[93] Mittnik, S. and Rachev, S. (1991). Alternative multivariate stable distributions and their applications to financial modeling. In Cambanis, S., Samorodnitsky, G., and Taqqu, M. (Eds.), *Stable Processes and Related Topics: A Selection of Papers from the Mathematical Sciences Institute Workshop, January 9–13, 1990*, pp. 107–119. Boston: Birkhäuser.

[94] Mittnik, S., Rachev, S., Doganoglu, T., and Chenyao, D. (1999). Maximum likelihood estimation of stable Paretian models. *Mathematical and Computer Modelling* **29**(10) 275–293.

[95] Monin, A. (1955). Equations of turbulent diffusion. *Proceedings of the USSR Academy of Sciences* **105** 256–259. In Russian.

[96] Nikias, C. L. and Shao, M. (1995). *Signal Processing with Alpha-Stable Distributions and Applications*. New York: Wiley.

[97] Nolan, J. P. (1997). Numerical calculation of stable densities and distribution functions. *Communications in Statistics: Stochastic models* **13**(4) 759–774.

[98] Nolan, J. P. (1998a). Multivariate stable distributions: approximation, estimation, simulation and identification. In Adler, R. J., Feldman, R. E., and Taqqu, M. S. (Eds.), *A Practical Guide to Heavy Tails*, pp. 509–526. Boston: Birkhäuser.

[99] Nolan, J. P. (1998b). Parameterizations and modes of stable distributions. *Statistics & Probability Letters* **38**(2) 187–195.

[100] Nolan, J. P. (1999). An algorithm for evaluating stable densities in Zolotarev's (M) parameterization. *Mathematical and Computer Modelling* **29**(10-12) 229–233.

[101] Nolan, J. P. (2001). Maximum likelihood estimation and diagnostics for stable distributions. In Barndorff-Nielsen, O., Resnick, S., and Mikosch, T. (Eds.), *Lévy Processes*, pp. 379–400. Boston: Birkhäuser.

[102] Nolan, J. P. (2003). Modeling financial data with stable distributions. In Rachev, S. (Ed.), *Heavy Tailed Distributions in Finance, Handbooks in Finance*, pp. 105–130. Amsterdam: Elsevier Science.

[103] Nolan, J. P. (2005). *STABLE Program for Windows. Version 3.14.02*. Washington, DC. `http://academic2.american.edu/~jpnolan/stable/stable.html`.

[104] Nolan, J. P. (2018a). Bibliography on stable distributions, processes and related topics. Original version August 4, 2003 Revised June 1, 2018. Online at `http://fs2.american.edu/jpnolan/www/stable/StableBibliography.pdf`.

[105] Nolan, J. P. (2018b). *Stable Distributions - Models for Heavy Tailed Data*. Boston: Birkhäuser. In progress, Chapter 1 online at `http://fs2.american.edu/jpnolan/www/stable/stable.html`.

[106] Nolan, J. P. (2018c). Truncated fractional moments of stable laws. *Statistics & Probability Letters* **137** 312 – 318.

[107] Pad, P., Alishahi, K., and Unser, M. (2017). Optimized wavelet denoising for self-similar α-stable processes. *IEEE Transactions on Information Theory* **63**(9) 5529–5543.

[108] Parzen, E. (1962). On estimation of a probability density function and mode. *The Annals of Mathematical Statistics* **33**(3) 1065–1076.

[109] Paulson, A. S., Holcomb, E. W., and Leitch, R. A. (1975). The estimation of the parameters of the stable laws. *Biometrika* **62**(1) 163–170.

[110] Peng, Y., Chen, J., Xu, X., and Pu, F. (2013). SAR images statistical modeling and classification based on the mixture of alpha-stable distributions. *Remote Sensing* **5** 2145–2163.

[111] Pitman, E. J. G. and Pitman, J. (2016). A direct approach to the stable distributions. *Advances in Applied Probability* **48**(A) 261–282.

[112] Press, J. S. (1972). Estimation in univariate and multivariate stable distributions. *jasa* **67**(340) 842–846.

[113] R Core Team (2018). *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing. `https://www.R-project.org/`.

[114] Rao, M. M. and Swift, R. J. (2006). *Probability Theory with Applications* (2nd ed.). New York: Springer.

[115] Reynkens, T., Verbelen, R., Beirlant, J., and Antonio, K. (2017). Modelling censored losses using splicing: A global fit strategy with mixed Erlang and extreme value distributions. *Insurance: Mathematics and Economics* **77** 65 – 77.

[116] Robust Analysis Inc. (2017). *STABLE 5.3 R Version for Windows*. Washington, DC, USA: Robust Analysis Inc. `http://www.robustanalysis.com/`.

[117] Rosiński, J. (2007). Tempering stable processes. *Stochastic Processes and their Applications* **117**(6) 677 –707.

[118] Salas-Gonzalez, D., Kuruoglu, E. E., and Ruiz, D. P. (2009). Finite mixture of α-stable distributions. *Digital Signal Processing* **19**(2) 250 – 264.

[119] Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. New York: Chapman & Hall.

[120] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

[121] Simon, T. (2014). Comparing Fréchet and positive stable laws. *Electronic Journal of Probability* **19** 25 pp.

[122] Steutel, F. and van Harn, K. (2004). *Infinite divisibility of probability distributions on the real line*. Boca Raton, FL: CRC Press.

[123] Swihart, B. and Lindsey, J. (2018). *rmutil: Utilities for Nonlinear Regression and Repeated Measurements Models. R package version 1.1.1*. `https://CRAN.R-project.org/package=rmutil`.

[124] Teimouri, M., Rezakhah, S., and Mohammadpour, A. (2018). EM algorithm for symmetric stable mixture model. *Communications in Statistics - Simulation and Computation* **47**(2) 582–604.

[125] Thornton, J. C. and Paulson, A. S. (1977). Asymptotic distribution of characteristic function-based estimators for the stable laws. *Sankhyā: The Indian Journal of Statistics, Series A* **39**(4) 341–354.

[126] Tsihrintzis, G. A. and Nikias, C. L. (1996). Fast estimation of the parameters of alpha-stable impulsive interference. *IEEE Transactions on Signal Processing* **44**(6) 1492–1503.

[127] Uchaikin, V. and Zolotarev, V. (1999). *Chance and Stability: Stable Distributions and their Applications*. Utrecht: Walter de Gruyter.

[128] Ushakov, N. G. (1999). *Selected Topics in Characteristic Functions*. Utrecht: Walter de Gruyter.

[129] van der Vaart, A. (1998). *Asymptotic Statistics*. Cambridge: Cambridge University Press.

[130] von Holtsmark, J. (1919). Über die Verbreiterung von Spektrallinien. *Annalen der Physik* **363**(7) 577–630.

[131] Wang, R., Xiong, J., and Xu, L. (2017). Irreducibility of stochastic real Ginzburg–Landau equation driven by α-stable noises and applications. *Bernoulli* **23**(2) 1179–1201.

[132] Weron, R. (1996). On the Chambers-Mallows-Stuck method for simulating skewed stable random variables. *Statistics & Probability Letters* **28**(2) 165 – 171.

[133] Würtz, D. and Mächler, M. (2016). *stabledist: Stable Distribution Function. R package version 0.7-1*. `https://CRAN.R-project.org/package=stabledist`.

[134] Xu, W., Wu, C., Dong, Y., and Xiao, W. (2011). Modeling Chinese stock returns with stable distribution. *Mathematical and Computer Modelling* **54**(1) 610 –617.

[135] Yanushkevichiene, O. and Saenko, V. (2017). Estimation of the characteristic exponent of stable laws. *Lithuanian Mathematical Journal* **57** 266–281.

[136] Yu, J. (2004). Empirical characteristic function estimation and its applications. *Econometric Reviews* **23**(2) 93–123.

[137] Zolotarev, V. (1983). *Odnomernye Ustoichivye Raspredeleniya*. Moscow: Nauka. In Russian.

[138] Zolotarev, V. (1986). *One-dimensional Stable Distributions*, Volume 65 of *Translations of mathematical monographs*. American Mathematical Society.

[139] Zolotarev, V. (1995). On representation of densities of stable laws by special functions. *Theory of Probability & Its Applications* **39**(2) 354–362.

Sisukokkuvõte

Stabiilsete jaotuste parameetrite hindamine empiirilise kumulantfunktsiooni kaudu

Stabiilsed jaotused on lai tõenäosusjaotuste klass, millel on palju matemaatiliselt huvitavaid omadusi. Klassi eristas Lévy (1925) oma uurimuses sõltumatute sama jaotusega juhuslike suuruste summa piirjaotustest. Ühemõõtmelised stabiilsed jaotused moodustavad 4-parameetrilise jaotuste klassi, mis kaasab erineva sümmeetria ja sabaraskusega jaotuseid ning mille erijuhtudeks on normaal-, Levy ja Cauchy jaotus. Viimastel aastakümnetel on stabiilsetest jaotustest esitatud mitmeid uurimusi, kuid jaotuse parameetrite hindamine (andmete põhjal) on senini väljakutset pakkuv. Tavapärased hindamismeetodid ei sobi stabiilsete jaotuste korral, kuna tihedusfunktsioonil puudub analüütiline kuju ja momendid on üldjuhul määramatud. Välja on pakutud mitmeid algoritmilisi protseduure, kuid need on arvutuslikult üsna keerulised ning vastavad rakendused ei ole statistikapakettides (tarkvaras) tihtipeale kättesaadavad. See omakorda takistab stabiilsete jaotuste laialdasemat kasutamist. Samas on stabiilsed jaotused hinnatud asjakohasteks mudeliteks erinevatele protsessidele mitmetes eluvaldkondades, näiteks klimatoloogias, füüsikas ning kindlustus- ja finantssektoris, kuna võimaldavad kirjeldada nii sümmeetrilisi kui ebasümmeetrilisi protsesse kui ka arvesse võtta protsessides esineva varieeruvuse dünaamika ja ekstreemsed kõikumised.

Doktoritöö aluseks on Press (1972) meetod, mis põhineb stabiilsete jaotuste parameetrite avaldamisel karakteristliku funktsiooni logaritmi ehk kumulantfunktsiooni kaudu. Meetodi idee on arvutuslikult lihtne ja hinnangud on esitatavad analüütiliste avaldistena, kuid sõltuvad neljast vabalt valitavast reaalarvust (kui funktsioonid empiirilise kumulantfunktsiooni reaal- ja imaginaarosast). Erinev argumentide valik mõjutab tulemusi märgatavalt ja võib anda ka sobimatuid tulemusi (parameetrite ruumi mõistes). Argumentide valikuks seni lahendusi pakutud ei ole ning seetõttu ei ole ka see hindamismeetod rakendustes kasutust leidnud. Doktoritöös esitatakse kõnealusest meetodist parendatud versioon - tõestatakse, et meetodi rakendamiseks piisab kahest vabalt valitud reaalarvust, ning antakse soovitusi argumentide valikuks meetodi rakendamisel. Täpsemalt,

- esitatakse uus versioon Press (1972) meetodist: tõestatakse, et meetod on rakendatav kahe vabalt valitud reaalarvu $u_1 > 0, u_2 > 0, u_1 \neq u_2$ korral;

- iseloomustatakse hinnangute mõjusust ja nihet; tõestatakse, et hinnangud on asümpootilise normaaljaotusega;
- tõestatakse empiirilise kumulantfunktsiooni reaal- ja imaginaarosa asümptootiline normaaljaotus;
- viiakse läbi Monte–Carlo simulatsioonidel põhinev empiiriline uurimus argumentide u_1, u_2 valikuks;
- pakutakse välja valimil (andmetel) põhinev argumentide u_1, u_2 valikumeetod;
- viiakse läbi mahukad Monte–Carlo simulatsioonid hindamaks meetodi efektiivsust ja võrreldakse tulemusi teiste meetoditega (stabiilsete jaotuste parameetrite hindamiseks);
- rakendatakse väljatöötatud ECuF hindamismeetodit kahele kahjukindlustuse andmestikul ja võrreldakse tulemusi teiste meetoditega.

Doktoritöös väljapakutud ECuF meetod stabiilsete jaotuste parameetrite hindamiseks on arvutuslikult lihtne, mõjus ja saadavd hinnagud asümpootilise normaaljaotusega. Meetod ei oma piiranguid parameetrite ruumis; ei vaja andmete standardiseerimist, annab samaväärseid või paremaid tulemusi võrreldes keerukamate algoritmiliste protseduuridega. Kokkuvõttes saab öelda, et doktoritöös antud soovitusi järgides on meetod praktikas kasutatav. Seega saab seni kirjanduses esitatud vastupidised (e.g., Paulson et al. (1975), Borak et al. (2005)) arvamused ümber lükata.

Annika Krutto

Institute of Mathematics and Statistics
J. Liivi Str 2 Tartu 50409 ESTONIA
University of Tartu
www.math.ut.ee

E-mail: annika.krutto@ut.ee

General data

Citizenship: Estonian
Date of Birth: 28.06.1978
Place of Birth: Viljandi, Estonia

Education

University of Tartu
2005 – 2019 Ph.D. studies, Mathematical Statistics
2005 – 2006 Ph.D. studies, Mathematical Statistics
2004 – 2005 Teacher of Mathematics in Upper Secondary School
2001 – 2003 M.Sc., Mathematical Statistics
1996 – 2000 B.Sc., Mathematical Statistics

Occupation

2008 – 2018 University of Tartu,
Instructor (Life Insurance Mathematics, Data Analysis)
2014 – 2015 Statistics Estonia,
Methodologist
2003 – 2009 Estonian Entrepreneurship University of Applied Sciences,
Adjunct Lecturer (Statistics, Data Analysis)
2003 – 2006 Estonian Entrepreneurship University of Applied Sciences,
Academic Affairs Senior Specialist
2001 – 2006 Fr. Tuglas' Ahja Upper Secondary School,
Teacher of Informatics

Languages and Skills

Estonian (native), English
LaTeX, R, SAS, SPSS

Field of Research

Natural Sciences and Engineering; Statistics; Actuarial Mathematics

Annika Krutto

Matemaatika ja statistika instituut
J. Liivi 2 Tartu 50409 Eesti
Tartu Ülikool
www.math.ut.ee

e-post: annika.krutto@ut.ee

Isikuandmed

Kodakondsus: Eesti
Sünniaeg: 28.06.1978
Sünnikoht: Viljandi, Eesti

Haridus

Tartu Ülikool
2005 – 2019 doktoriõpe, matemaatiline statistika
2003 – 2004 gümnaasiumi matemaatikaõpetaja
2001 – 2003 M.Sc., matemaatiline statistika
1996 – 2000 B.Sc., matemaatiline statistika

Teenistuskäik

2008 – 2018 Tartu Ülikool,
õppeülesande täitja:
elukindlustusmatemaatika, andmeanalüüs
2014 – 2015 Eesti Statistikaamet,
metoodik
2003 – 2009 Eesti Ettevõtluskõrgkool Mainor,
lepinguline õppejõud:
statistika, andmeanalüüs
2003 – 2006 Eesti Ettevõtluskõrgkool Mainor,
õppekorralduse peaspetsialist
2001 – 2006 Fr. Tuglase nim. Ahja Keskkool,
informaatikaõpetaja

Keeled ja oskused

Eesti (emakeel), Inglise
LaTeX, R, SAS, SPSS

Teadustöö põhisuunad

Loodusteadused ja tehnika: Statistika:
finants- ja kindlustusmatemaatika

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

1. **Mati Heinloo.** The design of nonhomogeneous spherical vessels, cylindrical tubes and circular discs. Tartu, 1991, 23 p.
2. **Boris Komrakov.** Primitive actions and the Sophus Lie problem. Tartu, 1991, 14 p.
3. **Jaak Heinloo.** Phenomenological (continuum) theory of turbulence. Tartu, 1992, 47 p.
4. **Ants Tauts.** Infinite formulae in intuitionistic logic of higher order. Tartu, 1992, 15 p.
5. **Tarmo Soomere.** Kinetic theory of Rossby waves. Tartu, 1992, 32 p.
6. **Jüri Majak.** Optimization of plastic axisymmetric plates and shells in the case of Von Mises yield condition. Tartu, 1992, 32 p.
7. **Ants Aasma.** Matrix transformations of summability and absolute summability fields of matrix methods. Tartu, 1993, 32 p.
8. **Helle Hein.** Optimization of plastic axisymmetric plates and shells with piece-wise constant thickness. Tartu, 1993, 28 p.
9. **Toomas Kiho.** Study of optimality of iterated Lavrentiev method and its generalizations. Tartu, 1994, 23 p.
10. **Arne Kokk.** Joint spectral theory and extension of non-trivial multiplicative linear functionals. Tartu, 1995, 165 p.
11. **Toomas Lepikult.** Automated calculation of dynamically loaded rigid-plastic structures. Tartu, 1995, 93 p, (in Russian).
12. **Sander Hannus.** Parametrical optimization of the plastic cylindrical shells by taking into account geometrical and physical nonlinearities. Tartu, 1995, 74 p, (in Russian).
13. **Sergei Tupailo.** Hilbert's epsilon-symbol in predicative subsystems of analysis. Tartu, 1996, 134 p.
14. **Enno Saks.** Analysis and optimization of elastic-plastic shafts in torsion. Tartu, 1996, 96 p.
15. **Valdis Laan.** Pullbacks and flatness properties of acts. Tartu, 1999, 90 p.
16. **Märt Põldvere.** Subspaces of Banach spaces having Phelps' uniqueness property. Tartu, 1999, 74 p.
17. **Jelena Ausekle.** Compactness of operators in Lorentz and Orlicz sequence spaces. Tartu, 1999, 72 p.
18. **Krista Fischer.** Structural mean models for analyzing the effect of compliance in clinical trials. Tartu, 1999, 124 p.
19. **Helger Lipmaa.** Secure and efficient time-stamping systems. Tartu, 1999, 56 p.
20. **Jüri Lember.** Consistency of empirical k-centres. Tartu, 1999, 148 p.
21. **Ella Puman.** Optimization of plastic conical shells. Tartu, 2000, 102 p.
22. **Kaili Müürisep.** Eesti keele arvutigrammatika: süntaks. Tartu, 2000, 107 lk.

23. **Varmo Vene.** Categorical programming with inductive and coinductive types. Tartu, 2000, 116 p.
24. **Olga Sokratova.** Ω-rings, their flat and projective acts with some applications. Tartu, 2000, 120 p.
25. **Maria Zeltser.** Investigation of double sequence spaces by soft and hard analitical methods. Tartu, 2001, 154 p.
26. **Ernst Tungel.** Optimization of plastic spherical shells. Tartu, 2001, 90 p.
27. **Tiina Puolakainen.** Eesti keele arvutigrammatika: morfoloogiline ühestamine. Tartu, 2001, 138 p.
28. **Rainis Haller.** *M(r,s)*-inequalities. Tartu, 2002, 78 p.
29. **Jan Villemson.** Size-efficient interval time stamps. Tartu, 2002, 82 p.
30. Töö kaitsmata.
31. **Mart Abel.** Structure of Gelfand-Mazur algebras. Tartu, 2003. 94 p.
32. **Vladimir Kuchmei.** Affine completeness of some ockham algebras. Tartu, 2003. 100 p.
33. **Olga Dunajeva.** Asymptotic matrix methods in statistical inference problems. Tartu 2003. 78 p.
34. **Mare Tarang.** Stability of the spline collocation method for volterra integro-differential equations. Tartu 2004. 90 p.
35. **Tatjana Nahtman.** Permutation invariance and reparameterizations in linear models. Tartu 2004. 91 p.
36. **Märt Möls.** Linear mixed models with equivalent predictors. Tartu 2004. 70 p.
37. **Kristiina Hakk.** Approximation methods for weakly singular integral equations with discontinuous coefficients. Tartu 2004, 137 p.
38. **Meelis Käärik.** Fitting sets to probability distributions. Tartu 2005, 90 p.
39. **Inga Parts.** Piecewise polynomial collocation methods for solving weakly singular integro-differential equations. Tartu 2005, 140 p.
40. **Natalia Saealle.** Convergence and summability with speed of functional series. Tartu 2005, 91 p.
41. **Tanel Kaart.** The reliability of linear mixed models in genetic studies. Tartu 2006, 124 p.
42. **Kadre Torn.** Shear and bending response of inelastic structures to dynamic load. Tartu 2006, 142 p.
43. **Kristel Mikkor.** Uniform factorisation for compact subsets of Banach spaces of operators. Tartu 2006, 72 p.
44. **Darja Saveljeva.** Quadratic and cubic spline collocation for Volterra integral equations. Tartu 2006, 117 p.
45. **Kristo Heero.** Path planning and learning strategies for mobile robots in dynamic partially unknown environments. Tartu 2006, 123 p.
46. **Annely Mürk.** Optimization of inelastic plates with cracks. Tartu 2006. 137 p.
47. **Annemai Raidjõe.** Sequence spaces defined by modulus functions and superposition operators. Tartu 2006, 97 p.
48. **Olga Panova.** Real Gelfand-Mazur algebras. Tartu 2006, 82 p.

49. **Härmel Nestra.** Iteratively defined transfinite trace semantics and program slicing with respect to them. Tartu 2006, 116 p.
50. **Margus Pihlak.** Approximation of multivariate distribution functions. Tartu 2007, 82 p.
51. **Ene Käärik.** Handling dropouts in repeated measurements using copulas. Tartu 2007, 99 p.
52. **Artur Sepp.** Affine models in mathematical finance: an analytical approach. Tartu 2007, 147 p.
53. **Marina Issakova.** Solving of linear equations, linear inequalities and systems of linear equations in interactive learning environment. Tartu 2007, 170 p.
54. **Kaja Sõstra.** Restriction estimator for domains. Tartu 2007, 104 p.
55. **Kaarel Kaljurand.** Attempto controlled English as a Semantic Web language. Tartu 2007, 162 p.
56. **Mart Anton.** Mechanical modeling of IPMC actuators at large deformations. Tartu 2008, 123 p.
57. **Evely Leetma.** Solution of smoothing problems with obstacles. Tartu 2009, 81 p.
58. **Ants Kaasik.** Estimating ruin probabilities in the Cramér-Lundberg model with heavy-tailed claims. Tartu 2009, 139 p.
59. **Reimo Palm.** Numerical Comparison of Regularization Algorithms for Solving Ill-Posed Problems. Tartu 2010, 105 p.
60. **Indrek Zolk.** The commuting bounded approximation property of Banach spaces. Tartu 2010, 107 p.
61. **Jüri Reimand.** Functional analysis of gene lists, networks and regulatory systems. Tartu 2010, 153 p.
62. **Ahti Peder.** Superpositional Graphs and Finding the Description of Structure by Counting Method. Tartu 2010, 87 p.
63. **Marek Kolk.** Piecewise Polynomial Collocation for Volterra Integral Equations with Singularities. Tartu 2010, 134 p.
64. **Vesal Vojdani.** Static Data Race Analysis of Heap-Manipulating C Programs. Tartu 2010, 137 p.
65. **Larissa Roots.** Free vibrations of stepped cylindrical shells containing cracks. Tartu 2010, 94 p.
66. **Mark Fišel.** Optimizing Statistical Machine Translation via Input Modification. Tartu 2011, 104 p.
67. **Margus Niitsoo**. Black-box Oracle Separation Techniques with Applications in Time-stamping. Tartu 2011, 174 p.
68. **Olga Liivapuu.** Graded q-differential algebras and algebraic models in noncommutative geometry. Tartu 2011, 112 p.
69. **Aleksei Lissitsin.** Convex approximation properties of Banach spaces. Tartu 2011, 107 p.
70. **Lauri Tart.** Morita equivalence of partially ordered semigroups. Tartu 2011, 101 p.
71. **Siim Karus.** Maintainability of XML Transformations. Tartu 2011, 142 p.

72. **Margus Treumuth.** A Framework for Asynchronous Dialogue Systems: Concepts, Issues and Design Aspects. Tartu 2011, 95 p.
73. **Dmitri Lepp.** Solving simplification problems in the domain of exponents, monomials and polynomials in interactive learning environment T-algebra. Tartu 2011, 202 p.
74. **Meelis Kull.** Statistical enrichment analysis in algorithms for studying gene regulation. Tartu 2011, 151 p.
75. **Nadežda Bazunova.** Differential calculus $d^3 = 0$ on binary and ternary associative algebras. Tartu 2011, 99 p.
76. **Natalja Lepik.** Estimation of domains under restrictions built upon generalized regression and synthetic estimators. Tartu 2011, 133 p.
77. **Bingsheng Zhang.** Efficient cryptographic protocols for secure and private remote databases. Tartu 2011, 206 p.
78. **Reina Uba.** Merging business process models. Tartu 2011, 166 p.
79. **Uuno Puus.** Structural performance as a success factor in software development projects – Estonian experience. Tartu 2012, 106 p.
80. **Marje Johanson.** $M(r, s)$-ideals of compact operators. Tartu 2012, 103 p.
81. **Georg Singer.** Web search engines and complex information needs. Tartu 2012, 218 p.
82. **Vitali Retšnoi.** Vector fields and Lie group representations. Tartu 2012, 108 p.
83. **Dan Bogdanov.** Sharemind: programmable secure computations with practical applications. Tartu 2013, 191 p.
84. **Jevgeni Kabanov.** Towards a more productive Java EE ecosystem. Tartu 2013, 151 p.
85. **Erge Ideon**. Rational spline collocation for boundary value problems. Tartu, 2013, 111 p.
86. **Esta Kägo.** Natural vibrations of elastic stepped plates with cracks. Tartu, 2013, 114 p.
87. **Margus Freudenthal.** Simpl: A toolkit for Domain-Specific Language development in enterprise information systems. Tartu, 2013, 151 p.
88. **Boriss Vlassov.** Optimization of stepped plates in the case of smooth yield surfaces. Tartu, 2013, 104 p.
89. **Elina Safiulina.** Parallel and semiparallel space-like submanifolds of low dimension in pseudo-Euclidean space. Tartu, 2013, 85 p.
90. **Raivo Kolde.** Methods for re-using public gene expression data. Tartu, 2014, 121 p.
91. **Vladimir Šor.** Statistical Approach for Memory Leak Detection in Java Applications. Tartu, 2014, 155 p.
92. **Naved Ahmed.** Deriving Security Requirements from Business Process Models. Tartu, 2014, 171 p.
93. **Kerli Orav-Puurand.** Central Part Interpolation Schemes for Weakly Singular Integral Equations. Tartu, 2014, 109 p.
94. **Liina Kamm.** Privacy-preserving statistical analysis using secure multi-party computation. Tartu, 2015, 201 p.

95. **Kaido Lätt.** Singular fractional differential equations and cordial Volterra integral operators. Tartu, 2015, 93 p.
96. **Oleg Košik.** Categorical equivalence in algebra. Tartu, 2015, 84 p.
97. **Kati Ain.** Compactness and null sequences defined by ℓ_p spaces. Tartu, 2015, 90 p.
98. **Helle Hallik.** Rational spline histopolation. Tartu, 2015, 100 p.
99. **Johann Langemets.** Geometrical structure in diameter 2 Banach spaces. Tartu, 2015, 132 p.
100. **Abel Armas Cervantes.** Diagnosing Behavioral Differences between Business Process Models. Tartu, 2015, 193 p.
101. **Fredrik Milani.** On Sub-Processes, Process Variation and their Interplay: An Integrated Divide-and-Conquer Method for Modeling Business Processes with Variation. Tartu, 2015, 164 p.
102. **Huber Raul Flores Macario.** Service-Oriented and Evidence-aware Mobile Cloud Computing. Tartu, 2015, 163 p.
103. **Tauno Metsalu.** Statistical analysis of multivariate data in bioinformatics. Tartu, 2016, 197 p.
104. **Riivo Talviste.** Applying Secure Multi-party Computation in Practice. Tartu, 2016, 144 p.
105. **Md Raknuzzaman.** Noncommutative Galois Extension Approach to Ternary Grassmann Algebra and Graded q-Differential Algebra. Tartu, 2016, 110 p.
106. **Alexander Liyvapuu.** Natural vibrations of elastic stepped arches with cracks. Tartu, 2016, 110 p.
107. **Julia Polikarpus.** Elastic plastic analysis and optimization of axisymmetric plates. Tartu, 2016, 114 p.
108. **Siim Orasmaa.** Explorations of the Problem of Broad-coverage and General Domain Event Analysis: The Estonian Experience. Tartu, 2016, 186 p.
109. **Prastudy Mungkas Fauzi.** Efficient Non-interactive Zero-knowledge Protocols in the CRS Model. Tartu, 2017, 193 p.
110. **Pelle Jakovits.** Adapting Scientific Computing Algorithms to Distributed Computing Frameworks. Tartu, 2017, 168 p.
111. **Anna Leontjeva.** Using Generative Models to Combine Static and Sequential Features for Classification. Tartu, 2017, 167 p.
112. **Mozhgan Pourmoradnasseri.** Some Problems Related to Extensions of Polytopes. Tartu, 2017, 168 p.
113. **Jaak Randmets.** Programming Languages for Secure Multi-party Computation Application Development. Tartu, 2017, 172 p.
114. **Alisa Pankova.** Efficient Multiparty Computation Secure against Covert and Active Adversaries. Tartu, 2017, 316 p.
115. **Tiina Kraav.** Stability of elastic stepped beams with cracks. Tartu, 2017, 126 p.
116. **Toomas Saarsen.** On the Structure and Use of Process Models and Their Interplay. Tartu, 2017, 123 p.

117. **Silja Veidenberg.** Lifting bounded approximation properties from Banach spaces to their dual spaces. Tartu, 2017, 112 p.
118. **Liivika Tee.** Stochastic Chain-Ladder Methods in Non-Life Insurance. Tartu, 2017, 110 p.
119. **Ülo Reimaa.** Non-unital Morita equivalence in a bicategorical setting. Tartu, 2017, 86 p.
120. **Rauni Lillemets.** Generating Systems of Sets and Sequences. Tartu, 2017, 181 p.
121. **Kristjan Korjus.** Analyzing EEG Data and Improving Data Partitioning for Machine Learning Algorithms. Tartu, 2017, 106 p.
122. **Eno Tõnisson.** Differences between Expected Answers and the Answers Offered by Computer Algebra Systems to School Mathematics Equations. Tartu, 2017, 195 p.
123. **Kaur Lumiste.** Improving accuracy of survey estimators by using auxiliary information in data collection and estimation stages. Tartu, 2018, 112 p.
124. **Paul Tammo.** Closed maximal regular one-sided ideals in topological algebras. Tartu, 2018, 112 p.
125. **Mart Kals.** Computational and statistical methods for DNA sequencing data analysis and applications in the Estonian Biobank cohort. Tartu, 2018, 174 p.