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HINA ARIF

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Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Estonia.

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Errata in The Dissertation

Stability analysis of stepped nanobeams with defects

Hina Arif

August 30, 2021

1. **p. 14: line. 1, replace** "$k = 0, \dots, n$" **by** "$k = 1, \dots, n$"**.**
2. **p. 14: line. 3, replace "thickness** $h_k(k = 0, \dots, n)$**" by "width** b**".**
3. **p. 16: paragraph. 3, replace "**$(e_o a)^2 < 2$**" by "**$e_o a \leq 2$ **nm".**
4. **p. 22: R.H.S of equation (2.31) have an "alternate form"** $\pm M(a_k)$**.**
5. **p. 23: Figure. 3 should be "symmetric along x-axis" .**
6. **p. 52: paragraph. 1, replace** $h_o/l = 0.5nm$ **to** $h_o/l = 0.5$ **and** $h1/l = 0.4nm$ **to** $h_1/l = 0.4$**.**

Contents

List of original publications

I H. Arif and J. Lellep, Stability analysis of stepped nanobeams/columns with cracks, *Numerical Methods for Partial Differential Equations* (accepted).

II H. Arif and J. Lellep, Stability of nanobeams and nanoplates with defects, *Acta et Commentationes Universitatis Tartuensis de Mathematica* (accepted).

III H. Arif and J. Lellep, Buckling analysis of cantilever nanobeams with defects, *Applied Nanoscience* (2021), 1–7, DOI: 10.1007/s13204-021-01827-2.

IV H. Arif and J. Lellep, Buckling of nanobeams and nanorods with cracks, *In: Proceedings of the* 13^{th} *International Conference, Modern Building Materials, Structures and Techniques, Vilnius, Lithuania*, VGTU Press (2019), 722–727.

V H. Arif and J. Lellep, Buckling of stepped nanobeams with intermediate supports, *Results in Physics* (submitted).

Author's contribution

The author of this dissertation is responsible for majority of the research in all phases (including writing, simulation and preparing of images) of the papers I–V. The solution procedure was developed in co-operation with the supervisor; the statement of the problem belongs to the supervisor.

1 Introduction

1.1 Historical review of literature

Nanomaterials are of great importance in the field of physics, chemistry and engineering. Because of the special mechanical, electronic and electrical properties, nanomaterials are the fundamental components of various nanocomposites and nanoelectromechanical systems. Some well-known examples of nanomaterials are nanoparticles, nanotubes and nanowires. Beam-like nanostructures are widely used in civil, mechanical and aerospace engineering. Nanobeams play an important role in the field of nanotechnology (Eringen [44], Chen and Meguid [37], Akgöz and Civalek [2]).

Because of the various practical applications, static and dynamic analysis of nanobeams have been carried out by the researchers. The experimental and atomistic simulation results indicate that the small length scale may not be neglected at the nanoscale. Due to this reason, the classical local continuum theories fail to analyse the nanostructures. Hence, the use of nonlocal theories like strain gradient theory, couple stress theory, modified couple stress theory also called Eringen's elasticity theory presume the necessity to analyse nanoscale structures. In the present work, Eringen's nonlocal theory of elasticity is used to study the stability of Euler-Bernoulli nanobeams. Eringen [44, 45] developed a promising theory to investigate the nanomaterials without ignoring the internal length scale which accounts for the forces acting between atoms. Static and dynamic problems of stability were successfully treated by using the nonlocal theory of elasticity. Some of the dynamic problems involving the determination of frequencies of natural vibrations of nanobeams/rods are carried out by Bagdatli [17], Lellep, Lenbaum [71, 72], Roostai, Haghpanahi [98], Lu et al. [81, 82] and also by Li et al. [78, 79]. The buckling analysis of nanobeams/rods was carried out with the help of nonlocal theory of elasticity by Arif, Lellep [10–14], Emam [43], Reddy [93, 94], Challamel et al. [31, 33, 35, 36], Xu [119] and others.

Anderson [4], Broberg [22], Broek [23], Freund and Hermann [50], Jones [60], Gdoutos [51, 52], Gross [54], Wen [116] and others presented the basic ideas of the fracture mechanics which are followed in the present study. It is known in the engineering that discontinuities like cracks, slots and also steps with various boundary conditions affect the stability of structural elements. Under unfavourable conditions defects may result in the loss of stability. Hence, It is of great importance to study the influence of discontinuities on the stability of structures. The loss of stability is investigated by many researchers like Alfutov [3], Atanackovic [15], Bazant and Cedolin [19], Farshad [48], Iyengar [58], Simitses [99–101], Thomsen [105], Timoshenko and Gere [106], Ziegler [124] and others. Gross and Seelig [55] presented the fracture mechanics for micro-structures. The review of methods of design and optimization including the financial cost optimization has presented in the books by Farkas and Jármai [46, 47].

The paper by Okamura et al. [90] is one of the first papers dedicated to the problem of stability of cracked beams under compression. Gross and Srawley [56] conducted experiments in order to calculate the stress intensity factor at the crack tip of the cracked beams. The idea of the distributed line-spring method was suggested by Rice and Levy [96] for modelling part through cracks in rectangular plates. Rice and Levy treated the stress distribution in the neighbourhood of a crack in an elastic plate. This concept presents the key for calculation of additional compliance due to the crack and for determination of the energy release rate and the stress intensity factor for the plate weakened with crack. Later this idea was coupled with the additional local compliance due to crack. The rotating spring model was used by Anifantis, Dimarogonas [5–7], [42], Chondros, Dimarogonas [38, 39], Nikpour [88, 89], Murakami [85] and other researchers for the determination of the stress intensity factor for the beam elements with cracks subjected to generalized stresses. Cicirello and Palmeri [40] studied the case of multiple unilateral cracks under combined axial and transverse loads.

Polizzotto [92] employed variational principles for the derivation of main equations of the nonlocal theory of elasticity. Caddemi, Calio and others [24–29] treated the case of a beam weakened with the finite number of cracks. Akgöz and Civalek [2] employed the strain gradient theory of elasticity for treating the micro-sized beams, Ansari and Sahmani [8] presented the buckling analyses of nanobeams with the help of different beam theories. Functionally graded materials are considered by Yang and Chen [120] in the case of beams having open cracks. Wang et al. [114] derived exact solutions for beam problems employing the two-phase nonlocal model. Aydogdu [16] has developed a general nonlocal beam theory with applications in buckling and vibration. Wang et al. [108–114] presented exact solutions for the buckling and vibration of beams and columns with the internal hinge. Xiang et al. [118] employed the Levy method to derive the exact buckling solutions of rectangular plates with the internal hinge. The analysis of cracked functionally graded strip made of a piezo-electric material is undertaken by Mousavi and Paavola [83, 84]. The electrically impermeable crack face assumption is employed as the dislocation condition and the distributed dislocation technique is used for the antiplane analysis of the cracked strip.

Wang et al. [108–115] applied Timoshenko beam theory for the buckling analysis of micro and nanorods. Timoshenko nanobeams are studied with the help of modified couple stress theory also by Khorshidi and Shariati [61]. The model suggested in [61] admits to account for the discontinuities of the axial displacement. Banerjee and Williams [18] have defined the critical buckling loads of columns taking the effect of shear forces into account. Similarly Murmu, Pradhan [87] investigated carbon nanotubes embedded in an elastic medium by using Timoshenko beam theory. A review on the behaviour of thin-walled carbon nanotubes is presented by Arash and Wang [9]. A simple computational model for estimation of transversly cracked columns was developed by Skrinar [102]. The Euler-Bernoulli, Timoshenko, Reddy and Levinson beam theories were reformulated by Reddy [95]. In [93] Eringen's nonlocal constitutive relations are used to analyse the static and dynamic stability of beams and plates with different boundary conditions. Lim et al. [80], Thai [104], Reddy [93], Li et al. [78] and Wang et al. [115] used the analytical approach to analyse the dynamic behaviour of nanobeams. Ansari and Sahmani [8] presented the comparison of different beam theories applied to the analysis of nanobeams. Wang at al. [114], Zhang et al. [122], and Kumar et al. [66], Wang [108], also Viola and Marzani [107] examined the stability of nanobeams under conservative loading and study the impact of nonlocal parameters and different boundary conditions on the critical buckling loads and postcritical states of nanobeams. Zhou and Huang [123] studied the behaviour of eccentrically loaded columns under axial compression.

For the solution of the stability and vibration problems of stepped beams there are two alternatives. The first one which is used in the present study also consists of the partition of the beam or a plate strip into continuous segments. Each of these elements is treated separately. However, Yavari and Sarkani [121] suggest to formulate the eigenvalue problem in the space of generalized functions. Since the generalized piecewise continuous functions have derivatives the vibration and stability problems of beams and bars can be formulated without partitioning.

1.2 Aim of the dissertation

The dissertation aims to investigate the buckling behaviour of nonlocal elastic nanobeams weakened with irregularities like cracks, steps and internal supports. To analyse the buckling of nanobeams, an analytical approach has been developed within the framework of Eringen's nonlocal theory of elasticity to embrace the small size effect. The nonlocal theory of elasticity for Euler-Bernoulli nanobeams is combined with the concept of the stress intensity factor known in the linear elastic fracture mechanics. The crack effect is considered by coupling the local compliance of the structure with the stress intensity factor. Critical buckling loads for axially loaded stepped nanobeams and nanocolumns including cracks and internal supports are calculated. Simply supported, clamped and cantilever nanobeams are considered in the investigation. The influence of different physical and geometrical parameters on the stability of stepped nanobeams is also discussed. The numerical results calculated with the aid of MATLAB tools are presented for uniform and one-stepped nanobeams. Since conducting experiments at nanolevel is difficult to handle, the accuracy of the presented method is verified by the comparison of results with the available works in the literature.

1.3 Structure of the dissertation

The dissertation is organised as follows. Section 1 contains a historic background of the stability analysis of nonlocal beams, the aim and the structure of the dissertation. In section 2, the nonlocal physical model and the local flexibility of stepped nanobeams with cracks are described in detail. In sections 3, 4 and 5, the method is applied to the nanobeams with different support conditions and in section 6, to the nanobeams with additional internal supports. The concluding remarks of the dissertation are presented in section 7.

2 Physical Model

2.1 Stepped nanobeam with cracks

Consider a linearly elastic and isotropic nanobeam with finite steps including cracks at the re-entrant corners of the steps as shown in the Figure 1.

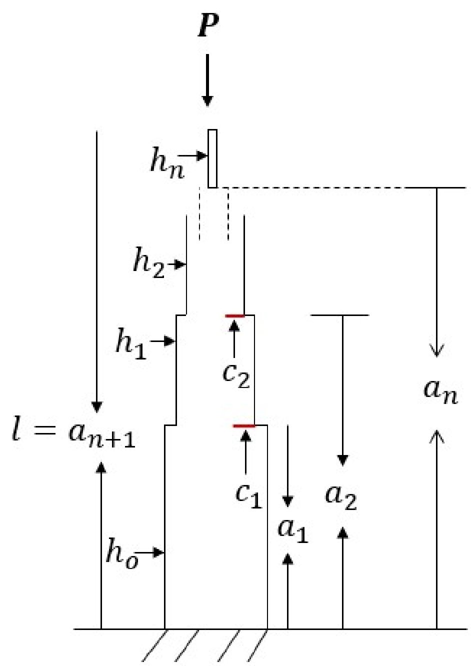

Figure 1: n-stepped nanobeam with cracks

We are going to investigate the stability of stepped nanobeams of length l subjected to the axial load P. The influence of cracks on the stability of stepped nanobeams will also be investigated. By making use of Eringen's nonlocal theory of elasticity [44,45], one can determine the constitutive equations for Euler-Bernoulli nanobeams. The solution of the governing equations is then used to study the buckling of stepped nanobeams weakened by stable surface cracks of length c_k at the step locations $x = a_k, k = 0, \ldots, n$, where $a_o = 0$ and $a_{n+1} = l$. Assume that the cracks are uniformly penetrated throughout the thickness $h_k (k = 0, \ldots, n)$ of the nanobeam. The nanobeams are considered to be of the rectangular cross-section of a constant width b and the height $h = h_k = const, k = 0, \ldots, n$. Let us assume that the origin of the coordinates is located at the centre of the bottom of the nanobeam.

According to Chondros et al. [38,39] and Dimarogonas [41,42], the cracks are assumed to be stable surface cracks during the elastic buckling. It was accepted already by Irwin [57] that the formation of cracks in any structural element involves a significant change of local flexibility due to the concentration of strain energy in the vicinity of the crack tip. Dimarogonas [42] employed this effect to study the influence of the cracks on the static and dynamic behaviour of beams and plates.

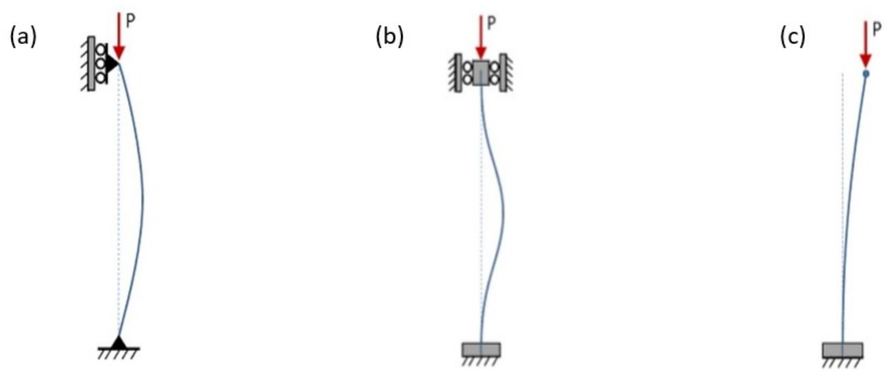

Figure 2: Nanobeams with different support conditions

(a) Simply supported
(b) Clamped
(c) Cantilever

We are interested in the determination of the critical buckling loads of stepped nanobeams weakened by cracks and we shall investigate the influence of discontinuities like cracks, steps and various boundary conditions on the critical buckling loads of nanobeams. Similarly, we will investigate the sensitivity of critical buckling loads to some other physical and geometrical parameters. Simply supported, clamped and cantilever nanobeams (plates) (see Figure 2) are the main objects of the investigation.

2.2 Nonlocal constitutive equations for Euler-Bernoulli nanobeams

According to Eringen's nonlocal theory of elasticity [44, 45], the stress state at a reference point x in a continuum medium depends not only on the strain state at x but also on the strain states at all other points x' of the medium. The constitutive equation in the nonlocal elasticity can be presented as (see Eringen [44, 45], Lellep and Lenbaum [71, 72], Arif and Lellep [11, 12], Ansari and Sahmani [8], Aydogdu [16], Reddy [93–95] and Polizzotto [92])

$$\sigma_{ij}(x) = \int_V \mathcal{K}(|x - x'|, \tau) t_{ij} dV(x'), \qquad \forall x \in V. \tag{2.1}$$

Here $\mathcal{K}$ is the nonlocal kernel function which shows the effect of the strain at the point x' on the stress at the point x. The quantities σ_{ij} and t_{ij} represent the nonlocal stress tensor and the local stress tensor, respectively. The integration covers the entire volume V of the medium, τ is the scale effect which is based on the internal and external length characteristics and is defined as

$$\tau = \frac{e_o a}{l} \tag{2.2}$$

In (2.2), e_o is the material constant, a is the lattice parameter and $\bar{l}$ is the size of the sample. A simplified version of the nonlocal constitutive relations for Euler-Bernoulli nanobeams can be described as (see Lellep and Kraav [63], [70], Kukla [65], Murmu and Adhikari [86], Skrinar [102])

$$\sigma_{xx} - (e_o a)^2 \frac{\partial^2}{\partial x^2}\sigma_{xx} = E\varepsilon_{xx}, \tag{2.3}$$

where ε_{xx} is the normal strain and E represents the Young's modulus of elasticity. Equation (2.3) can be written in terms of the bending moment M as

$$M - \eta \frac{\partial^2 M}{\partial x^2} = M_c, \tag{2.4}$$

where $\eta = (e_o a)^2$ is the nonlocal length scale parameter and the value of $(e_o a)^2 < 2$. Here M_c is the bending moment in the classical theory of elasticity and it can be defined as (see Timoshenko and Gere [106], Lellep et al. [71], [72])

$$M_c = -EI_k \frac{\partial^2 w}{\partial x^2}, \tag{2.5}$$

where I_k represents the second moment of the cross-section. For a beam of rectangular cross-section having width b and the height h_k, one has

$$I_k = \frac{b h_k^3}{12}, \tag{2.6}$$

for $k = 0, \ldots, n$.
In (2.5) w stands for the lateral displacement and it is assumed that

$$w = w(x), \tag{2.7}$$

so that equation (2.4) takes the form (henceworth primes denote the differentiation with respect to x)

$$M - \eta \frac{d^2 M}{dx^2} = -EI_k w''. \tag{2.8}$$

The bending moment M for nonlocal theory of elasticity can be calculated by combining (2.5) and (2.4) with the equilibrium equation. It can be shown that (see Lellep and Lenbaum [71], [72]) this yields finally

$$M = -(EI_k - \eta P)w'', \tag{2.9}$$

for $x \in (a_k, a_{k+1})$; $k = 0, \ldots, n$.
It is worthwhile to mention that P is the compressive load and it is treated here as a positive constant. Then the equilibrium conditions for the Euler-Bernoulli nanobeams have the form (see Wang et al. [113])

$$M' = Q, \tag{2.10}$$

and

$$Q' = Pw''. \tag{2.11}$$

In (2.10) and (2.11), Q represents the shear force. Eliminating Q from (2.10) and (2.11) results in

$$M'' - Pw'' = 0. \tag{2.12}$$

Substituting M from (2.9) to (2.12) yields a fourth order ordinary differential equation with respect to the lateral displacement w for $x \in (a_k, a_{k+1})$

$$w^{IV} + \lambda_k^2 w'' = 0, \tag{2.13}$$

where

$$\lambda_k = (\frac{P}{EI_k - \eta P})^{\frac{1}{2}} \tag{2.14}$$

and $k = 0, \ldots, n$.

In order to solve the equation (2.13), let us compile the characteristic equation

$$\omega_k^4 + \lambda_k^2 \omega_k^2 = 0. \tag{2.15}$$

The equation (2.15) has roots

$$(\omega_k)_{1,2} = \pm i\lambda_k, \quad (\omega_k)_{3,4} = 0, \tag{2.16}$$

where i stands for the imaginary unit.

The general solition of the fourth order equation (2.13) corresponding to roots of the characteristic equation (2.16) can be presented as

$$w = A_k cos\lambda_k x + B_k sin\lambda_k x + C_k x + D_k, \tag{2.17}$$

for $x \in (a_k, a_{k+1})$, $k = 0, \ldots, n$.
Here A_k, B_k, C_k and D_k are integration constants.
While constructing the solutions of the stability problems of nanobeams, we have to fit together the solutions (2.17) for segments (a_k, a_{k+1}) so that the corresponding continuity and jump conditions taken along with the boundary requirements are satisfied.

2.3 Continuity conditions and the local compliance

According to the physical considerations the displacement $w(x)$, the bending moment M and the shear force Q are continuous everywhere; particularly at $x = a_k, (k = 1, \ldots, n)$. Therefore,

$$w(a_k - 0) = w(a_k + 0). \tag{2.18}$$

Since the bending moment M and the shear force Q are continuous, it follows from (2.9) and (2.10) that

$$(EI_{k-1} - \eta P)w''(a_k - 0) = (EI_k - \eta P)w''(a_k + 0), \tag{2.19}$$

and

$$(EI_{k-1} - \eta P)w'''(a_k - 0) = (EI_k - \eta P)w'''(a_k + 0), \tag{2.20}$$

for every $k = 1, \ldots, n$.
However, there is a significant effect of cracks and other defects on the mechanical behaviour of structural elements. According to Dimarogonas et al. [42], Wang et al. [111,112], the slope w' has finite jumps passing the cross

sections with stable cracks. Denoting

$$\theta_k = w'(a_k + 0) - w'(a_k - 0), \tag{2.21}$$

one can consider θ_k as a generalized coordinate and M_k as the generalized force $M_k = M(a_k)$.

The influence of the cracks on the static and dynamic behaviour of beams is treated with the help of the weightless rotating spring model by Anifantis and Dimarogonas [5–7], [41]. The stiffness of the spring K_T is influenced by the crack length c_k of the beam. Here $K_k = K_T(a_k), \quad k = 1, \ldots, n$. It is known in the analytical mechanics that

$$\theta_k = C_{ok} M_k, \tag{2.22}$$

for $k = 1, \ldots, n$. In (2.22) C_{ok} represents the additional compliance due to the crack (defect) located at the cross section $x = a_k$ (k=1, ...,n).

It is already shown by many researchers (Lellep and Kraav [69, 70], Arif and Lellep [10, 12], Lellep and Liyvapuu [73, 74]) that

$$C_{ok} = \frac{72\pi(1 - \nu^2)}{Ebh_{ok}^2} f(s_k). \tag{2.23}$$

In (2.23), $h_{ok} = min(h_{k-1}, h_k)$ and $s_k = \frac{c_k}{h_{ok}}$, whereas ν stands for the Poisson ratio.

By the inversion of compliance C_{ok}, one can obtain the stiffness K_k of the spring, so that

$$K_k = \frac{1}{C_{ok}}, \tag{2.24}$$

where K_k and C_{ok} represent the elements of the local stifness matrix $[K]$ and compliance matrix $[C_{ok}]$, respectively. In the present case, C_{ok} is considered as a scalar function depending on the crack parameters. Evidently, one can define the spring stiffness by the following relation

$$K_k = \frac{EI_{k-1}}{6\pi h_{k-1} f(s_k)(1-\nu^2)}, \quad k = 1, \ldots, n. \tag{2.25}$$

The function $f(s_k)$ in (2.23), (2.25) is defined as (see Dimarogonas [42])

$$f(s_k) = \int_0^{s_k} yF^2(y)dy, \tag{2.26}$$

where $F = F(s_k)$ represents the stress correction function (also called shape function) which can be specified by the interpolation of experimental data. Efforts have been made by many researchers to develop the best experimental approximation for calculation of the stress intensity factor. A review of the results is presented in the handbook by Tada, Paris and Irwin [103]. Following these results, one can use the correction function $F(s_k)$ as,

$$F(s_k) = 1.93 - 3.07s_k + 14.53s_k^2 - 25.11s_k^3 + 25.80s_k^4. \tag{2.27}$$

Several other researchers have used the correction function $F(s_k)$ in the form of

$$F(s_k) = \frac{\sqrt{tan\pi\frac{s_k}{2}}}{\pi\frac{s_k}{2}} . \frac{0.923 + 0.199(1 - sin\pi\frac{s_k}{2})^4}{cos\pi\frac{s_k}{2}}. \tag{2.28}$$

It was established that the correction function (2.28) gives adequate values only for the cracks of length c, not exceeding $0.7h_{ok}$ (see Tada et al. [103]). Freund and Herrmann [50] suggested a shape correction function for larger cracks as

$$F(s_k) = \begin{cases} 1.99 - 2.47s_k + 12.97s_k^2 - 23.17s_k^3 + 24.8s_k^4, & 0 < s_k < 0.5; \\ 0.663(1 - s_k)^{-3/2}, & 0.5 < s_k < 1. \end{cases} \tag{2.29}$$

Many other researchers like Chondros et al. [38, 39], Freund and Hermann [50], Ostachowich and Krawczuk [91] have been suggested different forms of the stress correction function $F = F(s_k)$. A comparison of these functions was presented by Caddemi and Calio [24].

The approximation (2.27) has been used widely by many researchers to develop the solutions of particular problems. Among them Dimarogonas [41] studied the buckling of rings and tubes, Lellep and Kraav [69, 70] investigate the elastic buckling of stepped beams with cracks whereas Arif and Lellep [10,11], Binici [21], Rizos et al. [97], Li [77] and many others employed this concept for evaluation of the flexibility of beams under compression.

It is important to note that the correction functions for specimens of different shapes loaded in different manner should have different forms. Similarly, the type of crack (an intrinsic, a surface crack or an array of cracks) plays an important role in defining the shape of correction function (see Tada et al. [103]).

Thus, it follows from (2.21) – (2.26) that the jump of the slope of the displacement can be defined as

$$w'(a_k + 0) = w'(a_k - 0) - C_{ok}(EI_k - \eta P)w''(a_k + 0), \tag{2.30}$$

for $k = 1, \ldots, n$ where C_{ok} is defined by (2.22) – (2.29).

An alternative form of the jump conditions for the spring model can be presented as (see [11])

$$K_k[w'(a_k)] = -M(a_k), \quad k = 1, \ldots, n. \tag{2.31}$$

In (2.31), the square brackets denote the finite jumps of the corresponding quantities at the given point. The value of the bending moment M at the step location a_k can be defined from (2.9).

3 Buckling of cantilever nanobeams and nanocolumns with cracks

3.1 Problem formulation

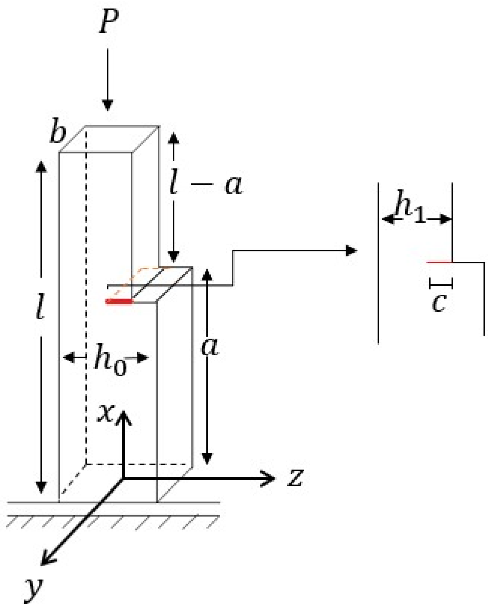

Figure 3: Stepped nanocantilever with a defect

Consider the stability of an elastic stepped cantilever nanobeam of length l subjected to the axial load P. Eringen's nonlocal theory of elasticity is used to determine the constitutive equations for Euler-Bernoulli nanobeams. The governing equations are solved for the case of buckling of stepped nanobeams weakened by stable surface cracks of length c_k at the step locations $x = a_k, k = 0, \ldots, n$, where $a_o = 0$ and $a_{k+1} = l$. It is assumed that the column is fully clamped at the bottom where $x = 0$. Let us assume that the origin of the coordinates is located at the centre of the bottom of the nanobeam.

This section aims in the investigation of the stability of stepped cantilever nanobeams weakened by stable surface cracks and in the sensitivity of critical buckling loads to cracks, steps and various physical and geometrical parameters. One-stepped cantilever nanobeam of length l having a defect of length c at the cross-section $x = a$ is shown in Figure 3.

3.2 Continuity and support conditions

The displacements of the central line of stepped nanocantilevers are presented by (2.17). While constructing the solution of the buckling problem of a nanobeam, one has to fit together the solutions (2.17) for segments (a_k, a_{k+1}) so that the corresponding continuity conditions, jump conditions and the boundary conditions are satisfied. The kinematic boundary conditions for nanocantilevers are presented as

$$w(0) = w'(0) = 0 \tag{3.1}$$

Since the upper end of the nanocantilever is free of moments of external loads, one has the boundary conditions

$$M(l) = 0, \tag{3.2}$$

and

$$Q(l) = Pw'(l). \tag{3.3}$$

The intermediate conditions can be derived from physical considerations. It is evident that quantities like the displacement $w(x)$, the bending moment M and the shear force Q must be continuous everywhere, in particular at $x = a_k, k = 1, \ldots, n$. Therefore one has

$$w(a_k - 0) = w(a_k + 0). \tag{3.4}$$

Due to the continuity of the bending moment M

$$(EI_{k-1} - \eta P)w''(a_k - 0) = (EI_k - \eta P)w''(a_k + 0). \tag{3.5}$$

The continuity of the shear force Q demands that

$$(EI_{k-1} - \eta P)w'''(a_k - 0) = (EI_k - \eta P)w'''(a_k + 0), \tag{3.6}$$

for every $k = 1, \ldots, n$.

Evidently, cracks and other defects affect the mechanical behaviour of structural elements. The impact of cracks on the stability and vibrations of beams and columns has been studied by many researchers. Among them, Dimarogonas et al. [42], Wang et al. [111, 112], Lellep and Kraav [70] modelled the influence of cracks on the buckling of beams and columns by making use of the concept of the weightless rotating spring. The stiffness of the spring K_T is influenced by the crack length c_k of the beam. Here $K_k = K_T(a_k), \quad k = 1, \ldots, n$.

The slope w' of the displacement has finite jumps passing through the cross sections with stable cracks. According to the papers by Lellep and Sakkov [76], Arif and Lellep [10,11], Lellep and Lenbaum [71,72], the jump conditions for the spring model can be presented by (2.31).

3.3 Critical buckling load

The critical buckling load for nanocantilevers can be determined by making use of (2.17) satisfying the corresponding support and intermediate conditions. For a one-stepped nanocantilever (2.17) can be presented as

$$w = A_o cos\lambda_o x + B_o sin\lambda_o x + C_o x + D_o, \tag{3.7}$$

for $x \in (0, a)$ and

$$w = A_1 cos\lambda_1 x + B_1 sin\lambda_1 x + C_1 x + D_1, \tag{3.8}$$

for $x \in (a, l)$.

The boundary conditions (3.1) and equality (3.7) lead to the relations

$$A_o = -D_o, \quad C_o = -\lambda_o B_o. \tag{3.9}$$

The requirements (3.2), (3.3) and the equation (3.8) result in

$$\begin{aligned} A_1 &= -B_1 tan\lambda_1 l, \\ C_1 &= -B_1(\mu\lambda_1^3 + \lambda_1)\{tan\lambda_1 l.sin\lambda_1 l + cos\lambda_1 l\}, \end{aligned} \tag{3.10}$$

where

$$\mu = -\frac{EI_1 - \eta P}{P} \tag{3.11}$$

Substituting (3.9), (3.10) in (3.7), (3.8) lead to the distribution of the displacement in the form

$$w = B_o(sin\lambda_o x - \lambda_o x) + D_o(1 - cos\lambda_o x), \tag{3.12}$$

for $x \in (0, a)$ and

$$\begin{aligned} w = \quad & B_1\{-x(\mu\lambda_1^3 + \lambda_1)(tan\lambda_1 l.sin\lambda_1 l + cos\lambda_1 l) \\ & +sin\lambda_1 x - tan\lambda_1 l.cos\lambda_1 x\} + D_1, \end{aligned} \tag{3.13}$$

for $x \in (a, l)$.

According to the continuity condition (3.4), the displacement $w(x)$ must be continuous at $x = a$, so that

$$B_o(\lambda_o a - sin\lambda_o a) + D_o(cos\lambda_o a - 1) + B_1 X + D_1 = 0, \tag{3.14}$$

where

$$X = -tan\lambda_1 l.cos\lambda_1 a + sin\lambda_1 a - a(\mu\lambda_1^3 + \lambda_1)\{tan\lambda_1 l.sin\lambda_1 l + cos\lambda_1 l\}. \tag{3.15}$$

The jump condition (2.31) can be presented as

$$\begin{aligned} &B_o\{-K(\lambda_o cos\lambda_o a - \lambda_o) + (EI_o + \eta N)\lambda_o^2 sin\lambda_o a\} + B_1 Y \\ &+D_o\{-K\lambda_o sin\lambda_o a - (EI_o + \eta N)\lambda_o^2 cos\lambda_o a\} = 0, \end{aligned} \tag{3.16}$$

where

$$Y = K\lambda_1\{tan\lambda_1 l.sin\lambda_1 a + cos\lambda_1 a\} - K(\mu\lambda_1^3 + \lambda_1)\{tan\lambda_1 l.sin\lambda_1 l + cos\lambda_1 l\}. \tag{3.17}$$

According to (3.5) and (3.6), the continuity of the bending moment M and the shear force Q lead to the equations

$$B_o\lambda_o^2 sin\lambda_o a - D_o\lambda_o^2 cos\lambda_o a + B_1\alpha\lambda_1^2\{tan\lambda_1 l.cos\lambda_1 a - sin\lambda_1 a\} = 0, \tag{3.18}$$

and

$$B_o\lambda_o^3 cos\lambda_o a + D_o\lambda_o^3 sin\lambda_o a - B_1\alpha\lambda_1^3\{tan\lambda_1 l.sin\lambda_1 a + cos\lambda_1 a\} = 0, \tag{3.19}$$

where

$$\alpha = \frac{EI_1 + \eta N}{EI_o + \eta N}, \tag{3.20}$$

where $N = -P$.
The system (3.14) – (3.20) represents a linear algebraic system of four unknowns. Equalizing the determinant Δ of the system to zero, one can solve the equation with respect to the critical buckling load P.
In the calculations, the relation

$$\lambda_{o,1}^2 = \frac{P}{EI_{o,1} - \eta P}, \tag{3.21}$$

is taken into account.

3.4 Numerical results

The numerical solutions are calculated by solving the eigenvalue problem presented above by equilizing the determinant Δ to zero. The numerical results for cantilever nanobeams of constant thickness and stepped nanobeams coincide with the available data in the literature. However, the literature on the investigation of stepped nanobeams with cracks is limited.

In the following, the numerical results for the presented method are shown by various graphical examples, which aims to study the influence of nonlocal parameter, crack, step height, step and crack location on the critical buckling load of the stepped nanobeam.

The influence of the length l and the constant height h on the critical buckling load P_{cr} of the cantilever nanobeams is shown in Figure 4. It can be seen from Figure 4 that the values of the critical buckling loads of cantilever nanobeams decrease monotonically by increasing the length of the nanobeam. It can also be seen that the critical buckling loads of the cantilever nanobeams increase by increasing the height of the nanobeam, as might be expected.

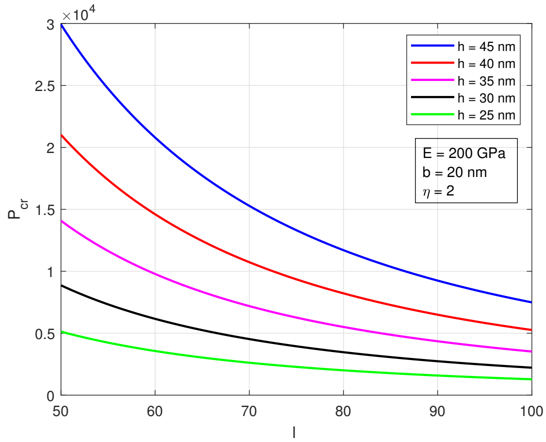

Figure 4: Critical buckling load versus length of nanocantilevers of various heights

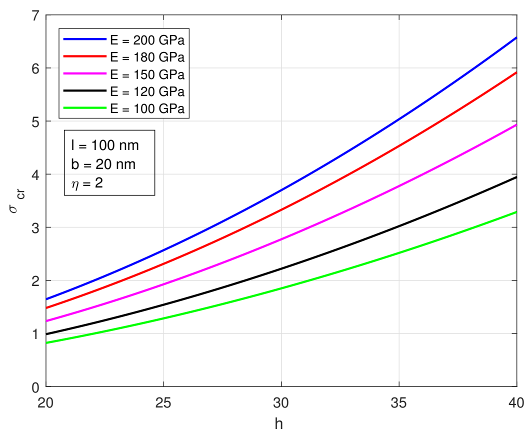

Figure 5: Critical stress versus height of nanocantilevers for different values of Young's modulus

While using nonlocal elasticity theory, investigation of the critical stress is one of the important concerns in the stability analysis of beams/columns. Figure 5 presents the critical stress σ_{cr} versus the height h of the cantilever nanobeams for different values of Young's modulus E. It can be seen that the value of the critical stress increases by increasing the thickness of the cantilever nanobeam. Figure 5 also reveals that the higher is the Young's modulus, the higher the stress is required to create the same amount of strain (for elastic buckling) in the case of the cantilever nanobeams. Therefore, more stress is needed for the elastic buckling of a stiff nanobeam in comparison to that of a soft nanobeam.

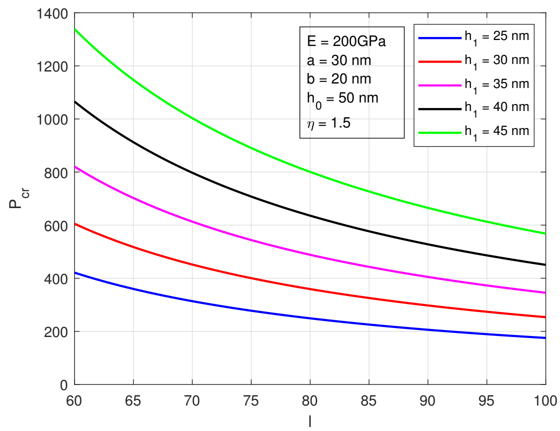

Figure 6: Critical buckling load versus length for different step heights of nanobeams

In Figure 6 the critical buckling load P_{cr} is presented versus length l for various step heights h_1 of the one-stepped cantilever nanobeam with a crack at the step location. Figure 6 reveals that the behaviour of the critical buckling load for various lengths of the stepped nanobeams is similar to that of the nanobeam without a step and a crack, i.e., the value of the critical buckling load decreases by increasing the length of the nanobeam. It can also be seen that the values of the critical buckling loads of nanobeams increase with increasing the step heights of the cantilever nanobeam.

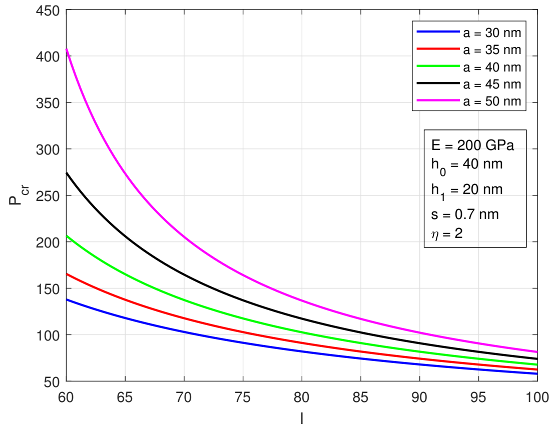

Figure 7: Critical buckling loads for different crack locations of nanocantilevers

Figure 7 presents the impact of crack locations a on the critical buckling loads of one-stepped cantilever nanobeams. One can see that by increasing the step/crack location a, the values of the critical buckling loads also increase. When, for instance, $a = 20\ nm$, the value of the critical buckling load is smaller than the value of critical buckling load for $a = 60\ nm$. Therefore, one can conclude that the values of critical buckling loads with higher values of a is higher.

Figure 8 and Figure 9 demonstrate the relationship between critical buckling load P_{cr} and the crack length s of the one-stepped cantilever nanobeams with various lengths and crack locations. It can be seen that the critical buckling load decreases by increasing the crack length of the stepped nanobeam. For instance, the value of the critical buckling load for $s = 0.4\ nm$ is higher than that for $s = 0.8\ nm$. Hence, one can observe that the values of critical buckling loads for $s = 0\ nm$ (nanobeams without a crack) provide the greater values of critical buckling loads, as might be expected.

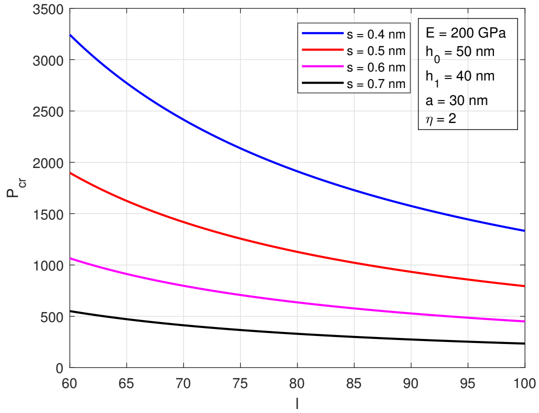

Figure 8: Critical buckling load versus length of cantilever nanobeams for different crack lengths

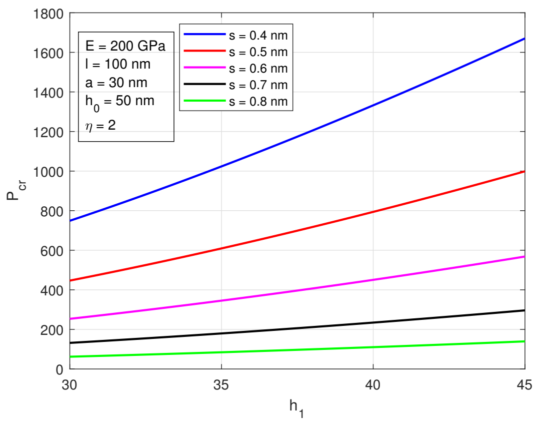

Figure 9: Relationship between the critical buckling load and the crack length of the stepped cantilever nanobeams

From the numerical examples, it can be summarized that the physical and the geometrical parameters such as length, step height, step location and the nonlocal parameter have significant effect on the critical buckling load of the stepped cantilever nanobeams.

4 Buckling of simply supported nanobeams and nanorods with defects

A general method for the determination of critical buckling loads of simply supported stepped nanobeams with cracks based on the Euler-Bernoulli concept and the nonlocal theory of elasticity is developed.

4.1 The nonlocal model formulation

Consider a nanobeam or nanorod of variable thickness. We assume that the nanobeam of length l is under the axial compression P (see Figure 10).

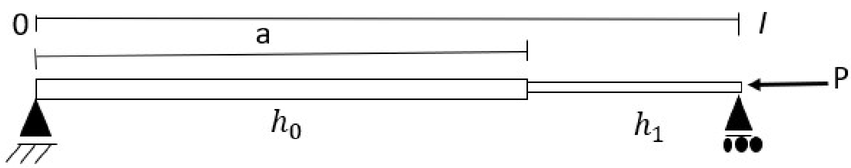

Figure 10: Simply supported stepped nanobeam with defect

We are interested in the determination of the critical value of the exerted pressure which corresponds to the loss of stability of the nanobeam. For the sake of simplicity, we confine our attention to the stepped nanobeams consisting of two or three segments only. Let the nanobeam have a rectangular cross-section with width b and thickness

$$h = \begin{cases} h_o, & x \in (0, a), \\ h_1, & x \in (a, l). \end{cases} \tag{4.1}$$

It is expected herein that the origin of the coordinate axis O_x is located at the left-hand end of the nanobeam and a, h_o, h_1 are given real numbers and of course, $a < l$. Let the nanobeam have a defect at the cross-section $x = a$.

The defect will be modelled as a stable crack of length c which is uniformly penetrated through the width of the nanobeam. The energy necessary for the formation of the crack will be captured in the local compliance (flexibility) of the nanobeam at the cross-section $x = a$. Together with the determination of the critical buckling load, we shall investigate its sensitivity concerning the geometrical and physical parameters of the nanorods.
According to (4.1) one has

$$I = \begin{cases} \frac{bh_o^3}{12}, & x \in (0, a), \\ \frac{bh_1^3}{12}, & x \in (a, l), \end{cases} \tag{4.2}$$

where b is the width of the nanobeam.
The displacement w has the form (3.7) and (3.8), where

$$\lambda_{o,1} = \sqrt{\frac{P}{-\eta P + EI_{o,1}}}. \tag{4.3}$$

The constants of integration will be specified according to the boundary and intermediate conditions. In the case of a nanobeam simply supported at both ends the transverse displacement w and the bending moment M must vanish at both ends. Taking (2.9) into account one can state that in this case

$$w(0) = w''(0) = 0, \tag{4.4}$$

and

$$w(l) = w''(l) = 0. \tag{4.5}$$

It is evident from the physical considerations that certain quantities (stresses, displacements) are continuous. Therefore, the continuity conditions (2.18) – (2.20) hold good in the present case, as well.

4.2 Critical buckling load

In order to determine the critical buckling load for simply supported nanobeams, we have to fit together the solutions of (3.7) and (3.8) for segments $(0, a)$ and (a, l) so that the corresponding continuity and jump conditions along with the boundary requirements are satisfied. For the one-stepped nanobeam, the critical buckling load can be determined with the aid of (4.4) and (4.5).
Using the boundary conditions (4.4) with (3.7) one obtains

$$A_o = D_o = 0. \tag{4.6}$$

The boundary requirements (4.5) and the equation (3.8) lead to the relations

$$\begin{aligned} A_1 &= -B_1 tan\lambda_1 l, \\ D_1 &= -C_1 l. \end{aligned} \tag{4.7}$$

Thus, the displacement

$$w = B_o sin\lambda_o x + C_o x, \tag{4.8}$$

for $x \in (0, a)$ and

$$w = -\frac{B_1}{cos\lambda_1 l} sin\lambda_1 (l - x) + C_1(x - l), \tag{4.9}$$

for $x \in (a, l)$.
The displacement $w(x)$ must be continuous at $x = a$, so that

$$B_o sin\lambda_o a + C_o a + B_2 sin\lambda_1 (l - a) - C_1(a - l) = 0. \tag{4.10}$$

The jump condition (2.30) can be presented as

$$B_2\{\lambda_1 cos\lambda_1(l-a) - C_{o1}(EI_1-\eta P)\lambda_1^2 sin\lambda_1(l-a)\}$$
$$-B_o\lambda_o cos\lambda_o a - C_o + C_1 = 0 \qquad (4.11)$$

The continuity conditions of the bending moment M and the shear force Q lead to the equations

$$-B_o\lambda_o^2 sin\lambda_o a(EI_o-\eta P) - B_2\lambda_1^2 sin\lambda_1(l-a)(EI_1-\eta P) = 0, \qquad (4.12)$$

and

$$-B_o\lambda_o^3 cos\lambda_o a(EI_o-\eta P) + B_2\lambda_1^3 cos\lambda_1(l-a)(EI_1-\eta P) = 0, \qquad (4.13)$$

where $B_2 = \frac{B_1}{cos\lambda_1 l}$.
The system (4.10) – (4.13) presents a linear algebraic system of four unknowns. Equalizing the determinant Δ of the system to zero, one can solve the equation with respect to the load P.

4.3 Numerical results

Calculations are carried out in the case of simply supported nanorods with constant dimensions of the cross-section and the case of one-stepped nanorods. The results are presented in Figure 11 to Figure 14.

The first three modes of buckling of the simply supported nanobeam of constant thickness $h = 50nm$, width $b = 30nm$, $l = 100nm$ and $E = 200GPa$ with the value of nonlocal parameter $\eta = 2$ is shown in Figure 11.

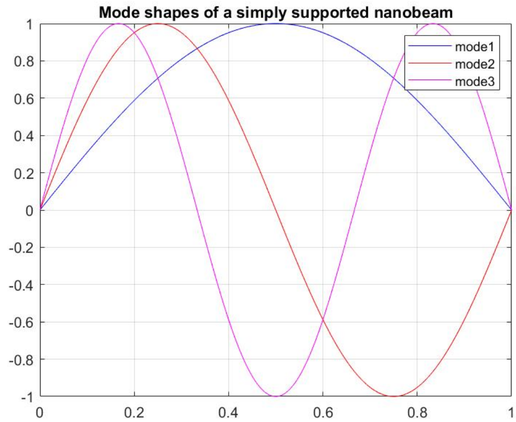

Figure 11: Modes of buckling of a simply supported nanobeam

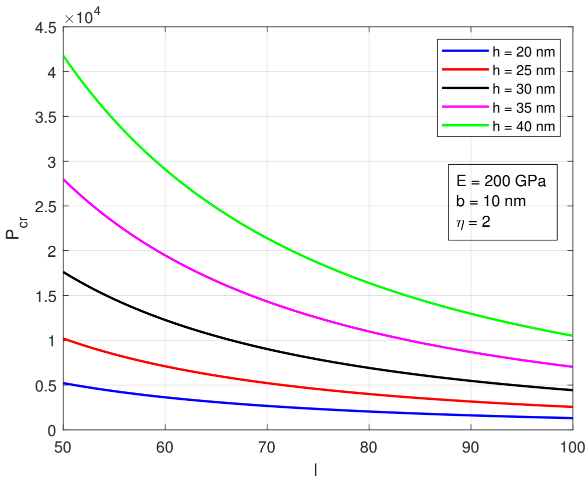

Figure 12: Critical buckling load versus the length of simply supported nanobeams

The relationship between the critical buckling load P and the length l of the nanobeam is shown in Figure 12 for different values of the constant height h. It can be seen that the values of the critical buckling load decrease monotonically with the increase of the length of the nanobeam. Figure 12 also reveals the matter that the thicker is the nanobeam, the higher will be the critical buckling load.

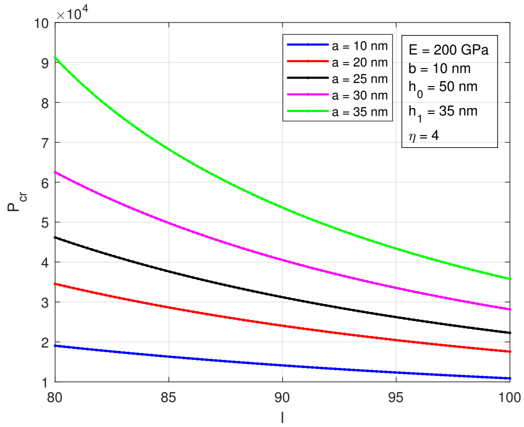

Figure 13: Critical buckling loads of stepped nanobeams with different step locations

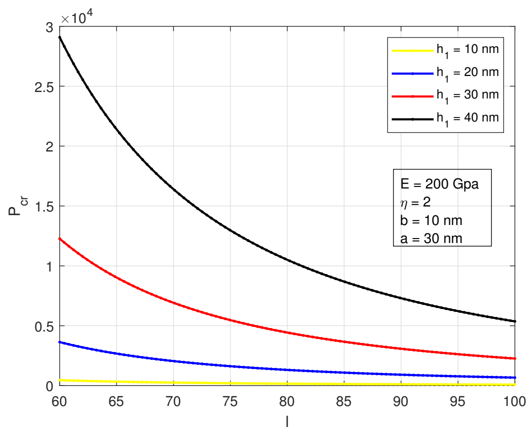

Figure 14: Critical buckling loads of stepped nanobeams with different thicknesses

In Figure 13 and Figure 14 the results regarding one-stepped simply supported nanobeams are presented. In Figure 13 the relationship between critical buckling load P and the length l of the nanobeam is shown for different crack locations a. It can be seen from Figure 13 that there is a direct relationship between critical buckling load and the step coordinate a of the nanobeam. Figure 13 reveals that together with the increase of the thicker part of the nanobeam, the value of the critical buckling load also increases.

In Figure 14 the results are presented for stepped nanobeams of various thicknesses. It is shown by the Figure 14 that the thicker is the stepped nanobeam the higher will be the value of the critical buckling load; as might be expected.

5 Buckling of clamped nanobeams and nanoplates with cracks

Clamped nanobeams and nanoplates subjected to the axial pressure are treated in the frameworks of the nonlocal theory of elasticity. Critical buckling loads of stepped nanobeams are defined under the condition that the nanoelements are weakened with stable crack-like defects. The influence of the crack on the critical buckling load of the nanorod is taken into account by the method developed earlier for buckling and vibration analysis of beams with cracks. The sensitivity of the critical load with respect to crack parameters is studied.

5.1 The problem formulation and critical buckling load

Let us treat the problem investigated in section 4 with different support conditions. Consider that the clamped nanobeam of length l is subjected to the axial compression P.

The aim of this section is to determine the critical buckling loads for stepped nanobeams with cracks and to elucidate the sensitivity of the critical buckling load with respect to the location of the crack, dimension of the nanobeam and other geometrical and physical parameters.

For the nanobeams clamped at both ends, the transverse displacement w and the slope of the transverse displacement w' must vanish at both ends. In such case

$$w(0) = w'(0) = 0, \tag{5.1}$$

and

$$w(l) = w'(l) = 0. \tag{5.2}$$

Consider the case of one-stepped nanobeam clamped at both ends. For the one-stepped nanobeam, the deflections can be calculated according to (3.7) – (3.9) where the constants A_k, B_k, C_k, D_k where $k = 0, 1$, must be defined so that the boundary conditions (5.1) and (5.2) are satisfied.
The boundary conditions (5.1) with the equality (3.7) furnish the relations (3.9). The support conditions (5.2) with (3.8) result in

$$\begin{aligned} A_1 cos\lambda_1 l + B_1 sin\lambda_1 l + C_1 l + D_1 &= 0, \\ \lambda_1(-A_1 sin\lambda_1 l + B_1 cos\lambda_1 l) + C_1 &= 0, \end{aligned} \tag{5.3}$$

which leads to the following relations

$$\begin{aligned} C_1 &= \lambda_1(A_1 sin\lambda_1 l - B_1 cos\lambda_1 l), \\ D_1 &= -A_1(cos\lambda_1 l + l\lambda_1 sin\lambda_1 l) + B_1(l\lambda_1 cos\lambda_1 l - sin\lambda_1 l). \end{aligned} \tag{5.4}$$

The relations (5.4) in (3.7) – (3.8) admit to present the deflection of the nanobeam in the form

$$w = A_o(cos\lambda_o x - 1) + B_o(sin\lambda_o x - \lambda_o x), \tag{5.5}$$

for $x \in (0, a)$ and

$$\begin{aligned} w &= A_1(cos\lambda_1 x + \lambda_1 sin\lambda_1 l.x - cos\lambda_1 l - l\lambda_1 sin\lambda_1 l) \\ &+ B_1(sin\lambda_1 x - \lambda_1 cos\lambda_1 l.x - sin\lambda_1 l + l\lambda_1 cos\lambda_1 l), \end{aligned} \tag{5.6}$$

for $x \in (a, l)$.
The displacement $w(x)$ is continuous at $x = a$, if

$$\begin{aligned} &A_1(cos\lambda_1 a + \lambda_1 sin\lambda_1 l.a - cos\lambda_1 l - l\lambda_1 sin\lambda_1 l) + B_1(sin\lambda_1 a - \lambda_1 a cos\lambda_1 l \\ &-sin\lambda_1 l + l\lambda_1 cos\lambda_1 l) - A_o(cos\lambda_o a - 1) - B_o(sin\lambda_o a - \lambda_o a) = 0. \end{aligned} \tag{5.7}$$

The continuity of the bending moment M and the shear force Q is satisfied, if

$$\lambda_1^2(\eta P - EI_1)(A_1 cos\lambda_1 a + B_1 sin\lambda_1 a) - \lambda_o^2(\eta P - EI_o)(A_o cos\lambda_o a + B_o sin\lambda_o a) = 0, \tag{5.8}$$

and

$$\lambda_1^3(\eta P - EI_1)(-A_1 sin\lambda_1 a + B_1 cos\lambda_1 a) - \lambda_o^3(\eta P - EI_o)(-A_o sin\lambda_o a + B_o cos\lambda_o a) = 0. \tag{5.9}$$

The jump conditions (2.21) – (2.23) lead to the equation

$$A_1\lambda_1(-sin\lambda_1 a + sin\lambda_1 l - \lambda_1 C_{o1}(EI_1 - \eta P)cos\lambda_1 a) + B_1\lambda_1(cos\lambda_1 a - cos\lambda_1 l - \lambda_1 C_{o1}(EI_1 - \eta P)sin\lambda_1 a) + A_o\lambda_o sin\lambda_o a - B_o\lambda_o(cos\lambda_o a - 1) = 0. \tag{5.10}$$

The system (5.7) – (5.10) is a linear homogeneous algebraic system with respect to A_o, B_o, A_1 and B_1. Equalizing its determinant Δ to zero, one can calculate the eigenvalues λ_o and λ_1. This leads to the values of critical buckling loads for clamped nanobeams by taking (2.14) into account.

5.2 Numerical results

Clamped nanobeams and nanoplate strips with constant thickness and one-stepped nanobeams including crack at the step location are considered here in calculations. Results are accomodated in the Figures 15 – 19.

The relationship between critical buckling load P_{cr} and length l of the clamped nanobeam is shown in Figure 15 for different thicknesses h of the nanobeam. Results presented here show that the value of critical buckling load decreases monotonically with increase in the length of of clamped nanobeam. It can also be seen that the thicker is the nanobeam, the higher is the value of critical buckling load.

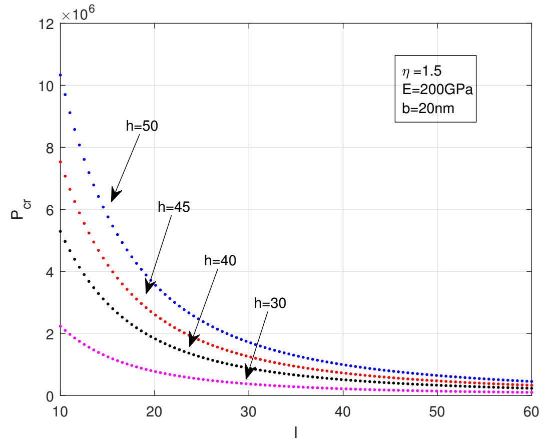

Figure 15: Critical buckling loads versus length of clamped nanobeams

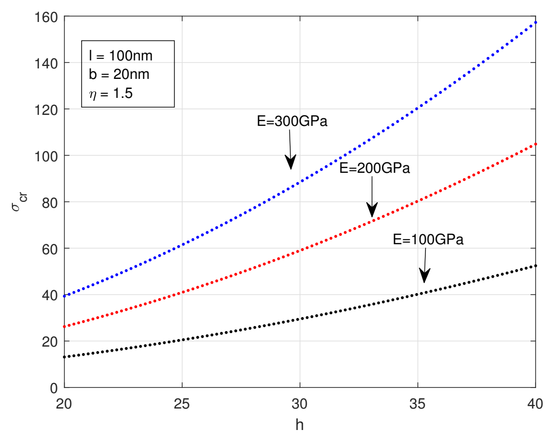

Figure 16: Critical stress versus thickness of clamped nanobeams

Critical stress σ_{cr} is one of the quantities of major interest while studying a stability problem in the nonlocal theory of elasticity. Figure 16 shows the relationship between the critical stress and thickness of the clamped nanobeam for different values of the Young's modulus E. Results presented in Figure 16 show that there is a direct relationship between critical stress and thickness of the clamped nanobeam. Its value increases by increasing the thickness of the nanobeam. Figure 16 also reveals that the Young's modulus has strong impact on the critical stress of a clamped nanobeam. Figure 16 shows that the greater is the value of Young's modulus, the higher is the value of critical stress.

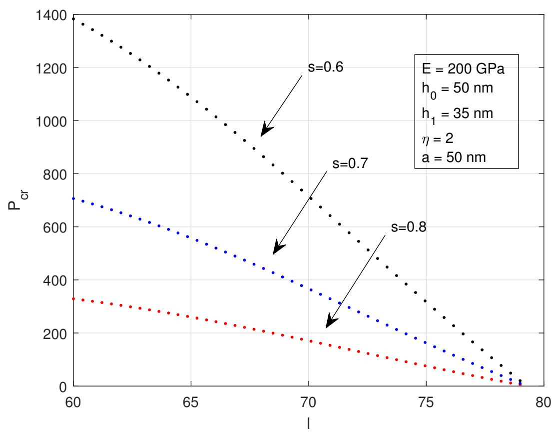

Figure 17: Relationship of critical buckling loads and crack lengths of stepped nanobeams clamped at the ends

In Figure 17 the relationship between the critical buckling load P_{cr} is presented for a stepped nanobeam including a crack of length $s.h_{o1}$ at the re-entrant corner of the step. It reveals that the value of critical buckling load decreases by increasing the crack length s of the stepped nanobeam.

Figure 18 reveals the impact of the crack location a on the value of critical buckling load of clamped nanobeams. It can be seen that the greater is the value of the coordinate a, the higher will be the critical buckling load. Evidently, the parameter a satisfies the inequality $a \leq l$.

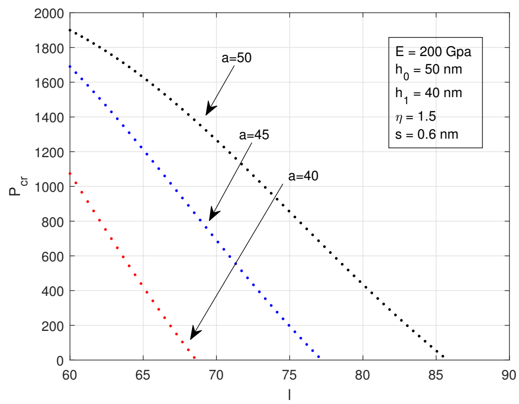

Figure 18: Critical buckling loads versus length of clamped nanobeams

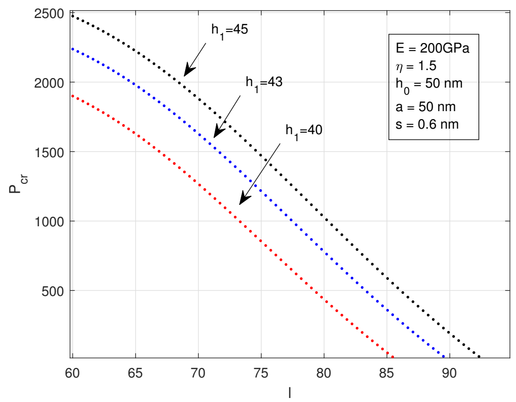

Figure 19: Impact of the thickness h_1 on the critical buckling load of a clamped nanobeam with defect

Results presented in Figure 19 are obtained for different thicknesses of stepped nanobeams having defect. Figure 19 reveals that the thicker is the step of the cracked nanobeam, the higher will be the value of critical buckling load.

5.3 Comparison of results and discussion

Calculations are carried out by considering the nanobeams of constant thickness and one-stepped nanobeams of length l. The nanobeams have a crack of length c at the step location a, uniformly penetrated throughout the thickness h_1 of the stepped nanobeams. Nanobeams with various boundary conditions are considered when calculating the values of critical buckling loads P_{cr} and critical stresses σ_{cr} to solve the stability problems. Influence of discontinuities like steps, cracks and various boundary conditions on the stability of nanobeams is investigated as well as the impact of physical and geometrical parameters on the sensitivity of critical buckling load is studied here. Clamped (C-C), simply supported (S-S) and cantilever (C-F) nanobeams are considered in the calculations. Results are accomodated in the Figures 20 – 25 and in the Tables 1 – 4. Additionally, the comparison of current results with the results of Wang et al. [112] is also accomodated in Figure 26 - 28.

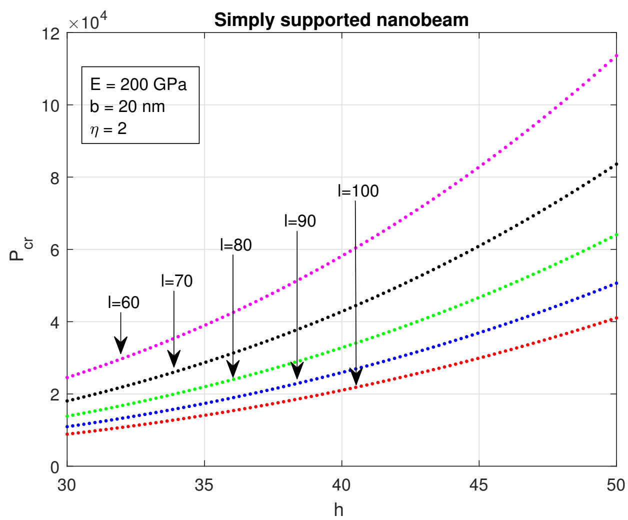

Figure 20: Critical buckling loads of simply supported nanobeams

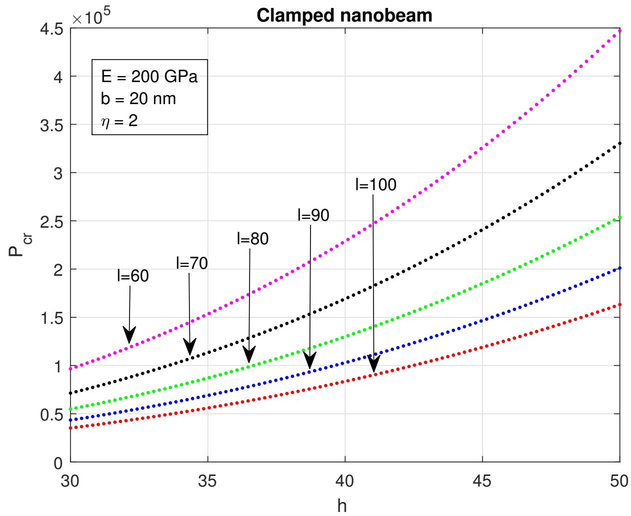

Figure 21: Critical buckling loads of clamped nanobeams

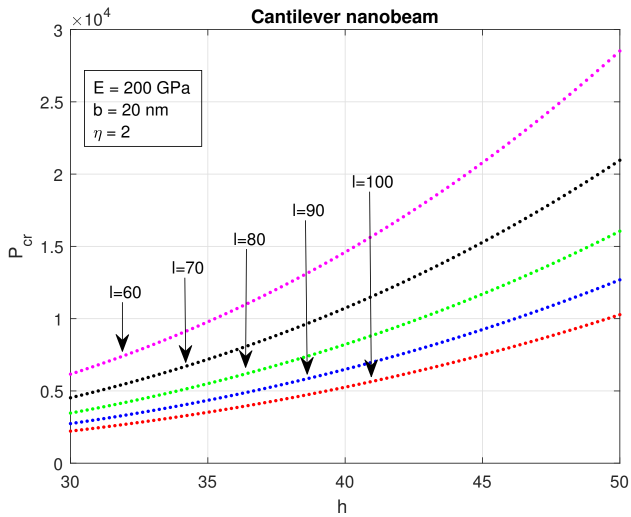

Figure 22: Critical buckling loads of cantilever nanobeams

In Figures 20 – 22, the relationship between the critical buckling load P_{cr} and the thickness h of the nanobeam is shown for nanobeams of different lengths. Various boundary conditions are applied to solve the problems of buckling. It can be seen from Figures 20 – 22 that the values of the critical buckling load increase with increasing the thickness of the nanobeam and the values of the critical buckling load decrease by increasing the length of the nanobeams. Another observation to be mentioned here is the impact of boundary conditions on the values of the critical buckling loads of nanobeams. Figure 20 corresponds to the nanobeam with simply supported edges, Figure 21 corresponds to the clamped nanobeams whereas Figure 22 corresponds to the cantilever nanobeams. Calculations are carried out by fixing the same physical parameters for nanobeams with various boundary conditions. It can be seen that the values of the critical buckling loads are highest in the case of clamped nanobeams whereas the cantilever nanobeams are much weaker than the clamped and simply supported nanobeams. The values of critical buckling loads for cantilever nanobeams are the lowest ones for these cases.

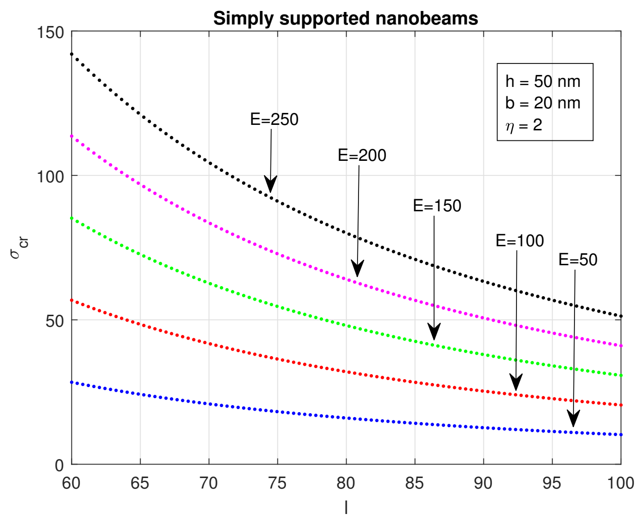

Figure 23: Critical stresses of simply supported nanobeams

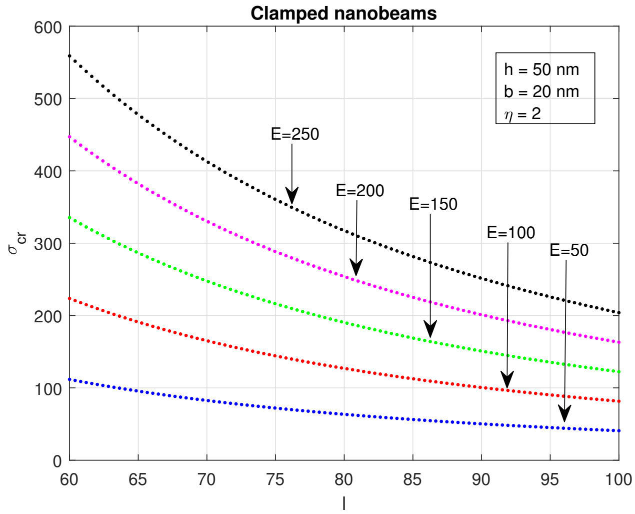

Figure 24: Critical stresses of clamped nanobeams

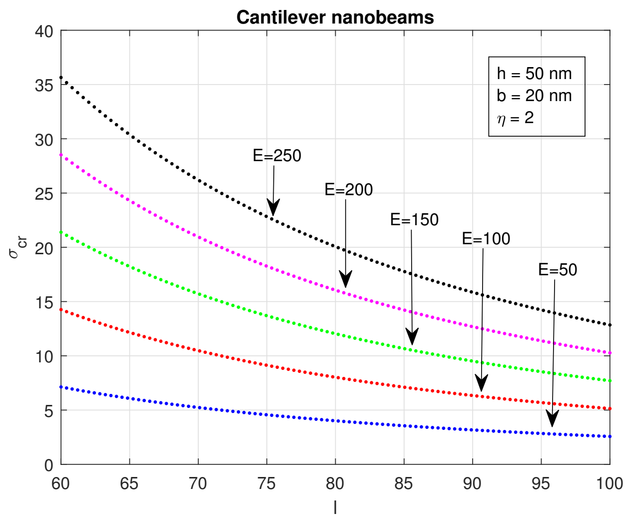

Figure 25: Critical stresses of cantilever nanobeams

Figures 23 – 25 reveals the effect of boundary conditions and parameters like length l and Young's modulus E on the values of critical stresses σ_{cr} of the nanobeams without irregularities. Results presented in Figures 23 – 25 show the strong impact of Young's modulus on the values of critical stresses for nanobeams with various boundary conditions. The values of the critical stresses increase significantly by increasing the value of Young's modulus and the values of the critical stresses decrease with increasing the length of the nanobeams. Additionally, it can be seen that the values of critical stresses are higher in the case of clamped nanobeams and these are the lowest in the case of cantilever nanobeams.

Table 1 Impact of height-to-length ratio h/l on the critical buckling loads of nanobeams with various boundary conditions

h/l	Clamped	Simply supported	Cantilever
0.5	1.6320e+05	4.1042e+04	1.0276e+04
0.4	8.3561e+04	2.1014e+04	5.2612e+03
0.3	3.5252e+04	8.8651e+03	2.2196e+03
0.2	1.0445e+04	2.6267e+03	0.6576e+03

Table 2 Relationship of non dimensional length scale parameter η to the critical buckling loads of nanobeams with various boundary conditions

η	Clamped	Simply supported	Cantilever
1	8.3889e+04	2.1034e+04	5.2625e+03
2	8.3561e+04	2.1014e+04	5.2612e+03
3	8.3235e+04	2.0993e+04	5.2599e+03
4	8.2911e+04	2.0972e+04	5.2586e+03

Critical buckling loads P_{cr} of nanobeams of constant thickness as well as of stepped nanobeams including a crack at the step location are presented in Tables 1 – 4. Nanobeams of length $l = 100nm$ and width $b = 20nm$ are considered in the numerical experiments, whereas the value of Young's modulus is $E = 200GPa$. Tables $1-4$ reveal the fact that the values of critical buckling loads for clamped nanobeams are always higher than those of simply supported and cantilever nanobeams. The values of critical buckling loads for cantilever nanobeams are always smaller than those of simply supported and clamped nanobeams. The results regarding to the nanobeams without a step and a crack are presented in Tables 1, 2. Table 1 reveals the impact of height-to-length ratio on the critical buckling loads of nanobeams with various boundary conditions. It can be seen that the value of critical buckling load decreases monotonically by decreasing the value of height-to-length ratio of the nanobeam/plate. Table 2 presents the relationship between the non-dimensional length scale parameter η and the critical buckling loads P_{cr} of nanobeam/plate with various boundary conditions. One can see that the higher is the value of the nonlocal parameter, the smaller will be the value of the critical buckling load. Although the value of the nonlocal parameter has not a severe impact on the critical buckling loads of nanobeams, however in the case of nano-structures one can never neglect the impact of nonlocal length scale parameter while solving the static or dynamic stability problems.

Table 3 Effect of height-to-length ratio h_1/l of the stepped column on the critical buckling loads of nanobeams with various boundary conditions

h_1/l	Clamped	Simply supported	Cantilever
0.4	2.6279e+4	1.9148e+3	50.7417
0.3	2.9668e+3	5.3561e+2	28.5147
0.2	1.7869e+3	14.5270	12.6488
0.1	2.3080e+2	9.1770	3.1440

Table 4 Relationship between the critical buckling load and the crack location of nanobeams with various boundary conditions

a/l	Clamped	Simply supported	Cantilever
0.4	2.6279e+4	1.9148e+3	50.7417
0.3	1.1010e+4	9.13232+2	43.4723
0.2	8.3906e+3	5.3891e+2	38.0203
0.1	4.8331e+3	3.2631e+2	33.7798

The results for one-stepped nanobeams including a crack at the step location are presented in Tables 3, 4. Here the height-to-length ratio of the first step is taken as $h_o/l = 0.5nm$, the value of the Poisson ratio as $\nu = 0.38$, and the crack length as $s = 0.6$. Table 3 reveals the effect of the height-to-length ratio of the stepped column on the value of critical buckling loads of nanobeams with various boundary conditions. One can see that the value of the critical buckling load monotonically decreases with decreasing the height-to-length ratio of the stepped column. In Table 4, the relationship between the critical buckling load and the crack location of the nanobeams with various boundary conditions is presented. The height-to-length ratio of the stepped column is chosen as $h_1/l = 0.4nm$. It can be seen that the values of the critical buckling loads decrease monotonically by decreasing the value of the crack location of the nanobeam. It can be observed that the impact of boundary conditions on the stability of stepped nanobeams with cracks remains similar to that in the case of nanobeams of constant thickness.

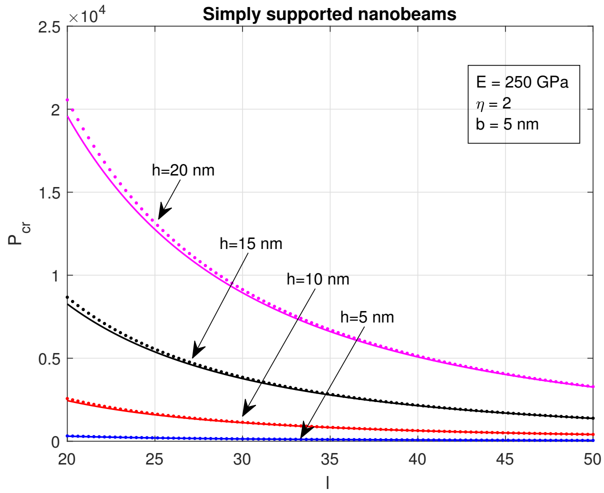

Figure 26: Comparison of results for simply supported nanobeams with Wang et al. [114]

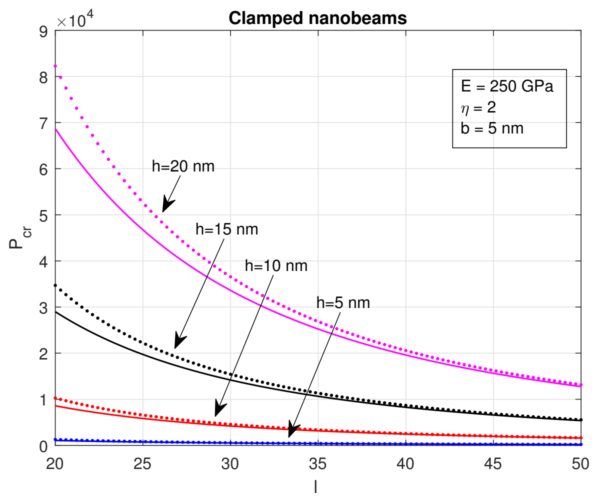

Figure 27: Comparison of results for clamped nanobeams with Wang et al. [114]

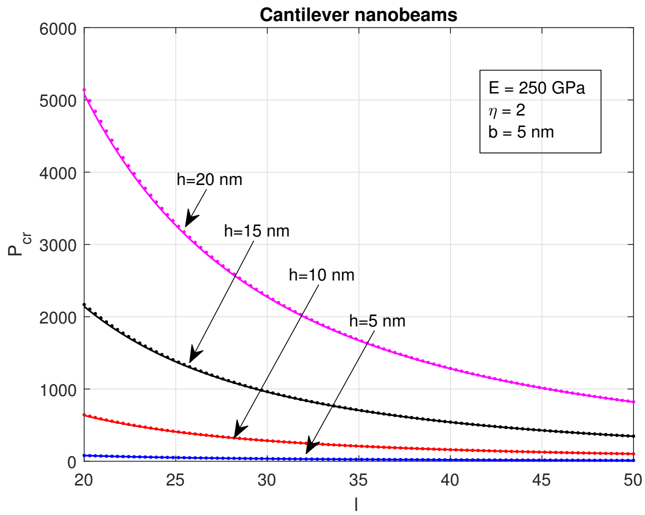

Figure 28: Comparison of results for cantilever nanobeams with Wang et al. [114]

The results are compared with those obtained by Wang et al. [114] in Figures 26 – 28. The results for simply supported nanobeams of constant thickness are presented in Figure 26. Similar results for clamped and cantilever nanobeams are presented in Figure 27 and Figure 28, respectively. Here the *dotted lines* correspond to the values of critical buckling loads of Wang et al. [114] and the *solid lines* correspond to the current solutions. It can be observed from Figure 26 – 28 that the presented results are quite close to those obtained by Wang et al. [114]. In the case of nanocantilevers the results practically coincide.

6 Buckling of stepped nanobeams with intermediate supports

The buckling behaviour of nonlocal elastic nanobeams with intermediate supports is studied in this section. An analytical method has been developed for the determination of the critical buckling loads for axially loaded nanobeams with intermediate supports. The results of the current study are compared with the available data in the literature.

6.1 The nonlocal constitutive equations for Euler-Bernoulli beam model

Consider an elastic and isotropic stepped nanobeam of uniform width b, length l and of flexural rigidity EI, which is simply supported at both ends and subjected to an axial compression P. Let the nanobeam be intermediately supported at $x = s_k, k = 1, \ldots, n$, by rigid supports as shown in Figure 29 (assume that $s_0 = 0$ and $s_{n+1} = l$). The current problem consists in the determination of the critical buckling load of a rectangular nanobeam of piecewise constant thickness $h = h_k = const$, for $x \in (s_k, s_{k+1}), k = 0, \ldots, n$.

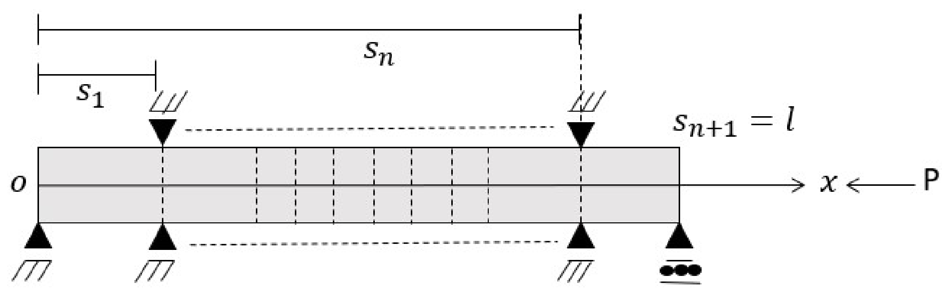

Figure 29: Simply supported nanobeam with intermediate support

The nonlocal constitutive equations corresponding to Eringen's nonlocal theory of elasticity are employed. Thus the equations (2.8) – (2.10) and (2.13) – (2.17) remain valid.
For the current model, the continuity conditions for slope and bending moment can be expressed by the relations

$$w'_{k-1}(s_k) = w'_k(s_k), \qquad k = 1, \dots, n, \tag{6.1}$$

and

$$M_{k-1}(s_k) = M_k(s_k), \qquad k = 1, \dots, n. \tag{6.2}$$

Here the value of the bending moment M is calculated by making use of (2.9), so that the continuity condition (2.19) can be applied for $x = s_k$.
However, there are no transverse displacements at the intermediate rigid supports at $x = s_k, k = 1, \dots, n$. Thus

$$w_{k-1}(s_k) = 0, \qquad k = 1, \dots, n, \tag{6.3}$$

and

$$w_k(s_k) = 0, \qquad k = 1, \dots, n. \tag{6.4}$$

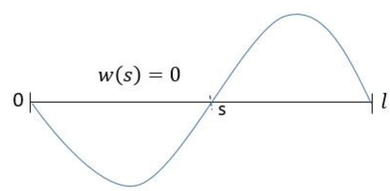

Figure 30: Transverse deflection of nanobeam in the vicinity of an intermediate support

For simply supported nanobeams, the transverse displacement $w(x)$ and the bending moment $M(x)$ must vanish at both ends. In the case of n-stepped nanobeams divided into $n+1$ segments $(s_k, s_{k+1}), k = 0, \dots, n$, the boundary conditions can be expressed as

$$\begin{aligned} w_0(s_0) &= 0, \\ M_0(s_0) &= 0, \\ w_n(s_{n+1}) &= 0, \\ M_n(s_{n+1}) &= 0. \end{aligned} \tag{6.5}$$

.

6.2 Critical buckling load

Consider a one-stepped simply supported nanobeam of length l onto an intermediate rigid support at $x = s$. Let us divide the nanobeam into two segments having deflections w_0 for $x \in (0, s)$ and w_1 for $x \in (s, l)$, respectively.

According to our notation

$$w = w_0, \qquad x \in (0, s), \tag{6.6}$$

and

$$w = w_1, \qquad x \in (s, l). \tag{6.7}$$

The boundary conditions (6.5) with (6.6) and (6.7) infer the following relations

$$w_0 = B_0 sin\lambda_0 x + C_0 x, \qquad x \in (0, s), \tag{6.8}$$

and

$$w_1 = -B_2 sin\lambda_1(l-x) + C_1(x-l), \qquad x \in (s,l), \tag{6.9}$$

where

$$B_2 = \frac{B_1}{cos\lambda_1 l}. \tag{6.10}$$

The continuity of slope w'_k and the bending moment M_k for $k=0,1$ is verified by making use of (6.1) and (6.2) as

$$-B_0\lambda_0 cos\lambda_0 s - C_0 + B_2\lambda_1 cos\lambda_1(l-s) + C_1 = 0, \tag{6.11}$$

and

$$B_0\lambda_0^2 sin\lambda_0 s + B_2\mu\lambda_1^2 sin\lambda_1(l-s) = 0, \tag{6.12}$$

where

$$\mu = \frac{EI_1 - \eta P}{EI_0 - \eta P}, \tag{6.13}$$

here $I_{1,2}$ stand for the area moment of inertia of rectangular beams and correspond to (2.6).

The intermediate conditions (6.3) and (6.4) lead to the equations

$$-B_2 sin\lambda_1(l-s) + C_1(s-l) = 0, \tag{6.14}$$

and

$$B_0 sin\lambda_0 s + C_0 s = 0. \tag{6.15}$$

The equations (6.11), (6.12), (6.14) and (6.15) form a linear algebraic system of equations with respect to B_o, C_o, B_1, C_1. Equalizing its determinant Δ to zero, we can calculate the eigenvalues λ_o and λ_1, which leads to the values of critical buckling load for simply supported nanobeams onto an intermediate rigid support.

6.3 Numerical results

The method developed in the current section is used to determine critical buckling loads of stepped nanobeams with intermediate supports. Numerical calculations are implemented for simply supported nanobeams of constant thicknesses, as well as for one-stepped nanobeams. The geometric and material properties of the nanobeams are taken as $b = 03nm$, $\nu = 0.3$ and $E = 200GPa$. The results of calculations are presented in Figures 31 – 35.

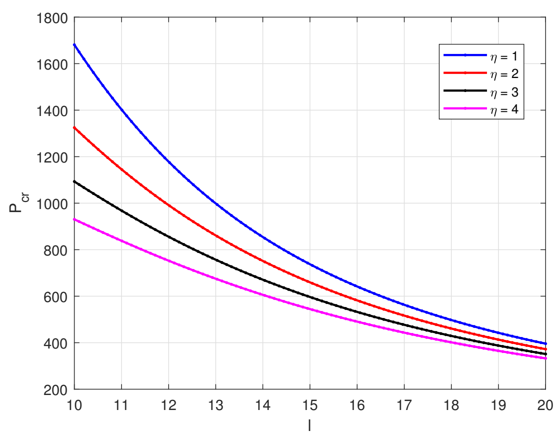

Figure 31: Critical buckling load versus length of simply supported nanobeams onto intermediate rigid supports for different values of nonlocal parameter

Figures 31 – 33 present the results for simply supported nanobeams of constant thickness with intermediate rigid supports. The effect of nonlocal parameter η on the buckling of the nanobeam is presented in Figure 31. Results are calculated for nanobeams of constant thickness $h = 05\ nm$ with intermediate support at the distance from the left edge $s = 4\ nm$. It is known from the nonlocal elasticity theory that $\eta \leq 4$, whereas, $\eta = 0$ corresponds to the classical theory of elasticity. Here, the nonlocal theory of elasticity is applied to embrace the small scale effect on the stability of nanobeams. It can be seen that the value of critical buckling load P_{cr} decreases by increasing the value of the nonlocal parameter η. The presented results also show that the value of critical buckling load decreases monotonically by increasing the length l of the nanobeam, as might be expected.

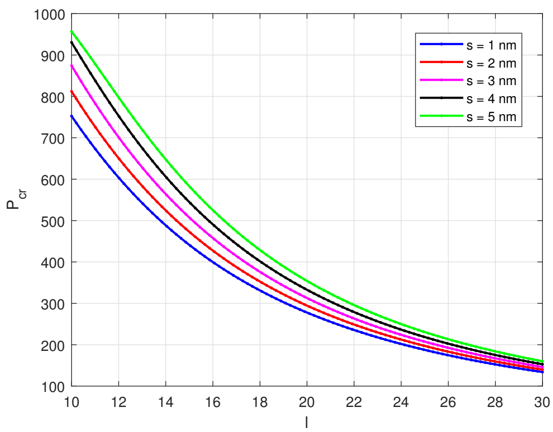

Figure 32: Critical buckling loads of simply supported nanobeams onto intermediate rigid supports for different values of s

Figure 32 illustrates the influence of the intermediate support location s on the critical buckling load of simply supported nanobeams of constant thickness $h = 05\ nm$ and $\eta = 4$. It is shown that the values of the critical buckling loads increase with increasing the value of s.

The results presented in Figure 33 reveal the significant impact of the thickness h on the buckling of the nanobeams with intermediate rigid supports at $s = 4\ nm$ (here $\eta = 4$). From Figure 33, one can observe that the thicker is the nanobeam, the higher is the value of critical buckling load.

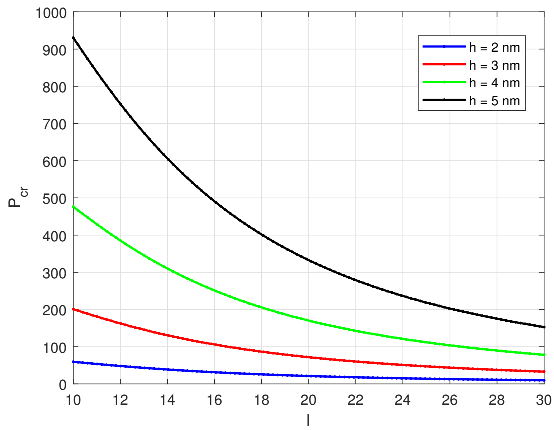

Figure 33: Critical buckling loads of nanobeams with intermediate supports for different thicknesses

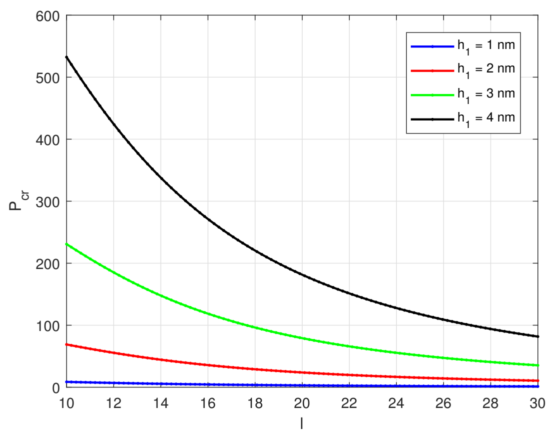

Figure 34: Critical buckling loads of stepped nanobeams with intermediate supports

The results for one-stepped nanobeams with intermediate rigid supports are presented in Figure 34 and Figure 35. In Figure 34, the calculations are carried out for one-stepped nanobeams having rigid supports at the step locations $s = 4\ nm$ (here $h_0 = 5\ nm$ and $\eta = 4$). The critical buckling load of stepped nanobeams exhibits the behaviour which is similar to the case of constant thicknesses. The values of critical buckling loads are much higher than those for stepped nanobeams.

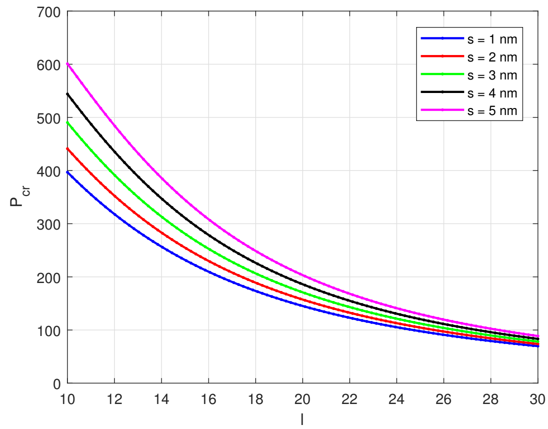

Figure 35: Critical buckling loads of stepped nanobeams onto intermediate rigid supports at different locations

It is also demonstrated in Figure 35 that the effect of the support location on the buckling of nanobeams is analogous. The parameters for stepped nanobeams under consideration are $h_0 = 6\ nm$, $h_1 = 4\ nm$ and $\eta = 4$. It can be seen from Figure 35 that the values of the critical buckling loads increase by increasing s as shown in Figure 32 for nanobeams of constant thicknesses.

By summarizing all the results, one can declare that there is a remarkable impact of the nonlocal parameter, steps, the support location and the other physical parameters on the stability of nanobeams with the intermediate rigid supports. The accuracy of the presented method is verified by the comparison of presented results with the available results in the literature ([109] and [118]), where the results are presented for beams of constant thicknesses with intermediate supports. Although, the literature on the stability of nanobeams with intermediate supports is poor, the results reveal the correlation with the buckling behaviour of macro beams presented in [109] and [118].

7 Conclusions

An analytical method for the determination of critical buckling loads for nonlocal elastic nanobeams weakened with several discontinuities caused by cracks, steps and internal supports has been developed. The nanobeams under compression are treated within the framework of Eringen's nonlocal theory of elasticity to investigate the small size effect on the stability of nanobeams. The nonlocal theory of elasticity for Euler-Bernoulli nanobeams is combined with the linear fracture mechanics to develop an approximate method for the stability analysis of nanobeams and nanocolumns.

Numerical calculations for different support conditions reveal that the critical buckling loads for clamped nanobeams are always higher than those of simply supported and cantilever nanobeams. It is found that there is a significant effect of crack parameters like crack length and the crack location on the stability of stepped nanobeams. The numerical studies show that the value of critical buckling load is higher for the thicker nanobeams as in the case of traditional materials. Also, the critical buckling load is higher in the case of shorter nanobeams.

Results obtained for nanobeams with intermediate supports show that the critical buckling load is sensitive with respect to the location of the intermediate support. The problem of optimization of the nanobeam with internal supports is a topic for the future work.

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Summary

In the current dissertation, the stability analysis of nonlocal elastic nanobeams is carried out. The nanobeams under consideration are of rectangular cross-section with piecewise constant thicknesses subjected to the axial compressions. It is assumed that the nanobeams are weakened by the defects like steps, cracks and internal supports. The cracks are considered to be stable surface cracks that are located at the re-entrant corners of the steps penetrated throughout the thickness of the nanobeam.

To analyse the buckling of nanobeams, an analytical approach has been developed within the framework of Eringen's nonlocal theory of elasticity to embrace the small size effect. The nonlocal theory of elasticity for Euler-Bernoulli nanobeams is combined with the linear fracture mechanics to develop an approximate method for the stability analysis of nanobeams and nanocolumns. The crack effect is considered by coupling the local compliance of the structure with the stress intensity factor which can be calculated by the methods of linear fracture mechanics. The critical buckling loads for axially loaded stepped nanobeams and nanocolumns with cracks and with the internal supports are calculated. The influence of nonlocal parameter, crack parameters, steps, intermediate support location and the other physical parameters on the stability of nanobeams is investigated.

The method developed for stepped nanobeams with cracks is applied to the simply supported, clamped and cantilever nanobeams. The case of stepped nanobeam with intermediate rigid support is studied separately. Since conducting experiments at nanolevel is difficult to handle, the accuracy of the presented methods is verified by the comparison of results with the available work in the literature. MATLAB tools are used to provide significant numerical results.

The dissertation is based on the five papers of the author (four of these are published during the last three years). The dissertation consists of the review of the obtained results, the copies of the paper, the list of the literature

and the CV of the author.

The dissertation is organised as follows. Section 1 contains a historic background of the stability analysis of nonlocal beams, the aim and the structure of the dissertation. In section 2, the nonlocal physical model and the local flexibility of stepped nanobeams with cracks are described in detail. In sections 3, 4 and 5, the method is applied to the nanobeams with different support conditions. The cases of simply supported, clamped and cantilever nanobeams are studied in great details. The influence of the crack parameters and the nonlocal parameters on the buckling of nanobeams is investigated through parametric studies. Finally, in section 6, the method is applied to the stepped nanobeams with additional internal supports. The influence of the support parameters on the stability of nanobeams has been analyzed.

Kokkuvõte

Defektidega astmeliste nanovarraste stabiilsus

Käesolev väitekiri on pühendatud elastsete astmeliste nanovarraste stabiilsuse analüüsile. Eeldatakse, et vaadeldavad nanovardad on ristkülikukujulise ristlõikega ja neile on rakendatud teljesuunaline rõhk ning et astmete nurkades asuvad stabiilsed praod.

Nanotala kriitilise koormuse määramiseks kasutatakse Eringeni poolt välja töötatud mittelokaalse elastsusteooria võrrandeid. Prao mõju hindamiseks seotakse nanotala järeleandlikkus pinge intensiivsuse koefitsiendiga prao tipus ning leitakse kriitilised koormused vabalt toetatud ja jäigalt kinnitatud astmeliste varraste, samuti konsooltala jaoks. Numbrilised tulemused on saadud programmi MATLAB abil. Tulemusi võrreldakse teiste autorite töödega homogeensete nanovarraste korral.

Väitekiri põhineb autori viiel artiklil (neist neli on trükis ilmunud). Väitekiri koosneb sissejuhatusest, kirjanduse ülevaatest ja kuuest peatükist, kirjanduse nimestikust ja autori elulookirjeldusest.

Töö esimeses peatükis esitatakse kirjanduse ülevaade, samuti töö eesmärk ning tema struktuur. Teises peatükis kirjeldatakse nanovarda füüsikalist mudelit ja lokaalse järeleandlikkuse kontseptsiooni detailsemalt. Peatükkides 3, 4 ja 5 rakendatakse seda meetodit vabalt toetatud ja jäigalt kinnitatud nanovarraste ja nanokonsoolide korral. Uuritakse prao pikkuse, samuti teiste parameetrite mõju nanovarraste kriitilisele koormusele. Viimases peatükis rakendatakse kirjeldatud meetodit lisatugedega astmeliste nanovarraste korral.

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List of publications

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