DISSERTATIONES PHYSICAE UNIVERSITATIS TARTUENSIS
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CLASSICAL AND QUANTUM ASPECTS OF GEODESIC MULTIPLICATION

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CLASSICAL AND QUANTUM ASPETS OF GEODESIC MULTIPLICATION

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TARTU UNIVERSITY
PRESS

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Contents

List of Original Publications

This thesis consists of an introductory review part, followed by five research publications. These are listed below and will be referred to in the text as Paper I to Paper V.

I. J. Örd,
The group properties of local loops and odules,
Algebras, Groups and Geometries **10** (1993) 77–86.

II. P. Kuusk, J. Örd, E. Paal,
Geodesic multiplication and the theory of gravity,
J. Math. Phys. **35** (1994) 321–334.

III. P. Kuusk, J. Örd, E. Paal,
Geodesic multiplication and geometrical BRST-like operators,
Proc. Estonian Acad. Sci. Phys. Math. **44** (1995) 437–449.

IV. P. Kuusk, J. Örd, E. Paal,
Quantum kinematics of a test particle in a curved spacetime,
Class. Quantum Grav. **14** (1997) 2917–2926.

V. P. Kuusk, J. Örd,
Kinematics and uncertainty relations of a quantum test particle in a curved spacetime,
Phys. Lett. B **421** (1998) 99–104.

1 Introduction

The four fundamental forces in physics are described with two different theories. The Standard Model, which describes electromagnetic, weak and strong interactions, is using quantum field theory techniques and studies particles and their interactions. On the other hand, gravity is described by the general theory of relativity, which is purely a classical theory. Such a theory is an excellent model for describing physics of large distances and cosmological issues. But what about the short distance behaviour of the gravitational interaction. Can we introduce gravitons as the quanta of the gravitational field? Can we describe gravity as a quantum field — i.e. consider creation, annihilation and exchange of gravitons? A large number of physicists have been attacking the problem of quantum gravity with a variety of different methods. However, these efforts have been unsuccessful already for more than half a century. One of the biggest conceptual problems is that in a quantum field theory one assumes a non-dynamical background space-time, while in the theory of gravity it is exactly the space-time that becomes the dynamical field. A related issue is the problem of time, as one needs to specify a time coordinate in order to construct a quantum field theory.

The problem is made more complicated by the lack of experimental evidence. In a quantum theory of gravitation based on general relativity, one would expect that the fundamental scale at which the classical description breaks down should be set by the Planck's constant $\hbar$, the speed of light c and the gravitational constant G. There is a unique combination of these constants which has the dimension of length, namely the Planck length $l_P = (G\hbar/c^3)^{1/2}$. The magnitude of the Planck length is about 10^{-33} cm. To investigate such short distances, we need energies of the order 10^{19} Gev, which is about 16 orders of magnitude higher than the energies available in today's experiments. Thus, at present the internal consistency of a theory seems to be the only reliable criteria for quantizing gravity.

To quantize gravity, an obvious approach is to apply first the known quantization methods. Although some progress has been achieved, most of these schemes run sooner or later into difficulties and the problem is still far from being solved. Therefore, people have started to search for new unconventional ways to quantize gravity. This thesis is also an attempt to propose such a new technique. However, the problem of quantizing gravity is not directly a subject of this thesis. Instead, a new approach for classical theory of gravity based on nonassociative algebraic systems is proposed and studied, with the further aim to find applications for it in quantum gravity. In this approach, space-time is described as a set of neighbourhoods. Each of these neighbourhoods is equipped with a binary operation called geodesic multiplication of space-time points and becomes thus an algebraic system called local geodesic loop. Space-time curvature turns then out to be equivalent with the nonassociativity of the multiplication.

This new technique allows to study certain quantum aspects like BRST quantization and quantum mechanics in the background gravitational field. Notice that gravity is not quantized here but acts as a nontrivial curved background. Construction of the full quantum theory of gravity based on this approach is by no means obvious. Several proposals can still be speculated. One possibility would be to study the representation theory of geodesic multiplication. This is not an easy task, since the representation theory for nonassociative algebraic systems is still poorly developed. One could also hope that the quantum mechanics and the quantum field theory based on nonassociative algebras would have better behaviour than ordinary theories.

The thesis is organized as follows. The introductory part is intended to give an overview of the related areas and the motivations of this research. In Section 2, a brief summary of the original papers is given. The results of these papers belong to three slightly different subjects, which are described in sections 4, 5 and 6, respectively. To understand the background, some of the main conventional approaches of constructing a quantum theory of gravity are overviewed in Section 3. These include perturbation theory, canonical quantization and string theory. The problems and contradictions that arise are discussed.

In Section 4, nonassociative algebraic systems called geodesic loops are introduced. The main concepts and results relevant to this work are reviewed. The equivalence of the algebraic and the geometric description of a manifold with an affine connection is shown.

BRST quantization is considered in Section 5. Basic features of the BRST operator and the physical states are overviewed. Using the geodesic multiplication, a BRST-like operator can be constructed. Its possible physical meaning is discussed.

Section 6 considers quantum mechanics in a curved space-time. Possible modifications to the canonical commutation relations are discussed. As a consequence of this generalisation minimal uncertainties in positions and momenta can arise, which may cause a regularizing effect in the field theory. This kind of commutation relations arise also when the momentum operators in curved space-time are defined using the geodesic multiplication.

The original papers are appended at the end of the thesis.

2 Summary of Papers

All the original papers are based on a new algebraic approach to the theory of gravity. Essentially this is a purely algebraic formulation of differential geometry, where an affine connection of a manifold is replaced by a certain algebraic structure on the manifold.

In Paper I, the noncommutativity and the nonassociativity of a nonassociative algebraic system called a local loop are studied using techniques familiar from the group theory. The left and the right commutators are calculated. The results have been applied to the local geodesic multiplication in a space-time with an affine connection. Some relations of this algebraic system to the general relativity are demonstrated.

Nonassociative algebraic systems called local geodesic loops and their tangent Akivis algebras are considered in Paper II. As the approach of the nonassociative algebraic systems is not very widely known among physicists, the necessary mathematical formalism is introduced and some recent results obtained in this field are reviewed. The relations between the geometric and the algebraic description of the curved space-time are investigated. The noncommutativity of the geodesic multiplication is studied in more detail, it turns out to be related to the geodesic deviation equation.

Paper III introduces the generators of the local geodesic translations. These are proposed to be the generalizations of the Poincaré translations in the curved space-time. Using these generators, the exterior differential operator in the differentiable manifold can be generalized to give two BRST-like nilpotent operators. As an example, the local geodesic translation matrices are calculated in the space-time of a weak plane gravitational wave.

In Paper IV, a possible model for the quantum kinematics of a test particle in the curved spacetime is proposed. The curved space-time is considered as a set of neighbourhoods, each of which is equipped with geodesic multiplication as a binary operation. The position operators are defined as multiplication by the Riemann normal coordinates and the momentum operators via infinitesimal geodesic translations. The commutation relations of the position and the momentum operators are taken as the quantum kinematic algebra. Using the momentum operators, a BRST-like operator is constructed and its physical meaning is discussed. Explicit form of the commutation relations is calculated for the space-time of a weak plane gravitational wave.

The commutation relations obtained for the space-time of a weak plane gravitational wave are studied more closely in Paper V. The uncertainty relations following from the commutation rules are derived. It turns out, that under certain conditions these imply a minimal uncertainty in momentum.

3 Quantization of Gravity

The basic motivation of this research is to develop a new approach for general relativity and to study the possibilities to apply it in quantum gravity. Up to now no satisfactory theories have been constructed, which would describe quantized gravitational interaction. Therefore, the purpose of this review section is to describe briefly several main attempts to construct the quantum theory of gravity and to illustrate the problems that have arised in these models.

3.1 Particle Physics Approach

One may try to apply the apparatus of the conventional quantum field theory for quantization of gravity. In this approach graviton is perceived as propagating on a background Minkowski spacetime, and is associated with a specific representation of the Poincaré group. From general principles it follows that the graviton should be massless and have spin 2. The absence of mass comes from the usual gravitational inverse-square law. Half-integral values of spin are impossible because of the Pauli exclusion principle, values greater than two will not produce a static force and value one gives a repulsive force between like particles. The remaining values zero and two correspond to Newtonian gravity and general relativity, respectively.

One starts with the Einstein action

$$S = \frac{1}{\kappa^2} \int d^4x R\left(g(x)\right)\left(-\det g(x)\right)^{\frac{1}{2}}, \tag{1}$$

where $\kappa = (16\pi G/c^2)^{1/2}$ is the (dimensional) coupling constant and G is the Newton's constant. The metric tensor is written as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x). \tag{2}$$

The lowest order contribution to the field equations for $h_{\mu\nu}(x)$ is a wave equation for a free massless spin-2 field

$$h_{\mu\nu,\alpha}{}^{\alpha} - h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\nu}{}^{\alpha}{}_{,\alpha\mu} + h^{\alpha}{}_{\alpha,\mu\nu} + \eta_{\mu\nu}(h^{\alpha\beta}{}_{,\alpha\beta} - h^{\alpha}{}_{\alpha,}{}^{\beta}{}_{\beta}) = 0. \tag{3}$$

The field equations are invariant under the gauge transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}, \tag{4}$$

which is just the effect induced on $h_{\mu\nu}$ by an infinitesimal diffeomorhism of Minkowski space generated by the vector field ξ. Using ordinary quantum field theory techniques a theory of gravitons propagating in the background Minkowski spacetime can be constructed [1, 2, 3]. Unfortunately several problems arise with this scheme.

There is no reason to suppose that the causal structure of the Minkowski metric is the physically correct one. It is difficult to discuss cosmological problems, spacetime singularities, black holes or any other features of classical general relativity that involve non-trivial topological structure. Also, the expansion of $g_{\mu\nu}$ is a poor one from the geometrical perspective of classical general relativity. The quantity $g_{\mu\nu}$ will be a genuine metric tensor only for small values of $h_{\mu\nu}$.

In addition to the previous conceptual difficulties, a much more severe problem is the non-renormalisability of the theory [4, 5]. If the expansion (2) is inserted into the Einstein–Hilbert action (1), the Lagrangian obtained has interactions that are non-polynomial, derivatively-coupled and with a dimensional coupling constant — each of which is a recipe for non-renormalisability for a quantum field theory in four spacetime dimensions.

3.2 Canonical Quantization

At the transition from classical to quantum theory the central role is played by the algebraic structure of a field theory. A classical theory is said to be quantized, when it is modified in a certain way, with the aim of making it applicable for quantum mechanical systems. One of the procedures used thereby is the canonical quantization. The idea is to replace canonical phase space coordinates p, q by hermitean operators $\hat{p}$, $\hat{q}$ acting on some Hilbert space. The Poisson brackets $\{p, q\} = 1$ of the phase space coordinates are replaced by the canonical commutation relations $[\hat{p}, \hat{q}] = i\hbar$. This means that the Poisson algebra of functions $f(p, q)$ is mapped into the algebra of hermitean operators.

In case of general relativity, the canonical quantization has been investigated already for a long time [1, 6]. In this case, spacetime is foliated into a one-parameter family of spacelike three-surfaces, each of which is considered to be a surface of constant time. This is called the $3+1$-decomposition. The foliation is introduced using the shift vector N^a, $a = 1, 2, 3$, and the lapse function N, given in every space-time point. The space-time metric can be expressed via the three-metric γ_{ab} by

$$ds^2 = \gamma_{ab}(dx^a + N^a dt)(dx^b + N^b dt) - (N dt)^2. \tag{5}$$

The canonical variables are the three-metric components γ_{ab} and the corresponding canonical momenta π_{ab}, which can be calculated from the Einstein action (1)

$$\pi^{ab} = \frac{\delta L}{\delta \dot{\gamma}_{ab}}. \tag{6}$$

The canonical momenta corresponding to the lapse function and to the shift vector are zero. Using the canonical variables the action can be rewritten in the following first order form

$$S = \int d^3x dt (\pi^{ab}\dot{\gamma}_{ab} - NC - N^a C_a). \tag{7}$$

Here N and N^a are in the role of the Lagrange multipliers, and C and C_a are the constraints

$$\begin{aligned} C_a(x) &= -\nabla_b \left(\gamma_{ac}(x)\pi^{cb}(x)\right), & (8)\\ C(x) &= \frac{\kappa^2}{e}\left(\gamma_{ab}(x)\gamma_{cd}(x) - \frac{1}{2}\gamma_{ac}(x)\gamma_{bd}(x)\right)\pi^{ac}(x)\pi^{bd}(x) \\ &- \frac{e}{\kappa^2}R(\gamma(x)), & (9)\end{aligned}$$

where $e = (\det \gamma_{ab}(x))^{1/2}$. Introducing the basic Poisson brackets

$$\{\gamma_{ab}(x), \pi^{cd}(x')\} = \delta^c_{(a}\delta^d_{b)}\delta(x,x') \tag{10}$$

leads to the following algebra of the constraints

$$\begin{aligned} \{C_a(x), C_b(x')\} &= -C_b(x)\partial_a^{x'}\delta(x,x') + C_a(x')\partial_b^x\delta(x,x'), & (11)\\ \{C_a(x), C(x')\} &= C(x)\partial_a^x\delta(x,x'), & (12)\\ \{C(x), C(x')\} &= \gamma^{ab}(x)C_a(x)\partial_b^{x'}\delta(x,x') - \gamma^{ab}(x')C_a(x')\partial_b^x\delta(x,x'). & (13)\end{aligned}$$

Notice that this is not the Lie algebra of the group of diffeomorphisms Diff(M) of the space-time manifold M, even though this was the invariance group of the original theory. The reason for this is that the space-time diffeomorphisms generally do not preserve the foliation, mixing space and time. The Dirac algebra (11)–(13) is essentially the Lie algebra of Diff(M) projected along, and normal to, a spacelike hypersurface.

To proceed, one has to impose the constraints either on the classical level or on the quantum level. Solving constraints on the classical level removes parts of the metric tensor and makes it very difficult to interpret the resulting quantum theory. The alternative is to impose the constraints as the operator equations for the physically allowed states. Unfortunately, several difficulties arise in completing this scheme, e.g. the non-linearity and the non-polynomiality of the constraints, which introduces severe regularization and operator ordering problems. One of the most promising ways of trying to solve the constraints has been via the use of the Ashtekar canonical variables [7, 8], which substantially simplify the constraints. Using these new variables, several solutions to the quantum constraints have been found. However, two important ingredients are still missing: the inner product and observables. Without these, one cannot calculate any physical quantities.

3.3 String Theory

String theory [9] is an approach of constructing quantum theory, including quantum gravity, where the system that is quantized is not general relativity. Historically, the starting point was the Polyakov action

$$S[q,X] = \frac{1}{4\pi\alpha}\int_W d^2\sigma\sqrt{\det q(\sigma)}q^{ij}(\sigma)\partial_i X^\mu(\sigma)\partial_j X^\nu(\sigma)g_{\mu\nu}(X(\sigma)), \quad (14)$$

which describes the propagation of a one-dimensional string in the d-dimensional space-time. Here q_{ij} is a metric on the two-dimensional string worldsheet W, X^μ are the string fields which map the worldsheet into the space-time manifold M, and $g_{\mu\nu}$ is a background metric on M. The constant α is related to the string tension and is assumed to be of the order of the Planck length. It turns out that from the physical point of view the most interesting strings are closed strings, whose worldsheet are tubes. Unlike in the ordinary quantum field theory, in case of interacting strings there is no well-defined point at which the splitting or merging of the strings occurs. This is suggested to smear out the local (ultraviolet) divergences in field theory.

How does gravity arise from strings? The low energy limit of the string theory is an ordinary field theory. In this limit the string tension tends to infinity, the strings degenerate into points, and the effect of the extended structure is negligible. The classical system is invariant under the conformal transformations

$$q_{ij}(\sigma) \to \Omega(\sigma)q_{ij}(\sigma). \quad (15)$$

In general, this invariance will be lost when the two-dimensional fields q_{ij} and $X^\mu(\sigma)$ are quantized, meaning that there is a conformal anomaly. The reason for this is that the quantum theory needs to be regularized, and the regularization procedure violates the conformal invariance. To cancel the anomalies, one needs to require the vanishing of the trace of the energy-momentum tensor of the two-dimensional quantum field theory. This calculation can be done only in string perturbation theory and gives a set of equations for the metric tensor $g_{\mu\nu}$ and other fields. To the lowest order in α these equations are the Einstein equations. However if we include the first order corrections, we obtain a correction to the vacuum Einstein equations

$$R_{\mu\nu} + \frac{\alpha}{2} R_{\mu\lambda\sigma\tau} R_{\nu}^{\lambda\sigma\tau} = 0. \tag{16}$$

Another way to see how gravity appears from string theory, is to consider the spectrum of quantum states in string theory. It turns out that it includes a massless spin-2 particle, gauge transformations for which coincide with (4). This is the graviton.

The Polyakov action corresponds to the bosonic string. As more realistic models, superstrings and heterotic strings are considered. Basic requirements for quantum theory such as demanding space-time supersymmetry and absence of conformal anomalies restrict the number of possible string theoretic models down to five. For all these the dimensionality of space-time appears to be $d = 10$. However, several important questions are still unsolved. It is unclear, how these different string theoretic models are related with each other. Which one of them is correct and what to do with the others? Also, to make contact with the 4-dimensional world, the redundant dimensions have to be compactified. Unfortunately, the number of possible compactifications is huge.

During the last years, numerous important results have been obtained in string theory, which suggest that all known string theoretic models are equivalent, describing perturbation theory around different vacuum states. For a review of these new developments see e.g. [10, 11, 12]. Whether these ideas will succeed to solve the problems mentioned above, remains yet to be seen.

4 Nonassociative Algebras

Essential role in modern theoretical physics and geometry has been played by the associative algebras, including group theory. However, algebraic properties of physical systems do not always satisfy the axiom of associativity.

As a generalization nonassociative algebraic systems can be considered. The first known nonassociative algebra was the algebra of octonions. Since then several classes of nonassociative algebras have been studied like Jordan algebras, alternative algebras and Maltsev algebras. However, it should be mentioned that in cases of most thoroughly studied nonassociative algebras the nonassociativity is restricted, i.e. there exists some kind of identity which controls the way associativity is broken.

There exist also several applications to physical models. Octonions are related to the 7-sphere S^7 and to the group $E_8 \times E_8$, which are relevant in the superstring theory and in the supergravity. The nonassociativity of octonions has even been suggested [13] to explain the confinement of quarks. An important application is also the theory of quasigroups of transformations [14], which can be used to construct generalized gauge theories.

4.1 Quasigroups and Loops

New perspectives for gravity opens the nonlinear geometric algebra — an approach formulated by Sabinin [15, 16] a few years ago. For physical applications the essential feature of this approach is that nonassociativity is algebraically equivalent to the differential geometrical notion of curvature.

Such nonassociative algebraic systems as quasigroups and loops have been considered by mathematicians a long time already. Quasigroup is a set G of elements with a binary operation (multiplication) which has the following property: in the equation $xy = z$, the knowledge of any two elements specifies the third one uniquely. Quasigroup with a unit element e is called the loop. The loop is a natural generalisation of a group: it satisfies all group axioms except associativity. Also, the left and the right inverse elements x_L^{-1}, x_R^{-1} defined by $x_L^{-1}x = e$, $x\,x_R^{-1} = e$ are in general not equal. A thorough treatment of the subject of quasigroups is given e.g. by Albert [17, 18] and Beloussov [19], that of loops is presented by Bruck [20]. In analogy with the Lie groups, we can also talk about differentiable loops, if the loop is a differentiable manifold and the loop operations are specified with differentiable functions. The loop is called local if the operations are determined only in some neighbourhood of the unit element.

In the group theory, a Lie group is characterized by its structure constants which describe the noncommutativity of the multiplication and fully determine the local structure of the group. In case of the loop there are two sets of structure constants [21], $C^{\mu}_{\nu\rho}$ and $A^{\mu}_{\nu\rho\sigma}$, which measure the deviation of the loop multiplication from commutativity and associativity. The first set is the same as in case of groups, the second set is related to the nonassociativity, in group theory these vanish identically. The structure constants are defined by

$$
\begin{aligned}
\left[(yx)_L^{-1}(xy)\right]^{\mu} &= C^{\mu}_{\nu\rho}x^{\nu}y^{\rho} + \ldots, &(17)\\
\left\{[x(yz)]_L^{-1}\,[(xy)z]\right\}^{\mu} &= A^{\mu}_{\nu\rho\sigma}x^{\nu}y^{\rho}z^{\sigma} + \ldots, &(18)
\end{aligned}
$$

where dots mean higher order terms.

The noncommutativity and nonassociativity of the loop were investigated in detail by Akivis [22]. He used techniques already familiar from the group theory [23] and generalized them for the nonassociative case.

As in the case of the Lie groups, the multiplication in the loop determines the following left (L) and right (R) infinitesimal translation matrices:

$$
\begin{aligned}
(xy)^{\mu} &= y^{\mu} + L^{\mu}_{\nu}(y)x^{\nu} + \ldots, \qquad L^{\mu}_{\nu}(y) \equiv \left.\frac{\partial (xy)^{\mu}}{\partial x^{\nu}}\right|_{x=e}, &(19)\\
(xy)^{\mu} &= x^{\mu} + R^{\mu}_{\nu}(x)y^{\nu} + \ldots, \qquad R^{\mu}_{\nu}(x) \equiv \left.\frac{\partial (xy)^{\mu}}{\partial y^{\nu}}\right|_{y=e}. &(20)
\end{aligned}
$$

Also, instead of left- and right-invariant vector fields as in the group theory, we have only two preferred frame fields

$$
L_i(x) \equiv L^{\mu}_{i}(x)\partial_{\mu}, \qquad R_i(x) \equiv R^{\mu}_{i}(x)\partial_{\mu}. \qquad (21)
$$

We can now introduce the tangent algebras of loops. In the group theory, the Lie algebra coincides with the tangent space at the unit element. Now consider the tangent space T_eM of the unit element e of a loop M. It turns out to coincide with a binary-ternary algebra. Consider vectors $X, Y, Z \in T_eM$. The binary operation $[X,Y]$ and the ternary operation (X,Y,Z) are defined by means of the structure constants C^{i}_{lm} and A^{i}_{lmn} of the loop [22, 24, 25]:

$$
\begin{aligned}
[X,Y]^i &:= C^{i}_{lm}X^lY^m, &(22)\\
(X,Y,Z)^i &:= A^{i}_{lmn}X^lY^mZ^n. &(23)
\end{aligned}
$$

The tangent algebra is thus a binary-ternary algebra, and it needs not be a Lie algebra.

In group theory, a representation of the group is a homomorphism into a group of linear transformations in a vector space. This construction cannot be realized in case of non-associative algebras. Instead, one considers the so-called birepresentations, where each element of the algebra is represented by two transformations, the left and the right action of this element. It should also be mentioned, that the representation theory of nonassociative algebras is still poorly developed, and cannot be used directly in the physical applications.

4.2 Geodesic Multiplication

In an affinely connected space there is a natural way to define a local loop in some neighbourhood M_e of every point e. In the neighbourhood M_e an operation of multiplication can be defined by means of parallel displacement of geodesic lines. The geodesic lines (autoparallels) $x^\mu(t)$ satisfy the following differential equations:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0, \tag{24}$$

where $\Gamma^\mu_{\nu\rho}(x)$ denote the affine connection coefficients. The differential equation for the parallel transport of a tangent vector X' reads:

$$\frac{dX'^\mu}{ds} + \Gamma^\mu_{\nu\rho}\frac{dy^\nu}{ds}X'^\rho = 0, \tag{25}$$

Two new mappings can be defined based on these equations. An exponential mapping

$$\exp_e : T_eM \to M_e, \quad X \mapsto x(1) \tag{26}$$

is determined by solutions of (24) at initial values $x^\mu(0) = e^\mu$, $\frac{dx^\mu}{dt}|_{t=0} = X^\mu$. A parallel transport mapping

$$\tau^e_y : T_eM \to T_yM \tag{27}$$

along a geodesic line $y(s)$ emerging from e is given as a solution of (25) at initial conditions $X'^\mu(0) = X^\mu$. Using these mappings the local geodesic multiplication of points $x, y \in M_e$ can be introduced [26]:

$$x \cdot y = (\exp_y \circ \tau^e_y \circ \exp_e^{-1})x. \tag{28}$$

With respect to this multiplication the neighbourhood M_e becomes a local loop. The point e becomes the unit element of the loop. Notice that the concept of locality is essential here. To guarantee the uniqueness of the multiplication, the given neighbourhood has to be such that the geodesic lines emerging from e do not intersect.

Akivis [21] has found an analytical expression for the binary operation in the local geodesic loop in terms of Taylor series up to the third order terms. He has also proved that in the Riemann normal coordinates the structure constants of the geodesic loop can be expressed in terms of the torsion and curvature tensors of the affinely connected space. See Paper II for a more detailed review.

Following these ideas, Sabinin [15, 16] investigated a differentiable manifold with an affine connection and proved that it is equivalent to a manifold with a geoodular structure. More precisely, using the properties of the affine connection, the geodesic loops attached to different points of the manifold can be shown to satisfy certain compatibility conditions called geoodular identities. The converse is also true. Consider a manifold, with a local loop defined in the neighbourhood of each point, such that the loops at different points satisfy geoodular identities. Then an affine connection can be uniquely introduced as a derived object. The precise mathematical formulation of a geoodular structure is reviewed in Paper II.

In general relativity, the gravitational interaction is considered to be a manifestation of the space-time curvature, and it is usually proclaimed to be geometrical theory. The algebraic reformulation of differential geometry described above allows us to study classical gravity using algebraic tools and methods. When studying local properties of space-time, the crucial notion is that of a local geodesic loop. However, when we want to discuss global issues concerning the whole space-time, the geoodular structure introduced by Sabinin has to be considered.

All the original papers of this thesis are based on this algebraic reformulation of differential geometry. Commutativity and associativity of the local geodesic loop are studied in Papers I and II. The third order terms in the expansions of the left and the right commutators are calculated In Paper I. These higher order terms turn out to be related to the geodesic deviation equation, which allows to give it an algebraic reformulation. This is proven in Paper II. Paper II also gives a rather detailed review for most of the concepts mentioned earlier in this section. Properties of the generators of the local geodesic translations $L_i(g)$ and $R_i(g)$ are considered in Papers III, IV and V. It turns out that these generators are useful in several physical applications and lead us also to quantum theory. Using these, two BRST-like operators are constructed in Paper III. In Paper IV, they are used to define the momentum operators of a test particle moving in a curved space-time, which leads to modifications of the canonical commutation relations.

5 Becchi–Rouet–Stora–Tyutin Quantization

The Becchi–Rouet–Stora–Tyutin (BRST) operators are powerful tools for quantizing gauge field theories, and have been extensively studied in recent years (see [27] and references therein). A gauge theory is a theory which is described in terms of non-physical variables. Different values of these field variables corresponding to one physical configuration are related by gauge transformations. The physical degrees of freedom are invariant under these gauge transformations but the field variables are not. This technique is used mainly because it makes the theory manifestly covariant and thus easier to describe. When doing canonical quantization, the gauge symmetry gives constraint equations in addition to the ordinary equations of motion. The presence of constraints reduces the number of degrees of freedom back to the appropriate physical ones. The BRST quantization is a modification of the canonical quantization. In the following, we concentrate mainly on the features of the BRST operator, and skip the details of the exact quantization procedure. As the constraints play crucial role in the BRST quantization of a gauge theory, we assume that there are m independent constraints ϕ_a, $a = 1, \ldots, m$. We also assume that all the constraints are first class, meaning that the second class constraints (if any) have been eliminated by means of the Dirac procedure. For simplicity we consider ϕ_a to be real and bosonic. More general situations can also be considered — e.g. fermionic constraints are assigned odd Grassmann parity, the Poisson bracket between two odd functions being symmetric.

5.1 The BRST Quantization Method

The basic idea of the BRST quantization is that as a first step, the phase space of the system is extended by introducing additional ghost fields. The resulting classical system is then quantized in a standard way, replacing the canonical variables and the ghosts by Hermitean operators. To restore the space of physical states the extended phase space is reduced using the BRST condition.

The phase space is extended in such a way that for each independent constraint ϕ_a a ghost field η^a is introduced, together with its conjugate momentum $\mathcal{P}_a$, such that

$$\{\eta^a, \mathcal{P}_b\} = \delta^a_b. \tag{29}$$

Both ghosts and their momenta have odd Grassmann parity, and commute with the original phase space variables. All the phase space variables are assigned a ghost number: the original variables have ghost number zero, the ghosts η^a have ghost number one, whereas their momenta $\mathcal{P}_a$ have ghost number minus one.

Let us consider a generator

$$Q = \phi_a \eta^a + \sum_{i=1}^{N} C^{a_1 \dots a_i} \mathcal{P}_{a_1} \dots \mathcal{P}_{a_i}. \quad (30)$$

It can be shown that the functions $C^{a_1 \dots a_i}$ can be chosen so that Q is nilpotent $\{Q, Q\} = 0$, real $Q^* = Q$, and has odd Grassmann parity. To zeroth order in the momenta $\mathcal{P}_a$, the transformation generated by Q is just a gauge transformation with parameter η^a. In the case of Yang-Mills theory, Q coincides with the generator of the supersymmetry transformation invented independently by Becchi, Rouet, Stora [28, 29] and Tyutin [30]. However, it is called the BRST generator even for more general systems associated with open algebras.

In (30), N is called the rank of the theory. As the constraints ϕ_a are all first class, they obey an algebra

$$\{\phi_a, \phi_b\} = U^c_{ab} \phi_c. \quad (31)$$

In general U^c_{ab} may be complicated functions involving the fields and one distinguishes between different cases using the concept of the rank. Rank zero is an Abelian theory. In case of rank one the structure functions can be chosen to be constants $U^c_{ab} = f^c_{ab}$, which happens e.g. in case of a Lie algebra. The corresponding BRST generator is expressed by

$$Q = \phi_a \eta^a - \frac{1}{2} f^c_{ab} \eta^a \eta^b \mathcal{P}_c. \quad (32)$$

Rank bigger than one occurs for theories with an open algebra.

In quantum theory, the BRST generator becomes by construction nilpotent, $\mathbf{Q}^2 = 0$, and Hermitian operator. It is convenient to define the ghost number operator $\mathbf{N_g}$ on the extended phase space, which is given by

$$\mathbf{N_g} = i \eta^a \mathcal{P}_a, \quad (33)$$

and has the following properties

$$[\mathbf{N_g}, \eta^a] = \eta^a \quad , \quad [\mathbf{N_g}, \mathcal{P}_a] = -\mathcal{P}_a. \quad (34)$$

From the relations (34), it follows that if a state $|\psi\rangle$ has definite ghost number n, then states $\eta^a|\psi\rangle$, $\mathcal{P}_a|\psi\rangle$ have ghost number $n+1$ and $n-1$, respectively.

Quantum observables are zero-ghost number operators $\mathbf{A}$ which commute with $\mathbf{Q}$, i.e. which are invariant under BRS transformations

$$[\mathbf{A}, \mathbf{Q}] = 0. \tag{35}$$

In classical theory this means that the zeroth-order term of A in the ghost variables has weakly vanishing brackets with the constraints and is thus gauge invariant. Because of the constraints there is some ambiguity in the construction of $\mathbf{A}$. Therefore, two observables $\mathbf{A}$ and $\mathbf{A}'$ differing by $[\mathbf{K}, \mathbf{Q}]$, for some $\mathbf{K}$, should be identified

$$\mathbf{A} \sim \mathbf{A}' = \mathbf{A} + [\mathbf{K}, \mathbf{Q}]. \tag{36}$$

For this identification to be possible, (36) should not change the expectation value of $\mathbf{A}$ between physical states. Therefore, we can consider only a subset of all states as physical states. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are two physical states, one must have

$$\begin{aligned} \langle\psi_1|\mathbf{A}|\psi_2\rangle &= \langle\psi_1|\mathbf{A}'|\psi_2\rangle \\ &= \langle\psi_1|\mathbf{A}|\psi_2\rangle + \langle\psi_1|\mathbf{KQ}|\psi_2\rangle - \langle\psi_1|\mathbf{QK}|\psi_2\rangle. \end{aligned} \tag{37}$$

This will be the case if any physical state $|\psi\rangle$ is annihilated by the BRST operator

$$\mathbf{Q}|\psi\rangle = 0. \tag{38}$$

Because of the nilpotency of $\mathbf{Q}$, any state of the form $\mathbf{Q}|\chi\rangle$, where $|\chi\rangle$ is any state, satisfies also (38). However, those states have zero norm, and the expectation value of any observable between such a state and another physical state is necessarily zero,

$$\langle\psi|\mathbf{AQ}|\chi\rangle = \langle\psi|\mathbf{QA}|\chi\rangle = 0. \tag{39}$$

The physical states are thus identified with the cohomology classes of $\mathbf{Q}$.

Notice that a representative can be chosen from each equivalence class which has zero ghost number. So formally, BRST quantization can be proven to be equivalent with the other quantization techniques. This has indeed been the case for all known quantized systems. The advantage of the BRST quantization is that it has been useful also in the cases where other techniques have proven to be insufficient.

The BRST quantization method can be used in the path integral representation of the evolution operator to calculate transition amplitudes between physical states. Batalin, Fradkin and Vilkovisky [31, 32, 33] have developed a general method for doing this, based on BRST invariance. The path integral is formulated in the extended phase space and contains a gauge fixing function ψ. It is then possible to show that this path integral is independent of the choice of ψ.

One may want to apply the BRST quantization method also in case of gravity. However, several problems arise when doing that. Firstly, the gauge group in gravity is the group of diffeomorphisms. This group has several features that make it extremely difficult to handle, e.g. the exponential map from the Lie algebra to the group is not a local homeomorphism. Secondly, the group of diffeomorphisms is not compatible with $3+1$-decomposition used in canonical quantization, because diffeomorphisms in general mix space and time. One may still consider the algebra of the constraints arising in $3+1$-decomposition as the starting point of the BRST-quantization. Application of the general procedure of [33] was first considered in [34, 35]. But the conceptual and technical problems mentioned in Sec. 3 are still present, and one has no advantages compared with the canonical quantization.

5.2 Generalized BRST Operators

When doing BRST quantization, as we saw earlier, ghost fields are introduced and a nilpotent BRST operator is constructed. An explicit construction of the BRST operator is based on the algebra of gauge transformations and its nilpotency is a consequence of the Jacobi identity.

Several authors have proposed also more general nilpotent BRST-like operators. These are introduced purely kinematically, without referring to any underlying action. Once constructed, one then studies the cohomologies of the BRST operator and assumes that the physical states can be identified with its cohomology classes.

The geometrical BRST operator proposed by Bars and Yankielowicz [36, 37] has been introduced keeping in mind string field theory, but can be used also in ordinary differential geometry. They considered the generators G_i, whose action may include differentiation with respect to some base space. In this case their commutation rules close generally with a set of structure functions F_{ij}^k

$$[G_i, G_j] = F_{ij}^k G_k. \tag{40}$$

The structure functions F^k_{ij} may be field dependent and do not generally commute with G_i. Whatever the meaning of G_i is, a generalized Bianchi identity can be derived by commuting three G's and taking a cyclic sum. Corresponding to each generator a ghost field c^i and an antighost field b_i are introduced, which satisfy the anticommutation rules

$$\{c^i, b_j\} = \delta^i_j. \tag{41}$$

The generalized BRST-operator can then be defined as

$$Q = c^i G_i - \frac{1}{2} F^k_{ij} c^i c^j b_k. \tag{42}$$

The nilpotency of Q turns out to be a consequence of the generalized Bianchi identities.

An analogous geometrical BRST operator for a differentiable manifold with zero curvature but non-zero torsion tensor has been given by Okubo [38]. It is expressed by

$$Q = c^\mu(x)\partial_\mu + \frac{1}{2} c^\mu(x) c^\nu(x) b_\lambda(x) \theta^\lambda_{\mu\nu}(x), \tag{43}$$

where $\theta^\lambda_{\mu\nu}(x)$ is the torsion tensor of the manifold, and $c^\mu(x)$ and $b_\mu(x)$ are anticommuting ghost fields and their conjugates. As in the previous case, the nilpotency of the BRST operator relies on the Bianchi identities. In general case, the cohomologies of Q are rather involved. However, it is possible to find a simpler sub-system, such that Q restricted to this defines a cohomology isomorhic to that of the standard de Rham cohomology.

It turns out that Okubo's construction can be followed also in case of local geodesic loops. Using the left or the right local frame vector fields a new affine connection can be defined. The coefficients of this affine connection depend in a sophisticated way on the metric connection. These frame fields imply that the local geodesic loop is parallelizable, and the Riemann curvature tensor of this new connection vanishes. Therefore we can apply Okubo's construction. Notice that the BRST-like operator is defined locally, and in general may not be determined globally. To study globally defined BRST-like operator, one possibly has to consider patching conditions of the geoodular structure mentioned in the previous section.

Two nilpotent BRST-like operators expressed in terms of the left and the right local frame vector fields with corresponding structure functions of the geodesic multiplication are constructed in Paper III. These BRST-like operators are discussed also in [39]. In Paper IV, where the momentum operators are introduced, it is argued that only one of them acquires a meaning analogous to the conventional BRST operator. Following the standard BRST-quantization, the cohomology classes of this operator can be conjectured to form the space of the physical states of a quantum test particle in a curved space-time.

6 Quantum Mechanics in Curved Space-time

Standard quantum mechanics and quantum field theory are usually formulated in the flat Minkowski space-time. In order to be consistent with gravity, they have to be somehow modified. Quantum field theory in curved space-time [40] can be used to calculate quantum effects in a gravitational field. Quantum mechanics in curved space-time can also be constructed, using e.g. the path integral approach, where the integration is defined by a time-slicing procedure. In this section, quantum mechanics of a single particle in a curved space-time is considered. Several aspects like modified uncertainty relations and regularization issues are discussed. Note that gravity itself is not quantized here, but acts only as a nontrivial curved background.

6.1 Generalized Commutation Relations

A possible generalisation of quantum mechanics in flat space is to consider certain modifications of the Heisenberg algebra. This kind of ideas have been developed e.g. in case of quantum groups [41, 42], where the algebra of functions on a manifold becomes noncommutative. Although quantum groups lead also to deformed Heisenberg algebras, here a more general modification of the Heisenberg algebra is considered, where the canonical commutation relations obtain small corrections

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar(\delta_{ij} + \alpha_{ijkl}\mathbf{x}_k\mathbf{x}_l + \beta_{ijkl}\mathbf{p}_k\mathbf{p}_l + \ldots). \tag{44}$$

In general, also

$$[\mathbf{x}_i, \mathbf{x}_j] \neq 0 \quad \text{and} \quad [\mathbf{p}_i, \mathbf{p}_j] \neq 0. \tag{45}$$

The resulting quantum mechanics and quantum field theory has been studied in [43]. One of the consequences of this generalisation is that also the Heisenberg uncertainty relations become modified. It is well-known that for any two non-commuting observables $\mathbf{A}$, $\mathbf{B}$ there exists an uncertainty relation

$$\Delta A \Delta B \geq \frac{\hbar}{2}|\langle[\mathbf{A}, \mathbf{B}]\rangle|, \tag{46}$$

where the uncertainty of the observable $\mathbf{A}$ in a quantum state $|\psi\rangle$ is defined by

$$(\Delta A)^2 = \langle\psi|(\mathbf{A} - \langle\psi|\mathbf{A}|\psi\rangle)^2|\psi\rangle. \tag{47}$$

Because of the modified commutation relations (44) there may now be minimal uncertainties in position and momenta. The mechanism can be most easily seen in one dimension. Consider the following commutation relations:

$$[\mathbf{x},\mathbf{p}] = i\hbar(1+\alpha\mathbf{x}^2+\beta\mathbf{p}^2). \tag{48}$$

The corresponding uncertainty relation is

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1+\alpha(\Delta x)^2+\alpha\langle\mathbf{x}\rangle^2+\beta(\Delta p)^2+\beta\langle\mathbf{p}\rangle^2). \tag{49}$$

This implies nonzero minimal uncertainties in **x**- as well as in **p**- measurements. Now e.g. Δx cannot be made arbitratrily small because of the $(\Delta p)^2$ term. The minimal uncertainties depend on the expectation value in position and momentum via

$$k = \alpha\langle\mathbf{x}\rangle^2+\beta\langle\mathbf{p}\rangle^2 \tag{50}$$

and are explicitly

$$\Delta x_0 = \sqrt{\frac{(1+k)\beta\hbar^2}{1-\alpha\beta\hbar^2}} \tag{51}$$

$$\Delta p_0 = \sqrt{\frac{(1+k)\alpha\hbar^2}{1-\alpha\beta\hbar^2}} \tag{52}$$

For fixed but sufficiently small α and β one finds ordinary quantum mechanical behaviour at medium scales. The term proportional to β contributes in the ultraviolet (for large $\langle\mathbf{p}^2\rangle$). Similarly the term proportional to α leads to an infrared effect.

6.2 Minimal Uncertainties

Physically, the idea is that nonvanishing minimal uncertainties in positions and momenta could be effects caused by gravity. The possible reasons for modifications in the ultraviolet and in the infrared are rather different and will be considered separately.

The presence of a finite Δp_0, relevant in the infrared, may be motivated from the absence of plane waves (i.e. of sharp localisations in momentum space) on generic curved spaces. The existence of lower bounds to the possible determination of momentum has been suggested in [43, 44]. Because of the quantum mechanical uncertainty relations, momentum is a nonlocal property of a particle. When we measure a momentum vector, in the tangent space of which point of the manifold should it live? Doing quantum mechanics in flat space, we can identify all tangent spaces, which cannot be done in the presence of curvature due to the path-dependence of parallel transport. This might have an effect which becomes more significant, the more the wave function is spread over spacetime.

The presence of a finite Δx_0, physically relevant at the Planck scale, can be motivated from studies in string theory and quantum gravity [45, 46, 47]. Intuitively, a quantum uncertainty in the position of a particle implies an uncertainty in its momentum and therefore, due to gravity-energy interaction, also implies an uncertainty in the geometry, which in turn introduces an additional uncertainty in the position of the particle. In other words, in order to resolve extremely small distances test particles need extremely high energies, the gravity effects of which will eventually significantly disturb the spacetime structure. In case of gravity, several considerations and thought experiments [47, 48, 49] suggest the existence of the minimal length, which is of the order of the Planck length. For example, consider a measurement of the distance by sending a light signal from a clock to a mirror and measuring its return time. According to the uncertainty principle of quantum mechanics, the clock has the following spread in velocities

$$\delta v = \frac{\delta p}{m} > \frac{\hbar}{2m\delta l}, \tag{53}$$

where m is the mass of the clock and δl is its spread in position. Nonvanishing spread in velocity of the clock causes an additional uncertainty in the length measurement:

$$\delta l(t) = t\delta v > \frac{\hbar t}{2m\delta l(0)} = \frac{\hbar l}{mc\delta l(0)}. \tag{54}$$

Minimizing $\delta l(0) + \delta l(t)$, the uncertainty in the length measurement is given by

$$(\delta l)^2 > \frac{\hbar l}{mc}. \tag{55}$$

However, one cannot make the clock arbitrarily massive, since this would turn it into the black hole, making it impossible to send a signal from the clock to the mirror. In particular, the mirror cannot be inside the Schwarzschild radius of the clock (for more precise estimate see [48])

$$l > \frac{Gm}{c^2}. \tag{56}$$

From (55) and (56) we conclude that

$$\delta l > l_P, \tag{57}$$

where $l_P = (G\hbar/c^3)^{1/2}$ is the Planck length.

Similar observations concerning minimum length can be done in string theory. There have been a series of calculations of high energy scattering amplitudes. The results of these suggest the existence of a minimum length that can be reached by string [45, 50, 51], and which has been associated with some sort of effective uncertainty relation in the form [52]

$$\Delta x > \frac{\hbar}{\Delta p} + \alpha' \Delta p. \tag{58}$$

Here the linear term is due to the behaviour of strings at high energy: they spread out, their size being proportional to their energy. Intuitively, it can be argued that the internal quantum fluctuations of the string as an extended object stop it degenerating to a point. Also, in some of the compactification schemes, the scattering amplitudes are invariant under the transformation $R \to 1/R$ where R is the radius of the internal space. This suggests the existence of a minimum value for this radius.

Maggiore [46] has obtained a relation similar to (58) without considering strings. Instead, he discussed a gedanken experiment in which the radius of the apparent horizon of an extremal black hole is measured using the Hawking radiation. The resulting uncertainty in the radius of the horizon indicates that the concept of the black hole is not well defined if the mass is smaller than the Planck mass. Also an underlying algebraic structure can be found [53, 54, 55], which reproduces the generalized uncertainty relations and implies the existence of the minimal observable length. This algebraic structure is given by the quantum deformation of the Poincaré algebra

$$[\mathbf{x}_i, \mathbf{x}_j] = -\frac{\hbar^2}{4\kappa^2} i\epsilon_{ijk}\mathbf{J}_k, \tag{59}$$

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\delta_{ij}\left(1 + \frac{\mathbf{p}^2 + m^2}{4\kappa^2}\right)^{1/2}, \tag{60}$$

where $\mathbf{J}_i$ is the angular momentum and κ is the deformation parameter.

The importance of the finite minimal uncertainties Δx_0, Δp_0 is that their presence is suggested to have ultraviolet and infrared regularizing effect in field theory. Infrared regularity of Euclidean propagators $1/(\mathbf{p}^2 + m^2c^2)$ for all generalized commutation relations which imply a minimal uncertainty in momentum Δp_0 has been proved by Kempf [56]. Analogously, the minimal length Δx_0 is assumed to free the quantum amplitudes from ultraviolet divergencies.

The representation theory of the minimal length uncertainty relation has been considered in [57]. In terms of the representation theory the generalized commutation relations (44) mean that there are no physical states which are eigenstates of $\mathbf{x}$ or $\mathbf{p}$, since an eigenstate would have zero uncertainty in position or momentum, respectively.

For this reason we have to distinguish between symmetric and self-adjoint operators. An operator $\mathbf{A}$ is called symmetric, if $\langle \mathbf{A}\psi_1|\psi_2\rangle = \langle\psi_1|\mathbf{A}\psi_2\rangle$. In order for $\mathbf{A}$ to be self-adjoint, it must have a basis of eigenvectors. This distinction arises only in case of unbound operators in Hilbert space [58]. As was shown in [59], the minimal uncertainty in $\mathbf{A}$ causes it to become merely symmetric and not self-adjoint. The symmetry of $\mathbf{A}$ alone is sufficient to ensure the realness of expectation values $\langle\psi|\mathbf{A}|\psi\rangle$. Giving up self-adjointness is essential, otherwise the existence of zero uncertainty (i.e. eigen-) states would be required.

6.3 Quantum Test Particle in Curved Space-time

In relativistic theories, a major role is played by the representations of the Poincaré group, which is the symmetry group of the flat space-time. A generic curved space-time doesn't allow symmetry groups. The Lorentz group can be considered as the symmetry group of flat tangent spaces, but the status of the Poincaré translations is unclear.

The idea of modifying canonical commutation relations in curved space-time is followed in Paper IV, where a possible model for kinematics of a quantum test particle in the background of the gravitational field is studied. This is based on the idea that in a curved space-time, the geodesic translations can be considered to be the generalizations of the Poincaré shifts [60] and reduce to these in case of the flat space-time.

Flat space-time with the global Cartesian coordinates is imitated by considering a set of neighbourhoods, where every neighbourhood is endowed with a local Riemann normal coordinate system. According to Sec.4, each of these neighbourhoods can be considered to be a (local) geodesic loop. Following non-relativistic kinematics, the coordinate operators are defined as multiplication with Riemann normal coordinates and the momentum operators are defined as the generators of local geodesic translations. Using the algebra of local frame fields the canonical commutation relations can be derived. The new commutation relations turn out to be more general than (44), but reduce to these when expanded in the neighbourhood of some space-time point. A somewhat similar construction for position and momentum operators in curved space-time was proposed by Kempf [43], but he did not notice its connection with geodesic loops.

As an example that can be analytically worked out, the explicit form of these commutation relations are calculated for the case when the background space-time is that of a weak plane gravitational wave. In paper V, the corresponding uncertainty relations are calculated. It is shown, that under certain conditions they imply minimal uncertainties in momenta.

The algebra of Paper IV can also be considered as a generalization of the algebra proposed by Krause [61, 62]. Krause considered coordinate and momentum operators **Q**, **P** acting on a Hilbert space that carries a regular representation of the group. The coordinate operators can be defined by multiplication with group parameters and the momentum operators using the left and the right translation operators

$$\mathbf{Q}^a \mid q\rangle = q^a \mid q\rangle, \tag{61}$$

$$\mathbf{P}_a \mid q\rangle = i\hbar R^a_b(q)\partial_b \mid q\rangle \tag{62}$$

These operators obey the following commutation relations

$$[\mathbf{Q}^a, \mathbf{P}_b] = i\hbar R^a_b(\mathbf{Q}), \tag{63}$$

$$[\mathbf{P}_a, \mathbf{P}_b] = i\hbar f^c_{ab}\mathbf{P}_c. \tag{64}$$

Group quantization construction is based on a regular representation of an Abelian or a non-Abelian group. It cannot be followed directly in case of geodesic loops, since the representation theory of non-associative algebras is still poorly developed. The non-Abelian group quantization introduced by Krause is a modification of the canonical quantization first proposed by Weyl [63].

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Summary

General relativity is known to be a valid theory of gravity on the classical level, but all the attempts to construct a quantum theory have so far been unsuccessful. The aim of this thesis is to study a new approach to general relativity, with the further aim of applying this technique to the quantum theory. The main idea is that the differential geometrical description of the manifold with an affine connection is replaced by the algebraic description. This technique has been rather long known for geometry, but is not widely known for physicists.

According to this approach, the space-time manifold is equipped with an algebraic structure in such a way that nonassociative algebraic systems called local geodesic loops are assigned to all reasonable (non-singular) points of the manifold. A local geodesic loop can be constructed only in such a neighbourhood of each point, where the geodesics emerging from the given point do not intersect in the given neighbourhood. The operation of multiplication in the local geodesic loop (geodesic multiplication) is constructed using geodesic lines and the parallel transport mapping. The nonassociativity of this multiplication is a direct consequence of the curvature of the space-time. Also, the local geodesic loops attached to different points of the manifold are in general not independent, but satisfy certain identities called the geo-odular identities.

In the framework of this algebraic description of differential geometry, several concepts of the general relativity can be given an algebraic formulation. It has been shown that the geodesic deviation equation is related to the noncommutativity of the geodesic multiplication and can be obtained from the expansion of the commutator of the local geodesic loop.

The geodesic translations are considered to be the generalizations of the Poincaré shifts of flat space-time. The generalized coordinate and momentum operators in curved space-time are introduced. The coordinate operators act as multiplication with the Riemann normal coordinates and the momentum operators are defined via infinitesimal geodesic translations. Using the construction of the geodesic multiplication and the algebra of the generators of the geodesic transformations modified canonical commutation relations for a quantum test particle in a curved space-time are proposed. This leads to a generalized kinematic algebra which coincides with the canonical Poisson algebra only in the case of flat space-time.

The generalized momentum operators can be used to define a BRST-like operator, the nilpotency of which follows from the differential geometrical

Bianchi identities. The BRST-like operator is constructed without referring to any underlying action. Its possible physical meaning is discussed.

To illustrate all these developments, detailed calculations are performed for the space-time of a weak plane gravitational wave. Working in the linear approximations in the wave amplitude, the equations of geodesics and of parallel transport can be easily integrated even without using expansions in coordinates. The explicit form of the generalized commutation relations and the BRST-like operator are calculated and the uncertainty relations following from the commutation relations are derived. It its shown that under certain conditions these uncertainty relations imply a minimal uncertainty in momentum.

Geodeetilise korrutamise klassikalised ja kvantaspektid

Kokkuvõte

Üldrelatiivsusteooria on sobiv gravitatsiooniteooria klassikaliste nähtuste kirjeldamiseks, kuid ükski gravitatsiooni kvantteooria loomise katse pole seni olnud edukas. Käesoleva töö eesmärk on uurida üldrelatiivsusteooria uut käsitlusviisi, kaugemaks eesmärgiks on rakendada seda kvantteoorias. Meetodi põhiidee seisneb selles, et afiinse seostusega muutkonna kirjeldamine diferentsiaalgeomeetria meetoditega asendatakse algebralise kirjeldusega. See käsitlusviis on olnud ammu teada geomeetrias, kuid pole eriti tuntud füüsikute hulgas.

Vastavalt sellele meetodile on aegruumi muutkonnale antud algebraline struktuur sellisel viisil, et iga mittesingulaarse punktiga on seotud lokaalseks geodeetiliseks luubiks nimetatav algebraline süsteem. Lokaalne geodeetiline luup on võimalik konstrueerida vaid punkti sellises ümbruses, kus punktist lähtuvad geodeetilised ei lõiku. Korrutamisoperatsioon lokaalses geodeetilises luubis (geodeetiline korrutamine) defineeritakse, kasutades geodeetilisi jooni ja paralleelse ülekande operaatorit. Aegruumi kõveruse tõttu on see korrutamine mitteassotsiatiivne. Üldjuhul ei ole eri punktides defineeritud geodeetilised luubid sõltumatud, vaid rahuldavad identsusi, mida nimetatakse geoodulaarseteks identsusteks.

Diferentsiaalgeomeetria sellise algebralise kirjelduse abil on võimalik anda mitme üldrelatiivsusteooria mõiste algebraline kirjeldus. Käesolevas töös on näidatud, et geodeetilise deviatsiooni võrrand on seotud geodeetilise korrutamise mittekommutatiivsusega ning on saadav lokaalse geodeetilise luubi kommutaatori kõrgemat järku liikmetest.

Geodeetilisi nihkeid kõveras aegruumis võib vaadelda tasase ruumi Poincaré nihete üldistusena. Seetõttu on kasutusele võetud üldistatud koordinaadi- ja impulsioperaatorid kõveras aegruumis. Koordinaadioperaatorite toimeks on korrutamine Riemanni normaalkoordinaatidega ja impulsioperaatorid on defineeritud infinitesimaalsete geodeetiliste translatsioonide kaudu. Kasutades geodeetilist korrutamist ja geodeetiliste teisenduste algebrat, on välja pakutud modifitseeritud kvantmehaanilised kommutatsioonieeskirjad testosakese jaoks kõveras aegruumis. Tulemuseks on üldistatud kinemaatiline algebra, mis ühtib kanoonilise Poissoni algebraga vaid tasase aegruumi korral.

Üldistatud impulsioperaatorite abil on võimalik defineerida BRST-sar-

nane operaator, mille nilpotentsus järeldub otseselt diferentsiaalgeomeetrilistest Bianchi identsustest. See BRST-sarnane operaator on konstrueeritud ilma mõjufunktsionaali kasutamata. On püütud analüüsida selle operaatori füüsikalist tähendust.

Nende tulemuste illustreerimiseks on teostatud detailsed arvutused nõrgale tasasele gravitatsioonilainele vastava aegruumi korral. Geodeetiliste ja paralleelse ülekande võrrandid on lineaarses lähenduses laineamplituudi järgi lihtsalt integreeritavad isegi ilma reaksarenduseta koordinaatide järgi. On arvutatud üldistatud kommutatsioonieeskirjade ja BRST-sarnase operaatori konkreetne kuju ning kommutatsioonieeskirjadest järelduvad määramatuse relatsioonid. On näidatud, et kindlatel tingimustel järeldub nendest määramatuse relatsioonidest minimaalimpulsi olemasolu.

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Publications

I

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THE GROUP PROPERTIES OF LOCAL LOOPS AND ODULES

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Abstract

A notion of a loop as a nonassociative generalisation of a group is briefly reviewed. The noncommutativity and the nonassociativity of the multiplication in a local loop is studied generalising the group theory. As a special case, a local geodesic loop in a manifold with affine connection is considered. Some relations of this algebraic system to general relativity are demonstrated.

1 Loops and odules

Quasigroup [1, 2] is a set G of elements with a binary operation (multiplication) which has the following property: in the equation $gh = k$, the knowledge of any two elements specifies the third one uniquely. Quasigroup with a unit element ϵ is called the *loop* [1, 2].

For each element g of the loop there exist the left and right inverse elements g_L^{-1}, g_R^{-1} defined by $g_L^{-1}g = \epsilon, g\,g_R^{-1} = \epsilon$, in general $g_L^{-1} \neq g_R^{-1}$.

The multiplication in loop needs not be commutative and associative. There may exist such a triple of elements g, h, k in G that

$$gh \neq hg \quad , \quad (gh)k \neq g(hk) \; . \tag{1}$$

An *odule* [3, 4] is a loop with an additional operation, multiplication of the elements of the loop with scalars, so that the following identities are satisfied:

$$(tu)g = t(ug) \; , \tag{2}$$

$$1g = g \; , \tag{3}$$

$$(t+u)g = (tg)(ug) \; . \tag{4}$$

Here $t, u \in \mathbf{R}$ and g is arbitrary element of the odule. From eq.(4) it follows that $(gg)g = g(gg)$, this property is called monoassociativity.

Due to the properties (3) and (4), the set $\{tg\}_{t\in R}$ for every fixed element $g \in G$ is a one-parameter subgroup of the odule. Therefore, the left and right inverse elements in the odule are equal:

$$g_L^{-1} = g_R^{-1} = -1g \; .$$

In the following, we restrict our attention only to *local differentiable* loops and odules, i.e. we assume that G is a differentiable manifold and that all operations are determined in some neighbourhood of the unit element and are represented by differentiable functions.

2 Commutator and associator in the loop

Following M.Akivis [5], let us investigate the local structure of a loop using expansions that are familiar from the group theory [6].

Let g, h be two elements of a local differentiable loop and let us choose the coordinate system so that the unit element e has coordinates $e^i = 0$. As in group theory [6] the product $k = gh$ can be given in terms of coordinates by the following expansion:

$$k^i = g^i + h^i + a^i_{mn} g^m h^n + b^i_{lmn} g^l g^m h^n + d^i_{lmn} g^l h^m h^n + \dots \quad (5)$$

where dots mean higher order terms with respect of g^i and h^i. It is easy to check that the expressions for the left and right inverse elements of g are:

$$(g_L^{-1})^i = -g^i + a^i_{jk} g^j g^k + (d^i_{jkl} - b^i_{jkl} - a^i_{mj} a^m_{kl}) g^j g^k g^l + \dots \quad (6)$$
$$(g_R^{-1})^i = -g^i + a^i_{jk} g^j g^k + (b^i_{jkl} - d^i_{jkl} - a^i_{jm} a^m_{kl}) g^j g^k g^l + \dots \quad (7)$$

Their equality in group theory follows from the requirement of the associativity of the multiplication.

The deviation of a loop from commutativity and associativity can be measured by the *left structure constants* $\underset{L}{C}{}^i_{lm}$ and $\underset{L}{A}{}^i_{lmn}$ defined by

$$[(hg)_L^{-1}(gh)]^i = \underset{L}{C}{}^i_{lm} g^l h^m + \dots \quad (8)$$
$$\left\{[g(hk)]_L^{-1}[(gh)k]\right\}^i = \underset{L}{A}{}^i_{lmn} g^l h^m k^n + \dots \quad (9)$$

or by the *right structure constants* $\underset{R}{C}{}^i_{lm}$ and $\underset{R}{A}{}^i_{lmn}$:

$$[(gh)(hg)_R^{-1}]^i = \underset{R}{C}{}^i_{lm} g^l h^m + \dots \quad (10)$$
$$\left\{[(gh)k][g(hk)]_R^{-1}\right\}^i = \underset{R}{A}{}^i_{lmn} g^l h^m k^n + \dots \quad (11)$$

M.Akivis [5] has demonstrated, that the left and right structure constants are equal and can be expressed by

$$C^i_{lm} = a^i_{lm} - a^i_{ml} \,. \quad (12)$$
$$A^i_{lmn} = 2b^i_{lmn} - 2d^i_{lmn} + a^i_{pl} a^p_{mn} - a^i_{lp} a^p_{mn} \,. \quad (13)$$

This is also true in group theory, only in that case $A^i_{lmn} \equiv 0$.

Here we shall study also the third order terms in the expansion of the commutators $[(hg)_L^{-1}(gh)]$ and $[(gh)(hg)_R^{-1}]$. Using eq.(6) for the point hg, we get

$$
\begin{aligned}
[(hg)_L^{-1}]^i \;=\; & -h^i - g^i + a^i_{jk} g^j h^k + a^i_{jk} g^j g^k + a^i_{jk} h^j h^k + \\
& + (d^i_{jkl} - b^i_{jkl} - a^i_{mj} a^m_{kl}) g^j g^k g^l + \\
& + (d^i_{jkl} - b^i_{jkl} - a^i_{mj} a^m_{kl}) h^j h^k h^l + \\
& + (2d^i_{jkl} - b^i_{jkl} - 2b^i_{jlk} - \\
& - a^i_{mj} a^m_{kl} + a^i_{jm} a^m_{lk} - a^i_{ml} a^m_{jk}) g^j g^k h^l + \\
& + (d^i_{jkl} + 2d^i_{klj} - 2b^i_{jkl} - 2b^i_{klj} + \\
& + a^i_{km} a^m_{lj} - a^i_{mj} a^m_{kl} - a^i_{mk} a^m_{jl}) g^j h^k h^l + \ldots \; .
\end{aligned}
$$

Analogous expressions can be found for $[(hg)_R^{-1}]^i$. Finally, after direct but lengthy calculations we obtain

$$
\begin{aligned}
[(hg)_L^{-1}(gh)]^i \;=\; & (a^i_{jk} - a^i_{kj}) g^j h^k - \\
& - (d^i_{kjl} - b^i_{jlk} + a^i_{jm} a^m_{lk} - a^i_{jm} a^m_{kl})(g^j h^k g^l - h^j g^k h^l) \quad (14) \\
[(gh)(hg)_R^{-1}]^i \;=\; & (a^i_{jk} - a^i_{kj}) g^j h^k - \\
& - (d^i_{kjl} - b^i_{jlk} + a^i_{mj} a^m_{lk} - a^i_{mj} a^m_{kl})(g^j h^k g^l - h^j g^k h^l) \quad (15)
\end{aligned}
$$

We see that although the first order terms on right-hand side of eqs.(14),(15) are equal and determine unique structure constants C^i_{jk} of the loop, the following terms are different for the left and the right commutator. In case of an odule or a group, where $g_L^{-1} = g_R^{-1}$, the difference of the commutators (14) and (15) expresses the noncommutativity of the multiplication.

3 Local geodesic multiplication

Let us consider a manifold M with an affine connection. We can now introduce a local loop in a neighbourhood of each of its points, defining the product of two points by means of parallel displacement of geodesics [7, 8].

Let us denote by $T_e M$ the tangent space of a fixed point $e \in M$. Consider a local geodesic (autoparallel) path $t \mapsto g(t; X)$ in M through the point e with a tangent vector $X \in T_e M$ at e such that $g^i(0; X) = e^i$. The exponential mapping $X \mapsto g := \mathrm{Exp}_e X := g(1; X)$ at e is known [9] to be a local diffeomorphism of a suitable neighbourhood of the origin of $T_e M$ onto the corresponding (normal) neighbourhood of $e \in M$. Choose in M another local geodesic arc $h(s; Y)$ through the point e in the direction of $Y \in T_eM$. Performing the parallel transport of X along this geodesic arc, we obtain at $h := \mathrm{Exp}_e Y$ the tangent vector $X' \in T_hM$. Now, draw the local geodesic arc through h in the direction of X', and mark point $\mathrm{Exp}_h X'$ on it. This point is called the product of g and h, and it will be denoted as gh. Explicitly, the multiplication formula reads [7]

$$gh \equiv R_h g = (\mathrm{Exp}_h \circ \tau_h^e \circ \mathrm{Exp}_e^{-1})g \,, \tag{16}$$

where $\tau_h^e : T_eM \mapsto T_hM$ denotes the parallel transport mapping of tangent vectors from T_eM into T_hM along the unique local geodesic arc joining the points e and h: $\tau_h^e(X) = X'$.

Analytically, the geodesic lines (autoparallels) are determined by the equation

$$\frac{d^2g^i}{dt^2} + \Gamma^i_{jk}\frac{dg^j}{dt}\frac{dg^k}{dt} = 0 \tag{17}$$

and parallel transport of a vector X along a curve $g^i(t)$ is given by the equation

$$\frac{dX^i}{dt} + \Gamma^i_{jk}\frac{dg^k}{dt}X^j = 0 \,. \tag{18}$$

The *local geodesic loop* at e can be constructed in such a neighbourhood M_e of e where all required exponential mappings are well defined local diffeomorphisms. Then the neighbourhood M_e of e with the multiplication rule (16) turns out to be a local differentiable loop [3, 4, 7, 8] with e as a unit element. The loop is denoted henceforth by M_e as well.

A local odule attached to the point e can be constructed by defining the multiplication with a scalar $t \in \mathbf{R}$ as follows [3, 4]:

$$tg = (\mathrm{Exp}_e t \mathrm{Exp}_e^{-1})g \,. \tag{19}$$

From the definition of exponential mapping it follows that the line $\{tg\}_{t\in R}$ coincides with a geodesic arc $\mathrm{Exp}_e(tX)$ with the tangent vector $X = \mathrm{Exp}_e^{-1}g$ and with t as a canonical parameter.

4 Commutator and associator in the local geodesic loop

Let us choose the Riemannian normal coordinates for local coordinates in geodesic loop M_e. In this case the equations of geodesic lines emerging from the unit element are simply

$$g^i(\tau) = X^i\tau \quad , \quad X^i \in T_e M \; . \tag{20}$$

Then it follows from the equation of geodesic lines (17) that

$$\Gamma^i_{jk} g^j g^k = 0 \; . \tag{21}$$

Let us denote the *torsion* and *curvature* tensors by S^i_{jk} and R^i_{jkl}, respectively. The sign conventions we use are:

$$\begin{aligned} S^i_{jk} &= -\Gamma^i_{[jk]} \; , \\ R^i_{jkl} &= -\Gamma^i_{j[k,l]} - \Gamma^m_{j[k}\Gamma^i_{|m|l]} \; . \end{aligned}$$

The square brackets and the parenthesis around indices denote antisymmetrization and symmetrization, for example:

$$\begin{aligned} \Gamma^i_{[jk]} &= \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) \; , \\ \Gamma^i_{(jk)} &= \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) \; . \end{aligned}$$

We follow now the procedure of the previous section and solve the equations of geodesic lines (17) and parallel transport of tangent vectors (18) by using Taylor expansion [8] to find the coefficients in expansion for the product of two points (5). The result is:

$$\begin{aligned} a^i_{jk} &= S^i_{jk}(e) \; , \\ b^i_{jkl} &= -\frac{1}{2}\Gamma^i_{(jk),l}(e) \; , \\ d^i_{jkl} &= -\frac{1}{2}\left(\Gamma^i_{j(k,l)}(e) - S^i_{m(k}(e) S^m_{|j|l)}(e)\right) \; . \end{aligned}$$

The formulae for the inverse elements reduce simply to

$$(g_L^{-1})^i = (g_R^{-1})^i = -g^i + o(\rho^3) \; . \tag{22}$$

We see that $g_L^{-1} = g_R^{-1}$ as expected in case of an odule.

It turns out that *noncommutativity and nonassociativity of the local geodesic loops are intimately related to the torsion and the curvature.* The direct computations [8] in Riemannian normal coordinates show that the structure constants are given by

$$C^i_{lm} = 2S^i_{lm}(\epsilon) . \tag{23}$$

$$A^i_{lmn} = R^i_{lmn}(\epsilon) - \nabla_n S^i_{lm}(\epsilon) . \tag{24}$$

where ∇_n is the covariant differentiation operator.

The commutators (14),(15) take the form:

$$\begin{aligned}
[(hg)_L^{-1}(gh)]^i &= C^i_{jk}g^j h^k - (A^i_{jkl} - \frac{1}{2}C^i_{mj}C^m_{lk})(g^j h^k g^l - h^j g^k h^l) + \ldots \\
&= 2S^i_{jk}(\epsilon)g^j h^k - \Big(R^i_{jkl}(\epsilon) - \nabla_l S^i_{jk}(\epsilon) - \\
&\quad - 2S^i_{mj}(\epsilon)S^m_{lk}(\epsilon)\Big)(g^j h^k g^l - h^j g^k h^l) + \ldots .
\end{aligned} \tag{25}$$

$$\begin{aligned}
[(gh)(hg)_R^{-1}]^i &= C^i_{jk}g^j h^k - (A^i_{jkl} + \frac{1}{2}C^i_{mj}C^m_{lk})(g^j h^k g^l - h^j g^k h^l) + \ldots \\
&= 2S^i_{jk}(\epsilon)g^j h^k - \Big(R^i_{jkl}(\epsilon) - \nabla_l S^i_{jk}(\epsilon) + \\
&\quad + 2S^i_{mj}(\epsilon)S^m_{lk}(\epsilon)\Big)(g^j h^k g^l - h^j g^k h^l) + \ldots .
\end{aligned} \tag{26}$$

In case of a Riemannian space with vanishing torsion, the only nontrivial structure constants are those of the associator, $A^i_{lmn} = R^i_{lmn}(\epsilon)$. This simplifies greatly calculations and now the third order terms in the expansions of the associators (9),(11) can also be found. For this purpose the knowledge of fourth order terms in the expansion of the product

$$\begin{aligned}
k^i &= g^i + h^i - \frac{1}{2}\Gamma^i_{jk,l}(\epsilon)g^j g^k h^l - \frac{1}{2}\Gamma^i_{j(k,l)}(\epsilon)g^j h^k h^l - \\
&\quad - \frac{1}{6}\Gamma^i_{(jk,l)m}(\epsilon)g^j g^k g^l h^m - \frac{1}{4}\Gamma^i_{jk,lm}(\epsilon)g^j g^k h^l h^m - \\
&\quad - \frac{1}{6}\Gamma^i_{j(k,lm)}(\epsilon)g^j h^k h^l h^m
\end{aligned} \tag{27}$$

and the inverse elements

$$(g_L^{-1})^i = (g_R^{-1})^i = -g^i + o(\rho^4) \tag{28}$$

is required. Direct calculations give us the following result:

$$\begin{aligned}\left\{[g(hk)]_L^{-1}[(gh)k]\right\}^i &= R^i_{jkl}(\epsilon)g^j h^k k^l + \\ &+ \frac{1}{2}\Big(-\Gamma^i_{(jk.l)m}(\epsilon) + \Gamma^i_{jk.lm}(\epsilon)\Big)g^j g^k h^l k^m + \\ &+ \frac{1}{2}\Big(-\Gamma^i_{(jk.l)m}(\epsilon) + \Gamma^i_{j(k.lm)}(\epsilon)\Big)g^j h^k h^l k^m + \\ &+ \frac{1}{2}\Big(-\Gamma^i_{jk.lm}(\epsilon) + \Gamma^i_{j(k.lm)}(\epsilon)\Big)g^j h^k k^l k^m \ . \qquad (29)\end{aligned}$$

The expression for $\left\{[(gh)k][g(hk)]_R^{-1}\right\}^i$ is the same up to fourth order terms.

In the Riemannian coordinates the derivatives of the affine connection coefficients can be expressed [10] in terms of curvature tensor only:

$$\begin{aligned}\Gamma^i_{jk.l}(\epsilon) &= -\frac{4}{3}R^i_{(jk)l}(\epsilon)\ , \\ \Gamma^i_{jk.lm}(\epsilon) &= -\frac{4}{3}R^i_{jk(l.m)}(\epsilon) - R^i_{(k|j|l.m)}(\epsilon)\ .\end{aligned}$$

Taking into account the Bianchi identity and other symmetry properties of curvature tensor. the associator can be transformed to the form

$$\begin{aligned}\left\{[g(hk)]_L^{-1}[(gh)k]\right\}^i &= R^i_{jkl}(\epsilon)g^j h^k k^l - \\ &- \frac{2}{3}R^i_{(jk)l,m}(\epsilon)g^j g^k h^l k^m + \\ &+ \frac{1}{3}\Big(R^i_{jkm.l}(\epsilon) - R^i_{kjl.m}(\epsilon)\Big)g^j h^k h^l k^m + \\ &+ \frac{2}{3}R^i_{jk(l.m)}(\epsilon)g^j h^k k^l k^m\ . \qquad (30)\end{aligned}$$

The third term in right-hand side also turns out to be symmetric with respect to indices k and l. though this symmetry is not obvious from the first sight.

From eq.(25) we see that the left commutator now reads

$$[(hg)_L^{-1}(gh)]^i = -R^i_{jkl}(\epsilon)(g^j h^k g^l - h^j g^k h^l) + \ldots \qquad (31)$$

and that the first nonvanishing term in the right commutator coincides with that in eq.(31):

$$[(gh)(hg)_R^{-1}]^i = -R^i_{jkl}(\epsilon)(g^j h^k g^l - h^j g^k h^l) + \ldots\ . \qquad (32)$$

Note that although in a Riemannian space without torsion the structure constants C^i_{jk} vanish, the geodesic loop is not commutative due to the non-vanishing associator $A^i_{jkl} = R^i_{jkl}(\epsilon)$ that determines the next terms in the commutators (8),(10).

The terms in right-hand side of eqs.(31),(32) are reminiscent of those in the geodesic deviation equation

$$\frac{\delta^2 V^i}{\delta t^2} = -2R^i_{jkl}U^j V^k U^l \ . \tag{33}$$

where $U^i = \frac{dx^i}{dt}$ is the tangent vector of a geodesic line and $V^i \delta v$ is the vector that connects a geodesic with the neighbouring geodesic. A more detailed investigation [11] reveals, that in a space with nonvanishing torsion only right commutator (26) contains terms that occur in the geodesic deviation equation [12]

$$\frac{\delta^2 V^i}{\delta t^2} = -\frac{\delta}{\delta t}(2S^i_{jk}U^k V^j) - 2R^i_{jkl}U^j V^k U^l \ .$$

The given approach gives us the possibility to reformulate some notions of general relativity in algebraic terms. The further consequences of this in classical and quantum theory of gravity need still closer investigation.

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II

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Geodesic multiplication and the theory of gravity

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Nonassociative algebraic systems called local geodesic loops and their tangent Akivis algebras are considered. The construction of geo-odular structure of a manifold with an affine connection is briefly reviewed. A possible role of these algebraic structures in classical and quantum gravity is discussed.

I. INTRODUCTION

In general relativity, the gravitational interaction is considered to be a manifestation of the space–time curvature, and it is usually proclaimed to be a geometrical theory. The algebraic structure used in constructing a quantum theory of gravity is the algebra of constraints obtained in the framework of the canonical (Arnowitt–Deser–Misner) approach.[1]

Recently a purely algebraic formulation of differential geometry has been elaborated by Sabinin[2,3] and considered by Akivis.[4] In the present article we investigate in more detail the relations between the geometric and the algebraic description of a curved space–time with the further aim of proposing a new framework for quantum gravity.

The article is organized as follows. In Sec. II, the basic algebraic notions of the quasigroup, the loop, and the odule are reviewed. In Sec. III, differentiable loops and odules and their tangent Akivis algebras are considered. In Sec. IV, the construction of the local geodesic multiplication of points of a manifold M with an affine connection is described. It seems to be long known for geometry,[2,5] but is not widely known for physicists. An algebraic system with the local geodesic multiplication as a binary operation is called the local geodesic loop. For affine spaces, all geodesic loops are Abelian groups and the geodesic multiplication can be represented by the ordinary vector addition rule. In Sec. V, the geodesic product of two points in a space–time describing a weak plane gravitational wave is calculated explicitly in a linear approximation. In Sec. VI, compatibility conditions[2,3] for geodesic loops attached to different points of a manifold M with an affine connection are presented. In Sec. VII, a covariant derivative is given in terms of the differential of the right translation map R_g of the geodesic loop. In Sec. VIII, the commutator and the associator of the local geodesic loop are calculated in Riemannian normal coordinates. The Akivis formulas[4] reveal that the structure constants of local geodesic loops are determined by the torsion and the curvature tensors. In Sec. IX, an algebraic formulation is given to the geodesic deviation equation. In Sec. X, a possible role of the tangent Akivis algebra of a geodesic loop in quantum gravity is briefly discussed.

II. LOOPS AND ODULES

At first let us introduce some basic algebraic notions. A quasigroup[6,7] is a set G of points with a binary operation (multiplication) which has the following property: in the equation $gh=k$, the knowledge of any two elements specifies the third one uniquely. A quasigroup with a unit element e is called a loop.[6,7] We also use the notation $g \overset{e}{\cdot} h$ for the product.

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For each element g of the loop there exist the left and the right inverse elements g_L^{-1}, g_R^{-1} defined by $g_L^{-1}g=e$, $gg_R^{-1}=e$.

As in the case of groups one can define the left (L) and the right (R) translations by

$$gh=L_gh=R_hg. \tag{1}$$

From the definition of the quasigroup it follows that these translations are bijections. The multiplication in the loop is not necessarily commutative and associative. There may exist such a triple of points g, h, k in the loop that

$$gh\neq hg, \quad (gh)k\neq g(hk). \tag{2}$$

The nonassociativity of the loop can be measured by the deviation from unity of the following transformations:

$$L_{gh}^{-1}L_gL_h, \tag{3}$$

$$R_{gh}R_g^{-1}R_h^{-1}, \tag{4}$$

$$R_gL_h^{-1}R_g^{-1}L_h. \tag{5}$$

An odule[2,3] is a loop with an additional operation, a multiplication of the points of the loop with scalars, so that the following identities are satisfied:

$$(tu)g=t(ug), \tag{6}$$

$$1g=g, \tag{7}$$

$$(t+u)g=(tg)(ug). \tag{8}$$

Here $t,u\in\mathrm{R}$ and g is an arbitrary point of the odule. Sabinin[2] calls property (8) monoassociativity. It follows immediately that the odule is power associative: $g^mg^n=g^{m+n}$ for all $m,n\in N$. This can easily be checked by considering that $g^n=ng$.

In the odule the left and right inverse elements are equal as a consequence of the monoassociativity property. Really, taking $t=1$ and $u=-1$ in Eq. (8), we obtain

$$e=g(-1g), \quad g_R^{-1}=-1g.$$

On the other hand, taking $t=-1$ and $u=1$, we get

$$e=(-1g)g, \quad g_L^{-1}=-1g.$$

III. ANALYTIC LOOPS AND AKIVIS ALGEBRAS

The quasigroup G is called differentiable (analytic) if G is a differentiable manifold and the knowledge of any two elements in the equation $gh=k$ specifies analytically the local coordinates of the third one. Analogously we also define an analytic loop and an analytic odule.

The loop (odule) is called local if the operations are determined only in some neighborhood of the unit element.

Let us choose coordinates in a local loop where $e^i=0$ for all i. The deviation of the loop from commutativity and associativity can be measured[4] by the structure constants C^i_{lm} and A^i_{lmn} defined by

$$[(hg)_L^{-1}(gh)]^i = C^i_{lm} g^l h^m + \cdots, \tag{9}$$

$$\{[g(hk)]_L^{-1}[(gh)k]\}^i = A^i_{lmn} g^l h^m k^n + \cdots, \tag{10}$$

where dots mean higher order terms.

We can now introduce the tangent algebras of loops. Geometrically, the tangent algebra A_e of M_e coincides with the tangent space T_eM of M_e at e. The product $[X, Y]$ of $X, Y \in A_e$ is defined in A_e by means of the structure constants C^i_{lm} of the geodesic loop (9)

$$[X, Y]^i := C^i_{lm} X^l Y^m = -[Y, X]^i. \tag{11}$$

We can equip A_e with a ternary operation as well.[8-10] For a triple $X, Y, Z \in A_e$, define their triple product (X, Y, Z) in A_e by using the other structure constants A^i_{lmn} of geodesic loop (10)

$$(X,Y,Z)^i := A^i_{lmn} X^l Y^m Z^n. \tag{12}$$

The tangent algebra A_e is thus a binary–ternary algebra, and it need not be a Lie algebra. In other words, there may be a triple $X, Y, Z \in A_e$, such that the Jacobi identity fails in A_e

$$J(X,Y,Z) := [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \neq 0. \tag{13}$$

Instead, for all $X, Y, Z \in A_e$, we have a more general identity[10]

$$J(X,Y,Z) = (X,Y,Z) + (Y,Z,X) + (Z,X,Y) - (X,Z,Y) - (Z,Y,X) - (Y,X,Z) \tag{14}$$

called the Akivis identity. The binary–ternary algebra A_e is called the Akivis algebra.

Let M be a manifold, generally without affine connection. Having an odule at each point of the manifold, we say that we have an odular structure on M. In general, odules at different points are independent. The odular structure on M is given by a ternary operation $R: M_e^3 \to M_e$ and a family of binary operations $\omega_t: M_e^2 \to M_e$, $t \in \mathbf{R}$, such that for each point $e \in M$ and for the points g, h from M_e, operations

$$R(g,e,h) = g \overset{e}{\cdot} h, \tag{15}$$

$$\omega_t(e,g) = t_e g \tag{16}$$

define an odule in M_e with e as the unit element.

IV. LOCAL GEODESIC MULTIPLICATION

Let us consider a manifold M with an affine connection. For a fixed point $e \in M$ choose a tangent vector X from the tangent space T_eM of M at e. Consider a local path $t \mapsto g(t;X)$ in M through the point e with the tangent vector X at e

$$g^i(0;X) = e^i, \quad \frac{dg^i(0;X)}{dt} = X^i. \tag{17}$$

It is well known that this path is a unique local geodesic path through e in the direction of X iff the following differential equation holds:

$$\frac{\delta}{\delta t}\frac{dg^i}{dt} \equiv \frac{d^2g^i}{dt^2} + \Gamma^i_{jk}\frac{dg^j}{dt}\frac{dg^k}{dt} = 0, \tag{18}$$

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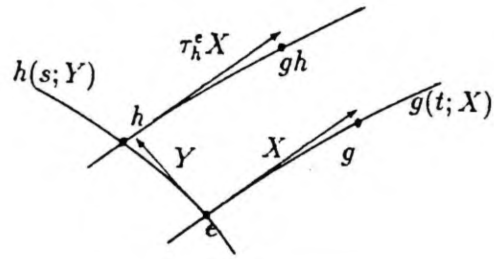

FIG. 1. Local geodesic multiplication.

where Γ^i_{jk} denote the affine connection coefficients. The exponential mapping $X \mapsto g := \exp_e X := g(1;X)$ at e is known[11] to be a local diffeomorphism of a suitable neighborhood of the origin of T_eM onto the corresponding (normal) neighborhood of $e \in M$. The local geodesic loop at e can be constructed (see Fig. 1) in such a neighborhood M_e of e where all required exponential mappings are well-defined local diffeomorphisms. Choose in M_e another local geodesic arc $h(s;Y)$ through the point e in the direction of $Y \in T_e M$. To perform a parallel transport of $X \in T_e M$ along this geodesic, we must solve the linear Cauchy problem

$$\frac{\delta X'^i}{\delta s} \equiv \frac{dX'^i}{ds} + \Gamma^i_{jk} \frac{dh^k}{ds} X'^j = 0, \quad X'(0) = X. \tag{19}$$

Performing the parallel transport of $X \in T_e M$, we obtain at $h := \exp_e Y$ the tangent vector $X' := X'(1)$ in $T_h M$. Now, draw the local geodesic arc through h in the direction of X', and mark point $\exp_h X'$ on it. This point is called the product of g and h, and it will be denoted as gh, or $g \overset{e}{\cdot} h$, where $e \in M_e$ is the unit element of the local geodesic loop. Explicitly, the multiplication formula reads[5]

$$g \overset{e}{\cdot} h \equiv gh \equiv R_h g = (\exp_h \circ \tau_h^e \circ \exp_e^{-1}) g, \tag{20}$$

where τ_h^e: $T_e M \mapsto T_h M$ denotes the parallel transport mapping of tangent vectors from $T_e M$ into $T_h M$ along the unique local geodesic arc joining the points e and h: $\tau_h^e(X) = X'$. In respect to multiplication (20) only the right translation can be seen explicitly

$$R_h = \exp_h \circ \tau_h^e \circ \exp_e^{-1}. \tag{21}$$

The left translations cannot be explicitly expressed, but they also play an essential role in describing nonassociativity of the geodesic multiplication [cf. Eqs. (3)–(5)].

The neighborhood M_e of e with multiplication rule (20) turns out to be a local differentiable loop[2–5] denoted henceforth by M_e as well. The unit element of M_e is e, and local geodesic paths through the unit element e are one-parameter subgroups of M_e. From defining formula (20) it follows that the local geodesic multiplication is power associative.

Note that the crucial part of the construction lies in the Cauchy problems (17)–(18) and (19), on the existence and uniqueness of their solutions, and also on the local diffeomorphism property of the exponential mapping.[11]

V. GEODESIC MULTIPLICATION IN THE SPACE–TIME OF A WEAK PLANE GRAVITATIONAL WAVE

One can easily check[12] that all geodesic loops of the Minkowski space–time are Abelian groups. In this particular case, the geodesic multiplications can be represented by the ordinary vector addition rule. The Abelian property manifests algebraically the fact that affine spaces are globally torsionless and flat.

As a more nontrivial example let us apply our mathematical constructions to the physical theory of a weak plane gravitational wave. In general relativity,[13] the space–time is considered to be a four-dimensional space M with a Riemannian metric $g_{\mu\nu}$ of the Lorentzian signature that determines the affine connection coefficients

$$g_{\sigma\rho}\Gamma^{\sigma}_{\mu\nu}=\tfrac{1}{2}(g_{\mu\rho,\nu}+g_{\nu\rho,\mu}-g_{\mu\nu,\rho}). \qquad (22)$$

The metric tensor describing a polarized weak plane gravitational wave moving in the direction of x can be given as follows:[13]

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}, \qquad (23)$$

$$\eta_{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1), \qquad (24)$$

$$h_{yy}=-h_{zz}=A\cos\omega(t-x). \qquad (25)$$

Here $A=$const, $A\ll 1$, is the wave amplitude, and Eqs. (23), (25) hold in linear approximation in A.

To calculate the product of points g, h with the coordinates (t_1,x_1,y_1,z_1) and (t_2,x_2,y_2,z_2) from a neighborhood of the unit element e, we must integrate the corresponding equations of geodesics (18) and of parallel transport (19) of the tangent vector X^μ. Direct but lengthy calculations give the following result:

$$t=t_1+t_2+\tfrac{1}{4}\omega^2A(t_1-x_1)[y_1y_2-z_1z_2+(y_2)^2-(z_2)^2],$$

$$x=x_1+x_2+\tfrac{1}{4}\omega^2A(t_1-x_1)[y_1y_2-z_1z_2+(y_2)^2-(z_2)^2],$$

$$y=y_1+y_2+\tfrac{1}{4}\omega^2A(t_1-x_1)[y_1(t_2-x_2)+(t_1-x_1)y_2+2(t_2-x_2)y_2],$$

$$z=z_1+z_2-\tfrac{1}{4}\omega^2A(t_1-x_1)[z_1(t_2-x_2)+(t_1-x_1)z_2+2(t_2-x_2)z_2].$$

We can see that in the case of the flat space–time, when $A=0$, the geodesic multiplication really reduces to the addition of orthogonal coordinates of the points.

VI. GEODESIC ODULES AND GEO-ODULAR IDENTITIES

Let us repeat the construction of Sec. IV and attach a local geodesic loop to all reasonable (nonsingular) points of the manifold M.

In the case of a local geodesic loop M_e, a local odule attached to the same point can be constructed by defining the multiplication with a scalar $t\in\mathbf{R}$ as follows:

$$t_e g=(\exp_e t\,\exp_e^{-1})g. \qquad (26)$$

From the definition of exponential mapping it follows that the line $\{t_e g\}_{t\in R}$ coincides with a geodesic arc $\exp_e(tX)$ with the tangent vector $X=\exp_e^{-1}g$ and with t as a canonical parameter.

It turns out that the geodesic odules obtained in this way at different points of the manifold M with affine connection are not independent. The reason for this is that we have used for the construction of geodesic odules an additional structure on the manifold M, an affine connection which determines the parallel transport of the tangent vectors.

To express explicitly the relations between geodesic odules we introduce, following Sabinin,[2,3] the notion of a geo-odular structure. An odular structure of the space M is said to be a geo-odular structure if it satisfies the first and the second geo-odular identities

$$R^{t_g h}_{u_g h} \circ R^g_{t_g h} = R^g_{u_g h}, \tag{27}$$

$$R^g_h \circ t_g = t_h \circ R^g_h, \tag{28}$$

where $R^g_h k \equiv R(k,g,h) \equiv k \overset{g}{\cdot} h$. It is easy to check using Eqs. (20), (15) for R^g_h and Eq. (26) for t_g that in the case of geodesic odules these identities are indeed satisfied. Really

$$\begin{aligned} R^{t_g h}_{u_g h} \circ R^g_{t_g h} k &= \exp_{u_g h} \tau^{t_g h}_{u_g h} \exp^{-1}_{t_g h} \exp_{t_g h} \tau^g_{t_g h} \exp^{-1}_g k \\ &= \exp_{u_g h} \tau^{t_g h}_{u_g h} \tau^g_{t_g h} \exp^{-1}_g k \\ &= \exp_{u_g h} \tau^g_{u_g h} \exp^{-1}_g k \\ &= R^g_{u_g h} k, \end{aligned}$$

$$\begin{aligned} R^g_h \circ t_g k &= (\exp_h \tau^g_h \exp^{-1}_g)(\exp_g t \exp^{-1}_g k) \\ &= \exp_h \tau^g_h t \exp^{-1}_g k \\ &= \exp_h t \tau^g_h \exp^{-1}_g k \\ &= (\exp_h t \exp^{-1}_h)(\exp_h \tau^g_h \exp^{-1}_g k) \\ &= t_h \circ R^g_h k. \end{aligned}$$

In a sense, the first geo-odular identity demonstrates additivity of the right translations of the local odules along the geodesic $\{t_g h\}_{t \in R}$. Having any three points g, $t_g h$, $u_g h$ on the geodesic, the right translation $R^g_{u_g h}$ of an odule at the point g [Fig. 2(i)] equals the superposition of two translations, the translation $R^g_{t_g h}$ and the translation $R^{t_g h}_{u_g h}$ of the resulting odule at the point $t_g h$ [Fig. 2,(ii)]. In other words it is the additivity property of the parallel displacement along a geodesic.

The second geoodular identity means that as the result of the right translation R^g_h, geodesic $\{t_g k\}_{t \in R}$ is transformed into the geodesic $\{t_h R^g_h k\}_{t \in R}$, preserving its canonical parameter t (Fig. 3).

VII. CONNECTION FROM GEO-ODULAR STRUCTURE

In this section, we consider the correspondence between affine connections and geo-odular structures.[2,3] Using the procedure of Sec. VI we can construct a uniquely determined geo-odular structure for every affine connection. We shall refer to it later as a natural geo-odular structure. Such geometric notions as parallel displacement and a covariant derivative can then

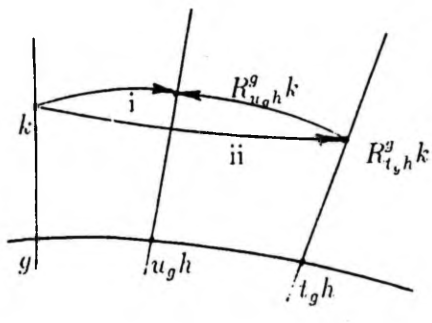

FIG. 2. First geo-odular identity.

be expressed purely in terms of algebraic operations. Explicitly, by differentiating the second geo-odular identity (28) with respect to the parameter t at $t=0$, we can see that the parallel displacement operator τ^g_h coincides with the differential of the right translation R^g_h at g (see Appendix A)

$$\tau^g_h=(R^g_h)_{*,g}. \tag{29}$$

Knowing the parallel displacement operator τ^g_h, we can give an explicit expression for the covariant derivative. Let $g(t;X)$ be a geodesic arc through the point $e=g(0;X)$ and let $X=\dot{g}(0;X)\in T_eM$ be the tangent vector of this arc at e. Let Y be a vector field. Parallel displacement along a geodesic is uniquely invertible, therefore $\tau_e^{g(t;X)}=[\tau^e_{g(t;X)}]^{-1}$ and

$$(\nabla_X Y)_e=\lim_{t\to 0}\left(\frac{\tau_e^{g(t;X)}Y_{g(t;X)}-Y_e}{t}\right)=\left(\frac{d}{dt}\{[\tau^e_{g(t;X)}]^{-1}Y_{g(t;X)}\}\right)_{t=0}. \tag{30}$$

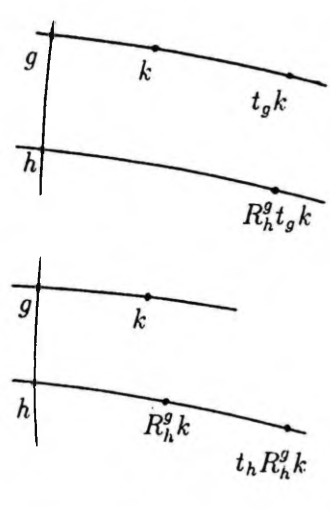

FIG. 3. Second geo-odular identity.

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Let us assume more generally that the geo-odular structure is defined without assuming the affine connection, i.e., there is a ternary operation R defined by Eq. (15) and a family of binary operations $\omega_t, t\in \mathrm{R}$ defined by Eq. (16), which satisfy identities (6)–(8) and geo-odular identities (27), (28). Then it turns out that an affine connection can be introduced as a derived object. It also turns out to be unique and the corresponding geo-odular structure is the natural geo-odular structure of the space M with this connection.[2] We can demonstrate it as follows. Let us define an object [cf. Eqs. (29) and (30)]

$$(\nabla_X Y)_e = \left(\frac{d}{dt}\{[(R^e_{g(t;X)})_{*,e}]^{-1} Y_{g(t;X)}\}\right)_{t=0}. \tag{31}$$

This object $\nabla_X Y$ is linear with respect to X and additive with respect to Y, and for every smooth function f the Leibniz rule holds

$$[\nabla_X(fY)]_e = [(Xf)Y + f\nabla_X Y]_e. \tag{32}$$

We shall call ∇_X the covariant derivative in the direction of X induced by the geo-odular structure. It will be checked in Appendix A that the operator $\tau^g_h := [(R^g_h)_*]_g$ is then the parallel transport mapping along the curve $\{t_g h\}_{t\in R}$ from the point g to the point h. The tangent vector field $(t_g h)^\bullet$ is transported in parallel along the curve $\{t_g h\}_{t\in R}$ as a consequence of monoassociativity (8). This implies that these curves are the geodesic lines of the connection ∇. Introducing the exponential mapping as in Sec. IV, we obtain the familiar expression for the binary operation ω_t, namely [cf. Eq. (26)]

$$\omega_t(e,g) \equiv t_e g = (\exp_e t \exp_e^{-1})g. \tag{33}$$

Finally, differentiating the second geo-odular identity (28) with respect to t at $t=0$, it follows that the given geo-odular structure is the natural geo-odular structure of the connection ∇

$$R^e_h g = (\exp_h \circ \tau^e_h \circ \exp_e^{-1})g. \tag{34}$$

The above considerations establish one-to-one correspondence between affine connections and geo-odular structures given in a space M.

VIII. AKIVIS FORMULAS

It turns out that noncommutativity and nonassociativity of the local geodesic loops are intimately related to the torsion and the curvature. Let the local coordinates be the Riemannian normal coordinates determined by the coordinate condition

$$\Gamma^i_{ab} g^a g^b = 0. \tag{35}$$

Then the equations of geodesic lines emerging from the unit element e are simply

$$g^i(\tau) = X^i \tau, \quad X^i \in T_e M. \tag{36}$$

Denote the torsion and the curvature tensors as S^i_{lm} and R^i_{lmn}, respectively,

$$S^i_{jk} = -\Gamma^i_{[jk]}, \quad R^i_{jkl} = -\Gamma^i_{j[k,l]} - \Gamma^m_{j[k}\Gamma^i_{|m|l]}.$$

The square brackets and parentheses around indices denote antisymmetrization and symmetrization

$$\Gamma^i_{[jk]}=\tfrac{1}{2}(\Gamma^i_{jk}-\Gamma^i_{kj}), \quad \Gamma^i_{(jk)}=\tfrac{1}{2}(\Gamma^i_{jk}+\Gamma^i_{kj}).$$

The direct computations[4] show that the structure constants (see Sec. III) of the geodesic loop are given by

$$C^i_{lm}=2S^i_{lm}(e), \tag{37}$$

$$A^i_{lmn}=R^i_{lmn}(e)-\nabla_n S^i_{lm}(e), \tag{38}$$

where ∇_n is the covariant differentiation operator. The next term in the expansion of commutator (9) contains both structure constants, C^i_{lm} and A^i_{lmn}

$$\begin{aligned}[(hg)^{-1}_L(gh)]^i&=C^i_{jk}g^jh^k-(A^i_{jkl}-\tfrac{1}{2}C^i_{mj}C^m_{lk})(g^jh^kg^l-h^jg^kh^l)+\cdots\\ &=2S^i_{jk}(e)g^jh^k-(R^i_{jkl}(e)-\nabla_l S^i_{jk}(e)-2S^i_{mj}(e)S^m_{lk}(e))(g^jh^kg^l-h^jg^kh^l)+\cdots.\end{aligned} \tag{39}$$

We also present here the expansion of a commutator of another type, which will be useful later

$$\begin{aligned}[(gh)(hg)^{-1}_R]^i&=C^i_{jk}g^jh^k-(A^i_{jkl}+\tfrac{1}{2}C^i_{mj}C^m_{lk})(g^jh^kg^l-h^jg^kh^l)+\cdots\\ &=2S^i_{jk}(e)g^jh^k-(R^i_{jkl}(e)-\nabla_l S^i_{jk}(e)+2S^i_{mj}(e)S^m_{lk}(e))(g^jh^kg^l-h^jg^kh^l)+\cdots.\end{aligned} \tag{40}$$

Commutators (39) and (40) differ because of the noncommutativity of the local geodesic loop.

In case of a Riemannian space without torsion, the only nontrivial structure constants are those of the associator, $A^i_{lmn}=R^i_{lmn}(e)$. This simplifies greatly the calculations and now the next term in the expansion of the associator can also be found. Equations (39) and (10) for the commutator and the associator now read

$$[(hg)^{-1}_L(gh)]^i=-R^i_{jkl}(e)(g^jh^kg^l-h^jg^kh^l)+\cdots, \tag{41}$$

$$\begin{aligned}\{[g(hk)]^{-1}_L[(gh)k]\}^i=&R^i_{jkl}(e)g^jh^kk^l-\tfrac{2}{3}R^i_{(jk)l,m}(e)g^jg^kh^lk^m+\tfrac{1}{3}(R^i_{jkm,l}(e)\\ &-R^i_{kjl,m}(e))g^jh^kh^lk^m+\tfrac{2}{3}R^i_{jk(l,m)}(e)g^jh^kk^lk^m+\cdots.\end{aligned} \tag{42}$$

In their right-hand sides, we recognize the terms occuring in the geodesic deviation equation. Let us investigate it more in detail.

IX. ALGEBRAIC FORMULATION OF GEODESIC DEVIATION

Let $x^i=x^i(u,v)$ be a two-surface in M parametrized by the pair of parameters (u,v). Let us denote the vectors tangent to the parametric lines $g_v(u)=x^i(u,v=\text{const})$, $h_u(v)=x^i(u=\text{const}, v)$ by

$$U^i=\frac{dg^i_v}{du}, \quad V^i=\frac{dh^i_u}{dv}. \tag{43}$$

Suppose that the lines $g^i_v(u)$ are geodesic lines with an affine parameter u

$$\frac{\delta U^i}{\delta u}=0. \tag{44}$$

For a constant infinitesimal dv, the vector $V^i\,dv$ is a vector joining a geodesic to a neighboring geodesic. Its rate of change along the geodesic $g(u)$ is given by the geodesic deviation equation[14]

$$\frac{\delta^2 V^i}{\delta u^2}+\frac{\delta}{\delta u}\,(2S^i_{jk}U^kV^j)+2R^i_{jkl}U^jV^kU^l=0. \tag{45}$$

In a space–time with torsion we have

$$\frac{\delta V^i}{\delta u}=\frac{\delta U^i}{\delta v}+2S^i_{jk}U^jV^k \tag{46}$$

and geodesic deviation equation (45) can be transformed to the form

$$\frac{\delta^2 V^i}{\delta u^2}=-2(R^i_{jkl}-\nabla_l S^i_{jk}-2S^i_{jm}S^m_{lk})\,U^jV^kU^l+2S^i_{jk}U^j\,\frac{\delta U^k}{\delta v}. \tag{47}$$

Suppose now that the parameters (u,v) are a part of the Riemannian coordinates with $x^i(0,0)=e$. Equation (47) holds also in the unit element e

$$\frac{\delta^2 V^i(e)}{\delta u^2}=-2(R^i_{jkl}(e)-\nabla_l S^i_{jk}(e)-2S^i_{jm}(e)S^m_{lk}(e))U^j(e)\,V^k(e)\,U^l(e)+2S^i_{jk}(e)\,U^j(e)\,\frac{\delta U^k(e)}{\delta v}. \tag{48}$$

Let us compare the result with Eq. (40) for the commutator of the geodesic loop. According to Eq. (43), for infinitesimal vectors g,h we have $g^i=dg^i_0(u)$, $h^i=dh^i_0(v)$ and

$$g^i=U^i(e)du,\quad h^i=V^i(e)dv. \tag{49}$$

Equation (40) can be written in the form of a Taylor expansion

$$[(gh)(hg)^{-1}_R]=\frac{1}{2}\,\frac{\partial^2}{\partial u\partial v}\,[(gh)(hg)^{-1}_R]du\,dv+\frac{1}{6}\,\frac{\partial^3}{\partial u^2\partial v}\,[(gh)(hg)^{-1}_R]du^2\,dv+\frac{1}{6}\,\frac{\partial^3}{\partial u\partial v^2}\times[(gh)(hg)^{-1}_R]du\,dv^2+\cdots. \tag{50}$$

We can identify

$$\frac{\partial^2}{\partial u\partial v}\,[(gh)(hg)^{-1}_R]^i_e=4S^i_{jk}U^jV^k, \tag{51}$$

$$\frac{\partial^3}{\partial u^2\partial v}\,[(gh)(hg)^{-1}_R]^i_e=-6(R^i_{jkl}-\nabla_l S^i_{jk}-2S^i_{jm}S^m_{lk})\,U^jV^kU^l, \tag{52}$$

$$\frac{\partial^3}{\partial u\partial v^2}\,[(gh)(hg)^{-1}_R]^i_e=6(R^i_{jkl}-\nabla_l S^i_{jk}-2S^i_{jm}S^m_{lk})\,V^jU^kV^l. \tag{53}$$

In the product gh, the tangent vector U is transported in parallel along the geodesic $h(v)$

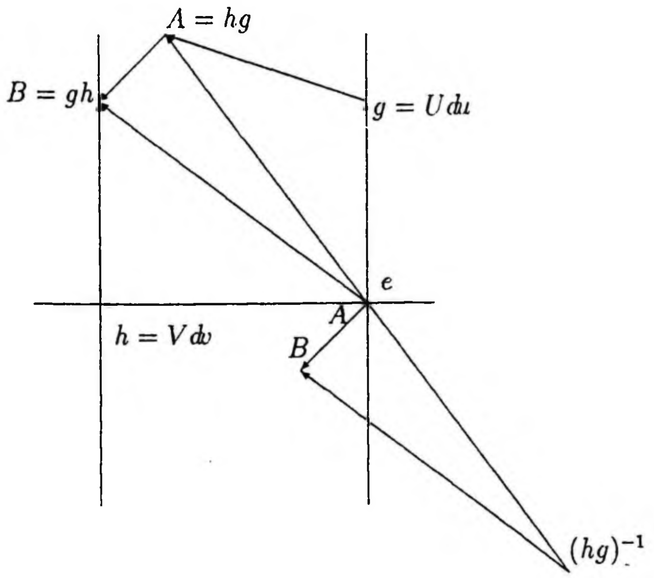

FIG. 4. Geodesic deviation equation.

$$\frac{\delta U^i}{\delta v}=0 \tag{54}$$

and the last term in Eq. (48) for geodesic deviation vanishes. As a result, the geodesic deviation in the unit element e together with an additional condition (54) can be given in terms of the commutator of the geodesic loop M_e

$$\frac{\delta^2 V^i(e)}{\delta u^2}=\frac{1}{3}\frac{\partial^3}{\partial u^2\,\partial v}\,[(gh)(hg)_R^{-1}]^i_e. \tag{55}$$

In the case of a Riemannian space–time without torsion, the geodesic deviation is determined by the associator $A^i_{jkl}=R^i_{jkl}(e)$ via Eqs. (55), (42)

$$\frac{\delta^2 V^i(e)}{\delta u^2}=-2A^i_{jkl}U^jV^kU^l. \tag{56}$$

The connection between the geodesic deviation equation and geodesic multiplication can be visualized as follows (Fig. 4). In the case of the infinitesimal vectors $g=Udu$, $h=Vdv$, the geodesic product $B=gh$ belongs to the geodesic line that goes through the point h in the direction of U. The geodesic product $A=hg$ corresponds to the vector V transported in parallel in the direction of U. The vector **AB** transported in parallel back to the unit element e can be given as $(gh)(hg)^{-1}$. According to the derivation of the geodesic deviation equation,[13] the following relation holds:

$$\frac{\delta^2}{\delta u^2}V=\lim_{\Delta u,\Delta v\to 0}\left(\frac{\mathbf{AB}}{(\Delta u^2)\Delta v}\right). \tag{57}$$

18

This reproduces our result (55).

Note that geodesic multiplication is defined not only for the infinitesimal vectors $g = U\,du, h = V dv$, but for all the points $g,h \in M_e$. This may be useful for deriving higher order geodesic deviation equations.[15,16]

X. DISCUSSION

In general relativity, the space–time is considered to be a (pseudo)Riemannian manifold, the affine connection of which is given by the metric tensor $g_{\mu\nu}$. In terms of physics, the role of the metric tensor is to introduce the causal structure of the space–time, the affine connection coefficients represent gravitational and inertial forces, and the Riemannian curvature tensor determines the relative acceleration between two freely falling point particles via the geodesic deviation equation.

We have presented the idea of how to reformulate differential geometry in algebraic terms given by Sabinin.[2,3] We have seen that the category of spaces with affine connection is equivalent to the category of spaces with a geo-odular structure, where an algebraic system with a nonassociative binary operation (geodesic multiplication) is given in a neighborhood of each point of the space. Now we propose some ideas for interpreting these algebraic structures in terms of physics.

There have been several attempts to quantize gravity. In a sense, every quantization is a representation of classical observables: algebraic properties of quantum observables are believed to imitate algebraic properties of the classical ones. Otherwise we are confronted with an anomaly (quantum mechanical symmetry breaking). For example, in canonical quantization, the canonical algebraic structure of observables is required to be preserved. Likewise, we can try to preserve the algebraic structure of classical events for the quantum ones as well.

Let us identify the tangent vectors $X \in T_e M$ with (classical) infinitesimal events and let us denote the corresponding quantum (infinitesimal) events as Q_X. If we believe that the infinitesimal quantum events preserve the structure of the geodesic Akivis algebras, introduced in Sec. III, then the geodesic quantum conditions can be proposed[12,17]

$$[Q_X, Q_Y] := Q_X Q_Y - Q_Y Q_X = q Q_{[X,Y]}, \tag{58}$$

$$(Q_X, Q_Y, Q_Z) := Q_X Q_Y \cdot Q_Z - Q_X \cdot Q_Y Q_Z = q^2 Q_{(X,Y,Z)}. \tag{59}$$

Here, the quantization constant (geodesic quantum deformation parameter) is denoted as q, and the classical infinitesimal events X, Y, Z must belong to the same geodesic Akivis algebra A_e. In Appendix B we show that, as a consequence of these quantum conditions, the quantum events satisfy the Akivis identity.

Another possibility for a physical interpretation of the geo-odular structure of a curved space–time may follow from the fact that the geodesic multiplication is a generalization of a rigid shift $x^\mu \to x^\mu + a^\mu$ of a flat space–time (cf. Sec. V). So the representation theory for geodesic loops must naturally embrace the representation theory of Poincaré translations.

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APPENDIX A: PARALLEL DISPLACEMENT FROM GEO-ODULAR STRUCTURE

In this appendix, we derive, following Sabinin,[2] an explicit formula for the parallel displacement operator in terms of algebraic operations of a natural geo-odular structure on M. By differentiating the second geo-odular identity (28) with respect to t, we get

$$(R_h^g)_{*,t_gk}(t_gk)^\bullet=(t_hR_h^g)^\bullet, \quad (t_gk)^\bullet\equiv\frac{d}{dt}(t_gk).$$

Using the expression for multiplication with scalar (26) and taking $t=0$, it follows that

$$(R_h^g)_{*,g}\exp_g^{-1}k=\exp_h^{-1}R_h^gk.$$

Finally, from the definition of R_h^g we obtain an explicit expression for the parallel displacement operator

$$\tau_h^g=(R_h^g)_{*,g}.$$

We also demonstrate here that, given an arbitrary geo-odular structure on M, i.e., a ternary operation R and a family of binary operations ω_t, $t\in\mathbf{R}$ satisfying identities (6)–(8) and geo-odular identities (27), (28), then $\tau_h^g=(R_h^g)_{*,g}$ is a parallel displacement operator along the curve $\{t_gh\}_{t\in R}$ from the point g to the point h. This can be checked by using the fact that the vector field Y is parallel along the curve $\{t_gh\}_{t\in R}$ iff

$$(\nabla_{(t_gh)^\bullet}Y)_{t_gh}=\frac{d}{du}\left[(\tau^{t_gh}_{(t+u)_gh})^{-1}Y_{(t+u)_gh}\right]_{u=0}=0.$$

Applying the operator $(\tau^g_{t_gh})^{-1}$ to both sides of the previous equation and using the first geo-odular identity (27), we get

$$\begin{aligned}
0&=(\tau^g_{t_gh})^{-1}\frac{d}{du}\left[(\tau^{t_gh}_{(t+u)_gh})^{-1}Y_{(t+u)_gh}\right]_{u=0}\\
&=\frac{d}{du}\left[(\tau^g_{t_gh})^{-1}(\tau^{t_gh}_{(t+u)_gh})^{-1}Y_{(t+u)_gh}\right]_{u=0}\\
&=\frac{d}{du}\left[(\tau^{t_gh}_{(t+u)_gh}\tau^g_{t_gh})^{-1}Y_{(t+u)_gh}\right]_{u=0}\\
&=\frac{d}{du}\left[(\tau^g_{(t+u)_gh})^{-1}Y_{(t+u)_gh}\right]_{u=0}\\
&=\frac{d}{dt}\left[(\tau^g_{t_gh})^{-1}Y_{t_gh}\right].
\end{aligned}$$

Therefore, $(\tau^g_{t_gh})^{-1}Y_{t_gh}=\text{const}=Y_g$. We found that if the vector field Y is parallel along the curve $\{t_gh\}_{t\in R}$, then (taking $t=1$) $Y_h=\tau_h^gY_g$ and so τ_h^g is really a parallel displacement operator from the point g to the point h along this curve.

APPENDIX B: THE AKIVIS IDENTITY FOR THE QUANTUM EVENTS

In this appendix we show that geodesic quantum conditions (58) and (59) preserve the structure of the geodesic Akivis algebra. From Eq. (58) it follows that

$$[[Q_X, Q_Y], Q_Z] = [qQ_{[X,Y]}, Q_Z] = q^2 Q_{[[X,Y],Z]}.$$

Using this result and its cyclic permutations and also the fact that the classical infinitesimal events X,Y,Z belong to the Akivis algebra A_e, we can write the following sequence of equalities:

$$\begin{aligned} J(Q_X,Q_Y,Q_Z) &= [[Q_X, Q_Y], Q_Z] + [[Q_Y, Q_Z], Q_X] + [[Q_Z, Q_X], Q_Y] \\ &= q^2 Q_{[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]} \\ &= q^2 Q_{J(X,Y,Z)} \\ &= q^2 (Q_{(X,Y,Z)} + Q_{(Y,Z,X)} + Q_{(Z,X,Y)} - Q_{(Y,X,Z)} - Q_{(X,Z,Y)} - Q_{(Z,Y,X)}) \\ &= (Q_X,Q_Y,Q_Z) + (Q_Y,Q_Z,Q_X) + (Q_Z,Q_X,Q_Y) - (Q_Y,Q_X,Q_Z) - (Q_X,Q_Z,Q_Y) \\ &\quad - (Q_Z,Q_Y,Q_X). \end{aligned}$$

We can see that for the quantum infinitesimal events Q_X,Q_Y,Q_Z the Akivis identity is indeed satisfied.

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III

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GEODESIC MULTIPLICATION AND GEOMETRICAL BRST-LIKE OPERATORS

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Abstract. A generalization of Poincaré translations into a nonassociative algebraic system called the local geodesic loop is proposed. Two BRST-like nilpotent operators based on the local geodesic left and right translation matrices are constructed. As an explicit example, the local geodesic translation matrices are calculated in the space-time of a weak plane gravitational wave.

Key words: nonassociative algebras, geodesic multiplication, BRST operator, quantum gravity.

1. INTRODUCTION

The BRST quantization [1], which is up to now the most advanced method of quantization, is a generalization of Dirac's scheme of quantization for physical systems with constraints and gauge freedom. Both methods coincide if the algebra of constraints closes and its structure constants do not involve fields. In the case of open gauge algebras and field-dependent structure functions only the BRST quantization is applicable. Usually an explicit construction of the BRST generator is based on the algebra of constraints. However, one of the main points of the BRST quantization lies in the statement that different physical states correspond to the different cohomology classes of the nilpotent BRST operator, and physical observables must commute with it.

There exist physical systems for which a consistent quantum theory cannot be constructed by means of any of the available quantization schemes, e. g. the gravitational field. Nevertheless, we can try to construct a reasonable nilpotent operator that can be used for defining cohomology classes and investigate the resulting quantum theory. A geometrical BRST operator Q has been proposed by Bars and Yankielowicz [2]. It involves an infinite-dimensional algebra of the modified

Poincaré group where the torsion tensor and the curvature tensor act as structure functions. The nilpotency of Q turns out to be a consequence of the differential geometrical Bianchi identities. An analogous geometrical BRST-like operator for a N-dimensional differentiable manifold with a zero curvature but a nonzero torsion tensor has recently been given also by Okubo [3]. In the present paper we investigate a possibility of using our earlier work [4] on the role of geodesic multiplication in the theory of gravity for constructing BRST-like operators according to Okubo's scheme, and for discussing their possible geometrical and physical meaning.

Geodesic multiplication of points of a differentiable manifold with an affine connection is a generalization of constant translations of a flat torsionless manifold which allows to transform straight lines (geodesics) into parallel straight lines. In the case of a flat spacetime, constant translations form a subgroup of the Poincaré group, the Abelian group of Poincaré translations. It can be gauged, e. g. constant translations $x \to x+a$, $a=\text{const.}$ can be replaced by point-dependent translations, $x \to x+a(x)$. The corresponding gauge group is the group of general coordinate transformations (diffeomorphisms), and the gauge field is the local frame of reference $e_A{}^\mu(x)$ [5]. But the mathematical structure of the following gauge theory differs in some essential points from that of a standard gauge theory. We propose to generalize the group of Poincaré translations into a geometrically defined algebraic system called the geodesic loop which has the geodesic multiplication as a binary operation. Due to the nonassociativity of the geodesic multiplication, it does not form a group and the methods of the conventional gauge theory cannot be applied. However, using Okubo's construction it is possible to write down two BRST-like operators containing the algebra of vector fields which generate left and right infinitesimal geodesic translations of space-time points. Cohomologies of these BRST-like operators turn out to be analogous to the de Rham cohomology of the space-time.

The paper is organized as follows. In Sec. 2, Okubo's construction of a BRST-like operator is briefly reviewed. In Sec. 3, the notions of the geodesic multiplication and the geodesic loop are introduced and their main algebraic properties are described. In Sec. 4, the left and the right geodesic translation matrices are used for constructing two geometrical BRST-like operators. Our main idea is to derive these operators not directly from the geometry of the space-time, but from the geometry of the geodesic loop. In Sec. 5, explicit expressions for the left and the right geodesic translation matrices are calculated in the case of a weak plane gravitational wave and the corresponding parallelizing torsions of the geodesic loop are determined. Section 6 is devoted to the problem of physical state vectors and their cohomologies.

2. OKUBO'S CONSTRUCTION FOR A BRST-LIKE OPERATOR

Okubo [3] has proposed the following formal construction of an anticommuting nilpotent operator for a N-dimensional differentiable manifold with local coordinates x^μ. Let us introduce a N-bein field $e^\mu_A(x)$ that is invertible, i.e. there exists also the inverse matrix $e^A_\mu(x)$,

$$e^\mu_A e^B_\mu = \delta^B_A, \quad e^\mu_A e^A_\nu = \delta^\mu_\nu. \tag{1}$$

In the framework of the Cartan formalism, $e^A_\mu(x)$ determines the basis 1-forms ω^A,

$$\omega^A = e^A_\mu dx^\mu. \tag{2}$$

Let us suppose that the connection 1-forms $\omega^A_B \equiv \Gamma^A_{BD}\,\omega^D$ vanish,

$$\omega^A_B=0, \quad \Gamma^A_{BD}=0. \tag{3}$$

From the Cartan structure equations

$$d\omega^A+\omega^A_B \wedge \omega^B=\Omega^A, \tag{4}$$

$$d\omega^A_B+\omega^A_D \wedge \omega^D_B=\Omega^A_B \tag{5}$$

it follows that the curvature 2-form Ω^A_B also vanishes, $\Omega^A_B=0$, and the components of the torsion 2-form $\Omega^A=\frac{1}{2}S^A_{\mu\nu}\,dx^\mu \wedge dx^\nu$ are determined by the inverse N-bein field e^A_μ,

$$\partial_\mu e^A_\nu - \partial_\nu e^A_\mu = S^A_{\mu\nu}\,. \tag{6}$$

Although the connection 1-forms ω^A_B and the Riemann curvature tensor vanish, the connection coefficients $\Gamma^\lambda_{\mu\nu}$ in local holonomic curvilinear coordinates x^μ may acquire nonvanishing values due to the coordinate transformation from anholonomic flat coordinates y_A to x_μ:

$$\frac{\partial y^A(x)}{\partial x^\mu}=e^A_\mu(x), \tag{7}$$

$$\Gamma^\lambda_{\mu\nu}\equiv e^\lambda_D\,(e^A_\mu e^B_\nu \Gamma^D_{AB}+\partial_\nu e^D_\mu)=e^\lambda_D\,\partial_\nu e^D_\mu\,. \tag{8}$$

For constructing a BRST-like operator Q, Okubo introduced coordinate-independent anticommuting ghost-like operators c^A, b_A satisfying

$$b_A b_B+b_B b_A=0, \quad c^A c^B+c^B c^A=0, \tag{9}$$

$$b_A c^B+c^B b_A=\delta^B_A\,. \tag{10}$$

They are covariantly constant if considered in holonomic coordinates x^μ:

$$b_\mu(x)=e^A_\mu(x)\,b_A, \quad b_{\nu;\mu}\equiv\partial_\mu b_\nu-\Gamma^\lambda_{\nu\mu}\,b_\lambda=0, \tag{11}$$

$$c^\nu(x)=e^\nu_A(x)\,c^A, \quad c^\nu_{;\mu}\equiv\partial_\mu c^\nu+\Gamma^\nu_{\lambda\mu}\,c^\lambda=0. \tag{12}$$

The definition of Q as given by Okubo [3] reads

Direct computations using the Bianchi identity

$$Q=c^\mu(x)\,\partial_\mu+\frac{1}{2}c^\mu(x)\,c^\nu(x)\,S^\lambda_{\mu\nu}(x)\,b_\lambda(x). \tag{13}$$

$$d\Omega^A+\omega^A_B \wedge \Omega^B=\Omega^A_D \wedge \omega^D \tag{14}$$

confirm that Q is nilpotent,

$$2Q^2\equiv\{Q,\ Q\}=0. \tag{15}$$

In a sense, operator Q is a generalization of the exterior differential operator. It has been known already for a long time that the BRST generator in the classical constrained dynamics can be considered as an exterior differential operator along orbits of the gauge group in the phase space of a mechanical system [6] or in the configuration space of a gauge field [7]. Okubo's operator Q has an essential difference from the latter ones: Q is a generalization of the exterior differential in the differentiable manifold (space-time) with holonomic coordinates x^μ, but the conventional BRST generator is a generalization of the exterior differential in the space of the gauge group.

3. GEODESIC MULTIPLICATION

Let us consider a 4-dimensional differentiable manifold M with an affine connection (the space-time). Its geodesic lines (autoparallels) $x^\mu(t)$ must satisfy the following differential equations:

$$\frac{d^2x^\mu}{dt^2}+\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}=0, \tag{16a}$$

where $\Gamma^\mu_{\nu\rho}(x)$ denote the affine connection coefficients. Let $M_e\subset M$ be such a neighbourhood of $e\in M$ where geodesic lines emerging from e do not intersect. In general, M_e is a finite region of M which does not contain singular points. Solutions of Eq. (16a) at initial values $x^\mu(0)=e^\mu$, $\left.\frac{dx^\mu}{dt}\right|_{t=0}=X^\mu$ determine the exponential mapping $T_eM\to$ $\to M_e: X\to x:=\exp_e X:=x(1;X)$. A parallel transport mapping $\tau^e_y: T_eM\to T_yM$ along a geodesic line $y(s)$ emerging from e is given as a solution of the Cauchy problem

$$\frac{dX'^\mu}{ds}+\Gamma^\mu_{\nu\rho}\frac{dy^\nu}{ds}X'^\rho=0,\quad X'^\mu(0)=X^\mu. \tag{16b}$$

Using the exponential mapping and the parallel transport mapping, the local geodesic multiplication of points $x,\ y\in M_e$ can be introduced [8,9]:

$$x\cdot y\equiv L_xy\equiv R_yx=(\exp_y\circ\tau^e_y\circ\exp^{-1}_e)x. \tag{17}$$

The local geodesic multiplication can be constructed in such a neighbourhood M_e where all required exponential mappings and parallel transport operations are well-defined local diffeomorphisms.

In general, the local geodesic multiplication need not be commutative and associative. In the Riemann normal coordinates with the origin in e, equations for geodesic lines (16a) and parallel transport (16b) can be solved, using expansions in local coordinates. Direct calculations [10] demonstrate that the commutator and the associator of the local geodesic multiplication are intimately related to the torsion $S^\mu_{\nu\rho}(x)$ and the curvature tensor $R^\mu_{\nu\rho\sigma}(x)$ of the space-time M:

$$\Big((y\cdot x)^{-1}_L\cdot(x\cdot y)\Big)^\mu=C^\mu_{\nu\rho}x^\nu y^\rho+\dots, \tag{18a}$$

$$C^\mu_{\nu\rho}=2S^\mu_{\nu\rho}(e), \tag{18b}$$

$$\Big((x\cdot(y\cdot z))^{-1}_L((x\cdot y)\cdot z)\Big)^\mu=A^\mu_{\nu\rho\sigma}x^\nu y^\rho z^\sigma+\dots, \tag{19a}$$

$$A^\mu_{\nu\rho\sigma}=R^\mu_{\nu\rho\sigma}(e)-\nabla_\sigma S^\mu_{\nu\rho}(e). \tag{19b}$$

Here x^{-1}_L denotes the left inverse element of x, $x^{-1}_L\cdot x=e$, ∇_ν is the covariant differentiation operator and dots mean higher-order terms.

The local geodesic multiplication converts the neighbourhood M_e into the space of an algebraic system called the local geodesic loop [11,12]. Point $e\in M$ is the unit element of the loop.

Local geodesic multiplication (17) determines the following infinitesimal left (L) and right (R) translation matrices:

$$(x\cdot y)^\mu = y^\mu + L^\mu_\nu(y)x^\nu + \dots,\quad L^\mu_\nu(y) \equiv \frac{\partial (x\cdot y)^\mu}{\partial x^\nu}\Big|_{x=e}, \tag{20a}$$

$$= x^\mu + R^\mu_\nu(x)y^\nu + \dots,\quad R^\mu_\nu(x) \equiv \frac{\partial (x\cdot y)^\mu}{\partial y^\nu}\Big|_{y=e}. \tag{20b}$$

At general coordinate transformations $x' = x'(x)$ they transform as bitensors, i.e. they are contravariant vectors in x and covariant vectors in e [13]:

$$L^{\mu'}_{\nu'}(x') = A^{\mu'}_{\mu}(x) L^\mu_\nu(x) (A^{-1})^\nu_{\nu'}(e), \tag{21a}$$

$$R^{\mu'}_{\nu'}(x') = A^{\mu'}_{\mu}(x) R^\mu_\nu(x) (A^{-1})^\nu_{\nu'}(e), \tag{21b}$$

where $A^{\mu'}_{\mu} = \partial x^{\mu'}/\partial x^\mu$. Although the upper and the lower indices of L^μ_ν, R^μ_ν do not transform independently, these matrices can be considered as defining two preferred local vierbein fields in the neighbourhood $M_e \subset M$.

Let the differentiable manifold M with an affine connection be torsionless, $S^\mu_{\nu\rho}(x) = 0$, and endowed with a metric $g_{\mu\nu}(x)$ that is compatible with the connection,

$$\nabla_\rho g_{\mu\nu} = 0, \tag{22a}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}). \tag{22b}$$

Then the main part of commutator (18) vanishes and the main part of associator (19) equals to the curvature tensor,

$$A^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma}(e).$$

If the metric tensor $g_{\mu\nu}(x)$ is a solution of the Einstein equations, then the associator $A^\mu_{\nu\rho\sigma}$ equals to the value of the curvature tensor $R^\mu_{\nu\rho\sigma}(e)$ in the point e of a physical (dynamical) space-time.

4. THE CONSTRUCTION OF LOCAL BRST-LIKE OPERATORS

Let us consider the space of the geodesic loop M_e with the left and the right infinitesimal geodesic translation operators $L_\mu(x)$, $R_\mu(x)$ as defined by Eq. (20),

$$L_\alpha(x) = L^\nu_\alpha(x)\frac{\partial}{\partial x^\nu},\quad R_\alpha(x) = R^\nu_\alpha(x)\frac{\partial}{\partial x^\nu}. \tag{23}$$

They generate two vector field algebras,

$$[L_\alpha(x), L_\beta(x)] = A^\gamma_{\alpha\beta}(x)L_\gamma(x),\quad [R_\alpha(x), R_\beta(x)] = B^\gamma_{\alpha\beta}(x)R_\gamma(x),$$

$$A^\gamma_{\alpha\beta} = (L^\mu_\alpha(\partial_\mu L^\nu_\beta) - L^\mu_\beta(\partial_\mu L^\nu_\alpha))L^\gamma_\nu,$$

$$B^\gamma_{\alpha\beta} = (R^\mu_\alpha(\partial_\mu R^\nu_\beta) - R^\mu_\beta(\partial_\mu R^\nu_\alpha))R^\gamma_\nu.$$

In the case of left (right) invariant vector fields on a group manifold, we have $A^\gamma_{\alpha\beta} = -B^\gamma_{\alpha\beta} = \mathrm{const} = -C^\gamma_{\alpha\beta}$, and $[L_\alpha, R_\beta] = 0$. In the case of a loop manifold these relations do not hold.

Both vector fields, $L_\alpha(x)$ and $R_\alpha(x)$, can be considered as a preferred local frame of reference that can be used for introducing a parallelizing torsion in the space of the geodesic loop. Let us suppose that the connection 1-forms and the curvature 2-forms vanish so that the second Cartan structure equation (5) is identically satisfied. Then the corresponding connection coefficients in local holonomic coordinates $L^\lambda_{\mu\nu}$, $R^\lambda_{\mu\nu}$ are given by vierbein fields according to Eq. (8):

$$L^\lambda_{\mu\nu}=L^\lambda_\sigma\partial_\nu(L^{-1})^\sigma_\mu,\quad R^\lambda_{\mu\nu}=R^\lambda_\sigma\partial_\nu(R^{-1})^\sigma_\mu. \tag{24}$$

This means that we can introduce three different connection coefficients to the same neighbourhood M_e, the affine connection coefficients $\Gamma^\lambda_{\mu\nu}$ of the space-time and the connection coefficients $L^\lambda_{\mu\nu}$, $R^\lambda_{\mu\nu}$ induced by the local geodesic multiplication. Note that $L^\lambda_{\mu\nu}$ and $R^\lambda_{\mu\nu}$ depend in a sophisticated way on $\Gamma^\lambda_{\mu\nu}$, since the local geodesic multiplication (17) is determined by geodesics (16a) and parallel transport operator (16b).

From Eqs. (24) two torsion tensors of the space of the local geodesic loop can be calculated as well:

$$\underset{L}{\theta}{}^\lambda_{\mu\nu}=L^\lambda_{\nu\mu}-L^\lambda_{\mu\nu},\quad \underset{R}{\theta}{}^\lambda_{\mu\nu}=R^\lambda_{\nu\mu}-R^\lambda_{\mu\nu}. \tag{25}$$

Note that the corresponding Cartan first structure equation (4) for them,

$$dL^\alpha=\underset{L}{\theta}{}^\alpha,\quad dR^\alpha=\underset{R}{\theta}{}^\alpha$$

can be considered as having been obtained from the structure equation of the Abelian group of Poincaré translations,

$$[P_\alpha, P_\beta]=0,\quad dP^\alpha=0$$

as soft-group manifolds [14], [15]. However, in the general case of a soft group, the left (right) invariant vector fields P_α are substituted by arbitrary vector fields. In our case, the vector fields L_α, R_β are not arbitrary but are generated by the geodesic multiplication.

Now, following Okubo's construction, two local anticommuting nilpotent BRST-like operators can be defined using definition (13):

$$\underset{L}{Q}=\underset{L}{c}{}^\mu(x)\partial_\mu+\frac{1}{2}\underset{L}{c}{}^\mu(x)\underset{L}{c}{}^\nu(x)\underset{L}{\theta}{}^\lambda_{\mu\nu}(x)\underset{L}{b}_\lambda(x), \tag{26}$$

$$\underset{R}{Q}=\underset{R}{c}{}^\mu(x)\partial_\mu+\frac{1}{2}\underset{R}{c}{}^\mu(x)\underset{R}{c}{}^\nu(x)\underset{R}{\theta}{}^\lambda_{\mu\nu}(x)\underset{R}{b}_\lambda(x). \tag{27}$$

They can be considered as generalizations of exterior derivative, not in the space-time but in the space of the geodesic loop. Okubo's construction implies

$$\underset{L}{Q}{}^2\equiv\{\underset{L}{Q},\underset{L}{Q}\}=0,\quad \underset{R}{Q}{}^2\equiv\{\underset{R}{Q},\underset{R}{Q}\}=0, \tag{28a}$$

and indicates that

$$\{\underset{L}{Q},\underset{R}{Q}\}\neq0. \tag{28b}$$

The possible physical meaning of BRST operators $\underset{L}{Q}$ and $\underset{R}{Q}$ must follow from the properties of their action on suitably defined space of state vectors.

5. AN EXAMPLE: THE SPACE-TIME OF A WEAK PLANE GRAVITATIONAL WAVE

For explicitly determining the expression of the local geodesic product of two arbitrary points $x, y \in M_e$ of a manifold M with a given affine connection $\Gamma^{\lambda}_{\mu\nu}(x)$, we need to integrate Eqs. (16a, b) for geodesic lines and parallel transport in arbitrary directions. It turns out to be analytically a rather complicated task even in the seemingly simple case of a 2-sphere. To give an example that can be analytically worked out, let us obtain explicit expressions for the left and the right translation operators and the corresponding parallelizing torsions in the case of the physical space-time of the weak plane gravitational wave.

The metric tensor can be given as perturbations around the Minkowski metric:

$$g_{\mu\nu}=\eta_{\nu\mu}+h_{\mu\nu},$$

$$\eta_{\mu\nu}=\mathrm{diag}(-1,\ +1,\ +1,\ +1).$$

In the case of a polarized weak plane gravitational wave moving in the direction of x the only nonzero components of $h_{\mu\nu}$ in the TT-gauge [16] are

$$h_{yy}=-h_{zz}=A\cos\omega(t-x).$$

Here $A=\mathrm{const.}$, $A\ll 1$, is the wave amplitude, and all subsequent equations hold in the linear approximation in A.

In these coordinates the equation of a geodesic line with a tangent vector X^{μ} at a point e can be easily integrated, yielding

$$g^{\mu}(t)=e^{\mu}+X^{\mu}t+AU^{\mu}[\sin\omega B(\cos\omega Ct-1)+\cos\omega B(\sin\omega Ct-\omega Ct)],$$

where we have denoted

$$B=e^{0}-e^{1},\qquad C=X^{0}-X^{1},$$

$$U^{0}=U^{1}=-\frac{1}{2\omega C^{2}}[(X^{2})^{2}-(X^{3})^{2}],$$

$$U^{2}=-\frac{X^{2}}{\omega C},\quad U^{3}=\frac{X^{3}}{\omega C}.$$

Let us choose the point e to be a unit element of the geodesic loop and let g, h be two points from its neighbourhood. We denote by X^{μ} and Y^{μ} the tangent vectors of geodesic lines, joining the point e with the points g and h, respectively.

To calculate the product of the points g, h, we must integrate the corresponding equations of geodesics (16a) and of parallel transport (16b) of the tangent vector X^{μ}. Direct but lengthy calculations give the following result:

$$\begin{aligned}k^{\mu}&\equiv(h\cdot g)^{\mu}=\\&=g^{\mu}+h^{\mu}-e^{\mu}+AU^{\mu}[(\sin\omega B'-\sin\omega B)(\cos\omega C-1)+\\&\qquad+(\cos\omega B'-\cos\omega B)(\sin\omega C-\omega C)]+\\&\qquad+\frac{AV^{\mu}}{2D}[\cos\omega B-\cos\omega(B+D)],\end{aligned}$$

where

$$B'=h^{0}-h^{1},\qquad D=Y^{0}-Y^{1},$$

$$V^{0}=V^{1}=X^{2}Y^{2}-X^{3}Y^{3},$$

$$V^2 = CY^2 + DX^2, \qquad V^3 = -CY^3 - DX^3.$$

The previous equations contain no singularities if $C \equiv X^0 - X^1 = 0$ or $D \equiv Y^0 - Y^1 = 0$. The equations corresponding to these special cases can be obtained by just taking the limit $C \to 0$ or $D \to 0$.

From the expression of the geodesic multiplication we can calculate the matrices of the left and right translations, respectively (we have taken $e = 0$ for simplicity):

$$L = I + A\xi \begin{pmatrix} 0 & 0 & \frac{y}{t-x} & -\frac{z}{t-x} \\ 0 & 0 & \frac{y}{t-x} & -\frac{z}{t-x} \\ \frac{y}{t-x} & -\frac{y}{t-x} & 1 & 0 \\ -\frac{z}{t-x} & \frac{z}{t-x} & 0 & -1 \end{pmatrix},$$

$$R = I + A\xi \begin{pmatrix} \frac{y^2-z^2}{(t-x)^2} & -\frac{y^2-z^2}{(t-x)^2} & 0 & 0 \\ \frac{y^2-z^2}{(t-x)^2} & -\frac{y^2-z^2}{(t-x)^2} & 0 & 0 \\ \frac{2y}{t-x} & -\frac{2y}{t-x} & 0 & 0 \\ -\frac{2z}{t-x} & \frac{2z}{t-x} & 0 & 0 \end{pmatrix},$$

where

$$\xi = \sin^2 \frac{\omega(t-x)}{2}$$

and I is the unit matrix. Taking into account that we are working in the linear approximation in A, their inverses differ only by the sign in front of the second terms.

Now let us calculate torsion tensors (25), corresponding to connections (24) in the space of the geodesic loop, obtained from the left and the right translations, respectively. To keep the expressions compact, we introduce some additional notations:

$$P = \frac{\omega \sin \omega(t-x)}{2(t-x)},$$

$$Q = \frac{\sin^2 \frac{\omega(t-x)}{2}}{(t-x)^2},$$

$$S = P - Q,$$

$$T = (2Q - P)\frac{y^2 - z^2}{t-x}.$$

Now we can write down the components of the torsion tensor obtained from the left translations

$$\underset{L}{\theta^1}=\underset{L}{\theta^2}=A\begin{pmatrix} 0 & 0 & yS & -zS \\ 0 & 0 & 0 & -Q(t-x) \\ -yS & 0 & 0 & -Q(t-x)+zS \\ zS & Q(t-x) & Q(t-x)-zS & 0 \end{pmatrix},$$

$$\underset{L}{\theta^3}=A\begin{pmatrix} 0 & -yS & P(t-x)+yS & -Q(t-x) \\ yS & 0 & -yS & Q(t-x) \\ -P(t-x)-yS & yS & 0 & 0 \\ Q(t-x) & -Q(t-x) & 0 & 0 \end{pmatrix},$$

$$\underset{L}{\theta^4}=A\begin{pmatrix} 0 & Q(t-x)+zS & -zS & -P(t-x) \\ -Q(t-x)-zS & 0 & zS & 0 \\ zS & -zS & 0 & P(t-x) \\ P(t-x) & 0 & -P(t-x) & 0 \end{pmatrix},$$

and the right translations

$$\underset{R}{\theta^1}=\underset{R}{\theta^2}=A\begin{pmatrix} 0 & 2Qz+T & -T & -2Qy \\ -2Qz-T & 0 & T & 2Qy \\ T & -T & 0 & 0 \\ 2Qy & -2Qy & 0 & 0 \end{pmatrix},$$

$$\underset{R}{\theta^3}=A\begin{pmatrix} 0 & -2yS & 2yS & -2Q(t-x) \\ 2yS & 0 & -2yS & 2Q(t-x) \\ -2yS & 2yS & 0 & 0 \\ 2Q(t-x) & -2Q(t-x) & 0 & 0 \end{pmatrix},$$

$$\underset{R}{\theta^4}=A\begin{pmatrix} 0 & 2Q(t-x)+2zS & -2zS & 0 \\ -2Q(t-x)-2zS & 0 & 2zS & 0 \\ 2zS & -2zS & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The BRST-like operators can be obtained in a straightforward way, substituting these values into Eqs. (26), (27). To ensure the correctness of the result, one can check that the Bianchi identities are indeed satisfied. Therefore it follows directly that $\underset{L}{Q}^2=0$, $\underset{R}{Q}^2=0$.

6. QUANTUM THEORY

In the conventional BRST quantization, the starting point is usually the Lagrangian of the physical system that determines the equations of motion and constraints. Upon quantization the Fourier coefficients in the solutions of the equations of motion are regarded as creation and annihilation operators. As distinct from this case, in our theory the solutions of the Einstein equations are not regarded as operators. Components of the classical curvature tensor act as structure functions of the geodesic loop. Its left and right translations determine two preferred local frames of reference that allow us to construct two BRST-like operators (26), (27). In a general case they do not commute, so the complete quantum theory must contain both of them. They possess a common system of ghost operators c, b but do not contain any creation or annihilation operators of gravitons (quanta of the gravitational field). Consequently, Fock states of the corresponding quantum system contain only ghost quanta. The background space-time remains to be a continuous differentiable manifold, possibly with a nontrivial topology.

Okubo considered in more detail the case where $b_\mu(x)$ are interpreted as annihilation operators. Then the vacuum state is defined by

$$b_\mu(x)\,|0>=0$$

and the Fock space consists of vectors

$$|\omega_n>=f_{\mu_1\dots\mu_n}(x)c^{\mu_1}(x)\dots c^{\mu_n}(x)\,|0>.$$

He demonstrated that ghosts c^μ can be identified with differential forms dx^μ and the resulting cohomology of the BRST-like operator Q turns out to be isomorphic to that of the standard de Rham cohomology of the underlying manifold M. In his theory, the local frame of reference $e_A{}^\mu(x)$ is arbitrary. Frames of reference $L^\mu_\alpha(x)$ and $R^\mu_\alpha(x)$ used here have a definite geometrical meaning as the infinitesimal operators of the left and the right geodesic translations.

Let us define the vacuum state as

$$b_A|0>0.$$

It is equivalent to Okubo's definition, since the coordinate-dependent operators $b_\mu(x)$ are obtained from b_A by means of multiplying it with a regular matrix, $b_\mu(x)=b_A e^A_\mu(x)$. In our case there are two preferred frames of reference, $L^\mu_\alpha(x)$ and $R^\mu_\alpha(x)$, and, respectively, two sets of annihilation operators $\underset{L}{b_\mu}(x)=b_\alpha(L^{-1})^\alpha_\mu(x)$ and $\underset{R}{b_\mu}(x)=b_\alpha(R^{-1})^\alpha_\mu(x)$

From the unique vacuum state

$$b_\alpha|0>=0$$

we obtain two conditions,

$$\underset{L}{b_\mu}(x)\,|0>=0,$$

$$\underset{R}{b_\mu}(x)\,|0>=0.$$

Owing to the regularity of the matrices $L^\mu_\alpha(x)$ and $R^\mu_\alpha(x)$, postulating any of these conditions forces the validity of the other two. The Fock space can now be defined by

$$|\omega_n>=f_{\alpha_1\dots\alpha_n}c^{\alpha_1}\dots c^{\alpha_n}|0>=$$
$$=\underset{L}{f_{\mu_1\dots\mu_n}}(x)\underset{L}{c^{\mu_1}}(x)\dots \underset{L}{c^{\mu_n}}(x)\,|0>=$$
$$=\underset{R}{f_{\mu_1\dots\mu_n}}(x)\underset{R}{c^{\mu_1}}(x)\dots \underset{R}{c^{\mu_n}}(x)\,|\,0>.$$

Therefore, any state can be represented in terms of either only operators $\underset{L}{c^\mu}(x)$ or only operators $\underset{R}{c^\mu}(x)$. The coefficients $f_{\mu_1\dots\mu_n}(x)$ are related through the transformation matrix from one preferred frame to another

$$(\Delta_{LR})^\nu_\mu=(L^{-1})^\alpha_\mu R^\nu_\alpha.$$

Note that indices μ, ν refer to the same coordinate system.

In the standard BRST quantization, the physical states are obtained as cohomologies of the BRST operator. By analogy, the cohomologies of our BRST-like operators are derived from the expressions for the action of nilpotent operators $\underset{L}{Q}$, $\underset{R}{Q}$ (see (26), (27)) in the space of state vectors $|\omega_n>$:

$$Q_{L}|\omega_n\rangle=\partial_\lambda f_{\underset{L}{\mu_1\ldots\mu_n}}\underset{L}{c^\lambda}\underset{L}{c^{\mu_1}}\ldots\underset{L}{c^{\mu_n}}|0\rangle,$$

$$Q_{R}|\omega_n\rangle=\partial_\lambda f_{\underset{R}{\mu_1\ldots\mu_n}}\underset{R}{c^\lambda}\underset{R}{c^{\mu_1}}\ldots\underset{R}{c^{\mu_n}}|0\rangle$$

that have the same structure as differential n-forms. Notice that it is important to have here the same kind of creation operators in each expression. Then the cohomologies for Q_L and Q_R are analogous to the standard de Rham cohomologies. However, the physical states defined by these cohomologies are different because of the different creation operators associated with the same equivalence classes of coefficient functions. The reason for this is that ghosts reflect the symmetry under transformations and these can be either left or right shifts in our case.

There is also another possibility for defining ghost operators. We can identify the coordinate-dependent ghost operators.

$$c^\mu(x)=\underset{L}{c^\mu}(x)=\underset{R}{c^\mu}(x),\quad b_\mu(x)=\underset{L}{b_\mu}(x)=\underset{R}{b_\mu}(x).$$

Then we have, in fact, started with two sets of coordinate-independent ghost operators, such that

$$L^\mu_\alpha(x)\underset{L}{c^\alpha}=c^\mu(x),\quad (L^{-1})^\alpha_\mu(x)\underset{L}{b_\alpha}=b_\mu(x),$$

$$R^\mu_\alpha(x)\underset{R}{c^\alpha}=c^\mu(x),\quad (R^{-1})^\alpha_\mu(x)\underset{R}{b_\alpha}=b_\mu(x).$$

The transition matrix between the left and right coordinate-independent ghost operators is now

$$(\nabla_{LR})^\alpha_\beta=(L^{-1})^\alpha_\sigma R^\sigma_\beta.$$

The Fock space is defined by

$$|\omega_n\rangle=f_{\mu_1\ldots\mu_n}(x)c^{\mu_1}(x)\ldots c^{\mu_n}(x)|0\rangle$$

and the action of Q_L, Q_R is

$$Q_L|\omega_n\rangle=Q_R|\omega_n\rangle=\partial_\lambda f_{\mu_1\ldots\mu_n}c^\lambda c^{\mu_1}\ldots c^{\mu_n}|0\rangle.$$

Its cohomology is analogous to the standard de Rham cohomology of differential forms.

7. DISCUSSION

There are several facts that indicate a possible role of geodesic multiplication in the theory of gravity [4, 17, 18]. In the case of a flat space-time with orthonormal coordinates x_μ, the right and left geodesic translations coincide, $R^{\text{flat}}_a=L^{\text{flat}}_a$. The geodesic multiplication $x\to x\cdot a=$ $=a\cdot x$ describes a rigid shift of the space-time, $x^\mu\to x^\mu+a^\mu$, $a^\mu=\text{const}$. The latter transformation is a Poincaré translation. In the case of a curved space-time, matrices of infinitesimal left and right translations are different, as has explicitly been demonstrated in Sec. 5. They also have different meanings. An infinitesimal right translation given by Eq. (20b) describes an infinitesimal shift of a point x (or a geodesic line $x(t)$) in the direction of y (Fig. 1). An infinitesimal left translation given by Eq. (20a) describes an infinitesimal shift of a point y in the direction that is parallel to the tangent vector $\frac{dx(e)}{dt}$ (Fig. 2).

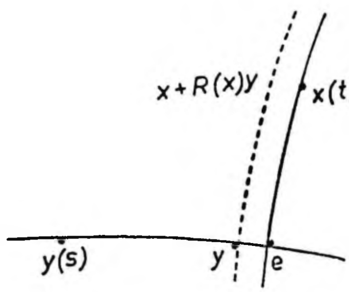

Fig. 1. Right translation as an infinitesimal shift of a geodesic line $x(t)$ in the direction of y.

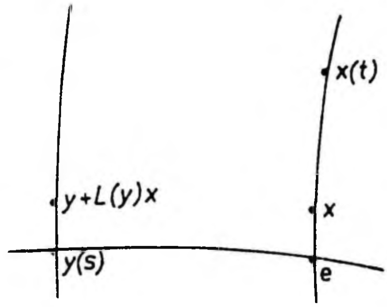

Fig. 2. Left translation as an infinitesimal shift of a point y in the direction that is parallel to the tangent vector $\frac{dx(e)}{dt}$.

The geodesic loop has a close connection with the group of general coordinate transformations (diffeomorphisms) which is sometimes considered as the gauge group for the theory of gravity [13]. Left and right translations of a loop generate a group that is called the Albert group of this loop [11, 12]. Nonassociativity of the loop can be measured by the deviation from the unity of the following elements of its Albert group,

$$L(g,h) \equiv L_{gh}^{-1} L_g L_h, \quad R(g,h) \equiv R_{gh} R_g^{-1} R_h^{-1}, \quad M(g,h) \equiv R_g L_h^{-1} R_g^{-1} L_h. \tag{29}$$

In particular, the left and right translations (17) L_x and R_x of the local geodesic loop M_e generate a subgroup in the group of space-time diffeomorphisms, the Albert group of M_e. Indeed, the left and right geodesic translations (17) determine diffeomorphisms given by

$$x \cdot y \equiv L_x y \equiv R_y x, \quad L_x, R_y \in \mathrm{Diff}\, M_e. \tag{30}$$

The pair (L, R) of the maps $x \to L_x$, $x \to R_x$ can be considered as a regular (bi)representation of the geodesic loop M_e. However, the analytical description of the Albert group of M_e is extremely complicated due to the definition (17) of geodesic multiplication that involves integrations of geodesic and parallel transport equations.

In a full theory, the space-time may contain also quantized matter fields. According to the idea that the geodesic loop is the most natural generalization of the Poincaré translations, matter fields may be described by suitable representations of the geodesic loop. In this way one can achieve the replacing of the infinite dimensional group of diffeomorphisms Diff M by finite dimensional geodesic loop and its Albert group.

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GEODEETILINE KORRUTAMINE JA GEOMEETRILISED BRST-SARNASED OPERAATORID

Piret KUUSK, Jüri ÕRD, Eugen PAAL

Poincaré translatsioonid on üldistatud mitteassotsiatiivseks algebraliseks süsteemiks, mida nimetatakse geodeetiliseks luubiks. Lokaalsete geodeetiliste vasak- ja paremnihete maatriksite abil on konstrueeritud kaks BRST-sarnast operaatorit. Näitena on arvutatud lokaalsete geodeetiliste nihete maatriksite ilmutatud kuju nõrka gravitatsioonilist tasalainet kirjeldavates aegruumides.

ГЕОДЕЗИЧЕСКОЕ ПРОИЗВЕДЕНИЕ И ГЕОМЕТРИЧЕСКИЕ БРСТ-ПОДОБНЫЕ ОПЕРАТОРЫ

Пирет КУУСК, Юрий ЭРД, Эуген ПААЛ

Предложено обобщение трансляций Пуанкаре, приводящее к неассоциативной алгебраической системе, называемой геодезической лупой. При помощи матриц правых и левых геодезических трансляций сконструированы два БРСТ-подобных оператора. В качестве примера вычислены матрицы локальных левых и правых геодезических трансляций в пространстве-времени слабой плоской гравитационной волны.

IV

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Quantum kinematics of a test particle in a curved spacetime

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Abstract. A possible model for the quantum kinematics of a test particle in curved spacetime is proposed. Every reasonable neighbourhood of a curved spacetime can be equipped with a non-associative binary operation called geodesic multiplication. Its infinitesimal right translations are used to define the (geodesic) momentum operators. The corresponding commutation relations are taken as the quantum kinematic algebra. It coincides with the usual canonical Poisson algebra (Weyl's kinematics) only in the case of flat spacetime. A BRST-like operator is constructed and its physical meaning is discussed. As an example, detailed calculations are performed for the spacetime of a weak plane gravitational wave.

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1. Introduction

The kinematics of a non-relativistic quantum test particle are characterized by its position operators $\boldsymbol{x}^i$ and momentum operators $\boldsymbol{p}_k$ which satisfy the canonical commutation relations:

$$[\boldsymbol{x}^i, \boldsymbol{x}^k] = 0, \qquad [\boldsymbol{p}_i, \boldsymbol{p}_k] = 0, \qquad [\boldsymbol{x}^i, \boldsymbol{p}_k] = i\hbar\delta^i_k, \qquad i, k = 1, 2, 3. \tag{1}$$

In the position-space Schrödinger representation the operators $\boldsymbol{x}^i, \boldsymbol{p}_k$ are

$$\boldsymbol{x}^i = x^i, \qquad \boldsymbol{p}_i = -i\hbar\frac{\partial}{\partial x^i}. \tag{2}$$

Let us now consider a quantum test particle in a curved spacetime M (possibly with torsion). There are no privileged coordinates in the curved spacetime which correspond to the global Cartesian coordinates used in (2). However, one can imitate flat spacetime by considering a set of neighbourhoods $M_e \subset M$, where every neighbourhood M_e is endowed with a local Riemann normal coordinate system with the origin of coordinates at $e \in M_e$. Coordinate lines in M_e are the geodesics emerging from e and the metric tensor at e is Minkowskian. Every neighbourhood M_e is such that the Riemann normal coordinates are well defined and geodesics emerging from e do not intersect, i.e. there is no gravitational lensing or singularities.

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Every neighbourhood M_e can be seen as a non-associative binary algebraic system called a (local) geodesic loop [1–4]. The local geodesic multiplication in M_e determines the left- and right-translation operators L_x, R_y (section 2). Local geodesic loops M_e, $M_{e'}$, attached to different origins e, e', must satisfy suitable patching (geo-odular) conditions [2].

Following non-relativistic kinematics, we define in M_e position operators $\boldsymbol{x}^i$ as multiplication by the Riemann normal coordinates x^i and momentum operators $\boldsymbol{p}_i$ via infinitesimal right geodesic translations (section 3). Evidently, in a curved spacetime these coordinate and momentum operators do not satisfy the canonical commutation relations (1). The modified commutators must be found from the geodesic loop representation. A somewhat analogous construction for position and momentum operators was recently proposed by Kempf [5], but he did not notice its connection with geodesic loops.

In the case of the Minkowski spacetime, left and right translations of the geodesic multiplication coincide and amount to rigid shifts of the spacetime; one in fact obtains the common vector addition rule in Minkowski spacetime. In a curved spacetime, the left and right translations of the geodesic loops are deformed and need not coincide, but remain as local diffeomorphisms of the spacetime. Diffeomorphisms can be considered as symmetries of the physical models in a curved spacetime. In the framework of the Becchi–Rouet–Stora–Tyutin (BRST) quantization [6], ghost fields are introduced and a nilpotent BRST operator constructed. The main point of the BRST quantization lies in the statement that different physical states correspond to the different cohomology classes of the nilpotent BRST operator and physical observables must commute with it. An explicit construction of the BRST operator is based on the algebra of gauge transformations and its nilpotency is a consequence of the Jacobi identity.

The group of diffeomorphisms is an infinite-dimensional Lie group with structure constants that differ essentially from the structure constants of infinite-dimensional (local) Lie groups used in Yang–Mills theories. There have been several attempts to find a nilpotent geometrical BRST operator Q for quantum gravity, e.g. by Bars and Yankielowicz [7]. It involves an infinite-dimensional algebra of a modified Poincaré group where the torsion and curvature tensors act as structure functions. The nilpotency $Q^2 = 0$ turns out to be a consequence of the differential geometrical Bianchi identities. An analogous geometrical BRST operator for a differentiable manifold with zero curvature but non-zero torsion tensor has been given by Okubo [8]. It is remarkable that both BRST operators can be introduced purely kinematically, without referring to any underlying action.

Following Okubo's construction it is possible to construct two nilpotent operators (we shall call them BRST-like operators) containing the left and right local frame vector fields with corresponding structure functions of the geodesic multiplication. However, only one of them acquires a meaning analogous to the conventional BRST operator (section 4).

As an example, the explicit form of the momentum operators and the quantum kinematic algebra for a weak plane gravitational wave is calculated (section 5).

2. Geodesic multiplication and translations

Let us consider a 4-dimensional differentiable manifold M with an affine connection (the spacetime). Its geodesics (autoparallels) $x^\mu(t)$ satisfy the differential equations

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0, \tag{3}$$

where $\Gamma^\mu_{\nu\rho}(x)$ denote the affine connection coefficients. The unique solution of (3) at initial conditions $x^\mu(0) = e^\mu$, $(dx^\mu/dt)\big|_{t=0} = X^\mu$ determines the exponential mapping

$T_eM \to M_e: X \mapsto x = \exp_e X \equiv x(1; X)$. A parallel transport mapping $\tau_y^e: T_eM \to T_yM$ along a geodesic $y(s)$ emerging from e is given by the unique solution of the Cauchy problem

$$\frac{\mathrm{d}X'^\mu}{\mathrm{d}s} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}y^\nu}{\mathrm{d}s}X'^\rho = 0, \qquad X'^\mu(0) = X^\mu. \tag{4}$$

Using the exponential mapping and the parallel transport mapping, the local geodesic multiplication of points $x, y \in M_e$ is defined by [1–4]

$$xy \equiv L_x y \equiv R_y x = \left(\exp_y \circ \tau_y^e \circ \exp_e^{-1}\right)x. \tag{5}$$

The local geodesic multiplication can be constructed in every neighbourhood M_e where all required exponential mappings and parallel transport operations are well defined; M_e is a finite region of M, where geodesics emerging from e do not intersect and which does not contain singular points.

By introducing the local geodesic multiplication, M_e can be seen to be a binary algebraic system called the local geodesic loop [1–4]. The point $e \in M$ is the unit element of the loop M_e and the (left) inverse x_L^{-1} of $x \in M_e$ is defined by $x_L^{-1}x = e$.

In general, the geodesic multiplication need not be commutative and associative. In the Riemann normal coordinates with the origin at e, equations for geodesics (3) and parallel transport (4) can be solved using expansions in local coordinates. By defining structure constants $C^\mu_{\nu\rho}$, $A^\mu_{\nu\rho\sigma}$ by

$$\left((yx)_L^{-1}(xy)\right)^\mu = C^\mu_{\nu\rho}x^\nu y^\rho + \cdots, \tag{6}$$

$$\left((x(yz))_L^{-1}((xy)z)\right)^\mu = A^\mu_{\nu\rho\sigma}x^\nu y^\rho z^\sigma + \cdots, \tag{7}$$

the direct calculations [3] demonstrate that the above commutator and associator of the local geodesic multiplication are intimately related to the torsion tensor $S^\mu_{\nu\rho}(x)$ and curvature tensor $R^\mu_{\nu\rho\sigma}(x)$ of M:

$$C^\mu_{\nu\rho} = 2S^\mu_{\nu\rho}(e), \qquad A^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma}(e) - \nabla_\sigma S^\mu_{\nu\rho}(e), \tag{8}$$

where ∇_ν is the covariant differentiation operator.

The local geodesic multiplication (5) determines the following left (L) and right (R) infinitesimal translation matrices:

$$(xy)^\mu = y^\mu + L^\mu_\nu(y)x^\nu + \cdots, \qquad L^\mu_\nu(y) \equiv \left.\frac{\partial(xy)^\mu}{\partial x^\nu}\right|_{x=e}, \tag{9}$$

$$(xy)^\mu = x^\mu + R^\mu_\nu(x)y^\nu + \cdots, \qquad R^\mu_\nu(x) \equiv \left.\frac{\partial(xy)^\mu}{\partial y^\nu}\right|_{y=e}. \tag{10}$$

Note that the lower indices of the matrices L^μ_ν, R^μ_ν are in fact flat and belong to the tangent space T_eM with the Lorentz group as a symmetry group. In the following let us denote flat indices by Latin letters. It follows from (9), (10) that

$$L^\mu_i(e) = \delta^\mu_i = R^\mu_i(e). \tag{11}$$

Both matrices are locally invertible, so L^μ_i and R^μ_i can be used for constructing two preferred local vierbein fields in M_e. The corresponding local frame fields read

$$L_i(x) \equiv L^\mu_i(x)\partial_\mu, \qquad R_i(x) \equiv R^\mu_i(x)\partial_\mu. \tag{12}$$

The algebraic meaning of the components of the frame fields (9), (10) can be seen via the corresponding differentials. For a given $X \in T_e M$ we have

$$L_X(x) \equiv X^i L_i(x) = (\mathrm{d}R_x)_e X \in T_x M, \tag{13}$$

$$R_X(x) \equiv X^i R_i(x) = (\mathrm{d}L_x)_e X \in T_x M. \tag{14}$$

It is well known that for two vector fields their commutator is again a vector field. We know that $L_i(x)$ and $R_i(x)$ are frame fields, so it is quite natural to define the structure functions $\lambda^k_{ij}(x)$ and $\rho^k_{ij}(x)$ by

$$[L_i(x), L_j(x)] = -\lambda^k_{ij}(x) L_k(x), \tag{15}$$

$$[R_i(x), R_j(x)] = +\rho^k_{ij}(x) R_k(x). \tag{16}$$

The structure functions need not coincide. Instead, we have the expansions

$$\lambda^k_{ij}(x) = C^k_{ij} - 2A^k_{[ij]n} x^n + \cdots, \tag{17}$$

$$\rho^k_{ij}(x) = C^k_{ij} + 2A^k_{n[ij]} x^n + \cdots, \tag{18}$$

so the initial conditions read

$$\lambda^k_{ij}(e) = \rho^k_{ij}(e) = 2S^k_{ij}(e) \equiv C^k_{ij}. \tag{19}$$

In (17) and (18), the associator A^k_{ijm} again appears, but in altered forms.

If M_e is a local (geodesic) Lie group, then $[L_i(x), R_j(x)] = 0$ and the Maurer–Cartan equations read

$$\lambda^k_{ij}(x) = \rho^k_{ij}(x) = C^k_{ij}. \tag{20}$$

In general, these two frame fields need not commute,

$$[L_i(x), R_j(x)] = A^k_{ij}(x)\partial_k, \tag{21}$$

this is caused by the non-associativity of the geodesic multiplication,

$$A^k_{ij}(x) = A^k_{inj} x^n + \cdots = \left(R^k_{inj}(e) - \nabla_j S^k_{in}(e)\right) x^n + \cdots. \tag{22}$$

3. Geodesic momentum operators and kinematic algebra

Let us introduce an action of the position operators $\boldsymbol{x}^\mu$ (on scalar valued functions) as multiplication with the Riemann normal coordinates x^μ. Then we have $[\boldsymbol{x}^\mu, \boldsymbol{x}^\nu] = 0$. We propose to define (geodesic) momentum operators $\boldsymbol{p}_i$ via infinitesimal right-geodesic translations.

By construction, the right-geodesic translations

$$x'^\mu = (x\alpha)^\mu = x^\mu + R^\mu_i(x)\alpha^i + \mathrm{O}(\alpha^2) \tag{23}$$

transform geodesics into adjacent geodesics (figure 1). In general, vectors $R_i(x)$ cannot be obtained from $R_i(e)$ via parallel transport. Left-geodesic translations

$$y'^\mu = (\beta y)^\mu = y^\mu + L^\mu_i(y)\beta^i + \mathrm{O}(\beta^2) \tag{24}$$

move points along geodesics (figure 2). Lines $y'(s) = y(s) + L_i(y(s))\beta^i$ need not be geodesics, but vectors $L_i(y)$ are tangent to the geodesics emerging from y and obtained from $L_i(e)$ via parallel transport along geodesics.

Define the geodesic momentum operators by $\boldsymbol{p}_k(x) = -\mathrm{i}\hbar R^\mu_k(x)\partial_\mu$. Then,

$$[\boldsymbol{x}^\mu, \boldsymbol{p}_k(x)] = [\boldsymbol{x}^\mu, -\mathrm{i}\hbar R^\nu_k(x)\partial_\nu] = \mathrm{i}\hbar R^\mu_k(x). \tag{25}$$

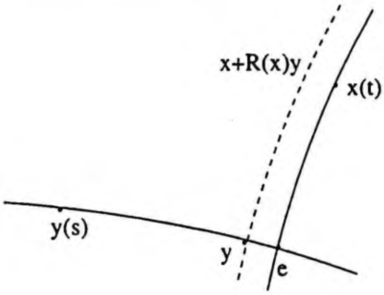

Figure 1. Right translation as an infinitesimal shift of a geodesic $x(t)$ in the direction of y.

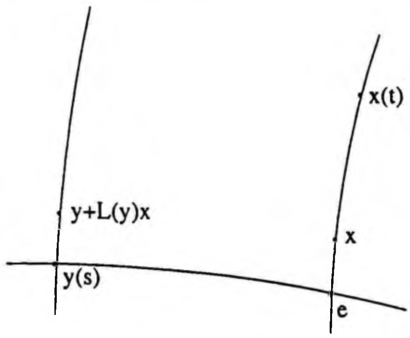

Figure 2. Left translation as an infinitesimal shift of a point y in the direction parallel to the tangent vector $\mathrm{d}x(e)/\mathrm{d}t$.

In a torsion-free spacetime, an expansion of $R_i^\mu(x)$ in the Riemann normal coordinates ($\Gamma^\mu_{\nu\rho}(e) = 0$) can be found by using (23) and the geodesic multiplication formula of Akivis [3]

$$(x\alpha)^\mu = x^\mu + \alpha^\mu - \tfrac{1}{2}\Gamma^\mu_{\nu\rho,\sigma}(e)x^\nu x^\rho \alpha^\sigma - \tfrac{1}{2}\Gamma^\mu_{\nu(\rho,\sigma)}(e)x^\nu \alpha^\rho \alpha^\sigma + \cdots . \tag{26}$$

As a result we get

$$[x^\mu, p_k(x)] = \mathrm{i}\hbar R_k^\nu(e)\big(\delta_\nu^\mu - \tfrac{1}{3}J^\mu{}_{\nu\alpha\beta}(e)x^\alpha x^\beta + \mathrm{O}(x^3)\big), \tag{27}$$

where $J^\mu{}_{\nu\alpha\beta}$ denotes the Jacobi curvature tensor,

$$J^\mu{}_{\nu\alpha\beta} \equiv \tfrac{1}{2}(R^\mu{}_{\alpha\nu\beta} + R^\mu{}_{\beta\nu\alpha}). \tag{28}$$

The same expression (27) for the commutator was presented by Kempf [5], but he introduced the momenta as generators of the passive coordinate transformations describing a change of the Riemann coordinates.

So, if we introduce the geodesic momentum operators $p_i = -\mathrm{i}\hbar R_i^\mu \partial_\mu$, the (geodesic) kinematic algebra of a quantum test particle in a curved spacetime is

$$[x^\mu, x^\nu] = 0, \qquad [x^\mu, p_k] = \mathrm{i}\hbar R_k^\mu, \qquad [p_j, p_k] = -\mathrm{i}\hbar \rho^n_{jk} p_n. \tag{29}$$

Note that the modification of commutation relations does not introduce any new dimensionful constants.

The algebra (29) can be considered as a generalization of the algebra proposed by Krause [11] in a non-Abelian group quantization, we use the structure functions of a geodesic multiplication instead of the structure constants of a non-Abelian Lie group. The group quantization (Krause kinematics) is a modification of the canonical quantization first proposed by Weyl [12]. Consider the position and momentum operators $\boldsymbol{x}, \boldsymbol{p}$ acting on a Hilbert space, which carries a regular representation of the group (e.g. the position operators can be defined by multiplication with group parameters and the momentum operators using the left and/or right translation operators). Consider the unitary transformations on this Hilbert space generated by $\boldsymbol{x}$ and $\boldsymbol{p}$. According to Weyl [12], in the case of an Abelian group, the statement that these unitary transformations commute up to a phase leads to the canonical commutator algebra (1). In the case of a non-Abelian group, the momentum operators need not commute and canonical commutation relations are modified. So the group quantization construction is based on the regular representation of an Abelian or a non-Abelian group. This idea cannot be directly followed in the case of geodesic loops, since the representation theory of non-associative algebras is still essentially lacking. However, by means of the above generalization of the ordinary position-space Schrödinger representation, we have a similar commutator algebra (29), which is a natural generalization of the Weyl and Krause kinematics. Both the Weyl and Krause kinematics are too rigid for a quantum test particle in the curved spacetime of general relativity.

4. BRST-like operators

Following Bars and Yankielowicz [7] and Okubo [8], in our previous papers [9, 10] we have proposed two BRST-like nilpotent operators where in the place of the structure constants of a non-Abelian Lie group are the structure functions $\lambda^k_{ij}(x)$ and $\rho^k_{ij}(x)$ of the geodesic multiplication defined by (15), (16):

$$Q_L(x) \equiv c^i L_i(x) + \tfrac{1}{2} c^i c^j b_k \lambda^k_{ij}(x), \tag{30}$$

$$Q_R(x) \equiv c^i R_i(x) - \tfrac{1}{2} c^i c^j b_k \rho^k_{ij}(x). \tag{31}$$

Here c^i and b_i denote auxiliary coordinate-independent operators (ghosts and antighosts) satisfying the anticommutation relations

$$b_i c^j + c^j b_i = \delta^j_i , \qquad b_i b_j + b_j b_i = 0 = c^i c^j + c^j c^i \tag{32}$$

and commuting with the left and right frame fields.

Now, using the geodesic momentum operators, the BRST-like operator can be redefined as follows:

$$Q(x) \equiv \frac{\mathrm{i}}{\hbar} c^j \boldsymbol{p}_j(x) - \tfrac{1}{2} c^j c^k b_n \rho^n_{jk}(x). \tag{33}$$

The nilpotency $\boldsymbol{Q}^2 = 0$ is guaranteed by the Jacobi (Bianchi) identities for the geodesic momentum operators $\boldsymbol{p}_i$,

$$\sum_{cyclic} \left[\boldsymbol{p}_i, [\boldsymbol{p}_j, \boldsymbol{p}_k]\right] = 0. \tag{34}$$

BRST-like cohomology is defined as the coset $\ker \boldsymbol{Q} / \operatorname{im} \boldsymbol{Q}$. Note that the identities (34) hold for all vector fields and not only for those given in terms of geodesic translations.

Vector fields are generators of the spacetime diffeomorphisms. The Lie algebras corresponding to our geodesic frame fields are given by (15), (16). In the case of a (local geodesic) Lie group, the structure functions are constant, the Lie algebras (15), (16) are finite-dimensional and (30), (31) give conventional BRST operators.

In section 3, we clarified the novel physical meaning of the structure functions ρ^k_{ij}, which occur in the BRST-like operator (33)—these arose in the structure relations of the kinematic algebra (29). In a standard BRST quantization, physical state space is claimed to be the cohomology of a BRST operator. By analogy, the cohomology of the BRST-like operator Q can be conjectured to be the physical state space of a quantum test particle in a curved spacetime.

The vacuum state $|0\rangle$ is defined by using the ghost annihilation operator b_i as in [8]: $b_i|0\rangle = 0$. Coordinate dependent ghost operators are introduced by means of the right frame field R_i^μ as

$$b_\mu(x) = b_i(R^{-1})^i_\mu(x), \qquad c^\mu(x) = c^i R_i^\mu(x). \tag{35}$$

Due to regularity of the matrix R_i^μ, the condition $b_\mu(x)|0\rangle = 0$ is equivalent to the above definition of the vacuum state. The state vectors of a quantum test particle in a curved spacetime are defined as

$$|\omega_n\rangle = f_{i_1\cdots i_n}c^{i_1}\cdots c^{i_n}|0\rangle = f_{\mu_1\cdots\mu_n}(x)c^{\mu_1}(x)\cdots c^{\mu_n}(x)|0\rangle. \tag{36}$$

Then, it turns out [8] that an action of the BRST-like operator on a state vector $|\omega_n\rangle$ reads

$$Q|\omega_n\rangle = \partial_\lambda f_{\mu_1\cdots\mu_n}c^\lambda c^{\mu_1}\cdots c^{\mu_n}|0\rangle. \tag{37}$$

So, the action of Q on physical states has the same structure as the action of the de Rham co-boundary operator on differential forms.

5. An example: quantum kinematics in a weak plane gravitational wave

In order to determine explicitly the expression of the local geodesic product of two arbitrary points $x, y \in M_e$ of a manifold M with a given affine connection $\Gamma^\lambda_{\mu\nu}(x)$, we need to integrate (3), (4) for geodesics and parallel transport in arbitrary directions. It turns out to be analytically a rather complicated task even in the seemingly simple case of a 2-sphere. To give an example which can be worked out analytically, let us present explicit expressions for a quantum test particle in a weak plane gravitational wave.

The metric tensor can be given as perturbations around the Minkowski metric (we take $c = 1 = \hbar$ here):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1).$$

In the case of a polarized weak plane gravitational wave moving in the direction of x^1, the only non-zero components of $h_{\mu\nu}$ in the TT-gauge are

$$h_{22} = -h_{33} = A\cos\omega(x^0 - x^1).$$

Here $A = \text{constant} \ll 1$ is the wave amplitude, and all subsequent equations hold in the linear approximation in A.

In these coordinates the equation of a geodesic $x^\mu(t)$ with a tangent vector X^μ at a point e can be easily integrated, yielding

$$x^\mu(t) = X^\mu t + AU^\mu \frac{\sin\omega Ct - \omega Ct}{\omega C^2},$$

where we have taken $e^\mu = 0$ and denoted

$$C = X^0 - X^1,$$

$$U^0 = U^1 = -\tfrac{1}{2}[(X^2)^2 - (X^3)^2],$$

$$U^2 = -X^2 C, \qquad U^3 = X^3 C.$$

The coordinates X^μ are also the Riemann normal coordinates of a point x with TT-coordinates $x^\mu \equiv x^\mu(1)$.

Let us choose the point e to be a unit element of the geodesic loop and let x, y be two points from its neighbourhood. We denote by X^μ and Y^μ the tangent vectors of geodesics joining the point e with the points x and y, respectively.

To calculate the product of the points x, y, we must integrate the corresponding equations of geodesics (3) and of parallel transport (4) of the tangent vector X^μ. The calculation can be most easily done in TT-coordinates and gives the following result:

$$(xy)^\mu = x^\mu + y^\mu + AU^\mu \frac{1}{\omega C^2}\left[\sin\omega D(\cos\omega C - 1) + (\cos\omega D - 1)(\sin\omega C - \omega C)\right] + AV^\mu \frac{1}{2D}(1 - \cos\omega D), \tag{38}$$

where

$$D = Y^0 - Y^1,$$

$$V^0 = V^1 = X^2Y^2 - X^3Y^3,$$

$$V^2 = CY^2 + DX^2, \qquad V^3 = -CY^3 - DX^3.$$

The previous equations contain no singularities if $C = 0$ or $D = 0$. The equations corresponding to these special cases can be obtained by just taking the limit $C \to 0$ or $D \to 0$.

Transforming the expression of the geodesic multiplication to the Riemann normal coordinates X^μ, we can evaluate the matrix of the right translations

$$R_j^\mu(X) = \delta_j^\mu + A\frac{\sin\omega C - \omega C}{\omega C^3}(2U^\mu \partial_j C - \partial_j U^\mu C).$$

Taking into account that if working in the linear approximation in A, we have $A\partial_j = A\delta_j^\mu \partial_\mu$ and the inverse $(R_j^\mu)^{-1}$ differs from R_j^μ only by the sign in front of the second term. According to our proposal, quantities $p_j = -\mathrm{i}R_j^\mu \partial_\mu$ represent momentum operators in the generalized Schrödinger representation.

The structure functions defined by (16) can be calculated to be

$$\rho_{jk}^n = \frac{2A}{\omega C^2}\sin^2\frac{\omega C}{2}(\partial_k U^n \partial_j C - \partial_j U^n \partial_k C). \tag{39}$$

The kinematic algebra (29) of a quantum test particle in the weak plane gravitational wave is thus

$$[\boldsymbol{x}^\mu, \boldsymbol{x}^\nu] = 0,$$

$$[\boldsymbol{x}^\mu, \boldsymbol{p}_k] = \mathrm{i}\delta_k^\mu + \mathrm{i}A\frac{\sin\omega C - \omega C}{\omega C^3}(2U^\mu \partial_k C - \partial_k U^\mu C),$$

$$[\boldsymbol{p}_j, \boldsymbol{p}_k] = \frac{2\mathrm{i}A}{\omega C^2}\sin^2\frac{\omega C}{2}(\partial_k U^n \partial_j C - \partial_j U^n \partial_k C)\boldsymbol{p}_n.$$

BRST-like operator can be written down using the structure functions (39) according to (33).

6. Discussion

We have presented the kinematical structure of a relativistic quantum mechanics of a test particle in a curved spacetime. In flat spacetime, the full first-quantized theory of a relativistic particle is plagued by negative probabilities and must be replaced by a second-quantized quantum field theory. Although probably unrealistic, our model has several interesting features. It allows one to introduce the modified commutation relations (29) of position and momentum operators. By expanding this idea, one can obtain a modified principle of uncertainty and analyse its consequences. Following the ideas of Bars and Yankielowicz [7] and Okubo [8], we have introduced a geodesic BRST-like operator (33) and undertook its physical interpretation.

In the case of a physical theory with an action functional given in flat spacetime, the significance of the momentum operator as the generator of spacetime translations is based on Noether's theorem and the existence of the corresponding conserved currents. A possible generalization of the Noether procedure to the case of a Lagrangian with non-associative Moufang symmetries has recently been presented in [13]. We have proposed a definition for the modified momentum operators of a quantum test particle in a curved spacetime based on a non-associative geodesic multiplication operation. Perhaps this is the proper reflection of the fact that in the framework of general relativity, (canonical) conserved energy and momentum cannot be introduced.

Finally, let us indicate an additional possibility for the interpretation of the above-presented approach. Modified parts of the commutation relations (29) and the BRST-like operator (33) are given via the geodesic infinitesimal translation operators R_i, which depend (in a sophisticated way) on the affine connection of the curved spacetime. Thus they characterize not only a possible kinematics of a freely moving quantum test particle in a curved spacetime but also the background gravitational field itself.

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Kinematics and uncertainty relations of a quantum test particle in a curved space-time

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Abstract

A possible model for quantum kinematics of a test particle in a curved space-time is proposed. Every reasonable neighbourhood V_e of a curved space-time can be equipped with a nonassociative binary operation called the geodesic multiplication of space-time points. In the case of the Minkowski space-time, left and right translations of the geodesic multiplication coincide and amount to a rigid shift of the space-time $x \rightarrow x + a$. In a curved space-time infinitesimal geodesic right translations can be used to define the (geodesic) momentum operators. The commutation relations of position and momentum operators are taken as the quantum kinematic algebra. As an example, detailed calculations are performed for the space-time of a weak plane gravitational wave. The uncertainty relations following from the commutation rules are derived and their physical meaning is discussed. © 1998 Elsevier Science B.V.

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Keywords: Geodesic multiplication; Commutator algebra; Uncertainty relations

1. Introduction

The Poincaré group – the symmetry group of the flat space-time M – and its representations are basic constitutive elements for relativistic theories. A generic curved space-time V doesn't allow symmetry groups and the Poincaré group looses its central role. The Lorentz group can be considered as the symmetry group of flat tangent spaces, but the status of the Poincaré translations is unclear.

The Poincaré translations form an Abelian group and describe rigid shifts along straight lines of a flat space-time, $x \rightarrow x + a$, $a = \text{const}$. Straight lines are geodesic lines of the Minkowski space. In a curved space-time analogous geodesic translations can be introduced in a (finite) neighbourhood V_e of $e \in V$ using the concept of geodesic multiplication of points $x, y \in V_e$. The neighbourhood V_e together with the binary operation of geodesic multiplication constitutes an algebraic system called local geodesic loop. In general, it is noncommutative and nonassociative [1–3]. As a result we obtain a novel generalization of Poincaré shifts to the case of a curved space-time (Section 2).

In the present paper we investigate some prospects of using local geodesic loops for constructing a

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quantum kinematics in the background of a curved space-time (Section 3). Let us introduce an action of the position operators x^i (on scalar valued functions) as multiplication with the Riemann normal coordinates x^i. We propose to define (geodesic) momentum operators p_k via infinitesimal right geodesic translations by $p_k(x) = -i\hbar R_k^s(x)\partial_s$. The corresponding commutation relations are taken as the quantum kinematic algebra [4]:

$$[x^i, x^k] = 0, \quad [x^i, p_k] = i\hbar R_k^i(x),$$

$$[p_j, p_k] = -i\hbar\, \rho_{jk}^n(x)\, p_n.$$

The uncertainty relations which follow from the modified $[x^i, p_k]$ commutator can put restrictions on minimal values of coordinates and momenta. Analogous modified uncertainty relations and restrictions on the measurability of distances have recently been presented in the framework of quantum group $SU_q(n)$ Heisenberg algebras [5–7], in the formalism of quantum κ-deformed Poincaré groups [8,9] and in the string theory [10,11].

As an example of our formalism, detailed calculations are performed in the case of a weak plane gravitational wave background (Section 4).

There seems to be also another possibility of using the concept of local geodesic loops for constructing quantum kinematics in the background of a curved space-time. In the flat space-time quantum field theory, one-particle states are introduced as representations of the Poincaré group and their momentum is identified as eigenvalues of the Poincaré translation operators P_μ. We could mimic it by defining one-particle states in a curved background via representations of geodesic loops. However, since the representation theory of general nonassociative structures is still essentially lacking, we cannot hope a quick progress along these lines (Section 5).

2. Geodesic loops and geodesic translations

Let us consider a manifold V with a symmetric (torsionless) affine connection $\Gamma_{\mu\nu}^\lambda(x) = \Gamma_{\nu\mu}^\lambda(x)$. Let $x, y \in M_e$ be two points of a neighbourhood V_e of $e \in V$ such that geodesic arches between each two points are uniquely determined. Geodesic multiplication in respect of the unit element e is defined by the following formula [1,2]

$$x \cdot y \equiv L_x y \equiv R_y x = \left(\exp_y \circ \tau_y^e \circ \exp_e^{-1}\right) x. \qquad (1)$$

Here $\exp_e X$ denotes exponential mapping $X \to x$, $X \in T_e V$, $x \in V$, and $\tau_y^e : T_e V \to T_y V$ is the parallel transport mapping of tangent vectors from $T_e V$ into $T_y V$ along the unique local geodesic arch joining the points e and y. By L_x and R_y we have defined the left (L) and the right (R) translation operators in analogy with the case of groups.

From the definition of the geodesic multiplication (1) it follows that in the case of the Minkowski space-time with orthonormal coordinates x, the right and left geodesic translations coincide, $R_a^{\text{flat}} = L_a^{\text{flat}}$ and the geodesic multiplication $x \to x \cdot a = a \cdot x$ describes a rigid shift of the space-time, $x \to x + a$.

Let us introduce the following infinitesimal right translation matrix:

$$(x \cdot y)^\mu = x^\mu + R_\nu^\mu(x) y^\nu + \dots,$$

$$R_\nu^\mu(x) \equiv \frac{\partial (x \cdot y)^\mu}{\partial y^\nu}\Big|_{y=e}. \qquad (2)$$

Matrix $R_\nu^\mu(x)$ can be used to introduce a local frame field [12,13]

$$R_\nu(x) \equiv R_\nu^\mu(x) \partial_\mu. \qquad (3)$$

From Eq. (2) it follows that in the unit element e we have $R_\nu^\mu(e) = \delta_\nu^\mu$. The commutator of vector fields $R_\nu(x)$ define the structure functions $\rho_{\mu\nu}^\sigma(x)$:

$$\left[R_\mu(x), R_\nu(x)\right] = \rho_{\mu\nu}^\sigma(x) R_\sigma(x). \qquad (4)$$

Let us specify the coordinates $x \in V_e$ to be the Riemann normal coordinates, i.e. the equations of geodesics emerging from e are

$$x^i(t) = X^i t, \quad X^i \in T_e V. \qquad (5)$$

Now the equations of exponential mapping and parallel transport which determine the geodesic multiplication (1) can be integrated in the neighbourhood of e as power series in x [14,3] and the following expansion for structure functions $\rho_{ij}^k(x)$ can be calculated:

$$\rho_{ij}^k(x) = 2 R_{n[ij]}^k(e) x^n + \dots. \qquad (6)$$

Here $R^k_{nij}(e)$ denote the components of the Riemann curvature tensor at the origin of coordinates e. Note that from the algebraic point of view, they are the main part of the associator of the geodesic multiplication [14]:

$$\left((x(yz))_L^{-1}((xy)z)\right)^m = R^m_{nrs}(e)x^n y^r z^s + \dots. \quad (7)$$

In this sense the emergence of structure functions instead of structure constants in the commutator (4) is caused by the nonassociativity of geodesic multiplication.

3. Kinematics of a quantum test particle

Let us introduce an action of the position operators x^i (on scalar valued functions) as multiplication with the Riemann normal coordinates x^i. Then we have $[x^i,x^k]=0$. We propose to define (geodesic) momentum operators p_i via infinitesimal right geodesic translations (2) by $p_k = -i\hbar R^s_k(x)\partial_s$. Then,

$$\left[x^i,p_k\right] = \left[x^i, -i\hbar R^s_k(x)\partial_s\right] = i\hbar R^i_k(x). \quad (8)$$

The full (geodesic) kinematic algebra of a quantum test particle in a curved space-time now reads [4]

$$[x^i,x^k]=0, \quad \left[x^i,p_k\right] = i\hbar R^i_k(x),$$
$$[p_j,p_k] = -i\hbar\rho^n_{jk}(x)p_n. \quad (9)$$

Note that the modification of commutation relations does not introduce any new dimensionful constants. However, according to Eqs. (6), (11), (12) functions $R^i_k(x)$ and $\rho^n_{jk}(x)$ contain curvature tensor of the space-time. When the latter is determined via the Einstein equations by some matter tensor, it is proportional to the gravitational constant G. Nontrivial terms on the r.h.s. of the commutation relations (9) are then proportional to the square of the Planck length $l^2_{Pl} \sim \hbar G$.

In a torsionless space-time, an expansion of $R^i_k(x)$ in the Riemann normal coordinates ($\Gamma^m_{nr}(e)=0$) can be found by using Eq. (2) and the geodesic multiplication formula of Akivis [14]

$$(x\alpha)^m = x^m + \alpha^m - \tfrac{1}{2}\Gamma^m_{nr,s}(e)x^n x^r\alpha^s$$
$$-\tfrac{1}{2}\Gamma^m_{n(r,s)}(e)x^n\alpha^r\alpha^s + \dots. \quad (10)$$

We get

$$\left[x^i,p_k\right] = i\hbar\left(\delta^i_k - \tfrac{1}{3}J^i_{kmn}(e)x^m x^n + O(x^3)\right), \quad (11)$$

where J^i_{kmn} denotes the Jacobi curvature tensor,

$$J^i_{kmn} = \tfrac{1}{2}\left(R^i_{mkn} + R^i_{nkm}\right). \quad (12)$$

Using Eq. (6) the expression for $[p_i,p_j]$ in the Riemann normal coordinates reads

$$[p_i,p_j] = -2i\hbar\left(R^k_{n[ij]}(e)x^n + O(x^2)\right)p_k. \quad (13)$$

Approximate expressions (11), (13) for the commutators were presented also by Kempf [7], who introduced momentum operators as generators of the change of geodesic coordinates at infinitesimal shift of their origin and used Synge's world function for calculating commutators. He demonstrated that representation (9) does not necessarily ensure the momentum operators to be hermitean (symmetric). To ensure hermiticity, an extra term might be needed in the representation of p_k which is a function of the x^i only and therefore does not play a role in the commutation relations.

4. Quantum kinematics and uncertainty relations in the background of a weak plane gravitational wave

To give an example that can be analytically worked out, let us consider the spacetime of a weak plane gravitational wave. The metric tensor for this space-time can be given as perturbations around the Minkowski metric [15]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1),$$

In case of a polarized weak plane gravitational wave moving in the direction of x^1 the only non-zero components of $h_{\mu\nu}$ in the TT-gauge are

$$h_{22} = -h_{33} = A\cos\omega(x^0 - x^1). \quad (14)$$

Here $A=$ const., $A \ll 1$, is the wave amplitude, and all subsequent equations hold in the linear approximation in A.

In these coordinates the equation of a geodesic line $x^\mu(t)$ with a tangent vector X^μ at a point e can be easily integrated, yielding [12]

$$x^\mu(t) = X^\mu t + AU^\mu \frac{\sin\omega Ct - \omega Ct}{\omega C^2},$$

where we have taken $e^\mu = 0$ and denoted

$$C = X^0 - X^1,$$
$$U^0 = U^1 = -\tfrac{1}{2}\big((X^2)^2 - (X^3)^2\big),$$
$$U^2 = -X^2 C, \quad U^3 = X^3 C.$$

The coordinates X^μ are also the Riemann normal coordinates of a point x with TT-coordinates $x^\mu \equiv x^\mu(1)$.

Let us choose the point e to be the unit element of the geodesic loop and let x, y be two points from its neighbourhood. According to (1), for calculating the product of the points x, y, the corresponding equations of geodesics and of parallel transport of the tangent vector X^μ must be integrated. From the expression for the geodesic product the matrix of the right translations and the corresponding structure functions can be determined [12].

According to our proposal, leaving out the hermiticity issues, canonical momentum operators can be represented by $p_i = -iR_i^s\partial_s$. The generalized commutation relations (9) in the background of a weak plane gravitational wave read

$$[x^i, x^k] = 0,$$
$$[x^k, p_i] = i\hbar\left(\delta_i^k + A\frac{\sin\omega C - \omega C}{\omega C^3}\big(2U^k\partial_i C - \partial_i U^k C\big)\right),$$
$$[p_i, p_j] = \frac{2i\hbar A}{\omega C^2}\sin^2\frac{\omega C}{2}\big(\partial_j U^k\partial_i C - \partial_i U^k\partial_j C\big)p_k.$$

Let us consider more in detail $[x^k, p_i]$ commutator. Near the light-cone emerging from the unit element or in the long wavelength approximation, it can be expanded as a series in ωC, giving a polynomial function on its r.h.s.:

$$[x^k, p_i] = i\hbar\big(\delta_i^k - a(2U^k\partial_i C - C\partial_i U^k)\big), \tag{15}$$

where $a \equiv \frac{1}{6}A\omega^2 > 0$. Introducing operators in light-cone coordinates $u = X^0 - X^1$, $v = X^0 + X^1$, $y = X^2$, $z = X^3$, $p_u = \frac{1}{2}(p_0 - p_1)$, $p_v = \frac{1}{2}(p_0 + p_1)$, the nonvanishing commutators read

$$[u, p_u] = i\hbar, \quad [v, p_u] = 2i\hbar a(y^2 - z^2),$$
$$[y, p_u] = i\hbar ayu, \quad [z, p_u] = -i\hbar azu,$$
$$[v, p_v] = i\hbar,$$
$$[v, p_y] = -2i\hbar ayu, \quad [y, p_y] = i\hbar(1 - au^2),$$
$$[v, p_z] = 2i\hbar azu, \quad [z, p_z] = i\hbar(1 + au^2).$$

From these commutators the following uncertainty relations can be derived:

$$\Delta u\Delta p_u \ge \frac{\hbar}{2},$$
$$\Delta v\Delta p_u \ge \hbar a\big((\Delta y)^2 - (\Delta z)^2 + \langle y\rangle^2 - \langle z\rangle^2\big),$$
$$\Delta y\Delta p_u \ge \frac{\hbar a}{2}(\Delta y\Delta u + \langle y\rangle\langle u\rangle),$$
$$\Delta z\Delta p_u \ge \frac{\hbar a}{2}(\Delta z\Delta u + \langle z\rangle\langle u\rangle),$$
$$\Delta v\Delta p_v \ge \frac{\hbar}{2},$$
$$\Delta v\Delta p_y \ge \hbar a(\Delta y\Delta u + \langle y\rangle\langle u\rangle),$$
$$\Delta y\Delta p_y \ge \frac{\hbar}{2}\big(1 - a((\Delta u)^2 + \langle u\rangle^2)\big),$$
$$\Delta v\Delta p_z \ge \hbar a(\Delta z\Delta u + \langle z\rangle\langle u\rangle),$$
$$\Delta z\Delta p_z \ge \frac{\hbar}{2}\big(1 + a((\Delta u)^2 + \langle u\rangle^2)\big).$$

Consider a state which has $\langle u\rangle = 0$, $\langle y\rangle = 0$ or $\langle z\rangle = 0$. An example of such a state is a test particle moving along with the wave front. Two of the uncertainty relations involving p_u then read

$$\Delta p_u \ge \frac{\hbar}{2\Delta u}, \quad \Delta p_u \ge \frac{\hbar a}{2}\Delta u, \tag{16}$$

the other two do not imply further restrictions. The relations (16) entail a minimal uncertainty for p_u

$$(\Delta p_u)_{\min} = \frac{\hbar\sqrt{a}}{2}, \tag{17}$$

which is achieved when $\Delta u = 1/\sqrt{a}$. The uncertainty relations for the transverse components of the momentum operator are more complex, those for

Δp_t remain unmodified. No minimal uncertainties can be obtained for these components.

As was shown in [5], the minimal uncertainty in momentum causes the momentum operator to become merely symmetric and not self-adjoint. This subtle distinction allows the expectation values to be real without implying the existence of zero uncertainty (i.e. eigen-) states [16]. It is not necessarily the case here where we have shown the minimal uncertainty condition to be valid only for certain states and under certain conditions.

5. Discussion

We have presented possible kinematics of a relativistic quantum test particle in a curved space-time. In a flat space-time, the full first-quantized theory of a relativistic particle is plagued by negative probabilities and must be replaced by a second-quantized quantum field theory.

We may try to mimic the flat space-time quantum field theory by replacing the momentum operators $P_k = -i\hbar\partial_k$ with the generalized momentum operators defined via the infinitesimal right geodesic translation operators $p_k = -i\hbar R_k^s(x)\partial_s$. In a flat space-time, components of the momentum operator commute, $[P_j, P_k] = 0$. In a curved background we have instead,

$$[p_j, p_k] = -i\hbar\,\rho_{jk}^n(x)\,p_n. \qquad (18)$$

For instance, in the case of a weak plane gravitational wave background, in the approximations considered in the last section we have two nonvanishing commutators:

$$[p_y, p_u] = \frac{i\hbar}{2} A\,\omega(2yp_v - up_y),$$

$$[p_z, p_u] = -\frac{i\hbar}{2} A\,\omega(2zp_v + up_z).$$

It follows that the time-like component of the momentum $p_t = p_u + p_v$ doesn't commute with the transverse components of the momentum, $[p_t, p_y] \neq 0$, $[p_t, p_z] \neq 0$. If p_t could be interpreted as the energy operator (hamiltonian) and if an analog of the Noether theorem for geodesic loops could be established, this means that the transverse momentum of the quantum field is not conserved in the background of a weak plane gravitational wave. There has been some progress in establishing generalized conservation laws in the case of Moufang loops [17], but nothing can be said in the case of more general geodesic loops. The representation theory of general nonassociative structures is also essentially lacking and we cannot introduce one- and many-particle states as suitable representations of geodesic loops. It seems that some novel mathematical ideas and developments are needed for continuing our investigations in this direction.

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