DISSERTATIONES
MATHEMATICAE
UNIVERSITATIS
TARTUENSIS
106

ALEXANDER LIYVAPUU

Natural vibrations of elastic stepped arches with cracks

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
106

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
106

ALEXANDER LIYVAPUU

Natural vibrations of elastic stepped arches with cracks

UNIVERSITY OF TARTU
Press

Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Estonia

Dissertation has been accepted for the commencement of the degree of Doctor of Philosophy (Mathematics) on June 17, 2016 by the Council of the Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu.

Supervisor:
Prof. Jaan Lellep
University of Tartu
Tartu, Estonia

Opponents:
Prof. Rimantas Kačianauskas
Vilnius Gediminas Technical University
Vilnius, Lithuania

Prof. Aleksander Klauson
Tallinn University of Technology
Tallinn, Estonia

Commencement will take place on August 24, 2016, at 13.00 in J. Liivi 2-403.

Publication of this dissertation has been granted by University of Tartu.

ISSN 1024-4212
ISBN 978-9949-77-175-2 (print)
ISBN 978-9949-77-176-9 (pdf)

University of Tartu Press
www.tyk.ee

Contents

List of original publications

I J.Lellep, A.Liyvapuu, Natural vibrations of elastic arches with cracks. *In: Proceedings of the 2nd International Conference Optimization and Analysis of Structures.* Editors: J.Lellep, E.Puman, University of Tartu Press, pp. 64–69, 2013.

II J.Lellep, A.Liyvapuu, Free vibrations of elastic laminated arches. *In: Proceedings of the 3nd International Conference Optimization and Analysis of Structures.* Editors: J.Lellep, E.Puman, University of Tartu Press, pp. 52–58, 2015.

III A.Liyvapuu, J.Lellep, Natural vibrations of stepped arches. *International Journal of Advancements in Mechanical and Aeronautical Engineering* 3(1), pp. 68–72, 2016.

IV J.Lellep, A.Liyvapuu, Natural vibrations of stepped arches with cracks. *Agronomy Research* 14(3), pp. 821–830, 2016.

V J.Lellep, A.Liyvapuu, Free vibrations of uniform and hollow-sectional elastic arches, *Elsevier, Procedia Engineering, Modern Building Materials, Structures and Techniques*, 2016, (accepted).

Author's contribution

The author of this dissertation is responsible for majority of the research in all phases (including writing, simulation and preparing of images) of the papers I–V. The solution procedure was developed in co-operation with supervisors; the statement of the problem belongs to the supervisor.

1 Introduction

1.1 Overview of the literature

Plates, shells and beams (straight and curved) are the main structural elements in more complex structures. Modern construction techniques provide for the production of basic elements with the subsequent installation. Extremely important is the quality control during manufacturing and installation. Here the important role play non-destructive testing methods. One possible method might be offered via an analysis of natural vibrations. Mindlin shows in the [55] that the rotational inertia effect is small for the lower plate modes. Huang [28] studied the effects of rotational inertia and shear deformation for the natural frequencies and mode shapes in uniform Timoshenko beam with boundary conditions for free supported ends. He has shown that the effect of these two effects for the natural frequencies increases with the number of modes or cross-sectional dimensions. But the comparative impact on the normal modes of vibration, seems to be very little. Murty [56] developed a linear approximate equation for the transverse vibrations of uniform short beams, including shear deformation and rotary inertia effects. Its values of natural frequencies are in better agreement with the experimental data for comparison with the results obtained using shift correction coefficients proposed by Timoshenko [72], [70], [71] and Cowper [18]. Adams and Bacon [1] showed that the effect of the shear deformation is a function of the aspect ratio (i.e., ratio of length to thickness) and less than 1% for isotropic materials with an aspect ratio of more than twenty.

The effect of the shear deformation and rotatory inertia on the vibrations of beams, rings and arches are investigated also by Rao [64], Lee *et al.* [40], [42], [41], Wu and Chiang [78], [79], [80]. The work by Shames [68] gives us a reason to use the Euler-Bernoulli beam theory in the dissertation and to assume that plane cross sections, normal to the neutral axis before deformation, continue to remain plane and normal to the neutral axis and do not undergo any strain in their planes. Free vibrations of curved beams or arches in term of Euler-Bernoulli theory are the main subject of research in the dissertation. We will use the first modes of vibrations only in our study. The frequencies of free vibrations of curved beams and arches of constant dimensions of the cross section of the arch are calculated by Petyt and Fleischer [61], Markus and Nanosi [53], Chidaparam and Leissa [15], Cerri *et al.* [12], [13], Auciello and DeRosa [4], DeRosa [20], Viola *et al.* [77], [76], [75]

making use of analytical and numerical methods.
Finite element and finite strip methods have been used by Cheung *et al.* [14], Yang and Chen [84], Yang *et al.* [83], Ishaguddin *et al.* [29], Karaagac *et al.* [32]. A method based on curved beam elements is developed by Wu and Chiang [78], [79], [80].
Due to their high stiffness and strength and other mechanical properties compared to the weight the composite and laminate structures have gained popularity during last decades. This involves the need for investigation of the behavior of structures made of composites and laminates. The foundation of the mechanics of composite materials are presented in the books by Herakovich [27], Jones [30], Daniel [19], Tuttle [73]. Various numerical and semi-analytical models for the analysis of granular materials are developed by Kacianauskas *et al.* [31].
Orthotropic beams and plates are studied by Bui *et al.* [9], Ogierman and Kokot [59], Bao *et al.* [6] and others.

Thin-walled plate and shell structures are treated by Kollar and Springer [36], Reddy [65], Vinson and Sierakovski [74], Qatu [62], Gürdal *et al.* [25], Xiang and Wang [81].
The vibrations and stability of structural elements containing cracks and other defects have deserved the attention of many researchers. However, the most of attention is paid to the vibrations of beams (see Dimarogonas [21], [22], [23], Nandwana and Maiti [58], Nahvi and Jabbari [57], Kisa and Brandon [34], Kisa *et al.* [34], Zheng and Kessissoglou [86]). Lellep and Kägo [43] investigated the influence of defects on the eigenfrequencies of classic stretched strips.
The presence of cracks and other defects in structural elements is a source of additional compliance. Anifantis and Dimarogonas [3], Dimarogonas [22], Chondros *et al.* [16], [17], Rizos *et al.* [66], Kukla [37] explored the idea of an elastic spring modelling the additional flexibility due to a crack in the cases of vibrating beams and bars weakened with crack. This approach was extended to the case of elastic plate strips by Lellep and Kägo [43], [38]. Axisymmetric vibrations of elastic circular cylindrical shells with piece wise constant thickness were treated by Lellep and Roots [49], [50]. In the 1990s scientists have paid attention to the development of methods for describing the vibrations of beams containing a crack. Rizos *et al.* [66], Liang *et al.* [51], [52], Nandwana and Maiti [58], Kisa *et al.* [34], [35], Yang and Chen [84], Ostachowicz and Krawczuk [60], Bamnios and Trochides [5] studied the behavior of a beam having a crack at a certain place. Rizos *et al.* [66],

Ostachovicz and Krawczuk [60], Kisa *et al.* [34] and other attempted to simulate the presence of a crack in a beam using the analogy of rotational springs. According to this approach considered cracked beam is consisting of two parts connected by a rotational spring in place of a crack location. This model has been successful in order to describe the local compliance due to the presence of cracks. In the works Dimarogonas [22], Dimarogonas and Paipetis [23] calculations showed a very good approximation methods of linear elastic fracture mechanics (see Anderson [2], Broberg [7] and Broek [8]) experimental data.

This idea is explored in the non-destructive detection of defects. Another method for non-destructive evaluation of plate-like structures is based on Lamb waves (see Ratassepp *et al.* [63]). Lamb waves are used also in the determination of material constants in [39].

The influence of the axial force on the vibrations of Euler-Bernoulli beams was revealed by Caddemi and Calio [10], also by Matsunaga [54]. The Timoshenko arches are investigated by Calio *et al.* [11]. Here, as well as in the paper by Kawakami *et al.* [33] out-of-plane vibrations of curved beams are considered. Functionally graded materials are considered by Eroglu [24].

The foundations of vibrations of beams, plates and shells are presented in the books by Qatu [62], Reddy [65], Vinson and Sierakowski [74], Henrych [26]. The book by Soedel [69] contains the governing equations of the dynamic analysis of beams, plates, shells and of certain non-shell structures. The main principles of the analysis of composite and laminated structures can be found in the books by Tuttle [73], Qatu [62], Daniel and Ishai [19], Jones [30], Herakovich [27], Sadeghpour *et al.* [67] and others. Arches are considered by Xu *et al.* [82]. In the paper by Lellep and Liyvapuu [45] a method of determination of natural frequencies of elastic laminated arches was developed. These results are extended to the case of composite and laminated arches of variable thickness.

During last decades a lot of attention has been paid to the natural vibrations of beams, plates and shells with defects (see Kukla [37], Yang and Chen [84], Lellep and Roots [49], [50]). In the investigations of this type the key problem is the modelling of influence of the defect on the structural behavior. Dimarogonas [21], also Chondros *et al.* [17] have used the model of "massless rotational spring". According to this concept a beam with a defect or a crack is modelled as a mechanical system consisting of two bars which are connected at the cross section where the defect occurs. This approach was employed by Zheng and Fan [85] and others. Lellep and Liyvapuu [46][47],

[48] extended this method to the case of natural vibrations of circular arches having rectangular cross sections with piece-wise constant dimensions. In the present study the natural frequencies of elastic arches with step wise varying cross sections are investigated.

1.2 Aim of the dissertation

The main goal of the research is to investigate the frequency of natural vibrations of elastic stepped arches with crack-like defects and to analyse the sensitivity of the frequency to the geometrical, mechanical and physical properties of the arch. The reasons to choose the methodology that is used in the study and the detailed description of its development are revealed in the introductory section and in Sections 2, 3 and 4 of the present dissertation.

1.3 Structure of the dissertation

The thesis is organized as follows. The first section (Introduction) provides an overview of the study and analysis of vibration, in particular the analysis of free vibrations, in recent decades; it is followed by the explanation of the aim and structure of the dissertation. The main part of the thesis (Sections 2, 3 and 4) describes in detail our method, which is then applied to the arches having both isotropic and laminated structure. The second section discusses two particular cases of stepped arches. At first, the arch of constant thickness simply supported at both ends is studied. The second case is related to laminated arches of constant thickness without cracks. The cases of simply supported arches and arches clamped at both ends are investigated. The third section addresses the issue of the dependence of the frequency of free vibrations on the parameters of the stepped arches. The fourth section examines the frequency of free vibrations of stepped arches with cracks clamped at both ends. The arches are supposed to be with an internal cavity. Finally, the conclusions in English and Estonian are presented, copies of published articles and the author's CV are enclosed.

2 Free vibrations of arches with cracks

In the second section of the dissertation we consider arches with constant thickness. The determination of natural frequencies of these arches is the goal of the research. At first we consider a simply supported at the both ends circular arch with a crack. Then we will study natural vibrations of a laminated arch simply supported at the both ends and arches clamped at the both ends.

2.1 Elastic arch with a crack

Formulation of the problem

Let us consider natural vibrations of an elastic arch of radius $R = const$. It is assumed that the arch is simply supported at both ends (see Fig.1). We use polar coordinates. When using polar coordinates the position of current cross section is defined by the angle φ whereas the edges of the arch correspond to $\alpha = 0$ and $\alpha = \beta$, respectively. It is assumed that at $\varphi = \alpha$ a crack-like defect is located. The defect is treated as a stable crack; no attention will be paid to its extension. Let the depth (length) of the crack be $c < h$. It is assumed herein that the arch has rectangular cross section with dimensions h (thickness) and b (width of the arch). The aim of the section is to determine the frequencies of natural vibrations and to reveal the sensitivity of eigenfrequencies to crack parameters. Note that the arch will be treated as a curved beam with the neutral curved axis lying wholly in one plane. It is assumed that the motion of every point of the neutral curve takes place in this plane only.

Solution of the equation of motion

Equilibrium conditions of an element of the arch lead to the equations (Soedel [69]):

$$\begin{aligned} \frac{\partial M}{\partial s} - Q &= 0, \\ \frac{\partial N}{\partial s} + \frac{Q}{R} + p_s &= \bar{\rho} h \ddot{U}, \\ \frac{\partial Q}{\partial s} - \frac{N}{R} + p_n &= \bar{\rho} h \ddot{W}, \end{aligned} \tag{2.1}$$

where dots denote the differentiation with respect to time t and s is the length of the arch. In (2.1) U and W denote the displacements in the cir-

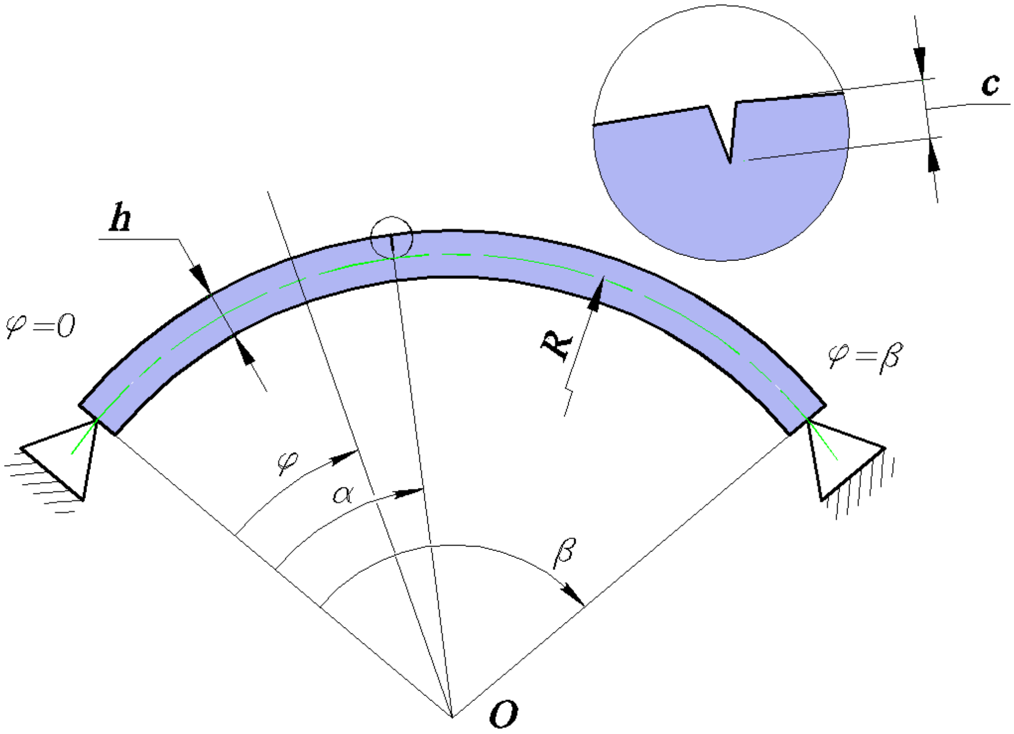

Figure 1: Elastic arch with the crack

cumferential and transverse direction, respectively, whereas p_s and p_n stand for the external loads in these directions. Here N and M are the membrane force and bending moment, Q denotes the shear force; $\bar{\rho}$ is the mass per unit width and h — the thickness of the arch. According to the classical approach $p_s = u = 0$. Assuming that

$$M(0,t) = 0, \quad M(\beta,t) = 0 \tag{2.2}$$

and the membrane force N vanishes at the edges of the arch it follows from (2.1) that $M = -NR$ and

$$M'' + M + R^2(p_n - \bar{\rho}h\ddot{W}) = 0, \tag{2.3}$$

where prims denote the differentiation with respect to the angle φ and $s = \varphi R$.

In the case of curved beams and arches the strain components are the relative extension (see Soedel [69])

$$\varepsilon = \frac{1}{R}(U' + W) \tag{2.4}$$

and the curvature

$$\varkappa = -\frac{1}{R^2}(-U' + W''). \tag{2.5}$$

According to the Hooke's law (see [44], [45], [69])

$$N = Ebh\varepsilon, \quad M = \frac{Eh^3b}{12}\varkappa. \tag{2.6}$$

Here E stands for the Young's modulus.
Thus

$$\begin{aligned} N &= \frac{Ehb}{R}(U' + W), \\ M &= \frac{Eh^3b}{12R^2}(U' - W''), \end{aligned} \tag{2.7}$$

Since $N = 0$ the equation (2.17) yields $U' = -W$. Therefore

$$\varkappa = -\frac{1}{R^2}(W + W'') \tag{2.8}$$

and

$$M = \frac{Eh^3b}{12R^2}(-W - W''). \tag{2.9}$$

We will focus on the free vibrations of the arch. Taking $p_n = 0$ and substituting (2.9) in (2.3) one obtains the equation

$$\frac{Eh^3}{12R^2}(W^{IV} + 2W'' + W) + \bar{\rho}hR^2\ddot{W} = 0. \tag{2.10}$$

In order to solve the differential linear equation (2.10) with partial derivatives the method of Fourier' will be employed. According to this method let us assume that

$$W(\varphi, t) = w(\varphi) \cdot \sin(\omega t), \tag{2.11}$$

where ω stands for the frequency of natural vibrations and $w(\varphi)$ is an unknown function of the variable φ. The substitution (2.11) in (2.10) leads to the equation

$$w^{IV} + 2w'' + w(1 - k^2) = 0, \tag{2.12}$$

where

$$k = \sqrt{\frac{12\bar{\rho}R^4\omega^2}{Eh^2}}. \tag{2.13}$$

The roots of the characteristic equation of the linear differential equation (2.12) are

$$\lambda = \pm\sqrt{-1 \pm k}. \tag{2.14}$$

We assume that $-1 + k > 0$ and $-1 - k < 0$. Thus the general solution of (2.12) takes form

$$w = A_1 \cosh(\mu\varphi) + A_2 \sinh(\mu\varphi) + A_3 \cos(\nu\varphi) + A_4 \sin(\nu\varphi) \tag{2.15}$$

for $\varphi \in [0, \alpha]$ and

$$w = B_1 \cosh(\mu\varphi) + B_2 \sinh(\mu\varphi) + B_3 \cos(\nu\varphi) + B_4 \sin(\nu\varphi) \tag{2.16}$$

for $\varphi \in [\alpha, \beta]$.
Here the notation $\mu^2 = 1 - k,\ \ \nu^2 = 1 + k$ is used.

Arbitrary constants A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 in (2.15), (2.16) have to be determined from boundary and intermediate conditions.

Additional flexibility caused by the crack

Let us use the method developed by Chondros *et al.* [17], Dimarogonas [22], Kukla [37]. The authors of [17] supposed that a slope of the deflection can be considered as a discontinuous quantity at the cross section with cracks.

Let us denote the jump of the discontinuous quantity at the cross section $\varphi = \alpha$ as

$$\theta = W'(\alpha + 0, t) - W'(\alpha - 0, t). \tag{2.17}$$

It is recognized that there exists a relationship between the local compliance of the arch C and the stress intensity factor K known in the linear elastic fracture mechanics.

The quantity θ can be treated as a generalized displacement corresponding to the generalized force (bending moment) $M_c = M(\varphi, t)|_{\varphi=\alpha}$ one can write

$$\theta = C \cdot M_c, \tag{2.18}$$

where C is the additional compliance caused by the crack.

On the other hand, it is known in the fracture mechanics that the energy release rate can be calculated as (here $A = bc$)

$$G = \frac{M_c^2}{2}\frac{dC}{dA} \tag{2.19}$$

or as (see Anderson [2], Broek [8])

$$G = \frac{K^2}{E'}. \tag{2.20}$$

In (2.20) $E' = E$ for plane stress state and $E' = \frac{E}{(1-\nu^2)}$ for the plane deformation state.
The stress intensity factor K itself is defined as

$$K = \sigma\sqrt{\pi c}F(s). \tag{2.21}$$

Here $s = \frac{c}{h}$. If the cracked element is subjected to pure bending then

$$\sigma = \frac{6M_c}{bh^2}. \tag{2.22}$$

The shape function F in (2.21)must be defined on the basis of experimental results. It depends on the type of a cracked element. Following Dimarogonas [21], Rizos *et al.* [66] one can take

$$F(s) = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.80s^4, \tag{2.23}$$

where $s = c/h$.

Combining the relations (2.18)—(2.23) after algebraic manipulations one obtains

$$\frac{dC}{ds} = \frac{72\pi}{E'h^2b}sF^2(s). \tag{2.24}$$

The integration of (2.24) leads to the result

$$C = \frac{72\pi}{E'h^2b}f(s), \tag{2.25}$$

where

$$f(s) = \int_0^s \xi F^2(\xi)d\xi. \tag{2.26}$$

Thus the jump of the slope can be evaluated as

$$w'(\alpha + 0) - w'(\alpha - 0) = p\Big(w''(\alpha) + w(\alpha)\Big), \tag{2.27}$$

where

$$p = -\frac{6\pi h(1 - \nu^2)}{R^2} \cdot f(s). \tag{2.28}$$

Boundary and intermediate conditions for a simply supported arch
Boundary conditions for an arch simply supported at both ends are

$$\begin{aligned} w(0) &= 0, \quad w''(0) = 0, \\ w(\beta) &= 0, \quad w''(\beta) = 0. \end{aligned} \tag{2.29}$$

At the internal points of the interval $(0, \beta)$ the displacement W and its derivative, also bending moment M and shear force Q must be continuous. Therefore one has

$$\begin{aligned} w(\alpha - 0) &= w(\alpha + 0), \\ w''(\alpha - 0) &= w''(\alpha + 0), \\ w'''(\alpha - 0) &= w'''(\alpha + 0). \end{aligned} \tag{2.30}$$

The quantity w' is discontinuous at $\varphi = \alpha$; the jump condition for it (2.27) must be fulfilled.

The conditions (2.29) furnish at $\varphi = 0$ the equations

$$\begin{aligned} A_1 + A_3 &= 0, \\ \mu^2 A_1 - \nu^2 A_3 &= 0. \end{aligned} \tag{2.31}$$

It immediately follows from (2.31) that

$$A_1 = A_3 = 0. \tag{2.32}$$

The conditions (2.29) furnish at $\varphi = \beta$ the equations

$$B_1 = -B_2 \tanh{(\mu\beta)}, \quad B_3 = -B_4 \tan{(\nu\beta)}. \tag{2.33}$$

Taking (2.27), (2.30), (2.32), (2.33) into account the continuity requirements can be presented in the form of the following system

$$
\begin{aligned}
A_2 \sinh(\mu\alpha) + A_4 \sin(\nu\alpha) &= \\
= B_2\Big(\sinh(\mu\alpha) &- \tanh(\mu\beta)\cosh(\mu\alpha)\Big) + \\
&+ B_4\Big(\sin(\nu\alpha) - \tan(\nu\beta)\cos(\nu\alpha)\Big),
\end{aligned}
$$

$$
\begin{aligned}
A_2\mu \cosh(\mu\alpha) + A_4\nu \cos(\nu\alpha) &= \\
= p\Big[B_2(\mu^2+1)\Big(\sinh(\mu\alpha) &- \tanh(\mu\beta)\cosh(\mu\alpha)\Big) + \\
&+ B_4(1-\nu^2)\Big(\sin(\nu\alpha) - \tan(\nu\beta)\cos(\nu\alpha)\Big)\Big],
\end{aligned} \tag{2.34}
$$

$$
\begin{aligned}
A_2\mu^2 \sinh(\mu\alpha) - A_4\nu^2 \sin(\nu\alpha) &= \\
= B_2\mu^2\Big(\sinh(\mu\alpha) &- \tanh(\mu\beta)\cosh(\mu\alpha)\Big) + \\
&+ B_4\nu^2\Big(\tan(\nu\beta)\cos(\nu\alpha) - \sin(\nu\alpha)\Big),
\end{aligned}
$$

$$
\begin{aligned}
A_2\mu^3 \cosh(\mu\alpha) - A_4\nu^3 \cos(\nu\alpha) &= \\
= B_2\mu^3\Big(\cosh(\mu\alpha) &- \tanh(\mu\beta)\sin(\mu\alpha)\Big) + \\
&+ B_4\nu^3\Big(-\cos(\nu\alpha) - \tan(\nu\beta)\sin(\nu\alpha)\Big).
\end{aligned}
$$

Evidently, we have the linear homogeneous system of four algebraic equations (2.34) with the unknowns A_2, A_4, B_2, B_4. This system has a non-trivial solution if and only if its determinant Δ equals to zero.

Numerical results and discussion

The equation $\Delta = 0$ is solved up to the end numerically making use of the computer code MATLAB. We found the lowest frequency corresponding to the first mode of vibration. Some examples are shown in Fig.2, Fig.3. The results presented in Fig.2, Fig.3 correspond to the simply supported arch made of steel with moduli $E = 2.1 \cdot 10^{11}\ Pa,\ \nu = 0.3,\ \rho = 7865\ kg/m^3$.

The geometrical dimensions of the arch are $R = 1\ m$, $b = 0.05\ m$, $h = 0.05\ m$. In Fig.2 $\beta = 1.5\ rad$ and in Fig.3 $\beta = 1\ rad$.

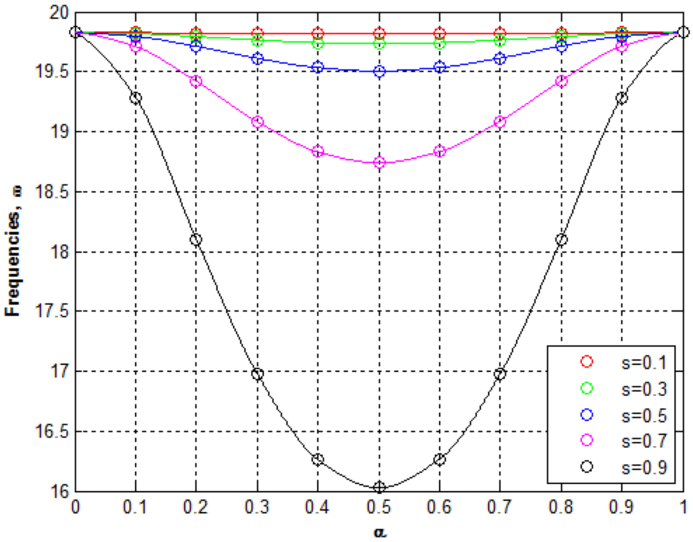

Figure 2: Natural frequency vs. length of the crack, $\beta = 1.5$ rad.

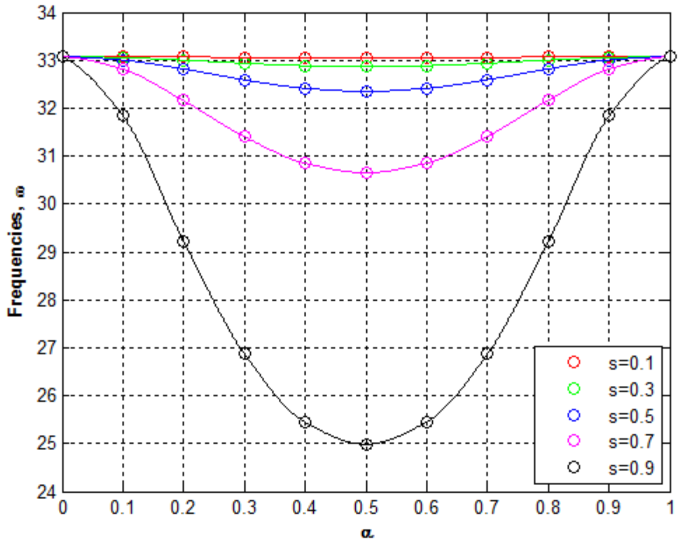

Figure 3: Natural frequency vs. length of the crack, $\beta = 1$ rad.

Different curves in Fig.2 and Fig.3 correspond to different extensions of the crack. Evidently, the higher frequencies correspond to the larger angle of the arch. It can be seen from Fig.2 and Fig.3 that the eigenfrequency has the highest level for the arch without any defects.

Results obtained by the method presented above are compared with the results of Wu, Chang [78] and Petyt, Fleischer [61] in the case of simply supported arches in Table 2.1. and Table 2.2.

Here K indicates the number of natural frequency.
Table 2.1 corresponds to the arch with $\beta = 1\ rad$, $R = 30\ in \approx 0.762\ m$, $h = 0.01289\ in \approx 3.27 \cdot 10^{-4}\ m$
The material of the arch is a mild steel with characteristics:
$E = 10^7\ lb/in^2 \approx 2.871 \cdot 10^{11}\ Pa,\ \rho = 0.1\ lb/in^3 \approx 2768.2\ kg/m^3$.
Here $\omega_0 = \omega\sqrt{\frac{AR^4}{EI}}$, $I = \frac{1}{12}bh^3,\ A = bh$.

Table 2.1. Comparison of results, ($\beta = 1\ rad$)

K	1	2	3	4	5
ω_0	0.366	1.590	3.672	6.481	10.159
Petyt, Fleischer [61]	0.349	1.571	3.612	6.470	10.144

Table 2.2. Comparison of results, ($\beta = \pi/2$)

K	1	2	3	4	5
ω_0	2	15	35	63	99
Wu, Chiang [78]	—	13.773	32.426	61.610	96.375

It can be seen from Table 2.1 and Table 2.2 that the results of the current study are quite close to results obtained by other researchers.

2.2 Laminated arches

Basic assumptions
Free vibrations of an elastic arch made of a multi layered material will be considered. It assumed that the cross-section of the arch is rectangular with the width b and total height (thickness) $H = const.$

The cross section of the arch consists of layers with thickness h_j ($j = 0, 1, \ldots, n$). Each layer is assumed to be an elastic layer with material pa-

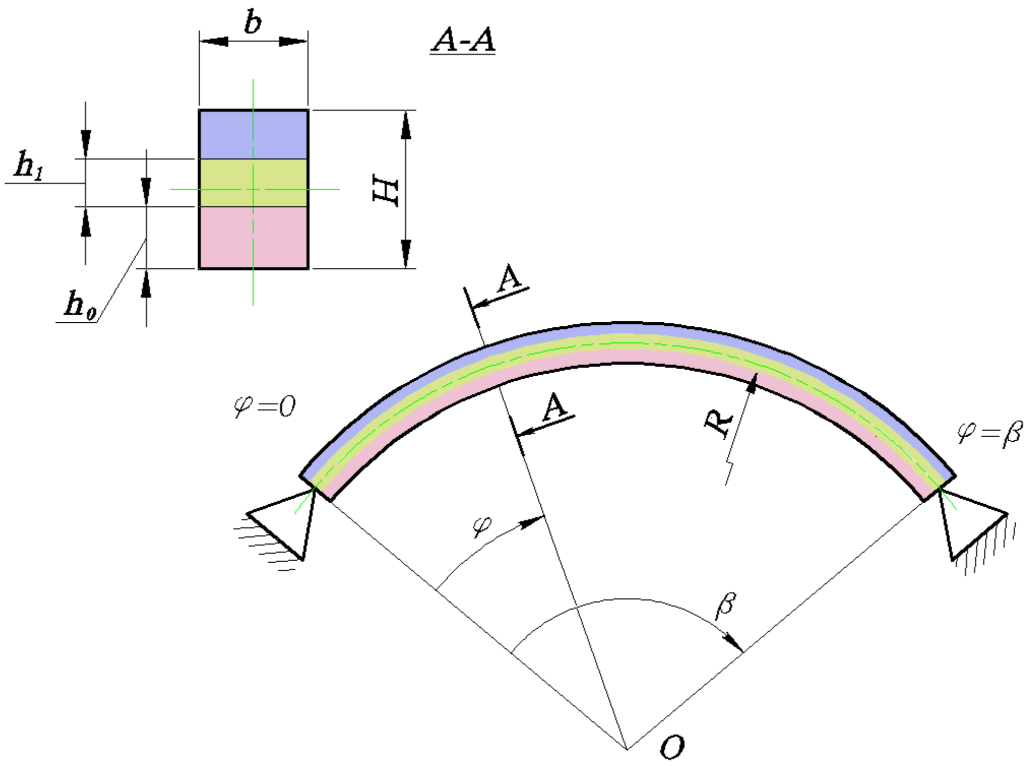

Figure 4: Elastic laminated arch

rameters ρ_j (density), E_j (Young's modulus), ν_j (Poisson's ratio). The layers are treated as orthotropic laminae consisting of a matrix material and of uniformly embedded fibers. However, in the case of an orthotropic lamina these engineering constants are not independent. For a plane stress situation one has (see Gürdal *et al.*[25]) for the neighbouring layers must be satisfied restrictions

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \tag{2.35}$$

for $i = 0, 1, \ldots, n-1, \quad j = i + 1.$

The goal of the study is to determine the natural frequencies of free vibrations of laminated elastic arches and to elucidate the sensitivity of eigenfrequencies on the geometrical parameters of the laminate.

Governing equations

In the theory of laminated elastic plates the physical relations can be presented as (see Kollar and Springer [36], Reddy [65], Vinson and Sierakovski [74])

$$N = A\varepsilon + B\varkappa \tag{2.36}$$

and

$$M = B\varepsilon + D\varkappa. \tag{2.37}$$

Here N and M stand for vectors of membrane forces and principal moments, respectively, whereas ε and $\varkappa$ are corresponding vectors of strain components and curvatures. The elements of matrices A, B and D can be calculated as (Vinson and Sierakowski [74])

$$\begin{aligned}
A_{ij} &= \sum_{k=1}^{n} Q_{ij}^{(k)} \left(z_k - z_{k-1}\right), \\
B_{ij} &= \frac{1}{2}\sum_{k=1}^{n} Q_{ij}^{(k)} \left(z_k^2 - z_{k-1}^2\right), \\
D_{ij} &= \frac{1}{3}\sum_{k=1}^{n} Q_{ij}^{(k)} \left(z_k^3 - z_{k-1}^3\right),
\end{aligned} \tag{2.38}$$

where $|z_k - z_{k-1}| = h_k$.

In the case of laminates for which each lamina is reinforced with a unidirectional array of fibers one has

$$Q_{11}^{(k)} = \frac{E_1^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \quad Q_{12}^{(k)} = \frac{\nu_{12}^{(k)}E_2^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \quad Q_{22}^{(k)} = \frac{E_2^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \tag{2.39}$$

where the superscript (k) indicates the number of the lamina. In the case of beams and arches the quantities A, B and C are scalars. Also the vectors ε and $\varkappa$ are one-dimensional strain components.

These can be expressed as (2.6),(2.5).

The Hooke's law furnishes with (2.36), (2.37) the relations

$$N = \frac{A}{R}\left(U' + W\right) + \frac{B}{R^2}\left(U' - W''\right) \tag{2.40}$$

and

$$M = \frac{B}{R}\left(U' + W\right) + \frac{D}{R^2}\left(U' - W''\right). \tag{2.41}$$

According to the Euler-Bernoulli beam theory there is no extension of the middle surface of the arch. Therefore, one can take $\varepsilon = 0$ and $U' = -W$. Thus one obtains

$$M = D\varkappa, \tag{2.42}$$

where

$$D = \frac{Eh^3 b}{12}. \tag{2.43}$$

Evidently, (2.42) coincides with (2.9). Similarly, the the equilibrium conditions lead to the equation (2.10), as in the previous case.

Let the general solution on the equation of motion be

$$w = C_1 \cosh(\mu\varphi) + C_2 \sinh(\mu\varphi) + C_3 \cos(\nu\varphi) + C_4 \sin(\nu\varphi), \tag{2.44}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants and

$$\mu = \sqrt{1 - \omega R^2 \sqrt{\frac{\bar{\rho}}{D}}}, \quad \nu = \sqrt{1 + \omega R^2 \sqrt{\frac{\bar{\rho}}{D}}}. \tag{2.45}$$

In the case of an arch simply supported at both ends the boundary requirements can be presented by (2.29).

It follows from (2.29), (2.44), (2.45) that

$$\omega = \pm \sqrt{\frac{D}{\bar{\rho}}} \cdot \frac{\pi^2 k^2 - \beta^2}{\beta^2 R^2}, \tag{2.46}$$

where $k = 1, 2, \ldots, n$.

Arches clamped at the both ends

In the case of an arch clamped at both ends the boundary requirements are

$$w(0) = 0, \quad w'(0) = 0 \tag{2.47}$$

and

$$w(\beta) = 0, \quad w'(\beta) = 0. \tag{2.48}$$

Conditions (2.47), (2.48) with (2.44) lead to the linear algebraic system with the determinant

$$\Delta = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \mu & 0 & \nu \\ \cosh(\mu\beta) & \sinh(\mu\beta) & \cos(\nu\beta) & \sin(\nu\beta) \\ \mu\sinh(\mu\beta) & \mu\cosh(\mu\beta) & -\nu\sin(\nu\beta) & \nu\cos(\nu\beta) \end{vmatrix}. \tag{2.49}$$

Evidently, the determinant (2.49) equals to zero.

Numerical results

The results of calculations are presented in Fig.5—Fig.7 and Table 2.3. The obtained results correspond to the three-layers arch with thicknesses h_0, h_1, h_2.

The dependence of the natural frequency on the order of stacking of layers with different materials is shown in Fig.5. Here as in the previous case $h_0 = h_1 = h_2 = 0.01\ m = const.$

The natural frequency versus radius of the arch is depicted in Fig.6 for different values of the thickness h_2. Here $h_0 = h_1 = 0.01\,m$.

It can be seen from Fig.6 that the natural frequency decreases with increasing the radius of the arch.

The sensitivity of the natural frequency on the angle β is shown in Fig.7 for different values of the radius of the arch. Here $h_0 = h_1 = h_2 = 0.01\ m$. It can be seen from Fig.7 that the larger is the radius of the arch the lower is the natural frequency.

The frequency decreases if the angle β increases.

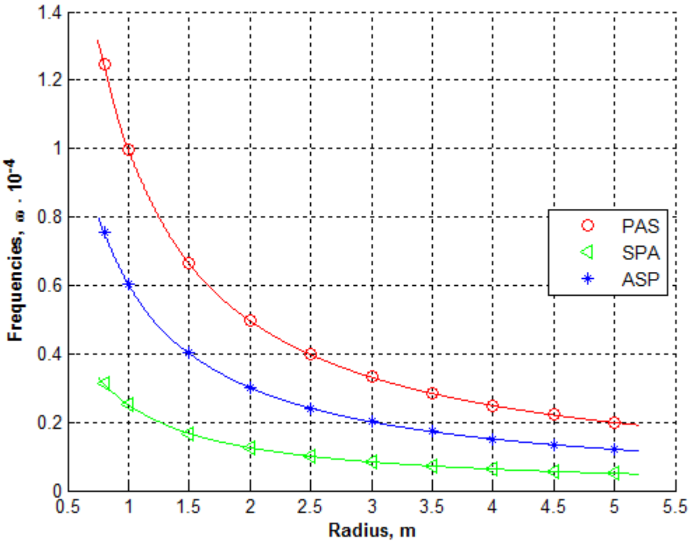

Figure 5: Natural frequency vs. radius of the arch

(A– aluminum, P– polystirol, S– steel)

Table 2.3. Materials of layers

		E, Pa	$\rho, kg/m^3$	ν
(h_2)	Steel	$2.1 \cdot 10^{11}$	7865	0.3
(h_1)	Aluninum	$7 \cdot 10^9$	2700	0.35
(h_0)	Polystirol	$1.5 \cdot 10^9$	30	0.1

It can be seen from Fig.5 that the lowest natural frequency is achieved in the case of stacking materials in the order Steel—Polystirol—Aluminum.

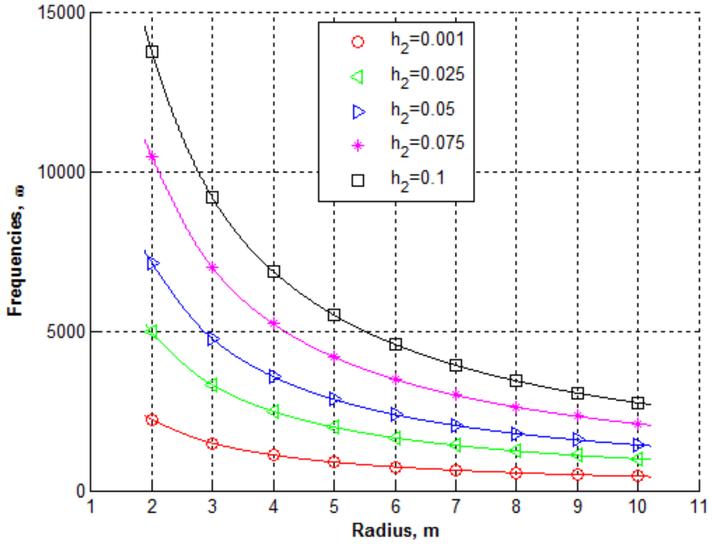

Figure 6: Natural frequency vs. radius of the arch

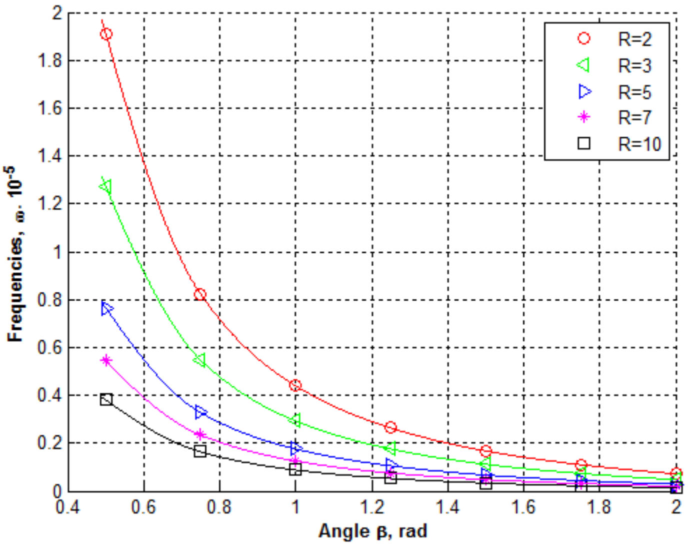

Figure 7: Natural frequency vs. angle of the arch

3 Natural frequencies of stepped arches

The aim of this section is to determine the eigenfrequencies of elastic stepped arches and to study the sensitivity of free vibrations on the crack location and depth. This section is based on the papers Lellep and Liyvapuu [46], [47].

3.1 Stepped arches without defects

Let us consider an elastic circular arch of radius R. The arch is simply supported at the both ends (see Fig.8). As in the previous sections the current angle φ defines the position of the cross-section of the arch. It is assumed that the arch has rectangular cross-section with a constant width b and the height $h = h_j$ for $\varphi \in (\alpha_j, \alpha_{j+1})$. The quantities h_j are assumed to be given constants.

It is assumed that the due to the initial excitation the arch is performing free vibrations around its equilibrium position. The goal is to determine the natural frequencies of the free vibrations and to reveal the sensitivity of parameters. We can assume that the arch under consideration is made of a composite of laminated material.

In the theory of elastic plates (see Kollar and Springer [36], Reddy[65], Vinson and Sierakowski[74]) membrane force N and bending moment M are presented as follows

$$N = \sum_{k=1}^{K_j} \int_{z_k}^{z_{k+1}} \sigma_s dz \tag{3.1}$$

and

$$M = \sum_{k=1}^{K_j} \int_{z_k}^{z_{k+1}} \sigma_s z dz,$$

$$|z_{k+1} - z_k| = h_{kj}$$

is the thickness of layer number k; here z_k is the local coordinate directed through the thickness of the arch and K_j is the number of layers.

Shear force

$$Q_s = \int_{-h/2}^{h/2} \sigma_{s\varphi} dz, \tag{3.2}$$

where $\sigma_{s\varphi}$ — tangential stress component

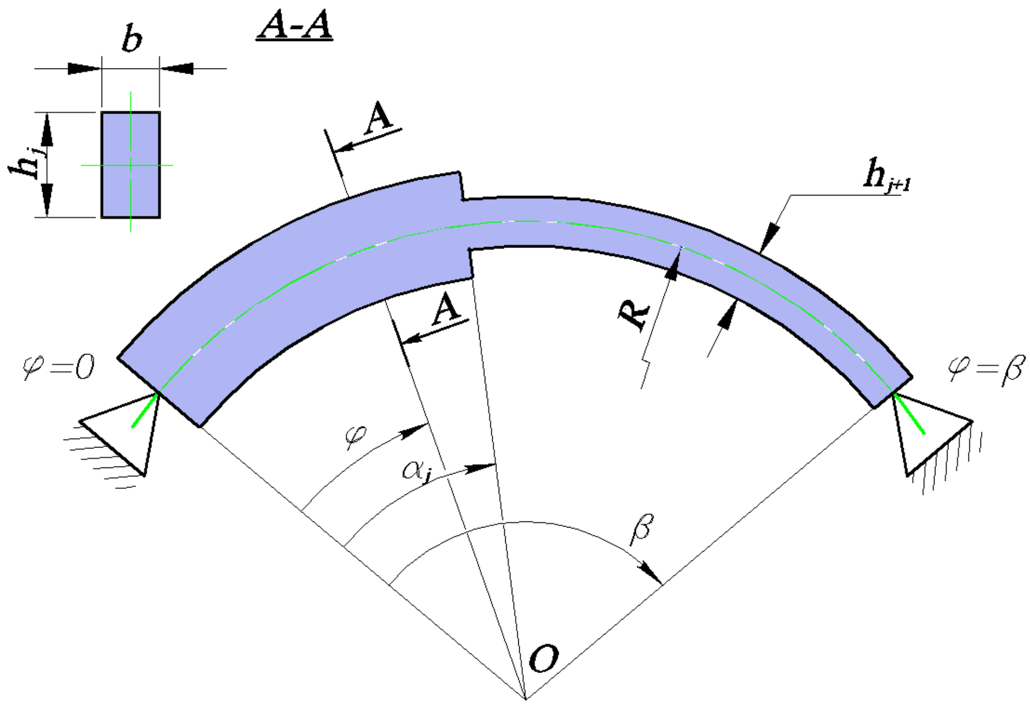

Figure 8: Elastic stepped arch

The Hooke's law for a multilayered arch is given by (2.36), (2.37). Here ε and $\varkappa$ are vectors of strain components and curvatures, respectively, expressed by (2.6), (2.5).

The elements of matrices A, B and D (see Vinson and Sierakowski [74]) given by (2.38).

Also the vectors ε and $\varkappa$ are one-dimensional strain components.

It easily follows from governing equations that in the case of stepped arches and

$$M_j = -D_j \varkappa, \tag{3.3}$$

where $\varkappa$ is defined by (2.8). In the case of free vibrations of stepped arches one has

$$\frac{D_j}{R^2}\left(W^{IV} + 2W'' + W\right) + \bar{\rho}_j R^2 \ddot{W} = 0, \tag{3.4}$$

where $\bar{\rho}_j$ — mass per unit area of the middle surface of the arch in different regions.

Here $j = 0$ for $\varphi \in (0, \alpha)$ and $j = 1$ for $\varphi \in (\alpha, \beta)$.

The general solution of equation (3.4) is

$$w = C_{j1} \cosh(\mu_j \varphi) + C_{j2} \sinh(\mu_j \varphi) + C_{j3} \cos(\nu_j \varphi) + C_{j4} \sin(\nu_j \varphi), \quad (3.5)$$

where C_{j1}, C_{j2}, C_{j3} and C_{j4} are arbitrary constants and

$$\mu_j = \sqrt{1 - \omega R^2 \sqrt{\frac{\bar{\rho}_j}{D_j}}}, \quad \nu_j = \sqrt{1 + \omega R^2 \sqrt{\frac{\bar{\rho}_j}{D_j}}}. \quad (3.6)$$

Boundary conditions for an arch simply supported at the both edges are presented by (2.29) and for clamped arches by (2.47), (2.48).

Thus one can write

$$\begin{aligned} w(\alpha_j - 0) &= w(\alpha_j + 0), \\ w'(\alpha_j - 0) &= w'(\alpha_j + 0), \\ D_{j-1}\Big(w(\alpha_j - 0) + w''(\alpha_j - 0)\Big) &= D_j\Big(w(\alpha_j + 0) + w''(\alpha_j + 0)\Big), \\ D_{j-1}\Big(w'(\alpha_j - 0) + w'''(\alpha_j - 0)\Big) &= D_j\Big(w'(\alpha_j + 0) + w'''(\alpha_j + 0)\Big) \end{aligned} \quad (3.7)$$

for each $j = 1, 2, \ldots, n$,
where

$$D_j = \frac{E h_j^3 b}{12 R^2}. \quad (3.8)$$

In order to determine the frequency of the free vibrations of stepped arch one has to determine the constants $C_{j1}, C_{j2}, C_{j3}, C_{j4}$ so that all boundary and intermediate conditions are satisfied. Thus the system of boundary conditions with intermediate conditions (3.7) consists of $4n + 4$ algebraic equations.

In the case $n = 1$ the determinant of this system can be expressed as

$$\Delta = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \mu_0 \xi_5 & \nu_0 \xi_6 & \mu_1 \xi_7 & -\nu_1 \xi_8 \\ \mu_0^2 \xi_1 & -\nu_0^2 \xi_2 & \mu_1^2 \xi_3 & -\nu_1^2 \xi_4 \\ \mu_0^3 \xi_5 & -\nu_0^3 \xi_6 & \mu_1^3 \xi_7 & \nu_1^3 \xi_8 \end{vmatrix}, \quad (3.9)$$

where

$\xi_1 = \sinh(\mu_0 \alpha)$, $\quad \xi_5 = \cosh(\mu_0 \alpha)$,
$\xi_2 = \sin(\nu_0 \alpha)$, $\quad \xi_6 = \cos(\nu_0 \alpha)$,
$\xi_3 = \tanh(\mu_1 \beta) \cosh(\mu_1 \alpha) - \sinh(\mu_1 \alpha)$, $\quad \xi_7 = \tan(\nu_1 \beta) \sin(\nu_1 \alpha) + \cos(\nu_1 \alpha)$,
$\xi_4 = \tan(\nu_1 \beta) \cos(\nu_1 \alpha) - \sin(\nu_1 \alpha)$, $\quad \xi_8 = \tanh(\mu_1 \beta) \sinh(\mu_1 \alpha) - \cosh(\mu_1 \alpha)$.

The equation $\Delta = 0$ is solved numerically using the computer code of MATLAB. The results of calculations are presented in Fig.9—Fig.11.

The lowest frequency of free vibrations is presented in Fig.9 for various values of the angle β. The arch is made of a mild steel with $E = 2.1 \cdot 10^{11}\ Pa$. The dimensions of the arch are $R = 1\ m,\ h_0 = 0.02\ m,\ h_1 = 0.01\ m$.

The natural frequency versus the step coordinate α is presented in Fig.10. Different curves in Fig.10 correspond to different values of the ratio of thicknesses $\gamma = h_1/h_0$.
Here $\beta = 1.5\ rad,\quad b = 0.01\ m,\quad h_0 = 0.05\ m$.

The relationship between ω and α is shown in Fig.11 for different materials. Here $R = 1\ m,\ h_0 = 0.02\ m,\ h_1 = 0.01\ m$. It can be seen from Fig.11 that curves corresponding to bronze and concrete are quite near to each other.

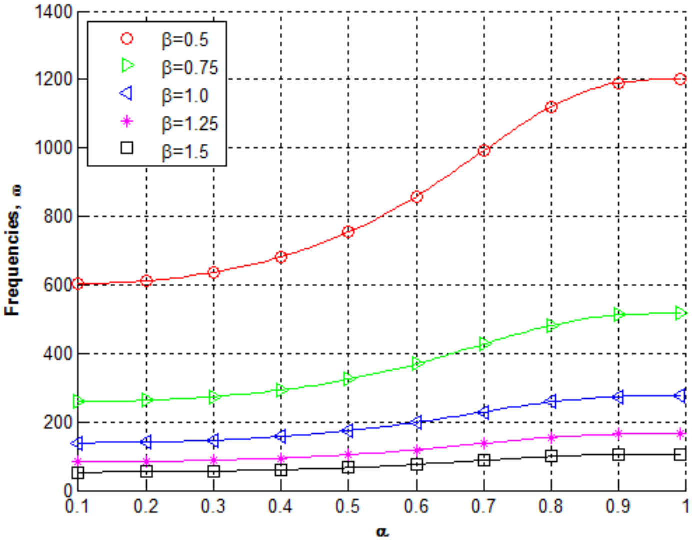

Figure 9: Natural frequency vs. angle of the arch

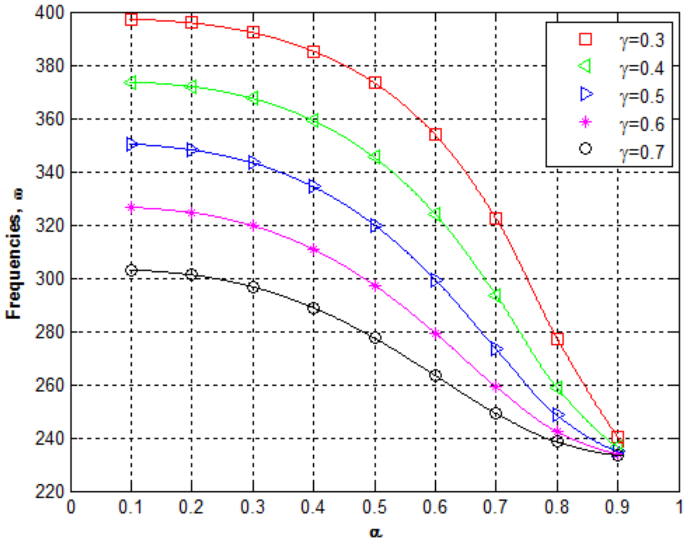

Figure 10: Natural frequency vs. ratio of the thicknesses, $\gamma = \frac{h_1}{h_0}$

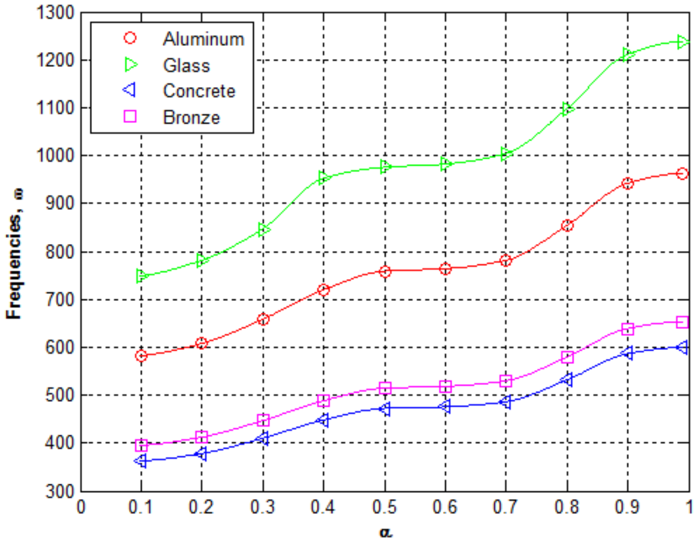

Figure 11: Natural frequency vs. materials of the arch made

3.2 Stepped arch with cracks

Let us consider an elastic stepped arch with cracks. We assume that the arch has piece-wise constant thickness

$$h = h_j, \quad \varphi \in (\alpha_j, \alpha_{j+1}) \tag{3.10}$$

for $j = 0, 1, \ldots, n$. Cracks with length c_j are placed in re-entrant corners of steps (see Fig.12). Let other geometrical, physical and mechanical properties of the arch be the same as in the previous section 3.1.

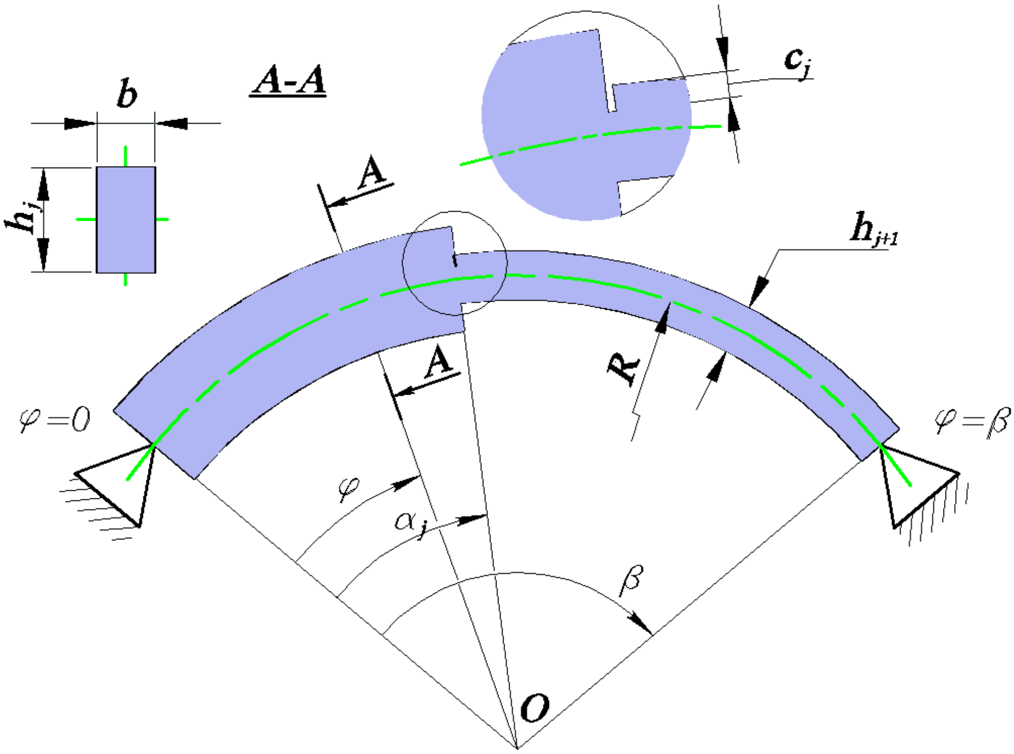

Figure 12: Elastic stepped arch with cracks

Let us study an impact of a surface crack in the cross section $\varphi = \alpha_j$ on the behavior of the arch. Evidently, the continuity conditions and boundary requirements used in the previous section remain valid.

The goal of this section is to study the behavior of an elastic stepped arch with surface cracks.

Equations of equilibrium of an element of a vibrating arch can be transformed into (3.4) for $\varphi \in (\alpha_j, \alpha_{j+1}) \quad j = 0, 1, \ldots, n$.

Now the general solution of (3.4) is to be taken as

$$w = C_{1j} \cosh (\mu_j \varphi = + C_{2j} \sinh (\mu_j \varphi) + C_{3j} \cos (\nu_j \varphi) + C_{4j} \sin (\nu_j \varphi). \quad (3.11)$$

For determination of constants C_1, C_2, C_3 and C_4 one can use the boundary conditions (2.29), also intermediate conditions (2.30) and (2.27). The set of intermediate conditions can be presented as (see Lellep and Liyvapuu [44]) in form

$$\begin{aligned}
[w(\alpha_j)] &= 0, \\
[w'(\alpha_j)] &= \frac{6\pi h_{j+1}}{R^2} \cdot f(s_j) \Big(w''(\alpha_j + 0) + w(\alpha) \Big), \\
[w''(\alpha_j)] &= 0, \\
[w'''(\alpha_j)] &= 0.
\end{aligned} \quad (3.12)$$

In (3.12) the square brackets denote the jump of corresponding quantities and

$$\begin{aligned}
f(s_j) &= 1.86 s_j^2 - 3.95 s_j^3 + 16.37 s_j^4 - 34.23 s_j^5 + \\
&+ 76.81 s_j^6 - 126.9 s_j^7 + 172 s_j^8 - 143.97 s_j^9 + 66.56 s_j^{10}. \quad (3.13)
\end{aligned}$$

Here $s_j = c_j / h_{j+1}, \quad c_j$ being the crack depth.

The requirements (3.7) form a linear algebraic system with the determinant Δ. Since this is a homogeneous system it has a non-trivial solution if and only if $\Delta = 0$. This equation enables to determine the natural frequencies for an arch containing a crack.

The results of calculations are presented in Fig.13—Fig.16 for the arch with $R = 1\ m, \quad \beta = 1\ rad, \quad h_0 = 0.02\ m, \quad h_1 = 0.01\ m$.

The natural frequency versus α is depicted in Fig.13 for different extensions of the crack. The upper curve corresponds to the undamaged arch (with $s = 0$).

The dependence of natural frequency on the radius of the arch is shown in Fig.14 for different radii of the arch.

The natural frequency versus the step location is presented in Fig.15 and Fig.16 for the cases $s = 0$ and $s = 0.8$, respectively.

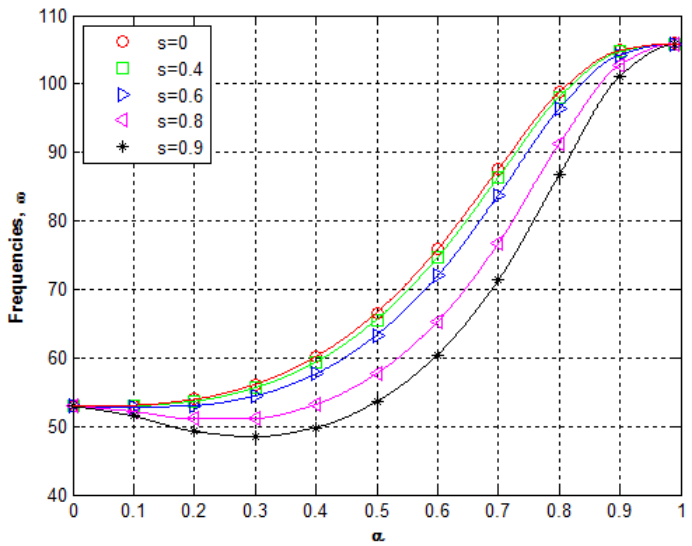

Figure 13: Natural frequency vs. depth of the crack

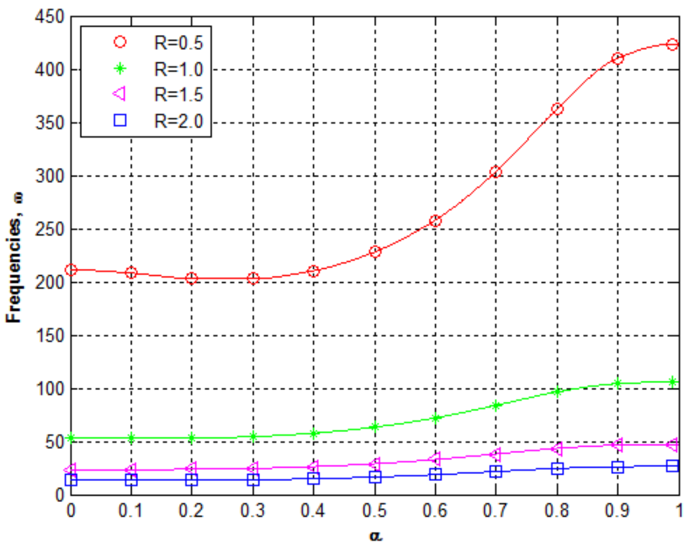

Figure 14: Natural frequency vs. radius of the arch

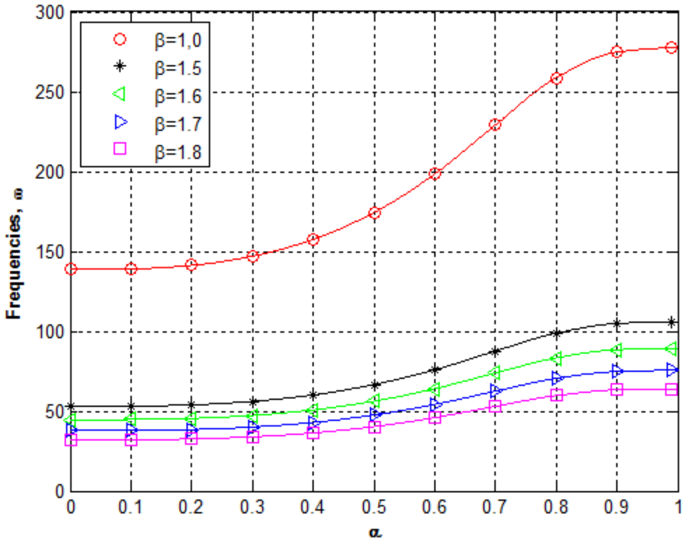

Figure 15: Natural frequency vs. step location ($s = 0$)

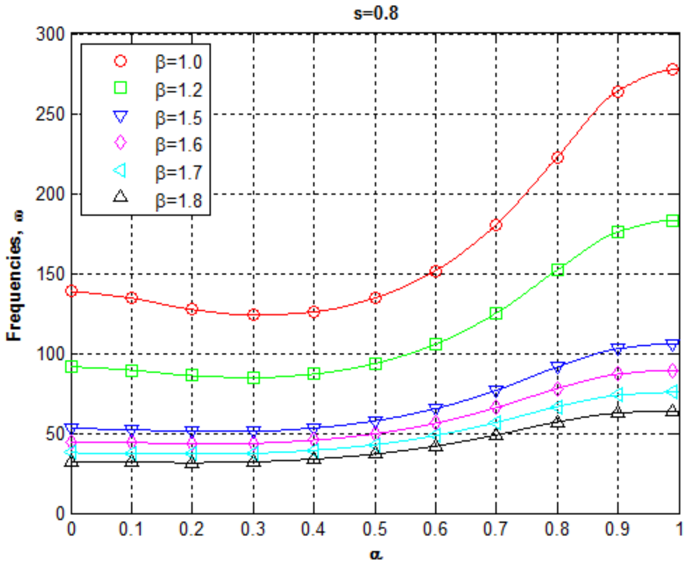

Figure 16: Natural frequency vs. step location ($s = 0.8$)

4 Free vibrations of tubular elastic arches

Let us consider the dynamic behavior of an elastic arch of rectangular uniform or non-uniform cross section. Here we will use capital letters to denote external dimensions of the cross-section of the arch and small letter for the internal dimensions. It is assumed that the external dimensions of the cross section are B (width) and H (height). Let b and h be for internal width and height, respectively.

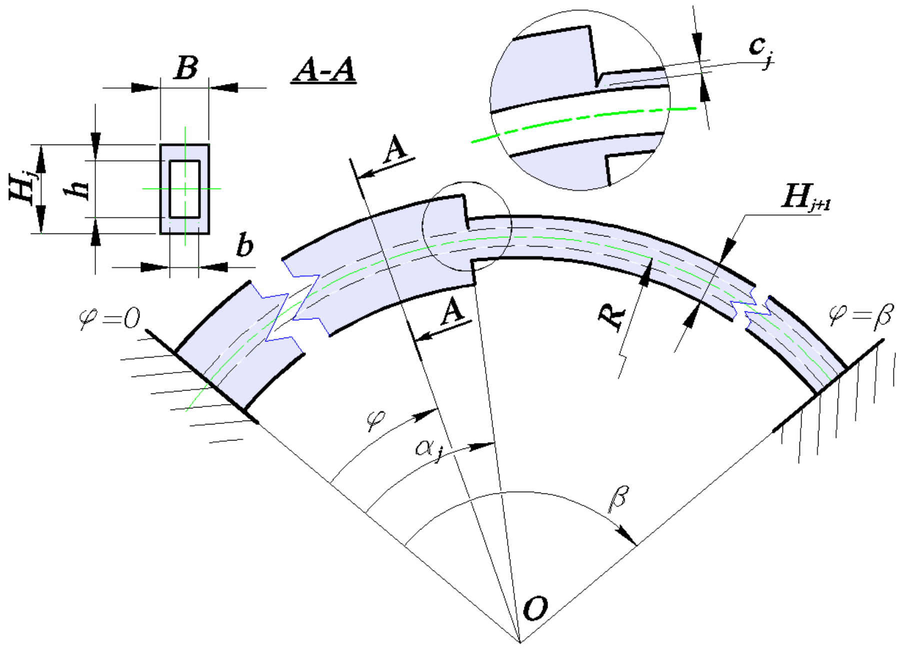

Figure 17: Stepped arch with tubular cross-sections

We assume that the height H is piece-wise constant, e.g.

$$H = H_j, \quad \varphi \in (\alpha_j, \alpha_{j+1}) \tag{4.1}$$

for $j = 0, \ldots, n$. However, the radius of the arch $R = const$ and $B = const$. Let the arch be a circular arch with radius R and central angle β. (see Fig.17). For the sake of simplicity one can take $\alpha_0 = 0$, $\alpha_{n+1} = \beta$.

The cross sectional area of the arch is $S_j = BH_j$ for $j = 0, \ldots, n$. However, the area of these parts which are occupied with the material is

$$S_j = BH_j - bh \tag{4.2}$$

for $\varphi \in (\alpha_j, \alpha_{j+1})$ $j = 0, \ldots, n$.
The arch is weakened with cracks of length c_j locating at $\varphi = \alpha_j$, $(j = 1, \ldots, n)$.

The goal of the study is to define frequencies of natural vibrations and to analyze the sensitivity of eigenfrequencies on the crack parameters.

The basic equations used in the case remain valid.
The Hooke's law for a stepped arch can be presented as

$$M = -\frac{D_j}{R^2}(W + W'') \tag{4.3}$$

for $\varphi \in (\alpha_j, \alpha_{j+1})$ $j = 0, \ldots, n$. Here

$$D_j = -E \iint_A z^2 dxdz, \tag{4.4}$$

where A is domain of integration and depends on the configuration of the cross section of the arch.

It appears that the system of equilibrium equations (2.1) can be converted in a single third order equation. According to the first equation in (2.1) and (2.6)

$$N' = -\frac{M'}{R}. \tag{4.5}$$

Differentiating the last equality in (2.1) with respect to φ and taking $Q = -N'$ and (4.5) into account one obtains

$$M''' + M' = \bar{\mu}_j R^2 \ddot{W}' \tag{4.6}$$

for $\varphi \in (\alpha_j, \alpha_{j+1})$ $j = 0, \ldots, n$.

Substituting the bending moment from (4.3) into (4.6) one reaches to the equation

$$D_j(W^V + 2W''' + W') + \rho S_j R^4 \ddot{W}' = 0 \tag{4.7}$$

for $\varphi \in (\alpha_j, \alpha_{j+1})$ $j = 0, \ldots, n$; where $\bar{\mu}_j = \rho S_j$, ρ being the density of the material.

The equation (4.7) can be solved by the method of separation of variables (see section 2.). Substituting (2.11) in (4.7) results in the equation

$$D_j(w^V + 2w''' + w') - w' \cdot \rho S_j R^4 \omega^2 = 0. \tag{4.8}$$

The latter can be presented as

$$w^V + 2w''' + w'(1 - k_j^2) = 0, \tag{4.9}$$

where

$$k_j = \sqrt{\frac{\rho S_j R^4 \omega^2}{D_j}}. \tag{4.10}$$

The characteristic equation of (4.9) has the form

$$\lambda_j^5 + 2\lambda_j^3 + \lambda_j(1 - k_j^2) = 0. \tag{4.11}$$

The roots of (4.11) are

$$\begin{aligned} \lambda_{j1} &= 0, \\ \lambda_{j2,3} &= \pm\sqrt{1 - k_j}, \\ \lambda_{j4,5} &= \pm i\sqrt{1 + k_j}, \end{aligned} \tag{4.12}$$

where i is the imaginary unit.

The general solution of (4.9) is

$$C_{1j} \cosh(\mu_j \varphi) + C_{2j} \sinh(\mu_j \varphi) + C_{3j} \cos(\nu_j \varphi) + C_{4j} \sin(\nu_j \varphi) + C_{5j}. \tag{4.13}$$

Here $\mu_j = \sqrt{1 - k_j}, \quad \nu_j = \sqrt{1 + k_j}$.

The slope of the deflection is considered as a discontinuous quantity at the cross section with cracks.

In the case of hollow sectional beams instead of (2.22) one has

$$\sigma_j = \frac{6 M_j H_j}{B_j H_j^3 - b h^3}. \tag{4.14}$$

The differential equation (here $s_j = c_j / H_j$) (2.22) takes the form

$$\frac{dC_j}{ds_j} = \frac{72 H_j^4 B_j}{E'(B_j H_j^3 - b h^3)^2} s_j F^2(s_j). \tag{4.15}$$

The equation (4.15) with the initial condition $C_j(0) = 0$ has the solution

$$C_j = \frac{72 H_j^4 B_j}{E'(B_j H_j^3 - b h^3)^2} f(s_j) \tag{4.16}$$

in the case where $c_j \le \frac{1}{2}(H_j - h)$. Here the function $f(s_j)$ is defined by (3.13). Summarizing the results obtained above one can present the jump conditions for the slope of the deflection at $\varphi = \alpha_j$ as

$$\theta_j = -\frac{1}{R^2} C_j D_j(\alpha_j + 0)\Big(w(\alpha_j + 0) + w''(\alpha_j + 0)\Big), \tag{4.17}$$

where D_j and C_j are defined by (4.4) and (4.16), respectively. Note that (4.15), (4.16) apply for small cracks; in the case of larger cracks (4.15) and (4.16) must be modified suitably.

Some numerical results of calculation using the computer code MATLAB are presented in Fig. 18—Fig. 21.

The frequency of free vibrations is presented in Fig. 18 for various radii of the arch. The arch is made of a mild steel with $E = 2.1 \cdot 10^{11}\ Pa$.

The natural frequency versus depth of the crack can be seen in Fig.19. The dimensions of the arch are $R = 1\ m,\ h_0 = 0.02\ m,\ h_1 = 0.01\ m$; material mild steel.

Natural frequencies of arches made of various materials are depicted in Fig. 20. Here $R = 1\ m,\ h_0 = 0.02\ m,\ h_1 = 0.01\ m$. Characteristics of the materials are shown in Table 4.1 below.

Table 4.1. Constants of different materials

	$E, (Pa)$	$\rho, (kg/m^3)$	ν
Steel	$2.1 \cdot 10^{11}$	7865	0.3
Aluninum	$7 \cdot 10^{9}$	2700	0.33
Glass	$7 \cdot 10^{9}$	2600	0.25
Concrete	$1.7 \cdot 10^{9}$	1500	0.15
Polystirol	$1.5 \cdot 10^{9}$	30	0.1

It can be seen that curves corresponding to aluminum, glass and steel are quite close to each other.

Different curves in Fig.21 correspond to different values of the ratio of thicknesses $\gamma = h_1/h_0$.

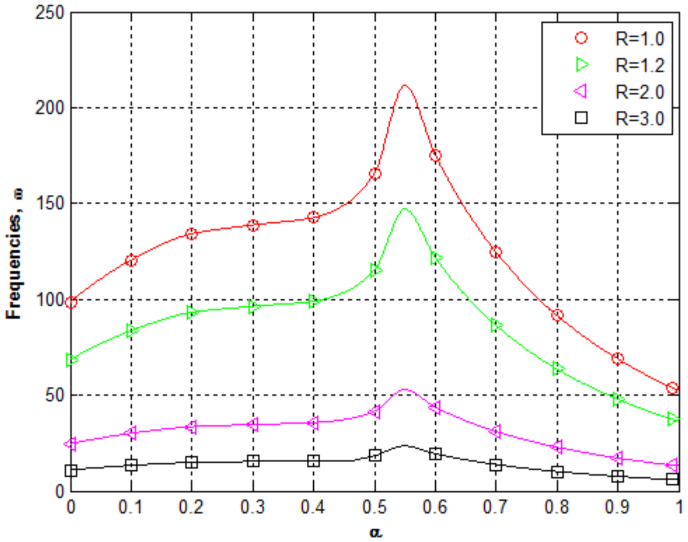

Figure 18: Natural frequency vs. radius of the arch

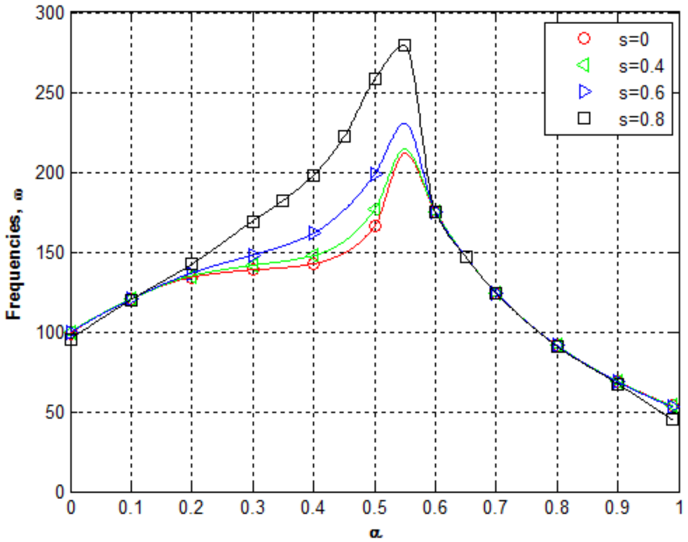

Figure 19: Natural frequency vs. depth of the crack

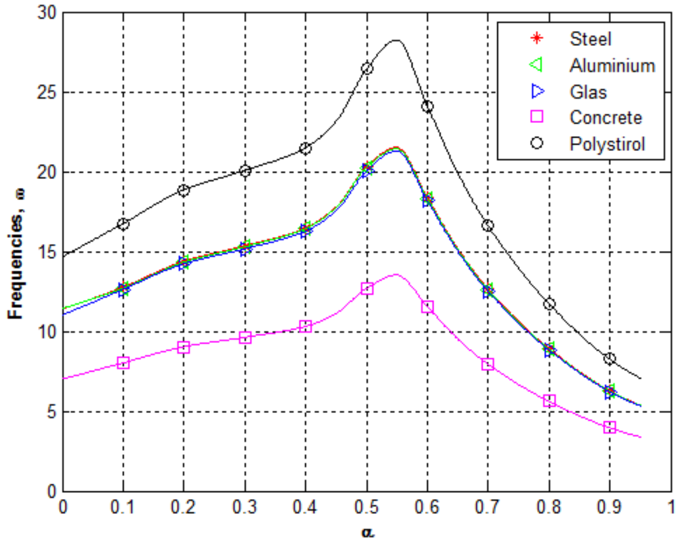

Figure 20: Natural frequency vs. material of the arch made

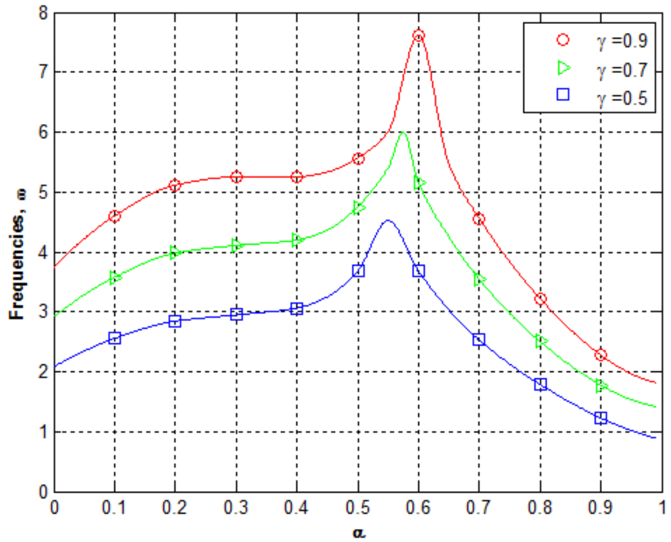

Figure 21: Natural frequency vs. ratio of the step $\gamma = \frac{h_1}{h_0}$

References

[1] R.D. Adams and D.G.C. Bacon. Measurement of the flexural damping capacity and dynamic Young's modulus of metals and reinforced plastics. *Journal of Physics D: Applied Physics*, 6: pp. 27–41, 1973.

[2] T.L. Anderson. *Fracture Mechanics: Fundamentals and Applications.* CRC Press, 2005.

[3] N. Anifantis and A. Dimarogonas. Stability of columns with a single crack subjected to follower and vertical loads. *International Journal of Solids and Structures,* 19(4): pp. 281–291, 1983.

[4] N.M. Auciello and M.A. DeRosa. Free vibrations of circular arches: a review. *Journal of Sound and Vibration.* 176(4): pp. 433–458, 1994.

[5] G. Bamnios and A. Trochides. Dynamic behaviour of cracked cantilever beam. *Applied Acoustics.* 45: pp. 97–112, 1995.

[6] G. Bao, S. Ho, Z. Suo, and B. Fan. The role of material orthotrophy in fracture specimens for composites. *International Journal of Solids and Structures,* 29(9): pp. 1105–1116, 1992.

[7] K.B. Broberg. *Cracks and Fracture.* Academic Press, New York, 1999.

[8] D. Broek. *The Practical Use of Fracture Mechanics.* Kluwer Academic Publishers, 1989.

[9] T.Q. Bui, M.N. Nguyen and C. Zhang. An efficient meshfree method for vibration analysis of laminated composite plates. *Computational Mechanics,* 48(2): pp. 175–193, 2011.

[10] S. Caddemi and I. Calio. The influence of the axial force on the vibration of the Euler-Bernoulli beam with an arbitrary number of cracks. *Archive of Applied Mechanics.* 82(6): pp. 827–839, 2012.

[11] I. Calio, A. Greco and D. D'Urso. Free vibrations of spatial Timoshenko arches. *Journal of Sound and Vibration.* 333: pp. 4543-4561, 2014.

[12] M.N. Cerri, M. Dilena and G.C. Ruta. Vibration and damage detection in undamaged and cracked circular arches: experimental and analytical results. *Journal of Sound and Vibration.* 314: pp. 83–94, 2008.

[13] M.N. Cerri and G.C. Ruta. Detection of localized damage in plane circular arches by frequency data. *Journal of Sound and Vibration.* 270: pp. 39–59, 2004.

[14] Y.K. Cheung, F.T.K. Au and D.Y. Zheng. Finite strip method for the free vibration and buckling analysis of plates with abrupt changes in thickness and complex support conditions. *Thin-Walled Structures,* 36(2): pp. 89–110, 2000.

[15] P. Chidaparam and A.W. Leissa. Influence of centerline extensibility on the in-plane free vibrations of loaded circular arches. *Journal of Sound and Vibration.* 183(5): pp. 779–795, 1995.

[16] T.G. Chondros, A.D. Dimarogonas and J. Yao. A continuous cracked beam vibration theory. Journal of Sound and Vibration, 215(1): pp. 17–34, 1998.

[17] T.G. Chondros, A.D. Dimarogonas and J. Yao. Vibration of a beam with a breathing crack. *Journal of Sound and Vibration*, 239(1): pp. 57–67, 2001.

[18] G.R. Cowper. The shear coefficient in Timoshenko's beam theory. *Journal of Applied Mechanics*, 33: pp. 335–340, 1966.

[19] I.M. Daniel and O. Ishai. *Engineering Mechanics of Composite Materials.* Oxford University Press, 2006.

[20] M.A. DeRosa. Free vibration of stepped beams with flexible ends, in the presence of follower forces at the step. *Journal of Sound and Vibration,* 194(4): pp. 631–635, 1996.

[21] A.D. Dimarogonas. Vibration of cracked structures: A state of the art review. *Engineering Fracture Mechanics,* 55(5): pp. 831–857, 1996.

[22] A.D. Dimarogonas and S.A Paipetis. *Analytical Methods In Rotor Dynamics.* Elsevier, London, 1983.

[23] A.D. Dimarogonas. *Vibration Engineering.* St.Paul, West Publishers, 1976.

[24] U. Eroglu. In-plane free vibrations of circular beams made of functionally graded material in thermal environment: Beam theory approach. *Composite Structures*. 122: pp. 217–228, 2015.

[25] Z. Gürdal, R.T. Haftka and P. Hajela. *Design and Optimization of Laminated Composite Materials*. Wiley, New York, 1999.

[26] J. Henrych. *Dynamics of Arches and Frames*. Elsevier, Amsterdam, 1981.

[27] C.T. Herakovich. *Mechanics of Fibrous Composites*. Wiley, New York, 1998.

[28] T.C. Huang. The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions. *Journal of Applied Mechanics*, 28: pp. 579–584, 1961.

[29] M. Ishaguddin, P. Raveendranath and J.N. Reddy. Efficient coupled polynomial scheme for out-of-plane free vibration analysis of curved beams. *Finite Element Analysis and Design*. 110: pp. 58–66., 2016.

[30] R.M. Jones. *Mechanics of Composite Materials*. Taylor and Francis, Philadelphia, 1999.

[31] R. Kacianauskas, A. Ramirez-Gomez and U. Radvilaite. Semi-analytical model for non-spherical particles. *AIP Conf. Proc.*. Proc. ICNAAM-2014. 1648, UNSP400008, 2015.

[32] C. Karaagac, H. Ozturk and M. Sabuncu. Crack effects on the in-plane static and dynamic stabilities of a curved beam with edge crack. *Journal of Sound and Vibration*. 330: pp. 1718–1736, 2011.

[33] M. Kawakami, T. Sakiyama, H. Matsuda and C. Morita. In-plane and out-of-plane free vibrations of curved beams with variable sections. *Journal of Sound and Vibration*. 187: pp. 381–401, 1995.

[34] M. Kisa, J. Brandon and M. Topcu. Free vibration analysis of cracked beams by a combination of finite elements and component mode synthesis methods. *Computers and Structures*. 67: pp. 215–223, 1998.

[35] M. Kisa and J. Brandon. The effect of closure cracks on the dynamics of a cracked cantilever beam. *Journal of Sound and Vibration*. 238(1): pp. 1–18, 2000.

[36] L. Kollar and G.S. Springer. *Mechanics of Composite Structures*. Cambridge University Press, Cambridge, 2003.

[37] S. Kukla. Free viibration and stability of stepped columns with cracks. *Journal of Sound and Vibration*. 319: pp. 1301–1311, 2009.

[38] E. Kägo and J. Lellep. Vibrations of elastic stretched trips with cracks. *Proceedings of the 2d International Conference. Optimization and Analysis of Structures*. J.Lellep, E.Puman (Ed-s), Tartu Univ. Press, Tartu, pp. 59–63, 2013.

[39] K. Lasn, A. Klauson, F. Chati and D. Decultot. Experimental determination of elastic constants of an orthotropic composite plate by using lamb waves. *Mechanics of Composite Materials*. 47(4): pp. 435–446, 2011.

[40] B.K. Lee, K.K. Park, S.J. Oh and T.E. Lee. Planar free vibrations of horseshoe elliptic arch. *KSCE Journal of Civil Engineering*. 20(4): pp. 1411–1419, 2016.J. Lellep and E. Kägo. Vibrations of elastic stretched strips with cracks, *International Journal of Mechanics*. 1(5): pp. 27–34, 2011.

[41] S-Y Lee and Q-Z Yan. Exact static analysis of in-plane curved Timoshenko beams with strong nonlinear boundary conditions. *Mathematical Problems in Engineering*. ID64391, 2015.

[42] S-Y Lee aand S. M Lin. Dynamic analysis of nonuniform beams with time-dependent elastic boundary conditions. *Journal of Applied Mechanics*. 63(2): pp. 474–478, 1996.

[43] J. Lellep and E. Kägo. Vibrations of elastic stretched strips with cracks. *International Journal of Mechanics*. 5(1): pp. 27–34, 2011.

[44] J. Lellep and A. Liyvapuu. Natural vibrations of elastic arches with cracks. *Proceedings of the 2nd International Conference. Optimization and Analysis of Structures*. J.Lellep, E.Puman (Ed-s), Tartu Univ. Press, Tartu, pp. 64–69, 2013.

[45] J. Lellep and A. Liyvapuu. Free vibrations of elastic laminated arches. *Proceedings of the 3nd International Conference. Optimization and Analysis of Structures.* J.Lellep, E.Puman (Ed-s), Tartu Univ. Press, Tartu, pp. 52–58, 2015.

[46] J. Lellep and A. Liyvapuu. Natural vibrations of stepped arches. *Proceedings of the 3nd International Conference on Advances in Mechanical and Automation Engineering* — MAE, Rome, Chief Editor: Dr.Rakesh Kumar, IRED: pp. 68–72, 2015.

[47] J. Lellep and A. Liyvapuu. Natural vibrations of stepped arches with cracks. *Agronomy Research.* 14(3): pp. 821–830, 2016.

[48] J. Lellep and A. Liyvapuu. Free vibrations of uniform and hollow-sectional elastic arches, *Modern Building Materials, Structures and Techniques.* Procedia Engineering Elsevier, 2016, (accepted).

[49] J. Lellep and L. Roots. Vibrations of cylindrical shells with circumferential cracks. *WSEAS Transactions on Mathematics,* 9(9): pp. 689–699, 2010.

[50] J. Lellep and L. Roots. Axisymmetric vibrations of orthotropic circular cylindrical shells with cracks. Part I.*Mechanics of Composite Materials,* 49(1): pp. 5968, 2013.

[51] R.Y. Liang, F.K. Choy and J. Hu. Detection of cracks in beam structures using measurements of natural frequencies. *Journal of Franklin Institute.* 328(4): pp. 505–518, 1991.

[52] R.Y. Liang, J. Hu and F.K. Choy. Theoretical study of crack-induced eigenfrequency changes on beam structures. *Journal of Mechanical Engineering.* 118(2): pp. 384–395, 1992.

[53] S. Markus and T. Nanosi. Vibrations of curved beams. *Shock and Vibration Digest.* 13(4): pp. 3–14, 1981.

[54] H. Matsunaga. In plane vibration and stability of shallow arches subjected to axial forces. *International Journal of Solids and Structures.* 33(4): pp. 469–482, 1996.

[55] R.D. Mindlin. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *Journal of Applied Mechanics,* 18: pp. 31–38, 1951.

[56] A.V.K. Murty. Vibrations of short beams. *AIAA Journal*, 8: pp. 34–38, 1970.

[57] H. Nahvi and M. Jabbari. Crack detection in beams using experimental modal data and finite element model, *International Journal of Mechanical Sciences.* 10(47): pp. 1477–1495, 2005.

[58] B.P. Nandwana and S.K. Maiti. Detection of the location and size of crack in stepped cantilever beams based on measurements of natural frequencies. *Journal of Sound and Vibration.* 203(3): pp. 435–446, 1997.

[59] W.Ogierman and G. Kokot. A study on fiber orientation influence on the mechanical response of a short fiber composite structure. *Acta Mechanica.*227(1): pp. 173–183, 2016.

[60] W. Ostachowicz and M. Krawczuk. Analysis of the effect of cracks on the natural frequencie of a cantilever beam. *Journal of Sound and Vibration.* 150(2): pp. 191–201, 1991.

[61] M. Petyt and C.C. Fleischer. Free vibration of a curved beam. *Journal of Sound and Vibration.* 18(1): pp. 17–30, 1971.

[62] M.S. Qatu. *Vibrations of Laminated Shells and Plates.* Academic Press, New York, 2004.

[63] M. Ratassepp, A. Klauson, F. Chati, F. Leon, D. Decultot, G. Maze and M. Fritzsche. Application of orthogonality-relation for the separation of Lamb modes at a plate edge: Numerical and experimental predictions. *Ultrasonics.* 57: pp. 90–95, 2015.

[64] S.S. Rao. Effects of transverse shear and rotatory inertia on the coupled twist-bending vibrations of circular rings. *Journal of Sound and Vibration.* 16(4): pp. 551–566, 1971.

[65] J.N. Reddy. *Mechanics of Laminated Composite Plates and Shells. Theory and Analysis.* CRC Press, 2004.

[66] P.F. Rizos, N. Aspragathos and A.D. Dimarogonas. Identification of crack location and magnitudein a cantilever beam from the vibration modes. *Journal of Sound and Vibration*. 138(3): pp. 381–388, 1990.

[67] E. Sadeghpour, M. Sadighi and A. Ohadi. Free vibration analysis of a debonded curved sandwich beam. *European Journal of Mechanics, 1 (Solids)*. 57: pp. 71–84, 2016.

[68] I.H. Shames and C.L. Dym. *Energy and Finite Element Methods in Structural Mechanics*. McGraw-Hill, New York, 1985.

[69] W. Soedel. *Vibration of Shells and Plates*. Marcel Dekker, 2004

[70] S.P. Timoshenko. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Philosophical Magazine and Journal of Science*, 41: pp. 744–747, 1921.

[71] S.P. Timoshenko. On the transverse vibrations of bars of uniform cross-section, *Philosophical Magazine and Journal of Science,* 43: pp. 125–131, 1922.

[72] S.P. Timoshenko, D.H. Young and W. Weaver Jr. *Vibration Problems in Engineering*. 4th ed. Wiley, New-York, 1974.

[73] M.E.Tuttle. *Structural Analysis of Polyneric Composite Materials*. Marcel Dekker, New-York, 2004.

[74] J.R. Vinson and R.L. Sierakowski. *The Behavior of Structures Composed of Composite Materials*. Kluwer, 2002.

[75] E. Viola and F. Tornabene. Vibration analysis of damaged circular arches with varying cross-sectionn. *Structural Integrity and Durability.* 1(2): pp. 155–169, 2005.

[76] E. Viola, E. Artioli and M. Dilena. Analytical and differential quadrature results for vibration analysis of damaged circular arches. *Journal of Sound and Vibration*. 288: pp. 887–906, 2005.

[77] E. Viola, M. Miniaci, N. Fantuzzi and Alessandro Marzani. Vibration analysis of multi-stepped and multi-damaged parabolic arches using GDQ. *Curved and Layered Structures*. 2: pp. 28–49, 2015.

[78] J.S. Wu and L.K. Chiang. A new approach for free vibration analysis of arches with effect of shear deformation and rotary inertia considered. *Journal of Sound and Vibration*. 277: pp. 49–71, 2004.

[79] J.S. Wu and L.K. Chiang. Free vibration analysis of arches using curved beam elements. *International Journal for Numeric Methods in Engineering*. 58: pp. 1907–1936, 2003.

[80] J.S. Wu and L.K. Chiang. Free vibration of a circularly curved Timoshenko beam normal to its initial plain usin finite curved beam elements. *Computers and Structures*. 82: pp. 2525–2540, 2004.

[81] Y. Xiang and C.M. Wang. Exact buckling and vibration solutions for stepped rectangular plates. *Journal of Sound and Vibration*. 250: pp. 503–517, 2002.

[82] Y. Xu, X. Gu, B. Zhao and R. Zhou. In-plane elastic stability of arches under a radial concentrated load. *Engineering*. 6: pp.572–583, 2014.

[83] F. Yang R. Sedaghati and E. Esmailzadeh. Free in-plane vibration of general curved beams using finite element method. *Journal of Sound and Vibration*. 318: pp. 850–867, 2008.

[84] J. Yang and Y. Chen. Free vibrations and buckling analyses of functionally graded beams with edge cracks. *Composite Structures*. 83: pp. 48–60, 2008.

[85] D.Y. Zheng and S.C. Fan. Vibration and stability of cracked hollow-sectional beams. *Journal of Sound and Vibration*. 267:, pp. 933–954, 2003.

[86] D.Y. Zheng and N.J. Kessissoglou. Free vibration analysis of a cracked beam by finite element method. *Journal of Sound and Vibration*. 273(3): pp. 457–475, 2004.

Summary

In the present dissertation, the natural frequencies of free vibrations of circular elastic arches are studied. The circular elastic arches, in which each segment has a piece-wise constant thickness, are under consideration. It is assumed that stable crack-like surface defects occur at the re-entrant corners of the steps. The propagation of the cracks is neglected. The cracks are considered to be stationary surface cracks which have not fully penetrated the arch thickness. Combining the methods of the theory of elastic plates and shells and the theory of linear elastic fracture mechanics, a new method for determining the natural frequencies of elastic structures is developed in the thesis. The dissertation is based on the five articles of the author (four of which have been published). The dissertation consists of the review of the obtained results, the copies of the papers, the list of literature and the CV of the author. The review starts with the consideration of the background of the study of free vibrations, then overview of the used literature is provided and the aim of the thesis is presented (these parts form the Introduction), which is followed by the three main sections. In the second section of the dissertation, the method of finding the natural frequencies of arches with a constant thickness is developed. Two problems are discussed: the first one regards the arches made of homogeneous materials and containing surface cracks; the second case covers laminated arches without cracks. In the both cases, simply supported and clamped arches are studied. Because of the small amplitude of the oscillations in the case of free vibration, the material of the arches is assumed to be a purely elastic material and the hypothesis of Kirchhoff is considered applicable. A refined version of the classical bending theory is employed. The influence of cracks on the vibrational characteristics is taken into account with the use of the model of distributed line springs. The latter employs the stress intensity coefficient known in the elastic fracture mechanics. In the third section, stepped arches without and with cracks are studied. In the fourth section of the dissertation, the developed method is used for determination of the natural frequency of free vibration in case of hollow elastic arches. The stepped arches under consideration are assumed to be clamped at both ends. The influence of the geometrical parameters and material properties on the vibration of the arches is analysed.

The method developed in the dissertation can be used in the non-destructive testing of structures.

Kokkuvõte

Pragudega elastsete astmeliste kaarte omavõnkumised

Käesolevas väitekirjas vaadeldakse elastsete astmeliste kaarte omavõnkumisi. Vaatluse all olevad kaared on tükiti konstantse paksusega ja konstantse laiusega. Kaarte tükiti konstantse paksuse muutumise kohtades asuvad praod. Eeldatakse, et praod on stabiilsed ja konstantse pikkusega. Kasutades elastsete plaatide ja koorikute teooria ning lineaarse purunemismehaanika meetodeid on välja töötatud analüütilis-numbriline meetod kaarte vabavõnkumiste sageduste määramiseks. Antud dissertatsioon põhineb autori viiel teaduslikul artiklil, millest neli on avaldatud ja üks ametlikult vastu võetud. Väitekiri koosneb kokkuvõtvast ülevaateartiklist, publitseeritud ja trükki suunatud artiklite koopiatest ja kokkuvõttest. Lisatud on autori elulookirjeldus.

Väitekirja sissejuhatavas osas on esitatud kirjanduse ülevaade ning on kirjeldatud väitekirja eesmärke ja struktuuri. Töös uuritakse kaarte vabavõnkumisi konstantse ja tükiti konstantse paksuse korral. Esiteks vaadeldakse pinnalt lähtuva praoga homogeensest materjalist kaart, seejärel uuritakse lamineeritud elastse kaare vabavõnkumisi nii vabalt toetatud kui ka jäigalt kinnitatud kaare korral. Eeldatakse, et kehtivad Kirchhoffi hüpoteesid. Prao mõju vabavõnkumiste sagedusele modelleeritakse jaotatud lineaarse vedru mudeli abil. Vastavalt sellele on prao mõju kaare omavõnkumisele seotud lokaalse järeleandlikkuse koefitsiendi ning pinge intensiivsuse koefitsiendiga, mis arvutatakse purunemismehaanika meetoditega. Uuritakse ka astmeliste kaarte vabavõnkumisi. Vaadeldud on nii pragudega kaari kui ka ilma defektideta kaari. Dissertatsiooni viimane osa on pühendatud jäigalt kinnitatud õõnsustega kaarte vabavõnkumiste uurimisele. Kõikidel juhtudel on omavõnkumiste uurimiseks konstrueeritud analüütilis-numbrilised meetodid, mis põhinevad klassikalise plaatide ja koorikute teooria ning purunemismehaanika võrranditel ja kriteeriumitel.

Välja töötatud meetod võib leida kasutamist konstruktsioonide mittepurustaval katsetamisel.

Acknowledgements

I wish to express my deepest gratitude to my supervisor Prof. Jaan Lellep for all the ideas, advice and continuous support he has generously provided throughout my doctoral study. I especially appreciate the personal wisdom and composure he showed in all these years of my studying.

My whole-hearted thanks go to my colleagues at the University of Tartu and the Estonian University of Life Sciences. Without their support and participation this work would not be possible.

I want to thank all the people, who have assisted me in this important period of my life the doctoral study. They have been many, and each of them has contributed to my professional formation.

Publications

Curriculum vitae

Name: Alexander Liyvapuu
Date of birth: 02.12.1971
Citizenship: Russian
E-mail: liyvapuu@ut.ee

Education:

1987–1991	Technical Colledge of Road-Transport, Habarovsk Technician-mechanic 1991
1998–2003	University of Tartu, bachelor studies in mathematics B.Sc. in mathematics 2003
2007–2009	University of Tartu, MBA studies Entrepreneurship and technology management
2011–	University of Tartu, doctoral studies in mathematics

Employment:

02/2012 –	Estonian University of Life Sciences, Institute of Technology, lecturer of theoretical mechanics
2007 –2008	Tarkon AS, Tartu Technologist of CNC-lathe turning
2005 –2007	AT Metalli Grupp OÜ, Tartu region Production manager – technologist
2003 –2005	Estre AS, Tartu Production engineer

Member of EstORS – Estonian Operational Research Society.

Languages: Russian, Estonian, German, English

List of publications

1 J.Lellep, A.Liyvapuu, Natural vibrations of elastic arches with cracks. *In: Proceedings of the 2nd International Conference Optimization and Analysis of Structures.* Editors: J.Lellep, E.Puman, University of Tartu Press, pp. 64–69, 2013.

2 J.Lellep, A.Liyvapuu, Free vibrations of elastic laminated arches. *In: Proceedings of the 3nd International Conference Optimization and Analysis of Structures.* Editors: J.Lellep, E.Puman, University of Tartu Press, pp. 52–58, 2015.

3 A.Liyvapuu, J.Lellep, Natural vibrations of stepped arches. *International Journal of Advancements in Mechanical and Aeronautical Engineering* 3(1), pp. 75–79, 2016.

4 J.Lellep, A.Liyvapuu, Natural vibrations of stepped arches with cracks. *Agronomy Research* 14(3), pp. 821–830, 2016.

5 J.Lellep, A.Liyvapuu, Free vibrations of uniform and hollow-sectional elastic arches, *Elsevier, Procedia Engineering, Modern Building Materials, Structures and Techniques*, 2016, (accepted).

6 J.Lellep, A.Liyvapuu, Natural vibrations of stepped arches. *In: Proceedings of the 3nd International Conference on Advances in Mechanical and Automation Engineering* — MAE, Rome, Chief Editor: Dr.Rakesh Kumar, IRED, pp. 68–72, 2015.

7 J.Olt, V.Maksarov, A.Liyvapuu, M.Madissoo. Dynamical Simulation of Chip Formation in the Process of Cutting. *International Journal of Materials and Product Technology*, 53(1), pp.1–14, 2016.

8 J.Olt, O.Liivapuu, V.Maksarov, A.Liyvapuu, T.Tärgla. Mathematical Modelling of Cutting Process System. *Springer Proceedings in Physics. Engineering Mathematics: Applications and Methods of Algebra, Analysis and Probability*, pp. 73–86, 2016.

9 V.Adamchuk, I.Petrychenko, J.Olt, A.Liyvapuu. Study of plane-parallel motion of machine combination. *In: Proceedings of the 3nd International Conference Optimization and Analysis of Structures.* Editors: J.Lellep, E.Puman, University of Tartu Press, pp.10–16, 2015.

Elulookirjeldus

Nimi: Alexander Liyvapuu
Sünniaeg: 02.12.1971
Kodakondsus: vene
E-mail: liyvapuu@ut.ee

Haridus:

1987–1991	Habarovski Autotehnikum, tehnik–mehaanik 1991
1998–2003	Tartu Ülikool, matemaatika bakalaureuseõpe, BSc matemaatika erialal 2003
2007–2009	Tartu Ülikool, rakendusmagistriõpe Ettevõtluse ja tehnoloogia juhtimise erialal
2011–	Tartu Ülikool, matemaatika doktoriõpe

Teenistuskäik:

02/2012 –	Eesti Maaülikool, Tehnikainstituut, teoreetilise mehaanika lektor
2006–2008	AS Tarkon, CNC-treimise jaoskonna tehnoloog
2005–2006	AT Metalli Grupp OÜ, tootmisjuht–tehnoloog
2003–2005	AS Estre, tehnikaosakonna insener

EstORS – Eesti Operatsioonianalüüsi Seltsi liige.

Keeleoskus: vene, eesti, saksa, inglise

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

1. **Mati Heinloo.** The design of nonhomogeneous spherical vessels, cylindrical tubes and circular discs. Tartu, 1991, 23 p.
2. **Boris Komrakov.** Primitive actions and the Sophus Lie problem. Tartu, 1991, 14 p.
3. **Jaak Heinloo.** Phenomenological (continuum) theory of turbulence. Tartu, 1992, 47 p.
4. **Ants Tauts.** Infinite formulae in intuitionistic logic of higher order. Tartu, 1992, 15 p.
5. **Tarmo Soomere.** Kinetic theory of Rossby waves. Tartu, 1992, 32 p.
6. **Jüri Majak.** Optimization of plastic axisymmetric plates and shells in the case of Von Mises yield condition. Tartu, 1992, 32 p.
7. **Ants Aasma.** Matrix transformations of summability and absolute summability fields of matrix methods. Tartu, 1993, 32 p.
8. **Helle Hein.** Optimization of plastic axisymmetric plates and shells with piece-wise constant thickness. Tartu, 1993, 28 p.
9. **Toomas Kiho.** Study of optimality of iterated Lavrentiev method and its generalizations. Tartu, 1994, 23 p.
10. **Arne Kokk.** Joint spectral theory and extension of non-trivial multiplicative linear functionals. Tartu, 1995, 165 p.
11. **Toomas Lepikult.** Automated calculation of dynamically loaded rigid-plastic structures. Tartu, 1995, 93 p, (in Russian).
12. **Sander Hannus.** Parametrical optimization of the plastic cylindrical shells by taking into account geometrical and physical nonlinearities. Tartu, 1995, 74 p, (in Russian).
13. **Sergei Tupailo.** Hilbert's epsilon-symbol in predicative subsystems of analysis. Tartu, 1996, 134 p.
14. **Enno Saks.** Analysis and optimization of elastic-plastic shafts in torsion. Tartu, 1996, 96 p.
15. **Valdis Laan.** Pullbacks and flatness properties of acts. Tartu, 1999, 90 p.
16. **Märt Põldvere.** Subspaces of Banach spaces having Phelps' uniqueness property. Tartu, 1999, 74 p.
17. **Jelena Ausekle.** Compactness of operators in Lorentz and Orlicz sequence spaces. Tartu, 1999, 72 p.
18. **Krista Fischer.** Structural mean models for analyzing the effect of compliance in clinical trials. Tartu, 1999, 124 p.
19. **Helger Lipmaa.** Secure and efficient time-stamping systems. Tartu, 1999, 56 p.

20. **Jüri Lember.** Consistency of empirical k-centres. Tartu, 1999, 148 p.
21. **Ella Puman.** Optimization of plastic conical shells. Tartu, 2000, 102 p.
22. **Kaili Müürisep.** Eesti keele arvutigrammatika: süntaks. Tartu, 2000, 107 lk.
23. **Varmo Vene.** Categorical programming with inductive and coinductive types. Tartu, 2000, 116 p.
24. **Olga Sokratova.** Ω-rings, their flat and projective acts with some applications. Tartu, 2000, 120 p.
25. **Maria Zeltser.** Investigation of double sequence spaces by soft and hard analitical methods. Tartu, 2001, 154 p.
26. **Ernst Tungel.** Optimization of plastic spherical shells. Tartu, 2001, 90 p.
27. **Tiina Puolakainen.** Eesti keele arvutigrammatika: morfoloogiline ühestamine. Tartu, 2001, 138 p.
28. **Rainis Haller.** *M(r,s)*-inequalities. Tartu, 2002, 78 p.
29. **Jan Villemson.** Size-efficient interval time stamps. Tartu, 2002, 82 p.
30. **Eno Tõnisson.** Solving of expession manipulation exercises in computer algebra systems. Tartu, 2002, 92 p.
31. **Mart Abel.** Structure of Gelfand-Mazur algebras. Tartu, 2003. 94 p.
32. **Vladimir Kuchmei.** Affine completeness of some ockham algebras. Tartu, 2003. 100 p.
33. **Olga Dunajeva.** Asymptotic matrix methods in statistical inference problems. Tartu 2003. 78 p.
34. **Mare Tarang.** Stability of the spline collocation method for volterra integro-differential equations. Tartu 2004. 90 p.
35. **Tatjana Nahtman.** Permutation invariance and reparameterizations in linear models. Tartu 2004. 91 p.
36. **Märt Möls.** Linear mixed models with equivalent predictors. Tartu 2004. 70 p.
37. **Kristiina Hakk.** Approximation methods for weakly singular integral equations with discontinuous coefficients. Tartu 2004, 137 p.
38. **Meelis Käärik.** Fitting sets to probability distributions. Tartu 2005, 90 p.
39. **Inga Parts.** Piecewise polynomial collocation methods for solving weakly singular integro-differential equations. Tartu 2005, 140 p.
40. **Natalia Saealle.** Convergence and summability with speed of functional series. Tartu 2005, 91 p.
41. **Tanel Kaart.** The reliability of linear mixed models in genetic studies. Tartu 2006, 124 p.
42. **Kadre Torn.** Shear and bending response of inelastic structures to dynamic load. Tartu 2006, 142 p.
43. **Kristel Mikkor.** Uniform factorisation for compact subsets of Banach spaces of operators. Tartu 2006, 72 p.

44. **Darja Saveljeva.** Quadratic and cubic spline collocation for Volterra integral equations. Tartu 2006, 117 p.
45. **Kristo Heero.** Path planning and learning strategies for mobile robots in dynamic partially unknown environments. Tartu 2006, 123 p.
46. **Annely Mürk.** Optimization of inelastic plates with cracks. Tartu 2006. 137 p.
47. **Annemai Raidjõe.** Sequence spaces defined by modulus functions and superposition operators. Tartu 2006, 97 p.
48. **Olga Panova.** Real Gelfand-Mazur algebras. Tartu 2006, 82 p.
49. **Härmel Nestra.** Iteratively defined transfinite trace semantics and program slicing with respect to them. Tartu 2006, 116 p.
50. **Margus Pihlak.** Approximation of multivariate distribution functions. Tartu 2007, 82 p.
51. **Ene Käärik.** Handling dropouts in repeated measurements using copulas. Tartu 2007, 99 p.
52. **Artur Sepp.** Affine models in mathematical finance: an analytical approach. Tartu 2007, 147 p.
53. **Marina Issakova.** Solving of linear equations, linear inequalities and systems of linear equations in interactive learning environment. Tartu 2007, 170 p.
54. **Kaja Sõstra.** Restriction estimator for domains. Tartu 2007, 104 p.
55. **Kaarel Kaljurand.** Attempto controlled English as a Semantic Web language. Tartu 2007, 162 p.
56. **Mart Anton.** Mechanical modeling of IPMC actuators at large deformations. Tartu 2008, 123 p.
57. **Evely Leetma.** Solution of smoothing problems with obstacles. Tartu 2009, 81 p.
58. **Ants Kaasik.** Estimating ruin probabilities in the Cramér-Lundberg model with heavy-tailed claims. Tartu 2009, 139 p.
59. **Reimo Palm.** Numerical Comparison of Regularization Algorithms for Solving Ill-Posed Problems. Tartu 2010, 105 p.
60. **Indrek Zolk.** The commuting bounded approximation property of Banach spaces. Tartu 2010, 107 p.
61. **Jüri Reimand.** Functional analysis of gene lists, networks and regulatory systems. Tartu 2010, 153 p.
62. **Ahti Peder.** Superpositional Graphs and Finding the Description of Structure by Counting Method. Tartu 2010, 87 p.
63. **Marek Kolk.** Piecewise Polynomial Collocation for Volterra Integral Equations with Singularities. Tartu 2010, 134 p.
64. **Vesal Vojdani.** Static Data Race Analysis of Heap-Manipulating C Programs. Tartu 2010, 137 p.
65. **Larissa Roots.** Free vibrations of stepped cylindrical shells containing cracks. Tartu 2010, 94 p.

66. **Mark Fišel.** Optimizing Statistical Machine Translation via Input Modification. Tartu 2011, 104 p.
67. **Margus Niitsoo**. Black-box Oracle Separation Techniques with Applications in Time-stamping. Tartu 2011, 174 p.
68. **Olga Liivapuu.** Graded q-differential algebras and algebraic models in noncommutative geometry. Tartu 2011, 112 p.
69. **Aleksei Lissitsin.** Convex approximation properties of Banach spaces. Tartu 2011, 107 p.
70. **Lauri Tart.** Morita equivalence of partially ordered semigroups. Tartu 2011, 101 p.
71. **Siim Karus.** Maintainability of XML Transformations. Tartu 2011, 142 p.
72. **Margus Treumuth.** A Framework for Asynchronous Dialogue Systems: Concepts, Issues and Design Aspects. Tartu 2011, 95 p.
73. **Dmitri Lepp.** Solving simplification problems in the domain of exponents, monomials and polynomials in interactive learning environment T-algebra. Tartu 2011, 202 p.
74. **Meelis Kull.** Statistical enrichment analysis in algorithms for studying gene regulation. Tartu 2011, 151 p.
75. **Nadežda Bazunova.** Differential calculus $d^3 = 0$ on binary and ternary associative algebras. Tartu 2011, 99 p.
76. **Natalja Lepik.** Estimation of domains under restrictions built upon generalized regression and synthetic estimators. Tartu 2011, 133 p.
77. **Bingsheng Zhang.** Efficient cryptographic protocols for secure and private remote databases. Tartu 2011, 206 p.
78. **Reina Uba.** Merging business process models. Tartu 2011, 166 p.
79. **Uuno Puus.** Structural performance as a success factor in software development projects – Estonian experience. Tartu 2012, 106 p.
80. **Marje Johanson.** $M(r, s)$-ideals of compact operators. Tartu 2012, 103 p.
81. **Georg Singer.** Web search engines and complex information needs. Tartu 2012, 218 p.
82. **Vitali Retšnoi.** Vector fields and Lie group representations. Tartu 2012, 108 p.
83. **Dan Bogdanov.** Sharemind: programmable secure computations with practical applications. Tartu 2013, 191 p.
84. **Jevgeni Kabanov.** Towards a more productive Java EE ecosystem. Tartu 2013, 151 p.
85. **Erge Ideon**. Rational spline collocation for boundary value problems. Tartu, 2013, 111 p.
86. **Esta Kägo**. Natural vibrations of elastic stepped plates with cracks. Tartu, 2013, 114 p.
87. **Margus Freudenthal.** Simpl: A toolkit for Domain-Specific Language development in enterprise information systems. Tartu, 2013, 151 p.
88. **Boriss Vlassov.** Optimization of stepped plates in the case of smooth yield surfaces. Tartu, 2013, 104 p.

89. **Elina Safiulina.** Parallel and semiparallel space-like submanifolds of low dimension in pseudo-Euclidean space. Tartu, 2013, 85 p.
90. **Raivo Kolde.** Methods for re-using public gene expression data. Tartu, 2014, 121 p.
91. **Vladimir Šor.** Statistical Approach for Memory Leak Detection in Java Applications. Tartu, 2014, 155 p.
92. **Naved Ahmed.** Deriving Security Requirements from Business Process Models. Tartu, 2014, 171 p.
93. **Kerli Orav-Puurand.** Central Part Interpolation Schemes for Weakly Singular Integral Equations. Tartu, 2014, 109 p.
94. **Liina Kamm.** Privacy-preserving statistical analysis using secure multi-party computation. Tartu, 2015, 201 p.
95. **Kaido Lätt.** Singular fractional differential equations and cordial Volterra integral operators. Tartu, 2015, 93 p.
96. **Oleg Košik.** Categorical equivalence in algebra. Tartu, 2015, 84 p.
97. **Kati Ain.** Compactness and null sequences defined by ℓ_p spaces. Tartu, 2015, 90 p.
98. **Helle Hallik.** Rational spline histopolation. Tartu, 2015, 100 p.
99. **Johann Langemets.** Geometrical structure in diameter 2 Banach spaces. Tartu, 2015, 132 p.
100. **Abel Armas Cervantes.** Diagnosing Behavioral Differences between Business Process Models. Tartu, 2015, 193 p.
101. **Fredrik Milani.** On Sub-Processes, Process Variation and their Interplay: An Integrated Divide-and-Conquer Method for Modeling Business Processes with Variation. Tartu, 2015, 164 p.
102. **Huber Raul Flores Macario.** Service-Oriented and Evidence-aware Mobile Cloud Computing. Tartu, 2015, 163 p.
103. **Tauno Metsalu.** Statistical analysis of multivariate data in bioinformatics. Tartu, 2016, 197 p.
104. **Riivo Talviste.** Applying Secure Multi-party Computation in Practice. Tartu, 2016, 144 p.
105. **Md Raknuzzaman.** Noncommutative Galois Extension Approach to Ternary Grassmann Algebra and Graded q-Differential Algebra. Tartu, 2016, 110 p.