DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
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AFFINE MODELS IN MATHEMATICAL FINANCE: AN ANALYTICAL APPROACH

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Minu vanematele
To My Parents

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List of original publications

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2. Kangro, R., Pärna, K., Sepp, A., (2004), "Pricing European-Style Options under Jump Diffusion Processes with Stochastic Volatility: Applications of Fourier Transform," *Acta et Commentationes Universitatis Tartuensis de Mathematica, vol. 8*, 123-133.

3. Sepp, A., (2004) "Analytical Pricing of Double-Barrier Options under a Double-Exponential Jump Diffusion Process: Applications of Laplace Transform," *International Journal of Theoretical and Applied Finance 2*, 151-175.

4. Sepp, A., (2006), "Extended CreditGrades Model with Stochastic Volatility and Jumps," *Wilmott Magazine, September*, 50-62.

5. Sepp, A., (2007a), "Variance Swaps under no Conditions," *Risk, March, vol.20*, 82-87.

6. Sepp, A., (2007b), "Pricing Options on Realized Volatility in Heston Model with Volatility Jumps," *Manuscript*, submitted to *Journal of Computational Finance.*

7. Sepp, A., (2007c), "Viable VIX," *Manuscript*, submitted to *Risk.*

Frequently Used Notations

1. $\mathbb{R}^n$ - n-dimensional Euclidean space

2. $\mathbb{R}^+$ - the subset of non-negative real numbers

3. $\mathbb{C}$ - the complex plane

4. $C^k(\mathbb{R}^n)$ $(C_0^k(\mathbb{R}^n))$ - the functions with continuous derivatives up to order k (with compact support in $\mathbb{R}^n$)

5. $C^{1,k}(\mathbb{R}, \mathbb{R}^n)$ $(C_0^{1,k}(\mathbb{R}, \mathbb{R}^n)$) - the functions $f(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ which are C^1 with respect to $t \in \mathbb{R}^+$ and C^k (C_0^k) with respect to $x \in \mathbb{R}^n$

6. $L^k(\mathbb{C}^n),\ k = 1, 2, ..$ - the space of absolutely integrable functions $f : \mathbb{C}^n \to \mathbb{C}^n$ such that $\int |f(x)|^k dx < \infty$, $x \in \mathbb{C}^n$ (in Riemann sense)

7. $Q(t,T)$ $(Q^c(t,T))$ - conditional survival (default) probability of survival (default event) up to time T given survival at time t, $0 \leq t < T$

8. ι - the random default time

9. $\aleph$ - the post-default state

10. $W(t)$ - Brownian motion or Wiener process

11. $N^d(t)$ - Poisson process driving the jump-to-default

12. $N^j(t)$ - Poisson process driving simultaneous jump in the underlying dynamics

13. $S(t)$ - asset price process at time t

14. $V(t)$ - the instantaneous variance of asset price process at time t

15. $I(t)$ - the realized variance of asset price process at period $[0,t]$

16. J - random jump in price dynamics

17. $\mathcal{F}(t)$ - filtration representing the information generated by price dynamics available at time t, $0 \leq t < \infty$

18. $\mathcal{G}(t)$ - filtration representing the information generated by price dynamics and the information generated by the default time variable available at time t, $0 \le t < \infty$

19. $\mathbb{Q}$ - the risk neutral or martingale probability measure

20. $\mathbb{E}^{\mathbb{Q}}[Y(T)|\mathcal{F}(t)]$ - the expectation of random variable Y at time T under the measure $\mathbb{Q}$ given information set $\mathcal{F}(t)$ at time t, $0 \le t < T < \infty$

21. $\mathbb{E}^{Y(T)}[F(Y(T))|\mathcal{F}(t)]$ - the expectation of function $F(y)$ with respect to probability density of random variable $Y(T)$ given $\mathcal{F}(t)$ at time t, $0 \le t < T < \infty$

22. $\mathbf{1}_{\{\text{condition}\}} = \begin{cases} 1, & \text{if condition is true} \\ 0, & \text{otherwise} \end{cases}$ - indicator function

23. $\mathbb{P}^{\mathbb{Q}}[\text{event}|\mathcal{F}(t)] = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{\text{event}\}}|\mathcal{F}(t)]$ - the probability of event under the measure $\mathbb{Q}$ given information set $\mathcal{F}(t)$

24. $G^{Y}(t, T, Y, Y')$ - Green function or transition probability function of the stochastic process $Y(t)$, $G^{Y}(t, T, Y, Y') = \mathbb{P}^{\mathbb{Q}}[Y(T) = Y'|\mathcal{F}(t)]$

25. $M_1^v(t, T)$ ($\overline{M}_k^v(t, T)$) - the raw (central) moments of the process $V(T)$

26. $\Phi \in \mathbb{C}$ - the transform parameter with $\Phi_R = \Re[\Phi] \in \mathbb{R}$, $\Phi_I = \Im[\Phi] \in \mathbb{R}$ so that $\Phi = \Phi_R + i\Phi_I$, $i = \sqrt{-1}$

27. $\mathbf{F}_{-}[W(x)](\Phi)$ - the forward Fourier transform of function $W \in L^1(\mathbb{R})$ with respect to variable $x \in \mathbb{R}$ with transform parameter $\Phi \in \mathbb{C}$

28. $\mathbf{F}_{+}[\widehat{W}(\Phi)](x)$ - the inverse Fourier transform of function $\widehat{W} \in L^1(\mathbb{C})$ with transform variable $\Phi \in \mathbb{C}$

29. $\mathcal{L}[W(\tau)](p)$ - the forward Laplace transform of function $W(\tau)$, $0 < \tau < \infty$, with transform parameter $p \in \mathbb{R}$

30. $\mathcal{L}^{-1}[\widehat{W}(p)](\tau)$ - the inverse Laplace transform

31. (T)PDF - (transition) probability density function

32. CDF - cumulative probability density function

33. P(I)DE - partial (integro) differential equation

34. O(I)DE - ordinary (integro) differential equation

35. SDE - stochastic differential equation

36. FFT - Fast Fourier Transform

37. CDS - credit default swap

Introduction

This thesis is devoted to solving some practical problems arising in mathematical finance. In particular, we address the following questions:

1) How to design a model that takes into account the random evolution of the asset price and its volatility, sudden jumps in the asset price and its volatility, and a possible default of the issuer of this asset. All these effects are observed in the market, thus it is important to have a model consistent with these effects.

2) How to find fair values of derivative claims on the underlying asset in this model including call and put options on the asset price, contingent claims on the realized volatility of the asset price, and claims that are linked to a possible default of the issuer of the underlying asset, along with forward-start versions of these claims. By now, all these instruments are actively traded by market participants, so that it is important to develop a unified methodology for pricing and risk-managing of these derivative securities.

To tackle the first issue, we introduce a two-factor affine model, where the first factor is the price of the underlying asset and the second factor is the variance of the asset price. We supplement the asset price process with random jumps in returns and a random jump-to-default. We also include random jumps in the variance dynamics.

To solve the second problem, we develop robust solution methods based on Fourier and Laplace transforms. In general, we proceed as follows: we obtain closed-form solutions to Green (transition probability density) functions for the underlying variables in our model, and then we use convolution methods to price contingent claims on these variables.

Model

As a generic pricing model, we consider the following affine model to describe the joint evolution of the underlying asset price, denoted by $S(t)$, its variance, $V(t)$, its realized variance, $I(t)$, and the default intensity, $\lambda(t)$, under the risk-neutral martingale (pricing) measure $\mathbb{Q}$ as follows:

$$\begin{cases} \frac{dS(t)}{S(t-)} &= \mu(t)dt + \sqrt{V(t-)}dW^s(t) + (e^{J^s}-1)dN^j(t) - dN^d(t), \\ dV(t) &= \kappa(\theta - V(t-))dt + \varepsilon\sqrt{V(t-)}dW^v(t) + J^v dN^j(t), \\ dI(t) &= V(t-)dt + J^{s2}dN^j(t), \\ d\lambda(t) &= \alpha dt + \beta V(t-)dt, \end{cases} \tag{0.0.1}$$

where $S(t-)$ $(V(t-))$ stands for the value of the asset price (variance) process $S(t)$ $(V(t))$ just before either price jump J (variance jump J^v) or jump-to-default occur, and the remaining notations are defined in the following way.

$W^s(t)$ and $W^v(t)$ are standard Brownian motions with correlation ρ, $\rho \in [-1,1]$.

$N^j(t)$ is Poisson process with intensity γ, $\gamma \in (0,\infty)$. Following arrival of jump in $N^j(t)$, jump magnitude in the asset price, J^s, and its variance, J^v, are drawn from the bi-variate PDF $\varpi(J^v, J^s)$, $J^v \in (0,\infty)$, $J^s \in (-\infty,\infty)$, and given the realization of price jump J^s the jump in the realized variance is square of it, $J^{s2} = (J^s)^2$.

$N^d(t)$ is (doubly stochastic) Poisson process that drives the jump-to-default with stochastic intensity $\lambda(t)$, which is chosen to be a linear function of the variance with coefficients α, $\alpha \in [0,\infty)$, and β, $\beta \in [0,\infty)$, to capture the link between the asset variance and the likelihood of the default; following the arrival of the first jump in $N^d(t)$, the asset price drops to zero.

Both Poisson processes are assumed to be independent from each other and from Brownian motions.

$\mu(t)$, $\mu(t) = r(t) - d(t) - \gamma m^j + \lambda(t)$, is risk-neutral drift of the asset price, where $r(t)$, $r(t) \in [0,\infty)$, and $d(t)$, $d(t) \in [0,\infty)$, are respectively deterministic forward interest rate and dividend yields applicable for time period $[t, t+dt]$. Asset drift is adjusted by compensator γm^j, with m^j defined by:

$m^j = \int_0^\infty \int_{-\infty}^\infty \left(e^{J^s} - 1\right) \varpi(J^v, J^s) dJ^s dJ^v$, and default intensity rate $\lambda(t)$ to make the asset price process a martingale under the measure $\mathbb{Q}$.

θ, $\theta \in (0,\infty)$, is the long-term mean of the variance.

κ, $\kappa \in (0,\infty)$, is the reversion speed to the long-term mean.

ε, $\varepsilon \in (0,\infty)$, is the volatility of variance.

We consider two applications of general model (0.0.1):

1) A two-factor model with jump-to-default, variance and price jumps. This model is the best suited for pricing equity options, options on the realized volatility, and default-contingent claims on assets issued by institutions subject to the default risk. The purpose of introducing the jump-to-default along with price and variance jumps is to make this model consistent with market quotes for liquid European options, variance swap prices, and/or credit default swap spreads. This model is also directly applicable to indexes, which are subject to no default risk, by setting the intensity of the jump-to-default process to zero.

2) A one-factor model for the asset price dynamics with double-exponential price-jumps, constant volatility and no jump-to-default process. This model is applied to study path-dependent pricing problems which arise by pricing of path-dependent options, such as barrier and lookback options, and in the so-called structural models of the default. These problems depend on the history of the asset price process and, as a result, are very hard to deal with by means of analytical methods. Also, we apply this model to extend the CreditGrades model which plays an important role among market models designed to explain the risk of default.

Literature Overview

The ground-breaking approach to pricing and risk-managing of contingent claims was developed by Black-Scholes and Merton (1973), which still remains an important tool today. This model assumes that the asset price is driven by a geometric Brownian motion with constant volatility with no jumps and default risk, so, in principal, it cannot be applied in its original form for pricing and hedging of many derivative products traded in today's marketplace, such as options on realized volatility or credit default swaps. The Black-Scholes-Merton model can be viewed as a reduced form of dynamics (0.0.1), with $V(t) = \text{const}$ and $\gamma = \lambda \equiv 0$. Merton (1976) extended the Black-Scholes-Merton model to incorporate log-normally distributed price jumps. Heston (1992) made an important step towards introducing the stochastic volatility dynamics to the Black-Scholes-Merton model making the latter more consistent with empirical observations. Bates (1996) augmented Heston model with log-normal price-jumps in order to improve model consistency with prices of options with shorter-term maturities.

Duffie-Pan-Singleton (2000) introduced generalized version of affine model for asset dynamics with price and volatility jumps along with stochastic volatility. The model was termed affine since its coefficients are linear functions of the underlying dynamics. To incorporate the default risk, Hull-White-Nelken (2004) proposed an extension of Black-Scholes-Merton model with a random jump-to-default driven by Poisson process with constant intensity. Carr-Wu (2007) presented an affine model with the jump-to-default driven by Poisson process with a stochastic arrival intensity. Mikhailov-Nogel (2003) and Wu-Zhang (2006) studied affine models with piece-wise constant parameters, which are important to capture the term structure effects observed in the market.

Furthermore, the square-root diffusion, which is a particular example of the affine model, became the central part of many important financial models including the CIR interest rate model (Cox-Ingersoll-Ross (1985)) and the above mentioned models. The attractiveness of the square-root diffusion is motivated by several essential properties including positivity, mean-reversion, and closed-form solution for the transition density function. In particular, the availability of the closed-from solution for European option prices makes the calibration to market prices relatively quick and efficient. Combined with the ability to capture volatility smiles and skews, all this makes the affine model a viable tool in many pricing applications, including equity and foreign exchange (Lipton (2001) and (2002a), Lewis (2000) and (2001)), interest rates (Andreasen (2006)), and credit (Carr-Wu (2007)).

Finally, in addition to the jump-to-default or, in other words, intensity-based approach, another very popular approach to manage credit risk is Merton (1974) structural model of the default. In this model, the company defaults if the value of its assets becomes less than its promised debt repayment at maturity time T. Among others, Black-Cox (1976) and Leland-Toft (1996) extended Merton's model to account for the possibility that default may happen prior to maturity date T. Other extensions propose stochastic default barriers (Finger at al (2002)) and jumps in the firm's value dynamics (Zhou (2001), Hilberink-Rogers (2002), Lipton (2002b)). One of the drawbacks of the Merton's model is that it provides no connection to equity market, in particular, to equity options. This drawback was circumvented by CreditGrades (equity-to-credit) model, which became quite popular in the credit derivatives market. The CreditGrades model was jointly developed by CreditMetrics, JP Morgan, Goldman Sachs, and Deutsche Bank, and it has subsequently been copywritten. A detailed description of the CreditGrades model is presented in Stamicar *et al* (2005), Finger *et al* (2002), Finkelstein (2001).

Contributions

The key contributions of this thesis are as follows:

1) We present general version of the affine model and obtain explicit solutions for Green functions (transition probability densities) of the joint evolution of underlying dynamics in this model, including the asset price, its variance, and its realized variance. Our contribution in this respect is the extension of the affine model to develop general solution to the pricing problems of equity and volatility options along with credit contingent claims.

The formal approach to pricing contingent claims with jump-to-default is given in Chapter 2, while Chapters 3 and 4 are devoted to applying the affine dynamics for pricing contingent claims. An early version of these results for the affine model with no jump-to-default has appeared in Kangro-Pärna-Sepp (2004).

2) We derive closed-form solutions for pricing options on realized volatility and variance under the affine model with price and volatility jumps and the default risk. An analytical solution through Fourier transform for the Green function of the realized variance in the Heston model was obtained by Lipton (2001). However, he considered a pure diffusion model for the asset variance with no price or variance jumps. In addition, his method is based on computing the TPDF of the realized variance, so that by employing it we need to discretisize the state space of future realized variance, evaluate its probability by inverting the Fourier integral, and finally compute the expected value of the option payoff yielding. We apply the generalized Fourier transform and reduce the computation of a single option price to numerical inversion of a single Fourier integral. Pricing formulas for volatility derivatives under the Heston model with asset and volatility jumps, and default risk are original.

These results are developed in Chapters 2, 3 and 5, and a summary of this method for pricing options on realized volatility under the Heston model with variance jumps is presented in Sepp (2007b).

The application of the above setting to pricing conditional variance swaps, which are now liquid products in volatility trading, is given in Sepp (2007a). Also, an application of these results for pricing claims on VIX futures (as well as similar volatility indices) is given in Sepp (2007c). For brevity, in the thesis we do not consider these applications in details, noting that they are particular extensions of the general framework developed here.

3) We pay particular attention to pricing equity and volatility options under the default risk. Although pricing of equity options under the default risk was considered among others by Carr-Wu (2007) and Hull-White-Nelken (2004), our unified treatment of equity and volatility claims including forward-start analogs of these claims is original. The forward-start options are especially sensitive to the default risk and they require appropriate approach. More importantly, forward-start options are indispensable tools to model forward volatility surfaces and having a model with a sufficient number of parameters is important to impose a plausible evolution of these surfaces. A closed-form solution for forward-start equity options under the Heston model was obtained by Lipton (2001), however his method requires numerical integration of European call or put values over the probability density of asset variance. Our original results are explicit solutions, which are expressed by one-dimensional Fourier integrals, for pricing of forward-start equity and volatility options.

These results are presented in Chapters 2, 3 and 4.

4) We derive explicit formulas for pricing double (single) barrier and touch options with time-dependent rebates assuming that the asset price follows a double-exponential jump diffusion process. Our method extends previous results obtained by Kou-Wang (2001, 2003), who worked out formulas for the distribution of the first exit time and single barrier options using memoryless property of the exponential distribution; and Lipton (2002b), who derived pricing formulas for single barrier options relying on fluctuation identities.

These results are discussed in Chapter 5, and main conclusions are presented in Sepp-Skachkov (2003) and Sepp (2004).

5) We extend the CreditGrades model by incorporating jumps and stochastic volatility. Although Merton structural model (1974) with jumps in the firm's value dynamics has already been considered in a number of studies, there was no reference on how to estimate default probabilities using equity option data. Secondly, we extend the CreditGrades model by introducing the stochastic variance in the firm value dynamics and making connection to equity options. This model is original.

These results are obtained in Chapter 6, and a summary of them is given in Sepp (2006).

Chapter 1

Background and Basic Tools

In this Chapter we will provide the background for our subsequent developments and fix some notations. We first consider some basic derivative securities, next we describe non-defaultable dynamics driving the asset price process, then we state the fundamental result on how to obtain the fair value of a contingent claim, and finally we connect the problem of option pricing with the methods of partial differential equations and Green functions. Also, we consider an important concept of quadratic variation. For this purpose, we adopt results from Lipton (2001), Øksendal (2003), and Duffie (2001) for our specific developments.

1.1. Financial Derivative Securities

First we consider specific contingent claims and highlight their key features. Later we will introduce some notions from stochastic calculus and provide a more rigorous treatment of financial derivatives.

1.1.1. European Vanilla Options

The most basic derivative securities are European call and put options on the underlying asset whose market price at valuation time t is denoted by $S(t)$. Let us consider a European call option and let us denote its value function by $U(t, S)$. This call option gives its holder the right, but no obligation, to buy one share of the underlying asset at maturity time T for contract delivery (strike) price K. In other words, the value or payoff function of call option at time T is specified by:

$$U(T, S) = \max(S(T) - K, 0), \tag{1.1.1}$$

where the maximum function represents the optionality of this derivative security indicating that the holder will exercise his right only if the strike (his buying) price is below the current market price of the underlying asset.

Now let us consider the position of a seller of a call option. Suppose that the current valuation time is t, $0 \le t < T$. The main issues of the seller are the following: 1) for how much to sell this call option, that is what is the fair price of this contract, and 2) how to eliminate or hedge the risk associated with this contract, which apparently has only limited up-side potential and might result in an unlimited pay-out in case the asset price would soar up during the life of this contract.

To tackle these issues from a mathematical point of view, our first principle is to describe the distribution of the underlying asset at time T, then by using this distribution to evaluate the fair or expected value of the call contract at maturity, and, finally, to apply a discount factor to get time-t value of this claim. Obviously, the critical issue is how to determine the mean, standard deviation and, more generally, the shape of this distribution. The mean would be associated with the expected return of holding this asset, while standard deviation would indicate the variability or risk associated with the return on the asset.

Clearly, the shape of this distribution would evolve in time depending on the current asset price and (possibly) on the current value of the volatility of returns on this asset, so that for dynamical pricing and hedging we need to specify the evolution of the asset price in time. The latter objective is achieved by means of stochastic calculus and stochastic differential equations. Feynman-Kac theorem enables us to represent expectations with respect to random variables through solutions to partial (integro) differential equations (P(I)DE-s), so that the option pricing problem can be reduced to solving certain P(I)DE-s.

1.1.2. Options on Realized Variance

Claims on the realized variance derive their value from the annualized variance of asset spot price $S(t)$ realized during time period from t_0 to t_N:

$$I_N(t_0, t_N) = \frac{AF}{N} \sum_{n=1}^{N} \left(\ln \frac{S(t_n)}{S(t_{n-1})} \right)^2, \tag{1.1.2}$$

where $S(t_n)$ is the asset closing price observed at times t_0 (contract inception), .., t_N (maturity), N is the number of observations from contract inception up to expiration date, $\ln(S(t_n)/S(t_{n-1}))$ is a return realized over the time period between t_{n-1} and t_n. AF is annualization factor.

In practice, annualization factor AF is specified in contract terms (typically, fixings are made on a daily basis and $AF = 252$), number of observations N is calculated based on the fixing schedule adjusted to the day count conversion.

In continuous-time setting, assuming that the asset price follows a diffusion process with stochastic variance and price jumps, $I_N(t_0, T)$ can be very accurately approximated by its continuous time limit denoted by $\hat{I}(t_0, T)$ (we refer to the quadratic variation given by (1.3.4)):

$$\hat{I}(t_0, T) = \frac{1}{T - t_0} \left(\int_{t_0}^{T} V(t')dt' + \sum_{k=N^j(t_0)}^{N^j(T)} J_k^2 \right) = \frac{1}{T - t_0} I(t_0, T), \tag{1.1.3}$$

where $V(t)$ is instantaneous variance of assets returns, $N^j(t)$ is Poisson process driving the arrival of asset price jumps and J_k^2 is the squared realization of the k-th asset price jump, $\hat{I}(t_0, T)$ is cumulative annualized variance realized over the period (t_0, T), $I(t_0, T)$ is cumulative de-annualized variance. Here, we take into account the fact that the number of fixings is typically proportional to the annualization factor: $N \sim (T - t_0)AF$. For options on future realized variance the averaging starts at some future time t_F ($t_0 < t_F < T$) and the payoff function is dependent on future realized variance $I(t_0, T)$.

Widely spread are swaps on realized variance and volatility, swaps with cap and floor levels, and options on these swaps.

1.1.3. Forward-Start Options

In general, a forward-start claim derives its value from a given underlying claim. The underlying claim initiates at some future start time t_F and expires at time T, and the forward start contract allows the investor to enter the underlying claim at time t_F. We are interested in calculating the fair value of the forward start claim at time t_0, $0 \leq t_0 < t_F < T < \infty$.

1.1.4. Default Contingent Claims

The most common security whose value is affected by a possible default of its issuer is a corporate bond. As a result, the market price of corporate debt reflects the possibility of default by having a positive spread over the risk-free interest rate. Nowadays, one of the most liquid claims, whose payoffs are directly linked to the default of the underlying asset, is the credit default swap (CDS), which provides a protection against the default of the referenced company. Typically, the notional amount of CDS swaps on the referenced company is a few times greater than the notional amount of all outstanding corporate debt issued by this company.

1.2. Option Valuation

Now we consider the problem of option evaluation from a modeling point of view, and for this purpose we first fix the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ augmented with filtration $\mathcal{F}(t)$ that supports the default-free asset price dynamics. Here, Ω is the sample space, $\mathbb{Q}$ is a probability measure, and filtration $\mathcal{F}(t)$ represents the information flow of asset price. Subsequently, all expectations are taken with respect to the measure $\mathbb{Q}$.

1.2.1. Financial Option

Formally, a financial options gives the right to its holder to execute a certain type of trade on a specific underlying asset at a certain future time along with the right to receive a possible income stream before the contract maturity time. Additional contract terms specify the payoff of a defaultable security upon the default event.

For brevity, below we give a formal definition for one-dimensional and non path-dependent option payoffs, that is when the payoff at maturity time T is determined only by the value of the underlying asset at time T, which are typically called payoffs of European type. Generalization to multi-factor and path-dependent payoffs and options is straightforward.

Now, we consider one-dimensional process $Y(t)$ adopted to $\mathcal{F}(t)$ which drives the price dynamics of default-free security, respectively.

Definition 1.2.1. ***[Financial Option on a Non-Defaultable Security]***

A financial option with the value function denoted by $F(t, Y)$, whose payoff and reward functions are dependent on a non-defaultable security $Y(t)$, provides its holder with a payoff in amount $u_1(Y(T))$ at maturity time T and with instantaneous income stream given by a reward function $u_2(t, Y(t))$.

Optionality is enforced by non-negativity of payoff and reward functions assuming $u_1 : \mathbb{R} \rightarrow \mathbb{R}^+$ and $u_2 : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$.

1.2.2. Fundamental Pricing Theorems

The key problem in mathematical finance is how to derive a fair value of financial options. Fundamental results are stated in the following two theorem. Before stating them, we introduce the following definition.

Definition 1.2.2. ***[Discount Factor and Money Market Account]***

We denote the deterministic risk-free interest by $r(t)$.

The discount factor for the present value at time t of one currency unit of a riskless cash flow at time T is denoted by $DF(t, T)$ and defined by:

$$DF(t, T) = e^{-\int_t^T r(t')dt'}. \tag{1.2.1}$$

The money market account is defined by:

$$DF^{-1}(0, t) = e^{\int_0^t r(t')dt'}. \tag{1.2.2}$$

Assumption 1.1. ***[Martingale Pricing Measure]***

We assume that there exists probability measure $\mathbb{Q}$ under which the asset price process $Y(t)$, adopted to $\mathcal{F}(t)$, satisfies the following martingale properties:

i) $\mathbb{E}^{\mathbb{Q}}[|Y(t)|] < \infty$,

ii) $\mathbb{E}^{\mathbb{Q}}[DF(t, T)Y(T)|\mathcal{F}(t)] = Y(t)$.

Such a measure $\mathbb{Q}$ is called martingale (alternatively, pricing or risk-neutral) measure. For brevity, we skip the definition of an arbitragable market and notice that we mean by no arbitrage value of the contingent claim its fair value under measure $\mathbb{Q}$. If the derivative claim is sold by its fair value then the the expected returns on both investment strategies - buying the derivative security or replicating it by trading in the underlying security and money market account - are equal to the risk-free rate of return.

The next two theorems are fundamental to calculate fair values of contingent claims. More detailed versions of these theorems and proofs are given in Øksendal (2003), Chapter 12.

Theorem 1.2.1. ***[First Fundamental Theorem of Option Pricing]***
Under assumption (1.1), the discounted no-arbitrage price processes of all contingent claims are martingales under the measure $\mathbb{Q}$*.*

Corollary 1.2.1. *The fair (non-arbitragable) value of contingent claim* $F(t, Y)$ *on non-defaultable asset* $Y(t)$ *at time* t*,* $0 \le t \le T < \infty$*, is given by:*

$$F(t, Y) = \mathbb{E}^{\mathbb{Q}} \left[\left(e^{-\int_t^T r(t')dt'} u_1(Y(T)) + \int_t^T e^{-\int_t^{t'} r(t'')dt''} u_2(t', Y(t'))dt' \right) \middle| \mathcal{F}(t) \right]. \tag{1.2.3}$$

The next theorem postulates the existence of a unique replicating strategy for derivative securities.

Theorem 1.2.2. ***[Second Fundamental Theorem of Option Pricing]***
If and only if there exists a unique measure $\mathbb{Q}$ *that satisfies the requirements i) and ii) of assumption (1.1), then every financial contingent claim on asset* $Y(t)$ *is uniquely replicable by a hedging portfolio consisting of positions in asset* $Y(t)$ *and in money market account, that is the payoff of every contingent claim can be achieved by taking positions only in the underlying security and money market account.*

We note that finding a unique measure $\mathbb{Q}$ that satisfies the requirement of the first theorem is extremely involved when there is a few risky factors. However for practical purposes, we can assume that the measure $\mathbb{Q}$ is already fixed by market participants and it is reflected in market prices of traded derivative securities. Accordingly, the problem of finding measure $\mathbb{Q}$ comes down to enforcing the martingale condition for the model implied evolution of the asset price process under the measure $\mathbb{Q}$ and calibrating parameters of our chosen pricing model to market prices of traded securities. This estimated measure $\mathbb{Q}^*$ is sometimes called empirical or pricing martingale measure, and this approach to specify $\mathbb{Q}$ is used by a majority of market participants to mark-to-market and risk-manage their positions.

The replication strategy (under the measure $\mathbb{Q}^*$) is typically achieved by assembling many (hundreds of) individual option contracts in a portfolio (the so-called option book) and then hedging aggregated risks of these portfolios (books).

1.3. Stochastic Calculus

Now, we introduce some important modeling tools to study the problem of pricing and hedging financial derivative securities. For the "diffusion part" we follow Øksendal (2003) and for "jump part" we follow Duffie (2001).

Definition 1.3.1. ***[Ito Jump-Diffusion]***

Stochastic process $Y(t)$ is called Ito jump-diffusion if its dynamics are driven by the following SDE (stochastic differential equation):

$$dY(t) = \mu(t, Y(t-))dt + \sigma(t, Y(t-))dW(t) + j(t, Y(t-), J)dN(t), \; Y(0) = Y_0, \tag{1.3.1}$$

where $Y(t-)$ stands for the value of the process $Y(t)$ just before jump J occurs. $W(t)$ is standard Wiener process and $N(t)$ is Poisson process with stochastic intensity $\gamma(t, Y(t))$, processes $W(t)$ and $N(t)$ are assumed to be independent and adopted to $\mathcal{F}(t)$. The random variable J measurable on $\mathcal{F}(t)$ with PDF $\varpi(J)$ describes the magnitude of the jump when it occurs, and $j(t, Y(t-), J)$ maps the jump size to post-jump value of $Y(t)$.

We assume that J has finite first and second moment and that coefficients of this SDE satisfy the regularity conditions:

$$\begin{aligned}
&\mathbb{P}^{\mathbb{Q}}\left(\int_0^t \sigma^2(t', Y(t'))dt' < \infty\right) = 1, \; \forall\, t, \; 0 \le t < \infty, \\
&\mathbb{P}^{\mathbb{Q}}\left(\int_0^t \left|\mu(t', Y(t'))\right| dt' < \infty\right) = 1, \; \forall\, t, \; 0 \le t < \infty, \\
&\mathbb{P}^{\mathbb{Q}}\left(\int_0^t j^2(t', Y(t'), J)dt' < \infty\right) = 1, \; \forall\, t, \; 0 \le t < \infty.
\end{aligned} \tag{1.3.2}$$

To study the dynamics of functions of $Y(t)$, the following lemma is fundamental.

Lemma 1.3.1. ***[Ito lemma]***

Assume that $F(t, y) \in C^2([0, \infty) \times \mathbb{R})$ and let $F_t(t, y)$, $F_Y(t, y)$, $F_{YY}(t, y)$ stand for partial derivatives with respect to t and y, respectively. Let $Y(t)$ be Ito jump-diffusion (1.3.1). Then the dynamics of $F(t, Y(t))$ are also driven by Ito jump-diffusion:

$$\begin{aligned}
dF(t, Y(t)) = &\left(F_t(t, Y(t-)) + \mu(t, Y(t-))F_Y(t, Y(t-)) + \frac{1}{2}\sigma^2(t, Y(t-))F_{YY}(t, Y(t-))\right) dt \\
&+ \sigma(t, Y(t-))F_Y(t, Y(t-))dW(t) + \left(F(t, Y(t-) + j(t, Y(t-), J)) - F(t, Y(t-))\right) dN(t),
\end{aligned} \tag{1.3.3}$$

where $Y(t-)$ is the value of the process $Y(t)$ just before jump J occurs.

1.3.1. Quadratic Variation

Now we introduce an important concept of the quadratic variation, which is connected to the asset realized volatility. General reference for quadratic variation of jump-diffusions is Cont-Tankov (2004), here we state the main result.

Theorem 1.3.1. ***[Quadratic Variation]*** *Let $Y(t)$ be driven by the SDE (1.3.1) and satisfy assumptions of definition (1.3.1). Then the quadratic variation or the* *realized variance of $Y(t)$, realized during time period $[t_0, T]$, is denoted by $I(t_0, T)$ and given by:*

$$I(t_0, T) = \lim_{k\to\infty} \sum_{t_i \in \pi^k} (Y(t_{i+1}) - Y(t_i))^2 = \int_{t_0}^{T} \sigma^2(t', Y(t'))dt' + \sum_{n=N(t_0)}^{N(T)} j^2(t_n, Y(t_n-), J_n), \tag{1.3.4}$$

where $\pi^k = \{t_0^k < t_1^k < .. < t_{k+1}^k\}$, with $t_0^k = t_0$ and $t_{k+1}^k = T$, is a sequence of partitions of interval $[t_0, T]$ such that $|\pi^k| = \sup_p |t_p^k - t_{p-1}^k| \to 0$ as $k \to \infty$, and convergence in (1.3.4) is in probability sense. Here, $j^2(t_n, Y(t_n-), J_n)$ is squared realization of n-th jump J_n that occurred at time t_n and $Y(t_n-)$ is the value of process just before time t_n.

At valuation time t, $t_0 \le t \le T$, the realized variance can be represented by:

$$I(t_0, T) = I(t_0, t) + I(t, T), \tag{1.3.5}$$

where $I(t_0, t)$ is a known quantity at time t and $I(t, T)$ represents future realization. If the asset price dynamics follow a jump-diffusion process with stochastic variance then asset realized variation $I(t, T)$ is a stochastic process itself. In particular, it represents process with a stochastic drift, no diffusion part, and a jump process.

The quadratic variation serves as an underlying "asset" for a derivative security on the asset realized volatility (1.1.2) computed as a sum of squared daily returns. Our Monte Carlo simulation analysis of affine model (0.0.1) has revealed that continuous time limit (1.3.4) serves as a very accurate approximation for the discrete sum quantity (1.1.2) (see Figure (3.3.5) for an example), so that pricing of claims on the asset realize volatility can be done by using TPDF of $I(t, T)$. We will pay a particular attention to studying the TPDF and Green function of this process.

1.3.2. Convexity Adjustment Formula

The following formula allows us to study the expectation of a smooth non-linear function of stochastic process $V(t)$ with respect to TPDF of $V(t)$ by using moments of $V(t)$. We will use this important result on a number of occasions.

Theorem 1.3.2. ***[Convexity Adjustment Formula]***

Let $V(t)$ be a stochastic process with Green function (TPDF) denoted by $G^v(t, T, V, V')$ and finite moments up to order 4. Assume that function $U(t, V)$ is at least in $C^{1,4}((0, \infty), \mathbb{R})$. Then

$$\begin{aligned} &\mathbb{E}^{V(T)}[U(t, V(T))] = \int_{-\infty}^{\infty} U(t, V')G^v(t, T, V, V')dV' \approx \\ &\approx U(t, \overline{M}_1^v(t, T)) + \frac{1}{2}\overline{M}_2^v(t, T)U_{VV}(t, \overline{M}_1^v(t, T)) + \\ &+ \frac{1}{6}\overline{M}_3^v(t, T)U_{VVV}(t, \overline{M}_1^v(t, T)) + \frac{1}{24}\overline{M}_4^v(t, T)U_{VVVV}(t, \overline{M}_1^v(t, T)), \end{aligned} \tag{1.3.6}$$

where $\overline{M}_k^v(t,T)$ are central moments of $V(T)$ given the value of $V(t)$, which can be computed by means of raw moments as follows:

$$\begin{aligned}
&\overline{M}_1^v(t,T) = M_1^v(t,T),\\
&\overline{M}_2^v(t,T) = -(M_1^v(t,T))^2 + M_2^v(t,T),\\
&\overline{M}_3^v(t,T) = 2(M_1^v(t,T))^3 - 3M_1^v(t,T)M_2^v(t,T) + M_3^v(t,T),\\
&\overline{M}_4^v(t,T) = -3(M_1^v(t,T))^4 + 6(M_1^v(t,T))^2 M_2^v(t,T) - 4M_1^v(t,T)M_3^v(t,T) + M_4^v(t,T),
\end{aligned} \tag{1.3.7}$$

with

$$M_k^v(t,T) = \mathbb{E}[V^k(T)|V(t)] = \int_{-\infty}^{\infty} V'^k G^v(t,T,V,V')dV'. \tag{1.3.8}$$

Proof. We expand $U(t,V)$ in Taylor series around the expected value of $V(T)$, $\overline{M}_1^v(t,T)$, and compute the expectation. □

1.4. Connection to PDE Methods

According to the results of Theorem (1.2.1), the option pricing problem comes down to computing the integral (1.2.3). This task is less than trivial for flexible and interesting dynamics driving $Y(t)$, and the main subject of our thesis is how to solve this task efficiently. For this purpose, we will need to obtain TPDF or Green function of the process $Y(t)$. We note that in the presents of jump-to-default, the integral over TPDF at some future point in time is less than one, so that it would be inappropriate to use the term TPDF in this case. The term Green function is adopted from the physics and engineering terminology and it can be used in a broader sense.

The Kolmoroff (Fokker-Plank) backward and forward equations are our key tool to study the pricing problem from the P(I)DE standpoint. We state a few important results from Lipton (2001) and Øksendal (2003) for our developments.

1.4.1. Kolmogoroff Equations

Theorem 1.4.1. ***[Kolmogoroff Backward Equation]***

Let the process $Y(t)$ satisfy assumptions of definition (1.3.1). Then the Green (transition probability density) function of $Y(t)$, $G^y(t,T,Y,Y')$, which is assumed to be at least $C_0^{1,2}(\mathbb{R}^+,\mathbb{R})$ function of "backward" arguments t and Y, respectively, solves the following Kolmogoroff backward equation:

$$\begin{aligned}
&G^y{}_t + \mathcal{D}^0 G^y = 0,\\
&G^y(T,T,Y,Y') = \delta(Y - Y'),
\end{aligned} \tag{1.4.1}$$

where $\delta()$ is Dirac delta function defined by (1.5.1), and $\mathcal{D}^0$ is the infinitesimal generator corresponding to the SDE of $Y(t)$ and defined by:

$$\begin{aligned}\mathcal{D}^0 G^y(t,T,Y,Y') &= \mu(t,Y)G^y_Y + \frac{1}{2}\sigma^2(t,Y)G^y_{YY} \\ &+ \gamma(t,Y)\int_{-\infty}^{\infty}[G^y(Y+j(t,Y,J)) - G^y]\varpi(J)dJ.\end{aligned} \tag{1.4.2}$$

Theorem 1.4.2. ***[Kolmogoroff Forward Equation]***

Let the process $Y(t)$ satisfy assumptions of definition (1.3.1). Then the Green (transition probability density) function of $Y(t)$, $G^y(t,T,Y,Y')$, which is now assumed to be at leat $C_0^{1,2}(\mathbb{R}^+,\mathbb{R})$ function of "forward" arguments T and Y', respectively, solves the following Kolmogoroff forward equation:

$$\begin{aligned} &{G^y}_T + \bar{\mathcal{D}}^0 G^y = 0, \\ &G^y(t,t,Y,Y') = \delta(Y'-Y), \end{aligned} \tag{1.4.3}$$

where $\bar{\mathcal{D}}^0$ is the infinitesimal generator adjoint to $\mathcal{D}^0$ and defined by:

$$\begin{aligned}\bar{\mathcal{D}}^0 G^y(t,T,Y,Y') &= (\mu(t,Y')G^y)_{Y'} - \frac{1}{2}(\sigma^2(t,Y')G^y)_{Y'Y'} \\ &- \gamma(t,Y')\int_{-\infty}^{\infty}[G^y(Y'-j(t,Y',J)) - G^y]\varpi(J)dJ.\end{aligned} \tag{1.4.4}$$

For practical purposes we solve the Kolmogoroff forward equation to construct TPDF for the entire space of the underlying price process up to time T, so that we can evaluate a continuum of European call and put options across all strikes and maturity times up to time T. This procedure is extremely useful by model calibration to available call and put options quoted in the market. In general, the backward Kolmogoroff equation is applied by valuing derivative securities, which might also include some optionality features, such as American options which can be exercised by the holder at any time up to maturity time T. For option pricing purposes we state an important result relating expectations with respect to realizations of stochastic processes to specific PIDE-s.

Definition 1.4.1. *We denote by $D^g(t,T,Y)$ the stochastic discount factor for payoff at time T, $0 \le t \le T < \infty$:*

$$D^g(t,T,Y) = e^{-\int_t^T g(t',Y(t'))dt'}, \tag{1.4.5}$$

where $g(t,y) \in C([0,\infty) \times \mathbb{R})$.

Theorem 1.4.3. ***[Feynman-Kac Theorem]***

Let $Y(t)$ be Ito jump-diffusion driven by SDE (1.3.1).

Assume that the function $F(t,y)$ is at least in $C^{1,2}((0,\infty),\mathbb{R})$ and it is defined as follows:

$$F(t,Y) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T g(t',Y(t'))dt'} u_1(Y(T)) + \int_t^T e^{-\int_t^{t'} g(t'',Y(t''))dt''} u_2(t',Y(t'))dt' \middle| \mathcal{F}(t) \right]. \tag{1.4.6}$$

Then $F(t,Y)$ solves the following equation:

$$\begin{aligned} &F_t(t,Y) + \mathcal{D}^g F(t,Y) = -u_2(t,Y), \\ &F(T,Y) = u_1(Y), \end{aligned} \tag{1.4.7}$$

where $\mathcal{D}^g$ is the infinitesimal generator, corresponding to the SDE of $Y(t)$ and the discount term $g(t,y)$, which is defined by:

$$\begin{aligned} \mathcal{D}^g F(t,Y) &= \mu(t,Y)F_Y(t,Y) + \frac{1}{2}\sigma^2(t,Y)F_{YY}(t,Y) \\ &+ \gamma(t,Y)\int_{-\infty}^{\infty} [F(t,Y+j(t,Y,J)) - F(t,Y)]\varpi(J)dJ - g(t,Y)F(t,Y). \end{aligned} \tag{1.4.8}$$

In particular, for non-defaultable assets $g(t,Z) = r(t)$.

The following theorem establishes the link between Feynman-Kac formula and the Green function.

Theorem 1.4.4. ***[Duhamel's Principle]***
The solution to PIDE (1.4.7) can be computed by:

$$F(t,Y) = \int_{-\infty}^{\infty} u_1(Y')G^y(t,T,Y,Y')dY' + \int_t^T \int_{-\infty}^{\infty} u_2(t',Y')G^y(t,t',Y,Y')dY'dt'. \tag{1.4.9}$$

where $G^y(t,T,Y,Y')$ is the Green function corresponding to the dynamics of $Y(t)$ and solving Kolmogoroff backward equation:

$$\begin{aligned} &G^y_t(t,Y) + \mathcal{D}^g G^y(t,Y) = 0, \\ &G^y(T,T,Y,Y') = \delta(Y-Y'), \end{aligned} \tag{1.4.10}$$

and generator $\mathcal{D}^g$ is defined by (1.4.8).

1.5. Transform Methods

Transform methods, including Fourier and Laplace transforms, are one of the classical and powerful methods for solving ordinary and partial differential equations as well as integral equations. The idea behind these methods is to transform the problem to a space where the solution is relatively easy to obtain. The corresponding solution is referred to as the solution in the Fourier or Laplace space. The original function can be retrieved either by means of computing the inverse transform analytically or, in complicated cases, by methods of numerical inversion.

1.5.1. Generalized Functions

Generalized functions, in particular, the Dirac function and its derivative of real-valued and complex valued arguments are important for our developments.

Definition 1.5.1 (Real-Valued Dirac Function). *Let $F(x) \in C_0^n(\mathbb{R})$ be a function of real argument x and let $a \in \mathbb{R}$. Delta function $\delta(x)$ of real valued argument x and its $n-th$ derivative, $\delta^{(n)}(x)$, $n = 0, 1, ..,$ are defined as follows:*

$$\begin{aligned} &\int_{-\infty}^{\infty} \delta(x' - a) F(x') dx' = F(a), \\ &\int_{-\infty}^{\infty} \delta^{(n)}(x' - a) F(x') dx' = (-1)^n F^{(n)}(x)|_{x=a}. \end{aligned} \tag{1.5.1}$$

Definition 1.5.2 (Complex-Valued Exponential). *We introduce the following formula for the exponential of complex valued argument $\Phi = \Phi_R + i\Phi_I$ with $i = \sqrt{-1}$:*

$$\begin{aligned} I_{\pm}(\Phi) &= \pm \int_0^{\pm\infty} e^{(\Phi_R + i\Phi_I)x'} dx' \\ &= \mp \frac{1}{\Phi_R + i\Phi_I} + \begin{cases} 0, & \pm\Phi_R < 0, \\ \pi\delta(\Phi_I), & \pm\Phi_R = 0, \\ 2\pi\delta(\Phi_I \mp i\Phi_R), & \pm\Phi_R > 0. \end{cases} \end{aligned} \tag{1.5.2}$$

1.5.2. Generalized Fourier Transform

Definition 1.5.3 (1-D Generalized Fourier Transform). *We define the generalized forward and inverse Fourier transform operator denoted by $\mathbf{F}_{\mp}$, respectively, of an $L^1(\mathbb{R})$ function $W(x)$ with transform variable $\Phi = \Phi_R + i\Phi_I$, where Φ_R, $\Phi_I \in \mathbb{R}$, as follows:*

$$\begin{aligned} \widehat{W}(\Phi) &= \mathbf{F}_{-}[W(x)](\Phi) = \int_{-\infty}^{\infty} e^{-\Phi x'} W(x') dx', \\ W(x) &= \mathbf{F}_{+}[\widehat{W}(\Phi)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\Phi x} \widehat{W}(\Phi) d\Phi_I, \end{aligned} \tag{1.5.3}$$

where $\widehat{W}(\Phi) \in L^1(\mathbb{C})$.

The range of possible values for Φ_R is determined by the problem at hand: for the transformed initial condition of the Green function Φ_R is unrestricted, while for transforms of the payoffs considered here we need to take $\Phi_R < 0$ to make the pricing problem well-defined. Representation (1.5.3) is a slightly modified version of that introduced by Lewis (2001) and it is written in terms of transform variable $\Phi \in \mathbb{C}$, which simplifies our notations and formulas.

Definition 1.5.4 (n-dimensional Generalized Fourier Transform)**.** *Assume that $W(x^n)$ is an $L^1(\mathbb{R}^n)$ function of n arguments. We define the generalized forward and inverse Fourier transform, $\mathbf{F}^n_{\mp}$, $n = 1, 2, ..,$ with transform variables $\Phi^n = (\Phi_1, .., \Phi_n)$, $\Phi_k = \Phi^k_R + i\Phi^k_I$ with Φ^k_R, $\Phi^k_I \in \mathbb{R}$, $k = 1, .., n$, as follows:*

$$\begin{aligned}\widehat{W}(\Phi^n) &= \mathbf{F}^n_-[W(x^n)](\Phi^n) = \int_{-\infty}^{\infty} e^{-\Phi\cdot x} W(x^n) dx^n,\\ W(x^n) &= \mathbf{F}^n_+[\widehat{W}(\Phi^n)](x^n) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} e^{\Phi\cdot x}\widehat{W}(\Phi^n) d\Phi^n_I,\end{aligned} \tag{1.5.4}$$

where $\widehat{W}(\Phi^n) \in L^1(\mathbb{C}^n)$ and $\Phi \cdot x = \sum_{k=1}^n \Phi_k x_k$.

A very useful method to compute the moments of a random variable given its Fourier transform is given by the following theorem.

Theorem 1.5.1. ***[Moment Generating Function]***

Let $M^v_k(t,T)$ be k-th moment of $V(T)$ as defined in (1.3.8) and let $\widehat{G}^v(t,T,V,\Phi)$ be the forward generalized Fourier transform of the Green function of $V(T)$, denoted by $\widehat{G}^v(t,T,V,\Phi)$. Assume that $\widehat{G}^v(\Phi) \in C^k(\mathbb{C})$, then

$$M^v_k(t,T) = (-1)^k \frac{\partial^k}{\partial \Phi^k_R}\widehat{G}^v(t,T,V,\Phi)\mid_{\Phi_R=0,\Phi_I=0} = (-i)^k \frac{\partial^k}{\partial \Phi^k_I}\widehat{G}^v(t,T,V,\Phi)\mid_{\Phi_R=0,\Phi_I=0}. \tag{1.5.5}$$

1.5.3. Laplace Transform

Let $F(\tau, x)$ be piecewise continuous as function of τ on every finite interval $\tau \in (a, b]$, $a, b \in \mathbb{R}^+$, $0 \le a < b < \infty$, satisfying the growth condition as function of τ:

$$|F(\tau, x)| \le M e^{-c\tau}, \tag{1.5.6}$$

for all $\tau \in (a, b]$ and $x \in (-\infty, \infty)$ with $M, c \in \mathbb{R}^+$ constant as functions of τ.

The Laplace transform of function $F(\tau, x)$ with respect to τ is defined by

$$U(p, x) = \mathcal{L}[F(\tau, x)](p) = \int_0^{\infty} F(\tau', x) e^{-p\tau'} d\tau' \tag{1.5.7}$$

where p is a transform variable with $\Re[p] > c$. To be specific, in subsequent analysis we assume that $p \in \mathbb{R}^+$. The standard rules yield:

$$\mathcal{L}[\frac{\partial}{\partial\tau}F(\tau,x)] = pU(p,x) - F(0,x),\ \mathcal{L}[\frac{\partial^n}{\partial x^n}F(\tau,x)] = \frac{\partial^n}{\partial x^n}U(p,x). \tag{1.5.8}$$

Given that $U(p, x)$ is Laplace transform of $F(\tau, x)$, the original function $F(\tau, x)$ is computed by the inverse transform:

$$F(\tau, x) = \mathcal{L}^{-1}[U(p, x)](\tau) = \frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty} e^{p\tau} U(p, x) dp, \tag{1.5.9}$$

where υ is a contour in $\mathbb{C}$ chosen so that $\upsilon > \max\{\Re[p_s]\}$ for set of all singular points $\{p_s\}$ of $U(p,x)$.

It can be very hard to compute (1.5.9) directly. However, there are a few robust algorithms to invert Laplace transform numerically (we refer to a survey by Abate *et al* (1999)). In this thesis, we will use the algorithm by Stehfest (1970) to compute $F(\tau,x)$ given its Laplace transform $U(p,x)$ as follows:

$$F(\tau,x) \approx \frac{\ln 2}{\tau}\sum_{j=1}^{N}\Lambda_j U(j\frac{\ln 2}{\tau},x) \tag{1.5.10}$$

where coefficients Λ_j are given by

$$\Lambda_j = (-1)^{N/2+j}\sum_{k=(j+1)/2}^{\min\{j,N/2\}}\frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(j-k)!(2k-j)!}, \tag{1.5.11}$$

N is an even number and k is computed using integer arithmetic.

A remarkable property of Stehfest algorithm is that since constants Λ_j depend neither on τ nor on x they can be tabulated and used for inversion of $U(p,x)$. In Table (1.5.1), we report values of Λ_j for $N = 14$ using 8-digit accuracy.

j	Λ_j	j	Λ_j
1	0.00277778	8	-63944913.04444440
2	-6.40277778	9	127597579.55000000
3	924.05000000	10	-170137188.08333300
4	-34597.92777778	11	150327467.03333300
5	540321.11111111	12	-84592161.50000000
6	-4398346.36666666	13	27478884.76666660
7	21087591.77777770	14	-3925554.96666666

Table 1.5.1: Coefficients Λ_j of the Stehfest algorithm for $N = 14$.

The Stehfest algorithm is very efficient inversion method and it allows obtaining accuracy up to 8-10 significant digits. VBA code for the numerical inversion of the Laplace transform by means of Stehfest algorithm is given in Sepp-Skachkov (2003).

1.6. Black-Scholes-Merton Model

The ground-breaking approach to pricing and hedging financial derivatives was proposed by Black-Scholes and Merton (1973), who suggested to use the log-normal diffusion to model the asset price evolution in time:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ S(0) \text{ is given,} \tag{1.6.1}$$

where μ is asset drift (measuring the expected return associated with $S(t)$), σ is asset volatility (measuring variability of returns on $S(t)$), and $W(t)$ is a standard Brownian motion used as a tool to model the variability of the asset price dynamics.

The main contribution of the Black-Scholes-Merton approach was an establishment of hedging strategy to eliminate the risk associated with a contingent claim through a dynamic trading in the stock and borrowing or lending money at a risk-free rate of return r. Since the latter strategy is supposed to be the same for all market participants independently of their risk preferences, it follows that options need to be priced under the so-called risk-neutral or martingale measure $\mathbb{Q}$, under which the discounted price of any derivative security, including the asset price itself, is a martingale and the risk-neutral drift of any security is equal to risk-free rate of return r. The latter can be thought of as a guaranteed return rate on investments in short-term government bonds.

Under this assumption, the call option with value function denoted by $U(t,S;T,K)$, where T is maturity time and K is strike price, is evaluated as a discounted expectation of the future payoff at maturity time T under martingale measure $\mathbb{Q}$:

$$U(t,S;T,K) = e^{-(T-t)r}\int_0^\infty \max(S'-K,0)G^S(t,T,S,S')dS', \tag{1.6.2}$$

where $G^S(t,T,S,S')$ is the transition density or Green function of log-normal random variable associated with risk-neutralized dynamics (1.6.1), where risk-neutral drift is $\mu = r-d$ with d being a rate of instantaneous dividend yield on the asset, which is given by:

$$G^S(t,T,S,S') = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}S'}\exp\left\{-\frac{\left(\ln\left(\frac{S}{S'}\right)+\left(r-d-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right\}. \tag{1.6.3}$$

By applying Feynman-Kac theorem (1.4.3), we can also represent the value function of call, $U(t,S;T,K)$, as solution to the celebrated Black-Scholes-Merton PDE:

$$\begin{aligned} &U_t + \frac{1}{2}\sigma^2S^2U_{SS} + (r-d)SU_S - rU = 0,\\ &U(T,S;T,K) = \max(S-K,0), \end{aligned} \tag{1.6.4}$$

where we assume that $U(t,S) \in C^{1,2}((0,\infty),\mathbb{R}^+)$.

Solving PDE (1.6.4), yields the famous Black-Scholes-Merton formula for a call option value:

$$\begin{aligned} &U(t,S;T,K) = e^{-(T-t)d}S\mathcal{N}(d_+) + e^{-(T-t)r}K\mathcal{N}(d_-),\\ &d_{+,-} = \frac{\ln(S/K)+(r-d\pm\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \end{aligned} \tag{1.6.5}$$

where $\mathcal{N}(x)$ is the CDF of standard normal random variable.

In formula (1.6.5), $S(t)$, r and d are specified by available market data (in case the asset pays discrete dividends, the standard approach is to imply the instantaneous dividend yield using the asset forward price), T and K are specified by the contract terms, and asset volatility σ is the only "free" parameter which needs to be somehow estimated or back-out from available data.

Finally, we note that since the call option value solves both integral formula (1.6.2) and PDE (1.6.4) we can find its value (and values of similar claims) in two ways: 1) by evaluating integral (1.6.2) analytically or through Monte-Carlo simulations, or 2) by solving PDE (1.6.4) explicitly by means of analytical methods including Fourier and Laplace transforms or numerically by means of finite-difference methods.

In practice, we choose a suitable solution method based on the complexity of the problem: for simpler problems we would prefer to have explicit solutions, while dealing with harder problems we would resort to finite-differences and Monte-Carlo methods. When we calibrate model parameters to the market data using some sort of optimization procedure, a robust and sufficiently fast solution method is necessary. That is why we need analytics for liquid flow products including European call and put options, options on realized asset volatility, and forward-start options, whose available market prices can be used in model calibration.

1.7. Volatility Skew

Apart from being theoretically robust and mathematically convenient to deal with, the Black-Scholes-Merton approach has its major limitation by assuming that the asset volatility σ is a constant. However in practice, investors are typically concerned with big negative movements in asset prices and hedge against them by buying puts with low strikes (unlike the call option, the put option gives to its holder the right to buy at maturity time T one share of the underlying asset at pre-determined delivery price K). In addition, investors, who want to realize their returns at maturity time T by selling the asset, are better off to buy a call option with a higher strike to lock in their returns at maturity time T.

The demand for puts with low strikes and supply of calls with higher strikes introduces the so-called volatility smile or skew. To quantify the skew effect, for a given market price of a call option, $U^M(T,K)$, with strike K and maturity time T, we calculate its implied volatility, $\sigma_{imp}(T,K)$, by solving the following non-linear equation in $\sigma_{imp}(T,K)$:

$$U(T,S;T,K,\sigma_{imp}(T,K)) = U^M(T,K), \tag{1.7.1}$$

where $U(t,S,;T,K,\sigma_{imp}(T,K))$ is the theoretical value of the call computed by (1.6.5) with volatility $\sigma = \sigma_{imp}(T,K)$.

The skew effect can be observed when for a given maturity time T we back out the implied volatility $\sigma_{imp}(T,K)$ from the market price of an out-of-the-money put with strike K_p, an out-of-the-money call with strike K_c, and at-the-money call with strike equal to the forward asset price $K_0 = e^{(r-d)T}S$ (with this strike both puts and calls have the same value), where $K_p < K_0 < K_c$, to obtain a remarkable result:

$$\sigma_{imp}(T,K_p) > \sigma_{imp}(T,K_0) > \sigma_{imp}(T,K_c), \tag{1.7.2}$$

which is in contradiction with Black-Scholes-Merton assumption about the constant volatility. Moreover, in the market we also observe so-called term structure effect meaning that the sharpness of inequality (1.7.2) also depends on maturity time T.

In Figure (1.7.1) we show implied volatilities of options on General Motor equity hare from the market data collected from Bloomberg and observed on November 8, 2005. We use three maturity times - short term (1.2 months), mid term (2.4 months), and longer term (1.2 years) - to plot the implied volatility as a function of normalized strikes, $K/S(0)$. The skew effect is very pronounced for out-of-the money puts with short maturities, then for longer maturities the skew effect flattens.

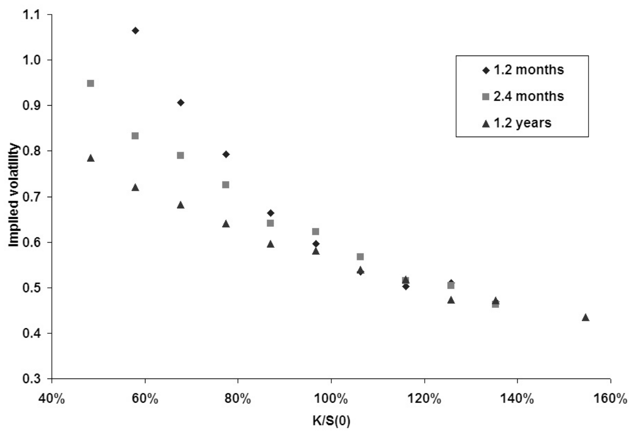

Figure 1.7.1: Implied volatility skew of options on General Motors equity shares

Typically, we have only a few option prices per maturity and market data are noisy, especially for strikes above and below the forward stock price and also for long maturity options. Market practice is to apply various interpolation and extrapolation methods to obtain a smooth volatility surface across all possible strikes and maturity times. Then this "static" representation of market uncertainty is used to price and hedge liquid European options.

To deal with the skew and term structure effects in dynamical setting for robust modeling of path-dependent and volatility products, we need to modify the Black-Scholes-Merton assumption about the log-normal asset price dynamics with constant volatility (1.6.4). We are interested in employing asset price dynamics resulting in so-called skewed distributions meaning that the values of asset price below its expected mean have higher likelihood compared to the log-normal distribution while the values above the mean have lower likelihood. This effects make puts (calls) with low (high) strikes to be more (less) valuable compared to the Black-Scholes-Merton price, naturally introducing the skew effect observed in the market.

Our general model (0.0.1) is fundamental generalization of the Black-Scholes-Merton framework, where we assume that the asset price follows the jump-diffusion process with stochastic volatility and jump-to-default.

Part I: Jump-to-Default Models

In the first part of the thesis we will study the pricing problem under the so-called reduced-form or intensity-based default approach, where conceptually the issuer of the asset defaults following a random default jump in the asset price dynamics. In this part we will formulate general pricing problem of call and put options, options on the realized asset volatility, and forward-start options, and then we will apply the Fourier transform to obtain closed form solutions for the pricing problem. Finally, we will illustrate Green functions of the underlying dynamics implied by particular specifications of general model (0.0.1) and provide an example of estimating model parameters by using General Motors option data.

Chapter 2

General Results

In this chapter we will first describe defaultable dynamics driving the asset price process, then we will state the fundamental result on how to obtain the fair value of defaultable asset. For mathematical formulation of the jump-to-default setup we follow the text by Schönbucher (2003).

Then we will derive our fundamental theorem (2.3.1) for pricing defaultable financial claims using generalized Fourier transform. Unlike previous results obtained, for example, by Carr-Madan (1999), Lewis (2000), Kangro-Pärna-Sepp (2004), this theorem provides a general solution for option pricing problem, including options on the asset realized volatility and forward-start options, by means of connecting the generalized Fourier transform (1.5.4) of Green (TPDF) function of the option underlying variable with the transformed option payoff function on this variable.

2.1. Stochastic Calculus with Jump-to-Default

We will work with the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ augmented with the filtration $\mathcal{F}(t)$. Here, the filtration $\mathcal{F}(t)$ represents the information flow of asset price, its realized and instantaneous variances, and default intensity.

Definition 2.1.1. ***[Jump-to-Default]***

Let $\overline{\xi}$ be unit-mean exponential random variable independent of $\mathcal{F}$. Given the non-negative default intensity process $\lambda(t)$ adopted to $\mathcal{F}$, we define hazard function $\Lambda(t)$ as follows:

$$\Lambda(t) = \int_0^t \lambda(t')dt'. \tag{2.1.1}$$

Stopping time defined by:

$$\iota = \min(t : \Lambda(t) \geq \overline{\xi}), \tag{2.1.2}$$

is called the jump-to-default (or just default) time.

We note that the flow of information generated by ι and denoted by $\sigma(\{\iota < \tau\}, \tau \leq t)$ contains information whether default happened up to time t or not.

Definition 2.1.2. ***[Default-Extended Filtration]***
The default-extended filtration is denoted by $\mathcal{G}(t)$ and defined as:

$$\mathcal{G}(t) = \mathcal{F}(t) \vee \sigma(\{\iota < \tau\}, \tau \leq t). \tag{2.1.3}$$

In this content, $\mathcal{G}(t)$ represents the information flow on both default-free and defaultable dynamics. We emphasize that the model prices of defaultable securities are computed as conditional expectations with respect to $\mathcal{G}(t)$.

Corollary 2.1.1. ***[Survival and Default Probability]*** *Conditioned that the jump-to-default has not occurred before time t, the survival and default probabilities for time T, $t < T$, are respectively given by:*

$$\begin{aligned} Q(t,T) &= \mathbf{1}_{\{\iota>t\}}\mathbb{P}^{\mathbb{Q}}\left[\iota > T|\mathcal{G}(t)\right] = \mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(t')dt'}|\mathcal{G}(t)\right], \\ Q^c(t,T) &= \mathbf{1}_{\{\iota>t\}}\mathbb{P}^{\mathbb{Q}}\left[\iota \leq T|\mathcal{G}(t)\right] = \mathbf{1}_{\{\iota>t\}}\left(1 - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(t')dt'}|\mathcal{G}(t)\right]\right). \end{aligned} \tag{2.1.4}$$

Definition 2.1.3. ***[Infinitesimal Default Probability]***
The infinitesimal conditional default probability at time $T + dt$ given no default at T is denoted by $q(t,T)$ and given by:

$$\begin{aligned} q(t,T) &= \mathbf{1}_{\{\iota>t\}}\mathbb{P}^{\mathbb{Q}}\left[T < \iota \leq T + dt|\mathcal{G}(t)\right] = \frac{\partial}{\partial T}\left[1 - Q(t,T)\right] \\ &= \mathbf{1}_{\{\iota>t\}}\mathbb{E}\left[e^{-\int_t^T \lambda(t')dt'}\lambda(T)|\mathcal{G}(t)\right]. \end{aligned} \tag{2.1.5}$$

To formulate the default event of the jump-to-default, we introduce a modified version of the process $Y(t)$ which is constructed by stopping $Y(t)$ at a random stopping (default) time independent from $\mathcal{F}(t)$.

Definition 2.1.4. ***[Defaultable Ito Jump-Diffusion]***
Let $Y(t)$ satisfy the assumptions of definition (1.3.1) and let ι be a random exponential stopping time independent from $\mathcal{F}(t)$. We define a defaultable process $X(t)$ as follows:

$$X(t) = \begin{cases} Y(t), & t < \iota, \\ \aleph, & t \geq \iota, \end{cases} \tag{2.1.6}$$

where $\aleph$ is a post-default state.

For technical reasons, the artificial post-default state $\aleph$ is not defined on the real line, so that the integral over the TPDF of the process $X(t)$ is less than one. The leak of the probability mass represents the probability of $X(t)$ attaining the post-default state $\aleph$.

The dynamic version of the stopping event is attained by augmenting the SDE (1.3.1) with Poisson process $N^d(t)$ adopted to $\mathcal{G}(t)$ with a stochastic intensity rate $\lambda(t, X)$, where $\lambda(t, x) : [0, \infty)\times(-\infty, \infty) \to (0, \infty)$ is assumed to be at least piecewise continuous in t and x. Upon the arrival of the first jump in $N^d(t)$, i.e. following the jump-to-default, $X(t)$ immediately achieves its post-default state $\aleph$. Accordingly, (2.1.6) can be represented as follows:

$$X(t) = \begin{cases} Y(t), \ N^d(t) = 0, \\ \aleph, \ N^d(t) > 0, \end{cases} \tag{2.1.7}$$

Definition 2.1.5. ***[Jump-to-Default Process]*** *Let $N^d(t)$ be a Poisson process with intensity $\lambda(t, X)$ and let $N^d(t)$ be adopted to $\mathcal{G}(t)$. Then $N^d(t)$ is called jump-to-default process. Upon the first jump in $N^d(t)$, $X(t)$ immediately attains its post-default state.*

Corollary 2.1.2. *Stopping time defined by (2.1.2) and the time of the first jump in $N^d(t)$ are equivalent:*

$$\iota = \min(t > 0 : \Lambda(t) \geq \overline{\xi}) \equiv \min(t > 0 : N^d(t) = 1). \tag{2.1.8}$$

2.2. Defaultable Securities

Defining default event for single stocks is challenging from both practical and modeling point of view. This issue is especially pronounced while dealing with options on the realized variance and forward-start options.

From a modeling point of view, we note that by switching to logarithmic coordinates we face the problem of defining the jump-to-default for logarithmic price, which would formally mean jump to minus infinity upon the jump-to-default event. As a result, defining the realized variance after the jump-to-default is quite involved since in this case it virtually becomes infinite.

From a practical point of view, we need to specify in contract terms what is defined by default event and what happens to both asset price and its realized variance upon the default. A standard approach is to assume that after the default event stock price jumps to zero. As a result, the call option becomes worthless and the put option pays off its strike price at the maturity date.

In case of the realized stock variance, a typical contract includes a cap on the realized variance. This cap is usually chosen to be two or three times greater than the fair variance calculated using the market implied volatility skew corresponding to a given maturity (the market standard approach for calculation of the fair variance is explained in details by Demeterfi *et al* (1999)). In normal circumstances, this cap is quite unlikely to be reached. But in case of the default, this cap will certainly be reached. Accordingly, in our modeling framework we specify that, following the default event, the realized variance jumps to a specified level. As a result, the contract payoff upon the default is known and it is paid at claims maturity.

To conclude, we assume that after the jump-to-default the spot price drops to zero while the realized variance jumps to its pre-specified cap level. As a result, by solving the pricing problem we take into account the pre-default dynamics and supply the pricing equation with an additional term $\lambda(t)$ in the discount factor, (1.4.5), to reflect the fact that the payoff is conditioned on the survival up to maturity time (which can also be interpreted as applying an additional discount factor given by (2.1.4)).

Formally, a defaultable security can be characterized as follows.

Definition 2.2.1. ***[Financial Option on a Defaultable Security]***

A financial option with the value function denoted by $F(t, X)$, whose payoff and reward functions are dependent on a defaultable security $X(t)$, provides its holder with the following:

a) if no default has occurred before maturity time T, the holder receives a payoff in amount $u_1(X(T))$ at maturity time T, and in addition, if default has not occurred before time t', $t_0 \leq t' \leq T$, where t_0 is contract inception time, the holder receives at time $t = t'$ an instantaneous income stream given by a reward function $u_2(t, X(t))$.

b) if default does occur at time ι, $t_0 < \iota \leq T$, the holder receives either a payoff in amount $u_3(\aleph)$ at maturity time T or/and a payoff in amount $u_4(\iota, \aleph)$ that is payable immediately after the default. Here, we assume that the values of $u_3(\aleph)$ and $u_4(t, \aleph)$ are defined in contracts terms and mathematically they are defined on the post-default state $\aleph$.

Again, optionality is enforced by non-negativity of payoff and reward functions.

Corollary 2.2.1. *Applying the results of theorem (1.2.1), we obtain that the fair value of contingent claim $F(t, X)$ on a defaultable asset $X(t)$ at time t, $0 \leq t \leq T < \infty$, is given by:*

$$\begin{aligned} F(t, X) = &\mathbf{1}_{\{\iota > t\}} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(t')dt'} u_1(X(T)) \mathbf{1}_{\{N^d(T) - N^d(t) = 0\}} \middle| \mathcal{G}(t) \right] \\ &+ \mathbf{1}_{\{\iota > t\}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^{t'} r(t'')dt''} u_2(t', X(t')) \mathbf{1}_{\{N^d(t') - N^d(t) = 0\}} dt' \middle| \mathcal{G}(t) \right] \\ &+ \mathbf{1}_{\{\iota > t\}} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(t')dt'} u_3(\aleph) \mathbf{1}_{\{N^d(T) - N^d(t) > 0\}} \middle| \mathcal{G}(t) \right] \\ &+ \mathbf{1}_{\{\iota > t\}} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{\iota} r(t')dt'} u_4(\iota, \aleph) \mathbf{1}_{\{N^d(\iota) - N^d(t) > 0\}} \middle| \mathcal{G}(t) \right]. \end{aligned} \tag{2.2.1}$$

Here, the event $\{N^d(T) - N^d(t) = 0\}$ means that the payoff is conditioned by survival during time period $(t, T]$ and condition $\mathbf{1}_{\{\iota > t\}}$ means that by pricing the claim at evaluation time t, $t_0 \leq t \leq T$, we assume that the asset has not defaulted at this time.

In particular, using the independence of ι we simplify (2.2.1) as follows.

Corollary 2.2.2. *By independence of the stopping time ι and tower property of* *conditional expectation, we obtain:*

$$
\begin{aligned}
F(t,X) = &\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t')dt'}u_1(X(T))\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{N^d(T)-N^d(t)=0\}}|\mathcal{G}(T)\right]\middle|\,\mathcal{G}(t)\right]\\
&+\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{t'} r(t'')dt''}u_2(t',X(t'))\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{N^d(t')-N^d(t)=0\}}|\mathcal{G}(T)\right]dt'\middle|\,\mathcal{G}(t)\right]\\
&+\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t')dt'}u_3(\aleph)\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{N^d(T)-N^d(t)>0\}}|\mathcal{G}(T)\right]\middle|\,\mathcal{G}(t)\right]\\
&+\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{t'} r(t'')dt''}u_4(t',\aleph)\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{N^d(t'+dt')-N^d(t')>0\}}|\mathcal{G}(T)\right]dt'\middle|\,\mathcal{G}(t)\right].
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
F(t,X) = &\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(t')+\lambda(t'))dt'}u_1(X(T))\middle|\,\mathcal{G}(t)\right]\\
&+\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{t'} (r(t'')+\lambda(t''))dt''}u_2(t',X(t'))dt'\middle|\,\mathcal{G}(t)\right]\\
&+\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t')dt'}u_3(\aleph)\left[1-e^{-\int_t^T \lambda(t')dt'}\right]\middle|\,\mathcal{G}(t)\right]\\
&+\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{t'} (r(t'')+\lambda(t''))dt''}u_4(t',\aleph)\lambda(t')dt'\middle|\,\mathcal{G}(t)\right].
\end{aligned}
\tag{2.2.2}
$$

Eq.(2.2.2) provides us with general pricing formula for financial derivative securities on defaultable dynamics. However, the direct computation of the expected values in (2.2.2) can be very involved because the need to know TPDF of $X(t)$. In next section, we show how to apply Fourier transform method to consider the pricing problem in Fourier space, which can considerably simplify the pricing problem.

2.3. General Solution by Fourier Transform

To develop a general approach for pricing contingent claims under model (0.0.1), we first present the solution to pricing problem by means of Fourier transform, and then in Chapter 3 we find the Green function for the model (0.0.1).

Below we obtain general formula (2.3.7) to value a contingent claim $F(t,X)$ on $X(t)$ using Fourier transformed Green function of $X(t)$. We emphasize that process $X(t)$ need not represent the asset price, for example, it can represent the asset realized variance, the instantaneous variance, the forward-start asset price and so on. Another feature of formula (2.3.7) is that it allows to generalize pricing of claims on $X(t)$ with different payoff functions by using Fourier transformed Green function of $X(t)$ along with Fourier transformed payoff functions. That is, given the former we just need to find transformed payoff functions of claims under consideration.

Theorem 2.3.1. *[Valuation Formula]*

Let $X(t)$ be a stochastic process adopted to $\mathcal{G}(t)$ with the pre-default dynamics driven by Ito jump-diffusion (1.3.1) and whose infinitesimal generator $\mathcal{D}$ is defined by (1.4.2). Assume that the value function of a derivative claim on $X(t)$ denoted by $F(t,X)$ satisfies the assumptions of theorem (1.4.3), so that it can be presented as follows:

***1)** as a solution to the expected value problem:*

$$F(t,X) = \mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T g(t',X(t'))dt'}u_1(X(T)) + \int_t^T e^{-\int_t^{t'} g(t'',X(t''))dt''}u_2(t',X(t'))dt' \middle| \mathcal{G}(t)\right], \tag{2.3.1}$$

***2)** and a solution to P(I)DE problem:*

$$\begin{aligned} &F_t(t,X) + \mathcal{D}^g F(t,X) = -u_2(t,X), \\ &F(T,X) = u_1(X), \end{aligned} \tag{2.3.2}$$

where $\mathcal{D}^g F(t,X) = \mathcal{D}F(t,X) - g(t,X)F(t,X)$.

Furthermore, let $G(t,T,X,X')$ denote the Green (transition probability density) function of $X(t)$ with discounting term $g(t,X)$ defined as a solution to P(I)DE (1.4.10), and let $\widehat{G}(t,T,X,\Phi)$ denote its forward Fourier transform:

$$\widehat{G}(t,T,X,\Phi) = \mathbf{F}_-[G(X')](\Phi) = \int_{-\infty}^{\infty} e^{-\Phi X'}G(t,T,X,X')dX', \tag{2.3.3}$$

$$G(t,T,X,X') = \mathbf{F}_+[\widehat{G}(\Phi)](X') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\Phi X'}\widehat{G}(t,T,X,\Phi)d\Phi_I. \tag{2.3.4}$$

Finally, let $\widehat{u}_1(\Phi)$ denote the transform of the payoff function:

$$\widehat{u}_1(\Phi) = \int_{-\infty}^{\infty} e^{\Phi X'}u_1(X')dX', \tag{2.3.5}$$

where we assume that $e^{\Phi x}u_1(x) \in L^1(\mathbb{C})$, and let $\widehat{u}_2(t,\Phi)$ denote the transform of the reward function:

$$\widehat{u}_2(t,\Phi) = \int_{-\infty}^{\infty} e^{\Phi X'}u_2(t,X')dX', \tag{2.3.6}$$

where we assume that $e^{\Phi x}u_2(t,x) \in L^1(\mathbb{C})\ \forall\ t,\ 0 \le t \le T$.

Then the value function of the claim $F(t,X)$ can be computed by:

$$F(t,X) = U_1(t,X) + U_2(t,X), \tag{2.3.7}$$

where

$$U_1(t,X) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{G}(t,T,X,\Phi)\widehat{u}_1(\Phi)d\Phi_I = \frac{1}{\pi}\int_0^{\infty}\Re\left[\widehat{G}(t,T,X,\Phi)\widehat{u}_1(\Phi)\right]d\Phi_I, \tag{2.3.8}$$

and

$$\begin{aligned} U_2(t,X) &= \frac{1}{2\pi}\int_t^T \int_{-\infty}^{\infty} \widehat{G}(t,t',X,\Phi)\widehat{u}_2(t',\Phi)d\Phi_I dt' \\ &= \frac{1}{\pi}\int_t^T \int_0^{\infty} \Re\left[\widehat{G}(t,t',X,\Phi)\widehat{u}_2(t',\Phi)\right] d\Phi_I dt'. \end{aligned} \tag{2.3.9}$$

Proof. According to Kolmogoroff equation (1.4.1), The Green (transition density) function of $X(t)$ with discounting term $g(t,X)$, denoted by $G(t,T,X,X')$, solves the following problem:

$$\begin{aligned} &G_t + \mathcal{D}^g G = 0 \\ &G(T,T,X,X') = \delta(X-X'), \end{aligned} \tag{2.3.10}$$

Once the Green function is known we use Duhamel's principle (1.4.9) to solve the final value problem as follows:

$$\begin{aligned} &F(t,X) = U_1(t,X) + U_2(t,X), \\ &U_1(t,X) = \int_{-\infty}^{\infty} G(t,T,X,X')u_1(X')dX', \\ &U_2(t,X) = \int_t^T \int_{-\infty}^{\infty} G(t,t',X,X')u_2(t',X')dX'dt'. \end{aligned} \tag{2.3.11}$$

Now, applying transform (1.5.3) to the Green function (2.3.10), $\widehat{G}(t,T,X,\Phi) = \mathbf{F}_-[G(X')](\Phi)$, and exchanging infinitesimal and transform operators yields:

$$\begin{aligned} &\widehat{G}_t + \mathcal{D}^g \widehat{G}_{XX} = 0, \\ &\widehat{G}(T,T,X,\Phi) = e^{-\Phi X}. \end{aligned} \tag{2.3.12}$$

We assume that the closed-form solution to the above problem is found. As a result, we evaluate the original Green function $G(t,T,X,X')$ by computing the inverse transform:

$$\begin{aligned} G(t,T,X,X') = \mathbf{F}_+[\widehat{G}(\Phi)](X') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\Phi X'}\widehat{G}(t,T,X,\Phi)d\Phi_I \\ &= \frac{1}{\pi}\int_0^{\infty} \Re\left[e^{\Phi X'}\widehat{G}(t,T,X,\Phi)\right] d\Phi_I. \end{aligned} \tag{2.3.13}$$

Furthermore, we can compute the solution to terminal value problem given by (2.3.11) in the following way:

$$U_1(t,X) = \int_{-\infty}^{\infty} G(t,T,X,X')u_1(X')dX' = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\Phi X'}\widehat{G}(t,T,X,\Phi)u_1(X')d\Phi_I dX'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{\Phi X'}u_1(X')dX'\right]\widehat{G}(t,T,X,\Phi)d\Phi_I$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{G}(t,T,X,\Phi)\widehat{u}_1(\Phi)d\Phi_I = \frac{1}{\pi}\int_0^{\infty} \Re\left[\widehat{G}(t,T,X,\Phi)\widehat{u}_1(\Phi)\right] d\Phi_I,$$

where

$$\widehat{u}_1(\Phi) = \int_{-\infty}^{\infty} e^{\Phi X'} u_1(X') dX',$$

is the transformed terminal condition, and to make valid the interchange of the integration we require $e^{\Phi X'} u_1(X')$ to be in $L^1(\mathbb{C})$.

Similarly, the source term can be computed as follows:

$$U_2(t, X) = \int_t^T \int_{-\infty}^{\infty} G(t, t', X, X') u_2(t', X') dX' dt' =$$

$$= \frac{1}{2\pi} \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\Phi X'} \widehat{G}(t, t', X, \Phi) u_2(t', X') d\Phi_I dX' dt'$$

$$= \frac{1}{2\pi} \int_t^T \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{\Phi X'} u_2(t', X') dX' \right] \widehat{G}(t, t', X, \Phi) d\Phi_I dt' =$$

$$= \frac{1}{2\pi} \int_t^T \int_{-\infty}^{\infty} \widehat{G}(t, t', X, \Phi) \widehat{u}_2(t', \Phi) d\Phi_I dt' = \frac{1}{\pi} \int_t^T \int_0^{\infty} \Re \left[\widehat{G}(t, t', X, \Phi) \widehat{u}_2(t', \Phi) \right] d\Phi_I dt',$$

where

$$\widehat{u}_2(t', \Phi) = \int_{-\infty}^{\infty} e^{\Phi X'} u_2(t', X') dX',$$

and also we require $e^{\Phi X'} u_2(t,' X')$ to be in $L^1(\mathbb{C})\ \forall\ t,\ 0 \leq t \leq T$.

Here, we also employed the fact that the Fourier transform, $\widehat{G}(t, T, X, \Phi)$, of real valued function with a conjugate transformed parameter $\overline{\Phi} = \Phi_R - i\Phi_I$ is given by $\overline{\widehat{G}(t, T, X, \Phi)}$, so that

$$\Re \left[\int_{-\infty}^{\infty} \widehat{G}(t, T, X, \Phi) d\Phi_I \right] = \Re \left[\int_{-\infty}^{0} \widehat{G}(t, T, X, \Phi) d\Phi_I \right] + \Re \left[\int_0^{\infty} \widehat{G}(t, T, X, \Phi) d\Phi_I \right]$$

$$= \Re \left[\int_0^{\infty} \overline{\widehat{G}(t, T, X, \Phi)} d\Phi_I \right] + \Re \left[\int_0^{\infty} \widehat{G}(t, T, X, \Phi) d\Phi_I \right] = 2 \int_0^{\infty} \Re \left[\widehat{G}(t, T, X, \Phi) \right] d\Phi_I.$$

□

Chapter 3

Square Root Model

In this Chapter we will obtain general solution to the Green function in the affine model (0.0.1) summarized by Theorem (3.2.1). The result of this theorem is original in the sense that it relates the joint distribution of the log-price, its variance, and its realized variance with the corresponding three-dimensional Fourier transform. Some early result are published in Kangro-Pärna-Sepp (2004).

3.1. General Notations

We will use the following (pre-default) variables:

$X_K(t) = \ln \frac{S(t)}{K}$ - normalized asset log-price, where $K > 0$ is a constant;

$\bar{X}_{t_F}(t)$ - stands for the so-called forward-start asset log-price (this variable arises when we deal with the pricing problem of forward-start options) defined as follows:

$$\bar{X}_{t_F}(t) = \int_0^t \chi(t' - t_F) \ln \frac{S(t')}{S(t_F)} dt' = \begin{cases} 0, \ 0 \leq t < t_F \\ \ln \frac{S(t)}{S(t_F)} = X_1(t) - X_1(t_F), \ t_F \leq t \leq T \end{cases}, \tag{3.1.1}$$

where $\chi(t)$ is translation function defined as:

$$\int_0^t \chi(t' - t_F) f(t') dt' = f(t) \mathbf{1}_{\{t_F \leq t\}}.$$

As a result, the dynamics of $\bar{X}_{t_F}(t)$ can be written as:

$$d\bar{X}_{t_F}(t) = \chi(t - t_F) X_{X(t_F)}(t) dt, \ t_F < t, \ \bar{X}_{t_F}(t) = 0, \ 0 \leq t \leq t_F.$$

$I(t_0, t)$ represents realized (pre-default) variance of the log-spot asset price during time period $[t_0, t]$ in case it is driven by jump-diffusion dynamics (0.0.1).

$\bar{I}_{t_F}(t) = \int_0^t \chi(t' - t_F) I(t') dt'$ - is the so-called forward-start process of realized variance $I(t)$ with the ollowing dynamics:

$$d\bar{I}_{t_F}(t) = \chi(t - t_F) I(t_0, t) dt, \ 0 \leq t_0 < t_F < t, \ \bar{I}_{t_F}(t) = 0, \ 0 \leq t_0 < t \leq t_F.$$

For brevity, subsequently we will use simplified notations:
$X := X_K(t)$, $\bar{X} := \bar{X}_{t_F}(t)$, $I := I(t_0, t)$, $\bar{I} := \bar{I}_{t_F}(t)$.

Using the dynamics (0.0.1), we obtain the following pre-default dynamics for these variables:

$$\begin{cases} dX(t) &= \left(r(t) - d(t) - \frac{V(t-)}{2} - \gamma m^j + \lambda(t)\right) dt + \sqrt{V(t-)}dW^s(t) + J^s dN^j(t), \\ dV(t) &= \kappa(\theta - V(t-))dt + \varepsilon\sqrt{V(t-)}dW^v(t) + J^v dN^j(t), \\ dI(t) &= V(t-)dt + J^{s2}dN^j(t), \\ d\bar{X}(t) &= \chi(t - t_F)(X(t-) - X(t_F))dt,\ \bar{X}_{t_F}(t) = 0,\ 0 \le t \le t_F, \\ d\bar{I}(t) &= \chi(t - t_F)(I(t-) - I(t_F))dt,\ \bar{I}_{t_F}(t) = 0,\ 0 \le t \le t_F, \\ d\lambda(t) &= \alpha dt + \beta V(t-)dt, \end{cases} \tag{3.1.2}$$

with given values of $X(0) = \ln \frac{S(0)}{K}$, $V(0)$, and $I(0) = I_0$, where I_0 is the variance realized from inception time t_0 up to valuation time t.

To effectively solve the pricing problem, we will study the joint and marginal evolutions of these processes using Fourier transform technique. For this purpose, we introduce Green's functions, $G(')$, (transition probability densities) describing the distribution of these variables at a future point of time T given their current values at time t as well as their Fourier transforms (moment generating functions) $\widehat{G}(\cdot) = \mathbf{F}_{-}[G(')](\cdot)$ defined by (1.5.3) as follows:

$G^v(t, T, V, V')$, $\widehat{G}^v(t, T, V, \Theta)$ - for the variance process V;

$G^x(t, T, V, X, X')$, $\widehat{G}^x(t, T, V, X, \Phi)$ - the log-spot price X;

$G^{\bar{x}}(t, t_F, T, V, X, \bar{X}')$, $\widehat{G}^{\bar{x}}(t, t_F, T, V, X, \Phi)$ - the forward-start log-spot price $\bar{X}$;

$G^I(t, T, V, I, I')$, $\widehat{G}^I(t, T, V, I, \Psi)$ - the realized variance I;

$G^{\bar{I}}(t, t_F, T, V, I, \bar{I}')$, $\widehat{G}^{\bar{I}}(t, t_F, T, V, I, \Psi)$ - the forward-start realized variance $\bar{I}$;

$G^{vx}(t, T, V, V', X, X')$, $\widehat{G}^{vx}(t, T, V, \Theta, X, \Phi)$ - the joint evolution of V and X;

$G^{vp}(t, T, V, S, V')$, $\widehat{G}^{vp}(t, T, V, S, \Theta)$ - the price denominated variance process;

$G^{\bar{x}p}(t, t_F, T, V, S, X, \bar{X}, \bar{X}')$, $\widehat{G}^{\bar{x}p}(t, t_F, T, V, S, X, \bar{X}, \Phi)$ - for the price denominated forward-start log-spot price;

We assume that both $G(')$ and $\widehat{G}(\cdot)$ are in L^1 so that operations of integration are well defined and interchangeable.

We assume that the risk-free interest rate, $r(t)$, and dividend yield, $d(t)$, are deterministic (in case of discrete dividends, the typical approach is to appropriately adjust $d(t)$ using the term structure of asset forward prices) and introduce the following discount factors for time period (t, T):

$$\begin{aligned} DF(t, T) &= e^{-\int_t^T r(t')dt'},\ CF(t, T) = e^{-\int_t^T (r(t') - d(t')dt'}, \\ DF^{-1}(t, T) &= e^{\int_t^T r(t')dt'},\ CF^{-1}(t, T) = e^{\int_t^T (r(t') - d(t')dt'}. \end{aligned} \tag{3.1.3}$$

3.2. Solution to the Green Function

To deal with the pricing problem in its generality, we need to solve for the Green function of the joint dynamics of $X(t)$, $V(t)$, and $I(t)$. This function, denoted by $G(t,T,V,V',X,X',I,I')$, solves the following backward Kolmogoroff equation:

$$\begin{aligned}
&G_t + (r(t) - d(t) + \alpha + (-\frac{1}{2} + \beta)V)G_X + \frac{1}{2}VG_{XX} \\
&\quad + \rho\varepsilon VG_{XV} + \kappa(\theta - V)G_V + \frac{1}{2}\varepsilon^2 VG_{VV} + VG_I \\
&\quad + \gamma\int_0^\infty\int_{-\infty}^\infty \left(G(V+J^v, X+J^s, I+J^{s2}) - G - m^j G_X\right)\varpi(J^v, J^s)dJ^s dJ^v \\
&\quad - (\alpha + \beta V)G = 0, \\
&G(T,T,V,V'X,X',I,I') = \delta(X-X')\delta(V-V')\delta(I-I').
\end{aligned} \tag{3.2.1}$$

We note the non-local integral part in PIDE (3.2.1) stems from the fact that we assume simultaneous jumps in all three underlying dynamics. Discount term represents the probability of default (loss of the probability mass).

Applying 3-dimensional generalized Fourier transform (1.5.4) to (3.2.1) ,

$$\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi) = \mathbf{F}^3_-[G(V',X',I')](\Theta,\Phi,\Psi), \tag{3.2.2}$$

and exchanging the integration order yields:

$$\begin{aligned}
&\widehat{G}_t + (r(t) - d(t) + \alpha + (-\frac{1}{2} + \beta)V)\widehat{G}_X + \frac{1}{2}V\widehat{G}_{XX} \\
&\quad + \rho\varepsilon V\widehat{G}_{XV} + \kappa(\theta - V)\widehat{G}_V + \frac{1}{2}\varepsilon^2 V\widehat{G}_{VV} + V\widehat{G}_I \\
&\quad + \gamma\int_0^\infty\int_{-\infty}^\infty \left(\widehat{G}(V+J^v, X+J^s, I+J^{s2}) - \widehat{G} - m^j \widehat{G}_X\right)\varpi(J^v, J^s)dJ^s dJ^v \\
&\quad - (\alpha + \beta V)\widehat{G} = 0, \\
&\widehat{G}(T,T,V,\Theta,X,\Phi,I,\Psi) = e^{-\Theta V - \Phi X - \Psi I}.
\end{aligned} \tag{3.2.3}$$

Theorem 3.2.1. ***[Fundamental Solution]***

I) The solution to the transformed Green function (3.2.3) is given by:

$$\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi) = e^{-\Phi(X+\int_t^T(r(t')-d(t'))dt') - \Psi I + A(t,T) + B(t,T)V + \Gamma(t,T)}, \tag{3.2.4}$$

where functions $A(t,T)$, $B(t,T)$, $\Gamma(t,T)$ satisfy the following system of ODE-s:

$$\left\{\begin{array}{l}
\dot{A} + a_1 B + a_0 = 0,\ A(T,T) = 0, \\
\dot{B} + b_2 B^2 + b_1 B + b_0 = 0,\ B(T,T) = -\Theta, \\
\dot{\Gamma} + \gamma\int_0^\infty\int_{-\infty}^\infty \left(e^{-\Phi J^s - \Psi J^{s2} + BJ^v} + m^j\Phi - 1\right)\varpi(J^s, J^v)dJ^s dJ^v = 0,\ \Gamma(T,T) = 0,
\end{array}\right. \tag{3.2.5}$$

where dot stands for derivative with respect to t, and
$b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa - \rho\varepsilon\Phi$, $b_0 = \frac{1}{2}\Phi^2 - (-\frac{1}{2}+\beta)\Phi - \Psi - \beta$, $a_1 = \kappa\theta$, $a_0 = -\Phi\alpha - \alpha$.

II) We assume that the following inequalities hold:

$$
\begin{aligned}
&0)\ \Re\left[b_2\right] > 0,\ \ \Re[\Theta] \le 0;\\
&1)\ \Re\left[b_1^2 - 4b_0 b_2\right] > 0;\\
&2)\ \Re[\psi_+] = \Re\left[b_1 + \sqrt{b_1^2 - 4b_0 b_2}\right] < 0;\\
&3)\ \Re[\psi_-] = \Re\left[-b_1 + \sqrt{b_1^2 - 4b_0 b_2}\right] > 0;
\end{aligned}
\tag{3.2.6}
$$

Then the solution to the system (3.2.5) is well-defined and it is given by:

$$
\begin{aligned}
A(t,T) &= -\frac{a_1}{b_2}\left(\frac{1}{2}\psi_+\tau + \ln\left(C_+ e^{-\zeta\tau} + C_-\right)\right) + \tau a_0,\\
B(t,T) &= -\frac{-\psi_- C_+ e^{-\zeta\tau} + \psi_+ C_-}{2b_2\left(C_+ e^{-\zeta\tau} + C_-\right)},\\
\Gamma(t,T) &= \gamma \int_t^T \int_0^\infty \int_{-\infty}^\infty \left(e^{-\Phi J^s - \Psi J^{s2} + B(t',T)J^v} + m\Phi - 1\right) \varpi(J^v, J^s) dJ^s dJ^v dt',
\end{aligned}
\tag{3.2.7}
$$

where

$$
C_\pm = \frac{1}{\zeta}\left(\frac{\psi_\pm}{2} \mp b_2\Theta\right),\ \ \psi_\pm = \pm b_1 + \zeta,\ \ \zeta = \sqrt{b_1^2 - 4b_0 b_2},\ \ \tau = T - t.
$$

Formula (3.2.7) is one of our key results which enable to efficiently compute Green functions in the affine model. We discuss specific choices for jump-size distribution and computation of function $\Gamma(t,T)$ in the next section.

Remark 3.1. ***[Continuity of Complex-Valued Square Root and Logarithm]***
Assumption 1) ensures that the real part of the expression $b_1^2 - 4b_0 b_2$ *is always positive, thus* ζ *has no branch cut and it is always continuous as a function of* Θ, Φ, *and* Ψ.

Assumptions 1), 2), 3) ensure that the logarithm in the expression for $A(t,T)$ *is well-defined and, in particular, that the real part of the expression inside the logarithm is always positive, so that there is no branch cut and the above logarithm is continuous as a function of* Θ, Φ, *and* Ψ.

Proof. Assuming the affine specification (3.2.4) and plugging it into equation (3.2.3), we get the system of ODE-s (3.2.5). We solve this system imposing appropriate terminal conditions:

$$
A(T,T) = 0,\ \ B(T,T) = -\Theta,\ \ \Gamma(T,T) = 0. \tag{3.2.8}
$$

First, we linearize ODE for $B(t,T)$ as follows:

$$B(t,T) = -\frac{\dot{C}(t,T)}{b_2 C(t,T)}, \tag{3.2.9}$$

with

$$\dot{C}(t,T) = -b_2 B(t,T) C(t,T), \ \dot{C}(T,T) = b_2\Theta, \ C(T,T) = 1, \tag{3.2.10}$$

and obtain the second order ODE for $C(t,T)$:

$$\begin{aligned} &\ddot{C}(t,T) - b_1\dot{C}(t,T) + b_0 b_2 C(t,T) = 0, \\ &C(T,T) = 1, \ \dot{C}(T,T) = b_2\Theta. \end{aligned} \tag{3.2.11}$$

The general solution to this equation has the form:

$$C(t,T) = C_+ e^{-\frac{1}{2}\psi_-\tau} + C_- e^{\frac{1}{2}\psi_+\tau},$$

where

$$\psi_\pm = \pm b_1 + \zeta, \ \zeta = \sqrt{b_1^2 - 4b_0 b_2}, \ \tau = T - t.$$

We note that $\psi_+ + \psi_- = 2\zeta$, $\psi_+ \cdot \psi_- = -4b_0 b_2$, and $\Re[\zeta] > 0$ due to the assumption 1).

The boundary conditions:

$$\begin{cases} C(T,T) &= C_+ + C_- = 1, \\ \dot{C}(T,T) &= -\frac{1}{2}\psi_- C_+ + \frac{1}{2}\psi_+ C_- = b_2\Theta, \end{cases}$$

yield:

$$\begin{cases} C_+ &= \frac{1}{\zeta}\left(\frac{\psi_+}{2} - b_2\Theta\right) \\ C_- &= \frac{1}{\zeta}\left(\frac{\psi_-}{2} + b_2\Theta\right) \end{cases}.$$

Based on assumptions 1), 2), 3) we can write:

$$C(t,T) = C_+ e^{-\frac{1}{2}\psi_-\tau} + C_- e^{\frac{1}{2}\psi_+\tau} = e^{\frac{1}{2}\psi_+\tau}\left(C_+ e^{-\zeta\tau} + C_-\right),$$

and we point out that $C(t,T)$ is an increasing function of t thus $\dot{C}(t,T) > 0$.

Accordingly,

$$B(t,T) = -\frac{\dot{C}(t,T)}{b_2 C(t,T)} = -\frac{-\psi_- C_+ e^{-\zeta\tau} + \psi_+ C_-}{2b_2\left(C_+ e^{-\zeta\tau} + C_-\right)}. \tag{3.2.12}$$

and $B(t,T) < 0$ based on the above remark and assumption 0).

Next, we solve ODE for $A(t,T)$ by integration from t to T:

$$
\begin{aligned}
&\dot{A}(t,T) = -a_1 B(t,T) - a_0,\ A(T,T) = 0,\\
&A(T,T) - A(t,T) = \frac{a_1}{b_2}\int_t^T \frac{\dot{C}(t',T)}{C(t',T)}dt' - (T-t)a_0,\\
&A(t,T) = -\frac{a_1}{b_2}\ln\frac{C(T,T)}{C(t,T)} + (T-t)a_0 =\\
&= -\frac{a_1}{b_2}\left(\frac{1}{2}\psi_+\tau + \ln\left(C_+ e^{-\zeta\tau} + C_-\right)\right) + (T-t)a_0,
\end{aligned}
\tag{3.2.13}
$$

where the manipulation in the last row is justified by assumptions 1), 2), and 3).

Finally, we solve ODE for $\Gamma(t,T)$ by integration from t to T.

□

The above theorem is also directly applicable to one factor Green functions for the dynamics of only one underlying factor. In particular, we have the following result.

Corollary 3.2.1. ***[One-Factor Green Function]***

The transformed Green function for the dynamics of $X(t)$, $\widehat{G}^x(t,T,V,X,\Phi) = \mathbf{F}_-[G^x(X')](\Phi)$, *is obtained by taking limits in (3.2.4):*

$$\widehat{G}^x(t,T,V,X,\Phi) = \lim_{\Theta\to 0,\Psi\to 0}\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.2.14}$$

The transformed Green function for the dynamics of $V(t)$, $\widehat{G}^v(t,T,V,\Theta) = \mathbf{F}_-[G^v(V')](\Theta)$, *is obtained by:*

$$\widehat{G}^v(t,T,V,\Theta) = \lim_{\Phi\to 0,\Psi\to 0}\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.2.15}$$

Finally, the transformed Green function for the dynamics of $I(t)$, $\widehat{G}^I(t,T,V,I,\Psi) = \mathbf{F}_-[G^I(I')](\Psi)$, *is obtained by:*

$$\widehat{G}^I(t,T,V,I,\Psi) = \lim_{\Theta\to 0,\Phi\to 0}\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.2.16}$$

Furthermore, two-dimensional joint Green functions for these variables are also obtained by taking appropriate limits.

Remark 3.2. ***[Time-dependent Parameters]***

The results of theorem (3.2.1) can easily be extended to the case of time-dependent parameters. In general, we need to solve system (3.2.5) to compute values of $A(t,T)$, $B(t,T)$, and $\Gamma(t,T)$ numerically. However, in case of piecewise constant parameters the corresponding solution can conveniently be computed by recursion as shown, for example, in Mikhailov-Nogel (2003) or Wu-Zhang (2006). All of our results are directly applicable for the affine model with piecewise constant and, more generally, time-dependent parameters.

3.2.1. Jump Size Distribution

Now we will consider possible specifications of jump-size distributions and obtain the appropriate representation of function $\Gamma(t,T)$ specified by (3.2.7). In particular, Merton (1976) proposed jump-diffusions where the logarithm of jump size is normally distributed and Duffie-Pan-Singleton (2000) considered bivariate jump size distribution, where the jump in variance is exponentially distributed, and the jump in the asset price is normally distributed on conditional on the variance jump.

For our purposes, we consider the case where asset and variance jumps are independent of each other and variance jumps are exponentially distributed:

$$\varpi(J^v, J^s) = \frac{1}{\eta} e^{-\frac{1}{\eta}J^v} \varpi(J^s), \tag{3.2.17}$$

where η is variance jump mean and $\varpi(J^s)$ is the PDF of asset jumps. Then we obtain the following result.

Corollary 3.2.2. *Function $\Gamma(t,T)$ in formula (3.2.7) using notations of the theorem (3.2.1) is computed by:*

$$\Gamma(t,T) = \gamma(E(t,T)Z(\Phi,\Psi) + \tau(m^j\Phi - 1)), \tag{3.2.18}$$

where

$$E(t,T) = -\frac{4b_2\eta}{\overline{\psi}_+\overline{\psi}_-}\left(\ln\left(\frac{\overline{\psi}_- C_+ e^{-\zeta\tau} + \overline{\psi}_+ C_-}{\overline{\psi}_- C_+ + \overline{\psi}_+ C_-}\right) - \frac{1}{2}\tau\overline{\psi}_-\right), \tag{3.2.19}$$

with $\overline{\psi}_\pm = 2b_2 \pm \eta\psi_\pm$, and

$$Z(\Phi,\Psi) = \int_{-\infty}^{\infty}\left(e^{-\Phi J^s - \Psi J^{s2}}\right)\varpi(J^s)dJ^s. \tag{3.2.20}$$

In addition, the real part of the complex-valued logarithm in $E(t,T)$ is positive assuming:

$$\Re\left[-\frac{\psi_+}{2b_2}\right] = \Re\left[-\frac{b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2b_2}\right] < \frac{1}{\eta}. \tag{3.2.21}$$

Proof. First, we use Eq.(3.2.7) and represent $\Gamma(t,T)$ as:

$$\Gamma(t,T) = \gamma\left(\left(\int_{-\infty}^{\infty} e^{-\Phi J^s - \Psi J^{s2}}\varpi(J^s)dJ^s\right)\int_t^T \frac{1}{\eta}\int_0^{\infty} e^{B(t',T)J^v - \frac{1}{\eta}J^v} dJ^v dt' + \tau(m^j\Phi - 1)\right),$$

Then, we introduce functions $Z(\Phi,\Psi)$ and $E(t,T)$, where $Z(\Phi,\Psi)$ is defined by (3.2.20) and $E(t,T)$ solves the following equation:

$$E(t,T) = \frac{1}{\eta}\int_t^T\int_0^{\infty} e^{B(t',T)J^v - \frac{1}{\eta}J^v} dJ^v dt' = \int_t^T \frac{1}{1 - \eta B(t',T)}dt', \tag{3.2.22}$$

where we need to check that $\Re[B(t,T)] < \frac{1}{\eta}$. This statement follows from the fact that for $t = T$ this inequality is satisfied by the virtue of the assumption 0) of theorem (3.2.1), and then we ensure that the following asymptotic

$$\lim_{T\to\infty} \Re[B(t,T)] = -\frac{\psi_+}{2b_2}, \tag{3.2.23}$$

is correct by making assumption (3.2.21).

Finally, by integrating Eq (3.2.21) in time variable we obtain:

$$E(t,T) = -\frac{4b_2\eta}{\overline{\psi}_+\overline{\psi}_-}\left(\ln\left(\frac{\overline{\psi}_-C_+e^{-\zeta\tau} + \overline{\psi}_+C_-}{\overline{\psi}_-C_+ + \overline{\psi}_+C_-}\right) - \frac{1}{2}\tau\overline{\psi}_-\right). \tag{3.2.24}$$

□

Thus, assuming that variance jumps are exponentially distributed, for a given PDF of asset jumps we need to evaluate integral (3.2.20). In particular, we assume normally distributed log-asset jumps:

$$\varpi(J^s) = \frac{1}{\sqrt{2\pi\delta^2}}e^{-\frac{(J-\nu)^2}{2\delta^2}}, \tag{3.2.25}$$

where ν is asset jump mean, δ is its standard deviation.

Corollary 3.2.3. *For PDF (3.2.25), function $Z(\Phi,\Psi)$ in formula (3.2.18) is computed by:*

$$\begin{aligned} Z(\Phi,\Psi) &= \frac{1}{\sqrt{2\pi\delta^2}}\int_{-\infty}^{\infty} e^{-\Phi J^s - \Psi J^{s2} - \frac{(J-\nu)^2}{2\delta^2}}dJ^s \\ &= \frac{1}{\sqrt{2\Psi\delta^2+1}}e^{\frac{-\nu^2\Psi-\nu\Phi+\frac{1}{2}\delta^2\Phi^2}{2\Psi\delta^2+1}}, \end{aligned} \tag{3.2.26}$$

with $m^j = e^{\nu+\frac{1}{2}\delta^2} - 1$ and provided that $\Psi_R > -\frac{1}{2\delta^2}$.

Formula (3.2.18) along with (3.2.26) provides general solution under both price and variance jumps for all three underlying variables. We also consider some particular cases of formula (3.2.18).

Corollary 3.2.4. *We introduce limiting cases of formula (3.2.26):*

$$\begin{aligned} Z^x(\Phi) &= e^{-\nu\Phi+\frac{1}{2}\delta^2\Phi^2}, \\ Z^I(\Psi) &= \frac{1}{\sqrt{2\Psi\delta^2+1}}e^{\frac{-\nu^2\Psi}{2\Psi\delta^2+1}}, \end{aligned} \tag{3.2.27}$$

and obtain the following modifications of (3.2.18).

1) For dynamics with only exponential variance jumps, we use:

$$\Gamma(t,T) = \gamma(E(t,T) - \tau), \tag{3.2.28}$$

2) For dynamics with only price jumps, we use:

$$\Gamma(t,T) = \gamma\tau(Z(\Phi,\Psi) + (m^j\Phi - 1)), \tag{3.2.29}$$

3) For one-dimensional Green function of X with variance and price jumps, we use:

$$\Gamma(t,T) = \gamma\left(E(t,T)Z^x(\Phi) + \tau(m^j\Phi - 1)\right), \tag{3.2.30}$$

4) For one-dimensional Green function of X with price jumps, we use:

$$\Gamma(t,T) = \gamma\tau\left(Z^x(\Phi) + (m^j\Phi - 1)\right), \tag{3.2.31}$$

5) For one-dimensional Green function of I with variance and price jumps, we use:

$$\Gamma(t,T) = \gamma\left(E(t,T)Z^I(\Psi) - \tau\right). \tag{3.2.32}$$

6) For one-dimensional Green function of I with price jumps, we use:

$$\Gamma(t,T) = \gamma\tau\left(Z^I(\Psi) - 1\right). \tag{3.2.33}$$

Finally, we mention the double-exponential PDF for asset jumps proposed by Kou (2002):

$$\varpi(J) = \varpi^-(J) + \varpi^+(J) = q^-\frac{1}{\eta^-}e^{\frac{1}{\eta^-}J}\mathbf{1}_{\{J<0\}} + q^+\frac{1}{\eta^+}e^{-\frac{1}{\eta^+}J}\mathbf{1}_{\{J\geq 0\}} \tag{3.2.34}$$

where $1 > \eta^+ > 0$ and $\eta^- > 0$ are means of positive and negative jump sizes, respectively; constants q^+ and q^- represent the probabilities of positive and negative jumps, respectively, q^+, $q^- \geq 0$, $q^+ + q^- = 1$. Requirement that $\eta^+ < 1$ is needed to ensure that $\mathbb{E}^{\mathbb{Q}}[e^J] < \infty$ and $\mathbb{E}^{\mathbb{Q}}[S] < \infty$.

We do not study this jump-diffusion in this part, while we will pay a particular attention to it in the second part of this thesis. For the sake of completeness, we provide the following result.

Corollary 3.2.5. ***[Double-Exponential Jump-Size Distribution]***

In case of double-exponential jumps, assuming only jumps in the log-spot dynamics, formula (3.2.18) is given by:

$$\begin{aligned}\Gamma(t,T) &= \gamma\tau\left(\frac{q^+}{1+\Phi\eta^+} + \frac{q^-}{1-\Phi\eta^-} - 1 + \Phi m^j\right),\\ m^j &= \frac{q^+}{1-\eta^+} + \frac{q^-}{1+\eta^-} - 1,\end{aligned} \tag{3.2.35}$$

provided that $-\frac{1}{\eta^+} < \Phi_R < \frac{1}{\eta^-}$.

3.3. Underlying Dynamics

Now we consider the dynamics implied by the general SDE (3.1.2) in details and study their TPDFs. We also consider the limiting behavior of the transformed Green function to ensure that it is in $L^1(\mathbb{C}^n)$ as a function of n transform parameters and to obtain estimates for truncating the infinite integration bounds by employing numerical inversion of Fourier transform by means of FFT methods and Gaussian quadratures.

Throughout this section we will use four prototype dynamics to illustrate the impact of model components on the implied Green functions of the evolution of the underlying variables. These dynamics include:

1) SV - the asset price process with stochastic variance,

2) SV+JD - the asset price process with stochastic variance and jump-to-default with constant intensity α,

3) SV+VJ - the asset price process with stochastic variance and exponentially distributed variance jumps,

4) SV+PJ - the asset price process with stochastic variance and normally distributed price jumps.

Model parameters for these prototype processes are summarized in table (3.3.1). We will use $T = 1$ for the maturity time, the scaled asset price with $S(0) = 1$ and $r(t) = d(t) = 0$. We also scale the Green functions so that they integrate to one (under the default-free dynamics).

Finally, we note that given the Green function of the evolution of the log-asset price, $G^X(t, T, X, X')$, we obtain the Green function of the asset price, $G^S(t, T, S, S')$, by computing:

$$G^S(t, T, S, S') = \frac{1}{S'} G^X(t, T, X, X'), \tag{3.3.1}$$

where $X = \ln S$ and $X' = \ln S'$. Similar result is also valid for two and three-dimensional Green functions involving the asset price variable.

	SV	SV+JD	SV+VJ	SV+PJ
$V(0)$	0.04	0.04	0.04	0.04
θ	0.04	0.04	0.04	0.04
κ	2	2	2	2
ε	0.2	0.2	0.2	0.2
ρ	-0.8	-0.8	-0.8	-0.8
α		0.1		
β				
η			0.01	
γ			1	1
ν				-0.1
δ				0.1

Table 3.3.1: Parameters of four prototype dynamics.

We compute the corresponding Green functions by inverting Fourier integrals numerically by means of Gaussian quadratures (we use a standard routine from

Press *et al* (1992) with an appropriate discretization and localization as well as an extended routine for two-dimensional inversion problems) truncating the infinite integral using asymptotic bounds which we obtain in this section by analyzing the limiting behavior of corresponding Green functions.

3.3.1. Variance

Using formula (3.2.4), the transformed Green function for the dynamics of $V(t)$, $\widehat{G}^v(t,T,V,\Theta) = \mathbf{F}_-[G^v(V')](\Theta)$, is obtained by:

$$\widehat{G}^v(t,T,V,\Theta) = \lim_{\Phi\to 0,\Psi\to 0} \widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.3.2}$$

Corollary 3.3.1. ***[Defaultable Dynamics]***
The solution to (3.3.2) is given by formula:

$$\widehat{G}^v(t,T,V,\Theta) = e^{A^v(t,T,\Theta)+B^v(t,T,\Theta)V+\Gamma^v(t,T,\Theta)}, \tag{3.3.3}$$

where for computing $A^v(t,T,\Theta)$, $B^v(t,T,\Theta)$, $\Gamma^v(t,T,\Theta)$, we specify parameters in general formula (3.2.4) as follows: $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa$, $b_0 = -\beta$, $a_1 = \kappa\theta$, $a_0 = -\alpha$.

Corollary 3.3.2. ***[Default-Free Dynamics]***
The solution is given by general formula (3.2.4), where we specify parameters as follows: $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa$, $b_0 = 0$, $a_1 = \kappa\theta$, $a_0 = 0$. In case of exponentially distributed variance jumps we obtain the explicit solution given by (3.3.3) with:

$$\begin{aligned}
A^v(t,T,\Theta) &= -\frac{2\kappa\theta}{\varepsilon^2}\ln\left(\frac{\varepsilon^2}{2\kappa}\Theta(1-e^{-\kappa\tau})+1\right),\\
B^v(t,T,\Theta) &= -\frac{\Theta e^{-\kappa\tau}}{\frac{\varepsilon^2}{2\kappa}\Theta(1-e^{-\kappa\tau})+1},\\
\Gamma^v(t,T,\Theta) &= -\frac{\vartheta\eta}{\frac{1}{2}\varepsilon^2-\eta\kappa}\ln\left(\frac{(2\kappa+\varepsilon^2\Theta)-(\varepsilon^2-2\kappa\eta)\Theta e^{-\kappa\tau}}{2\kappa(1+\eta\Theta)}\right),
\end{aligned} \tag{3.3.4}$$

Proposition 3.1. ***[First and Second Moments]***
For default-free dynamics, we obtain:

$$\begin{aligned}
\overline{M}^v_1(t,T) &= e^{-\kappa(T-t)}V + \left(1-e^{-\kappa(T-t)}\right)\theta + \frac{\vartheta\eta}{\kappa}\left(1-e^{-\kappa(T-t)}\right),\\
\overline{M}^v_2(t,T) &= \frac{\varepsilon^2}{\kappa}\left(1-e^{-\kappa(T-t)}\right)\left(e^{-\kappa(T-t)}V+\frac{1}{2}(1-e^{-\kappa(T-t)})\theta\right)+\\
&\quad + \frac{\eta\vartheta}{2\kappa^2}\left(1-e^{-\kappa(T-t)}\right)\left(2\eta\kappa+\varepsilon^2+(2\eta\kappa-\varepsilon^2)e^{-\kappa(T-t)}\right).
\end{aligned} \tag{3.3.5}$$

Proof. The result is obtained using theorem (1.5.1) along with formula (3.3.4). □

From (3.3.5), we see that the long-term mean of the implied variance is $\theta + \frac{\lambda_\infty \eta}{\kappa}$ and the long-term variance of the implied variance is $\frac{\varepsilon^2\theta}{2\kappa} + \frac{\eta\vartheta}{2\kappa^2}\left(2\eta\kappa + \varepsilon^2\right)$.

Finally, we note that for non-defaultable variance dynamics with no jumps and constant parameters there is an analytical solution for the Green function (see, for example, Lipton (2001)):

$$G^v(t,T,V,V') = \overline{M} e^{-\overline{M}\left(e^{-\kappa\tau}V + V'\right)} \left(\frac{V'}{e^{-\kappa\tau}V}\right)^{\frac{\nu}{2}} \mathcal{I}_\vartheta\left(2\overline{M}\sqrt{e^{-\kappa\tau}VV'}\right), \quad (3.3.6)$$

where $\overline{M} = \frac{2\kappa}{\varepsilon^2(1-e^{-\kappa\tau})}$, $\mathcal{I}_\vartheta(\cdot)$ is the modified Bessel function of order ϑ, and ϑ is the so called Feller parameter defined by:

$$\vartheta = \frac{2\kappa\theta}{\varepsilon^2} - 1. \quad (3.3.7)$$

Then the corresponding CDF is given by:

$$\mathfrak{G}^v(t,T,V,y) = \int_0^y G^v(t,T,V,V')dV' = \chi^2(2\overline{M}y, 2\vartheta + 2, 2\overline{M}Ve^{-\kappa\tau}), \quad (3.3.8)$$

where $\chi^2(y,d,n)$ is the CDF of non-central chi-squared variable Y with d degrees of freedom and non-centrality parameter n.

Limiting Behavior

For brevity, we consider default-free dynamics and to ensure the continuity of logarithm in $A^v(t,T,\Theta)$, we consider the limit:

$$\lim_{\tau\to\infty}\left(\frac{(\psi_+ - \varepsilon^2\Theta)e^{-\kappa\tau} + (\psi_- + \varepsilon^2\Theta)}{2\kappa}\right) = \frac{(\psi_- + \varepsilon^2\Theta)}{2\kappa}. \quad (3.3.9)$$

Accordingly, we take $\Theta_R > -\frac{2\kappa}{\varepsilon^2}$.

Next, to ensure the continuity of logarithm in $\Gamma^v(t,T,\Theta)$, we consider the limit

$$\lim_{\tau\to\infty}\left(\frac{\overline{\psi}_-(\psi_+ - \varepsilon^2\Theta)e^{-\kappa\tau} + \overline{\psi}_+(\psi_- + \varepsilon^2\Theta)}{\overline{\psi}_-(\psi_+ - \varepsilon^2\Theta) + \overline{\psi}_+(\psi_- + \varepsilon^2\Theta)}\right) = \frac{\overline{\psi}_+(\psi_- + \varepsilon^2\Theta)}{2\kappa\varepsilon^2(1+\eta\Theta)}. \quad (3.3.10)$$

and we take $\Theta_R > -\frac{1}{\eta}$.

Accordingly, for the variance dynamics we take

$$\Theta_R > \max\left(-\frac{2\kappa}{\varepsilon^2}, -\frac{1}{\eta}\right). \quad (3.3.11)$$

Next, we take $\Theta = i\Theta_I$ and compute the following limit in (3.3.3):

$$\lim_{\Theta_I\to\pm\infty} \Re\left[\widehat{G}^v(t,T,V,i\Theta_I)\right] \sim e^{-\frac{2\kappa\theta}{\varepsilon^2}\ln\left(\frac{\varepsilon^2}{2\kappa}(1-e^{-\kappa\tau})|\Theta_I|\right)} \sim e^{-\frac{2\kappa\theta}{\varepsilon^2}\ln(|\Theta_I|)}. \quad (3.3.12)$$

Finally, given a precision level ϵ we determine the cut-off value $\overline{\Theta}_I$ as follows:

$$\int_{\overline{\Theta}_I}^{\infty} e^{-\frac{2\kappa\theta}{\varepsilon^2}\ln(\Theta_I)} d\Theta_I < \epsilon. \tag{3.3.13}$$

and obtain:

$$\overline{\Theta}_I > \left(\left(\frac{2\kappa\theta}{\varepsilon^2} - 1 \right) \epsilon \right)^{-\frac{1}{\frac{2\kappa\theta}{\varepsilon^2}-1}} = (\vartheta\epsilon)^{-\frac{1}{\vartheta}}, \tag{3.3.14}$$

provided $\vartheta > 0$.

Remark 3.3. ***[Feller Condition]***

We see that the dimensionless Feller parameter $\vartheta = \frac{2\kappa\theta}{\varepsilon^2} - 1$ plays a particular role for convergence of the transformed Green function. The following result is due to Feller (1971): in case $\vartheta \leq 0$, the origin is attainable for V' (the Green function does converge anyway). In this case zero is a reflecting boundary for the variance process, while if $\vartheta > 0$ then the origin is the natural boundary.

When we calibrate the model parameters to market data, as a rule, we get $\vartheta < 0$. To avoid unnecessary complications we enforce the non-negativity of ϑ by the model calibration.

Illustration

In Figure (3.3.1) we plot implied Green function for three prototype dynamics (price jumps do not affect the variance dynamics).

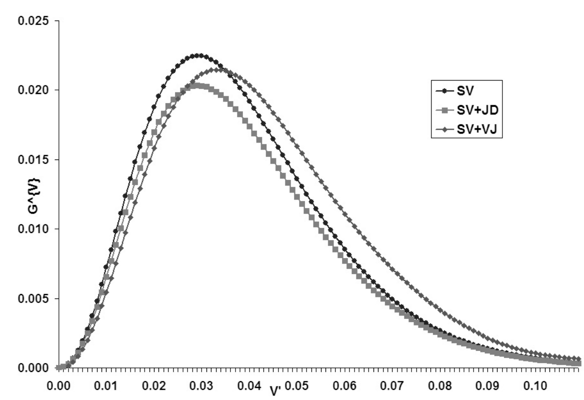

Figure 3.3.1: Green functions of the variance process

We see that the Green function of the process with jump-to-default does not integrate to one and it represents a "scaled" version of the original process with no jump-to-default. Adding variance jumps leads to a remarkable right tail to the TPDF of the variance. The TPDF of the variance is positively skewed.

3.3.2. Log-Spot Price

The transformed Green function for the dynamics of $X(t)$,
$\widehat{G}^x(t,T,V,X,\Phi) = \mathbf{F}_-[G^x(X')](\Phi)$, is obtained by:

$$\widehat{G}^x(t,T,V,X,\Phi) = \lim_{\Theta\to 0,\Psi\to 0} \widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.3.15}$$

Corollary 3.3.3. ***[Defaultable Dynamics]***
The solution is given by formula:

$$\widehat{G}^x(t,T,V,X,\Phi) = e^{-\Phi(X+\int_t^T(r(t')-d(t'))dt')+A^x(t,T)+B^x(t,T)V+\Gamma^x(t,T)}, \tag{3.3.16}$$

where $A^x(t,T)$, $B^x(t,T)$, $\Gamma^x(t,T)$ are computed using general formula (3.2.4) with: $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa - \rho\varepsilon\Phi$, $b_0 = \frac{1}{2}\Phi^2 - (-\frac{1}{2}+\beta)\Phi - \beta$, $a_1 = \kappa\theta$, $a_0 = -\Phi\alpha - \alpha$, $\Theta = 0$.

Corollary 3.3.4. ***[Default-Free Dynamics]***
The solution is given by formula (3.3.16) with $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa - \rho\varepsilon\Phi$, $b_0 = \frac{1}{2}\Phi^2 + \frac{1}{2}\Phi$, $a_1 = \kappa\theta$, $a_0 = 0$, $\Theta = 0$.

Limiting Behavior

Now, for brevity we consider the default-free dynamics and check that

$$f(\Phi) := f(\Phi_R,\Phi_I) = \Re[(\kappa+\rho\varepsilon\Phi)^2 - \varepsilon^2(\Phi^2+\Phi)] > 0. \tag{3.3.17}$$

First, we note that

$$\lim_{\Phi_I\to\infty} f(\Phi_R,\Phi_I) \sim \varepsilon^2(1-\rho^2)\Phi_I^2 > 0, \tag{3.3.18}$$

so that we need to ensure that $f(\Phi_R,0) > 0$, i.e:

$$(\kappa+\rho\varepsilon\Phi_R)^2 - \varepsilon^2(\Phi_R^2+\Phi_R) > 0, \tag{3.3.19}$$

which is true provided $-\frac{1}{(1-\rho^2)} < \Phi_R < \frac{1}{(1-\rho^2)}$.

First, to ensure the continuity of logarithm in $A^x(t,T)$, we consider the limit:

$$\lim_{\tau\to\infty}\left(\frac{\psi_+ e^{-\zeta\tau}+\psi_-}{2\zeta}\right) = \frac{\psi_-}{2\zeta}. \tag{3.3.20}$$

Accordingly, the real part of this limit is positive provided $-\frac{\kappa}{\varepsilon|\rho|} < \Phi_R < \frac{\kappa}{\varepsilon|\rho|}$.

Then, we ensure that the continuity of logarithm in $\Gamma(t,T)$ by considering:

$$\Re[B^x(t,T)] < \frac{1}{\eta}, \tag{3.3.21}$$

which hold if $-\frac{1}{\eta} < \Phi_R < \frac{1}{\eta}$.

Accordingly, we take

$$\max\left(-\frac{1}{(1-\rho^2)}, -\frac{1}{\eta}, -\frac{\kappa}{\varepsilon|\rho|}\right) < \Phi_R < \max\left(\frac{1}{(1-\rho^2)}, \frac{1}{\eta}, \frac{\kappa}{\varepsilon|\rho|}\right). \tag{3.3.22}$$

Next, we take $\Phi = i\Phi_I$ and compute the following limit in (3.3.16):

$$\lim_{\Phi_I \to \pm\infty} \left|\widehat{G}^x(t, T, V, X, i\Phi_I)\right| \sim e^{-\frac{(\kappa\theta\tau+V)\sqrt{(1-\rho^2)}}{\varepsilon}|\Phi_I|}. \tag{3.3.23}$$

Finally, given a precision level ϵ we determine the cut-off value $\overline{\Phi}_I$ as follows:

$$\int_{\overline{\Phi}_I}^{\infty} e^{-\frac{(\kappa\theta\tau+V)\sqrt{(1-\rho^2)}}{\varepsilon}\Phi_I} d\Phi_I < \epsilon. \tag{3.3.24}$$

and obtain:

$$\overline{\Phi}_I > -\frac{\varepsilon}{(\kappa\theta\tau+V)\sqrt{(1-\rho^2)}} \ln\left(\frac{(\kappa\theta\tau+V)\sqrt{(1-\rho^2)}}{\varepsilon}\epsilon\right). \tag{3.3.25}$$

Illustration

In Figure (3.3.2) we plot the implied Green function for four prototype dynamics driving the asset price process.

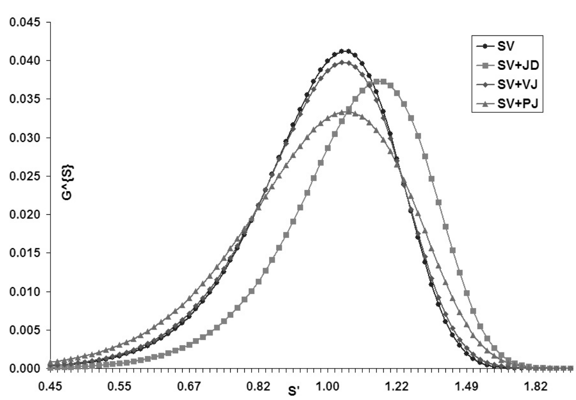

Figure 3.3.2: Green functions of the asset price process

We see that the Green function of the process with jump-to-default shifts the process to the right by augmenting the drift by the jump-to-default intensity rate and, at the same time, it implies a non-zero probability of attaining the post-default state. Adding variance jumps does not significantly affect the TPDF of the asset process, unlike the variance process itself, so that by model calibration to European options we may not consider jumps in variance at all. However adding price jumps leads to a heavy left tail of the TPDF, which help us explain the volatility skew effect. The TPDF of the asset process is negatively skewed.

3.3.3. Forward-Start Log-Spot Price

The Green function of the forward-start log-price process $G^{\bar{x}}(t, t_F, T, V, X, \bar{X}, \bar{X}')$ satisfies general equation (3.2.1) represented as follows:

$$\begin{aligned} & G^{\bar{x}}_t + \kappa(\theta - V)G^{\bar{x}}_V + \frac{1}{2}\varepsilon^2 V G^{\bar{x}}_{VV} + \left(r(t) - d(t) + \alpha + (\beta - \frac{1}{2})V \right) G^{\bar{x}}_X \\ & \quad + \frac{1}{2} V G^{\bar{x}}_{XX} + \rho\varepsilon V G^{\bar{x}}_{XV} \\ & \quad + \gamma \int_0^\infty \int_{-\infty}^\infty \left(G^{\bar{x}}(V + J^v, X + J^s) - G^{\bar{x}} - m^j G^{\bar{x}}_X \right) \varpi(J^v, J^s) dJ^s dJ^v \\ & \quad + \chi(t - t_F)(X(t) - X(t_F)) G^{\bar{x}}_{\bar{X}} - (\alpha + \beta V) G^{\bar{x}} = 0, \\ & G^{\bar{x}}(T, t_F, T, V, X, \bar{X}, \bar{X}') = \delta(\bar{X} - \bar{X}'). \end{aligned} \tag{3.3.26}$$

The corresponding Fourier transform, $\widehat{G}^{\bar{x}}(t, t_F, T, V, X, \bar{X}, \Phi) = \mathbf{F}_-[G^{\bar{x}}(\bar{X}')](\Phi)$, satisfies:

$$\begin{aligned} & \widehat{G}^{\bar{x}}_t + \kappa(\theta - V)\widehat{G}^{\bar{x}}_V + \frac{1}{2}\varepsilon^2 V \widehat{G}^{\bar{x}}_{VV} + \left(r(t) - d(t) + \alpha + (\beta - \frac{1}{2})V \right) \widehat{G}^{\bar{x}}_X \\ & \quad + \frac{1}{2} V \widehat{G}^{\bar{x}}_{XX} + \rho\varepsilon V \widehat{G}^{\bar{x}}_{XV} \\ & \quad + \gamma \left(\int_{-\infty}^\infty \left(\widehat{G}^{\bar{x}}(V + J^v, X + J^s) - \widehat{G}^{\bar{x}} \right) \varpi(J^v, J^s) dJ^s dJ^v - \widehat{G}^{\bar{x}} - m^j \widehat{G}^{\bar{x}}_X \right) \\ & \quad + \chi(t - t_F)(X(t) - X(t_F)) \widehat{G}^{\bar{x}}_{\bar{X}} - (\alpha + \beta V) \widehat{G}^{\bar{x}} = 0, \\ & \widehat{G}^{\bar{x}}(T, t_F, T, V, X, \bar{X}, \Phi) = e^{-\Phi \bar{X}}. \end{aligned} \tag{3.3.27}$$

Proposition 3.2. *The solution to PIDE (3.3.27) is given by:*

$$\widehat{G}^{\bar{x}}(t, t_F, T, V, X, \bar{X}, \Phi) = e^{-\Phi \bar{X} - \Phi \int_t^T \chi(t' - t_F)(X(t') - X(t_F)) dt' + A^{\bar{x}}(t,T) + B^{\bar{x}}(t,T)V + \Gamma^{\bar{x}}(t,T)}. \tag{3.3.28}$$

where

$$\begin{aligned} & A^{\bar{x}}(t, T) = -\Phi \int_{t_F}^T (r(t') - d(t')) dt' + A^x(t_F, T) + A^v(t, t_F, -B^x(t_F, T)), \\ & B^{\bar{x}}(t, T) = B^v(t, t_F, -B^x(t_F, T)), \\ & \Gamma^{\bar{x}}(t, T) = \Gamma^x(t_F, T) + \Gamma^v(t, t_F, -B^x(t_F, T)). \end{aligned} \tag{3.3.29}$$

Proof. We try solution of the form:

$$\widehat{G}^{\bar{x}}(t, t_F, T, V, X, \bar{X}, \Phi) = e^{-\Phi\bar{X} - \Phi\int_t^T \chi(t'-t_F)(X(t')-X(t_F))dt' + A^{\bar{x}}(t,T) + B^{\bar{x}}(t,T)V + \Gamma^{\bar{x}}(t,T)}, \tag{3.3.30}$$

where $\int_t^T \chi(t'-t_F)(X(t')-X(t_F))dt' = 0$ if $0 \le t \le t_F$ and $\int_t^T \chi(t'-t_F)(X(t')-X(t_F))dt' = X(T)$ otherwise with

$$\begin{cases} \widehat{G}_t^{\bar{x}} = \chi(t-t_F)(X(t)-X(t_F))\Phi\widehat{G}^{\bar{x}} + \dot{A}^{\bar{x}} + \dot{B}^{\bar{x}}V + \dot{\Gamma}^{\bar{x}}, \ \widehat{G}_{\bar{X}}^{\bar{x}} = -\Phi\widehat{G}^{\bar{x}}, \ 0 \le t < T, \\ \widehat{G}_X^{\bar{x}} = -\Phi\widehat{G}^{\bar{x}}, \widehat{G}_{XX}^{\bar{x}} = \Phi^2\widehat{G}^{\bar{x}}, \ t_F \le t \le T, \\ \widehat{G}_X^{\bar{x}} = \widehat{G}_{XX}^{\bar{x}} = 0, \ 0 \le t < t_F. \end{cases} \tag{3.3.31}$$

Accordingly, for $t_F \le t \le T$, $A^{\bar{x}}(t,T)$, $B^{\bar{x}}(t,T)$, $E^{\bar{x}}(t,T)$, $\Gamma^{\bar{x}}(t,T)$ solve:

$$\begin{cases} \dot{A}^{\bar{x}}(t) + \kappa\theta B^{\bar{x}}(t) - (1+\Phi)\alpha - (r(t)-d(t))\Phi = 0, \ B^{\bar{x}}(T,T) = 0, \ A^{\bar{x}}(T,T) = 0, \\ \dot{B}^{\bar{x}}(t) + \frac{1}{2}\varepsilon^2 B^{\bar{x}2}(t) - (\kappa + \rho\varepsilon\Phi)B^{\bar{x}}(t) - (1+\Phi)\beta + \frac{1}{2}(\Phi^2+\Phi) = 0, \ B^{\bar{x}}(T,T) = 0 \\ \dot{\Gamma}^{\bar{x}}(t) + \gamma\int_0^\infty\int_{-\infty}^\infty \left(e^{J^v B^{\bar{x}} - J^s\Phi} + m^j\Phi - 1\right)\varpi(J^v, J^s)dJ^s dJ^v, \ \Gamma^{\bar{x}}(T,T) = 0. \end{cases} \tag{3.3.32}$$

The solution to this equation is given by $A^x(t_F,T) - \Phi\int_{t_F}^T (r(t')-d(t'))dt'$, $B^x(t_F,T)$, and $\Gamma^x(t_F,T)$ as defined in formula (3.3.16).

Next, for $t_0 \le t \le t_F$, $A^{\bar{x}}(t,T)$, $B^{\bar{x}}(t,T)$, $\Gamma^{\bar{x}}(t,T)$ solve:

$$\begin{cases} \dot{A}^{\bar{x}}(t) + \kappa\theta B^{\bar{x}}(t) - \alpha = 0, \ A^{\bar{x}}(t,t_F) = A^{\bar{x}}(t_F,T), \\ \dot{B}^{\bar{x}}(t) + \frac{1}{2}\varepsilon^2 B^{\bar{x}2}(t) - \kappa B^{\bar{x}}(t) - \beta = 0, \ B^{\bar{x}}(t,t_F) = B^{\bar{x}}(t_F,T), \\ \dot{\Gamma}^{\bar{x}}(t) + \gamma\int_0^\infty \left(e^{B^{\bar{x}}(t,T)J^v} - 1\right)\varpi^v(J^v)dJ^v = 0, \ \Gamma^{\bar{x}}(t,t_F) = \Gamma^{\bar{x}}(t_F,T), \end{cases} \tag{3.3.33}$$

where $\varpi^v(J^v)$ is the marginal density of the variance jumps and where the corresponding terminal condition impose the continuity of the solution in time variable.

Solution for this ODE system is given by $A^v(t,T,-B^x(t_F,T))$, $B^v(t,T,-B^x(t_F,T))$, and $\Gamma^v(t,T,-B^x(t_F,T))$ as defined in formula (3.3.3).

Final statement follows from two observations:

1) $A^v(t,T,\Theta)$, $B^v(t,T,\Theta)$, and $\Gamma^v(t,T,\Theta)$ are solved with boundary condition $B^v(T,T,\Theta) = -\Theta$, so that now we can replace it by $\Theta \to -B^{\bar{x}}(t_F,T)$;

2) Since ODE-s for $A(t,T)$, and $\Gamma(t,T)$ are invariant to the shift $A(t,T) \to A(t,T) + \tilde{A}$, $\Gamma(t,T) \to \Gamma(t,T) + \tilde{\Gamma}$, we first find the solution to the problem with zero boundary conditions and then just add these conditions to the final solution. We also note that for $t_0 \le t \le t_F$, the dynamics is affected only by thumps in variance so that we apply only the marginal PDF of variance jumps $\varpi^v(J^v)$.

□

Illustration

We do not consider the limiting case of the corresponding Green function by noticing that it can be obtained by employing results (3.3.14) and (3.3.25). In Figure (3.3.3) we plot the implied Green function for four prototype dynamics driving the forward-start asset price process for the process starting at $t_F = 1$ with $T = 2$.

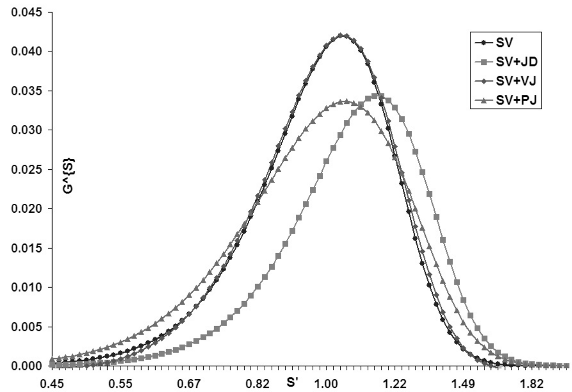

Figure 3.3.3: Green functions of the forward-start asset price process

We notice that the TPDF-s of the asset price and the forward-start asset price look almost the same. This observation is attributed to the fact that in our prototype models the variance process starts from its long-term mean, $V(0) = \theta$. In general, for time in-homogeneous processes, TPDF-s of the asset price and the forward-start asset price can differ remarkably. However, TPDF-s of the process with jump-to-default do differ because of the probability of defaulting before forward-start time t_F. Again, we see that TPDF-s of the process with variance jumps is only marginally different from the process with no variance jumps.

3.3.4. Asset Realized Variance

Using formula (3.2.4), the transformed Green function for the dynamics of $I(t)$, $\widehat{G}^I(t,T,V,I,\Psi) = \mathbf{F}_-[G^I(I')](\Psi)$, is obtained by:

$$\widehat{G}^I(t,T,V,I,\Psi) = \lim_{\Theta\to 0,\Phi\to 0} \widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.3.34}$$

Corollary 3.3.5. ***[Defaultable Dynamics]***
The solution is given by formula :

$$\widehat{G}^I(t,T,V,I,\Psi) = e^{-\Psi I + A^I(t,T) + B^I(t,T)V + \Gamma^I(t,T)}, \tag{3.3.35}$$

where for computing $A^I(t,T)$, $B^I(t,T)$, $\Gamma^I(t,T)$, we specify parameters in general formula (3.2.4) as follows: $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa$, $b_0 = -\beta - \Psi$, $a_1 = \kappa\theta$, $a_0 = -\alpha$.

Corollary 3.3.6. ***[Default-Free Dynamics]***
The solution is given by (3.3.35), where for computing $A^I(t,T)$, $B^I(t,T)$, $\Gamma^I(t,T)$, we specify parameters in general formula (3.2.4) as follows: $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa$, $b_0 = 0$, $a_1 = \kappa\theta$, $a_0 = 0$.

Proposition 3.3. *[First Moment]*

Under the default-free dynamics assuming independent normal price jumps with PDF (3.2.25), the first moment $M_1^I(t,T)$ is given by:

$$\overline{M}_1^I(t,T) = I + \left(\frac{\theta}{\kappa} + \frac{\eta\gamma}{\kappa^2}\right)\left((T-t)\kappa + e^{-\kappa(T-t)} - 1\right) + \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)V + (T-t)\gamma(\nu^2+\delta^2) \tag{3.3.36}$$

Proof. The result is obtained using theorem (1.5.1) along with formula (3.3.35). □

Limiting Behavior

Now, we check that

$$\Re[\kappa^2 + 2\varepsilon^2\Psi] > 0 \tag{3.3.37}$$

yielding $\Psi_R > -\frac{\kappa^2}{2\varepsilon^2}$.

First, to ensure the continuity of logarithm in $A^I(t,T)$, we consider the limit:

$$\lim_{\tau\to\infty}\left(\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right) = \frac{\psi_-}{2\zeta}. \tag{3.3.38}$$

which is true provided the above restriction is valid.

Then, we ensure that

$$\Re[B^I(t,T)] < \frac{1}{\eta}, \tag{3.3.39}$$

by taking limit $T \to \infty$ and obtaining $\Re[\frac{2\Psi}{\psi_-}] < \frac{1}{\eta}$, which hold if $-\frac{\kappa}{2\eta} < \Psi_R$.

Accordingly, we take

$$\Psi_R > \max\left(-\frac{\kappa}{2\eta}, -\frac{\kappa}{2\varepsilon^2}\right). \tag{3.3.40}$$

Next, we take $\Psi = i\Psi_I$ and compute the following limit in (3.3.35):

$$\lim_{\Psi_I\to\pm\infty}\left|\widehat{G}^I(t,T,V,I,i\Psi_I)\right| \sim e^{-\frac{V+\kappa\theta\tau}{\varepsilon}\sqrt{|\Psi_I|}}. \tag{3.3.41}$$

Finally, given a precision level ϵ we determine the cut-off value $\overline{\Psi}_I$ as follows:

$$\int_{\overline{\Psi}_I}^{\infty} e^{-\frac{V+\kappa\theta\tau}{\varepsilon}\sqrt{\Psi_I}}d\Psi_I < \epsilon. \tag{3.3.42}$$

and obtain:

$$\overline{\Psi}_I > \left(\frac{\varepsilon}{(\kappa\theta\tau + V)}\ln\left(\frac{(V+\kappa\theta\tau)^2}{2\varepsilon^2}\epsilon\right)\right)^2. \tag{3.3.43}$$

Illustration

In Figure (3.3.4) we plot the implied Green function of the realized variance process. We see a remarkable affect on the realized variance when the asset price process includes price jumps. The corresponding process of the realized variance has a very heavy right tail. Adding variance jumps affects the realized variance to much lesser degree, nevertheless, it results in a heavier right tail compared to the pure diffusion process. In general, for pure diffusion process the TPDF of the variance process has a slight positive skew and, when scaled by T, it is peaked at the long term mean level θ. Sepp (2007b) obtained an accurate approximation for the density of realized variance by means of matching its implied moments to the moments of a log-normal random variable.

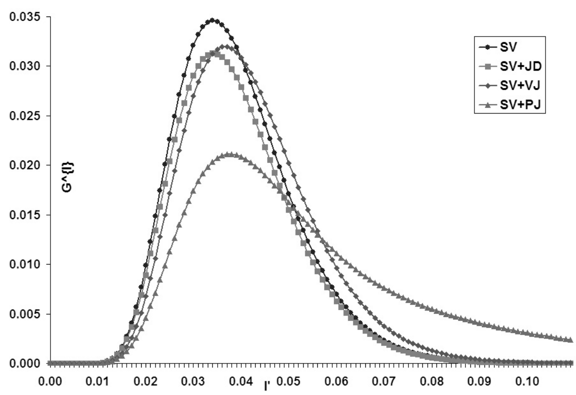

Figure 3.3.4: Green functions of the realized variance process

In Figure (3.3.5) we analyze how accurately the continuous time limit of quadratic variation of asset price process, defined by (1.3.4) and represented by process $I(t)$, approximates the actual discrete sum of daily squared asset returns defined by (1.1.2). For this purpose we apply Monte-Carlo simulation of affine model (0.0.1) using prototype parameters of SV+PJ process with $100,000$ paths to simulate the evolution of the asset price on daily intervals with the total number of days (within one path) equal to 252. Then for each path we calculate the discrete one-year realized variance using formula (1.1.2) with $N = AF = 252$. Finally, in Figure (3.3.5) we plot the histogram, scaled to sum up to one, of these realizations (denoted by SV+PJ (MC)), which represents implied "empirical" TPDF, along with Green function of theoretical TPDF of $I(t)$ with $T = 1$ (denoted by SV+PJ (Green)). From Figure (3.3.5) we see that our theoretical process for asset realized variance $I(t)$ adequately describes the evolution of the actual asset price realized variance computed by the sum (1.1.2).

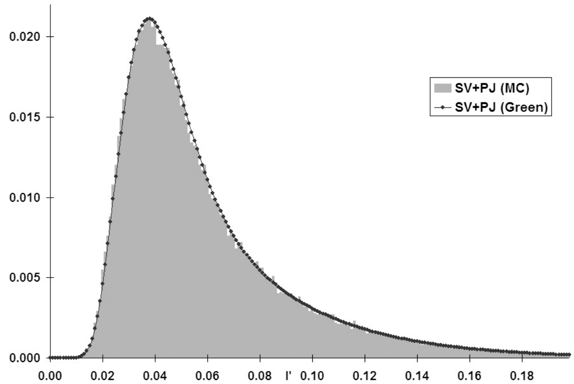

Figure 3.3.5: Histogram of simulated realized variance versus theoretical TPDF

3.3.5. Forward-Start Realized Variance

The Green function of the forward-start realized variance process $G^{\bar{I}}(t, t_F, T, V, I, \bar{I}, \bar{I}')$ satisfies general equation (3.2.1) represented as follows:

$$\begin{aligned}
&G_t^{\bar{I}} + \kappa(\theta - V)G_V^{\bar{I}} + \frac{1}{2}\varepsilon^2 V G_{VV}^{\bar{I}} + V G_I^{\bar{I}} \\
&\quad + \gamma \int_0^\infty \int_{-\infty}^\infty \left(G^{\bar{I}}(V + J^v, \bar{I} + J^{s2}) - G^{\bar{x}} \right) \varpi(J^v, J^s) dJ^s dJ^v \\
&\quad + \chi(t - t_F)(I(t) - I(t_F))G_{\bar{I}}^{\bar{I}} - (\alpha + \beta V)G^{\bar{I}} = 0, \\
&G^{\bar{I}}(T, t_F, T, V, I, \bar{I}, \bar{I}') = \delta(\bar{I} - \bar{I}').
\end{aligned} \tag{3.3.44}$$

The corresponding Fourier transform, $\widehat{G}^{\bar{I}}(t, t_F, T, V, I, \bar{I}, \Psi) = \mathbf{F}_{-}[G^{\bar{I}}(\bar{I}')](\Psi)$, satisfies:

$$\begin{aligned}
&\widehat{G}_t^{\bar{I}} + \kappa(\theta - V)\widehat{G}_V^{\bar{I}} + \frac{1}{2}\varepsilon^2 V \widehat{G}_{VV}^{\bar{I}} \\
&\quad + \gamma \int_0^\infty \int_{-\infty}^\infty \left(\widehat{G}^{\bar{I}}(V + J^v, \bar{I} + J^{s2}) - \widehat{G}^{\bar{x}} \right) \varpi(J^v, J^s) dJ^s dJ^v \\
&\quad + V\widehat{G}_I^{\bar{I}} + \chi(t - t_F)(I(t) - I(t_F))G_{\bar{I}}^{\bar{I}} - (\alpha + \beta V)\widehat{G}^{\bar{I}} = 0, \\
&\widehat{G}^{\bar{I}}(T, t_F, T, V, I, \bar{I}, \Psi) = e^{-\Psi \bar{I}}.
\end{aligned} \tag{3.3.45}$$

Proposition 3.4. *The solution is given by:*

$$\widehat{G}^{\bar{I}}(t, t_F, T, V, X, \bar{X}, \Psi) = e^{-\Psi \bar{X} - \Psi \int_t^T \chi(t' - t_F)(I(t') - I(t_F))dt' + A^{\bar{I}}(t,T) + B^{\bar{I}}(t,T)V + \Gamma^{\bar{I}}(t,T)}. \tag{3.3.46}$$

where

$$
\begin{aligned}
A^{\bar{I}}(t,T) &= A^{I}(t_F,T) + A^{v}(t,t_F,-B^{I}(t_F,T)),\\
B^{\bar{I}}(t,T) &= B^{v}(t,t_F,-B^{I}(t_F,T)),\\
\Gamma^{\bar{I}}(t,T) &= \Gamma^{I}(t_F,T) + \Gamma^{v}(t,t_F,-B^{I}(t_F,T)).
\end{aligned}
\tag{3.3.47}
$$

Proof. We assume the following form of the transformed Green function:

$$
\widehat{G}^{\bar{I}}(t,t_F,T,V,X,\bar{I},\Psi) = e^{-\Phi\bar{I}-\Psi\int_t^T \chi(t'-t_F)(I(t')-I(t_F))dt'+A^{\bar{I}}(t,T)+B^{\bar{I}}(t,T)V+\Gamma^{\bar{I}}(t,T)},
\tag{3.3.48}
$$

with

$$
\begin{cases}
\widehat{G}^{\bar{I}}_t = \Phi\chi(t-t_F)(I(t)-I(t_F))\widehat{G}^{\bar{I}} + A + BV + \Gamma,\ \widehat{G}^{\bar{I}}_{\bar{I}} = -\Psi\widehat{G}^{\bar{I}},\ 0 \le t < T,\\
\widehat{G}^{\bar{I}}_I = \widehat{G}^{\bar{I}}_{\bar{I}} = -\Psi\widehat{G}^{\bar{I}},\ t_F \le t \le T,\\
\widehat{G}^{\bar{x}}_X = \widehat{G}^{\bar{x}}_{XX} = 0,\ 0 \le t < t_F.
\end{cases}
\tag{3.3.49}
$$

Accordingly, for $t_F \le t \le T$, $A^{\bar{I}}(t,T)$, $B^{\bar{I}}(t,T)$, and $\Gamma^{\bar{I}}(t,T)$ solve:

$$
\begin{cases}
\dot{A}^{\bar{I}}(t) + \kappa(t)\theta(t)B^{\bar{I}}(t) - \alpha = 0,\ B^{\bar{I}}(T,T) = 0,\ A^{\bar{I}}(T,T) = 0,\\
\dot{B}^{\bar{I}}(t) + \frac{1}{2}\varepsilon^2 B^{\bar{I}2}(t) - \kappa B^{\bar{I}}(t) - (\Psi+\beta) = 0,\\
\dot{\Gamma}^{\bar{I}}(t) + \gamma\int_0^\infty\int_{-\infty}^\infty \left(e^{B^{\bar{I}}(t,T)J^v-\Psi J^{s2}} - 1\right)\varpi(J^v,J^s)dJ^v dJ^s = 0,\ E^{\bar{I}}(T,T) = 0
\end{cases}
\tag{3.3.50}
$$

The solution to this equation is given by $A^{I}(t_F,T)$, $B^{I}(t_F,T)$, $\Gamma^{I}(t_F,T)$ as defined in formula (3.3.35).

Next, for $t_0 \le t \le t_F$, $A^{\bar{I}}(t,T)$, $B^{\bar{I}}(t,T)$, $\Gamma^{\bar{I}}(t,T)$ solve:

$$
\begin{cases}
\dot{A}^{\bar{I}}(t) + \kappa\theta(t)B^{\bar{I}}(t) = 0,\ A^{\bar{I}}(t,t_F) = A^{\bar{I}}(t_F,T),\\
\dot{B}^{\bar{I}}(t) + \frac{1}{2}\varepsilon^2 B^{\bar{I}2}(t) - \kappa B^{\bar{I}}(t) = 0,\ B^{\bar{I}}(t,t_F) = B^{\bar{I}}(t_F,T),\\
\dot{\Gamma}^{\bar{I}}(t) + \gamma\int_0^\infty \left(e^{B^{\bar{I}}(t,T)J^v} - 1\right)\varpi^v(J^v)dJ^v = 0,\ \Gamma^{\bar{I}}(t,t_F) = \Gamma^{\bar{I}}(t_F,T),
\end{cases}
\tag{3.3.51}
$$

where the corresponding terminal condition impose the continuity of the solution.

Solution for this ODE system is given by $A^{v}(t,T,-B^{I}(t_F,T))$, $B^{v}(t,T,-B^{I}(t_F,T))$, and $\Gamma^{v}(t,T,-B^{I}(t_F,T))$ as defined in formula (3.3.3).

Final statement follows from the same two observations as stated in the proof of proposition (3.2).

□

Finally, we notice that the truncation limits can be obtained combining results (3.3.14) and (3.3.43) and the implied TPDF-s of the forward-start variance are not very different from the corresponding TPDF-s of the realized variance shown in Figure (3.3.4) (because the variance process starts from its mean level θ).

3.3.6. Price Denominated Green Function

In general, these Green functions arise by solving the pricing problem of forward-start options and their name follows from the terminal conditions.

Variance

The corresponding Green function $G^{vp}(t,T,V,S,V')$ solves:

$$\begin{aligned}
&G^{vp}_t + \kappa(\theta - V)G^{vp}_V + \frac{1}{2}\varepsilon^2 V G^{vp}_{VV} + (r(t) - d(t) + \alpha + \beta V)\, S G^{vp}_S \\
&\quad + \frac{1}{2} V S^2 G^{vp}_{SS} + \rho\varepsilon V S G^{vp}_{SV} - (\alpha + \beta V) G^{vp} \\
&\quad + \gamma \int_0^\infty \int_{-\infty}^\infty \left(G^{vp}(V + J^v, S e^{J^s}) - G^{vp} - m^j S G^{vp}_S\right) \varpi(J^v, J^s) dJ^s dJ^v \\
&G^{vp}(T,T,V,S,V') = S\delta(V - V').
\end{aligned} \tag{3.3.52}$$

We introduce $G^{vp*}(t,T,V,V') = CF(t,T)S^{-1}G^{vp}(t,T,V,S,V')$, which solves

$$\begin{aligned}
&G^{vp*}_t + \kappa^*(\theta^* - V)G^{vp*}_V + \frac{1}{2}\varepsilon^2 V G^{vp*}_{VV} + \gamma \int_0^\infty \left(G^{vp*}(V + J^v) - G^{vp*}\right) \varpi^v(J^v) dJ^v = 0, \\
&G^{vp*}(T,T,V,V') = \delta(V - V').
\end{aligned} \tag{3.3.53}$$

where $\kappa^* = \kappa - \rho\varepsilon$ and $\theta^* = \frac{\kappa\theta}{\kappa - \rho\varepsilon}$.

Corollary 3.3.7. ***[Transformed Greens Function]***
The solution to $\widehat{G}^{vp}(t,T,V,S,\Theta) = \mathbf{F}_-[G^{vp}(V')](\Theta)$ is given by formula :

$$\widehat{G}^{vp}(t,T,V,S,\Theta) = CF^{-1}(t,T) S e^{A^{vp}(t,T,\Theta) + B^{vp}(t,T,\Theta)V + \Gamma^{vp}(t,T,\Theta)}, \tag{3.3.54}$$

where $A^{vp}(t,T,\Theta)$, $B^{vp}(t,T,\Theta))$, and $\Gamma^{vp}(t,T,\Theta)$ are computed using general formula (3.2.4), where we specify parameters as follows: $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa^*$, $b_0 = 0$, $a_1 = \kappa^*\theta^*$, $a_0 = 0$, $\gamma = 0$.

Forward-Start Log-Price

The corresponding Green function $G^{\bar{x}p}(t, t_F, T, V, S, X, \bar{X}, \bar{X}')$ solves:

$$\begin{aligned}
&G^{\bar{x}p}_t + \kappa(\theta - V)G^{\bar{x}p}_V + \frac{1}{2}\varepsilon^2 V G^{\bar{x}p}_{VV} + (r(t) - d(t) + \alpha + \beta V)\, S G^{\bar{x}p}_S \\
&\quad + \frac{1}{2} V S^2 G^{\bar{x}p}_{SS} + \rho\varepsilon V S G^{\bar{x}p}_{SV} + \left(r(t) - d(t) + \alpha + \left(\beta - \frac{1}{2}\right) V\right) G^{\bar{x}p}_X \\
&\quad + \frac{1}{2} V G^{\bar{x}p}_{XX} + \rho\varepsilon V G^{\bar{x}p}_{XV} + \chi(t - t_F)(X(t) - X(t_F)) G^{\bar{x}}_{\bar{X}} - (\alpha + \beta V) G^{\bar{x}p} \\
&\quad + \gamma \int_0^\infty \int_{-\infty}^\infty \left(G^{\bar{x}p}(V + J^v, S e^{J^s}, X + J^s) - G^{\bar{x}p} - m^j S G^{\bar{x}p}_S - m^j G^{\bar{x}p}_X\right) \varpi^s(J^v, J^s) dJ^s dJ^v \\
&G^{\bar{x}p}(T, t_F, T, V, S, X, \bar{X}, \bar{X}') = S\delta(\bar{X} - \bar{X}').
\end{aligned}$$

We introduce $G^{\bar{x}p*}(t,t_F,T,V,X,\bar{X},\bar{X}') = CF(t,T)S^{-1}G^{\bar{x}p}(t,T,V,S,X,\bar{X},\bar{X}')$, which solves

$$
\begin{aligned}
&G_t^{\bar{x}p*} + \kappa^*(\theta^* - V)G_V^{\bar{x}p*} + \frac{1}{2}\varepsilon^2 V G_{VV}^{\bar{x}p*} + \chi(t-t_F)(X(t)-X(t_F))G_{\bar{X}}^{\bar{x}} \\
&\quad + \left(r(t) - d(t) + \alpha + \left(\beta - \frac{1}{2}\right)V\right)G_X^{\bar{x}p*} + \frac{1}{2}VG_{XX}^{\bar{x}p*} + \rho\varepsilon V G_{XV}^{\bar{x}p*} \\
&\quad + \gamma\int_0^\infty\int_{-\infty}^\infty \left(G^{\bar{x}p*}(V+J^v, X+J^s) - G^{\bar{x}p*}\right)\varpi(J^v,J^s)dJ^s dJ^v \\
&G^{\bar{x}p*}(T,t_F,T,V,\bar{X},\bar{X}') = \delta(\bar{X}-\bar{X}').
\end{aligned}
\tag{3.3.55}
$$

Proposition 3.5. *The solution to* $\widehat{G}^{\bar{x}p}(t,T,V,S,X,\bar{X},\Phi) = \mathbf{F}_-[G^{\bar{x}p}(\bar{X}')](\Phi)$ *is given by:*

$$
\begin{aligned}
&\widehat{G}^{\bar{x}p}(t,t_F,T,V,S,X,\bar{X},\Phi) = \\
&= CF^{-1}(t,T)Se^{-\Phi\bar{X}-\Phi\int_t^T\chi(t'-t_F)(X(t')-X(t_F))dt' + A^{\bar{x}p}(t,T)+B^{\bar{x}p}(t,T)V+\Gamma^{\bar{x}p}(t,T)},
\end{aligned}
\tag{3.3.56}
$$

with

$$
\begin{aligned}
&A^{\bar{x}p}(t,T) = -\Phi\int_{t_F}^T (r(t')-d(t'))dt' + A^{xp}(t_F,T) + A^{vp}(t,t_F,-B^{xp}(t_F,T)), \\
&B^{\bar{x}p}(t,T) = A^{xp}(t_F,T) + A^{vp}(t,t_F,-B^{xp}(t_F,T)), \\
&\Gamma^{\bar{x}p}(t,T) = \Gamma^{xp}(t_F,T) + \Gamma^{vp}(t,t_F,-B^{xp}(t_F,T)),
\end{aligned}
\tag{3.3.57}
$$

where $A^{vp}(t,T)$, $B^{vp}(t,T)$, $\Gamma^{vp}(t,T)$ *are computed as defined in formula (3.3.54); and* $A^{xp}(t,T))$, $B^{xp}(t,T))$, $\Gamma^{xp}(t,T)$, *are computed using general formula (3.2.4), where we specify parameters as follows:* $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa^* - \rho\varepsilon\Phi$, $b_0 = \frac{1}{2}\Phi^2 - (-\frac{1}{2}+\beta)\Phi$, $a_1 = \kappa^*\theta^*$, $a_0 = -\Phi\alpha$, $\Theta = 0$.

Proof. The proof is similar to the proof of proposition (3.2). □

3.4. Two-Dimensional Green Functions

Now we apply results of the theorem (3.2.1) to study the joint TPDF-s of the underlying variables, which are important for studying certain two-dimensional pricing problems. For example, Sepp (2007) uses the Green function of the joint evolution of the log-asset price, $X(t)$, and its variance, $V(t)$, to obtain pricing formulas for conditional variance swaps, which derive their values from both the asset price and its variance.

In particular, we state the following results.

Corollary 3.4.1. ***[Two-Factor Green Function]***

The transformed Green function for the dynamics of $X(t)$ and $V(t)$:

$$\widehat{G}^{xv}(t,T,V,\Theta,X,\Phi) = \mathbf{F}^2_{-}[G^{xv}(V',X')](\Theta,\Phi) \tag{3.4.1}$$

is obtained by taking limits in general formula (3.2.4):

$$\widehat{G}^{xv}(t,T,V,\Theta,X,\Phi) = \lim_{\Psi\to 0}\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.4.2}$$

The transformed Green function for the dynamics of $X(t)$ and $I(t)$:

$$\widehat{G}^{xI}(t,T,V,X,\Phi,I,\Psi) = \mathbf{F}^2_{-}[G^{xI}(X',I')](\Phi,\Psi), \tag{3.4.3}$$

is obtained by:

$$\widehat{G}^{xI}(t,T,V,X,\Phi,I,\Psi) = \lim_{\Theta\to 0}\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.4.4}$$

Finally, the transformed Green function for the dynamics of $V(t)$ and $I(t)$:

$$\widehat{G}^{vI}(t,T,V,\Theta,X,I,\Psi) = \mathbf{F}^2_{-}[G^{vI}(V',I')](\Theta,\Psi) \tag{3.4.5}$$

is obtained by:

$$\widehat{G}^{vI}(t,T,V,\Theta,X,I,\Psi) = \lim_{\Phi\to 0}\widehat{G}(t,T,V,\Theta,X,\Phi,I,\Psi). \tag{3.4.6}$$

Given the solution for the transformed Green function, we compute the original by numerically inverting the two-dimensional inverse Fourier transform.

Illustration

In Figure (3.4.1) we plot the implied two-dimensional Green function of joint evolution of the asset price, $S(t)$, and its variance, $V(t)$, using protopype parameters of the process with the stochastic volatility. We see that the negative correlation between the asset price and its volatility implies that a high (low) value of the variance will make low (high) values of the asset price to be more likely.

3.5. Survival Probability

Definition 3.5.1 (Survival Probability)**.** *Survival probability represents the probability of survival at time T conditioned on survival at time t, $0 \le t \le T < \infty$. Formally:*

$$Q(t,T,V) = \mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{N^d(T)-N^d(t)=0\}}|\mathcal{G}(t)\right]. \tag{3.5.1}$$

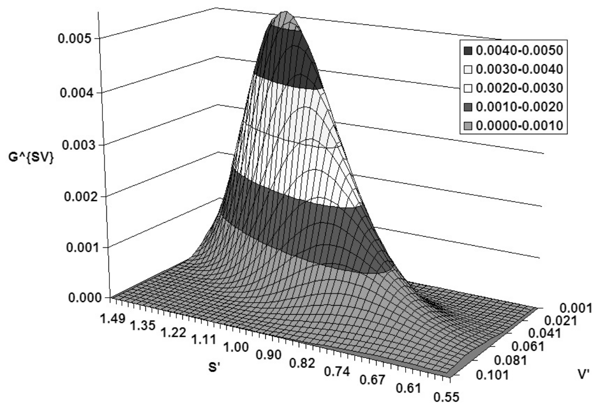

Figure 3.4.1: Green function of the joint evolution of the asset price and its variance

Assumption 3.1. *At this point we will make an assumption that the default event has not happened up to evaluation time t, so that indicator $\mathbf{1}_{\{\iota>t\}}$ is true, and subsequently in this section we will be using:*

$$\mathbf{1}_{\{\iota>t\}}Q(t,T,V) \to Q(t,T,V), \tag{3.5.2}$$

for all quantities introduced in this section.

Applying Feynman-Kac theorem (1.4.3) for affine model (0.0.1), we obtain that survival probability $Q(t,T,V)$ satisfies the following equation:

$$\begin{aligned} &Q_t + \kappa(\theta - V)Q_V + \frac{1}{2}\varepsilon^2(t)VQ_{VV} - (\alpha + \beta V)Q \\ &\quad + \gamma\int_0^\infty \left(Q(V+J^v) - Q\right)\varpi^v(J^v)dJ^v = 0, \\ &Q(T,T,V) = 1. \end{aligned} \tag{3.5.3}$$

Proposition 3.6. ***[Survival Probability]***
The solution to (3.5.3) is given by:

$$Q(t,T,V) = e^{A^Q(t,T)+B^Q(t,T)V+\Gamma^Q(t,T)} \tag{3.5.4}$$

where for computing $A^Q(t,T)$, $B^Q(t,T)$, and $\Gamma^Q(t,T)$ by means of formula (3.2.7) we specify parameters as follows: $b_2 = \frac{1}{2}\varepsilon^2$, $b_1 = -\kappa$, $b_0 = 0$, $a_1 = \kappa\theta$, $a_0 = -\alpha$, $\Phi = 0$, $\Theta = 0$, $\Psi = -\beta$, $I = 0$.

As a result, we obtain the following explicit formula:

$$\begin{aligned}
&A^Q(t,T) = -\frac{\kappa\theta}{\varepsilon^2}\left[\psi_+\tau + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right)\right] - \alpha\tau,\\
&B^Q(t,T) = -2\beta\frac{1-e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}},\\
&\Gamma^Q(t,T) = -\gamma\frac{4b_2\eta}{\overline{\psi}_+\overline{\psi}_-}\left(\ln\left(\frac{\overline{\psi}_- C_+ e^{-\zeta\tau} + \overline{\psi}_+ C_-}{\overline{\psi}_- C_+ + \overline{\psi}_+ C_-}\right) - \frac{1}{2}\tau\overline{\psi}_-\right) - \gamma\tau,\\
&\psi_\pm = \mp\kappa + \zeta,\ \zeta = \sqrt{\kappa^2 + 2\varepsilon^2\beta},\ \overline{\psi}_\pm = \varepsilon^2 \pm \psi_\pm.
\end{aligned} \tag{3.5.5}$$

Definition 3.5.2. ***[Forward-Start Survival Probability]***

We call the expectation at time t of survival probability during the period $[t_F, T]$ conditioned on survival during time $[t, t_F]$, $0 \le t \le t_F \le T$, as forward-start survival probability and denote it by $Q^{fs}(t, t_F, T, V)$. Formally:

$$Q^{fs}(t,t_F,T,V) = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{N^d(T)-N^d(t_F)=0|N^d(t_F)-N^d(t)=0\}}|\mathcal{G}(t)\right], \tag{3.5.6}$$

or, in other words:

$$Q^{fs}(t,t_F,T,V) = \mathbb{E}^{V(t_F)}\left[Q(t_F,T,V)|\,\mathcal{G}(t)\right] = \int_0^\infty Q(t_F,T,V')G^v(t,t_F,V,V')dV' \tag{3.5.7}$$

where $G^v(t, t_F, V, V')$ is the Green function of defaultable variance process.

Proposition 3.7. ***[Forward-Start Survival Probability]***

Forward-start survival probability $Q^{fs}(t, t_F, T, V)$ can be computed by:

$$Q^{fs}(t,t_F,T,V) = e^{A^{Q^{fs}}(t,t_F,T)+B^{Q^{fs}}(t,t_F,T)V+\Gamma^{Q^{fs}}(t,t_F,T)}, \tag{3.5.8}$$

where

$$\begin{aligned}
&A^{Q^{fs}}(t,t_F,T) = A^Q(t_F,T) + A^v(t,t_F,-B^Q(t_F,T)),\\
&B^{Q^{fs}}(t,t_F,T) = B^v(t,t_F,-B^Q(t_F,T)),\\
&\Gamma^{Q^{fs}}(t,t_F,T) = \Gamma^Q(t_F,T) + \Gamma^v(t,t_F,-B^Q(t_F,T)).
\end{aligned} \tag{3.5.9}$$

Proof.

$$\begin{aligned}
&Q^{fs}(t,t_F,T,V) = \int_0^\infty Q(t_F,T,V')G^v(t,t_F,V,V')dV'\\
&= \int_0^\infty e^{A^Q(t_F,T)+B^Q(t_F,T)V'+\Gamma^Q(t_F,T)}G^v(t,t_F,V,V')dV'\\
&= e^{A^Q(t_F,T)+\Gamma^Q(t_F,T)}\int_0^\infty e^{B^Q(t_F,T)V'}G^v(t,t_F,V,V')dV'\\
&= e^{A^Q(t_F,T)+\Gamma^Q(t_F,T)}\widehat{G}^v(t,t_F,V,-B^Q(t_F,T))\\
&= e^{A^Q(t_F,T)+\Gamma^Q(t_F,T)+A^v(t,t_F,-B^Q(t_F,T))+B^v(t,t_F,-B^Q(t_F,T))V+\Gamma^v(t,t_F,-B^Q(t_F,T))},
\end{aligned} \tag{3.5.10}$$

where we use solution for $\widehat{G}^v(t, t_F, V, \Phi)$ given in formula (3.3.3). $\square$

Definition 3.5.3. ***[Forward-Start Default Probability]***
We call the expectation at time t of default probability during the period $(t_F, T]$ conditioned on survival during time $(t, t_F]$, $0 \leq t \leq t_F \leq T$, as forward-start default probability and denote it by $Q^{cfs}(t, t_F, T, V)$. Formally:

$$Q^{cfs}(t, t_F, T, V) = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{N^d(T)-N^d(t_F)>0|N^d(t_F)-N^d(t)=0\}}|\mathcal{G}(t)\right]. \tag{3.5.11}$$

We compute this quantity using proposition (3.7) and formula (3.5.8),

$$Q^{cfs}(t, t_F, T, V) = 1 - Q^{fs}(t, t_F, T, V) = 1 - e^{A^{Q^{fs}}(t,t_F,T)+B^{Q^{fs}}(t,t_F,T)V+\Gamma^{Q^{fs}}(t,t_F,T)}. \tag{3.5.12}$$

Definition 3.5.4. ***[Price Denominated Forward-Start Survival Probability]***
We call the expectation at time t of the spot price at time t_F conditioned on survival during time $(t, t_F]$ and conditioned on survival during the period $(t_F, T]$, $0 \leq t \leq t_F \leq T$, as price denominated forward-start survival probability and denote it by $Q^{fsp}(t, t_F, T, V)$. Formally:

$$Q^{fsp}(t, t_F, T, V) = \mathbb{E}^{\mathbb{Q}}\left[S(t_F)\mathbf{1}_{\{N^d(T)-N^d(t_F)=0|N^d(t_F)-N^d(t)=0\}}|\mathcal{G}(t)\right]. \tag{3.5.13}$$

Proposition 3.8. ***[Price Denominated Forward-Start Survival Probability]***
Price denominated forward-start survival probability $Q^{fsp}(t, t_F, T, V)$ can be computed by:

$$Q^{fsp}(t, t_F, T, V) = CF^{-1}(t, t_F)Se^{A^{Q^{fsp}}(t,t_F,T)+B^{Q^{fsp}}(t,t_F,T)V+\Gamma^{Q^{fsp}}(t,t_F,T)}, \tag{3.5.14}$$

where

$$\begin{aligned}
A^{Q^{fsp}}(t, t_F, T) &= A^Q(t_F, T) + A^{vp}(t, t_F, -B^Q(t_F, T)), \\
B^{Q^{fsp}}(t, t_F, T) &= B^{vp}(t, t_F, -B^Q(t_F, T)), \\
\Gamma^{Q^{fsp}}(t, t_F, T) &= \Gamma^Q(t_F, T) + \Gamma^{vp}(t, t_F, -B^Q(t_F, T)).
\end{aligned} \tag{3.5.15}$$

Proof. Using definition and price denominated Green function (3.3.54), we obtain;

$$\begin{aligned}
Q^{fsp}(t, t_F, T, V) &= \int_0^\infty Q(t_F, T, V')G^{vp}(t, t_F, V, S, V')dV' \\
&= \int_0^\infty CF^{-1}(t, t_F)Se^{A^Q(t_F,T)+B^Q(t_F,T)V'+\Gamma^Q(t_F,T)}G^{vp}(t, t_F, V, S, V')dV' \\
&= CF^{-1}(t, t_F)Se^{A^Q(t_F,T)+\Gamma^Q(t_F,T)}\int_0^\infty e^{B^Q(t_F,T)V'}G^{vp}(t, t_F, V, S, V')dV' \\
&= CF^{-1}(t, t_F)Se^{A^Q(t_F,T)+\Gamma^Q(t_F,T)}\widehat{G}^{vp}(t, t_F, V, S, -B^Q(t_F, T)) \\
&= CF^{-1}(t, t_F)Se^{A^Q(t_F,T)+\Gamma^Q(t_F,T)+A^{vp}(t,t_F,-B^Q(t_F,T))+B^{vp}(t,t_F,-B^Q(t_F,T))V+\Gamma^{vp}(t,t_F,-B^Q(t_F,T))},
\end{aligned} \tag{3.5.16}$$

where we use solution for $\widehat{G}^{vp}(t, t_F, V, S, \Phi)$ given in formula (3.3.54). □

Definition 3.5.5. ***[Price Denominated Forward-Start Default Probability]*** *We call the expectation at time t of the spot price at time t_F conditioned on the survival during time $(t, t_F]$ and the default event during the period $(t_F, T]$, $0 \le t \le t_F \le T$, as price denominated forward-start default probability and denote it by $Q^{cfsp}(t, t_F, T, V)$. Formally:*

$$Q^{cfsp}(t, t_F, T, V) = \mathbb{E}^{\mathbb{Q}}\left[S(t_F)\mathbf{1}_{\{N^d(T)-N^d(t_F)>0|N^d(t_F)-N^d(t_0)=0\}}|\mathcal{G}(t)\right]. \quad (3.5.17)$$

We compute this quantity by using formula (3.5.15) as follows:

$$\begin{aligned} Q^{cfsp}(t, t_F, T, V) &= \mathbb{E}^{V(t_F)}\left[S(t_F)\, Q^c(t_F, T, V)|\,\mathcal{G}(t)\right] \\ &= CF(t, t_F)S - Q^{fsp}(t, t_F, T, V). \end{aligned} \quad (3.5.18)$$

Illustration

In Figure (3.5.1) we plot the term structure of implied default probability as a function of T which is implied by SV+JD and SV+VJ+JD processes with our prototype parameters, where we take different values of default intensity coefficients α and β in such a way that the initial default intensity is $\lambda(0) = 0.1$ for each model.

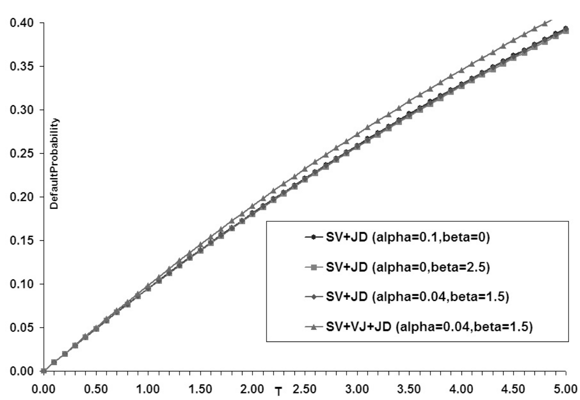

Figure 3.5.1: The term structure of the default probability.

We see that our specific choices of linear parameters α and β result in almost identical probability of the default, only the process with variance jumps has a little bit higher default probability due to an additional correction induced by the jump part. This implies that for practical purposes we can take $\beta = 0$, so that the default probability is not affected by the variance dynamics. This situation is similar to using the stochastic interest rate for the most of equity products, where the stochasticity of the interest rate has a little impact on the price, while its term structure, that is its drift, does affect the price through discount factors.

Chapter 4

Pricing Applications

In this Chapter we will use the formalism developed in previous two chapters to analyze and obtain closed-form solutions for pricing equity and volatility derivatives along with their forward-start versions. Our general strategy is as follows:

1) We assume that the transformed Green function (3.2.2) corresponding to the problem at hand is found by the procedure developed in Chapter 3.

2) To value a derivative security, we invoke theorem (2.3.1) from Chapter 2 and, in particular, formula (2.3.7), for which we compute the transformed payoff (2.3.5) or (2.3.6) of the given derivative security making sure that $e^{\Phi X'}u_1(X')$ and $e^{\Phi X'}u_2(t', X')$, $t \leq t' \leq T$, are in $L^1(\mathbb{C})$, which may require imposing some conditions on Φ_R.

3) We invert integral (2.3.8) or (2.3.9) numerically by means of FFT methods (the application of FFT for this type of problems is considered by Carr-Madan (1999)) or Gaussian quadratures truncating the infinite integral using asymptotic bounds obtained in Chapter 3 by analyzing the limiting behavior of corresponding Green functions.

Finally, we will illustrate some of model specifications by calibrating their parameters to the implied volatility surface of General Motors equity options and analyzing implied distributions of the underlying variables.

The result of pricing equity options with no default risk appears in Kangro-Pärna-Sepp (2004). The application of this methodology for pricing options on realized variance and forward-start realized variance is given in Sepp (2007b). Applications for pricing VIX futures and options on VIX appear in Sepp (2007c). The unified treatment of forward-start options in the presence of default risk is original.

4.1. Equity Options

4.1.1. Call and Put Options

The payoffs of a call option, $\hat{W}^c(t,T,V,S,K)$, and a put option, $\hat{W}^p(t,T,V,S,K)$, can be represented as follows:

$$\begin{aligned}\hat{W}^c(T,T,V,S,K) &= \mathbf{1}_{\{\iota>t_0\}}\max(S-K,0)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}}\\ &= \mathbf{1}_{\{\iota>t_0\}}\left(S-\min(S,K)\right)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}},\\ \hat{W}^p(T,T,V,S,K) &= \mathbf{1}_{\{\iota>t_0\}}\Big(\max(K-S,0)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}}+K\mathbf{1}_{\{N^d(T)-N^d(t_0)>0\}}\Big)\\ &= \mathbf{1}_{\{\iota>t_0\}}\Big(\left(K-\min(S,K)\right)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}}+K\mathbf{1}_{\{N^d(T)-N^d(t_0)>0\}}\Big)\\ &= \mathbf{1}_{\{\iota>t_0\}}K-\min(S,K)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}},\end{aligned} \tag{4.1.1}$$

where event $\{N^d(T)-N^d(t_0)=0\}$ means survival from the contract inception at time t_0 up to its maturity time T, and $\mathbf{1}_{\{\iota>t_0\}}$ ($\mathbf{1}_{\{\iota>t\}}$) means that the asset has not defaulted before the inception (valuation) time t_0 (t).

Accordingly, the value functions of these options at time t can be represented using the value function of covered call, $W^{cc}(t,T,V,S,K)$, as follows:

$$\begin{aligned}\hat{W}^c(t,T,V,S,K) &= \mathbf{1}_{\{\iota>t\}}\left(e^{-\int_t^T d(t')dt'}S-\hat{W}^{cc}(t,T,V,S,K)\right),\\ \hat{W}^p(t,T,V,S,K) &= \mathbf{1}_{\{\iota>t\}}\left(e^{-\int_t^T r(t')dt'}K-\hat{W}^{cc}(t,T,V,S,K)\right),\end{aligned} \tag{4.1.2}$$

where $\hat{W}^{cc}(t,T,V,S,K)$ has the following payoff function:

$$\hat{\mathcal{W}}(S,K)=\min(S,K)\mathbf{1}_{\{N^d(T)-N^d(t)=0\}}, \tag{4.1.3}$$

and from switching from (4.1.1) to (4.1.2) we employed the fact that asset price process $S(t)$ is a martingale under the measure $\mathbb{Q}$.

From formula (4.1.2) it is clear that the put-call parity is still valid under the defaultable dynamics.

We introduce logarithmic spot price normalized by the strike: $X=\ln\frac{S}{K}$, and new function $\hat{W}^{cc}(t,T,V,S,K)\rightarrow W^{cc}(t,T,V,X,K)$ supplied with the following payoff function $\mathcal{W}^{cc}(X,K)=\min(e^X,1)$.

Using the first fundamental pricing theorem (1.2.1) along with Duhamel's formula (1.4.9), we compute the value of $W^{cc}(t,T,V,X,K)$ as follows:

$$W^{cc}(t,T,V,X,K)=DF(t,T)\int_{-\infty}^{\infty}\mathcal{W}^{cc}(X',K)G^x(t,T,V,X,X')dX', \tag{4.1.4}$$

where $G^x(t,T,V,X,X')$ is Green function of the normalized defaultable price process $X(t)$ and $DF(t,T)$ is discount factor for a risk-free cash flow at time T.

To compute the above expectation we apply theorem (2.3.1) and, in particular, formula (2.3.8) along with transformed Green function of log-asset price (3.3.16) to obtain:

$$W^{cc}(t,T,V,X,K) = DF(t,T)\frac{1}{\pi}\int_0^{\infty} \Re\left[\widehat{G}^x(t,T,V,X,\Phi)\widehat{u}(\Phi,K)\right] d\Phi_I, \tag{4.1.5}$$

where the transformed payoff function $\widehat{u}(\Phi,K)$ is given by:

$$\widehat{u}(\Phi,K) = \int_{-\infty}^{\infty} e^{\Phi X'}\mathcal{W}^{cc}(X',K)dX' = \int_{-\infty}^{\infty} e^{\Phi X'}\min(e^{X'},1)dX' = -\frac{1}{\Phi(\Phi+1)}, \tag{4.1.6}$$

provided $-1 < \Phi_R < 0$. An appropriate choice of Φ_R for numerical evaluation of formula (4.1.5) is $\Phi_R = -\frac{1}{2}$.

Once the value of $W^{cc}(t,T,V,X,K)$ is obtained, we compute the values of call and put options using (4.1.2).

4.2. Claims on Realized Variance

To generalize, we denote the value function of a claim on annualized realized variance by $\hat{U}(t,T,V,S,\hat{I},\hat{K},o)$, where $o = 1$ for a claim on realized variance and $o = 1/2$ for a claim on realized volatility, and its payoff function by $\hat{\mathcal{U}}(\hat{I}^o,\hat{K}^o)$, where $\hat{K}^1$ is delivery price measured in variance points and $\hat{K}^{1/2}$ is delivery price measured in volatility points.

We consider some of the most common derivative claims on realized variance with the following payoffs at time T:

1) Swaps on realized variance:

$$\hat{U}^{swp}(T,T,V,S,\hat{I},\hat{K},o) = \hat{I}^o - \hat{K}^o. \tag{4.2.1}$$

2) Swaps on realized variance with cap $\hat{C}^o$:

$$\hat{U}^{swc}(T,T,V,S,\hat{I},\hat{K},o) = \min\{\hat{I}^o,\hat{C}^o\} - \hat{K}^o. \tag{4.2.2}$$

4) Call options on variance swaps:

$$\hat{U}^{swcl}(T,T,V,S,\hat{I},\hat{K},o) = \max\{\hat{I}^o - \hat{K}^o, 0\}. \tag{4.2.3}$$

5) Put options on variance swaps:

$$\hat{U}^{swpt}(T,T,V,S,\hat{I},\hat{K},o) = \max\{\hat{K}^o - \hat{I}^o, 0\}. \tag{4.2.4}$$

6) Covered all options on variance swaps with cap $\hat{C}^o$ and strike $\hat{K}^o$, $\hat{K}^o < \hat{C}^o$:

$$\hat{U}^{swcc}(T,T,V,S,\hat{I},\hat{K},o) = \max\{\min\{\hat{I}^o,\hat{C}^o\} - \hat{K}^o, 0\}. \tag{4.2.5}$$

These payoffs are further multiplied by the notional amount of the contract measured in currency units per variance (volatility) point for options on the realized variance (volatility). For brevity, we take the notional amount to be one unit.

In case the price process $S(t)$ is subject to the default risk and it is driven by a defaultable dynamics, if the default event occurs during the contract inception and maturity, we assume that the realized variance attains its pre-specified value determined in contract terms and the corresponding payment is made at the contract maturity time. We denote this post-default value of the realized variance by $\hat{\mathcal{I}}$.

We denote the value function of a general claim on realized volatility by $\hat{U}(t,T,V,S,\hat{I},\hat{K},o)$ and its payoff function by $\hat{\mathcal{U}}(\hat{I},\hat{K})$. At maturity time T this claim yields the following payoff:

$$\hat{U}(T,T,V,S,I,K,o) = \mathbf{1}_{\{\iota>t_0\}}\Big(\hat{\mathcal{U}}(\hat{I}^o,\hat{K}^o)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}} + \hat{\mathcal{U}}(\hat{\mathcal{I}}^o,\hat{K}^o)\mathbf{1}_{\{N^d(T)-N^d(t_0)>0\}}\Big). \tag{4.2.6}$$

Applying the results of theorem (1.2.1), we obtain that the option value function is given by:

$$\begin{aligned}\hat{U}(t,T,S,V,I,K,o) &= \mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t')dt'}\hat{\mathcal{U}}(\hat{I}^o,\hat{K}^o)\mathbf{1}_{\{N^d(T)-N^d(t)=0\}}\Big|\,\mathcal{G}(t)\right]\\ &+ \mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t')dt'}\hat{\mathcal{U}}(\hat{\mathcal{I}}^o,\hat{K}^o)\mathbf{1}_{\{N^d(T)-N^d(t)>0\}}\Big|\,\mathcal{G}(t)\right]\\ &= DF(t,T)\left(\mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(t')dt'}\hat{\mathcal{U}}(\hat{I}^o,\hat{K}^o)\Big|\,\mathcal{G}(t)\right] + Q^c(t,T,V)\hat{\mathcal{U}}(\hat{\mathcal{I}}^o,\hat{K}^o)\right),\end{aligned} \tag{4.2.7}$$

where $Q^c(t,T,V) = 1 - Q(t,T,V)$ is default probability during time interval $(t,T]$, and survival probability $Q(t,T,V)$ is specified by (3.5.4).

To simplify the problem, we denote by $U(t,T,V,S,I,K,o)$ the undiscounted value of the contingent claim on realized de-annualized variance $(I(t))$ for $o=1$ or volatility $(\sqrt{I(t)})$ for $o=1/2$, and if applicable with de-annualized delivery price, $K^o = (T-t_0)^o\hat{K}^o$ along with de-annualized cap level $C^o = (T-t_0)^o\hat{C}^o$, post-default value $\mathcal{I}^o = (T-t_0)^o\hat{\mathcal{I}}^o$ and final payoff $\mathcal{U}(I^o,K^o)$.

Using pricing formula (4.2.7) along with Duhamel's formula (1.4.9), we compute the option value as follows:

$$U(t,T,V,S,I,K,o) = \int_0^\infty \mathcal{U}(I'^o,K^o)G^I(t,T,V,I,I')dI' + \hat{\mathcal{U}}(\hat{\mathcal{I}}^o,\hat{K})Q^c(t,T,V), \tag{4.2.8}$$

where $G^I(t,T,V,I,I')$ is Green function of the defaultable realized variance $I(t)$.

To compute the above expectation we apply formula (2.3.8) along with transformed Green function of realized variance (3.3.35) to obtain:

$$U(t,T,V,S,I,K,o) = \frac{1}{\pi}\int_0^\infty \Re\left[\widehat{G}^I(t,T,V,I,\Psi)\widehat{\mathcal{U}}(\Psi,K)\right]d\Psi_I, \tag{4.2.9}$$

where the transformed payoff function $\widehat{u}(\Psi, K)$ is given by:

$$\widehat{\mathcal{U}}(\Psi, K) = \int_0^\infty e^{\Psi I'}\mathcal{U}(I', K)dI'. \tag{4.2.10}$$

Calculating the transform (4.2.10) for option payoffs on realized variance with $o = 1$ yields

$$\begin{aligned}
\widehat{\mathcal{U}}^{swp}(\Psi, K) &= \frac{1}{\Psi^2} + \frac{K}{\Psi},\\
\widehat{\mathcal{U}}^{swc}(\Psi, K) &= \frac{1}{\Psi^2}\left(1 - e^{C\Psi}\right) + \frac{K}{\Psi},\\
\widehat{\mathcal{U}}^{swcl}(\Psi, K) &= \frac{1}{\Psi^2}e^{K\Psi},\\
\widehat{\mathcal{U}}^{swpt}(\Psi, K) &= \frac{1}{\Psi^2}\left(e^{K\Psi} - 1\right) - \frac{K}{\Psi},\\
\widehat{\mathcal{U}}^{swcc}(\Psi, K) &= \frac{1}{\Psi^2}\left(e^{K\Psi} - e^{C\Psi}\right) - \frac{1}{\Psi}(C - K)e^{C\Psi},
\end{aligned} \tag{4.2.11}$$

provided $\Psi_R < 0$.

Similarly, for option payoffs on realized volatility with $o = \frac{1}{2}$ we obtain:

$$\begin{aligned}
\widehat{\mathcal{U}}^{swp}(\Psi, K) &= \int_0^\infty e^{\Psi I'}[\sqrt{I'} - K]dI' = \frac{\sqrt{\pi}}{2\,(-\Psi)^{3/2}} + \frac{K}{\Psi},\\
\widehat{\mathcal{U}}^{swcl}(\Psi, K) &= \int_0^\infty e^{\Psi I'}[\max\{\sqrt{I'} - K, 0\}]dI' = \frac{\sqrt{\pi}\left(1 - \mathrm{erf}(K\sqrt{-\Psi})\right)}{2\,(-\Psi)^{3/2}},\\
\widehat{\mathcal{U}}^{swc}(\Psi, K) &= \frac{\sqrt{\pi}\mathrm{erf}(C\sqrt{-\Psi})}{2\,(-\Psi)^{3/2}} + \frac{K}{\Psi},\\
\widehat{\mathcal{U}}^{swpt}(\Psi, K) &= \frac{-\sqrt{\pi}\mathrm{erf}(K\sqrt{-\Psi})}{2\,(-\Psi)^{3/2}} - \frac{K}{\Psi},\\
\widehat{\mathcal{U}}^{swcc}(\Psi, K) &= \frac{\sqrt{\pi}\left(\mathrm{erf}(C\sqrt{-\Psi}) - \mathrm{erf}(K\sqrt{-\Psi})\right)}{2\,(-\Psi)^{3/2}},
\end{aligned} \tag{4.2.12}$$

where $\Psi_R < 0$ and $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$ is the error function of a complex-valued argument (to evaluate this function we employ the algorithm given in Abramowitz-Stegun (1972)).

A robust choice of Ψ_R for numerical evaluation of formula (4.2.9) is $\Psi_R = -1$.

By linearity, once the value of $U(t, T, V, S, I, K, o)$ is found we retrieve the value of $\hat{U}(t, T, V, S, \hat{I}, \hat{K}, o)$ by normalizing and discounting:

$$\hat{U}(t, T, V, S, \hat{I}, \hat{K}, o) = \mathbf{1}_{\{\iota > t\}}\frac{DF(t, T)}{(T - t_0)^o}U(t, T, V, S, I, K, o). \tag{4.2.13}$$

4.2.1. Variance Swaps

Here, we analyze the value function of the variance swap under the model (0.0.1) in more details. Nowadays variance swaps are very liquid instruments and, similarly to the term structure of interest rates, market participants observe the term structure of the fair variance, where the fair variance quantifies such strike price of the variance swap that equates the value of the swap at its inception time to zero. Accordingly, it is important to make the model consistent with the market curve of fair variance.

It is clear that in case of default-free dynamics, the value function for the fair variance is obtained by using the formula (3.3.36):

$$\hat{U}^{swp}(t_0, t, T, V, S, \hat{I}, \hat{K}, 1) = \frac{DF(t,T)}{T - t_0}\left(I(t_0, t) + \overline{M}^I(t,T)\right), \tag{4.2.14}$$

with

$$\overline{M}^I(t,T) = \left(\frac{\theta}{\kappa} + \frac{\eta\gamma}{\kappa^2}\right)\left(\tau\kappa + e^{-\kappa\tau} - 1\right) + \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right)V + \tau\gamma(\nu^2 + \delta^2), \tag{4.2.15}$$

and $\tau = T - t$.

We see that under the default free dynamics there is three sources of contribution to the fair variance: the first is the stochastic dynamics of the variance $V(t)$, the second is jump process in $V(t)$ and the last one is the jump process in the asset price.

Expanding $\hat{U}^{swp}$ in Taylor series around $\tau = 0$, we obtain:

$$\begin{aligned} U^{swp}(t_0, t, T, S, V, I, K) = \frac{DF(t,T)}{T - t_0} \times \\ \left(I(t_0, t) + (V + \gamma(\nu^2 + \delta^2))\tau + \frac{1}{2}\kappa(\theta - V)\tau^2 + \frac{1}{2}\eta\gamma\tau^2 + O(\tau^3)\right). \end{aligned} \tag{4.2.16}$$

Expansion (4.2.16) allows to analyze the behavior of the expected realized variance in short term. We see that the first term of order $O(\tau^2)$ represents the effect of mean-reversion: it is positive if $V < \theta$ and negative otherwise, while the second term reflects the impact of jumps, which is always positive.

We also see that all being the same, the stochastic variance model with variance jumps implies higher prices for swaps on realized variance with contribution of order $O(\tau^2)$, and asset price jump process significantly affect the value of variance swap with contribution of order $O(\tau)$ and with the fair variance being proportional to jump intensity rate and squares of the jump mean and volatility parameters.

As a result, a stochastic volatility model with price and/or variance jumps can improve a pure diffusion model with no jumps, which is typically not consistent with available market prices by implying lower fair strikes than those observed in the market and also implying almost the same prices for plain swaps and swaps with cap levels that are sufficiently above the fair strike.

Under the defaultable dynamics with no price and variance jumps assuming $\mathbf{1}_{\{\iota>t\}} = 1$, we obtain:

$$
\begin{aligned}
&\hat{U}^{swp}(t_0, t, T, V, S, \hat{I}, \hat{K}, 1) \approx \frac{DF(t,T)}{T-t_0} \times \\
&\left(Q(t,T,V) \left(\frac{\theta}{\kappa} \left(\tau\kappa + e^{-\kappa\tau} - 1 \right) + \frac{1}{\kappa} \left(1 - e^{-\kappa\tau} \right) V \right) + Q^c(t,T,V)\mathcal{I} \right) \\
&\approx \frac{DF(t,T)}{T-t_0} \left((1 - (\alpha + \beta V)\tau + O(\tau^2))(I(t_0,t) + V\tau + O(\tau^2)) + ((\alpha + \beta V)\tau + O(\tau^2))\mathcal{I} \right) \\
&\approx \frac{DF(t,T)}{T-t_0} \left(I(t_0,t) + V\tau + (\alpha + \beta V)(\mathcal{I} - I(t_0,t))\tau + O(\tau^2) \right).
\end{aligned}
$$

Accordingly, we see that the default risk introduces noticeable contribution of order τ to the value of variance swap, so that the diffusion process with the jump-to-default can also be made consistent with an observable curve of fair variance.

4.3. Forward-Start Options

In general, a forward start option derives its value from two future observations of the underlying variable at times t_F and T, $t < t_F < T$. At time $t = t_F$, the forward-start asset price is fixed and the forward-start option becomes a standard option with only one underlying variable. Accordingly, our main ussue is to compute values of these contracts at time t, $t < t_F$.

4.3.1. Equity Options

For equity products, forward-start option derives its value from two future spot prices S_{t_F} and S_T at future times t_F and T, $t < t_F < T$. The most common contracts are the non-dimensional and dimensional forward-start calls with value functions denoted respectively by $\hat{W}^{fsnc}(t, t_F, T, S, V, K)$ and $\hat{W}^{fsdc}(t, t_F, T, S, V, K)$ with payoffs given respectively by:

$$
\begin{aligned}
\hat{W}^{fsnc}(T, t_F, T, S, V, K) &= \mathbf{1}_{\{\iota>t_0\}} \max\left(\frac{S(T)}{S(t_F)} - K, 0 \right) \mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}}, \\
\hat{W}^{fsdc}(T, t_F, T, S, V, K) &= \mathbf{1}_{\{\iota>t_0\}} \max\left(S(T) - KS(t_F), 0 \right) \mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}} \\
&= \mathbf{1}_{\{\iota>t_0\}} S(t_F) \max\left(\frac{S(T)}{S(t_F)} - K, 0 \right) \mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}}.
\end{aligned}
$$

The corresponding non-dimensional, $\hat{W}^{fsnp}(t, t_F, T, S, V, K)$, and dimensional,

$\hat{W}^{fsdp}(t, t_F, T, S, V, K)$, put options have the following payoffs:

$$\hat{W}^{fsnp}(T, t_F, T, S, V, K) = \mathbf{1}_{\{\iota > t_0\}} \left(\max\left(K - \frac{S(T)}{S(t_F)}, 0 \right) \mathbf{1}_{\{N^d(T) - N^d(t_0) = 0\}} + \right.$$
$$\left. + K \mathbf{1}_{\{N^d(T) - N^d(t_F) > 0 | N^d(t_F) - N^d(t_0) = 0\}} \right),$$
$$\hat{W}^{fsdp}(T, t_F, T, S, V, K) = \mathbf{1}_{\{\iota > t_0\}} \left(\max\left(K S(t_F) - S(T), 0 \right) \mathbf{1}_{\{N^d(T) - N^d(t_0) = 0\}} \right.$$
$$\left. + K S(t_F) K \mathbf{1}_{\{N^d(T) - N^d(t_F) > 0 | N^d(t_F) - N^d(t_0) = 0\}} \right).$$

Pricing by Convexity Adjustment

In general, using valuation theorem (1.2.1) and logarithmic variables with $X = \ln \frac{S(T)}{S(t_F)}$, we can represent the value of a forward-start option, $\hat{W}^{fs}(t, t_F, T, S, V, K)$, with payoff function $\hat{\mathcal{W}}(e^X, K)$ as follows:

$$\hat{W}^{fs}(t, t_F, T, S, V, K) = \mathbf{1}_{\{\iota > t\}} DF(t, T) \int_0^\infty \int_{-\infty}^\infty \hat{W}(t_F, T, X', V') G^{vx}(t, t_F, V, V', X, X') dX' dV', \tag{4.3.1}$$

where

$$\hat{W}(t_F, T, X', V') = DF(t_F, T) \int_{-\infty}^\infty \hat{\mathcal{W}}(e^{X''}, K) G^x(t, t_F, V', X', X'') dX'', \tag{4.3.2}$$

is the value function of the given underlying claim, and $G^{vx}(t, t_F, V, V', X, X')$ is the Greens function of the joint evolution of V and X.

Accordingly, we obtain for non-dimensional call option:

$$\hat{W}^{fsnc}(t, t_F, T, S, V, K) = \mathbf{1}_{\{\iota > t\}} DF(t, t_F) \int_0^\infty \hat{W}^c(t_F, T, 0, V', K) G^v(t, t_F, V, V') dV', \tag{4.3.3}$$

and for non-dimensional put option we obtain:

$$\hat{W}^{fsnp}(t, t_F, T, S, V, K) = \mathbf{1}_{\{\iota > t\}} DF(t, t_F) \times \left(\int_0^\infty \hat{W}^p(t_F, T, 0, V', K) G^v(t, t_F, V, V') dV' + K Q^{cfs}(t, t_F, T, V) \right), \tag{4.3.4}$$

where $\hat{W}^c(t, T, X, V, K)$ ($\hat{W}^p(t, T, X, V, K)$) is the value function of a call (put) option with inception at t, maturity at T, and zero initial log-price; $Q^{cfs}(t, t_F, T, V)$ is the forward-start default probability defined by (3.5.12).

Next, for dimensional call option we get:

$$\hat{W}^{fsdc}(t, t_F, T, S, V, K) = \mathbf{1}_{\{\iota > t\}} D(t, t_F) \int_0^\infty \hat{W}^c(t_F, T, 0, V', K) G^{vp}(t, t_F, V, S, V') dV', \tag{4.3.5}$$

and for dimensional put option we derive:

$$\hat{W}^{fsdp}(t,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}D(t,t_F)\left(\int_0^\infty \hat{W}^p(t_F,T,0,V',K)G^{vp}(t,t_F,V,S,V')dV' \right.$$
$$\left. +KQ^{cfsp}(t,t_F,T,V)\right), \tag{4.3.6}$$

where $G^{vp}(t,T,V,S,V')$ is price denominated Green function of the variance process whose Fourier transform is defined by (3.3.54) and $Q^{cfsp}(t,t_F,T,V)$ is the price denominated forward-start default probability defined by (3.5.18).

Accordingly, formulas (4.3.3) - (4.3.6) imply that the value of a forward-start option can be represented as the expected value of a call or put option with inception at time t_F and maturity T, where expectation is taken at time t_0 over all possible states of future variance at time t_F.

These formulas can also be represented employing the convexity adjustment formula (1.3.6). For example, the non-dimensional call option can be approximated as follows:

$$\hat{W}^{fsnc}(t,t_F,T,S,V,K)) \approx \mathbf{1}_{\{\iota>t\}}DF(t,t_F)\left(\hat{W}^c(t_F,T,0,\overline{M}^v_1(t,t_F),K)+ \right.$$
$$\left. +\frac{1}{2}\overline{M}^v_2(t,T)\hat{W}^c_{VV}(t_F,T,0,\overline{M}^v_1(t,t_F),K)\right), \tag{4.3.7}$$

where the second order partial derivative with respect to V, $\hat{W}^c_{VV}$, can be computed semi-analytically by differentiating the transformed option value and inverting it. Similar formulas can be obtained for other variants of forward-start options.

The above approach to value forward-start options was developed by Lipton (2001). However, since it requires an additional integration with respect to the TPDF of V it is not well suited for numerical evaluation. We develop an alternative way to solve the pricing problem by modeling the logarithm of the forward-start asset price $\bar{X}$ directly.

Analytical Pricing

First, we simplify the problem as much as possible and represent payoffs of these options as follows:

$$\hat{W}^{fsnc}(T,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}\left(\frac{S(T)}{S(t_F)} - \min\left(\frac{S(T)}{S(t_F)},K\right)\right)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}},$$
$$\hat{W}^{fsdc}(T,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}\left(S(T) - S(t_F)\min\left(\frac{S(T)}{S(t_F)},K\right)\right)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}},$$

$$\hat{W}^{fsnp}(T,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}\left(K - \min\left(K,\frac{S(T)}{S(t_F)}\right)\right)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}}$$
$$+\mathbf{1}_{\{\iota>t\}}K\mathbf{1}_{\{N^d(T)-N^d(t_F)>0|N^d(t_F)-N^d(t_0)=0\}},$$
$$\hat{W}^{fsdp}(T,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}\left(KS(t_F) - S(t_F)\min\left(K,\frac{S(T)}{S(t_F)}\right)\right)\mathbf{1}_{\{N^d(T)-N^d(t_0)=0\}}$$
$$+\mathbf{1}_{\{\iota>t\}}KS(t_F)\mathbf{1}_{\{N^d(T)-N^d(t_F)>0|N^d(t_F)-N^d(t_0)=0\}}.$$

Taking expectation at time t, $t < t_F$ and applying discounting, we obtain:

$$\hat{W}^{fsnc}(t,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}DF(t,T)\left(CF^{-1}(t_F,T)Q(t,t_F,V) - W^{fsncc}(t,t_F,T,\bar{X},V,K)\right)$$
$$\hat{W}^{fsdc}(t,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}DF(t,T)\left(CF^{-1}(t,T) - W^{fsdcc}(t,t_F,T,\bar{X},V,K)\right),$$
$$\hat{W}^{fsnp}(t,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}DF(t,T)$$
$$\left(KQ(t,T,V) - W^{fsncc}(t,t_F,T,\bar{X},V,K) + KQ^{cfs}(t,t_F,T,V)\right),$$
$$\hat{W}^{fsdcc}(t,t_F,T,S,V,K) = \mathbf{1}_{\{\iota>t\}}DF(t,T)$$
$$\left(KS(t)CF(t,t_F)Q(t_F,T,V) - W^{fsdp}(t,t_F,T,\bar{X},V,K) + KQ^{cfsp}(t,t_F,T,V)\right).$$

where $Q^{cfs}(t,t_F,T,V)$ ($Q^{cfsp}(t,t_F,T,V)$) is (price-denominated) forward-start default probability defined by (3.5.8) ((3.5.17)), $W^{fsncc}(t,t_F,T,\bar{X},V,K)$ is the value function of non-dimensional forward-start covered call, and $W^{fsdcc}(t,t_F,T,\bar{X},V,K)$ is its dimensional analog. These value functions are computed using our methodology (theorem (2.3.1) and formula (2.3.8)) as follows:

$$\begin{aligned} W^{fsncc}(t,T,V,X,K) &= \int_{-\infty}^{\infty}\mathcal{W}(\bar{X}',K)G^{\bar{x}}(t,T,V,X,\bar{X},\bar{X}')d\bar{X}' \\ &= \frac{1}{\pi}\int_0^{\infty}\Re\left[\widehat{G}^{\bar{x}}(t,T,V,X,\Phi)\widehat{u}(\Phi,K)\right]d\Phi_I, \\ W^{fsdcc}(t,T,V,X,K) &= \int_{-\infty}^{\infty}\mathcal{W}(\bar{X}',K)G^{\bar{x}p}(t,T,V,X,\bar{X},\bar{X}')d\bar{X}' \\ &= \frac{1}{\pi}\int_0^{\infty}\Re\left[\widehat{G}^{\bar{x}p}(t,T,V,X,\Phi)\widehat{u}(\Phi,K)\right]d\Phi_I, \end{aligned} \tag{4.3.8}$$

where $G^{\bar{x}}(t,T,V,X,\bar{X},\bar{X}')$ is Green function of the future log-spot price $\bar{X}$, and $G^{\bar{x}p}(t,T,V,X,\bar{X},\bar{X}')$ is Green function of the price denominated forward-start log-spot price, and $\widehat{G}$ are their respective transformed Green functions given respectively by (3.3.28) and (3.3.56). Finally, the payoff function is $\mathcal{W}(X,K) = \min(X,K)$ and its transform, $\widehat{W}(\Phi,K)$, is given by:

$$\widehat{W}(\Phi,K) = \int_{-\infty}^{\infty}e^{\Phi X'}\mathcal{W}(X',K)dX' = \int_{-\infty}^{\infty}e^{\Phi X'}\min(e^{X'},K)dX' = -\frac{Ke^{\Phi\ln K}}{\Phi(\Phi+1)}, \tag{4.3.9}$$

provided $-1 < \Phi_R < 0$.

4.3.2. Options on Realized Variance

Forward-start claim on forward-start realized volatility or variance derives its value from a specified underlying claim. Let $\hat{U}^{fs}(t, t_F, T, V, \hat{I}, \hat{K}, o)$ denote the value function of the forward start claim; and let $\hat{U}(t_F, T, V, \hat{I}, \hat{K}, o)$ and $\mathcal{U}(I^o, K^o)$ denote the value and payoff functions of the underlying claim, respectively. We note that in the present context $\hat{I}(t_F, T) = \frac{1}{T-t_F} I(t_F, T)$. The underlying claim initiates at some future time t_F and expires at time T; and the forward start contract allows the investor to enter the underlying claim at time t_F in amount proportional to $\hat{U}(t_F, T, V, \hat{I}, \hat{K}, o)$. We are interested in calculating the fair value of the forward start claim $\hat{U}^{fs}(t, t_F, T, V, \hat{I}, \hat{K}, o)$ at time t_0, $t_0 < t_F < T$.

Pricing by Convexity Adjustment

The forward-start claim has the following payoff at time maturity T:

$$\hat{U}(T, t_F, T, V, \hat{I}, \hat{K}, o) = \mathbf{1}_{\{\iota > t\}} \Big(\hat{\mathcal{U}}(\hat{I}^o, \hat{K}^o) \mathbf{1}_{\{N^d(T) - N^d(t_F) = 0 | N^d(t_F) - N^d(t_0) = 0\}}$$
$$+ \hat{\mathcal{U}}(\hat{\mathcal{I}}^o, \hat{K}^o) \mathbf{1}_{\{N^d(T) - N^d(t_F) > 0 | N^d(t_F) - N^d(t_0) = 0\}} \Big) . \tag{4.3.10}$$

Accordingly, employing theorem (1.2.1), we can present the value of the forward-start claim at time $t = t_0$ as follows:

$$\hat{U}^{fs}(t_0, t_F, T, V, \hat{I}, \hat{K}, o) = \mathbf{1}_{\{\iota > t\}} DF(t_0, t_F)$$
$$\int_0^\infty \int_0^\infty \hat{U}(t_F, T, V', \hat{I}', \hat{K}, o) G^{Iv}(t, t_F, V, V', I, I') dI' dV', \tag{4.3.11}$$

where

$$\hat{U}(t_F, T, V, \hat{I}, \hat{K}, o) =$$
$$\frac{DF(t_F, T)}{(T - t_F)^o} \int_0^\infty \int_0^\infty \left(\mathcal{U}(I'^o, K^o) + \mathcal{U}(\mathcal{I}^o, K^o) Q^c(t_F, T, V') \right) G^{Iv}(t_F, T, V, V', I, I') dI' dV', \tag{4.3.12}$$

and $G^{Iv}(t, T, V, V', I, I')$ is the defaultable Green function of the joint evolution of variance and realized variance.

Using the fact that the payoff function does not depend on $V(T)$ and that averaging starts only at time t_F, we simplify Eq(4.3.11) to obtain:

$$\hat{U}(t_0, t_F, T, V, \hat{I}, \hat{K}, o) = \mathbf{1}_{\{\iota > t\}} DF(t_0, t_F) \int_0^\infty \hat{U}(t_F, T, V, \hat{I}, \hat{K}, o) G^v(t_0, t_F, V, V') dV', \tag{4.3.13}$$

where $\hat{U}(t_F, T, V, \hat{I}, \hat{K}, o)$ is the value function of the underlying claim with inception time t_F and maturity T:

$$\hat{U}(t_F, T, V, \hat{I}, \hat{K}, o) = \frac{DF(t_F, T)}{(T - t_F)^o} \int_0^\infty \mathcal{U}(I'^o, K^o) G^I(t_F, T, V, I, I') dI' + \frac{DF(t_F, T)}{(T - t_F)^o} \mathcal{U}(\mathcal{I}^o, K^o) Q^c(t_F, T, V). \tag{4.3.14}$$

Accordingly, formula (4.3.13) implies that the value of forward-start option can be represented as the expected value of the underlying claim on realized variance with inception at time t_F and maturity T, where expectation is taken at time t_0 over all possible states of variance at time t_F. We can also approximate this expectation applying convexity adjustment formula (1.3.6) similarly to formula (4.3.7).

Analytical Pricing

To avoid using integration with respect to the TPDF of V, an alternative solution method is to calculate option values using the notion of the forward-start realized variance $\bar{I}$ and its Green function, $G^{\bar{I}}(t, t_F, T, V, I, \bar{I}')$, directly. Applying formula (2.3.8) along with transformed Green function of forward-start realized variance (3.3.46), we obtain:

$$\hat{U}(t_0, t_F, T, V, \hat{I}, \hat{K}, o) = \mathbf{1}_{\{\iota > t\}} \frac{D(t, T)}{(T - t_F)^o} \left(U^{fs}(t, t_F, T) + \mathcal{U}(\mathcal{I}^o, K^o) Q^{cfs}(t, t_F, T, V) \right). \tag{4.3.15}$$

where,

$$U^{fs}(t, t_F, T) = U^{fs}(t, t_F, T, V, I, K, o) = \int_0^\infty \mathcal{U}(\bar{I}'^o, K^o) G^{\bar{I}}(t, t_F, T, V, I, \bar{I}') d\bar{I}' = \frac{1}{\pi} \int_0^\infty \Re \left[\widehat{U}(\bar{I}'^o, K^o) \widehat{G}^{\bar{I}}(t, t_F, T, V, I, \Psi) \right] d\Psi_I,$$

where $\widehat{U}(I^o, K^o)$ is the transformed payoff function given by formulas (4.2.11)-(4.2.12), and $Q^{cfs}(t, t_F, T, V)$ is forward-start default probability defined by (3.5.8).

4.3.3. VIX Futures and Options on VIX

Derivative securities on VIX are particular examples of claims on forward-start asset realized volatility. VIX stands for Chicago Board Options Exchange Volatility Index which measures the implied volatility of S&P500 stock index options with maturities 30 days. In other words, VIX represents the market's expectation of the annualized volatility over the next 30 day period. The most popular VIX-based derivative instrument is VIX futures contract, which began trading in 2004. By now, investors can trade with other exchange-listed instruments including VIX options, which began trading in February 2006. More details on these derivatives can be found in

VIX futures white paper (CBOE 2006). Similar contracts deriving their value from implied volatilities of other stock indices are also actively traded.

In essence, derivatives on VIX are contracts on the asset variance $V(t)$. Here, we briefly apply our general framework to value these contracts. More details with illustrations are given in Sepp (2007b) and (2007c). We note that S&P500 index is virtually a default-free asset so that we omit the jump-to-default process.

We denote by $F(t,T)$ the value of VIX futures at time t with settlement at time T and by $F(t)$ the spot value of VIX, respectively. Let $F_2(t,T)$ and $F_2(t)$ denote the squares of these quantities, respectively. $F_2(t)$ measures the expected annualized variance for options with maturity time Δ_T, while $F_2(t,T)$ measures the expected annualized variance at future time T for options with maturity time $T+\Delta_T$, where Δ_T corresponds to year fraction of 30 days ($\Delta_T = \frac{30}{365}$). Thus, at future time $t=T$, $F(t,T)$ is the square root of expected variance realized over time period $[T, T+\Delta_T]$:

$$F(T,T) = \sqrt{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Delta_T}\int_T^{T+\Delta_T} V(t')dt' \mid \mathcal{F}(T)\right]} = \sqrt{\overline{M}^{\hat{I}}(T, T+\Delta_T)}, \qquad (4.3.16)$$

where $\overline{M}^{\hat{I}}(t,T)$ is obtained using (4.2.15) as follows:

$$\overline{M}^{\hat{I}}(t,T) = m_1(t,T) + m_2(t,T)V(T), \qquad (4.3.17)$$

with

$$m_1(t,T) = \frac{1}{\Delta_T}\left(\left(\frac{\theta}{\kappa} + \frac{\eta\gamma}{\kappa^2}\right)\left(\Delta_T\kappa + e^{-\kappa\Delta_T} - 1\right)\right) + \gamma(\nu^2+\delta^2),$$
$$m_2(t,T) = \frac{1}{\Delta_T}\frac{1}{\kappa}\left(1 - e^{-\kappa\Delta_T}\right).$$

Accordingly, the VIX squared spot value $F_2(t)$ can be represented as a linear function of the variance at time t:

$$F_2(t) = m_1(t,t) + m_2(t,t)V(t), \qquad (4.3.18)$$

while the VIX futures $F(t,T)$ with settlement time T can be represented as time-t expectation of square root of a linear function of the variance at time T:

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\sqrt{m_1(t,T) + m_2(t,T)V(T)} \mid \mathcal{F}(t)\right]. \qquad (4.3.19)$$

From equations (4.3.18) and (4.3.19), we see that a derivative security on VIX is essentially an option on the future variance $V(T)$.

Since $F_2(t)$ is a linear function of the implied variance, to solve the pricing problem we apply Green function, $G^v(t,T,V,V')$, of the default-free variance process $V(t)$ and its Fourier transform, $\widehat{G}^v(t,T,V,\Theta)$, given by (3.3.4), to obtain Fourier transform, $\widehat{G}^{F_2}(t,T,F_2,\Theta)$, of Green function of $F_2(t)$, $G^{F_2}(t,T,F_2,F_2')$, in the following way:

$$\widehat{G}^{F_2}(t,T,F_2,\Theta) = e^{-m_1(t,t)\Theta}\widehat{G}^v(t,T,F_2,m_2(t,t)\Theta). \qquad (4.3.20)$$

As a result, we apply our general pricing formula (2.3.8) along with Fourier transform (4.3.20) to get an explicit pricing formula for the value function of a claim on VIX denoted by $U(t,T,V,F_2,K)$, with payoff function $\mathcal{U}(F,K)$ expressed as a function of F_2 and with maturity time T and strike price K, as follows:

$$\begin{aligned} U(t,T,V,F_2,K) &= DF(t,T)\int_0^{\infty}\mathcal{U}(\sqrt{F_2'},K)G^{F_2}(t,T,F_2,F_2')dF_2' \\ &= DF(t,T)\frac{1}{\pi}\int_0^{\infty}\Re\left[\widehat{G}^{F_2}(t,T,F_2,\Theta)\widehat{\mathcal{U}}(\Theta,K)\right]d\Theta_I \end{aligned} \tag{4.3.21}$$

where

$$\widehat{\mathcal{U}}(\Theta,K) = \int_0^{\infty} e^{\Theta F_2'}\mathcal{U}(\sqrt{F_2'},K)dF_2', \tag{4.3.22}$$

is the transformed payoff function.

Formulas in (4.2.12) provide explicit transforms of payoff function, $\widehat{\mathcal{U}}(\Theta,K)$ with $\Theta \to \Psi$, for swaps (a swap with zero strike corresponds to the VIX futures contract), for call and put swaptions (which now correspond to call and put options on VIX futures), for swaps with cap protections. In the present setting, we use pricing formula (4.3.21) along with transformed payoff (4.3.22) to value claims on VIX.

4.3.4. Forward Volatility

Here we briefly discuss a very important notion of the forward volatility. In practice, forward-start options play a significant role in modeling of the so-called forward volatilities. Literally, the forward volatility is an equivalent Black-Scholes volatility to be used today for pricing a forward-start option initiating at some future time t_F and with a final payoff taking place at time T, $T > t_F > 0$, assuming the Black-Scholes model with constant volatility. More formally, the forward volatility can be associated with a volatility perceived at time t of the following random variable χ:

$$\chi(t_F,T) = Y(T) - Y(t_F), \tag{4.3.23}$$

where $Y(T)$ ($Y(t_F)$) is the value of process $Y(t)$ observed at time T (t_F), and $0 \le t < t_F < T < \infty$. For time-homogeneous processes, the volatility of $\chi(t_F,T)$ is proportional to the square root of the "time-to-maturity", $T - t_F$, and the current volatility of process $Y(t)$. However, since asset prices do not exhibit stationarity features, for purposes of modeling path-dependent equity and volatility products we need advanced dynamics to model the forward volatility of asset prices.

We refer to Figure (4.4.5) for an example (implied by the SV+PJ+DJ process) how market participants would perceive the implied volatility surface one year from now. The quantity of forward volatility arises in pricing problems of path-dependent options, for example, cliquet options which broadly represent a series of forward start options with some local and global cap levels.

The market practice is to quantify the forward-start volatility by means of the so-called Black-Scholes forward-start volatility, denoted by $\sigma^{fs}(t_F,T,K)$, which is computed from a market price of non-dimensional forward-start call option by implying its Black-Scholes volatility by means of the theoretical Black-Scholes value function, $U_{BS}^{fsnc}(t, t_F, T, S, K)$, with the constant volatility parameter:

$$U_{BS}^{fsnc}(t, t_F, T, S, K) = e^{-\int_t^{t_F} d(t')dt'} U(t_F, 1; T, K), \ 0 \leq t < t_F, \tag{4.3.24}$$

where $U(t, S; T, K)$ is the call price given by Black-Scholes-Merton formula (1.6.5) with average interest rate and dividend yield calculated respectively by $r = \frac{\int_t^T r(t')dt'}{T-t}$, and $d = \frac{\int_t^T d(t')dt'}{T-t}$, and volatility parameter $\sigma^{fs}(t_F, T, K)$.

We see that the value function of a non-dimensional forward-start call does not depend on the asset price up to forward-start time t_F, so that its delta and gamma (first and second partial derivatives with respect to the spot price) are zero, but its vega is not (partial derivatives with respect to the volatility parameter), so that forward-start calls and puts virtually represent pure volatility products and their pricing and hedging requires considerable efforts.

4.4. Calibration to Implied Volatilities of General Motors Options

To illustrate some particular specifications of our general model (0.0.1), their estimated parameters and implications, we calibrate four specifications to General Motors (GM) option data. GM is a good candidate for our study of the default problem since, once creditworthy, this company faced big financial problems in 2005 leading to the sharp downgrade of its ratings and soaring spreads on its debt obligations. At that point of time the possibility of the financial default of this company was very likely. In addition, out-of-the money put options on GM equity were very liquid at that time with a particular interest to longer maturity puts with strike prices 5 and 10 USD (on November 8, 2005, the stock price was 25.86 USD). As a result, the volatility skew of GM options was very pronounced.

Option implied volatilities were collected from the Bloomberg terminal on November 8, 2005. The data represent listed options with six maturity times ranging from 1.2 months up to 2.2 years and with a range of strike prices ranging from 10 to 40 USD. A snapshot of implied volatilities per three maturity times is given in Figure (1.7.1).

Following our implications obtained from analyzing implied TPDF-s in Section 3, we chose the following robust model specifications:

1) SV - the asset price process with stochastic volatility,

2) SV+JD - the asset price process with stochastic volatility and jump-to-default with constant intensity α,

3) SV+PJ - the asset price process with stochastic volatility and normally distributed price jumps.

4) SV+PJ+JD - the asset price process with stochastic volatility and with jump-to-default and normally distributed price jumps.

4.4.1. Implications

Model parameters, obtained by calibration of theoretical model prices to the corresponding market prices of call and put options, are reported in table (4.4.1).

	SV	SV+JD	SV+PJ	SV+PJ+JD
$V(0)$	0.5265	0.3647	0.4982	0.3568
θ	0.4854	0.1563	0.4871	0.1579
κ	4.0183	2.8018	4.3739	2.6306
ε	1.9751	0.9359	2.0642	0.9115
ρ	-0.9453	-0.7372	-0.9900	-0.9629
α		0.1024		0.0977
β				
η				
γ			0.0159	0.0039
ν			0.6902	1.6523
δ			0.0069	0.0071

Table 4.4.1: Parameter estimates for 4 prototype dynamics.

Our results are illuminating in a few respects.

1) SV process implies big values of initial and long-term variance parameters as well as the volatility of variance parameter, which are necessary for the model to capture the skew. Also, it produces a very strong correlation between the asset price and the variance. As a result, this process implies a very heavy left tail of the asset price TPDF. The fit to the volatility surface is not particularly good with rather big price differences for short maturity options.

2) SV+JD process implies a significant default probability, about 10% chance that GM will default in one year. The fit to the volatility surface is very good unlike pure diffusion process with close fit to short and longer maturities and with a small mismatching with options with the longest maturity. The jump-to-default helps to explain the skew, so that the diffusion part has lower initial and long-term variance as well as volatility of variance parameter. The implied correlation between the asset price and the variance is less than that of SV process.

3) SV+PJ process with normally distributed price jumps produces a rather interesting result - while its diffusion parameters look very close to SV process, its jumps process implies a big positive jump with almost zero volatility. Although the intensity of this huge price jump, with expected magnitude of asset price jump equal to about 100%, is tiny (it implies that we can expect one jump per 63 years), it indicates that some of the market participants had expected the GM to recover from its difficulties and raised some demand for GM out-of-the-money calls. (As a matter of fact, GM did improve its performance in subsequent years following some restructuring of its assets. Its highest price in year 2007 reached the level of 37.24 USD). The quality of fit to the implied volatility surface is moderate - it is better than that of the SV process in explaining the short terms skew but it is noticeably poorer compared to SV process with jump-to-default.

4) SV+PJ+JD process combines both jump-to-default and price jump features and has very similar to SV+JD model parameter of the diffusion part and jump to the default and, like SV+PJ process, it also has price jump part with a huge expected value and small probability. It implies that we can expect one huge price jump per 257 years leading to about 400% jump in the asset price. The specification results in the best fit to the data reproducing implied volatilities across all strikes and maturities.

As a result, we have arrived at an interesting conclusion that, in addition to the jump-to-default, a pricing model should also include a "jump-to-recovery" process, whose jump size PDF can be chosen to be the delta function with jump size implied from the market option data. A financial interpretation of this jump-to-recovery event is that distressed companies might be a target for acquisition by financially solid companies in which case their debt becomes upgraded and financed with lower costs. As a result, the price of company equity becomes more valuable for investors.

4.4.2. Illustrations

Now we illustrate model specifications and their estimated parameters by examining their implied TPDF-s for the underlying variables. We use model parameters from table (4.4.1) and maturity time $T = 1$. The state of the asset price process is scaled by the initial stock price $S(0)$. Green functions are scaled to integrate to one (under the default free dynamics).

In Figure (4.4.1) we plot the implied Green function for the asset price process.

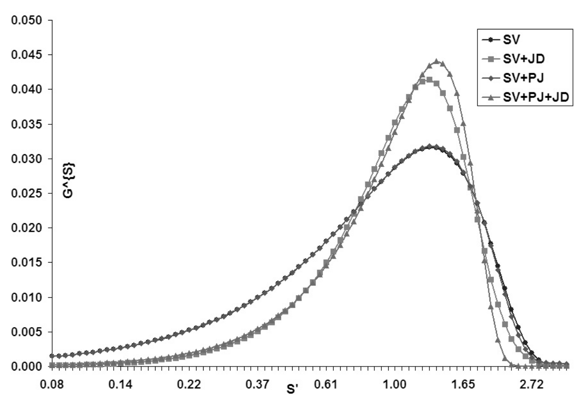

Figure 4.4.1: Green functions of the implied asset price process

It follows that SV and SV+PJ processes imply almost identical distribution for the asset price with a heavy left tail, virtually implying the default event. SV+PJ has a little bit more sizable right tail due to the possibility of a huge asset price jump. SV+JD and SV+PJ+JD processes have similar left tails but different right tails indicating for different implied recovery scenarios.

In Figure (4.4.2) we plot Green functions for the implied TPDF of variance V. We note that all four models imply zero Feller parameter ϑ defined in (3.3.7) (we imposed non-negativity of ϑ by model calibration), meaning that variance can hit zero. When we use market data to imply model parameters, the Feller parameter is typically negative or close to zero. This is explained by the fact that in pure diffusion models, the variance process needs to have a sizable variability to reproduce heavy tails of asset price distributions implied by market option prices. As a result, the volatility of variance parameter becomes high violating the Feller condition.

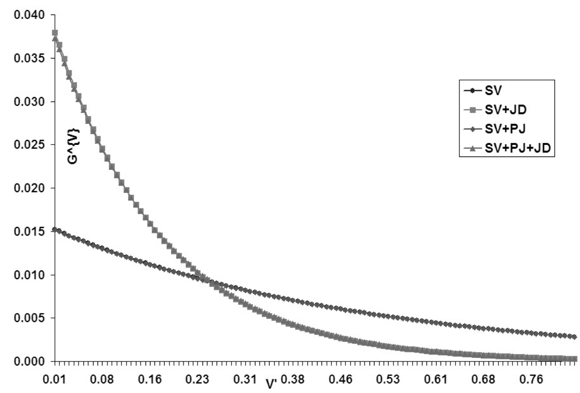

Figure 4.4.2: Green functions of the implied variance process

From Figure (4.4.2) we see that SV and SV+PJ processes have heavy right tails for the implied variance. SV+JD and SV+PJ+JD processes imply higher probabilities of observing small values of variance, which is explained that their long-term mean variance parameter is less in value compared to that of the SV and SV+PJ processes.

In Figure (4.4.3) we analyze the implied Green functions of the realized variance process. We see that since the magnitude of the price jump is small it does not affect the TPDF of the realized variance and SV and SV+PJ processes as well as SV+JD and SV+PJ+JD processes have similar shapes between each other. SV and SV+PJ processes imply a huge right tail for the realized variance as a reflection of the heavy tail for the variance process.

The shape of the TPDF of the realized variance is peaked at the long-term variance parameter and it is not concentrated close to the origin, so that it serves as a better representative of the market expected variability rather than the variance process itself.

In Figure (4.4.4) we show the term structure of fair variance swaps implied by these models. For the fair variance of SV and SV+PJ models, denoted by $K^{sv}(T)$, we use the following formula (for brevity we apply no discounting factor) obtained

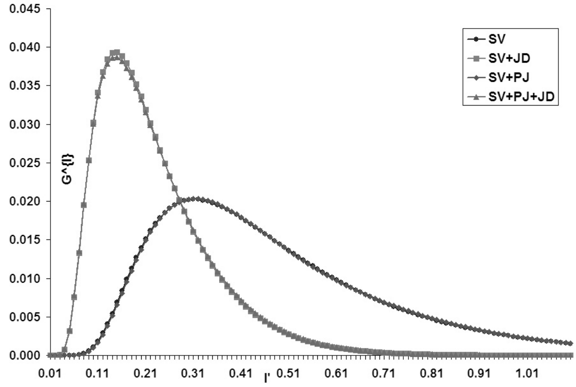

Figure 4.4.3: Implied Green functions of the realized variance process

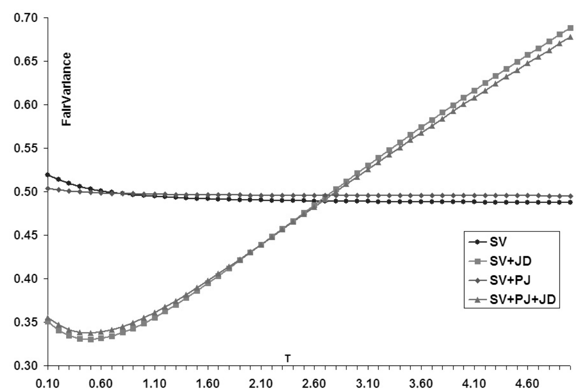

Figure 4.4.4: The term structure of the implied fair variance.

from (4.2.14) with $t_0 = t = 0$:

$$K^{sv}(T) = \frac{1}{T}\left(\frac{\theta}{\kappa}\left(T\kappa + e^{-\kappa T} - 1\right) + \frac{1}{\kappa}\left(1 - e^{-\kappa T}\right)V + T\gamma(\nu^2 + \delta^2)\right). \qquad (4.4.1)$$

To calculate the fair variance of SV+JD and SV+PJ+JD models, denoted by $K^{jd}(T)$, we assume that following the default event the post default state of the realized variance swap with maturity T is given by $\hat{\mathcal{I}}(T) = 3K^{sv}(T)$ (this is a standard market practice to set the cap on the realized variance equal to the fair variance observed for the given maturity and multiplied by a coefficient of 2.5 or 3), where $K^{sv}(T)$ is computed by means of formula (4.4.1) using parameters of the SV model. Then the fair variance under defaultable dynamics is computed by:

$$K^{jd}(T) = Q(0,T)K^{sv}(T) + (1 - Q(0,T))\hat{\mathcal{I}}(T), \qquad (4.4.2)$$

where the fair variance $K^{sv}(T)$ is computed by means of formula (4.4.1) using parameters of the SV+JD and SV+PJ+JD models and survival probability $Q(0,T)$ assuming constant intensity α is given by $Q(0,T) = e^{-\alpha T}$.

From Figure (4.4.4), we see that the term structure of the fair variance of SV and SV+PJ processes is almost constant because the initial variance is very close to its long-term mean. Also, the fair variance of SV+PJ model is only marginally more expensive to that of the SV process because the probability of experiencing the huge price jump is very small. However the term structure of fair variance for SV+JD and SV+PJ+JD models is increasing in maturity time T because it becomes almost linear in the default probability, which nearly linearly increases in maturity time parameter. The implied shape of the variance curve is consistent with market observations of having the non-flat term structures of the fair variance.

Finally, in Figure (4.4.5), left side, we show the volatility surface implied by SV+PJ+JD process across strike prices and maturity times (denoted by "Maturity" in the figure). The model implies a remarkable volatility skew for short-maturity options, then the skew flattens but remains persistent for longer maturities (which is a good sign).

In Figure (4.4.5), right side, we show the forward volatility surface, implied using Black-Scholes formula for forward-start call (4.3.24) from the corresponding model price, of one-year forward-start options with inception time at T (denoted by "Maturity" in the figure) and maturity at $T+1$. General remark is that the implied volatility of forward start options is higher than the corresponding implied volatility of standard options, because of uncertainty about volatilities of forward-start asset prices. A good sign is that the shape of the forward-volatility does not flatten out for out-of-the-money options and it remains stable across all maturities. These feature is very important by modeling complex path-dependent options, for example cliquet options, whose value functions depend on the evolution of the implied volatility surface in time.

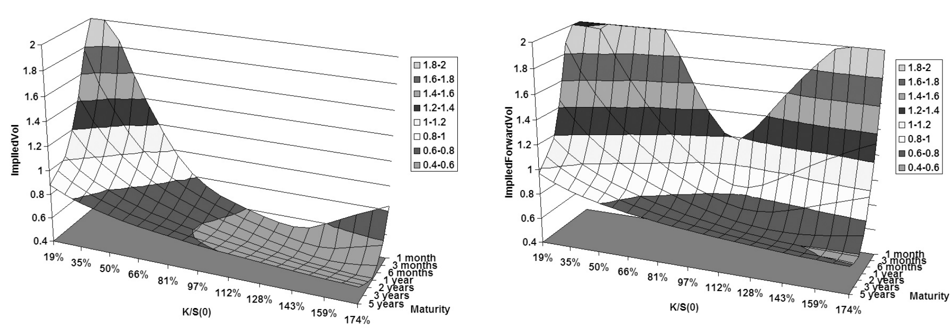

Figure 4.4.5: Left side: The volatility surface implied by SV+PJ+JD process. Right side: The volatility surface of forward-start options implied by SV+PJ+JD process

Part II: Pricing Problems with Barriers

In this part we will study pricing problems in the presence of barriers. In mathematical finance, these problems arise by dealing with path-dependent options, such as barrier options, and in the so-called structural models of the default. These problems are closely related to first passage times of stochastic processes and they are very challenging to deal with analytically. In this part we will mainly concentrate on the jump-diffusion with double-exponential jumps and our key solution tool is now the Laplace transform.

Chapter 5

Double-Barrier Options

This Chapter is based on the article "Analytical Pricing of Double-Barrier Options under a Double-Exponential Jump Diffusion Process: Applications of Laplace Transform" by Sepp (2004). In this chapter we will derive explicit formulas for pricing double (single) barrier and touch options with time-dependent rebates assuming that the asset price follows a double-exponential jump diffusion process. We will also consider incorporating time-dependent volatility.

By risk-neutral pricing, the value of a barrier option satisfies the generalized Black-Scholes equation with appropriate boundary conditions. We take the Laplace transform of this equation in time and solve it explicitly. Option price and risk parameters are computed via the numerical inversion of the corresponding solution. Numerical examples reveal that the pricing formulas are easy to implement and they result in accurate prices and risk parameters. Proposed formulas allow fast computing of skew-consistent prices of barrier and touch options.

The key contributions of this chapter are formula (5.4.2) for pricing double (single) barrier and formula (5.4.8) for pricing double (single) touch options with time-dependent rebates under a double-exponential jump diffusion. These formulas are inexpensive to compute and they allow to alleviate the problem of model calibration to market data. As a result, these formulas allow pricing and hedging barrier options consistently with the volatility skews observed in the markets.

5.1. Problem Formulation

The problem of pricing vanilla options consistent with the volatility skew has been attracting much attention in financial studies. Various models have been proposed to explain the skew effect. One of the well-accepted models are jump diffusions or, more generally, Lévy processes. Merton (1976) studied pricing of vanilla options on assets driven by jump diffusions with log-normally distributed price-jumps. Kou (2002) proposed a jump diffusion with log-double-exponential jumps. It turns out that this jump diffusion has a few appealing features which allow analytical pricing of barrier and lookback options. Significant papers in this direction include those of

Kou-Wang (2001, 2003), who worked out formulas for the distribution of the first exit time, single barriers, and floating strike lookback put options using memoryless property of the exponential distribution; and Lipton (2002), who derived pricing formulas for single barrier options relying on fluctuation identities. Boyarchenko-Levendorskii (2002) derived general pricing formulas for single barrier and touch options under a wide class of Lévy processes. The proposed approaches deal with the pricing problem in the Laplace space.

Here, we develop an alternative approach to the pricing problem of single and double barrier options and derive explicit pricing formulas for double-barrier and double-touch options with time-dependent rebates under a double-exponential jump diffusion process. Pricing formulas for single barriers and touches are obtained via a simplification of general formulas for double barrier options. Our approach is less probabilistic and deals with the PIDE satisfied by the option value value function and supplied with the appropriate boundary conditions directly by taking the Laplace transform of this PIDE in time and solving it analytically.

In this Chapter, we work with the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ augmented with filtration $\mathcal{F}(t)$ that supports standard Brownian motion $W(t)$ and Poisson process $N(t)$. $\mathbb{Q}$ is the risk-neutral martingale measure.

Throughout this Chapter, we assume that the SDE governing the underlying price under the risk-neutral measure $\mathbb{Q}$ is given by

$$dS(t)/S(t-) = (r - d - \gamma m^j)dt + (e^J - 1)dN(t) + \sigma dW(t),\ S(0) = S_0, \quad (5.1.1)$$

where r is a risk-free interest rate, d is a dividend yield, σ is the volatility, γ is intensity of Poison process $N(t)$. Upon arrival of jump in $N(t)$, price jump J is a random jump size with PDF $\varpi(J)$. We set $m^j = \mathbb{E}^{\mathbb{Q}}[e^J - 1]$ to make the discounted price process a martingale.

In this Chapter, we mainly consider the double-exponential PDF for jump sizes originally proposed by Kou (2002):

$$\varpi(J) = \varpi^-(J) + \varpi^+(J) = q^- \frac{1}{\eta^-} e^{\frac{1}{\eta^-}J} \mathbf{1}_{\{J<0\}} + q^+ \frac{1}{\eta^+} e^{-\frac{1}{\eta^+}J} \mathbf{1}_{\{J\geq 0\}} \quad (5.1.2)$$

where $1 > \eta^+ > 0$ and $\eta^- > 0$ are means of positive and negative jump sizes, respectively; constants q^+ and q^- represent the probabilities of positive and negative jumps, respectively, $q^+,\ q^- \geq 0,\ q^+ + q^- = 1$. Requirement that $\eta^+ < 1$ is needed to ensure that $\mathbb{E}^{\mathbb{Q}}[e^J] < \infty$ and $\mathbb{E}^{\mathbb{Q}}[S] < \infty$. We note that

$$\mathbb{E}^{\mathbb{Q}}[e^{\Phi J}] = \int_{-\infty}^{\infty} e^{\Phi J} \varpi(J) dJ = \frac{q^+}{1 - \Phi\eta^+} + \frac{q^-}{1 + \Phi\eta^-} \quad (5.1.3)$$

provided that $-\frac{1}{\eta^-} < \Re[\Phi] < \frac{1}{\eta^+}$.

A simple calculation yields

$$m^j = \int_{-\infty}^{\infty} [e^J - 1]\varpi(J) dJ = \frac{q^+}{1 - \eta^+} + \frac{q^-}{1 + \eta^-} - 1. \quad (5.1.4)$$

Merton (1976) proposed jump diffusion with lognormally distributed price jumps. In general, the solution method developed here for a double-exponential jump diffusion cannot be applied for a jump diffusion with normally-distributed jumps.

Based on the risk-neutral pricing theorem (1.2.1) and Feynman-Kac formula (1.4.3), the value function $F(t,S)$ of a European-style option then satisfies the so-called generalized Black-Scholes PIDE:

$$F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + (r - d - \gamma m^j) S F_S - rF + \gamma \int_{-\infty}^{\infty} [F(Se^J) - F(S)]\varpi(J)dJ = 0,$$
$$F(t,S) = \max\{\varphi[S-K], 0\} \tag{5.1.5}$$

where T is option maturity, K is strike price, and binary variable $\varphi = +1$ for a call option and $\varphi = -1$ for a put option.

In a double-barrier knock-out option, the contract becomes worthless if either of the barriers is reached before the option expiry date. If neither of the barriers is hit, the double-barrier knock-out call (put) pays off $\max\{S-K,0\}$ ($\max\{K-S,0\}$) at the expiry date. In a double-barrier knock-in option, one of the barriers must be reached before the expiry date, otherwise the option becomes worthless.

Let S_u and S_d be the up and down barrier levels, respectively. The value of a double-barrier knock-out option denoted by $F^{DB}(t,S)$ satisfies PIDE (5.1.5) for $S_d < S < S_u$ subject to the following boundary conditions for $0 \le t \le T$:

$$F^{DB}(T-t,S) = \phi_u^*(T-t),\ S \ge S_u;\ F^{DB}(T-t,S) = \phi_d^*(T-t),\ S \le S_d \tag{5.1.6}$$

where $\phi_u^*(t)$ and $\phi_d^*(t)$ are contract functions that determine payoffs if the corresponding up or down barrier is reached.

We usually have $\phi_u^*(t) = \phi_d^*(t) = 0$ for a standard double barrier option. Let R_u and R_d denote constant up and down rebates, respectively. If rebates are paid at the time when a barrier is hit, we have $\phi_u^*(t) = R_u$ and $\phi_d^*(t) = R_d$. If the payment is postponed until option maturity, we have $\phi_u^*(t) = e^{-rt}R_u$ and $\phi_d^*(t) = e^{-rt}R_d$.

We introduce new variables
$t \to \tau = T - t$, $S \to x = \ln\frac{S}{K}$, $S_{u,d} \to x_{u,d} = \ln\frac{S_{u,d}}{K}$, $\phi_{u,d}(t) = \frac{\phi_{u,d}^*(t)}{K}$,
and re-write the pricing PIDE (5.1.5) for $F^{DB}(t,S) \to V^{DB}(x,\tau) = \frac{1}{K}F^{DB}(t,S)$ as

$$-V_\tau^{DB} + \frac{1}{2}\sigma^2 V_{xx}^{DB} + \mu V_x^{DB} - rV^{DB} + \gamma \int_{-\infty}^{\infty} [V^{DB}(x+J) - V^{DB}(x)]\varpi(J)dJ = 0,$$
$$V^{DB}(0,x) = \max\{\varphi[e^x - 1], 0\},\ x_d < x < x_u;$$
$$V^{DB}(\tau,x) = \phi_u(\tau),\ x \ge x_u;\ V^{DB}(\tau,x) = \phi_d(\tau),\ x \le x_d, \tag{5.1.7}$$

where $\mu = r - d - \gamma m^j - \frac{1}{2}\sigma^2$.

A double-no-touch option pays off a constant amount of money if neither of the barriers is reached before the option expiry date. In opposite, a double-one-touch option pays off a fixed amount of money if either of the barriers is hit. The value of a double-touch option denoted by $F^{DT}(t,S)$ satisfies PIDE (5.1.5) subject to the boundary conditions (5.1.6) with the final payoff given by

$$F^{DT}(t,S) = K \tag{5.1.8}$$

where K is the contract payoff if neither of the barriers is hit. We usually have $K = 1$ without any rebates for double-no-touch options and $K = 0$ subject to rebates for double-one-touch options.

To allow pricing touch options with zero strike, we introduce variables $S \to y = \ln S$, $S_{u,d} \to y_{u,d} = \ln S_{u,d}$. The value of a double-one(no)-touch option $F^{DT}(t,S) \to V^{DT}(y,\tau)$ satisfies the following PIDE

$$\begin{aligned}
&-V_\tau^{DT} + \frac{1}{2}\sigma^2 V_{yy}^{DT} + \mu V_y^{DT} - rV^{DT} + \gamma\int_{-\infty}^{\infty}[V^{DT}(y+J) - V^{DT}(y)]\varpi(J)dJ = 0,\\
&V^{DT}(0,y) = K,\ y_d < y < y_u;\\
&V^{DT}(\tau,y) = \phi_u(\tau),\ y \geq y_u;\ V^{DT}(\tau,y) = \phi_d(\tau),\ y \leq y_d.
\end{aligned} \tag{5.1.9}$$

For pricing knock-in barrier options, we can use the following model-independent relationship

$$F_{in}(\varphi) = F_{vanilla}(\varphi) - F_{out}(\varphi) \tag{5.1.10}$$

where F_{in} is the value of double (single) barrier knock-in option, F_{out} is value of double (single) barrier knock-out option and $F_{vanilla}(\varphi)$ is value of vanilla option.

5.2. Pricing under the Black-Scholes model

We start with Black-Scholes model with constant volatility and solve the corresponding problem for pricing vanilla and barrier options in the Laplace space. We briefly review and generalize the framework proposed by Skachkov (2002) and then extend it to jump diffusion processes.

Below we will use the following characteristic equation:

$$\frac{1}{2}\sigma^2\xi^2 + \mu\xi - (r+p) = 0, \tag{5.2.1}$$

with $p \in \mathbb{R}^+$, which has two real roots ξ_1 and ξ_2:

$$\xi_{1,2} = \xi_{+,-} = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma^2(r+p)}}{\sigma^2}, \tag{5.2.2}$$

such that $\xi_2 < 0 < \xi_1$.

5.2.1. Vanilla Options

We set $\gamma = 0$ in PIDE (5.1.7) and omit the boundary conditions. Applying Laplace transform (1.5.7) in τ to the corresponding PDE defined for $x \in (-\infty, \infty)$, we obtain the following ODE for $U(p, x) = \mathcal{L}[V(\tau)](p)$:

$$\frac{1}{2}\sigma^2 U_{xx} + \mu U_x - (r+p)U = -\max\{\varphi[e^x - 1], 0\}. \tag{5.2.3}$$

The solution to this ODE is specified by the following proposition.

Proposition 5.1. *In the Laplace space, the value of a vanilla option is given by*

$$U(p,x) = \begin{cases} C_1 e^{\xi_1 x} + \frac{\varphi-1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x < 0 \\ C_2 e^{\xi_2 x} + \frac{\varphi+1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x \geq 0 \end{cases} \tag{5.2.4}$$

where

$$C_{1,2} = \frac{1}{\xi_1 - \xi_2}\left[\frac{\xi_{2,1}}{r+p} + \frac{1-\xi_{2,1}}{d+p}\right]. \tag{5.2.5}$$

Proof. First we solve the homogeneous ODE corresponding to Eq.(5.2.3). We guess that the solution has a form $U(p,x) = e^{\psi x}$. Plugging it into Eq.(5.2.3), we get the corresponding characteristic Eq. (5.2.1).

Now we find a particular solution for non-homogeneous ODE corresponding to (5.2.3) which holds if

$$\varphi\left[e^x - 1\right] > 0.$$

We guess that the solution corresponding to homogeneous ODE has a form $U^h(p,x) = ae^x + b$. Plugging it into Eq.(5.2.3) and equating terms of the guessed solution, we obtain that for a call option

$$a = \frac{1}{d+p} \text{ and } b = -\frac{1}{r+p}, \tag{5.2.6}$$

and for a put option

$$a = -\frac{1}{d+p} \text{ and } b = \frac{1}{r+p}. \tag{5.2.7}$$

Thus, the general solution has the form

$$U^g(p,x) = \begin{cases} C_1 e^{\psi_1 x} + C_3 e^{\psi_2 x} + \frac{\varphi-1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x \leq 0 \\ C_4 e^{\psi_1 x} + C_2 e^{\psi_2 x} + \frac{\varphi+1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x \geq 0 \end{cases}. \tag{5.2.8}$$

To proceed further, we consider the limiting behavior of the values of call and put options in time variable. We have

$$F(t,S) \sim e^{-r(T-t)}\frac{\varphi-1}{2}\left[Se^{(r-d)(T-t)} - K\right], \; S \to 0; \tag{5.2.9}$$

and

$$F(t,S) \sim e^{-r(T-t)}\frac{\varphi+1}{2}\left[Se^{(r-d)(T-t)}-K\right],\ S\to\infty. \tag{5.2.10}$$

For normalized option values in the Laplace space, formulas (5.2.9) and (5.2.10) correspond to the particular solution in formula (5.2.8). We require that the solution in Laplace space should behave according to (5.2.9) and (5.2.10). This implies that $C_3 = 0$ and $C_4 = 0$.

To determine C_1 and C_2, we require that $U(p,x)$ and $U_x(p,x)$ are continuous at point $x = 0$. This leads to a system of two equations:

$$\begin{cases} U(p,x)|_{x=0-} = U(p,x)|_{x=0+} \\ U_x(p,x)|_{x=0-} = U_x(p,x)|_{x=0+} \end{cases}.$$

Simple calculation yields the solution for C_1 and C_2 given by formula (5.2.5).

□

The option value in time variable is given by inverse Laplace transform (1.5.9):

$$F(t,S) = K\mathcal{L}^{(-1)}[U(p,x)]. \tag{5.2.11}$$

It is not difficult to find the original, which is exactly the Black-Scholes-Merton formula (1.6.1), using a table of Laplace transforms. Pricing formula (5.2.4) can serve as a benchmark for testing the quality of the numerical inversion of the Laplace transform by means of Stehfest algorithm (1.5.10), which is vital for our subsequent developments.

5.2.2. Barrier Options

Setting $\gamma = 0$ in PIDE (5.1.7) and applying the Laplace transform to the corresponding PDE, we obtain the boundary value problem for $U^{DB}(p,x) = \mathcal{L}[V^{DB}(\tau,x)]$:

$$\begin{aligned} &\frac{1}{2}\sigma^2 U^{DB}_{xx} + \mu U^{DB}_x - (r+p)U^{DB} = -\max\{\varphi[e^x-1],0\}, \\ &U^{DB}(p,x_u) = \overline{\phi_u},\ U^{DB}(p,x_d) = \overline{\phi_d} \end{aligned} \tag{5.2.12}$$

where $\overline{\phi_u} = \mathcal{L}[\phi_u(\tau)]$ and $\overline{\phi_d} = \mathcal{L}[\phi_d(\tau)]$.

We note that $\overline{\phi_u} = 0$ and $\overline{\phi_d} = 0$ for a standard barrier option. For rebates of exponential form $\phi_{u,d}(\tau) = e^{-\rho_{u,d}\tau}R_{u,d}$, the corresponding Laplace transform is given by $\overline{\phi}_{u,d} = \frac{R_{u,d}}{\rho_{u,d}+p}$.

The solution to ODE (5.2.12) can be represented as

$$U^{DB}(p,x) = U^u(p,x) + U^b(p,x) \tag{5.2.13}$$

where U^u is a solution of unbounded ODE (5.2.12) (the value of a vanilla option given by Eq. (5.2.4)) and U^b is a solution of homogeneous equation corresponding to ODE (5.2.12) with boundary conditions

$$\begin{cases} U^b(p,x_d) = \overline{\phi_d} - U^u(p,x_d) \\ U^b(p,x_u) = \overline{\phi_u} - U^u(p,x_u) \end{cases}. \tag{5.2.14}$$

For $x_d \le x \le x_u$, the solution to $U^b(p, x_d)$ has a form

$$U^b(p,x) = C_3 e^{\xi_1 x} + C_4 e^{\xi_2 x}. \qquad (5.2.15)$$

Plugging Eq.(5.2.15) into boundary conditions (5.2.14), we obtain a system of two equations to determine constants C_3 and C_4. Straightforward calculations yield the following result.

Proposition 5.2. *In the Laplace space, the value of double-barrier option is given by*

$$U^{DB}(p,x) = \begin{cases} (C_1 + C_3)e^{\xi_1 x} + C_4 e^{\xi_2 x} + \frac{\varphi-1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x < 0 \\ (C_2 + C_4)e^{\xi_2 x} + C_3 e^{\xi_1 x} + \frac{\varphi+1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x \ge 0 \end{cases} \qquad (5.2.16)$$

where

$$\begin{aligned} C_{3,4} = \pm \frac{1}{\chi}\Bigg(& -C_1 e^{\xi_1 x_d + \xi_{2,1} x_u} + C_2 e^{\xi_2 x_u + \xi_{2,1} x_d} - \frac{\varphi - 1}{2}\left[\frac{e^{x_d + \xi_{2,1} x_u}}{d+p} - \frac{e^{\xi_{2,1} x_u}}{r+p}\right] \\ & + \frac{\varphi+1}{2}\left[\frac{e^{x_u + \xi_{2,1} x_d}}{d+p} - \frac{e^{\xi_{2,1} x_d}}{r+p}\right] + e^{\xi_{2,1} x_u}\overline{\phi_d} - e^{\xi_{2,1} x_d}\overline{\phi_u}\Bigg), \\ \chi = {} & e^{\xi_1 x_d + \xi_2 x_u} - e^{\xi_1 x_u + \xi_2 x_d}, \end{aligned} \qquad (5.2.17)$$

and constants C_1, C_2 are given by (5.2.5).

The pricing problem for single barrier options can be solved via a simplification of the formula (5.2.16). In the presence of the down barrier, we let $x_u \to \infty$ and set $C_3 \equiv 0$. In the presence of the up barrier, we let $x_d \to -\infty$ and set $C_4 \equiv 0$.

The value of double-barrier option in time variable is computed via inversion of the Laplace transform (1.5.9):

$$F^{DB}(t,S) = K\mathcal{L}^{(-1)}[U^{DB}(p,x)]. \qquad (5.2.18)$$

Now, we solve the pricing problem for double touch options. Setting $\gamma = 0$ in PIDE (5.1.9) and applying the Laplace transform to the corresponding PDE, we obtain the boundary value problem for $U^{DT}(p,y) = \mathcal{L}[V^{DT}(y,\tau)]$:

$$\begin{aligned} & \frac{1}{2}\sigma^2 U^{DT}_{yy} + \mu U^{DT}_y - (r+p)U^{DT} = -K, \\ & U^{DT}(p, y_u) = \overline{\phi_u},\ U^{DT}(p, y_d) = \overline{\phi_d}. \end{aligned} \qquad (5.2.19)$$

We represent $U^{DT}(p,y)$ by analogy with Eq. (5.2.13), where we take $U^u(p,y) = \frac{K}{r+p}$. We omit details to obtain the solution.

Proposition 5.3. *In the Laplace space, the value of a double-one(no)-touch option is given by*

$$U^{DT}(p,y) = C_1 e^{\xi_1 y} + C_2 e^{\xi_2 y} + \frac{K}{r+p} \tag{5.2.20}$$

where

$$\begin{aligned} C_{1,2} &= \pm\frac{1}{\chi}\left(K\frac{e^{\xi_{2,1}y_u} - e^{\xi_{2,1}y_d}}{r+p} + e^{\xi_{2,1}y_d}\overline{\phi_u} - e^{\xi_{2,1}y_u}\overline{\phi_d}\right), \\ \chi &= e^{\xi_1 y_u + \xi_2 y_d} - e^{\xi_1 y_d + \xi_2 y_u}. \end{aligned} \tag{5.2.21}$$

If a touch option has only the down barrier, we let $y_u \to \infty$ and set $C_2 \equiv 0$. If a touch option has only the up barrier, we let $y_d \to -\infty$ and set $C_1 \equiv 0$.

We have tested the numerical inversion of formulas (5.2.16) and (5.2.20) with some alternative pricing formulas obtained via the method of images (see Lipton (2001) for an overview). The agreement between formulas is to the four decimals.

5.2.3. Illustration. European vs American Digital Options

We finish this section by illustrating the key difference between the so-called European and American digital options. Above we considered the double-no-touch option, also known as American digital option, which becomes worthless once the asset spot price hits either up or down barrier up to contract maturity time T and if no barrier is hit it pays its strike K. In opposite, European digital options becomes worthless only if the asset price at the contract maturity time T is either above the up barrier or below the down barrier and otherwise it pays its strike K.

Assuming risk-neutralized Black-Scholes dynamics (5.1.5) with no price jumps, the value function of the European digital option denoted by $U^{DD}(t,S;T,K)$ is computed as follows:

$$\begin{aligned} U^{DD}(t,S) &= e^{-(T-t)r}\mathbb{E}^{\mathbb{Q}}\left[K\mathbf{1}_{\{S_d<S(T)<S_u\}}|\mathcal{F}(t)\right] = Ke^{-(T-t)r}\left(\mathcal{N}(d_u) - \mathcal{N}(d_d)\right), \\ d_{u,d} &= \frac{\ln(S_{u,d}/S) + (r-d+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \end{aligned} \tag{5.2.22}$$

where $\mathcal{N}(x)$ is the CDF of standard normal random variable.

In Figure (5.2.1) we show value functions of American digital (DNT) (computed by inverting formula (5.2.20) using Stehfest algorithm (1.5.10) and European digital (DD) options as well as their terminal payoff function (PAYOFF) as functions of $S(0)$ assuming the Black-Scholes dynamics with $S_u = 1.3$, $S_d = 0.7$, $K = 1$, $T = 1$, $\sigma = 0.3$, $r = d = 0$, $\phi_u = \phi_d = 0$. We see that the American digital option is worth considerably less than its European version, since the latter contract does not terminate once the spot price is below or above the barriers and if the spot price is outside the range there is still a chance that the terminal price at maturity time T will be inside the range. A similar conclusion is also valid for other types of barrier options and this feature has profound pricing and hedging implications.

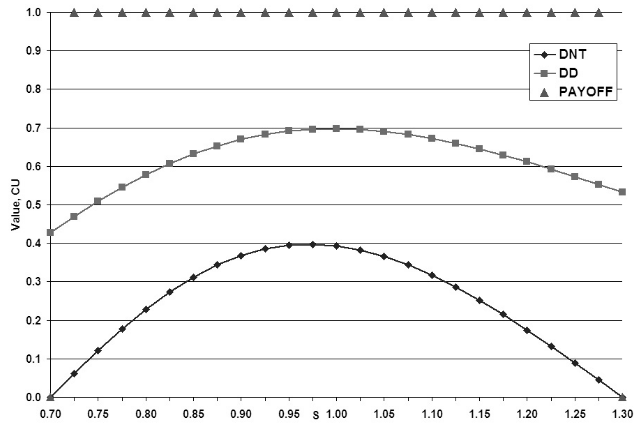

Figure 5.2.1: Values of American and European digital options and their payoff function

5.3. Pricing Vanilla Options under Double-Exponential Jump-Diffusion

We solve the pricing problem for barrier options under a double-exponential jump diffusion in a similar manner: first we solve the unbounded problem of vanilla options, and next we consider the bounded problem of barrier options.

At the beginning, we consider the pricing problem of vanilla options under jump diffusions and state an original formula (in Laplace space) for pricing vanilla options under a double-exponential jump diffusion. First, we will need the following lemma.

Lemma 4.1. *The equation*

$$\frac{1}{2}\sigma^2\psi^2 + \mu\psi - (r+p+\gamma) + \gamma\left[\frac{q^+}{1-\eta^+\psi} + \frac{q^-}{1+\eta^-\psi}\right] = 0 \qquad (5.3.1)$$

has four real roots $\psi_i, i = 0,1,2,3,$ *such that*

$$-\infty < \psi_3 < -\frac{1}{\eta^-} < \psi_2 < 0 < \psi_1 < \frac{1}{\eta^+} < \psi_0 < \infty. \qquad (5.3.2)$$

Proof. We consider Eq. (5.3.1) as a function $f(\psi)$. It follows that $f(\psi)$ is a convex function on interval $(-\frac{1}{\eta^-}, \frac{1}{\eta^+})$ with $f(0) = -(r+p) < 0$.
Since $\lim_{\psi\to(-\frac{1}{\eta^-})-} f(\psi) = +\infty$ and $\lim_{\psi\to(\frac{1}{\eta^+})-} f(\psi) = +\infty$, there is at least one root on the interval $(-\frac{1}{\eta^-}, 0)$ and another one on the interval $(0, \frac{1}{\eta^+})$.
Since $\lim_{\psi\to\frac{1}{\eta^+}+} f(\psi) = -\infty$ and $\lim_{\psi\to+\infty} f(\psi) = +\infty$, there is is at least one root

on $(\frac{1}{\eta^+}, \infty)$. By analogy, $\lim_{\psi\to-\infty} f(\psi) = -\infty$ and $\lim_{\psi\to(-\frac{1}{\eta^-})+} f(\psi) = +\infty$ imply that there is at least one root on the interval $(-\infty, -\frac{1}{\eta^-})$. Taking into the account that expression $(1-\eta^+\psi)(1+\eta^-\psi)f(\psi)$ represents a polynomial of the fourth order, it follows that the equation has exactly one root on each considered interval and the roots are real.

Finally, we consider expression $(1-\eta^+\psi)(1+\eta^-\psi)f(\psi)$ which yields polynomial (5.3.3). □

For computation purposes, it is better to represent Eq. (5.3.1) as the polynomial

$$\frac{1}{2}\sigma^2\eta^-\eta^+\psi^4 + (\mu\eta^-\eta^+ - \frac{1}{2}\sigma^2(\eta^- - \eta^+))\psi^3 - (\frac{1}{2}\sigma^2 + \mu(\eta^- - \eta^+) + (r+p+\gamma)\eta^-\eta^+)\psi^2$$
$$+ (-\mu + (r+p+\gamma)(\eta^- - \eta^+) - \gamma(q^+\eta^- - q^-\eta^+))\psi + (r+p) = 0. \tag{5.3.3}$$

Helpful algorithms for computing roots of this polynomial can be found, for example, in "Numerical recipes in C" [Press *et al* (1992)].

Taking the Laplace transform of the unbounded PIDE (5.1.7) and exchanging the order of integration, we obtain the following OIDE

$$\frac{1}{2}\sigma^2 U_{xx} + \mu U_x - (r+p+\gamma)U + \gamma\int_{-\infty}^{\infty}[U(x+J)]\varpi(J)dJ = -\max\{\varphi[e^x - 1], 0\} \tag{5.3.4}$$

defined for $x \in (-\infty, \infty)$.

The solution to the above equation is specified by the following proposition.

Proposition 5.4. *In the Laplace space, the value of a vanilla option under a double-exponential jump diffusion is given by*

$$U(p,x) = \begin{cases} C_0 e^{\psi_0 x} + C_1 e^{\psi_1 x} + \frac{\varphi-1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x < 0 \\ C_2 e^{\psi_2 x} + C_3 e^{\psi_3 x} + \frac{\varphi+1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], & x \geq 0 \end{cases} \tag{5.3.5}$$

where constants C_0, C_1, C_2, C_3 are solution of the system

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ \psi_0 & \psi_1 & -\psi_2 & -\psi_3 \\ \frac{1}{\psi_0\eta^-+1} & \frac{1}{\psi_1\eta^-+1} & -\frac{1}{\psi_2\eta^-+1} & -\frac{1}{\psi_3\eta^-+1} \\ \frac{1}{\psi_0\eta^+-1} & \frac{1}{\psi_1\eta^+-1} & -\frac{1}{\psi_2\eta^+-1} & -\frac{1}{\psi_3\eta^+-1} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{d+p} - \frac{1}{r+p} \\ \frac{1}{d+p} \\ \frac{1}{(d+p)(\eta^-+1)} - \frac{1}{r+p} \\ \frac{1}{(d+p)(\eta^+-1)} + \frac{1}{r+p} \end{pmatrix}. \tag{5.3.6}$$

Proof. A guess that the solution to the homogeneous OIDE (5.3.4) has the exponential form $U(p,x) = e^{\psi x}$ leads to the corresponding characteristic Eq. (5.3.1). We note that, although roots ψ_0 and ψ_3 do not satisfy the requirement of formula (5.1.3), the corresponding terms $e^{\psi_0 x}$ and $e^{\psi_3 x}$ are very important for solving OIDE (5.3.4). Below it will be clear that these terms also satisfy Eq. (5.3.1).

Now we introduce the operator

$$\Lambda(U) := \frac{1}{2}\sigma^2 U_{xx} + \mu U_x - (r+p+\gamma)U, \tag{5.3.7}$$

and define

$$\begin{aligned} U^-(p,x) &= \sum_{i=0}^{1} C_i e^{\psi_i x} + \frac{\varphi - 1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], \\ U^+(p,x) &= \sum_{i=2}^{3} C_i e^{\psi_i x} + \frac{\varphi + 1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right]. \end{aligned} \tag{5.3.8}$$

We make ansatz that $U(p,x)$ has the form

$$U(p,x) = U^-(p,x)\mathbf{1}_{\{x\le 0\}} + U^+(p,x)\mathbf{1}_{\{x\ge 0\}}. \tag{5.3.9}$$

We then determine constants C_j, $j = 0,1,2,3$ in such a way that OIDE (5.3.4) is satisfied. Two equations in system (5.3.6) arise from requiring the continuity of $U(p,x)$ and $U_x(p,x)$ at the point $x = 0$. Next we derive two additional equations.

We have two cases.

Firstly, if $x \le 0$. We consider the expectation

$$\begin{aligned} &\int_{-\infty}^{\infty} U(p,x+J)\varpi(J)dJ = \int_{-\infty}^{0} U(p,x+J)\varpi^-(J)dJ + \int_{0}^{\infty} U(p,x+J)\varpi^+(J)dJ \\ &= \int_{-\infty}^{0} U^-(p,x+J)\varpi^-(J)dJ + \int_{0}^{-x} U^-(p,x+J)\varpi^+(J)dJ + \int_{-x}^{\infty} U^+(p,x+J)\varpi^+(J)dJ \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Straightforward calculations using PDF (5.1.2) yield

$$I_1 = q^- \sum_{i=0}^{1} \frac{C_i e^{\psi_i x}}{\eta^- \psi_i + 1} + q^- \frac{\varphi - 1}{2}\left[\frac{e^x}{(d+p)(1+\eta^-)} - \frac{1}{r+p}\right]$$

provided that $\psi_0 > -\frac{1}{\eta^-}$ and $\psi_1 > -\frac{1}{\eta^-}$, which is satisfied,

$$I_2 = -q^+ \sum_{i=0}^{1} \frac{C_i e^{\psi_i x}}{\eta^+ \psi_i - 1} + q^+ \sum_{i=0}^{1} \frac{C_i e^{\frac{1}{\eta^+}x}}{\eta^+ \psi_i - 1} + q^+ \frac{\varphi - 1}{2}\left[\frac{e^{\frac{1}{\eta^+}x} - e^x}{(d+p)(\eta^+ - 1)} - \frac{1 - e^{\frac{1}{\eta^+}x}}{r+p}\right].$$

and finally

$$I_3 = -q^+ \sum_{i=2}^{3} \frac{C_i e^{\frac{1}{\eta^+}x}}{\eta^+ \psi_i - 1} - q^+ \frac{\varphi + 1}{2}\left[\frac{e^{\frac{1}{\eta^+}x}}{(d+p)(\eta^+ - 1)} + \frac{e^{\frac{1}{\eta^+}x}}{r+p}\right]$$

provided that $\psi_2 < \frac{1}{\eta^+}$ and $\psi_3 < \frac{1}{\eta^+}$, which is satisfied.

We make terms with $e^{\frac{1}{\eta^+}x}$ vanish which yields the fourth equation in system (5.3.6).

Next we consider the final expression

$$\Lambda(U^-) + \gamma(I_1 + I_2 + I_3). \tag{5.3.10}$$

It follows that terms with C_0 and C_1 vanish due to the characteristic Eq. (5.3.1).

We then verify that the remaining terms in (5.3.10) represent the right side of OIDE (5.3.4), which is indeed the case.

Secondly, if $x > 0$. We consider the expectation

$$\int_{-\infty}^{\infty} U(p, x+J)\varpi(J)dJ = \int_{-\infty}^{0} U(p, x+J)\varpi^-(J)dJ + \int_{0}^{\infty} U(p, x+J)\varpi^+(J)dJ$$
$$= \int_{-\infty}^{-x} U^-(p, x+J)\varpi^-(J)dJ + \int_{-x}^{0} U^+(p, x+J)\varpi^-(J)dJ + \int_{0}^{\infty} U^+(p, x+J)\varpi^+(J)dJ.$$

We calculate these integrals using PDF (5.1.2) and apply the same considerations to make vanish the terms with $e^{-\frac{1}{\eta^-}x}$ and those with C_2, C_3. As a result, we derive the third equation in system (5.3.6).

As a result, the representation (5.3.9) is valid and we obtain the system (5.3.6) for determination of constants C_0, C_1, C_2, and C_3.

□

Intuitively, the number of terms in the formula (5.3.5) is right. In the presence of positive jumps, it is clear that if $x < 0$ the point $x = 0$ can be crossed due to a positive jump, so it is necessary to account for such possibility. Let us consider the case when the jump amplitude becomes very small: $\eta^+ \to 0$ so that $\psi_0 \to \infty$. It is understandable that in this case the possibility is negligible and, accordingly, the corresponding term $C_0 e^{\psi_0 x}$ in formula (5.3.5) also becomes negligible. The same consideration applies for negative jumps in case $x > 0$.

The solution to system (5.3.6) can be computed using either the celebrated Cramer's rule or a purely numerical algorithm. We refer to "Numerical recipes in C" [Press *et al* (1992)] for helpful algorithms.

Finally, we consider some limiting cases of double-exponential jumps. In the presence of only positive jumps we let $\eta^- \to 0$, $\psi_3 \to -\infty$ and set $C_3 = q^- \equiv 0$. As a result, the third equation in system (5.3.6) vanishes. In the presence of only negative jumps we let $\eta^+ \to 0$, $\psi_0 \to \infty$ and set $C_0 = q^+ \equiv 0$. As a result, the fourth equation in system (5.3.6) vanishes. When $\eta^+ \to 0$, $\psi_0 \to \infty$, $\eta^- \to 0$ $\psi_3 \to -\infty$, the formula (5.3.5) reduces to the formula (5.2.4) which we use for pricing vanillas under constant volatility. The same is true if $\gamma \to 0$.

5.4. Pricing Barrier Options under Double-Exponential Jump-Diffusion

Here, we state the key contribution of this Chapter - an analytical formula in the Laplace space for pricing double-barrier options under a double-exponential jump-diffusion process.

The pricing problem for double-barrier options in the presence of double-exponential jumps has the following representation in Laplace space:

$$\begin{aligned}&\frac{1}{2}\sigma^2 U^{DB}_{xx} + \mu U^{DB}_{x} - (r+p+\gamma)U^{DB} + \gamma\int_{-\infty}^{\infty}[U^{DB}(x+J)]\varpi(J)dJ = \\ &= -\max\{\varphi[e^x-1],0\},\ x_d < x < x_d; \\ &U^{DB}(p,x) = \overline{\phi_d},\ x \le x_d;\ U^{DB}(p,x) = \overline{\phi_d},\ x \ge x_u,\end{aligned} \tag{5.4.1}$$

where the jump size PDF $\varpi(J)$ is defined by Eq. (5.1.2).

The solution to Eq.(5.4.1) is specified by the following proposition.

Proposition 5.5. *In the Laplace space, the value of a double-barrier knock-out option under a double-exponential jump diffusion is given by formula*

$$U^{DB}(p,x) = \begin{cases}(C_0+C_4)e^{\psi_0 x} + (C_1+C_5)e^{\psi_1 x} + C_6 e^{\psi_2 x} + C_7 e^{\psi_3 x} + \frac{\varphi-1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], x<0 \\ (C_2+C_6)e^{\psi_2 x} + (C_3+C_7)e^{\psi_3 x} + C_4 e^{\psi_0 x} + C_5 e^{\psi_1 x} + \frac{\varphi+1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right], x\ge 0\end{cases} \tag{5.4.2}$$

where constants $C_j,\ j=0,..,3,$ are solution of system (5.3.6) and constants $C_j,\ j=4,..,7,$ are solution of system

$$\begin{pmatrix} \frac{e^{\psi_0 x_d}}{\psi_0\eta^- +1} & \frac{e^{\psi_1 x_d}}{\psi_1\eta^- +1} & \frac{e^{\psi_2 x_d}}{\psi_2\eta^- +1} & \frac{e^{\psi_3 x_d}}{\psi_3\eta^- +1} \\ e^{\psi_0 x_d} & e^{\psi_1 x_d} & e^{\psi_2 x_d} & e^{\psi_3 x_d} \\ e^{\psi_0 x_u} & e^{\psi_1 x_u} & e^{\psi_2 x_u} & e^{\psi_3 x_u} \\ \frac{e^{\psi_0 x_u}}{\psi_0\eta^+ -1} & \frac{e^{\psi_1 x_u}}{\psi_1\eta^+ -1} & \frac{e^{\psi_2 x_u}}{\psi_2\eta^+ -1} & \frac{e^{\psi_3 x_u}}{\psi_3\eta^+ -1}\end{pmatrix}\begin{pmatrix}C_4\\C_5\\C_6\\C_7\end{pmatrix}$$
$$= \begin{pmatrix} -\frac{\varphi-1}{2}\left(\frac{e^{x_d}}{(d+p)(\eta^-+1)} - \frac{1}{r+p}\right) + \overline{\phi_d} - \frac{e^{\psi_0 x_d}}{\psi_0\eta^-+1}C_0 - \frac{e^{\psi_1 x_d}}{\psi_1\eta^-+1}C_1 \\ -\frac{\varphi-1}{2}\left(\frac{e^{x_d}}{(d+p)} - \frac{1}{r+p}\right) + \overline{\phi_d} - e^{\psi_0 x_d}C_0 - e^{\psi_1 x_d}C_1 \\ -\frac{\varphi+1}{2}\left(\frac{e^{x_u}}{(d+p)} - \frac{1}{r+p}\right) + \overline{\phi_u} - e^{\psi_2 x_u}C_2 - e^{\psi_3 x_u}C_3 \\ -\frac{\varphi+1}{2}\left(\frac{e^{x_u}}{(d+p)(\eta^+-1)} + \frac{1}{r+p}\right) - \overline{\phi_u} - \frac{e^{\psi_2 x_u}}{\psi_2\eta^+-1}C_2 - \frac{e^{\psi_3 x_u}}{\psi_3\eta^+-1}C_3\end{pmatrix}. \tag{5.4.3}$$

Proof. We present the solution of the value of a double-barrier option as a superposition of the unbounded solution $U^u(p,x)$ for the vanilla option given by formula (5.3.5) and the bounded solution denoted by $U^b(p,x)$ as

$$U^{DB}(p,x) = U^u(p,x) + U^b(p,x) \tag{5.4.4}$$

where

$$U^b(p,x) = \begin{cases} \overline{\phi_d} - U^u(p,x), \ x \le x_d \\ \sum_{i=4}^{7} C_i e^{\psi_{i-4}x}, \ x_d < x < x_u \\ \overline{\phi_u} - U^u(p,x), \ x \ge x_u \end{cases} . \tag{5.4.5}$$

We determine coefficients $C_i, i = 4,..,7$, in such a way that boundary conditions are matched and $U^b(p,x)$ satisfies the homogeneous OIDE (5.4.1).

First we consider the expectation

$$\int_{-\infty}^{\infty} U^b(p,x+J)\varpi(J)dJ = \int_{-\infty}^{0} U^b(p,x+J)\varpi^-(J)dJ + \int_{0}^{\infty} U^b(p,x+J)\varpi^+(J)dJ$$

$$= \int_{-\infty}^{x_d-x} [\overline{\phi_d} - U^u(p,x+J)]\varpi^-(J)dJ + \int_{x_d-x}^{0} \sum_{i=4}^{7} C_i e^{\psi_{i-4}(x+J)}\varpi^-(J)dJ$$

$$+ \int_{0}^{x_u-x} \sum_{i=4}^{7} C_i e^{\psi_{i-4}(x+J)}\varpi^+(J)dJ + \int_{x_u-x}^{\infty} [\overline{\phi_u} - U^u(p,x+J)]\varpi^+(J)dJ.$$

Next we calculate above integrals using PDF (5.1.2). We note that $U^u(p,x)$ for $x \le x_d$ is given by formula (5.3.5) for a vanilla option with $x \le 0$ and $U^u(p,x)$ for $x \ge x_u$ is given by formula (5.3.5) with $x > 0$. The final expression contains terms with $e^{\frac{1}{\eta^-}(x_d-x)}$ and $e^{-\frac{1}{\eta^+}(x_u-x)}$. We make them vanish by equating sums of corresponding coefficients to zero. As a result, we obtain the first and fourth equation in system (5.4.3), respectively.

Next we consider the sum of $\Lambda(U^b)$, where operator Λ is defined in (5.3.7), with remaining terms of the above integrals. It then follows that all remaining terms with $C_i, \ i = 4,..,7$, vanish due to characteristic Eq. (5.3.1).

The second and third equation in matrix (5.4.3) arise from the down and up boundary condition of the barrier option, respectively. Thus, representation (5.4.4) is valid and we obtained system (5.4.3) for determination of constants $C_i, \ i = 4,..,7$. □

Intuitively, the number of terms in the formula (5.4.2) is right. A jump diffusion process can cross the down barrier due to a negative jump in both cases $x < 0$ and $x \ge 0$, so we have one term to account for such possibility and another term to account for crossing the barrier continuously. It is clear that these terms should vanish when $x \to \infty$, so we have to add in our formula the corresponding terms $e^{\psi_2 x}$ and $e^{\psi_3 x}$. It is also clear that the term $e^{\psi_3 x}$ is related to negative jumps and it must vanish when the jump size magnitude becomes negligible, that is, when $\eta^- \to 0$ and, accordingly, $\psi_3 \to -\infty$. The term $e^{\psi_2 x}$ is related to the regular diffusion and it is needed to account for crossing the down barrier continuously. The same considerations apply for the up barrier.

We also consider some limiting cases of the formula (5.4.2). For the single down barrier, we let $x_u \to \infty$ and set $C_4 = C_5 \equiv 0$. As a result, two first equations in system (5.4.3) vanish. For the single up barrier, we let $x_d \to -\infty$ and set $C_6 = C_7 \equiv 0$. As a result, two last equations in system (5.4.3) vanish.

In the presence of only positive jumps, we let $\eta^- \to 0$, $\psi_3 \to -\infty$ and set $C_3 = C_7 = q^- \equiv 0$. As a result, the first equation in system (5.4.3) vanishes. In presence of only negative jumps, we let $\eta^+ \to 0$, $\psi_0 \to \infty$ and set $C_0 = C_4 = q^+ \equiv 0$. As a result, the fourth equation in system (5.4.3) vanishes. When $\eta^+ \to 0$ and $\eta^- \to 0$, pricing formula (5.4.2) reduces to the formula (5.2.16) which we use for pricing barriers under constant volatility. The same is true when $\gamma \to 0$.

To derive the formula for the value of a double-barrier knock-in option denoted by $U^{DBKI}(p,x)$, we use relationship (5.1.10) as well as formulas (5.3.5) and (5.4.2). As a result, we obtain the following formula for $x \in (x_d, x_u)$:

$$U^{DBKI}(p,x) = -\left(C_4 e^{\psi_0 x} + C_5 e^{\psi_1 x} + C_6 e^{\psi_2 x} + C_7 e^{\psi_3 x}\right). \tag{5.4.6}$$

Finally, we solve the pricing problem for double-touch options. Taking the Laplace transform of Eq.(5.1.9), we obtain the following OIDE for $U^{DT}(p,y) = \mathcal{L}[V^{DT}(\tau,y)]$:

$$\begin{gathered}\frac{1}{2}\sigma^2 U^{DT}_{yy} + \mu U^{DT}_y - (r+p+\gamma)U^{DT} + \gamma \int_{-\infty}^{\infty} U^{DT}(y+J)\varpi(J)dJ = -K, \\ y_d < y < y_u;\ U^{DT}(p,y) = \overline{\phi}_u,\ y \geq y_u;\ U^{DT}(p,y) = \overline{\phi}_d,\ y \leq y_d.\end{gathered} \tag{5.4.7}$$

The solution is specified by the following proposition.

Proposition 5.6. *In the Laplace space, the value of a double-touch option under a double-exponential jump diffusion is given by formula*

$$U^{DT}(p,y) = C_4 e^{\psi_0 y} + C_5 e^{\psi_1 y} + C_6 e^{\psi_2 y} + C_7 e^{\psi_3 y} + \frac{K}{r+p} \tag{5.4.8}$$

where constants C_4, C_5, C_6, C_7 are solution of system

$$\begin{pmatrix} \frac{e^{\psi_0 y_d}}{\psi_0\eta^- +1} & \frac{e^{\psi_1 y_d}}{\psi_1\eta^- +1} & \frac{e^{\psi_2 y_d}}{\psi_2\eta^- +1} & \frac{e^{\psi_3 y_d}}{\psi_3\eta^- +1} \\ e^{\psi_0 y_d} & e^{\psi_1 y_d} & e^{\psi_2 y_d} & e^{\psi_3 y_d} \\ e^{\psi_0 y_u} & e^{\psi_1 y_u} & e^{\psi_2 y_u} & e^{\psi_3 y_u} \\ \frac{e^{\psi_0 y_u}}{\psi_0\eta^+ -1} & \frac{e^{\psi_1 y_u}}{\psi_1\eta^+ -1} & \frac{e^{\psi_2 y_u}}{\psi_2\eta^+ -1} & \frac{e^{\psi_3 y_u}}{\psi_3\eta^+ -1} \end{pmatrix} \begin{pmatrix} C_4 \\ C_5 \\ C_6 \\ C_7 \end{pmatrix} = \begin{pmatrix} -\frac{K}{r+p} + \overline{\phi}_d \\ -\frac{K}{r+p} + \overline{\phi}_d \\ -\frac{K}{r+p} + \overline{\phi}_u \\ \frac{K}{r+p} - \overline{\phi}_u \end{pmatrix}. \tag{5.4.9}$$

Proof. We consider the bounded problem for $U^{DT}(p,y)$ which has the representation given by formula (5.4.4) with $U^u(p,y) = \frac{K}{r+p}$ and $x \equiv y$. Next we apply the same consideration as in Appendix A.4. and derive system (5.4.9).

□

We can treat limiting cases of the formula (5.4.8) by analogy with the aforementioned considerations for double-barrier options.

5.5. Pricing under Jump-Diffusion with Time-Dependent Volatility

Besides the skew effect, all derivative markets exhibit the term structure of implied volatility which indicates that at-the-money (ATM) implied volatility depends on option maturity time. Accordingly, the introduction of time-dependent volatility (or the term structure of model parameters) makes the pricing model more realistic. It is quite difficult to introduce stochastic volatility for analytical pricing of barrier options (see a survey by Lipton-McGhee (2002)), but time-dependent volatility can be treated analytically in some cases.

We assume that under risk-neutral measure $\mathbb{Q}$ the asset price is driven by the following SDE:

$$\begin{cases} dS(t)/S(t-) = (r - d - \widehat{\gamma} v(t) m^j)dt + (e^J - 1)dN(t) + \sqrt{v(t)}dW(t), \ S(0) = S_0; \\ dv(t) = \kappa(v_\infty - v(t))dt, \ v(0) = v_0 \end{cases} \tag{5.5.1}$$

where v_0 is a spot variance, v_∞ is a long-term variance, and κ is a reversion speed to the long-term variance, $N(t)$ is a Poisson process with time-dependent intensity $\widehat{\gamma} v(t)$, $\widehat{\gamma}$ is a leverage coefficient.

We emphasize that in case of deterministic variance it is important to let the jump intensity be proportional with the variance. Integration of the variance dynamics yields:

$$v(t) = v_\infty + (v_0 - v_\infty)e^{-\kappa t}. \tag{5.5.2}$$

Now, the pricing PIDE for the value of a barrier option becomes

$$F_t + \frac{1}{2}v(t)S^2 F_{SS} + (r - d - \widehat{\gamma} m^j v(t))SF_S - rF + \widehat{\gamma} v(t) \int_{-\infty}^{\infty} [F(Se^J) - F(S)]\varpi(J)dJ = 0,$$
$$F(t, S) = \max\{\varphi[S - K], 0\}.$$

subject to boundary conditions (5.1.6).

For subsequent analysis, we need to remove terms which are not proportional with the variance. We introduce new variables $\tau = T - t$, $\nu = r - d$, $\widehat{S} = Se^{\nu\tau}$, $\widehat{S}_{u,d} = S_{u,d}e^{\nu\tau}$, $\widehat{v}(\tau) = v(T - \tau)$, $\phi^*_{u,d}(T - t) = e^{-r\tau}\phi^\circ_{u,d}(\tau)$, and re-write PIDE for $F(t, S) = e^{-r\tau}\Psi(\widehat{S}, \tau)$ as

$$-\Psi_\tau + \frac{1}{2}\widehat{v}(\tau)\widehat{S}^2\Psi_{\widehat{S}\widehat{S}} - \widehat{\gamma} m^j \widehat{v}(\tau)\widehat{S}\Psi_{\widehat{S}} + \widehat{\gamma}\widehat{v}(\tau) \int_{-\infty}^{\infty} [\Psi(\widehat{S}e^J) - \Psi(\widehat{S})]\varpi(J)dJ = 0,$$
$$\Psi(0, \widehat{S}) = \max\{\varphi[\widehat{S} - K], 0\}, \ \widehat{S}_d < \widehat{S} < \widehat{S}_u;$$
$$\Psi(\tau, \widehat{S}) = \phi^\circ_u(\tau), \ \widehat{S} \geq \widehat{S}_u; \ \Psi(\tau, \widehat{S}) = \phi^\circ_d(\tau), \ \widehat{S} \leq \widehat{S}_d.$$

It is important to notice that now we have time-dependent barrier levels. It is known that we can remove non-dimensionalize this problem by introducing $\widehat{\tau} = \int_0^\tau \widehat{v}(s)ds$ [Lipton (2001)]. We calculate the corresponding integral and introduce

$$\widehat{\tau} = v_\infty \tau + \frac{v_0 - v_\infty}{\kappa} - (\frac{v_0 - v_\infty}{\kappa})e^{-\kappa\tau}, \text{ and } \widehat{\Psi}(\widehat{S}, \widehat{\tau}) = \Psi(\widehat{S}, \tau).$$

Using expression (5.5.2), we deduce that $\widehat{\Psi}(\widehat{S}, \widehat{\tau})$ must satisfy

$$-\widehat{\Psi}_{\widehat{\tau}} + \frac{1}{2}\widehat{S}^2\widehat{\Psi}_{\widehat{S}\widehat{S}} - \widehat{\gamma}m^j\widehat{S}\widehat{\Psi}_{\widehat{S}} + \widehat{\gamma}\int_{-\infty}^{\infty}[\widehat{\Psi}(\widehat{S}e^J) - \widehat{\Psi}(\widehat{S})]\varpi(J)dJ = 0 \tag{5.5.3}$$

subject to the corresponding initial and boundary conditions.

This is PIDE with constant coefficients and we have already developed the method for solving it. Finally, we introduce $x = \ln\frac{\widehat{S}}{K}$, $x_{u,d}(\widehat{\tau}) = \ln\frac{\widehat{S}_{u,d}}{K}$, and re-write the pricing PIDE for $\widehat{\Psi}(\widehat{\tau}, \widehat{S}) \to \Xi(\widehat{\tau}, x)$ as

$$\begin{aligned}
&-\Xi_{\widehat{\tau}} + \frac{1}{2}\Xi_{xx} + (-\widehat{\gamma}m^j - \frac{1}{2})\Xi_x + \widehat{\gamma}\int_{-\infty}^{\infty}[\Xi(x+J) - \Xi(x)]\varpi(J)dJ = 0,\\
&\Xi(0,x) = \max\{\varphi[e^x - 1], 0\},\ x_d < x < x_u;\\
&\Xi(\widehat{\tau}, x) = \widehat{\phi}_u(\widehat{\tau}),\ x \geq x_u(\widehat{\tau});\ \Xi(\widehat{\tau}, x) = \widehat{\phi}_d(\widehat{\tau}),\ x \leq x_d(\widehat{\tau}).
\end{aligned} \tag{5.5.4}$$

The solution of the unbounded PIDE (5.5.4) in Laplace space $\overline{\Xi}(p,x) = \mathcal{L}[\Xi(x,\widehat{\tau})]$ for pricing vanilla options under a double-exponential jump diffusion process with deterministic volatility and jump intensity is given by formula (5.3.5) where all expressions $d+p$ and $r+p$ have to be replaced by p and the corresponding variables have to be used.

However, we can solve bounded PIDE (5.5.4) for pricing barrier and touch options only if $r - d = 0$ because only in this case we have fixed (or flat) boundaries. In this case the general solution is given by formula (5.4.2) where all expressions $d+p$ and $r+p$ have to be replaced by p and the corresponding variables have to be used. In opposite case, we have moving boundaries which are rather difficult to handle analytically.

The value of a vanilla or barrier option is computed via numerical inversion of the Laplace transform (5.3.5) or (5.4.2) with corresponding variables:

$$F(t,S) = e^{-r\tau}K\mathcal{L}^{(-1)}[\overline{\Xi}(p,x)](\widehat{\tau}).$$

To illustrate the implied volatility surface of the double-exponential jump-diffusion with time-dependent volatility, we calibrated it to General Motors data used in Chapter 4. We obtained the following estimates: $v_0 = 0.4553$, $v_\infty = 0.01$, $\kappa = 9.905$, $\widehat{\gamma} = 17.06$, $\eta^+ = 0.008$, $\eta^- = 0.689$, $q^+ = 0.0004$. It follows that the spot intensity equals $\gamma_0 = \widehat{\gamma}v_0 = 7.76$ and long-term intensity equals $\gamma_\infty = \widehat{\gamma}v_\infty = 0.17$. The corresponding model implied volatility surface is shown in Figure (5.5.1).

It follows that the model can not be calibrated to the whole volatility surface. Although it can explain the volatility skew for a single maturity it lacks the power to explain the volatility skew across all maturities. Its long-term maturity is pretty flat. This is a common feature of all time-homogeneous jump-diffusions and Levý processes.

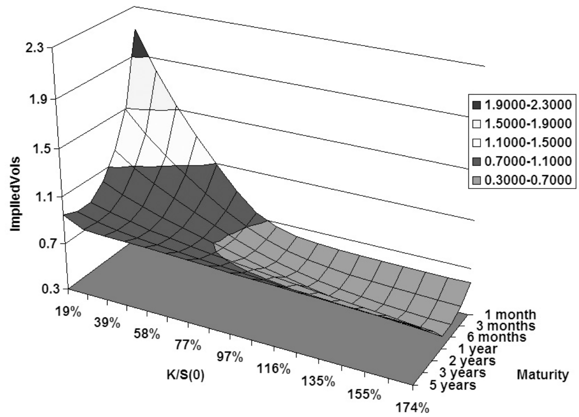

Figure 5.5.1: The volatility surface implied by double-exponential jump-diffusion

5.6. Alternative Jump Size Distributions

Here, we consider some alternative PDFs for modeling jumps. First, we note that our considerations can be applied to a mixture of positive and negative exponential jumps or the hyperexponential jump size PDF considered by Lipton (2002). In general, by pricing vanilla options, each additional term in the mixture of exponential jumps will increase the order of characteristic polynomial (5.3.3) and the number of equations in system (5.3.6) by one.

5.6.1. Analysis of Normally Distributed Jumps

Merton (1976) proposed jump-diffusion model where the logarithmic jump size is normally distributed with mean ν, volatility δ and PDF $\varpi(J) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(J-\nu)^2}{2\delta^2}}$.

The moment generating function is given by

$$\mathbb{E}[e^{\Phi J}] = e^{\nu\Phi + \frac{1}{2}\delta^2\Phi^2}.$$

By analogy, we obtain the following characteristic equation

$$\frac{1}{2}\sigma^2\psi^2 + \mu\psi - (r + p + \gamma) + \gamma e^{\nu\psi + \frac{1}{2}\delta^2\psi^2} = 0.$$

It is very difficult to analyze roots of this equation which is very important in our settings. Accordingly, the pricing problem in Laplace space becomes rather involved. We note that Merton (1976) derived a closed-form formula for pricing vanilla options under a jump diffusion with lognormally distributed price-jumps.

5.6.2. Comparison of Jump Size Distributions

To compare jump diffusions with double-exponential jumps and normally distributed jumps, we calibrated these two jump-diffusion models to General Motors implied volatility of options with maturity time $T = 0.2$ (about two and a half months). For this maturity, the market implied volatility depicts very pronounced skew effect. After calibration, both models are able to fit this skew leading to almost perfect replication of implied volatilities across all strike levels. We obtained the following estimates for jump diffusion process with
1) double-exponential jumps: $\sigma = 0.4138$, $\gamma = 1.2602$, $\eta^+ = 0.0076$, $\eta^- = 0.5304$, $q^+ = 0.0004$;
2) normally distributed jumps: $\sigma = 0.4036$, $\gamma = 0.8100$, $\nu = -0.6995$, $\delta = 0.5221$.

Based on these estimates it follows that both models imply sizable negative jumps. However, the double-exponential model has larger jump intensity. It is also interesting to have a look at logarithmic jump size distributions implied by model parameters. The corresponding densities are shown in Figure (5.6.1) only for negative values. We also multiply these densities by the corresponding jump rate intensity rates. We see that double-exponential jump-diffusion implies a large number of small jumps while it has a bigger right tail so that the expected probability of observing large negative jumps is higher.

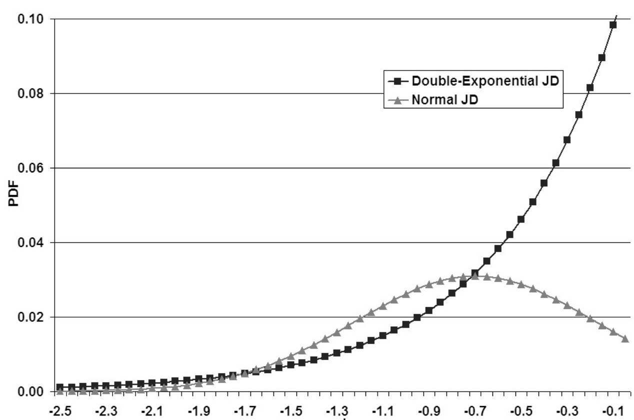

Figure 5.6.1: PDF-s of jump amplitudes implied by normal and double-exponential jump-diffusions

5.7. Numerical Results

As a numerical illustration, we shall present a few examples in Table (5.7.1) using the numerical inversion (NIL) of our derived formulas. We will also make a comparison with the Monte Carlo simulation. We use $K = 100$, $T = 1$, $t = 0$, $S_d = 80$,

$S_u = 120$, $\sigma = 0.2$, $r = 0.05$, $d = 0.02$, $\eta^+ = 0.1$, $\eta^- = 0.1$, $q^+ = 0.5$, no rebates, and different levels of spot S_0 and jump intensity rate γ for calculating prices of vanilla call (VC), double-barrier knock-out call (DBOC), double-barrier knock-in put (DBIP) and double-no-touch (DNOT) option with strike $K = 1$.

S_0	γ	VC		DBOC	
		NIL	Monte-Carlo	NIL	Monte-Carlo
90	0	4.3599	4.4316 (4.2999, 4.5633)	0.8283	0.8857 (0.8465, 0.9249)
	3	8.2049	8.2118 (7.9523, 8.4712)	0.3668	0.4061(0.3787, 0.4334)
	5	10.2478	10.3144 (9.9770, 10.6518)	0.2156	0.2528 (0.2307, 0.2749)
100	0	9.2270	9.2668 (9.0757, 9.4580)	1.0730	1.1112 (1.0681, 1.1544)
	3	13.3505	13.4221 (13.0885, 13.7557)	0.4743	0.4882 (0.4582, 0.5181)
	5	15.5462	14.8966 (14.4876, 15.3056)	0.2796	0.3084 (0.2844, 0.3323)
110	0	15.9613	15.8738 (15.6262, 16.1214)	0.6957	0.7422 (0.7061, 0.7782)
	3	19.7860	19.7516 (19.3493, 20.1539)	0.3309	0.3563 (0.3311, 0.3815)
	5	21.9267	22.5155 (22.0112, 23.0198)	0.2028	0.2197 (0.1996, 0.2398)
S_0	γ	DBIP		DNOT	
		NIL	Monte-Carlo	NIL	Monte-Carlo
90	0	9.9134	9.8088 (9.6407, 9.9769)	0.2940	0.3023 (0.2961, 0.3084)
	3	14.4758	14.2587 (14.0331, 14.4842)	0.1317	0.1380 (0.1334, 0.1427)
	5	16.7679	16.8082 (16.5588, 17.0577)	0.0780	0.0829 (0.0792, 0.0867)
100	0	4.7698	4.8320 (4.7029, 4.9611)	0.3579	0.3668 (0.3574, 0.3702)
	3	9.6648	9.7815 (9.5832, 9.9799)	0.1667	0.1750 (0.1699, 0.1801)
	5	12.1596	12.1392 (11.9127, 13.3657)	0.1000	0.1048 (0.1007, 0.1089)
110	0	2.3404	2.3218 (2.2336, 2.4100)	0.2211	0.2272 (0.2215, 0.2328)
	3	6.5537	6.4441 (6.2797, 6.6086)	0.1143	0.1174 (0.1131, 0.1218)
	5	8.8781	8.8210 (8.6225, 9.0194)	0.0720	0.0734 (0.0699, 0.0769)

Table 5.7.1: Barrier option prices using the numerical inversion of Laplace transform

All the computations are done on a Pentium 400 PC, 192 MB RAM. The Monte Carlo results are based on 20000 simulation paths with 2000 time steps. The computation time is about 0.05-0.07 seconds for the numerical inversion of the Laplace transform of vanilla options and 0.1-0.15 seconds for barrier options (with N=16 in the Stehfest algorithm). For the Monte-Carlo, the computation time is about 30-45 seconds. In addition to the point estimates, we also report the 95% confidence interval shown in parenthis.

Generally, results obtained by numerical inversion of Laplace transform and the Monte Carlo are comparable. We note that the price differences for vanilla options between the numerical inversion of the Laplace transform and the semi-analytical formula based on the Fourier transform, which we developed in the first part, are less than 10^{-4} for most realistic values of option and model parameters. Reported results for the double barrier knock-out call option under the double-exponential jump diffusion have been verified by the anonymous referee from the IJTAF using a numerical solver for the PIDE. The agreement is also to the four decimals.

Chapter 6

CreditGrades Model

In this Chapter, we will extend the framework developed in Chapters 3 and 5 for analytical evaluation of equity options in the CreditGrades model with double-exponential jumps and stochastic volatility.

This Chapter rests on the paper "Extended CreditGrades Model with Stochastic Volatility and Jumps" by Sepp (2006). Here, we present two robust extensions of the CreditGrades model: the first one assumes that the variance of returns on the firm's assets is stochastic, and the second one assumes that the firm's asset value process follows a double-exponential jump-diffusion. We derive closed-form formulas for pricing equity options on a reference firm in this setting and for calculating the survival probability of this firm during a finite time horizon. We apply these models for modeling credit default swap (CDS) spreads. We calibrate our models to General Motors options data and discuss the results. It follows that both models provide a good fit to the data and lead to non-zero short-term CDS spreads.

The contribution of Chapter 6 is threefold. First, we incorporate jumps into the CreditGrades model. Although the Merton's model with jump risk has already been considered in a number of studies, there was no reference on how to connect the default risk with equity risk, that is how to estimate default probabilities using equity options. Secondly, we consider the stochastic variance of the firm's value in the CreditGrades model and make a connection to equity options. This model seems to be new. Finally, we consider incorporating random default barriers and provide an alternative to the CreditGrades approach on how to deal with random default barriers by computing survival probabilities and option prices. This approach is based on the convexity adjustment and it can applied to diffusions with stochastic variance and jumps.

6.1. Structural Default Model

Credit risk is the risk that an obliger fails to honor its obligations. Coupled with a recent wave of bankruptcies, credit derivatives market has been sparking and raising demand for more sophisticated methods to evaluate and manage the credit risk.

In this Chapter we treat the credit risk of a single issuer and consider a fundamental model which takes into account issuer's debt and asset value processes and prescribes the default probability of this issuer. Our approach roots from the Merton's (1974) model. In Merton's model, the company defaults if the value of its assets becomes less than its promised debt repayment at maturity time T. Among others, Black-Cox (1976) and Leland-Toft (1996) extended Merton's model to account for the possibility that default may happen prior to maturity date T. Other extensions propose stochastic interest rates (Longstaff-Schwartz (1995)), stochastic default barriers (Finger at al (2002)), jumps in the firm's value dynamics (Zhou (1997, 2001), Hilberink-Rogers (2002), Lipton (2002)). For a comprehensive review of the Merton model and its ramifications we refer to the recent books by Bielecki-Rutkowski (2002), Schönbucher (2003), and Lando (2004).

One of the drawbacks of the Merton's model is that it provides no connection to equity market, in particular, to equity options. This drawback was circumvented by CreditGrades (equity-to-credit) model, which became quite popular in the credit derivatives market. The CreditGrades model was jointly developed by CreditMetrics, JP Morgan, Goldman Sachs, and Deutsche Bank, and it has subsequently been copywritten. A detailed description of the CreditGrades model is presented in Stamicar et al (2005), Finger et al (2002), Finkelstein (2001). In short, this approach supplements the Merton's model (1974) by providing a link to equity market and, in particular, to equity options. CreditGrades model is based on the assumption that the firms value follows a pure diffusion with a stochastic default barrier which is introduced to make the model consistent with high short-term CDS spreads.

Another essential problem in the Merton's model is the so-called predictability of default, which is discussed in details in Lando (2004) and Elizalde (2005). In a nutshell, since most of the structural models assume a continuous diffusion processes for the firms value dynamics and complete information about the firm's value and default barrier, the actual distance from the current value of the firm to the default barrier implies the "nearness of default". Accordingly, if the current market value of the firm is far away from its default barrier, the probability of default in the short-term is close to zero, because the firm's value process needs time to reach the default point. The knowledge of that distance to default and the fact that the firm's value follows a continuous diffusion process makes the firm's default a predictable event, that is the default does not come as a surprise, and it leads to very low short-term credit spreads, which are close to zero. In contrast, it is observed in the markets that even short-term credit spreads are non-zero, incorporating the possibility of an unexpected default or deterioration in the firms credit quality.

One of the approaches to deal with the predictability of the default is to include jumps in the firm's value process. Zhou (1997, 2001), Hilberink-Rogers (2002), Lipton (2002) deal with structural models in which firms dynamics incorporates jumps. However, in general this approach is not analytically tractable, since first-exit time densities for jump-diffusions are rather difficult to obtain in closed-form. In addition, it was not discussed how to estimate parameters of these models using equity option data.

6.2. Formulation

Here we fix notations and briefly present the CreditGrades model. Detailed description of this model can be found in Stamicar *et al* (2005), Finger *et al* (2002), Finkelstein (2001).

We assume a finite time horizon T and suppose a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ augmented with filtration $\mathcal{F}(t)$ supporting standard Brownian motions $W(t)$, $W^v(t)$ and a Poisson process $N(t)$.

We consider a certain reference firm and use the following notation: $S(t)$ denotes the firm's equity price per share, $F(t)$ denotes the firm's asset value per share, $B(t)$ denotes the firm's total debt per share, and R $(0 < R \leq 1)$ denotes the recovery rate of firm's debt.

We assume that the firm's debt $B(t)$ is deterministic:

$$B(t) = B(0)e^{\int_0^t (r(t')-d(t'))dt'}, \tag{6.2.1}$$

where $r(t)$ is a deterministic risk-free interest rate, and $d(t)$ is a deterministic dividend yield on the firm's assets.

We assume that the asset value of the firm, $F(t)$, is driven by a stochastic process under the risk-neutral measure $\mathbb{Q}$ and the firm defaults when its value falls below the deterministic barrier $D(t)$ defined by $D(t) = RB(t)$. Accordingly, the time of default on time interval $(t_0, T]$, $0 \leq t_0 < T < \infty$, is defined as

$$\iota = \inf\{\tau \in (t_0, T] : F(\tau) \leq D(\tau)\}, \tag{6.2.2}$$

with ι being an $\mathcal{F}$-stopping time.

We define the firm's equity per share $S(t)$ by:

$$\begin{cases} S(t) = F(t) - D(t), \text{ if } \iota > t \\ S(t) = 0, \text{ otherwise.} \end{cases} \tag{6.2.3}$$

From representation (6.2.3) we can induce an equivalent definition of default time expressed in terms of the firm's equity per share:

$$\iota = \inf\{\tau \in (t_0, T] : S(\tau) \leq 0\}. \tag{6.2.4}$$

The specification (6.2.3) provides a possible way to estimate parameters of the asset's value process which is not directly observable, by linking it to firm's equity, for which market data are typically available.

Let us also note that, in the CreditGrades model, the asset's value process $F(t)$ cannot be interpreted as the market value of the firm, since by the financial standpoint the market value of the firm is the sum of the market value of its equity and debt. Accordingly, in specification (6.2.3), $F(t)$ is rather interpreted as an underlying state variable related to the asset value process or the market value of the firm rather than its accounting value. The default barrier $D(t)$ can be interpreted as the recovery part of the debt or its market value. Since the initial value of $D(t)$ can be estimated from the balance sheet or market data, the initial value of $F(t)$ is calculated as $F(0) = S(0) + D(0)$.

Furthermore, the CreditGrades model assumes that the default barrier $D(t)$ is a random variable with log-normal distribution and suitably chosen parameters. This assumption leads to the unpredictability of the default event and thus to higher shorter term credit spreads. At this time we consider a deterministic barrier $D(t)$ and we will treat random barriers in Section 6.5, where we will employ a convexity adjustment formula (1.3.6) to deal with random barriers.

6.2.1. Firm's Value Dynamics

In this Chapter we consider three possible models for the firm's value dynamics: **1)** log-normal diffusion (regular diffusion), **2)** log-normal diffusion with stochastic variance uncorrelated with the firm's value dynamic, **3)** double-exponential jump-diffusion.

Our fundamental assumption is that the firm defaults when its equity price hits zero, so that zero is an absorbing barrier for the price process $S(t)$. This implies that European call and put options are priced as corresponding down-and-out barrier call and put options with the down barrier set at $S = 0$. We note that the assumption of the absorbing barrier is more complicated to deal with than under alternative assumption that the default can only happen at maturity time T, that is the firms default only if $S(T) \leq 0$. The former assumption is evidently more realistic but is requires a lot of analytical efforts since TPDF-s with absorbing barrier are available only for a limited (but nevertheless very interesting) class of stochastic processes.

Regular Diffusion

The basic version of the CreditGrades model assumes that the firm's follows a geometric Brownian motion with time-dependent parameters:

$$dF(t)/F(t) = (r(t) - d(t))dt + \sqrt{\nu(t)}dW(t), \ F(0) = S(0) + D(0), \qquad (6.2.5)$$

where $\nu(t)$ is variance of returns on the firm's assets. Under the above assumptions, prior to the default the stock price follows the so-called shifted log-normal diffusion:

$$dS(t) = (r(t) - d(t))S(t)dt + \sqrt{\nu(t)}(S(t) + D(t))dW(t), \ S(0) \text{ given.} \qquad (6.2.6)$$

Introducing a new process for $x = \ln(S(t) + D(t))$ and using Ito lemma (1.3.3), we can show that the solution to the stock price dynamics prior to the default is given by:

$$S(t) = (S(0) + D(0))e^{\int_0^t (r(t') - d(t') - \frac{1}{2}\nu(t'))ds + \int_0^t \sqrt{\nu(t')}dW(t')} - D(t). \qquad (6.2.7)$$

Thus, the distribution of stock price under the risk-neutral measure is shifted log-normal (equivalently, the distribution of $S(t) + D(t)$ is log-normal) with the transition probability density function given by

$$G^S(t,T,S(t),S'(T)) = \frac{1}{\sqrt{2\pi\tau}(S'(T)+D(T))}\times$$
$$\exp\left\{-\frac{\left(\ln\left(\frac{S(t)+D(t)}{S'(T)+D(T)}\right)+\int_t^T(r(t')-d(t'))dt'-\frac{1}{2}\tau\right)^2}{2\tau}\right\}, \tag{6.2.8}$$

with $\tau = \int_t^T \nu(t')dt'$.

The shifted distribution implies negative values of the stock price with positive probability. Using TPDF (6.2.8), we can check that

$$\mathbb{E}^{\mathbb{Q}}[S(T)|\mathcal{F}(t)] = S(t)e^{\int_t^T(r(t')-d(t'))dt'},$$

ensuring that the discounted price process is martingale under the measure $\mathbb{Q}$.

From (6.2.7), we see that the default does not occur up to time T as long as for all $t \in (0,T]$

$$(S(0)+D(0))e^{\int_0^t(r(t')-d(t')-\frac{1}{2}\nu(t'))dt'+\int_0^t\sqrt{\nu(t')}dW(t')} > D(0)e^{\int_0^t(r(t')-d(t'))dt'}, \tag{6.2.9}$$

which is equivalent to:

$$(S(0)+D(0))e^{-\frac{1}{2}\int_0^t\nu(t')dt'+\int_0^t\sqrt{\nu(t')}dW(t')} > D(0). \tag{6.2.10}$$

Accordingly, in this specification the default event does not depend on the risk-neutral drift. Another important conclusion is that default probabilities only depend on the ratio between firm's value $F(0) = S(0)+D(0)$ and its debt $D(0)$, and not on the firm's value and its debt separately.

In Figure (6.2.1), we illustrate the relationship between the probability density of $S(T)$ and the equity leverage coefficient $L = D(0)/S(0)$. If the firm does not have any debt the its leverage is zero and it has is no possibility of default on its obligation, so that $S'(T)$ is always positive. In opposite, the greater the leverage, the more obligations the firm has, so that the likelihood of $S'(T)$ attaining zero increases. We note that with an increased leverage, the TPDF of the equity price has also a bigger right tail so that, if things go well, equity holders can benefit from using an increased leverage. In Figure (6.2.1), we show the TPDF-s of $S'(T)$ corresponding to the four choices of the leverage coefficient L with $S(0) = 25$, $R = 0.5$, $B(0) = \{0.0, 25, 50, 100\}$, $D(0) = \{0.0, 12.5, 25.0, 50.0\}$, $F(0) = \{25.0, 37.5, 50.0, 75.0\}$, $L = \{0.0, 0.5, 1.0, 2.0\}$, $T = 1.0$, and flat term structure of rates and volatility: $r(0) = d(0) = 0$, with constant firm's asset variance $\nu = \nu(0) = 0.1$. The higher is the leverage, the greater is the likelihood that $S'(T)$ attains negative values.

As a direct implication of the dynamics (6.2.6), it follows that for a given firm's asset volatility, $\nu(t)$, the local stock volatility, $V(t)$, depends on the local stock price:

$$\sqrt{V(t)} = \sqrt{\nu(t)}\frac{S(t)+D(t)}{S(t)}. \tag{6.2.11}$$

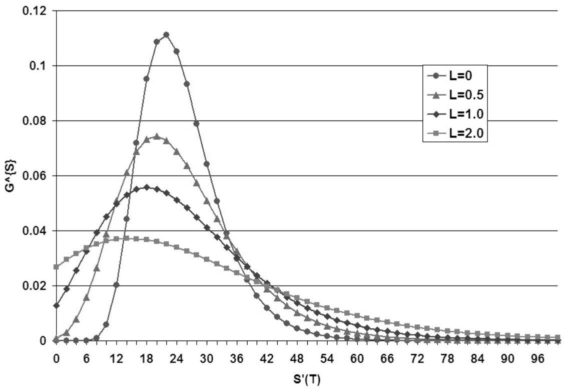

Figure 6.2.1: TPDF of the firm's equity price process

Thus, the CreditGrades model naturally introduces the volatility skew implying that lower equity prices are associated with higher volatility.

Diffusion with Stochastic Variance

In this model, which is based on the Heston (1993) approach to stochastic volatility, we assume that the firm's value is driven by a diffusion with stochastic variance that is uncorrelated with the firm's value process:

$$\begin{cases} dF(t)/F(t) = (r(t) - d(t))dt + \sqrt{\nu(t)}dW(t), \ F(0) = S(0) + D(0) \\ d\nu(t) = \kappa_\nu(\nu_\infty - \nu(t))dt + \varepsilon_\nu\sqrt{\nu(t)}dW^v(t), \ \nu(0) = \nu_0, \end{cases} \tag{6.2.12}$$

where $\nu(t)$ is now a stochastic variance on the firm's returns, ν_∞ is a long-term variance, κ_ν is a mean-reverting rate, ε_ν is a volatility of instantaneous variance, $W(t)$ and $W^v(t)$ are independent Brownian motions.

As a result, firm's equity price follows the shifted log-normal diffusion with stochastic variance:

$$\begin{cases} dS(t) = (r(t) - d(t))S(t)dt + \sqrt{\nu(t)}(S(t) + D(t))dW(t), \\ d\nu(t) = \kappa_\nu(\nu_\infty - \nu(t))dt + \varepsilon_\nu\sqrt{\nu(t)}dW^v(t), \ \nu(0) = \nu_0, \end{cases} \tag{6.2.13}$$

The motivation behind this model is that we cannot estimate the variance of the firm's asset returns for sure, thus we can allow some unpredictability for the variance of the firm's returns and deal with it appropriately. Another important consequence of this model is that, although we assume no correlation between the asset value and variance for the sake of analytical tractability, the leverage effect (which means that lower values of firm's assets result in an increased level of variance, while higher values result in a decreased level of variance) is introduced in the above model since the local variance of the firm's dynamics does depend on the leverage ratio, $(S(t) + D(t))/S(t)$.

We note that the original Heston model with zero correlation implies a symmetric smile (Lewis (2000)). However, in our process (6.2.13) the smile is asymmetric due to the the leverage ratio. The case of asymmetric smile is in agreement with empirical observations for equity options.

One of the possible drawbacks of this model is that not all parameters are directly observable on the markets, so we need to back out the model parameters from market prices of available securities. However, the availability of closed-form solutions for the values of call and put options as well as for the survival probability makes the calibration procedure less involved.

Double-Exponential Jump-Diffusion

Finally, we consider the firm's dynamics with double-exponential jumps:

$$\begin{cases} dF(t)/F(t-) = (r(t) - d(t) - \lambda m^j \nu(t))dt + \sqrt{\nu(t)}dW(t) + (e^J - 1)dN(t), \\ F(0) = S(0) + D(0), \end{cases} \tag{6.2.14}$$

where $N(t)$ is a Poisson process with deterministic intensity $\gamma\nu(t)$, and J is a random jumps with double-exponential distribution PDF $\varpi(J)$ is specified by (5.1.2) and compensator m^j is specified by (5.1.4).

The double-exponential jump-diffusion was introduced by Kou (2002) and it has subsequently been studied by Kou-Wang (2003), Lipton (2002), Sepp-Sckachkov (2003), Sepp (2004), who employed nice properties of this jump-diffusion, such as the availability of an analytical solution for the first-passage time density in Laplace space, to obtain closed-form formulas for barrier and lookback options. Here, we extend their results to the CreditGrades model.

In this model, the stock price follows the shifted log-normal diffusion with double-exponential jumps:

$$\begin{aligned} dS(t) = &\left((r(t) - d(t))S(t) - \gamma m^j \nu(t)(S(t) + D(t))\right) dt + \\ &+ \sqrt{\nu(t)}(S(t) + B(t))dW(t) + (S(t) + D(t))(e^J - 1)dN(t). \end{aligned} \tag{6.2.15}$$

The solution to the above SDE is given by:

$$S(t) = (S(0) + D(0))e^{\int_0^t (r(t') - d(t') - (\gamma m^j + \frac{1}{2})\nu(t'))dt' + \int_0^t \sqrt{\nu(t')}dW(t') + \sum_{n=1}^{N(t)} J_n} - D(t),$$

where J_n are independent random variables with PDF $\varpi(J)$.

By construction, the intensity of jumps depends on the variance and, as a result, on the local leverage ratio $\frac{S(t)+D(t)}{S(t)}$. This is a realistic assumption, since for lower asset prices returns are more volatile and thus they are subject to greater jump risk.

The motivation behind this model is that it introduces the unpredictability of the default event, that is the default can happen short after the contract inception. This leads to high credit spreads for even very short maturities. Although not all model parameters are directly observable, the availability of closed-form solutions for equity options and survival probabilities makes the model calibration feasible.

6.3. Equity Options

In this section, we will study the pricing of equity options under the above-described models. For brevity, we concentrate on call options. Put options can be handled via the same technique or they can be priced by the put-call parity.

Under risk-neutral valuation formula (1.2.3), the value of a call option, $W(t,S)$, can be represented using default time definitions (6.2.2)-(6.2.4) as follows:

$$\begin{aligned} W(t,S) &= \mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t')dt'}\max\{S(T)-K\}\mathbf{1}_{\{\iota>T\}}\,|\,\mathcal{F}(t)\right], \\ &= \mathbf{1}_{\{S(t)>0\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t')dt'}\max\{S(T)-K\}\mathbf{1}_{\{S(\tau)>0,\ t<\tau\le T\}}\,|\,\mathcal{F}(t)\right]. \end{aligned}$$

Accordingly, the value of a call $W(t,V)$ is equivalent to a down-and-out barrier call with strike K and lower barrier set at $S=0$.

6.3.1. Regular Diffusion

Applying Feynman-Kac theorem (1.4.3) to the dynamics (6.2.6), we obtain that $W(t,S)$ solves the following PDE:

$$\begin{cases} W_t + \frac{1}{2}\nu(t)(S+D(t))^2W_{SS} + (r(t)-d(t))\,SW_S - r(t)W = 0, \\ W(t,0)=0,\ W(T,S)=\max\{S-K,0\}. \end{cases} \tag{6.3.1}$$

Proposition 6.1. *The value of a call option under the regular diffusion is given by:*

$$\begin{aligned} W(t,S) =& U(t,S(t)+D(t);T,K+D(T)) \\ &- \frac{S(t)+D(t)}{D(t)}U(t,\frac{(D(t))^2}{S(t)+D(t)};T,K+D(T)). \end{aligned} \tag{6.3.2}$$

where $U(t,S;T,K)$ is the Black-Scholes price of a call option given by formula (1.6.5) with maturity T, strike K on the underlying with price S and average volatility, interest rate and dividend yield calculated by: $\bar{\sigma}^2 = \frac{\int_t^T \nu(s)ds}{T-t}$, $\bar{r} = \frac{\int_t^T r(s)ds}{T-t}$, and $\bar{d} = \frac{\int_t^T d(s)ds}{T-t}$, respectively.

Formula (6.3.2) resembles a standard formula for down-and-out barrier call (note there is no terms with $\bar{r}$ and $\bar{d}$ in the power of $\frac{S(t)+D(t)}{D(T)}$ since by construction the default barrier does not depend on the risk-neutral drift). As a result, CreditGrades model provides simple and robust solution to the value of equity options.

Proof. We solve PDE (6.3.1) in a few steps.

1) We introduce introduce new variables and value function as follows:

$$x = \ln\frac{S+D(t)}{D(t)},\ a = \ln\frac{D(T)+K}{D(T)},\ W(t,S)\to Z(t,x) = e^{\int_t^T r(t')dt'}\frac{W(t,S)}{D(T)}. \tag{6.3.3}$$

Then $Z(t,x)$ solves the following equation:

$$\begin{cases} Z_t + \frac{1}{2}\nu(t)Z_{xx} - \frac{1}{2}\nu(t)Z_x = 0, \\ Z(t,0) = 0, \ Z(T,x) = \max\{e^x - e^a, 0\}. \end{cases} \tag{6.3.4}$$

2) We introduce new variables and value function as follows:

$$x \to y = x - a, \ b = -a, \ t \to \tau = \int_t^T \nu(t')dt', \ Z(t,x) \to H(\tau,y) = e^{-a}Z(t,x). \tag{6.3.5}$$

Now, $H(\tau,y)$ solves:

$$\begin{cases} -H_\tau + \frac{1}{2}H_{yy} - \frac{1}{2}H_y = 0, \\ H(\tau,b) = 0, \ H(0,y) = \max\{e^y - 1, 0\}. \end{cases} \tag{6.3.6}$$

3) Then, we introduce $H(t,x) \to Y(\tau,y) = e^{\tau/8 - y/2}H(\tau,y)$:

$$\begin{cases} -Y_\tau + \frac{1}{2}Y_{yy} = 0, \\ Y(\tau,b) = 0, \ Y(0,y) = \max\{e^{\frac{y}{2}} - e^{-\frac{y}{2}}, 0\}. \end{cases} \tag{6.3.7}$$

4) We solve this equation by the method of images, which is discussed below, by introducing Green function $G^b(\tau,y,y^{'})$ corresponding to PDE (6.3.7) with absorption at $y' = b$:

$$G^b(\tau,y,y^{'}) = G(\tau,y-y^{'}) - G(\tau,y-(2b-y^{'})), \tag{6.3.8}$$

where $G(\tau,y)$ is Green function of PDE (6.3.7) defined for $y \in (-\infty,\infty)$, also known as the heat kernel, and given by:

$$G(\tau,y) = \frac{e^{-\frac{y^2}{2\tau}}}{\sqrt{2\pi\tau}}. \tag{6.3.9}$$

5 Finally, $Y(\tau,y)$ is computed by applying Duhamel's formula (1.4.9) as follows:

$$\begin{aligned} Y(\tau,y) &= \int_0^\infty (e^{\frac{y^{'}}{2}} - e^{-\frac{y^{'}}{2}})G^b(\tau,y,y^{'})dy^{'} = \\ &= e^{\tau/8+y/2}\mathcal{N}\left(\frac{y}{\sqrt{\tau}} + \frac{1}{2}\sqrt{\tau}\right) - e^{\tau/8-y/2}\mathcal{N}\left(\frac{y}{\sqrt{\tau}} - \frac{1}{2}\sqrt{\tau}\right) \\ &- e^{\tau/8+b-y/2}\mathcal{N}\left(\frac{2b-y}{\sqrt{\tau}} + \frac{1}{2}\sqrt{\tau}\right) + e^{\tau/8-b+y/2}\mathcal{N}\left(\frac{2b-y}{\sqrt{\tau}} - \frac{1}{2}\sqrt{\tau}\right). \end{aligned} \tag{6.3.10}$$

6) Substituting original variables and noting that $\ln\frac{D(T)}{D(t)} = \int_t^T (r(t') - d(t'))dt'$, we obtain:

$$\begin{aligned} W(t,S) &= e^{-\int_t^T d(t')dt'}(S(t)+D(t))\mathcal{N}(q_1^+) - e^{-\int_t^T r(t')dt'}(D(T)+K)\mathcal{N}(q_1^-) \\ &- e^{-\int_t^T d(t')dt'}D(t)\mathcal{N}(q_2^+) + e^{-\int_t^T r(t')dt'}\frac{(D(T)+K)(S(t)+D(t))}{D(t)}\mathcal{N}(q_2^-), \end{aligned}$$

where

$$q_1^{+,-} = \frac{\ln \frac{S(t)+D(t)}{K+D(T)} + \int_t^T (r(t') - d(t'))dt'}{\sqrt{\int_t^T \nu(t')dt'}} \pm \frac{1}{2}\sqrt{\int_t^T \nu(t')dt'},$$

$$q_2^{+,-} = \frac{\ln \frac{D^2(t)}{(K+D(T))(S(t)+D(t))} + \int_t^T (r(t') - d(t'))dt'}{\sqrt{\int_t^T \nu(t')dt'}} \pm \frac{1}{2}\sqrt{\int_t^T \nu(t')dt'}.$$

After a little rearrangement, we obtain:

$$W(t,S) = e^{-\int_t^T d(t')dt'}(S(t) + D(t))\mathcal{N}(q_1^+) - e^{-\int_t^T r(t')dt'}(D(T) + K)\mathcal{N}(q_1^-)$$
$$- \frac{S(t)+D(t)}{D(t)}\left(e^{-\int_t^T d(t')dt'}\frac{D^2(t)}{S(t)+D(t)}\mathcal{N}(q_2^+) - e^{-\int_t^T r(t')dt'}(D(T)+K)\mathcal{N}(q_2^-)\right).$$

The first two terms represent the value of a call option with spot price $S(t) + D(t)$ and strike $D(T) + K$, while the last two terms represent the value of a call option with spot price $\frac{D^2(t)}{S(t)+D(t)}$ and strike $D(T) + K$. Thus, we obtain formula (6.3.2).

□

Illustration. Method of images

Here, we briefly illustrate the idea behind the method of images, which is important for our present developments. A comprehensive reference for using this method to value options with barriers is Lipton (2001). The method of images allows us to construct solution to PDE (6.3.7) absorbed at point $y' = b$ by using the Green function $G(\tau, y)$ corresponding to the PDE (6.3.7) defined for $y \in (-\infty, \infty)$. Then by using a combination of two particular solutions given by (6.3.8), we construct function $G^b(\tau, y, y')$ absorbed at $y' = b$. The idea is illustrated in Figure (6.3.1) with $b = -1$, $y = 0$, $\tau = 1$, where the solution with source at $y' = -y + 2b$, $G(\tau, y-(2b-y'))$, is subtracted from the solution with source at $y' = y$, $G(\tau, y-y')$, so that the composite solution, $G^b(\tau, y, y')$, vanishes at $y' = b$.

6.3.2. Diffusion with Stochastic Variance

Now we consider the stock price dynamics with stochastic variance (6.2.13). For this model, the value function of a call option, $W(t, \nu)$, solves the following equation:

$$\begin{aligned} &W_t + \frac{1}{2}\nu(t)(S + D(t))^2 W_{SS} + (r(t) - d(t))\, SW_S + \kappa_\nu(\nu_\infty - \nu)W_\nu \\ &+ \frac{1}{2}\varepsilon_\nu^2 \nu W_{\nu\nu} - r(t)W = 0,\ W(t,0) = 0,\ W(T,S) = \max\{S - K, 0\}. \end{aligned} \tag{6.3.11}$$

We solve this PDE by a combination of the Fourier transform and the method of images discussed above. A similar technique was employed by Lipton (2001) for pricing double-barrier options in the Heston's model with zero correlation.

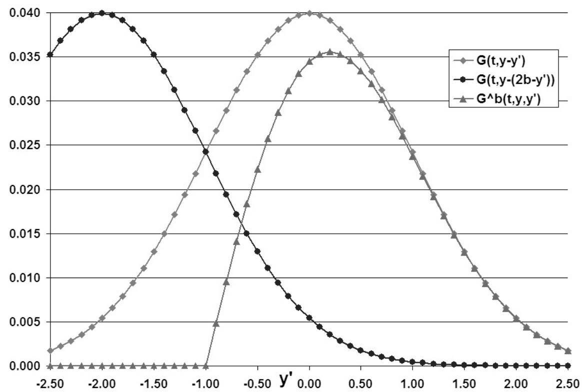

Figure 6.3.1: The construction of the method of images

Proposition 6.2. *The solution to PDE (6.3.11) is given by:*

$$W(t,S) = (D(T)+K)e^{-\int_t^T r(t')dt'} Z(\tau,y), \tag{6.3.12}$$

where $y = \ln\left(\frac{S+D(t)}{D(T)+K}\right) + \int_t^T (r(t') - d(t'))dt'$, $b = \ln\left(\frac{D(t)}{D(T)+K}\right) + \int_t^T (r(t') - d(t'))dt'$,

$$Z(\tau,y) = e^y - e^b - \frac{e^{\frac{1}{2}y}}{\pi} \int_0^\infty \frac{e^{A(t,T)+B(t,T)\nu}(\cos(y\Phi_I) - \cos((y-2b)\Phi_I)}{\Phi_I^2 + \frac{1}{4}} d\Phi_I, \tag{6.3.13}$$

with

$$\begin{aligned} A(t,T) &= -\frac{\kappa_\nu \nu_\infty}{\varepsilon_\nu^2}\left[(T-t)\psi_+ + 2\ln\left(\frac{\psi_- + \psi_+ e^{-(T-t)\zeta}}{2\zeta}\right)\right], \\ B(t,T) &= -(\Phi_I^2 + \frac{1}{4})\frac{1 - e^{-(T-t)\zeta}}{\psi_- + \psi_+ e^{-(T-t)\zeta}}, \\ \psi_\pm &= \mp\kappa_\nu + \zeta,\ \zeta = \sqrt{\kappa_\nu^2 + \varepsilon_\nu^2(\Phi_I^2 + \frac{1}{4})},\ \tau = T - t. \end{aligned} \tag{6.3.14}$$

Integral in (6.3.13) is pretty easy to evaluate numerically (for large values of Φ_I it is an exponentially decaying function of Φ_I). Thus, we obtain a closed-form solution for call option values in the extended CreditGrades model with stochastic variance.

Proof. We solve PDE (6.3.11) as follows.

1) We apply the sequence of transformations (6.3.3) and (6.3.5) with $t \to \tau = T - t$ to obtain the following equation for $H(\tau,y)$:

$$\begin{cases} -H_\tau + \frac{1}{2}\nu H_{yy} - \frac{1}{2}\nu H_y + \kappa_\nu(\nu_\infty - \nu)H_\nu + \frac{1}{2}\varepsilon_\nu^2 \nu H_{\nu\nu} = 0, \\ H(\tau,b) = 0,\ H(0,y) = \max\{e^y - 1, 0\}. \end{cases} \tag{6.3.15}$$

2) Next we remove the drift term by introducing $H(t,x) \to Y(\tau,y) = e^{-y/2}H(\tau,y)$:

$$\begin{cases} -Y_\tau + \frac{1}{2}\nu Y_{yy} + \kappa_\nu(\nu_\infty - \nu)Y_\nu + \frac{1}{2}\varepsilon_\nu^2 \nu Y_{\nu\nu} - \frac{1}{8}\nu Y = 0, \\ Y(\tau,b) = 0,\ Y(0,y) = \max\{e^{\frac{y}{2}} - e^{-\frac{y}{2}}, 0\}. \end{cases} \tag{6.3.16}$$

3) We solve PDE (6.3.16) via a combination of the Fourier transform and the method of images. A similar technique was employed by Lipton (2001) for pricing double-barrier options in the Heston's model with zero correlation. The Green function, corresponding to PDE (6.3.16) defined for $y, y' \in (-\infty,\infty)$ and denoted by $G(\tau,y,y',\nu)$, is obtained as follows:

i) We apply forward Fourier transform (1.5.3) to obtain the transformed Green function $\widehat{G}(\tau,y,\Phi,\nu) = \mathbf{F}_-[G(y')](\Phi)$ with $\Phi = i\Phi_I$, $\Phi_I \in \mathbb{R}$;

ii) We obtain solution for $\widehat{G}(\tau,y,y',\nu)$ using our affine solution formula (3.2.4) with $X \to y$, $X' \to y'$, $\rho \to 0$, $V \to \nu$, $\Phi \to i\Phi_I$, $r(t) = d(t) = 0$, $\Psi \to 0$, $\Theta \to 0$, and coefficients specified by $b_2 = \frac{1}{2}\varepsilon_\nu^2$, $b_1 = -\kappa_\nu$, $b_0 = -\frac{1}{2}\Phi_I^2 - \frac{1}{8}$, $a_1 = \kappa_\nu \nu_\infty$, $a_0 = 0$;

iii) We apply inverse Fourier transform to obtain $G(y') = \mathbf{F}_+[\widehat{G}(\Phi)](y')$ given by:

$$G(\tau,y,y',\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\Phi_I(y'-y)+A(t,T)+B(t,T)\nu} d\Phi_I, \tag{6.3.17}$$

with $A(t,T)$ and $B(t,T)$ defined in (6.3.14).

4) We note that Green function $G(\tau,y,y',\nu)$ is invariant with respect to the reflections $(y'-y), \Phi_I \to -(y'-y), -\Phi_I$, so we can employ the method of images to find a solution absorbed at $y' = b$, denoted by $G^b(\tau,y,y'\nu)$, by analogy to PDE (6.3.8) as follows:

$$G^b(\tau,y,y'\nu) = G(\tau,y,y',\nu) - G(\tau,y,y-(2b-y'),\nu). \tag{6.3.18}$$

5) Finally, applying Duhamel's formula (1.4.9) and exchanging the integration order, we compute the original function by:

$$\begin{aligned} Y(\tau,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} (e^{\frac{1}{2}y'} - e^{-\frac{1}{2}y'})G^b(\tau,y,y',\nu)dy'd\Phi_I \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{A(t,T)+B(t,T)\nu}\left(e^{-yi\Phi_I} - e^{(y-2b)i\Phi_I}\right)\left(-\frac{1}{i\Phi_I + \frac{1}{2}} + 2\pi\delta(\Phi_I - \frac{1}{2}i) + \frac{1}{i\Phi_I - \frac{1}{2}}\right)d\Phi_I \\ &= e^{\frac{1}{2}y} - e^{-\frac{1}{2}(y-2b)} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{A(t,T)+B(t,T)\nu}\left(e^{-yi\Phi_I} - e^{(y-2b)i\Phi_I}\right)}{\Phi_I^2 + \frac{1}{4}}d\Phi_I, \end{aligned} \tag{6.3.19}$$

where we employed the formula for the complex exponential given in (1.5.2). After some simplification, we obtain one-dimensional real-valued integral given in (6.3.13). □

6.3.3. Double-Exponential Jump-Diffusion

Finally we consider the stock price process driven by the double-exponential jump-diffusion (6.2.15). Under this model the value of a call option, $W(t,S)$, solves the following PIDE:

$$\left\{\begin{array}{l} W_t + \frac{1}{2}\nu(t)(S+D(t))^2 W_{SS} + (r(t)-d(t))\, SW_S - r(t)W \\ +\gamma\nu(t)\int_{-\infty}^{\infty}[W((S+D(t))e^J) - W(S+D(t))]\varpi(J)dJ = 0, \\ W(t,0)=0,\ W(T,S)=\max\{S-K,0\}. \end{array}\right. \tag{6.3.20}$$

We solve this equation as follows. First, we introduce $x = \ln(\frac{S+D(t)}{D(t)})$, $a = \ln\frac{D(T)+K}{D(T)}$, $W(t,V) \to H(t,x) = e^{\int_t^T r(t')dt'}W(t,V)/D(T)$:

$$\left\{\begin{array}{l} H_t + \frac{1}{2}\nu(t)H_{xx} + \mu\nu(t)H_x + \gamma\nu(t)\int_{-\infty}^{\infty}[H(x+J)-H]\varpi(J)dJ = 0, \\ H(t,0)=0,\ H(T,x)=\max\{e^x - e^a, 0\}. \end{array}\right. \tag{6.3.21}$$

where $\mu = -\frac{1}{2} - \gamma m^j$.

Next, we introduce $x \to y = x - a$, $b = -a$, $t \to \tau = \int_t^T \nu(t')dt'$, $H(t,x) \to Y(\tau,y) = e^{-a}H(t,x)$ to obtain:

$$\left\{\begin{array}{l} -Y_\tau + \frac{1}{2}Y_{yy} + \mu Y_y + \gamma\int_{-\infty}^{\infty}[Y(y+J)-Y]\varpi(J)dJ = 0, \\ Y(\tau,b)=0,\ Y(0,y)=\max\{e^y-1,0\}. \end{array}\right. \tag{6.3.22}$$

The equation of this type was solved in Chapter 5 through an application of Laplace transform with respect to τ, $U(p,y) = \mathcal{L}[Y(\tau,y)](p)$. We use the results of proposition (5.5) to present the solution for $U(p,y)$ as follows:

$$U(p,y) = \left\{\begin{array}{l} C_0e^{\psi_0 y} + C_1e^{\psi_1 y} + A_2^+e^{\psi_2 y} + A_3^-e^{\psi_3 y}, y \le 0 \\ C_2e^{\psi_2 y} + C_3e^{\psi_3 y} + A_2^+e^{\psi_2 y} + A_3^-e^{\psi_3 y} + \left[\frac{e^y}{p} - \frac{1}{p}\right], y > 0, \end{array}\right. \tag{6.3.23}$$

where constants C_0, C_1, C_2, C_3 are solution of the system

$$\left(\begin{array}{cccc} 1 & 1 & -1 & -1 \\ \psi_0 & \psi_1 & -\psi_2 & -\psi_3 \\ \frac{1}{\psi_0\eta^-+1} & \frac{1}{\psi_1\eta^-+1} & -\frac{1}{\psi_2\eta^-+1} & -\frac{1}{\psi_3\eta^-+1} \\ \frac{1}{\psi_0\eta^+-1} & \frac{1}{\psi_1\eta^+-1} & -\frac{1}{\psi_2\eta^+-1} & -\frac{1}{\psi_3\eta^+-1} \end{array}\right)\left(\begin{array}{c} C_0 \\ C_1 \\ C_2 \\ C_3 \end{array}\right) = \left(\begin{array}{c} 0 \\ \frac{1}{p} \\ \frac{1}{p(\eta^-+1)} - \frac{1}{p} \\ \frac{1}{p(\eta^+-1)} + \frac{1}{p} \end{array}\right),$$

and $\psi_3, \psi_2, \psi_1, \psi_0$ are solutions of the following quartic equation:

$$\frac{1}{2}\eta^-\eta^+\psi^4 + (\mu\eta^-\eta^+ - \frac{1}{2}(\eta^- - \eta^+))\psi^3 - (\frac{1}{2} + \mu(\eta^- - \eta^+) + (p+\gamma)\eta^-\eta^+)\psi^2$$
$$+ (-\mu + (p+\gamma)(\eta^- - \eta^+) - \gamma(q^+\eta^- - q^-\eta^+))\psi + p = 0.$$

which has four real roots $\psi_i, i = 0,1,2,3$, such that

$$-\infty < \psi_3 < -\frac{1}{\eta^-} < \psi_2 < 0 < \psi_1 < \frac{1}{\eta^+} < \psi_0 < \infty. \tag{6.3.24}$$

The constants A_2^+, A_3^- are computed by

$$A_{2,3}^{\pm} = \pm\frac{1+\psi_{2,3}\eta^-}{\psi_3-\psi_2}\left[\frac{(\psi_0-\psi_{3,2})C_0 e^{(\psi_0-\psi_{2,3})b}}{\psi_0\eta^-+1}+\frac{(\psi_1-\psi_{3,2})C_1 e^{(\psi_1-\psi_{2,3})b}}{\psi_1\eta^-+1}\right].$$

As a result, the option value in the time variable is given by inverse Laplace transform (1.5.9):

$$W(t,S) = (D(T)+K)e^{-\int_t^T r(t')dt'}\mathcal{L}^{(-1)}[U(p,y)]. \tag{6.3.25}$$

To invert the Laplace transform, we employ Stehfest algorithm (1.5.10). Computational speed of calculating call option prices for the double-exponential jump-diffusion is closely comparable to that for the diffusion with stochastic variance.

6.4. Survival Probability

In this section, we derive solutions for survival probability $Q(t,S)$ of the reference firm under the described models:

$$Q(t,T,S) = \mathbf{1}_{\{\iota>t\}}\mathbb{P}^{\mathbb{Q}}[\iota > T|\mathcal{F}(t)] = \mathbf{1}_{\{\iota>t\}}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{S(\tau)>0:\ t<\tau\leq T\}}|\mathcal{F}(t)], \tag{6.4.1}$$

which are necessary to calculate the values of credit default swaps.

6.4.1. Regular Diffusion

Invoking Feynman-Kac theorem (1.4.3) to the dynamics (6.2.6), we obtain that $Q(t,T,S)$ satisfies the following PDE:

$$\begin{cases} Q_t + \frac{1}{2}\nu(t)(S+D(t))^2 Q_{SS} + (r(t)-d(t))\,SQ_S = 0, \\ Q(t,T,0) = 0,\ Q(T,T,S) = 1. \end{cases} \tag{6.4.2}$$

We introduce $y = \ln(\frac{S+D(t)}{D(t)})$, $t \to \tau = \int_t^T \nu(t')dt'$, $Q(t,T,S) \to Y(\tau,y)$:

$$\begin{cases} -Y_\tau + \frac{1}{2}Y_{yy} - \frac{1}{2}Y_y = 0, \\ Y(\tau,0) = 0,\ Y(0,y) = 1. \end{cases} \tag{6.4.3}$$

We solve this equation by the method of images described by (6.3.8). The solution is given by:

$$F(\tau,y) = \mathcal{N}\left(\frac{y}{\sqrt{\tau}} - \frac{1}{2}\sqrt{\tau}\right) - e^y\mathcal{N}\left(\frac{-y}{\sqrt{\tau}} - \frac{1}{2}\sqrt{\tau}\right). \tag{6.4.4}$$

where $\mathcal{N}$ is the CDF of standard normal distribution.

For original variables, with $\tau = \int_t^T \nu(t')dt'$ we have:

$$Q(t,T,S) = \mathcal{N}\left(\frac{\ln(\frac{S+D(t)}{D(t)})}{\sqrt{\tau}} - \frac{1}{2}\sqrt{\tau}\right) - \frac{S+D(t)}{D(t)}\mathcal{N}\left(\frac{\ln(\frac{D(t)}{S+D(t)})}{\sqrt{\tau}} - \frac{1}{2}\sqrt{\tau}\right). \tag{6.4.5}$$

The formula (6.4.5) is one of the key results in the CreditGrades model. We see that in this setup the survival probability has a few nice properties: 1) it does not depend on the risk-neutral drift, 2) it depends on the level of leverage $\frac{F(t)}{D(t)}$ rather than on $F(t)$ and $D(t)$ directly.

6.4.2. Diffusion with Stochastic Volatility

Under this model, $Q(t,T,S)$ solves the following equation:

$$\left\{\begin{array}{l} Q_t + \frac{1}{2}\nu(t)(S+D(t))^2 Q_{SS} + (r(t)-d(t))\, SQ_S + \kappa_\nu(\nu_\infty - \nu)Q_\nu \\ +\frac{1}{2}\varepsilon_\nu^2 \nu Q_{\nu\nu} = 0,\ Q(t,T,0) = 0,\ Q(T,T,S) = 1. \end{array}\right. \tag{6.4.6}$$

Introducing $y = \ln(\frac{S+D(t)}{D(t)})$, $t \to \tau = T - t$, $Q(t,T,S) \to Z(\tau,y) = e^{-y/2}Q(t,S)$, we obtain:

$$\left\{\begin{array}{l} -Z_\tau + \frac{1}{2}\nu Z_{yy} + \kappa_\nu(\nu_\infty - \nu)Z_\nu + \frac{1}{2}\varepsilon_\nu^2 \nu Z_{\nu\nu} - \frac{1}{8}\nu Z = 0, \\ Z(\tau,0) = 0,\ Z(0,y) = e^{-\frac{y}{2}}. \end{array}\right. \tag{6.4.7}$$

We solve this PDE via a combination of the Fourier transform and the method of images described by (6.3.18). Applying Duhamel's formula (1.4.9) and exchanging the integration order, we compute the solution by:

$$\begin{aligned} Z(\tau,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-\frac{1}{2}y'} G^b(\tau,y,y',\nu)dy' d\Phi_I \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\frac{e^{-iy\Phi_I + A(t,T)+B(t,T)\nu}}{i\Phi_I - \frac{1}{2}} + \frac{e^{iy\Phi_I + A(t,T)+B(t,T)\nu}}{i\Phi_I - \frac{1}{2}}\right)d\Phi_I \\ &= \frac{2}{\pi}\int_0^{\infty}\frac{e^{A(t,T)+B(t,T)\nu}\Phi_I \sin(y\Phi_I)}{\Phi_I^2 + \frac{1}{4}}d\Phi_I, \end{aligned} \tag{6.4.8}$$

where $G^b(\tau,y,y',\nu)$ is defined by (6.3.18) with $b = 0$, and $A(t,T)$, $B(t,T)$ are defined by (6.3.14).

Given the value of $Z(\tau,y)$, we compute survival probability $Q(t,T,S)$ by:

$$Q(t,T,S) = e^{\frac{1}{2}y}Z(\tau,y). \tag{6.4.9}$$

Thus, the problem boils down to computing numerically integral in (6.4.8), which is straightforward to implement.

6.4.3. Double-Exponential Jump-Diffusion

Under this process, $Q(t, T, S)$ solves the following PIDE:

$$\begin{cases} Q_t + \frac{1}{2}\nu(t)(S + D(t))^2 Q_{SS} + (r(t) - d(t))\, SQ_S + \\ +\gamma\nu(t) \int_{-\infty}^{\infty} [Q((S + D(t))e^J) - Q(S + D(t))]\varpi(J)dJ = 0, \\ Q(t, T, 0) = 0,\ \ Q(T, T, S) = 1. \end{cases} \tag{6.4.10}$$

Similarly, we introduce $y = \ln(\frac{S+D(t)}{D(t)})$, $t \to \tau = \int_t^T \nu(t')dt'$, $Q(t, T, S) \to Z(\tau, y)$ to obtain:

$$\begin{cases} -Z_\tau + \frac{1}{2} Z_{yy} + \mu Z_y + \gamma \int_{-\infty}^{\infty} [Z(x + J) - Z]\varpi(J)dJ = 0, \\ Z(\tau, 0) = 0,\ \ Z(0, y) = 1. \end{cases}, \tag{6.4.11}$$

with $\mu = -\frac{1}{2} - \gamma m^j$.

Using the results of proposition (5.6), we obtain that the solution to the Laplace transform of this equation $U(p, y) = \mathcal{L}[Z(\tau, y)](p)$ is given by:

$$U(p, y) = A_2 e^{\psi_2 y} + A_3 e^{\psi_3 y} + \frac{1}{p}, \tag{6.4.12}$$

where

$$A_2 = -\frac{1}{p}\frac{\psi_3(1 + \eta_- \psi_2)}{\psi_3 - \psi_2},\ \ A_3 = \frac{1}{p}\frac{\psi_2(1 + \eta_- \psi_3)}{\psi_3 - \psi_2}, \tag{6.4.13}$$

and ψ_2, ψ_3 are corresponding roots of polynomial (5.3.2).

In the time variable, the survival probability is computed via inversion of Laplace transform:

$$Q(t, T, S) = \mathcal{L}^{(-1)}[U(p, y)], \tag{6.4.14}$$

where we employ Stehfest algorithm (1.5.10) to invert $U(p, y)$.

6.4.4. Default (First-Exit) Time Density

Having derived the closed-form solutions for the survival probability, we are now able to calculate the density of the firm's default time, which is useful to analyze. By definition, this density, $q(T, S)$, satisfies:

$$q(t, T, S)dt = \mathbb{P}^{\mathbb{Q}}[\iota \in (T, T + dt]|\mathcal{F}(t)] = \frac{d}{dT}\,(1 - Q(t, T, S)) = -\frac{d}{dT} Q(t, T, S). \tag{6.4.15}$$

For regular diffusion we use formula (6.4.5) to obtain:

$$q(t, T, S) = \frac{\nu(t)\ln(\frac{S+D(t)}{D(t)})}{\left(\int_t^T \nu(t')dt'\right)^{3/2}} n\left(\frac{\ln(\frac{S+D(t)}{D(t)})}{\sqrt{\int_t^T \nu(t')dt'}} - \frac{1}{2}\sqrt{\int_t^T \nu(t')dt'}\right), \tag{6.4.16}$$

where $n()$ is the PDF of standard normal random variable.

Similarly, for diffusion with stochastic volatility we use formula (6.4.9) (the differentiation inside the integral sign is allowed since the integral converges uniformly) to obtain:

$$q(t,T,S) = e^{\frac{1}{2}y}\frac{2}{\pi}\int_0^\infty \frac{(A'(T)+B'(T)\nu)e^{A(t,T)+B(t,T)\nu}\Phi_I\sin(y\Phi_I)}{\Phi_I^2+\frac{1}{4}}d\Phi_I, \quad (6.4.17)$$

with

$$\begin{aligned} B'(T) &= -(\Phi_I^2+\frac{1}{4})\frac{2\zeta^2 e^{-(T-t)\zeta}}{\left(\psi_- + \psi_+ e^{-(T-t)\zeta}\right)^2}, \\ A'(T) &= -\frac{\kappa_\nu \nu_\infty}{\varepsilon_\nu^2}\left[\psi_+ - 2\frac{\psi_+\zeta e^{-(T-t)\zeta}}{\psi_- + \psi_+ e^{-(T-t)\zeta}}\right]. \end{aligned} \quad (6.4.18)$$

Finally, for double-exponential jump-diffusion we employ the following relationship:

$$\mathcal{L}[q(t,T,S)] = -p\mathcal{L}[Q(t,T,S)], \quad (6.4.19)$$

and use formula (6.4.12) to compute $q(t,T,S)$ by inverting the Laplace transform:

$$q(t,T,S) = -\nu(T)\mathcal{L}^{(-1)}[pU(p,y)]. \quad (6.4.20)$$

6.5. Random Default Barriers

In this section we consider random default barriers. In the CreditGrades model, random barriers are intended to introduce uncertainty to the default event. In this formulation the default barrier is given by $D(t) = RB(t)$, where R is a log-normal random variable independent of the dynamics of $F(t)$ with the expected value $\overline{R}$, drift $-\frac{1}{2}\varrho^2$ and standard derivation ϱ. The formula for the firm's survival probability is based on an approximation of a Brownian motion with a random barrier by another Brownian motion with a flat barrier and an appropriately chosen diffusion parameter and initial value.

From (6.2.10), we see that the default does not occur if for all t, $0 \le t \le T$:

$$F(0)e^{-\frac{1}{2}\int_0^t \nu(t')dt' + \int_0^t \sqrt{\nu(t')}dW(t')} > \overline{R}B(0)e^{-\frac{1}{2}\varrho^2+\varrho Y}, \quad (6.5.1)$$

where Y is a standard normal random variable independent of the Brownian motion $Z(t)$. In this setting, Y is interpreted as a time-independent state variable whose value is revealed at time $t = 0$. By introducing process

$$X(t) = \int_0^t \sqrt{\nu(t')}dZ(t') - \varrho Y - \frac{1}{2}\int_0^t \nu(t')dt' - \frac{\varrho^2}{2}, \quad (6.5.2)$$

we can present (6.5.1) as:

$$X(t) > \ln\frac{\overline{R}B(0)}{F(0)} - \varrho^2 = -\ln\frac{F(0)e^{\varrho^2}}{\overline{R}B(0)}. \quad (6.5.3)$$

It follows that $X(t)$ is normally distributed with

$$\begin{aligned}\mathbb{E}[X(t)] &= -\frac{1}{2}\int_0^t \nu(t')dt' - \frac{1}{2}\varrho^2 = -\frac{\int_0^t \nu(t')dt'}{2t}\left(t + \frac{\varrho^2 t}{\int_0^t \nu(t')dt'}\right),\\ \mathbb{V}[X(t)] &= \int_0^t \nu(t')dt' + \varrho^2 = \frac{\int_0^t \nu(t')dt'}{t}\left(t + \frac{\varrho^2 t}{\int_0^t \nu(t')dt'}\right).\end{aligned} \tag{6.5.4}$$

Then process $X(t)$ is approximated with a Brownian motion $\widehat{X}(t)$ with drift $-\frac{\int_0^t \nu(t')dt'}{2t}$ and variance $\frac{\int_0^t \nu(t')dt'}{t}$ starting in the past at $-\Delta t = -\frac{\varrho^2 t}{\int_0^t \nu(t')dt'}$ with $\widehat{X}(-\Delta t) = 0$. Under these assumptions, to calculate the survival probability we can use formula (6.4.4) with $\tau = \int_t^T \nu(t')dt' + \varrho^2$ and $y = \frac{F(t)e^{\varrho^2}}{\overline{R}B(t)}$.

Finger et al (2002, p.8) mention that "this approximation replaces the uncertainty in the default barrier with an uncertainty in the level of the asset value at time $t = 0$". These approximation has its own limitations as pointed out by Finger et al: it includes the possibility of default in the period $(-\Delta t, 0]$, which leads to unrealistic result that there is a non-zero probability of default at $t = 0$. In addition, it is unclear how to implement this procedure for pricing equity options and for using diffusions with stochastic variance and jumps.

A viable alternative is to use the assumption that R is independent of the firm's value dynamics and employ the conditioning on R to represent the survival probability with a random barrier, $\Theta(t, T)$, by:

$$\begin{aligned}\Theta(t,T) &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{F(t)>RB(T)\}}|\mathcal{F}(t)] = \int_0^\infty \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{F(t)>R'B(t)\}}|\mathcal{F}(t)]\mu_R(R')dR'\\ &= \int_0^\infty Q(t,T,R')\mu_R(R')dR',\end{aligned} \tag{6.5.5}$$

where μ_R is PDF of R and $Q(t, T, R) \equiv Q(t, T, S)$ is the survival probability assuming a fixed value of R.

Next, we can use the convexity adjustment formula (1.3.6) and approximate $\Theta(t, V)$ by

$$\Theta(t,T) \approx \left(Q(t,T,\overline{R}) + \frac{1}{2}Q_{RR}(t,T,\overline{R})\vartheta_2 + \frac{1}{6}Q_{RRR}(t,T,\overline{R})\vartheta_3 + \frac{1}{24}Q_{RRRR}(t,T,\overline{R})\vartheta_4\right), \tag{6.5.6}$$

where ϑ_2, ϑ_3, ϑ_4 are the corresponding central moments of R given by:

$$\vartheta_2 = \overline{R}^2(e^{\varrho^2}-1),\ \vartheta_3 = \overline{R}^3(e^{\varrho^2}-1)^2(e^{\varrho^2}+2),\ \vartheta_4 = \overline{R}^4(e^{\varrho^2}-1)^2(e^{4\varrho^2}+2e^{3\varrho^2}+3e^{2\varrho^2}-3).$$

Partial derivatives in (6.5.6) can be approximated numerically with high accuracy given the analytical solutions for $Q(t, T, R)$. In general, the approximation of Nth order derivative by a differencing formula with second order accuracy requires $N +$

1 evaluations of the given function. Thus, by using formula (6.5.6), we need 5 evaluations of $Q(t, T, R)$, which is not burdensome since we use an explicit expression for $Q(t, T, R)$.

In Figure (6.5.1) we illustrate incorporating random default barrier using CreditGrades approximation (CG), convexity formula (6.5.6) with 2 terms (2T) and with 4 terms (4T) for calculating default probability under the regular diffusion. The spread implied by the diffusion with constant barrier $\overline{R}$ (RD) is given for comparison purposes. The term structure of variance is given by formula (6.7.1) and all relevant parameters are taken from Table (6.7.1) to make our analysis consistent with estimates obtained from General Motors. We use $\varrho = 0.2$ and other parameters are taken from Table (6.7.1): $\overline{R} = 0.5$, $S(0) = 25.86$, $B(0) = 65$, $V(0) = S(0) + \overline{R}B(0) = 58.36$, $\nu_0 = 0.11$, $\nu_\infty = 0.05$, $\kappa_\nu = 3.57$. It follows that In CreditGrades approximation, higher values of ϱ imply non-zero default probability at time $t = 0$ leading to unrealistic CDS spreads for very short maturities. For moderate values of ϱ, all three approximations are comparable, especially, for longer maturities.

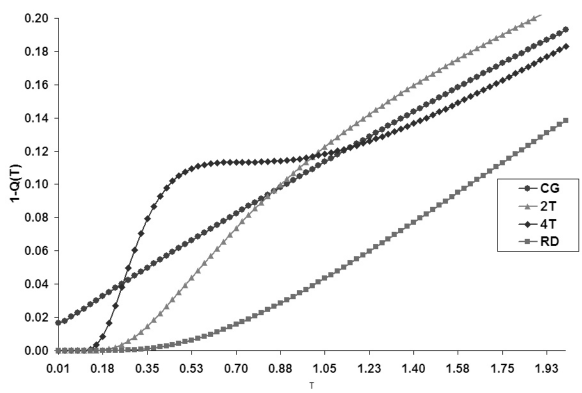

Figure 6.5.1: CreditGrades approximation for default probabilities with random barriers.

In addition, we see that all being the same random default barriers lead to higher shorter-term default probabilities compared to fixed barriers. Thus, random barriers naturally introduce the unpredictability of default and result in higher short-term spreads. Furthermore, mixing random barriers with stochastic variance or jumps can lead to a better modeling of shorter-term spreads.

The advantage of using the convexity adjustment formula (6.5.6) is that is can be extended in a straightforward way for calculating survival probabilities as well as option prices under the diffusion with stochastic variance and double-exponential jump-diffusion.

Let us mention a potential drawback of using log-normally distributed default barrier. Since log-normally distributed random variable can take any positive value, there is a positive probability that $R > 1$ or, equivalently, that the default barrier is greater than the firm's debt, which is unrealistic and counterintuitive consequence. To circumvent this problem, we can assume that R has beta distribution $B(a, b)$ with parameters a and b chosen to ensure that $\mathbb{E}[R] = \overline{R}$ and $\mathbb{V}[\ln(R)] = \varrho^2$. The beta distribution is easy to handle by using the convexity adjustment formula (6.5.6).

6.6. Credit Default Swaps

Now we briefly overview the credit default swap contract, whose active trading raised the need for quantitative models to deal with the default risk. For a more detailed overview of these contracts we refer to Brigo-Mercurio (2006).

A credit default swap (CDS) provides protection against the default of the swap reference name. We assume that the notional of CDS is one currency unit and we consider a CDS with the payer (premium) leg denoted by $P(t, T)$ and with the receiver (protection) leg denoted by $R(t, T)$, where t is annualized inception time of the swap and T is annualized maturity time. We denote the annualized payment schedule associated with this CDS by $\{T_i\}_{i=0..N}$, with $T_0 = 0$ and $T_N = T$, where N is the total number of observations (fixings), and we introduce annualized time periods $\Delta_i = T_i - T_{i-1}$, $i = 1..N$. Typically, the payment schedule is based on quarterly observations and standardized maturity times of a CDS contract at inception are 1,2,3,5,10 years.

Protection buyer of the CDS goes short the payer leg and pays the coupon (or spread) S at times $\{T_i\}_{i=1..N}$ conditioned that the underlying entity has not defaulted up to time $T_i, i = 0..N$, in exchange for the protection in the amount of $(1 - R)$ currency units paid following the default of the reference name. Protection seller of CDS goes short the protection leg and receives coupons S conditioned on the survival of the reference name and pays the protection in the amount $(1 - R)$ upon the default event.

In general, there are two specifications of the protection payments: firstly, in a running CDS, the protection is paid immediately following the default; secondly, in postponed running CDS protection is paid at time $T_{\varphi(\iota)}$, where $\varphi(\iota) = \{i : T_{i-1} < \iota \leq T_i, i = 1..N\}$, that is $T_{\varphi(\iota)}$ is the first payment time following the time of the default ι. For brevity, we consider postponed running CDS.

The cash flow of payer leg $P^p(t, T)$ for a postponed CDS can be presented as:

$$P^p(t, T) = S \sum_{i=1}^{N} \Delta_i \mathbf{1}_{\{\iota > T_i\}}, \tag{6.6.1}$$

while the cash payment of the default leg can be represented by:

$$P^p(t, T) = (1 - R) \sum_{i=1}^{N} \mathbf{1}_{\{T_{i-1} < \iota \leq T_i\}}. \tag{6.6.2}$$

To compute the expected values of the above legs, we employ the survival probability of the reference name $Q(t, T) = \mathbf{1}_{\{\iota > t\}} \mathbb{P}^{\mathbb{Q}}[\iota > T_i | \mathcal{F}(t)]$ under the pricing measure $\mathbb{Q}$. We also note that $Q(t,t) = 1$ given $\mathbf{1}_{\{\iota>t\}} = 1$. As a result, under $\mathbb{Q}$ the value function of the payer leg $P^p(t,T)$ is given by:

$$P^p(t,T) = S\sum_{i=1}^{N} DF(t,T_i)Q(t,T_i)\Delta_i, \tag{6.6.3}$$

while the protection leg is given by:

$$P^p(t,T) = (1-R)\sum_{i=1}^{N} DF(t,T_i)\left(Q(t,T_{i-1}) - Q(t,T_i)\right), \tag{6.6.4}$$

where $DF(t,T)$ is discount factor for a riskless cash flow at time T.

Finally, a fair coupon $S(T)$ for a swap with maturity T is computed by equating the corresponding payer and receiver legs. As a result, for a postponed running CDS, the fair coupon is given by:

$$S(T) = \frac{(1-R)\sum_{i=1}^{N} DF(t,T_i)\left(Q(t,T_{i-1}) - Q(t,T_i)\right)}{\sum_{i=1}^{N} DF(t,T_i)Q(t,T_i)\Delta_i}. \tag{6.6.5}$$

CDS are quoted in the market in terms of their fair coupons (spreads) $S(T)$. Given the term structure of CDS spread quotes we can also extract the market implied default probabilities (or implied hazard rates of the default time) and use them by model analysis and/or calibration (see, for example, Brigo-Mercurio (2006)).

6.7. Calibration to General Motors Options Data

By applying structural models, first we need to obtain accurate estimates for the current firm's asset value $F(0)$ and its volatility. Hull at al (2005) and Stamicar-Finger (2005) use equity option data to back out these estimates. Specifically, Hull at al (2005) use 50- and 25-delta implied put volatilities of options with maturity two month, while Stamicar-Finger (2005) suggest three methods to obtain these estimates: (a) estimate firm's debt per share $B(0)$ from balance sheet data and imply asset volatility from one-year at-the-money (ATM) implied volatility, (b) use one-year ATM volatility and a CDS spread quote to back out both $B(0)$ and asset volatility, (c) use two one-year implied volatilities to imply $B(0)$ and asset volatility.

The purpose of our calibration is to show the ability of presented structural models to fit the implied volatility surface. Since parameters for diffusions with stochastic volatility and double-exponential jumps are not directly observable, we need to use a calibration procedure to back out these parameter estimates from option data. Although this procedure is more involved, it has an advantage to fit a model to implied volatility surface across a few strikes and maturities in addition to some available CDS quotes, and thus it enables us to use all liquid available data to estimate and manage default risk.

To illustrate our models, we calibrate them to General Motors (GM) option data, this is the same data we used in Chapter 4. By calibration, we follow procedure (a) suggested by Stamicar-Finger (2005): we first estimate firm's debt per share $B(0)$ from balance sheet data and then back out model parameters from options implied volatilities. Option implied volatilities were collected from Bloomberg on November 8, 2005. The spot price is $S(0) = 25.86$ USD.

An estimate for ratio of total debt per equity, 12.4, is taken from GM balance sheet. This gives that an approximate debt per share is \$320. However, as pointed out by Stamicar-Finger (2005), over 80% of outstanding GM debt is issued by its financial services subsidiary (General Motors Acceptance Corporation (GMAC)), and since GMAC operates like a financial institution, much of its debt is secured. They also report that since July 2003 implied debt-per-share for GM has been approximately 20%-25% of its total debt-per-share which includes all liabilities of GM and its subsidiaries. Following these remarks, we take \$65 as an estimate of GM debt-per-share and assume that $R = 0.5$. Accordingly, debt-per-share is $D(0) = 32.5$.

For regular diffusion and double-exponential jump-diffusion, we assume that the firm's asset variance is time-inhomogeneous:

$$\nu(t) = \nu_\infty + (\nu_0 - \nu_\infty)e^{-\kappa_\nu t}, \tag{6.7.1}$$

where ν_0 is an initial variance, ν_∞ is a long-term mean, and κ_ν is a reversion speed to the long-term mean.

Parameter estimates are reported in Table (6.7.1). DEJD stands for double exponential jump-diffusion, RD stands for regular diffusion, SV stands for diffusion with stochastic variance.

	DEJD	RD	SV
ν_0	0.0260 (0.1612^2)	0.1051 (0.3242^2)	0.1081 (0.3287^2)
ν_∞	0.0151 (0.1229^2)	0.0541 (0.2345^2)	0.0704 (0.2654^2)
κ_ν	1.2433	3.5662	11.3099
γ	162.5382		
η^+	0.0443		
η^-	0.1181		
q^+	0.4894		
ε_ν			3.0833

Table 6.7.1: Parameter estimates backed out from GM options data

It follows that the regular diffusion and diffusion with stochastic variance imply approximately the same estimates for initial variance, ν_0, and long-term variance, ν_∞, while higher value for mean-reversion speed, κ_ν, in stochastic variance model accounts for a high volatility of variance, ε_ν. Lower values for ν_0 and ν_∞ in double-exponential model can be explained that some uncertainty is introduced by jump risk, whose initial intensity is $\gamma\nu_0 = 4.2308$ and long-term intensity is $\gamma\nu_\infty = 2.4539$.

Based on the quality of fit to market data, it follows that regular diffusion can not explain the pronounced volatility skew of GM options. SV model has a better fit and while fitting at-the-money options it produces sizable differences for option with higher and lower strikes. The double-exponential diffusion is capable of fitting the volatility skew relatively well. In Figure (6.7.1), we show the volatility surface implied by the double-exponential jump-diffusion.

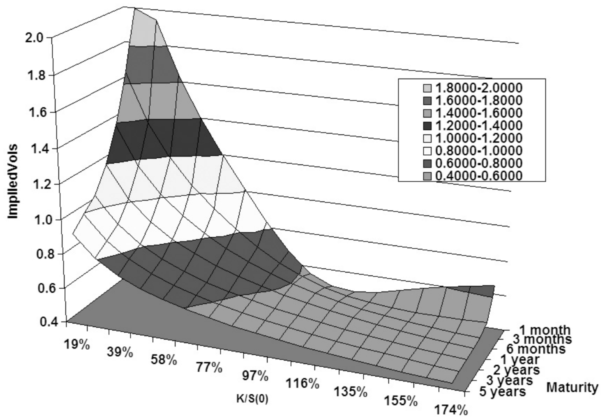

Figure 6.7.1: The volatility surface implied by the structural double-exponential jump-diffusion.

It is interesting to compare the implied volatility skew of the regular double-exponential jump-diffusion studied in Chapter 5 with the present structural model. Comparing figures (5.5.1) and (6.7.1), we see that the structural model implies more pronounced short-term skew, although the implied skew tends to flatten out unlike the skew of the regular double-exponential process.

6.8. Comparison of intensity based and structural models of default

We finish our presentation with making comparisons between the intensity-based default models considered in the first part and structural default models studied in the second part of this thesis. First, we compare some quantities implied by three models considered in this part and the process with stochastic volatility and jump-to-default, whose parameters are implied from the same market data and reported in table (4.4.1). We recall that it implies the default with a constant rate of α, estimated to be $\alpha = 0.1024$.

In Figure (6.8.1), we show the term structure of the default probabilities for the three diffusions with parameters from table (6.7.1) and the equity price process with stochastic volatility and jump-to-default rate $\alpha = 0.1024$ (SV+JD). DEJD stands for double-exponential jump-diffusion, RD stands for regular diffusion, and SV stands for diffusion with stochastic volatility. It also follows that, for a given maturity, the SV+JD process implies the highest default probability.

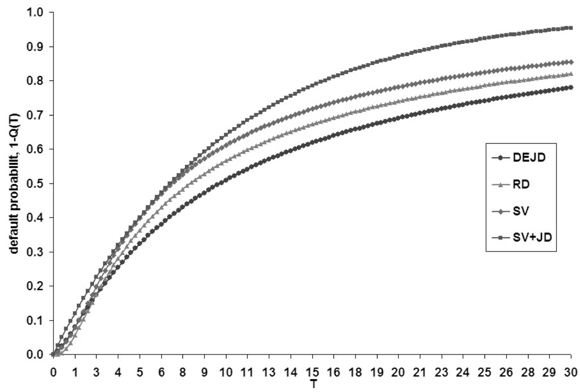

Figure 6.8.1: Implied default probabilities

Next we will look at the density of the default time implied by these processes. The explicit formulas for exit time density, $q(T, S)$, of RD, SV, and DEJD are given respectively by formulas (6.4.16) ,(6.4.17), and (6.4.20). For SV+JD process it is simply $q(T, S) = \alpha e^{-\alpha T}$. The corresponding densities are shown in Figure (6.8.2). We see that the default time density of SV+JD process is the density of exponential random variable with mean equal to $1/\alpha \approx 10$. The default time density of RD and SV processes have a similar shape. We note that the mode of the double-exponential jump-diffusion is less than the modes of RV and SV diffusions so that it implies higher default probabilities for short-term maturities.

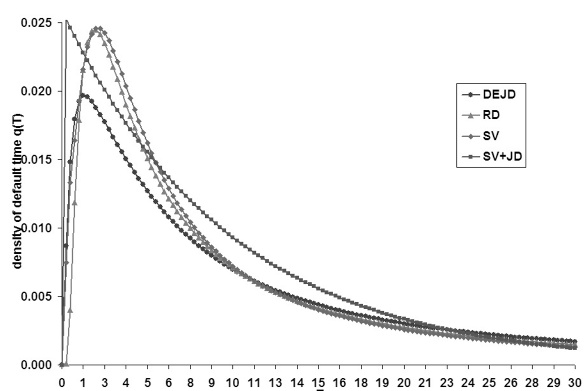

Figure 6.8.2: Implied densities of default time

Finally, we look at the term structure of the the implied CDS spreads shown in

Figure (6.8.3) as functions of T. We see that SV+JD process implied spreads rapidly increase with maturity time T, which is in agreement with empirical observations, and become almost flat after $T = 1$ (because the default intensity rate is constant in T). Spreads implied by DEJD also increase sharply with maturity time T but then they start decreasing. The spread of SV process almost reaches the spread implied by SV+JD process at maturity $T = 5$ and then it starts decreasing.

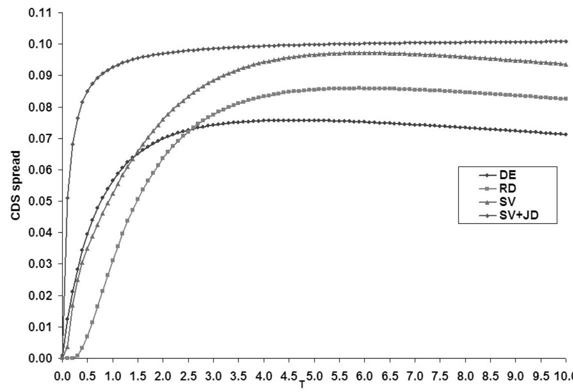

Figure 6.8.3: Implied credit default swap spreads

Figures (6.8.1), (6.8.2), and (6.8.3) demonstrate the key difference between the intensity based and structural approaches: since in the intensity based approach the default can happen at any time right from the contract inception, its CDS spread jumps very rapidly starting from very short maturities; while in the structural approach the probability that the firm's value value crosses the default barrier is small for small maturities, so that its implied CDS spread grows at a slower pace.

Kokkuvõte

Afiinsed mudelid finantsmatemaatikas: analüütiline lähenemine

Väitekiri on pühendatud mitmete praktikas üles kerkinud finantsmatemaatika probleemide lahendamisele. Konkreetsemalt, töös on käsitletud järgmisi olulisi küsimusi:

1) Kuidas ehitada aktiva hinna mudel, mis võtab arvesse hinna ning selle volatiil suse juhuslikku käitumist, hüppeid aktiva hinnas ja selle volatiilsuses, ning aktiva omaniku võimalikku laostumist? Kõik need nähtused esinevad finantsturgudel – seega on tähtis konstrueerida mudel, mis oleks kooskõlas nende ilmingutega.

2) Kuidas leida erinevate optsioonide õige hind? Meie poolt vaadeldavate optsioonide alusvaraks võib olla aktiva hind, aktiva hinna realiseerunud volatiilsus ning aktivaomaniku laostumise sündmus. Selliste optsioonidega kaubeldakse reaalselt turul ja järelikult on otstarbekas arendada välja ühtne metodoloogia nende optsioonide hindamiseks.

Esimese ülesande lahendamiseks on töös välja pakutud kahe faktoriga afiinne mudel, kus esimeseks faktoriks on aktiva hind ning teiseks faktoriks aktiva hinna volatiilsus. Matemaatilises mõttes on tegemist kahest stohhastilisest diferentsiaalvõrrandist koosneva süsteemiga, kuhu on lülitatud ka hinna ja volatiilsuse hüpped ning juhuslik laostumishüpe.

Teise probleemi lahendamiseks on välja töötatud robustsed lahendusmeetodid, mis tuginevad Fourier' ning Laplace'i teisendustel. Töös on leitud lähteprotsesside tihedused, mida on kasutatud nendest protsessidest sõltuvate optsioonide hindamiseks.

Antud töö peamised tulemused on järgmised.

1) On esitatud üldine afiinne mudel ning tuletatud Greeni funktsioonid (tihedusfunktsioonid ilmutatud kujul) selle mudeli alusel kulgevate protsesside jaoks. Töö tähtsaimaks panuseks on üldine lähenemine hindamisprobleemi lahendamiseks afiinse mudeli korral, sealhulgas aktiva optsioonide, realiseerunud volatiilsuse optsioonide ning krediidioptsioonide hindamiseks.

Matemaatiline ülesandepüstitus laostumishüppega hindamisprobleemi jaoks on toodud peatükis 2 ning peatükid 3 ja 4 on pühendatud afiinse mudeli rakendamisele hindamisprobleemi lahendamiseks. Mõned tulemused nendest peatükkidest on kajastatud artiklis Kangro-Pärna-Sepp (2004).

2) On tuletatud volatiilsuse optsioonide hindamise meetodid afiinses mudelis, mis sisaldavad volatiilsuse hüppeid ning laostumishüpet. See materjal on esitatud peatükides 2,3,4. Saadud tulemused on originaalsed ning nad on kajastatud artiklis Sepp (2007b).

Väljatöötatud metoodikat on kasutatud ka artiklis Sepp (2007a) nn tingliku volatiilsuse swap'pide hindamiseks, mis on praegu populaarsed instrumendid volatiilsusega kauplemisel. Lisaks on neid tulemusi rakendatud artiklis Sepp (2007c) nn VIX futuuride ning VIX optsioonide hindamiseks.

3) On uuritud optsioonide hindamist aktiva laostumisvõimaluse korral. Kuigi aktiva optsioonide hindamist laostumisriski olemasolul on hiljuti vaadeldud ka kirjanduses, on meie panuseks üldine metodoloogia, mida on rakendatud ka nn tuleviku-algusega (forward-start) optsioonide korral. Tuleviku-algusega optsioonid on eriti tundlikud volatiilsuse ning laostumissündmuse suhtes ning nad on väga kasulikud aktiva volatiilsuse tulevase evolutsiooni modelleerimiseks. Meie meetod põhineb aktiva tulevikuhinna ning realiseerunud volatiilsuse modellerimisel ning on sellisena uudne lähenemine.

Vastavad tulemused on esitletud peatükides 2,3,4.

4) On tuletatud valemid kahe barjääriga optsioonide hindamiseks juhul, kui aktiva hinna protsessiks on kaksik-eksponentjaotusega hüppe-difusioon. Meie poolt saadud originaalsed tulemused annavad vastuse mõnedele alusvara hinna trajektoorist sõltuvate optsioonide hindamise aktuaalsetele probleemidele.

Neid probleeme on käsitletud peatükis 5 ning tulemused on kajastatud artiklites Sepp-Skachkov (2003) ja Sepp (2004).

5) On laiendatud nn CreditGrades mudelit, mida kasutatakse krediidiriski juhtimiseks, eeldades, et firmaväärtuse protsessiks on kaksik-eksponentjaotusega hüppe-difusioon või difusioon juhusliku volatiilsusega. Tähelepanu on pööratud ka sel
lele, kuidas hinnata mudeli parameetreid, kasutades kaubeldavaid optsioone firma omakapitalile.

Vastavad tulemused on saadud peatükis 6 ning avaldatud artiklis Sepp (2006).

Bibliography

[1] Abate, J., Choudhury, G., Whitt W., (1999), "An introduction to numerical inversion and its application to probability models," *Kluger, Boston*, printed in Computational Probability, W. Grassman (ed.), 257-323.

[2] Abramowitz, M., Stegun, I., (1972), "Handbook of mathematical functions," *Dover*, New York.

[3] Andreasen, J., (2006), "Stochastic Volatility for Real," *Bank of America Fixed Income Quantitative Research*, available at papers.ssrn.com.

[4] Bates, D., (1996), "Jumps and stochastic volatility: exchange rate processes implicit in Deutsche mark options," *Review of Financial Studies 9*, 69-107.

[5] Black, F., Scholes, M., (1973), "The pricing of options and corporate liabilities," *Journal of Political Economy 81*, 637-659.

[6] Black, F., Cox, J. C., (1976), "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance 31*, 351-367.

[7] Boyarchenko, S., Levendorskii, S., (2002), "Barrier options and touch-and-out options under regular Lévy processes of exponential type," *Annals of Applied Probability 12(4)*, 1261-1298.

[8] Bielecki, T., Rutkowski, M., (2002), "Credit Risk: Modeling, Valuation and Hedging" *Springer, Berlin.*

[9] Brigo, D., Mercurio, F., (2006), "Interest Rate Models - Theory and practice - with Smile, Inflation, and credit" *Springer, Berlin*, 2nd edition.

[10] Carr, P., Wu, L., (2007), "Theory and Evidence on the Dynamic Interactions Between Sovereign Credit Default Swaps and Currency Options," *Journal of Banking and Finance,* forthcoming.

[11] Carr, P., Madan, D., (1999), " Option Valuation using the Fast Fourier Transform," *Journal of Computational Finance, 2, No.4, Summer*, 61-73.

[12] CBOE, (2006), "VIX futures white paper," *Chicago Board Options Exchange*, available at www.cboeoptionsinstitute.com/micro/vix/vixwhite.pdf

[13] Cont, R., Tankov, P., (2004), "Financial Modelling With Jump Processes" *Chapman & Hall.*

[14] Cox, J., Ingersoll, J., Ross, S., (1985), "A theory of the term structure of interest rates," *Econometrica, vol. 53, no. 2*, 385-407.

[15] Demeterfi, K., Derman, E., Kamal, M., Zou, J., (1999), "More than you ever wanted to know about volatility swaps," *Goldman Sachs Quantitative Strategies Research Notes.*

[16] Duffie, D., Pan, J., Singleton, K., (2000), "Transform analysis and asset pricing for affine jump-diffusion," *Econometrica, Vol.68, No.6*, 1343-1376.

[17] Duffie, D., (2001), "Dynamic Asset Pricing Theory" *Princeton university Press*, 3rd edition.

[18] Feller, W., (1971), "An Introduction to Probability Theory an its Applications" *Wiley, New York*, 2nd edition.

[19] Finger, C., Finkelstein, V., Pan, G., Lardy, J., Ta, T., Tierney, J., (2002), "CreditGrades technical document" *RiskMetircs Group*, available at www.riskmetrics.com.

[20] Finkelstein, V., (2001), "Assessing default probabilities from equity markets" *Presentation, Credit Risk Summit, NewYork.*

[21] Heston, S., (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *Review of Financial Studies 6*, 327-343.

[22] Elizalde, A., (2005), "Credit Risk Models II: Structural Models," *CEMFI and Universidad Pblica de Navarra working paper.*

[23] Hilberink, B., and Rogers, L. C. G., (2002), "Optimal Capital Structure and Endogenous Default," *Finance and Stochastics 6*, 237-263.

[24] Hull, J., Nelken, I., and White, A., (2005), "Mertons model, credit risk and volatility skews," *Journal of Credit Risk 1*, 328.

[25] Kangro, R., Pärna, K., Sepp, A., (2004), "Pricing European-Style Options under Jump Diffusion Processes with Stochastic Volatility: Applications of Fourier Transform," *Acta et Commentationes Universitatis Tartuensis de Mathematica, vol. 8*, 123-133.

[26] Kou, S., (2002), "A jump diffusion model for option pricing," *Management Science, vol. 48*, 1086-1101.

[27] Kou, S., Wang H (2003), "First passage times of a jump diffusion process," *Advances in Applied Probability, vol. 35*, 504-531.

[28] Lando, D., (2004), "Credit Risk Modelling: Theory and Applications," *Princeton University Press.*

[29] Leland, H., Toft, K., (1996), "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads" *Journal of Finance 51*, 987-1019.

[30] Lewis, A., (2000), "Option valuation under stochastic volatility," *Finance Press, Newport Beach, California.*

[31] Lewis, A., (2001), "A simple option formula for general jump-diffusion and other exponential Lévy processes," *www.optioncity.net, Working Paper.*

[32] Longstaff, F., Schwartz, E., (1995), "A simple approach to valuing risky fixed and floating rate debt," *Journal of Finance, 50(3)*, 789-819.

[33] Lipton, A., (2001), "Mathematical methods for foreign exchange: A Financial Engineer's Approach," *World Scientific.*

[34] Lipton, A., (2002a), "The vol smile problem," *Risk, February*, 81-85.

[35] Lipton, A. McGhee, W., (2002), "Universal Barriers," *Risk, May*, 81-85.

[36] Lipton, A., (2002b), "Assets with Jumps," *Risk, September*, 149-153.

[37] Merton, R., (1973), "Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science*, Vol. 4, No. 1, 141-183

[38] Merton, R., (1974), "On the pricing of corporate debt: the risk structure of interest rates," *Journal of Finance 29*, 449-470.

[39] Merton, R., (1976), "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics 3*, 125-144.

[40] Mikhailov, S., Nogel, U., (2003), "Heston's Stochastic Volatility Model Implementation, Calibration and Some Extensions," *Wilmott Magazine*, July 2003, 74-79.

[41] Øksendal, B., (2003), "Stochastic Differential Equationd," *Springer*, 6th edition.

[42] Press, W., Teukolsky, S., Vetterling, W., Flannery, B., (1992), "Numerical recipes in C: The Art of Scientific Computing," *Cambridge University Press.*

[43] Schönbucher, P., (2003), "Credit Derivatives Pricing Models," *Wiley Finance.*

[44] Sepp, A., Skachkov, I., (2003), "Option Pricing with Jumps," *Wilmott Magazine, November*, 50-58.

[45] Sepp, A., (2004) "Analytical Pricing of Double-Barrier Options under a Double-Exponential Jump Diffusion Process: Applications of Laplace Transform," *International Journal of Theoretical and Applied Finance 2*, 151-175.

[46] Sepp, A., (2006), "Extended CreditGrades Model with Stochastic Volatility and Jumps," *Wilmott Magazine, September*, 50-62.

[47] Sepp, A., (2007a), "Variance Swaps under no Conditions," *Risk, March, vol.20, N.3*, 82-87.

[48] Sepp, A., (2007b), "Pricing Options on Realized Volatility in Heston Model with Volatility Jumps," *Manuscript*, submitted for publication.

[49] Sepp, A., (2007c), "Viable VIX," *Manuscript*, submitted for publication.

[50] Stamicar, R., Finger, C., (2005) "Incorporating equity derivatives into the CreditGrades model," *RiskMetrics Group*, available at www.riskmetrics.com.

[51] Stehfest, H., (1970), " Numerical inversion of Laplace transforms," *Communications of the Association for Computing Machinery, v. 13, no.1*, 47-49.

[52] Skachkov, I., (2002), " Black-Scholes Equation in Laplace Transform Domain," *Wilmott.com, Technical Article (2002).*

[53] Wu, L., Zhang, F., (2006), " LIBOR Market Model with Stochastic Volatility," *Journal of Industrial and Management Optimization, v.2, no.2*, 199227.

[54] Zhou, C., (1997), "A Jump-Diffusion Approach to Modelling Credit Risk and Valuing Defaultable Securities," *Federal Reserve Board, Washington.*

[55] Zhou, C., (2001), "The Term Structure of Credit Spreads with Jump Risk," *Journal of Banking and Finance 25*, 2015-2040.

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