

TARTU UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
Institute of Mathematical Statistics

Maryna Tverdostup

**Optimization of Parameters of a Reinsurance Agreement in
Non-Life Insurance
Master's Thesis (15 cp)**

Supervisor: Meelis Käärik, Ph.D

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OPTIMIZATION OF PARAMETERS OF A REINSURANCE AGREEMENT IN NON-LIFE INSURANCE

The primary purpose of this thesis is to determine the methodologies of composing the optimal risk transfer mechanism from the direct insurer's point of view. The study aims to investigate reinsurance optimization approaches developed by the actuarial science with an emphasis on the derivation of mathematical formulation of the retention level, being a prior parameter of the reinsurance agreement. The application of derived techniques to quota share and excess of loss reinsurance treaties is discussed. Since it is usually admitted that reinsurance should ensure cedent's financial stability, the simulation model is composed to link the direct insurer's risk process and reinsurance parameter in order to analyze the effects of examined optimization methodologies on the insurer's general financial performance. Consequently, the conclusions based on obtained simulation results are provided.

Key terms: non-life insurance, reinsurance, insurance agreements, optimization

EDASIKINDLUSTUSLEPINGU PARAMEETRITE OPTIMEERIMINE KAHJUKINDLUSTUSES

Magistritöö uurib edasikindlustust ja sellega seotud probleeme kahjukindlustuses. Täpsemalt keskendutakse esmase kindlustaja põhiküsimustele edasikindlustega seoses: mis tüüpi edasikindlustusleping valida ja millised peaksid olema selle lepingu sobivaimad parameetrid esmase kindlustaja seisukohast. Töö esimeses osa tutvustatakse lugejale edasikindlustuse valdkonda üldiselt (taust, kontseptsioon, tüübid, juriidilised küsimused), teises osas kirjeldatakse matemaatiline mudel ning optimeerimisülesanne esmase kindlustaja jaoks kasutades kahte lähenemist: kasulikkusfunktsiooni maksimeerimine ja kasumi hajuvuse minimeerimine. Mõlema lähenemise korral vaadeldakse eraldi proportsionaalseid ja mitteproportsionaalseid edasikindlustuslepinguid. Töö teises osas rakendatakse vaadeldud meetodeid simuleeritud andmetele, võrreldakse erinevate edasikindlustuslepingute ja nende parameetrite mõju ning kirjeldatakse iga edasikindlustusmeetodi korral parameetrite muutmisega kaasnevat dünaamikat.

Märksõnad: kahjukindlustus, edasikindlustus, kindlustuslepingud, optimeerimine

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INTRODUCTION

The reinsurance activity can be viewed as a specific risk transfer operation, aiming to protect the insurance company against unpredictable losses and ensure its financial soundness. However, while setting as a goal a risk assurance of the company successfulness of reinsurance operations depends greatly on the reasonableness of the parameters stipulated by the contract. The principal determinant in this aspect is the retention level as the share of initial risk, which the ceding company keeps for its own account. Since the retention rate appears to influence the insurance company's performance in general, the problem of determining of the optimal retention level as a principal parameter of the reinsurance agreement gave a rise to numerous discussions over last years.

The evaluation of the optimal amount of risk retained within direct insurer's responsibility is a complex problem. As a part of overall insurance or in particular reinsurance politics, the estimate of the retention level should contribute to the general goals set by the company, however, being a subject to various specific criteria, the retention should be determined according to them. Hence, the reinsurance optimization techniques developed by actuarial science appeal to parameters and measures of general cedent's activity, setting them as a final target that should be fulfilled after implementation of the given retention estimation methodology. These goals defined by various techniques are commonly formulated in terms of a positive outcome resulting from setting of reinsurance contract with certain optimization approach applied and consequently the investigation of the optimization possibilities and derivation of retention level estimates will be performed with respect to maximization of the positive outcome.

Nevertheless, the implementation of the retention parameter results in a broad scope of possible consequences. The connection between the company's risk process and retention level defined by the chosen optimization technique is of the principal importance in this aspect. As a mechanism of risk reduction reinsurance is supposed to minimize expected insurer's losses, however this pattern of impact does not necessarily hold. Therefore, the analysis of examined methodologies will be accomplished with the investigation of the alterations in the behavior of the cedent's risk process in cases when different types of reinsurance agreements with retention parameters estimated by examined approaches are applied. Consequently, the investigation of optimization methodologies will include both theoretical derivation of retention level estimates and practical evaluation of reasonableness of implementation of given methodologies from the perspective of supporting the cedent's solvency and financial stability.

1. THE REINSURANCE ACTIVITY: THE ESSENCE, PURPOSE AND MAIN TYPES OF AGREEMENTS

1.1. The Concept of Reinsurance

The reinsurance as a specific kind of insurance activity appears to be an essential part of regular operations of insurance business entity. The reinsurance operation can be defined as a contractual arrangement between a reinsurer and a professional insurer, who alone is fully responsible to the policy holder, under which in return for reward the reinsurer bears all or part of the risks assumed by the direct insurer and agrees to reimburse according to specified conditions all or part of the sums due or paid by the cedent to the insured in case of claims [4, p.45]. In a reinsurance contract, as the result of successfully performed reinsurance operation, one party (the reinsurer) for a certain premium agrees to indemnify another party (the reinsured) for specified parts of its underwritten insurance risk [1].

So, in general, reinsurance agreement is an arrangement between an insurer and a reinsurer under which claims that occur in a fixed period of time (e.g. one year) are split between the ceding and reinsuring company in agreed manner. Thus, the insurer is effectively insuring the part of a risk with a reinsurer and provides a corresponding premium for reinsurer for this cover. The risk indemnified against, is the risk that the insurer will have to pay on the underlying insured risk as the insurable event occurs [6, p.11].

More precisely, reinsurance treaties are mutual agreements between different insurance companies with the aim to reduce the risk of the occurring followed by the reduction of the premium of the portfolio [15, p.142]. Along with sharing proportionally in premium and losses, the reinsurer typically pays a ceding commission to the ceding company to reimburse for expenses associated with issuing the underlying policy, i.e. a contribution towards the acquisition costs incurred by the direct insurer while setting the original agreement with insured party [9].

Despite the fact, that reinsurance is a form of risk sharing between insurance companies, the question of whether or not reinsurance can be considered insurance has given rise to numerous theoretical developments. Most of actuarial literature appeals to the idea that by the nature of operations reinsurance is the insurance of insurance companies. However, in European reality this statement appears to be highly controversial, since no definition of insurance business is given by the legal laws of member states of the European Union or by European law. For the instance, under French law reinsurance treaties are no insurance

contracts and reinsurers are not insurers, namely the Insurance Code specifies that reinsurance operations are excluded from the scope of insurance contract law. The same holds under most European laws.

Therefore, it may be not completely coherent to consider reinsurance activity identical to insurance. Under the law reinsurers are not insurers of insurers. Indeed, in the number of European states reinsurers are subject to far lighter licensing procedures and prudential rules than primary insurers and in some countries they are not even regulated. They are not insurers of the insured, as the Insurance Code stipulates that the cedent despite setting the reinsurance agreement remains alone liable to the holder of insurance policy [4, p.46].

The reinsurance relationship is usually structured so that the risk is firstly offered for coverage to insurer by insured and after the part of this risk or its full amount is transferred to reinsurer according to the rules of corresponding reinsurance activity accepted by cedent. However, the reinsurance scheme may include more than two participants. A direct insurer can make a decision to cede certain risk under one contract or purchase several contracts covering different aspects or portions of the same policy to achieve the desired degree of coverage. A layering process involving two or more reinsurance agreements is commonly employed to obtain sufficient monetary limits of reinsurance protection and, when a claim occurs the reinsurers respond in a predetermined order to cover the loss. On its turn, the reinsurance company may itself, again for some suitable premium, pass on parts of the reinsured risks to another reinsurer (both domestic and international), which is called retrocession [1]. Hence, the scale of reinsurance agreements related to certain risk or a group of risks may enlarge significantly.

1.2. Main Purpose and Objectives of Reinsurance Activity

Being an essential element of insurance operations, reinsurance reasonably comes into action, when an individual risk is too big for the given insurance company or the loss potential of the entire portfolio is too heavy. Therefore, reinsurance is an utter necessity for portfolios, which are subject to catastrophic risks such as earthquakes, hurricanes, floods, failures of nuclear power stations, tanker accidents, wars, riots [15, p.142].

In this case the risk of potential losses is adequately separated between parties of reinsurance agreement, i.e. the fundamental function of reinsurance – risk spreading – is realized. Risk spreading possibility allows the insurance company to maintain stability and certainty that its activity in some part is secured from the risk of unpredictable losses. Besides risk spreading

function reinsurance serves a number of other substantial functions for the direct insurance company, namely: increasing the capacity to write insurance (under prevailing insurance-regulatory law), stabilizing financial results through providing the protection against financial losses, protecting against catastrophic losses and financing growth [20, p.5].

1.3. The Legal Classification of Reinsurance Operations

In order to figure out all the possible kinds of the reinsurance contracts settled between ceding companies and reinsurers, the available reinsurance possibilities by the form of mutual obligations, i.e. by the rights and duties of the contracting parties, can be distinguished. In this aspect all arrangements can be classified as facultative individual reinsurance, facultative-obligatory agreements and compulsory treaty reinsurance contracts.

1) Facultative (individual) reinsurance is historically the first form of cover. The object of the treaty is a given risk already analyzed by the insurer who forwards his analysis to potential reinsurers. The cedent offers a cover for this risk, hence this kind of agreement implies that direct insurer has a right to offer a single risk to one of several reinsurers and the latter is free to decide whether to accept the suggested risk fully, partially or to decline it [20, p.70].

2) Facultative-obligatory reinsurance provides different rights and obligations of parties of reinsurance agreement. The decision concerning transferring of the particular share of the risk depends on the direct insurer, so, in this sense, for him the treaty remains optional. The reinsurer, on the contrary, is obliged to accept the cessions decided by its cedent, hence for the reinsurer the agreement is mandatory to accept.

3) Obligatory (treaty) reinsurance restores symmetry of rights and duties between the contracting parties. During such operations the ceding company agrees to cede according to given procedures all or part of the risks during a given period, very often equal to an accounting period. The reinsurer, on its turn, is obliged to accept all the cessions that are proposed to him under these conditions. Unlike facultative reinsurance, this kind of contract refers to the transferring of not a single risk, but the entire portfolio under the agreed upon conditions. This implies much less management efforts related to setting and tracing reinsurance arrangements, hence obligatory reinsurance agreement is the most widely used in practice. However, insurers sometimes purchase both facultative and treaty reinsurance to cover the same risk. Unless there are contract terms to the contrary, the facultative reinsurance will perform first and completely before any of the treaty reinsurance performs [4, p.48].

1.4. The Technical Classification of Reinsurance Agreements

The reinsurance activity is realized through a number of different types of agreements between ceding and accepting part. These contracts can be settled using different principles of dividing responsibility between direct insuring and reinsuring companies, namely proportional and non-proportional approach to estimation of the methodology of risk transfer. In all forms of reinsurance agreements both, proportional and non-proportional types of contract, are distinguished.

1) The proportional reinsurance, also known as “Pro Rata reinsurance”, implies that insurer pays a fixed proportion of each claim, that occur during the period of the reinsurance contract, and the remaining part of claim is reimbursed by the reinsurer. The distinctive characteristic of this type of insurance agreement is that the proportion between reinsurance premium and gross premium is equal to proportion between reinsurance and gross claim for each individual risk insured:

$$\frac{\text{Ceded Premiums}}{\text{Gross Premiums}} = \frac{\text{Ceded Claims}}{\text{Gross Claims}}.$$

In order to express the formulas of the most common types of reinsurance agreements, consider the following notations:

S – sum insured of the arbitrary risk in the portfolio;

X – amount of an individual claim;

Y – amount of claim paid by the direct insurer;

Z – amount of claim paid by the reinsurer.

So, in all cases the total claim is split between ceding company and reinsurer, i.e. $X = Y + Z$.

The proportional reinsurance is performed in the form of quota share and surplus reinsurance arrangements.

a) Under a quota share agreement being the most straightforward kind of reinsurance contract the ceding company retains a fixed percentage of each risk (say a proportion $\beta, 0 < \beta < 1$) of the covered portfolio and transfers to the reinsurer the remaining share (proportion $1 - \beta$), hence the reinsurer pays the same proportion of each claim and receives equal proportion of gross premium (with commissions deducted).

So, Y and Z as the amounts of a claim compensated by the direct insurer and reinsurer are defined as follows:

$$Y = \beta X, Z = (1 - \beta)X. \quad (1.1)$$

According to the definition of Y and Z , the main parameters of corresponding quantities are:

- expected value:

$$E[Y] = \beta E[X], E[Z] = (1 - \beta)E[X]; \quad (1.2)$$

- variance:

$$Var[Y] = \beta^2 Var[X], Var[Z] = (1 - \beta)^2 Var[X]; \quad (1.3)$$

- standard deviation of claims:

$$SD[Y] = \beta SD[X], SD[Z] = (1 - \beta)SD[X]. \quad (1.4)$$

Since the whole loss volume is split up between the ceding and reinsuring companies in certain proportion, the reinsurer does not carry the full responsibility for the “risky tail” of the claim distribution, thus this type of reinsurance is considered as the more attractive for the latter [11, p.222].

b) Surplus agreement implies that the ceding company retains certain maximum sum of each risk. In non-life insurance this limit is called “deductible” or, more generally, “retention level”, and, specifically in the analyzed type of reinsurance arrangement, – “one line” (denoted by m). This maximum retention may vary greatly from risk class to another.

If the incurred loss exceeds the insurer’s retention level, the amount above the limit is ceded to reinsurer, but only up to a certain multiple of retention (e.g. 10 lines). Once these cession rates are determined, the treaty operates as a quota share for each policy. Risks, whose insured value exceeds the underwriting limit, do not fall within the scope of the treaty. This leads to sole responsibility of the ceding company for the “risky tail” of claim amount distribution.

The cedent’s and reinsurer’s parts of reimbursement can be defined as

$$Y = \begin{cases} X & \text{if } S \leq m \\ \frac{m}{S}X & \text{if } S > m, \end{cases}$$

$$Z = \begin{cases} 0 & \text{if } S \leq m \\ \left(1 - \frac{m}{S}\right)X & \text{if } S > m. \end{cases}$$

The advantage of the surplus agreement over the quota share is that it allows to model the retained risk profile of insurer with greater precision: the higher is the cedent’s risks, the bigger share of it will be reinsured, i.e. the percentage of premium and liability ceded for each risk can vary. However, this type of treaty is relatively seldom used except for portfolios of

very reduced sizes, since it entails a more time-consuming administrative management than in the quota share case, since retention rates, consequently premiums and ceded claims, are determined on a policy-by-policy basis, which is unreasonably complex unless the number of risks is very small. Non-proportional reinsurance allows insurer to fix the same set of cessions more efficiently and with much less administrative efforts [4, p.54].

Proportional reinsurance, despite the fact that both insurer and reinsurer deal with all risks has significant advantages, namely simplicity of administrating and strong protection against severity of losses.

2) Non-proportional reinsurance includes all the treaties, which by their construction do not satisfy the property of similarity between the rates of ceded premiums and ceded claims. This kind of reinsurance agreement is realized through several types of contracts, specifically excess of loss, stop loss and largest claims reinsurance.

a) Excess of loss treaty refers to a term describing a reinsurance transaction, that being subject to a specified limit, indemnifies a ceding company against the amount of loss in excess of a specified retention. Hence on each claim exceeding the priority (the retention level of the direct insurer, known as a threshold or first risk denoted by $M, M > 0$), reinsurer will compensate the exceeding amount subject to a maximum (cover amount fixed by the parties of reinsurance contract, called a second risk). Therefore, under this type of agreement a claim is shared between the direct insurer and reinsurer only if the claim exceeds a fixed amount of retention, otherwise the insurer reimburse the full amount of claim [6, p.13].

So, the amounts paid by direct insurer and reinsurer can be expressed in the following way:

$$\begin{aligned} Y &= \begin{cases} X & \text{if } X \leq M \\ M & \text{if } X > M, \end{cases} \\ Z &= \begin{cases} 0 & \text{if } X \leq M \\ X - M & \text{if } X > M. \end{cases} \end{aligned} \quad (1.5)$$

The given expressions can be rewritten as

$$\begin{aligned} Y &= \min(X, M), \\ Z &= \max(0, X - M). \end{aligned} \quad (1.6)$$

Under the conditions of excess of loss reinsurance the direct insurer has limited liability on each claim (maximum threshold M), therefore no exposure to the “risky tail” of the claim amount distribution. It apparently causes a higher profitability of the given type of risk transfer from the ceding company’s point of view.

The expected value of the direct insurer's payoff under the excess of loss reinsurance treaty can be derived as follows:

$$\begin{aligned}
 E[Y] &= \int_0^M x f_X(x) dx + \int_M^\infty M f_X(x) dx = \int_0^\infty x f_X(x) dx - \int_M^\infty x f_X(x) dx + \int_M^\infty M f_X(x) dx \\
 &= E[X] - \int_M^\infty (x - M) f_X(x) dx,
 \end{aligned} \tag{1.7}$$

where f_X is the probability density function of the individual claim amount X and $E[Z] = \int_M^\infty (x - M) f_X(x) dx$ is the expected amount of reinsurer's payout of the claim representing an expected reduction in the payout of claims of the direct insurer [20, p.71].

Considerable advantage of the excess of loss reinsurance is that among all individual reinsurances with the same expected reinsurance loss and the same reinsurance premium excess of loss reinsurance is optimal, in the sense that it is not possible to find any other type of reinsurance with a retained risk less risky in the sense of the stop loss order. This explains why excess of loss reinsurance is so common on the reinsurance market [5].

b) The second type of non-proportional reinsurance agreement, stop loss treaty, aims to fulfill the main purpose of reinsurance – stabilization of the net result rate, i.e. limitation of the ceding's company yearly loss. With a stop loss the reinsurer pays all the claims exceeding a certain percentage of underlying premium volume (called as retention or stop loss point) up to a specified limit.

The direct insurer's and reinsurer's parts of reimbursement are given as:

$$\begin{aligned}
 Y &= \begin{cases} X & \text{if } X \leq \rho P \\ \rho P & \text{if } X > \rho P, \end{cases} \\
 Z &= \begin{cases} 0 & \text{if } X \leq \rho P \\ X - \rho P & \text{if } X > \rho P. \end{cases}
 \end{aligned}$$

Here ρP represents the stop loss point depending on the gross premium volume P and agreed upon limiting percentage ρ of the premium [20, p.74].

The expected claim payout of the direct insurer and reinsurer as a special case of excess of loss reinsurance with retention level denoted as $\vartheta = \rho P$ ($\vartheta > 0$), can be expressed in the following way:

$$E[Y] = E[X] - \int_\vartheta^\infty (x - \vartheta) f_X(x) dx,$$

$$E[Z] = \int_\vartheta^\infty (x - \vartheta) f_X(x) dx.$$

Hence by the way of performing stop loss reinsurance is similar to excess of loss case, however implying that the retention level depends directly on the gross premium level and the level of profitability the cedent aims to maintain in case of huge actual claim burden.

c) Largest claims reinsurance is designed for coverage against excessive claims and stipulates that the reinsurer pays a certain number of largest claims of one year. As a type of extreme value reinsurance, largest claims reinsurance is less frequent kind of reinsurance treaty. This kind of agreement is mostly settled in combinations with excess of loss or stop loss reinsurance [20, p.72].

Therefore, the proportional reinsurance has considerable advantages, namely: good protection against frequency or severity potential depending upon the retention level, a greater net premium retention allowed for direct insurer, higher reasonableness in terms of reinsurance premium and cost of administration [17, p.7].

1.5. The Question of Optimality of Reinsurance Contract

So, the insurance practice has developed a range of types of agreements available for setting between the direct insurer and reinsuring party. However, each reinsurance form has its particular advantages and disadvantages in terms of the type of protection it provides (frequency risk, large claim risk), premium calculation, practical handling, administration and processing of loss estimation (including issues like moral hazard and adverse selection) [1]. Moreover, the question of optimal reinsurance agreement contains not only the complexity of the choice between various kinds of reinsurance contracts, but, what is of the prior importance, the problem of estimation of parameters of such an agreement, namely the scale of insurer's (or reinsurer's) responsibility for a given risk or set of risks.

Since retention level appears to be a principal parameter in the relation between direct insurer and reinsurer, any type of reinsurance contract is in some way determined by cession rate. The estimation of this rate consequently needs significant preciseness and particular analysis. Hence in the past few years the problem of optimal reinsurance contract gained much interest in actuarial literature inducing the development of a number of numerical approaches aiming to determine the most reasonable parameters of reinsurance arrangements.

2. OPTIMIZING REINSURANCE CONTRACTS

When making a decision concerning settling the reinsurance agreement, direct insurance company bears a responsibility of choosing the type of reinsurance contract and fixing its main parameter – the level of retained risk. Hence, primary concerns in this aspect are related to the choice of an appropriate reinsurance type and the extent of coverage to be purchased. In particular, the direct insurer's management require an evaluation process that would be capable of aiding their assertions about what kind of contract to settle and under which conditions.

However, the general analysis of possible reinsurance agreement, as an important part of the decision making process, involves not only the consideration of the elements of the reinsurance contract, but also the relationship of the reinsurance decision to the other risk-related decisions of the ceding company, which include the choices of reserving policies, risk pooling, investment policies. And since all of these decisions affect the total risk picture of the firm, they should not be made independently [18].

The problem of fixing the reasonable retention level is a very complex one– on the one hand because of number of different criteria like solvency requirement, company's capacity, financing, services provided, the desirable level of net result, as a measure and objective of company's activity, and, on the other hand, because in practice insurers are not concentrated upon setting up one single retention, but rather how to design in the somehow optimal way an entire insurance or reinsurance program [20, p.76].

Nevertheless, the actuarial science developed several optimization approaches based on different ideas, but following the same aim – to determine the expression of insurer's retention such that certain positive outcome of insurer's activity will be achieved. When performing the optimization procedure, an insurer may have a concrete objective, such as to maximize expected total wealth by the end of certain period (subject to a certain security level condition), to minimize the volatility of expected revenue after reinsurance, to improve a control over the variability of total claim amount or to minimize the probability of company's insolvency. Hence, all available methodologies refer to different indexes, however are guided by the same prime purpose – estimation of the most reasonable retained share of risk under different kinds of arrangements.

Namely, the first examined optimization procedure – so called utility approach – appeals to maximization of the insurer's wealth resulting from arranging the reinsurance agreement,

using cedent's utility function. The other optimization approach investigates the best retention strategy from the insurer's point of view with respect to minimization of the variability of its net profit after covering all reinsurance related costs. Both procedures are performed from the cedent's perspective, hence the derived formulas of retention levels can be applied by the direct insurer when making the decisions concerning reinsurance agreements.

However, in practice the concrete form and amount of reinsurance choice for a certain portfolio are often influenced by experience, availability of reinsurance offers and current market prices as well [1]. Therefore, the reinsurance solutions are much more complex, including many factors beyond the reinsurance activity and are not limited only by the problem of the optimal cession rate.

2.1. Utility Maximization Approach

Utility approach appears to be one of the most developing technics of reinsurance optimization, rapidly gaining its popularity over last several decades. In particular, Friefelder [8] suggests that "the best method of determining property-liability insurance rates is through the use of utility theory". Friefelder's arguments can be applied equally strongly to the reinsurance decision, whereby the expected utility decision criterion can be usefully applied to the problem of the evaluation of reinsurance alternatives and estimation of the optimal share of the retained risk.

In order to investigate the possibilities of optimization of reinsurance contracts through the maximization of utility function consider the direct insurer performing insurance activity, which may result in gains or losses, with the initial capital W . Assume that the direct insurer adopts the utility function $u(x)$. In particular, we assume that insurer has an exponential utility function

$$u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x}), \quad (2.1)$$

where α ($\alpha > 0$) is an absolute risk aversion of the direct insurer. Since constants in the function are for scaling purpose, we consider the exponential utility function in the form $u(x) = -e^{-\alpha x}$, for some $\alpha > 0$.

If a certain action A leads to financial gain $Q(A)$, the direct insurer objectively chooses the action A , which maximizes

$$E[u(W + Q(A))] = E[-e^{-\alpha(W+Q(A))}].$$

It can be assumed that the ceding part is exposed to the risks with aggregate claims variable S , having the compound Poisson distribution with parameter λ (i.e. $S \sim CP(\lambda, F_X)$, where F_X is a cumulative distribution function of X).

The given methodology will be applied in case of two most common types of reinsurance agreements: proportional (quota share) and non-proportional (excess of loss).

2.1.1. The Quota Share Reinsurance

Consider a direct insurer setting a quota share reinsurance agreement with retention proportion β , incurring aggregate claim amount $S_I = \beta S$ ($S_I \sim CP(\lambda, F_Y)$, where $Y = \beta X$). The reinsurer, on its turn, takes a responsibility for the aggregate claims $S_R = (1 - \beta)S$ ($S_R \sim CP(\lambda, F_Z)$, where $Z = (1 - \beta)X$). Hence the moment generating function of S_R is

$$M_{S_R}(t) = e^{\lambda(M_Z(t)-1)} = e^{\lambda(M_X((1-\beta)t)-1)}.$$

Assume that the reinsurer charges a premium P_R estimated with the exponential principle with the loading factor η . The expression of the premium considering that the reinsurer adopts the utility function $u(x) = -e^{-\eta x}$ is the following:

$$\begin{aligned} P_R &= \frac{1}{\eta} \ln E[e^{\eta S_R}] = \frac{1}{\eta} \ln M_{S_R}(\eta) = \frac{1}{\eta} \left(\lambda (M_X((1-\beta)\eta) - 1) \right) \\ &= \frac{\lambda}{\eta} \left(\int_0^\infty e^{(1-\beta)\eta x} f_X(x) dx - 1 \right), \end{aligned} \quad (2.2)$$

where $f_X(x)$ is the probability density function of X and η is the reinsurer's premium loading factor.

So, by the end of insurance period the direct insurer's wealth will be $W + P - P_R - \beta S$, where P is the company's gross premium. Thus, the goal of the optimization is maximizing the following equality over all β ($0 \leq \beta \leq 1$)

$$E[u(W + P - P_R - \beta S)] = E[-e^{-\alpha(W+P-P_R-\beta S)}] = -e^{-\alpha(W+P)} e^{\alpha P_R} E[e^{\alpha \beta S}],$$

which is equivalent to the minimization of the direct insurer's expected payoff

$$e^{\alpha P_R} E[e^{\alpha \beta S}] = e^{\alpha P_R} M_S(\alpha \beta) = e^{\alpha P_R} M_N(\ln M_X(\alpha \beta))$$

and, considering that number of claims N has Poisson distribution, the previous expression can be rewritten as

$$e^{\alpha P_R} M_N(\ln M_X(\alpha \beta)) = e^{\alpha P_R} e^{\lambda(M_X(\alpha \beta)-1)}.$$

Thus, after taking logarithms, the minimizing problem can be stated as finding β that minimizes

$$h(\beta) = \alpha P_R + \lambda \left(\int_0^{\infty} e^{\alpha\beta x} f_X(x) dx \right). \quad (2.3)$$

This function can be differentiated with respect to β in order to find the optimal proportion of cedent's retention.

Given the formula for the reinsurer's premium P_R (2.2), the derivative of the function (2.3) with respect to β is the following

$$\begin{aligned} \frac{\partial h(\beta)}{\partial \beta} &= \alpha \left(\frac{\lambda}{\eta} (-\eta) \int_0^{\infty} x e^{(1-\beta)\eta x} f_X(x) dx \right) + \lambda \left(\alpha \int_0^{\infty} x e^{\alpha\beta x} f_X(x) dx \right) \\ &= -\alpha\lambda \int_0^{\infty} x e^{(1-\beta)\eta x} f_X(x) dx + \alpha\lambda \int_0^{\infty} x e^{\alpha\beta x} f_X(x) dx \\ &= -\alpha\lambda \int_0^{\infty} [e^{(1-\beta)\eta x} - e^{\alpha\beta x}] x f_X(x) dx. \end{aligned}$$

Therefore, the only case when $h'(\beta) = 0$ is if $(1 - \beta)\eta = \alpha\beta$. It follows, that

$$\beta = \frac{\eta}{\eta + \alpha}. \quad (2.4)$$

The distinctive feature of the obtained formula is that the retention level in case of quota-share reinsurance of the direct insurer is independent of the individual claim amount distribution. The proportion of the risk kept by the cedent is influenced only by the risk aversion of the insurer (α) and premium loading parameter of the reinsurer (η). Therefore, the more risk averse the direct insurer is, the higher is the risk aversion parameter coming from the utility function of the cedent and, consequently smaller share of the risk will be retained by the insurer. Thus, the result implies, that as the insurer's risk aversion increases, the insurer's share of each incurred claim decreases [6, p.191-192].

To conclude with application of utility function to the optimization of direct insurer's retention level, it should be mentioned that the analyzed approach has a set of limitations. As the analysis is based on the exponential utility function, the premium received by direct insurer does not effect the decision concerning risk retention level. However, if to assume that the reinsurance premium is paid from the original premium collected by cedent, it may seem unreasonable to ignore the impact of insurer's premium size.

2.1.2. The Excess of Loss Reinsurance

The second analyzed type of the reinsurance arrangements is non-proportional excess of loss contract. Hence, consider the direct insurer, adopting excess of loss reinsurance agreement with retention level M , collecting premium P for the given risk (group of risks). The reinsurer, accepting the risk, charges the premium P_R , determined with the expected value principle, assuming the loading factor ξ .

Hence, the reinsurer's premium can be expressed as follows

$$\begin{aligned} P_R &= (1 + \xi)\lambda E[Z] = (1 + \xi)\lambda \left(\int_0^\infty x f_X(x) dx - \int_0^M x f_X(x) dx - M(1 - F_X(M)) \right) \\ &= (1 + \xi)\lambda \int_M^\infty (x - M) f_X(x) dx. \end{aligned} \quad (2.5)$$

The direct insurer's aggregate claim amount S_I has a Compound Poisson distribution ($S_I \sim CP(\lambda, F_Y)$), where Y is the claim amount of the ceding company net reinsured claims.

Hence, the moment generating function of S_I is

$$M_{S_I}(t) = E[e^{tS_I}] = E[e^{\lambda(M_Y(t)-1)}].$$

Since the aggregate claim amount S_I depends on the threshold M , the moment generating function $M_{S_I}(t)$ depends on M as well.

As the ceding company aims to maximize its wealth, i.e. support and increase profitability, the optimization procedure implies the maximization of the current company's wealth at a given time moment. Therefore, by the end of the insurance period the ceding company's wealth is $W + P - P_R - S_I$, so M should be chosen such that

$$E[u(W + P - P_R - S_I)] = E[-e^{-\alpha(W+P-P_R-S_I)}] = -e^{-\alpha(W+P)} e^{\alpha P_R} E[e^{\alpha S_I}]$$

is maximized. This is equivalent to minimizing of the cedent's expected payout over all possible thresholds M , expressed by the following equality:

$$e^{\alpha P_R} E[e^{\alpha S_I}] = e^{\alpha P_R} e^{\lambda(M_Y(\alpha)-1)}.$$

Thus, after taking logarithms, we can state our minimization problem as finding threshold M , which minimizes

$$h(M) = \alpha P_R + \lambda(M_Y(\alpha) - 1). \quad (2.6)$$

Now taking into account the expression of P_R (2.5), and also considering that

$$M_Y(\alpha) = \int_0^M e^{\alpha x} f_X(x) dx + e^{\alpha M} (1 - F_X(M)),$$

the derivative of function (2.6) can be written as:

$$\begin{aligned}
\frac{\partial h(M)}{\partial M} &= \alpha\lambda(1 + \xi) \left(-Mf_X(M) - (1 - F_X(M)) + Mf_X(M) \right) \\
&\quad + \lambda \left(e^{\alpha M} f_X(M) + \alpha e^{\alpha M} (1 - F_X(M)) - e^{\alpha M} f_X(M) \right) \\
&= -\alpha\lambda(1 + \xi)(1 - F_X(M)) + \lambda\alpha e^{\alpha M} (1 - F_X(M)) \\
&= \alpha\lambda(1 - F_X(M))(e^{\alpha M} - (1 + \xi)).
\end{aligned}$$

Consequently, $h'(M) = 0$ when M satisfies one of the following cases:

- 1) $F_X(M) = 1$, meaning that the retention proportion is equal to 1, i.e. “no reinsurance” case.
- 2) $1 + \xi = e^{\alpha M}$, which, after taking logarithm, implies the following retention level

$$M = \frac{1}{\alpha} \ln(1 + \xi), \quad (2.7)$$

Thereby, optimal retention level depends on the parameter of the utility function, namely insurer’s risk aversion, and parameter of the reinsurer’s premium calculation principle, but does not depend on the individual claim amount distribution, similarly to the previous case. Retention level M appears as an increasing function of ξ implying that the direct insurer will keep a bigger share of the risk as the price of the risk transfer increases. The risk aversion parameter α , on the contrary, has an inverse relation with retention level: as the insurer’s risk aversion increases, the portion of risk retained decreases, hence the cedent’s share of each claim falls [6, p.193].

2.2. Direct Insurer’s Net Profit Variance Minimization Approach

In order to express the most reasonable retention rate another possible methodology can be considered. This approach is based on the idea of minimizing the variability of the cedent’s net profit, after covering all reinsurance expenses. The examined model was proposed by B. de Finetti and applied in cases of excess of loss and quota-share reinsurance arrangements (1940). In given optimization problem the direct insurer, holding the portfolio, consisting on n independent risks, is taken into consideration. For each of risks (group of risks) the direct insurer arranges a reinsurance contract (excess of loss or quota share agreement). However, in further analysis a certain group of risks will be assumed as a separate insurance portfolio being an object of reinsurance optimization procedure.

The cost of reinsurance is determined with the expected value principle (i.e. the security loading, fixed by the reinsurer, is added to its expected payout of each separate risk). The variance, as the target of minimization, is considered to be a subject to the fixed suitable

constraint, namely the expected value of cedent's net profit. Hence, given methodology implies the identifying of optimal retained proportions of risks, such that secure the direct insurer against the variability of achieved revenue and increase its financial stability.

2.2.1. The Quota Share Reinsurance

Firstly, assume the cedent with the insurance portfolio, settling the quota share reinsurance agreement. For the given portfolio of risks aggregate claim amount is S with compound Poisson distribution ($S \sim CP(\lambda, F_X)$), where X is the claim amount variable, with a distribution function F_X and probability density function f_X .

Since the aim of optimization is to minimize variance of cedent's expected net profit in respect to its fixed expected value (i.e. the desirable outcome defined by insurer), all the components of reinsurance process, influencing the insurer's financial outcome, should be considered.

For each insured risk (group of risks) the following parameters should be assumed:

- M – the cedent's retention level;
- P – the cedent's premium charged for the risk;
- Y – the direct insurer's payout of incurred claim;
- Z – the reinsurer's payoff of incurred claim;
- S_I – the insurer's total payout of claims for risk;
- S_R – the reinsurer's aggregate payout of claims for underlying risk;
- ξ – the reinsurer's security loading, referred to in the premium calculations.

Hence, to set up the optimization problem, assume the following quantities:

– the reinsurer's premium charged

$$(1 + \xi)E[S_R] = (1 + \xi)(1 - \beta)E[S]; \quad (2.8)$$

– the cedent's net profit

$$NP(\beta) = P - (1 + \xi)E[S_R] - S_I = P - (1 + \xi)(1 - \beta)E[S] - S_I; \quad (2.9)$$

– the expectation of insurer's net profit, as a suitable constraint for variance minimization,

$$\begin{aligned} E[NP(\beta)] &= P - (1 + \xi)(1 - \beta)E[S] - \beta E[S] = P - (1 - \beta - \xi\beta + \xi)E[S] - \beta E[S] \\ &= P - (1 - \xi\beta + \xi)E[S]; \end{aligned} \quad (2.10)$$

– variance of the net profit, following from (1.3) is, respectively,

$$Var[NP(\beta)] = Var[S_i] = \beta^2 Var[S]. \quad (2.11)$$

Applying the method of Lagrange multipliers to perform given optimization, define the function $h(\beta)$ as

$$h(\beta) = Var[NP(\beta)] - \gamma(E[NP(\beta)] - c), \quad (2.12)$$

where c denotes the constant value of expected net profit as a suitable constraint for the given optimization problem.

The derivatives of $E[NP(\beta)]$ and $Var[NP(\beta)]$ from (2.12) with respect to β are $\frac{\partial}{\partial \beta} E[NP(\beta)] = \xi E[S]$ and $\frac{\partial}{\partial \beta} Var[NP(\beta)] = 2\beta Var[S]$ accordingly. Thereby, the derivative of the function $h(\beta)$ with respect to β is given as

$$\frac{\partial h}{\partial \beta} = 2\beta Var[S] - \gamma \xi E[S],$$

and, consequently, the extremal value of h is reached if

$$2\beta Var[S] = \gamma \xi E[S] \Leftrightarrow \beta = \frac{\gamma \xi E[S]}{2Var[S]}. \quad (2.13)$$

The obtained formula of optimal retention display that the cedent's relative retention does not depend on total premium collected for the given risk i , but is in direct relation with the reinsurer's security loading (the higher ξ , the bigger premium the reinsurer will charge and the smaller share of risk will be passed to the reinsurer) [9].

The obtained optimization result appears to be indeed straightforward. The reverse relation between retained risk proportion and variance of direct insurer's total claims identifies the cedent's tending to increase the cession rate if its total claim amount has a higher variability, i.e. the riskiness of insurance operations is higher due to the increased probability of huge or frequent claim arrival.

2.2.2. The Excess of Loss Reinsurance

The same optimization idea can be implemented in case, when the direct insurer arranges a non-proportional (excess of loss) reinsurance treaty. In this aspect, consider the direct insurer holding the same portfolio of risks with corresponding parameters as in the previous section, settle the excess of loss reinsurance agreement with a threshold M .

To achieve the goal of optimization, i.e. fix the reasonable retained risk share, through the minimization of uncertainty of the expected net profit of direct insurer, the set of quantities should be considered.

Namely, the reinsurer's premium, charged for the portfolio of risks, can be given as

$$(1 + \xi)E[S_R] = (1 + \xi)\lambda E[Z] = (1 + \xi)\lambda \int_M^\infty (x - M)f_X(x)dx. \quad (2.14)$$

The direct insurer's payout on the given portfolio can be expressed as follows

$$E[S_I] = \lambda E[Y] = \lambda \left(\int_0^M x f_X(x) dx + M(1 - F_X(M)) \right), \quad (2.15)$$

where Y is the direct insurer's payout on individual incurred claim.

The expected net profit of the direct insurer, depending on the cedent's premium for the risk, reinsurance premium to be paid and total payout of claims, caused by the given risk, can be expressed (similarly to 2.9) as

$$NP(M) = P - (1 + \xi)E[S_R] - S_I. \quad (2.16)$$

The expected value of net profit, as one of major quantities required for estimation of optimal retention, considering the fact that $E[S_I] = E[S_I^I] + E[S_I^R]$, can be determined as

$$\begin{aligned} E[NP(M)] &= P - (1 + \xi)E[S_R] - E[S_I] = P - (1 + \xi)(E[S] - E[S_I]) - E[S_I] = \\ &= P - (1 + \xi)E[S] + \xi E[S_I], \end{aligned} \quad (2.17)$$

hence, the only source of randomness is the aggregate claim amount of given portfolio. Therefore, the uncertainty of the net profit is defined as

$$Var[NP(M)] = Var[S_I] = \lambda E[Y]^2 = \lambda \left(\int_0^M x^2 f_X(x) dx + M^2(1 - F_X(M)) \right). \quad (2.18)$$

The given optimization problem can be solved applying the method of Lagrange multipliers with a variable γ . Consider the following function, expressed using the method:

$$h(M) = Var[NP(M)] - \gamma(E[NP(M)] - c). \quad (2.19)$$

We set $\frac{\partial h}{\partial M} = 0$ and, by using the rule that $(\int_a^b h(x)dx)'_b = h(b)$ and product rule, estimate derivatives of summands of (2.19) with respect to M .

$$\begin{aligned} \frac{\partial}{\partial M} Var[NP(M)] &= \frac{\partial}{\partial M} \left[\lambda \left(\int_0^M x^2 f_X(x) dx + M^2(1 - F_X(M)) \right) \right] \\ &= \lambda [M^2 f_X(M) + 2M(1 - F_X(M)) - M^2 f_X(M)] = 2\lambda M(1 - F_X(M)). \end{aligned}$$

Since the total aggregate claim amount ($E[S]$) does not depend on M , the derivative of expected net profit $E[NP(M)]$ as it follows from formula (2.17) with respect to threshold M can be expressed as follows:

$$\begin{aligned}\frac{\partial}{\partial M} E[NP(M)] &= \frac{\partial}{\partial M} (\xi E[S_I]) = \frac{\partial}{\partial M} \xi \left[\lambda \left(\int_0^M x f_X(x) dx + M(1 - F_X(M)) \right) \right] \\ &= \xi \lambda (M f_X(M) + (1 - F_X(M)) - M f_X(M)) = \xi \lambda (1 - F_X(M)).\end{aligned}$$

Now substituting the derivatives of $E[NP(M)]$ and $Var[NP(M)]$ into (2.19), we obtain

$$\frac{\partial h}{\partial M} = 2\lambda M(1 - F_X(M)) - \gamma \xi \lambda (1 - F(M)),$$

and, consequently, the optimal retention level is reached if

$$2\lambda M(1 - F_X(M)) = \gamma \xi \lambda (1 - F_X(M)).$$

Since $1 - F(M) \neq 0$, the optimal retention level can be found as

$$2\lambda M = \gamma \xi \lambda \Leftrightarrow M = \frac{\gamma \xi}{2}. \quad (2.20)$$

As in previous section, the result is straightforward, since the cedent's relative retention depends on the reinsurer's security loading, which, respectively, determines the reinsurer's premium. Therefore, the higher the price of the risk transfer arrangement is, the higher proportion of risk will be retained by the direct insurer, i.e. the retention level and cost of reinsurance are in direct relation [11, p.248-250].

Hence, the obtained formulas of the optimal retention levels for cases of quota share and excess of loss reinsurance are likely to be quite similar in terms of their formulations. Namely, the only difference comes from the expected value and variance of insurer's total claim amount, which is considered in the final expression of the retained proportion within the quota share contract. Such circumstance is objectively responding to the nature of the quota share agreement, as the one implying that the cedent will have to participate in the reimbursement of all claims arriving to the company in a fixed proportion. Therefore, the expected value and variability of the claim amount falling under the cedent's responsibility are of considerable importance.

2.3. The Further Analysis of Reinsurance Optimization Methodologies within the Classical Risk Process Framework

The reinsurance operations, while aiming to limit the possible future losses of the direct insurer through the risk transfer mechanism, may influence on the company's general performance in various ways both beneficial and unfavorable. Certain goals set by the direct insurer may be fulfilled by application of optimization procedure, i.e. it can maximize the desirable total wealth or minimize the variability of net profit after reimbursement of all

reinsurance related costs. However, the effect of reinsurance operations is much more extensive, hence it should be linked with more general indicators of the company's efficiency.

In order to estimate the effectiveness of reinsurance optimization methodology thoroughly the insurance company should analyze the possible outcomes with respect to some principal index characterizing the insurer's performance. In this aspect the classical surplus process, as a basic generalization of the company's performance, can be applied as a visualization of the company's successfulness on the market. Indeed, the reinsurance operations and the slope of the surplus trajectory are directly related.

As mentioned in [13], the insurance company focuses on the impact that risk transfer instruments, such as reinsurance, have on the performance of insurers. While operating under the safety constraint, the company sets a target ruin probability (over a given time horizon), but if the firm cannot meet this level of insolvency risk (with a given strategy), then it must take steps to reduce the amount of risk in its portfolio and a natural technique is to purchase reinsurance. Moreover, reinsurance is able to offer additional underwriting capacity for cedents and also to reduce the probability of a direct insurer's ruin [7].

Since reinsurance plays an important role in reducing the risk in an insurance portfolio, the question of the optimal parameters of reinsurance contract may be studied through a classical risk process. Namely, the insurance company aims to stay solvent, i.e. to maintain the surplus trend on the positive side, therefore the goal of optimization procedure is to estimate the most reasonable retention level in terms of decreasing the probability of company's surplus process to ruin. Hence, the optimization problem can be extended and formulated as follows: the estimation of the optimal retention level with respect to insurer's positive surplus process.

Therefore, the aim is to find the optimization mechanism, i.e. setting a proportional or a non-proportional reinsurance treaty with optimal retained proportion of the risk, such that the general risk process will maintain the positive slope. Thus the reasonableness of optimization approaches can be experimentally determined through the impact of reinsurance activity on company's risk of ruin, i.e. on the intersection of two broad fields of insurance studies – optimization of reinsurance activity and insurer's ruin analysis.

3. INVESTIGATION OF THE OPTIMIZATION METHODOLOGIES THROUGH THEIR IMPACT ON THE COMPANY'S SURPLUS PROCESS

In order to evaluate the effect of reinsurance contracts determined with two analyzed optimization approaches on the risk process, consider the direct insurer providing motor insurance services. To illustrate both methodologies and to estimate their reasonableness in terms of influence they produce on the original surplus process of the cedent in the given conditions, simulated examples will be given, since the use of a real dataset is not possible due to the high confidentiality with which the insurance companies deal with their database.

Considering that the main objective is to derive results to estimate the reasonableness of practical implementation of optimization technics, simulation process, which will represent the company's operational cycle, should be built up. The framework of the modeling of objective process representing cedent's performance under a number of assumptions is the classical risk process possessing the following parameters:

1) $N(t)$ is a random variable characterizing the number of claims up to time t , driven by a Poisson process with parameter λ (claim intensity). In the classical risk process the claim number $\{N(t): t \geq 0\}$ is supposed to represent the homogeneous Poisson process implying that claims are arriving with the constant intensity $\lambda > 0$ and the times between events are independent and identically distributed. However, the homogeneous Poisson process does not give a fully reasonable description of insurance reality, namely the assumption about the constant claim intensity over the whole period of insurer's activity may be invalid, since frequency of claim occurrence is likely to depend on the time of the year [2].

For modeling such phenomena the non-homogeneous Poisson process (NHPP) is more acceptable than the homogeneous one. The NHPP can be presented as a Poisson process with a variable intensity defined by the deterministic intensity function $\lambda(t)$. In the special case when $\lambda(t)$ takes the constant value λ , the NHPP reduces to the homogeneous Poisson process with intensity [2].

Therefore, in the further analysis claim severity will be generated using a non-homogeneous Poisson process with average intensities λ_t for every month $t = 1, 2, \dots, 12$ depending on the given constant value of λ , namely:

- a) in January and February the average intensity is estimated with the function $\lambda_t = 1.2\lambda - 3t$, where t is 1 and 2 respectively;

- b) from March to June the average intensity is fixed, i.e. $\lambda_t = \lambda = 60$ for $t = 3, \dots, 6$;
- c) in July and August the jump in claim arrival severity is observed, the mean intensity is defined as $\lambda_t = 1.1\lambda + t$, with t equal to 7 and 8 respectively;
- d) in September and October the average claim arrival intensity retains a constant level $\lambda_t = \lambda = 60$, for $t = 9$ and $t = 10$ respectively;
- e) in November and December the frequency starts to enlarge and is described with the function $\lambda_t = \lambda + 3t$, where t is 11 and 12 respectively;

2) X_i ($i = 1, 2, \dots, N(t)$) are independent and identically distributed random variables denoting individual claim sizes in the insurance portfolio. To generate the claim severity for the case of motor insurance it may be suitable to attribute a lognormal distribution being realistic within the given type of insurance activity. For instance, in collision situations, where the individual claim values can increase almost without limits but nevertheless cannot fall below zero;

3) durations between consecutive claim arrival times are independent random variables;

4) claim size is independent of the claim arrival process.

3.1. Formulation of the Surplus Process Simulation Model

In order to perform the analysis of two possible retention estimation approaches applied to quota share and excess of loss reinsurance, the company's surplus simulation model should be composed. The previously stated assumptions of the classical risk process will be fulfilled within the model and specified below parameters will be considered in all examined reinsurance optimization cases.

Firstly, consider the cedent performing motor insurance activity over a period of twelve month ($t = 12$) under the following fixed parameters:

- initial capital at time $t = 0$ is equal to 0, i.e. $u = 0$;
- average individual claim size follows the log-normal distribution with mean $\mu = 6.5$ and standard deviation $\sigma = 1$, hence the average value of individual claim according to the formula of expected value of log-normally distributed random variable, can be determined as

$$E[X] = e^{\mu + \frac{\sigma^2}{2}} \approx 1096 \text{ units};$$

- premium loading factor is φ is constant ($\varphi = 10\%$) and total premium charged by the cedent is calculated with the expected value principle, i.e. $P = (1 + \varphi)E[X]$.
- the fixed intensity parameter λ representing the normal mean of claim arrival frequency is equal to 60 ($\lambda = 60$). In the following surplus process simulation it will be referred to, when estimating the average intensity of non-homogeneous Poisson process for every $t = 1, 2, \dots, 12$.

Secondly, the claim intensity is modeled in the following way:

- for every month t ($t = 1, 2, \dots, 12$) the average claim arrival intensity λ_t is defined according to the previously described non-homogeneous Poisson process $N(t)$;
- actual number of claims $N(t) - N(t - 1) = n_t$ for $t = 1, 2, \dots, 12$ is generated from a Poisson distribution with the expected intensity $E[N(t)] = \lambda_t$ of a particular month.

Since the operation period of one year is considered, the surplus amount will be estimated by the end of every month. Taking into account that at $t = 0$ we have $R_0 = u = 0$, for $t = 1, 2, \dots, 12$ the value of the surplus process can be formulated as

$$R_t = R_{t-1} + P_t - \sum_{i=1}^{n_t} X_i, \quad (3.1)$$

where

- 1) $P_t = \lambda_t(1 + \varphi)E[X]$ is the total premium received by the cedent during the month t . Since premiums are charged in advance the expected claim intensity $E[N(t)] = \lambda_t$ is considered in their estimation.
- 2) $X_t = \sum_{i=1}^{n_t} X_i$ is the total amount of claims incurred within month t with X_i being identically distributed random individual claims from log-normal distribution and n_t an actual number of claims over the month t for all $t = 1, 2, \dots, 12$.

The simulation of the surplus process can be performed using Monte Carlo approach [19, p.96-105] with the number of iterations $k = 100000$. Then for $j = 1, 2, \dots, k$ the terminal value of the surplus process R at $t = T = 12$ will be:

$$R_T^j = R_{T-1}^j + P_T^j - \sum_{i=1}^{n_t} X_i^j. \quad (3.2)$$

Based on the end values R_T^j ($j = 1, 2, \dots, k$) independently generated company's surplus paths a number of indicators can be computed in order to evaluate the final result of cedent's operations within one year period under specified conditions. Namely:

- the mean of terminal values over $j = 1, 2 \dots k$ iterations: $\overline{R_T} = \frac{1}{k} \sum_{j=1}^k R_T^j$; (3.3)

- the standard deviation of R_T : $s(R_T) = \sqrt{E[(R_T^j - E[R_T])^2]}$; (3.4)

- a set of specific indexes, e.g. quantiles, absolute differences based on the obtained selection of terminal values of surplus processes, which will be determined specifically for each particular case.

3.2. Proportional Reinsurance Agreement

In case of direct insurer setting a proportional reinsurance agreement with retention level β and the reinsurer's share of responsibility is $(1 - \beta)$, the cedent's surplus process obtains the following formulation:

$$R_t^\beta = R_{t-1}^\beta + P_t^\beta - \sum_{i=1}^{n_t} \beta X_i, \quad (3.5)$$

where $P_t^\beta = \beta P_t$ is the direct insurer's net premium less the reinsurer's share of total premiums collected over period. Given formula of the surplus process constitutes the basis of generation of surplus process for the case of quota share contract assuming reinsurance parameters stipulated by both utility and net profit variance minimization methodologies.

The underlying measures of the reasonableness of implementation of particular retention optimization methodology under the formulated conditions of cedent's activity will be:

a) Conclusions concerning changes caused in the general risk process with setting the proportional reinsurance agreement under certain retention level optimization approach, which can be traced through the comparison of the average surplus process trajectories $\{\overline{R^\beta}, \overline{R}\}$ based on values $\{R_t^\beta, t = 1, 2, \dots, 12\}$ and $\{R_t, t = 1, 2, \dots, 12\}$ respectively. Since the simulation is exercised over k iterations, the previously defined average processes can be determined as:

$$\overline{R^\beta} = \left\{ \frac{1}{k} \sum_{j=1}^k R_1^{\beta j}, \dots, \frac{1}{k} \sum_{j=1}^k R_{12}^{\beta j} \right\} \text{ and}$$

$$\overline{R} = \left\{ \frac{1}{k} \sum_{j=1}^k R_1^j, \dots, \frac{1}{k} \sum_{j=1}^k R_{12}^j \right\} \text{ respectively.} \quad (3.6)$$

Here it will be considered that both pairs of risk processes are taking the same random log-normally distributed claim amounts X_i and claim intensities from non-homogeneous Poisson process λ_t for all $t = 1, 2, \dots, 12$, i.e. both average trajectories are based on common stochastic process.

Since the principal purpose is to trace the changes in the original surplus process under certain reinsurance treaty applied, the case when the original surplus path goes negative is of particular interest in terms of the effect of the settled contract. As the average trajectories $\{\overline{R}^\beta, \overline{R}\}$ are expected to maintain positive slope due to specified conditions of the model, to investigate the behavior of reinsured risk process more precisely, the analysis of all individual trajectories can be considered. Therefore, the pattern of behavior of k individual generated surplus paths with and without the quota share treaty will be visualized with the respective graphs $\{R^{\beta j}, j = 1, 2, \dots, k\}$ and $\{R^j, j = 1, 2, \dots, k\}$.

b) To consider alterations in the reinsured surplus process caused by the changes in parameters of reinsurance agreement, the set of indexes based on terminal values $\{R_T^{\beta j}, j = 1, 2 \dots k\}$ and $\{R_T^j, j = 1, 2 \dots k\}$ can be computed. As the generation process for every iteration $j = 1, 2 \dots k$ returns a pair of terminal values $\{R_T^{\beta j}, R_T^j\}$ corresponding to reinsured and pure trajectory respectively, both based on the same stochastic process, the changes of surplus paths under different conditions of reinsurance agreement can be traced by adjusting the generation procedure to several retention parameters, namely the extended output of the generator will be of the form $\{R_T^{\beta_1 j}, R_T^{\beta_2 j} \dots R_T^{\beta_s j}, R_T^j\}$ where s is number of different values of β ($\beta = \{\beta_1, \beta_2, \dots, \beta_s\}$) computed under the examined methodology by changing determinant parameters of retention level. For the examined case $s = 7$. Since the simulation outputs are comparable in terms of random inputs and taking into account the dependence of retention parameter β and consequently the whole reinsured surplus path on the particular fixed parameters, their impact will be studied separately under the assumption that if one factor varies, another stays constant.

c) The probability of the reinsured surplus trajectories to go negative ($P\{\overline{R}^\beta < 0\}$) and the respective parameter for non-reinsurance case $P\{\overline{R} < 0\}$ will be computed as:

$$\begin{aligned} \bullet P\{\overline{R}^\beta < 0\} &= \frac{1}{k} \sum_{j=1}^k I_{\{R_T^{\beta j} < 0\}} \text{ for all } \beta = \{\beta_1, \beta_2, \dots, \beta_s\}, \\ \bullet P\{\overline{R} < 0\} &= \frac{1}{k} \sum_{j=1}^k I_{\{R_T^j < 0\}}. \end{aligned} \tag{3.7}$$

The comparison of the defined probabilities will be considered as well.

3.2.1. Utility Maximization Approach

Under the methodology of maximizing utility of direct insurer, the retention proportion is estimated with the formula

$$\beta = \frac{\eta}{\eta + \alpha},$$

hence the share of risk kept by the cedent is influenced by two parameters: the reinsurer's premium loading factor η and is the direct insurer's risk aversion coefficient α . Since both these factors are assumed to be known by the direct insurer, cession rate can be estimated directly.

The changes caused in the surplus process after implementing the quota share contract under utility approach can be visualized with the following graph:

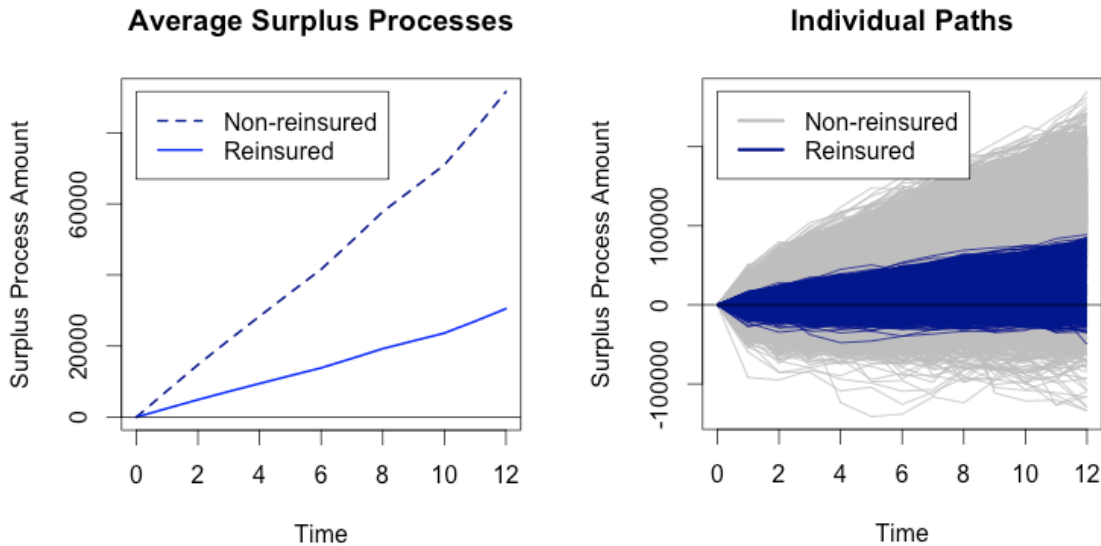


Figure 1: $\{\overline{R^\beta}, \overline{R}\}$ with utility approach applied

Figure 2: Individual reinsured trajectories $R^{\beta j}$ and non-reinsured $R^j, j = 1, 2, \dots, k$

$$(\eta = 0.1, \alpha = 0.2, \beta = 33.33\%)$$

The result obtained with the formula (3.6) illustrates the significant difference between the average pure \overline{R} and reinsured $\overline{R^\beta}$ surplus processes, namely $\overline{R^\beta}$ represents much smaller positive slope due to the fact that claims incurred by the cedent within the analyzed period are shared with reinsuring party in the same proportion as premiums collected, therefore the trajectory with settled quota share contract simply constitutes a certain percentage of the original process, equal to the retained proportion of underlying risk.

The selections of k individual trajectories for both reinsured and original case show, that the quota share contract operates as a risk reduction mechanism and prevent sharp decreases of the company's surplus in case the risk process obtains negative trend. Retention level β being

the underlying parameter of treaty defines the possible slope of the individual reinsured trajectories by specifying the proportion of underlying risks kept by the cedent, i.e. share of obtained losses retained within the cedent's responsibility. Consequently, the range of generated surplus reinsured paths appears to be much more narrow, stipulating a better predictability of operations and avoidance of severe losses.

The character of dependence between the retained amount of risk and parameters η and α is investigated on the basis of k independent surplus paths for every $\beta = \{\beta_1, \beta_2, \dots, \beta_s\}$ and non-reinsurance scenario by applying formulas (3.3) and (3.4).

η	β	Characteristics of the terminal value of reinsured surplus process (R_T^β)	
		$\overline{R_T^\beta}$	$s(R_T^\beta)$
0.1	0.3333	30135	17547
0.11	0.3548	32079	18679
0.12	0.3750	33902	19740
0.13	0.3939	35614	20737
0.14	0.4118	37225	21676
0.15	0.4286	38745	22561
0.2	0.5000	45202	26321

Table 1. The impact of reinsurance premium loading factor η ($\alpha = 0.2$)

α	β	Characteristics of the terminal value of reinsured surplus process (R_T^β)	
		$\overline{R_T^\beta}$	$s(R_T^\beta)$
0.01	0.9091	82979	47470
0.02	0.8333	76064	43514
0.03	0.7692	70213	40167
0.04	0.7143	65198	37298
0.05	0.6667	60851	34811
0.1	0.5000	45639	26108
0.2	0.3333	30426	17406

Table 2. The impact of the cedent's risk aversion α ($\eta = 0.1$)

From the obtained results it can be concluded, that the expected value of the selection $\{R_T^{\beta_j}\}$ is an increasing function of the retention level β . This pattern of dependence is observed due to the fact of proportionality in splitting both incurred claims and total underlying premium between initial insurer and reinsurer leading to the respective decrease in the cedent's net revenue amount by the percentage of risk transferred to reinsuring party.

In the first case, while the retention rate grows as the price of reinsurance contract represented by factor η increases, the mean of surplus amount along with the standard deviation of the

terminal values grows proportionally proving that proportion β influences expected surplus quantity directly. The same effect appears in the second case. As cedent's share of risk β declines with the raise of its absolute risk aversion, the mean of end values of surplus paths decreases and the standard deviation of examined selection decreases consequently.

However, the probability that surplus processes acquire negative values $P\{\overline{R}^\beta < 0\}$, for all $\beta = \{\beta_1, \beta_2, \dots, \beta_s\}$ and $P\{\overline{R} < 0\}$ (see formula 3.7) are the same, namely

$$P\{\overline{R}^\beta < 0\} = P\{\overline{R} < 0\} \approx 4.7 \% \text{ for all } \beta = \{\beta_1, \beta_2, \dots, \beta_s\}.$$

Therefore, the alterations in the cession rate under the proportional reinsurance agreement did not influence the general trend of the company's risk process. The quota share agreement in case of utility approach applied to retention estimation indeed does not reduce the probability of cedent's surplus process to ruin, but lessen amount of loss incurred.

3.2.2. Net Profit Variance Minimization Approach

The second approach to the estimation of retention parameter β implies that retained share of risk can be computed as

$$\beta = \frac{\gamma \xi E[S]}{2Var[S]}.$$

According to the mathematical formulation within the analyzed methodology the risk proportion β operates with the reinsurer's security loading ξ referred to in the premium calculations, a variable γ from the Lagrange multiplier method, expected value and variance of total claim amount S .

The only unknown factor of the given formulation of the retention proportion is Lagrange variable γ , since the remaining parameters are considered to be known or can be derived from generated data. Namely, premium loading factor ξ is assumed to be fixed by the reinsuring party, the mean value $E[S]$ and variance $Var[S]$ of total claim amount can be computed directly from the Compound Poisson distribution of S (with log-normally distributed individual claims $X_i, i = 1, 2, \dots, N(t)$).

Since the underlying idea of the given methodology is that variance of net profit $Var[NP(\beta)]$ is a subject to a fixed target level of expected net profit $E[NP(\beta)]$, the parameter γ can be estimated from formula (2.10) by setting it equal to some constant c (target level of expected net profit):

$$P - (1 - \xi\beta + \xi)E[S] = c.$$

From the expression of retention level (2.13) it follows that

$$\frac{\gamma\xi E[S]}{2\text{Var}[S]} = \frac{c - P + E[S](1 + \xi)}{\xi E[S]}.$$

Considering that in the given equation all arguments except parameter γ are known, the formula of Lagrange variable can be expressed in the following way

$$\gamma = \frac{2(c - P + E[S](1 + \xi))\text{Var}[S]}{\xi^2 E[S]^2}. \quad (3.8)$$

The changes in the trend of the cedent's general surplus process under quota share contract with β determined by net profit variance minimization approach can be visualized similarly to the case of utility approach, namely by generating a pair of average trajectories $\{\overline{R^\beta}, \overline{R}\}$ applying the formula (3.6) and through the comparison of the selection of k individual reinsured $R^{\beta j}$ and pure R^j trajectories ($j = 1, 2, \dots, k$). The obtained average surplus paths are:

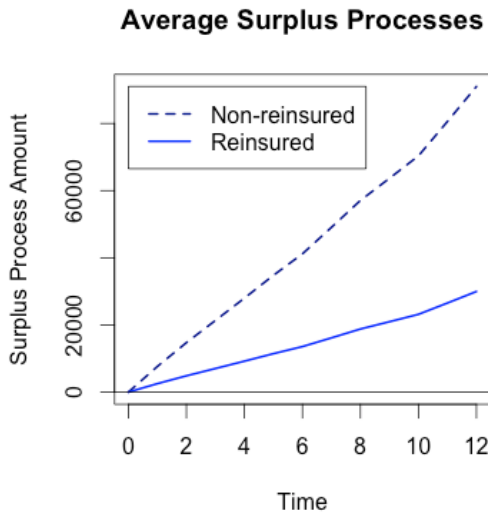


Figure 3: $\{\overline{R^\beta}, \overline{R}\}$ with net profit variance minimization approach applied
 $(\xi = 0.1, c = 30000, \beta = 32.92\%)$

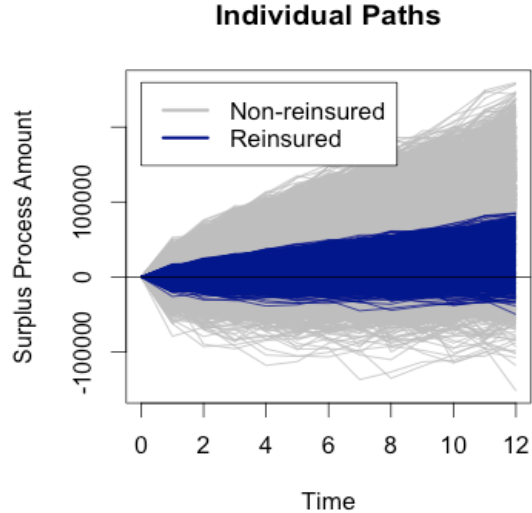


Figure 4: Individual reinsured trajectories $R^{\beta j}$ and non-reinsured R^j , $j = 1, 2, \dots, k$

The obtained results are similar to the utility approach case with the slight difference in the scope of direct insurer's responsibility. As the net profit variance minimization approach implies lower level of β that the previously examined methodology under the same given conditions, the average path $\overline{R^\beta}$ represents the smaller percentage of the original trajectory \overline{R} , in general repeating its slope.

The individual paths $R^{\beta j}$ and R^j verified that the intensity of losses in case of reinsurance contract settled is smaller due to the fact that claim amount only in the share β falls within the direct insurer's responsibility. Due to the fact that in the case of variance minimization approach the estimated retention rate is almost equal to the one derived with the utility technique, the extent of reduction in the surplus amount is the same as in previous case.

The influence of changes in retention parameter β on the risk process under quota share agreement can be investigated with assumption that reinsurer's premium loading factor ξ and fixed level of desirable net profit c are the underlying impact factors determining γ parameter (see formula 3.8) and consequently retention level β .

The results of simulations (see formulas 3.3 and 3.4) are the following:

ξ	γ	β	Characteristics of the terminal value of reinsured surplus process (R_T^β)	
			$\overline{R_T^\beta}$	$s(R_T^\beta)$
0.1	41777303	0.3292	29893	17000
0.11	45014781	0.3902	35430	20149
0.12	46637824	0.4410	40045	22774
0.13	47247958	0.4840	43949	24994
0.14	47214097	0.5209	47296	26898
0.15	46768984	0.5528	50197	28547
0.2	42170781	0.6646	60349	34320

Table 3. The impact of reinsurer's premium factor ξ ($c = 30000$ units)

c	γ	β	Characteristics of the terminal value of reinsured surplus process (R_T^β)	
			$\overline{R_T^\beta}$	$s(R_T^\beta)$
5000	3493616	0.0549	5053	2818
10000	6987232	0.1097	10107	5637
15000	10480847	0.1646	15160	8455
20000	13974463	0.2195	20213	11274
25000	17468079	0.2743	25267	14092
30000	20961695	0.3292	30320	16911
35000	24455310	0.3841	35373	19729

Table 4. The impact of expected amount of net profit c ($\xi = 0.1$)

So, in case when the net profit variance minimization approach is applied to the calculation of the cession rate, retention parameter β appears to be an increasing function of reinsurer's premium loading factor ξ and desirable level of net profit c . The alterations of the first factor imply the same changes in the retained share of risk as in the case of utility approach while the increasing desirable level of net profit stipulates the cedent to keep a bigger share of

expected surplus volume and consequently to implement the bigger reinsurance retention proportion β .

The expected average surplus amount \overline{R}_T^β is directly related to the retention parameter β , consequently an increase of cedent's retention level objectively implies a higher share of expected surplus amount of original risk process pertaining to the cedent. In particular, in the examined model original risk process maintains growing tendency and the average surplus amount increase proportionally to the rise of retention rate. However, if the pure surplus process ruins the reinsured path will go below zero as well, but less severely protecting the initial insurer against significant losses. Hence the proportional reinsurance contract does not eliminate the probability of ruin from the initial insurer's activity, but decline the amount of loss incurred in case surplus path goes negative in all cases.

The probabilities $P\{\overline{R}^\beta < 0\}$ and $P\{\overline{R} < 0\}$ following from formula (3.7) maintain the same value under all assumed levels of β and in the case without the reinsurance treaty, namely

$$P\{\overline{R}^\beta < 0\} = P\{\overline{R} < 0\} = 4.4 \% \text{ for all } \beta = \{\beta_1, \beta_2, \dots, \beta_s\}.$$

Hence, the results of application of net profit variance minimization approach confirmed that the general trend of insurer's surplus path remains constant despite the cession rate varies, i.e. alterations in the amount of cedent's responsibility cannot effect direct insurer's financial performance fundamentally under all other conditions staying the same.

3.3. Non-Proportional Reinsurance Agreement

In case of non-proportional reinsurance contract with the threshold $M > 0$ the surplus process of the insurance company obtains the following form:

$$R_t^M = R_{t-1}^M + P_t^M - \sum_{i=1}^{n_t} \min\{X_i, M\}. \quad (3.9)$$

One of the questions arising when setting an excess of loss reinsurance contract is the methodology of sharing of gross premium (P_t) at time $t = 1, 2 \dots 12$ between the initial insurer and reinsurer. Unlike a quota share agreement, for the non-proportional case there is no single rule of premium allocation under the reinsurance arrangement, therefore this issue is typically negotiated by parties and respective conditions are fixed in the contract. The methodology guiding the premium spreading process depends on certain principles accepted in the insurance and reinsurance practice. One of the most widely used is so called "Swing Rating" approach, according to which premium of the reinsurer varies with loss ratio [14].

The loss ratio of the reinsurer is directly related to the share of ceded risk and can be expressed as the expected total amount of claims under reinsurance multiplied by the reinsurer's premium loading factor (2.14), for all $t = 1, 2 \dots 12$. Consequently, the cedent's premium P_t^M under excess of loss contract can be estimated simply as a difference between the total premium volume (assuming a given premium loading factor φ) and the part of the premium transferred to the reinsuring party:

$$\begin{aligned} P_t^M &= P_t - (1 + \xi)E[S_R] = P_t - (1 + \xi)\lambda_t E[Z] \\ &= \lambda_t(1 + \varphi)E[X] - \lambda_t(1 + \xi)(E[X] - E[Y]) \end{aligned} \quad (3.10)$$

In order to evaluate the reasonableness of implementation of particular retention optimization methodology in case of excess of loss agreement, the following results will be estimated and considered:

a) The comparison between the average $\{\overline{R^M}, \bar{R}\}$ non-reinsurance and reinsured trajectories generated from $j = 1, 2 \dots k$ independent Monte-Carlo iterations can be executed, considering that:

- 1) $\overline{R^M} = \left\{ \frac{1}{k} \sum_{j=1}^k R_1^{Mj}, \dots, \frac{1}{k} \sum_{j=1}^k R_{12}^{Mj} \right\}$ represents average path of reinsured surplus process;
- 2) $\bar{R} = \left\{ \frac{1}{k} \sum_{j=1}^k R_1^j, \dots, \frac{1}{k} \sum_{j=1}^k R_{12}^j \right\}$ reflects average path of non-reinsured surplus process.

(3.11)

The selection of random reinsured trajectories $\{R^{Mj}, j = 1, 2, \dots, k\}$ and original ones $\{R^j, j = 1, 2, \dots, k\}$ will be generated to illustrate the behavior and the frequency of individual paths going negative with and without reinsurance cover.

b) The investigation of the behavior of selection or terminal values of k independent surplus processes generated in case when the excess of loss reinsurance contract is settled $\{R_T^{Mj}, j = 1, 2 \dots k\}$ and in case of pure surplus process $\{R_T^j, j = 1, 2 \dots k\}$ applying (3.3), (3.4).

c) The probability of the average reinsured surplus trajectories to go negative can be evaluated as

$$P\{\overline{R^M} < 0\} = \frac{1}{k} \sum_{j=1}^k I_{\{R_T^{Mj} < 0\}} \text{ for all } M = \{M_1, M_2, \dots, M_q\}, \quad (3.12)$$

The given probability will be compared with the respective parameter of the original surplus process (3.7), for all $M = \{M_1, M_2, \dots, M_q\}$.

3.3.1. Utility Maximization Approach

In case the direct insurer refers to utility approach when estimating its retention under the excess of loss agreement, the threshold can be computed in the following way:

$$M = \frac{1}{\alpha} \log(1 + \xi),$$

implying that level M depends on two factors: reinsurer's premium loading factor ξ and direct insurer's absolute risk aversion parameter α . Both indexes are assumed to be known by the cedent and remain fixed over the whole analyzed year.

Considering that under the excess of loss agreement the principle of sharing responsibility for compensation of claims does not infer proportionality, the consequences of operation with the excess of loss agreement settled may vary depending on the claim size and frequency: high claim intensity with small average amount can result in ruin of insurer, while the same claim arrival stochastic process can remain on the positive side without the reinsurance cover.

The relation between average pure \bar{R} and reinsured \bar{R}^M paths under given threshold estimation methodology, as well as the selection of k individual trajectories R^{Mj} and R^j , can be visualized in the following way:

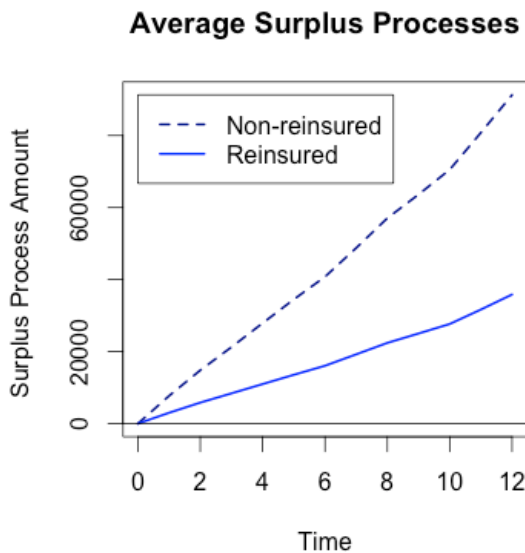


Figure 5: $\{\bar{R}^M, \bar{R}\}$ with utility approach applied

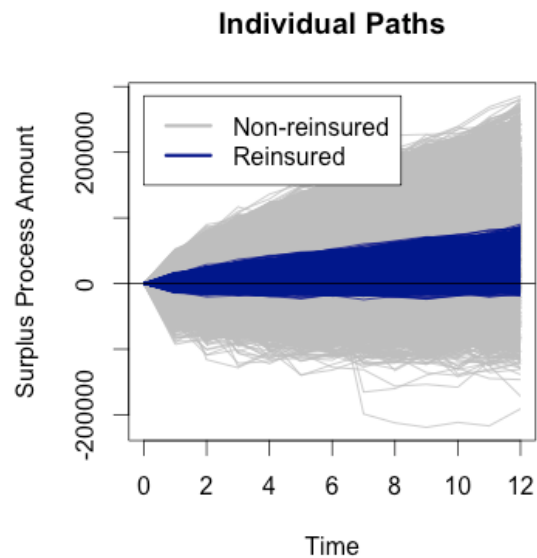


Figure 6: Individual reinsured trajectories R^{Mj} and non-reinsured R^j , $j = 1, 2, \dots, k$

($\xi = 0.1, \alpha = 0.2, M = 523$ units)

Unlike the case of quota share agreement, excess of loss surplus trajectory poses higher independency from the original one, since the relation between them is determined by the

maximum limit of claim payable by cedent (M). The generated over k iterations average trajectories $\{\overline{R^M}, \overline{R}\}$ verify that in general excess of loss treaty under the net profit variance minimization approach support the positive trend of surplus trajectory, however the slope of average reinsured path is smaller due to the outflow of the particular share of underlying premium resulting from setting the reinsurance contract.

The selection of the individual k paths proves that the excess of loss agreement settled under utility approach generally reduces the boundaries of the individual surplus trajectories both from the positive and negative side, defining the area of possible financial outcomes in case the treaty is applied. Consequently, the expected losses of the direct insurer are lessened and the surplus paths with the reinsurance cover are secured against severe claims, however in case of positive outcome by the end of the analyzed period the financial result in absolute value is expected to be smaller in case when the reinsurance agreement is settled due to the expenses associated with purchasing of the treaty.

The influence of the reinsurer's loading factor ξ and the absolute risk aversion α on the threshold M (with respect to formulas 3.3 and 3.4) can be investigated as follows:

ξ	M	Characteristics of the terminal value of surplus process (R_T^M)		Comparison of R_T^M and R_T	
		$\overline{R_T^M}$	$s(R_T^M)$	$\overline{R_T^M} - \overline{R_T}$	$\frac{s(R_T^M)}{s(R_T)}$
0.1	523	35532	13090	-55666	0.2510
0.11	572	32598	14060	-58600	0.2696
0.12	621	29946	14979	-61252	0.2872
0.13	670	27548	15852	-63651	0.3040
0.14	718	25374	16684	65824	0.3199
0.15	766	23403	17477	-67795	0.3351
0.2	1000	15967	20943	-75231	0.4015

Table 5. The impact of reinsurance premium loading factor ξ ($\alpha = 0.2$)

α	M	Characteristics of the terminal value of surplus process (R_T^M)		Comparison of R_T^M and R_T	
		$\overline{R_T^M}$	$s(R_T^M)$	$\overline{R_T^M} - \overline{R_T}$	$\frac{s(R_T^M)}{s(R_T)}$
0.01	10452	77410	49225	-12954	0.9318
0.02	5226	79264	43748	-11100	0.8282
0.03	3484	75370	38947	-14994	0.7373
0.04	2613	70706	35026	-19658	0.6631
0.05	2090	66066	31789	-24298	0.6018
0.1	1045	51333	21623	-39031	0.4093
0.2	522	33632	13058	-56732	0.2472

Table 6. The impact of the cedent's risk aversion α ($\xi = 0.1$)

The obtained simulation results stipulates two underlying conclusion regarding the dependence of the surplus process on the threshold M through the initial parameters, namely:

1) In case the reinsurer's loading factor ξ is growing the threshold M increase as well as the price of reinsurance treaty, i.e. reinsurer's premium, becomes higher. However, the premium retained by the cedent (3.10) is the increasing function of M and decreasing with respect to ξ , implying that the expected value of terminal values of the surplus process \overline{R}_T^M will decrease with the increase of ξ . This result appears to be straightforward, since even despite high threshold, i.e. small share of risks is transferred, the premium payable for reinsurance cover will increase considerably with the increase of loading factor ξ . Therefore, the standard deviation of the selection of terminal values $s(R_T^M)$ will grow due to the higher maximum limit of claims payable by cedent resulting in more significant fluctuations of amounts of incurred claims.

2) In the second case with cedent's risk aversion being variable the threshold is naturally a decreasing function of α , implying that the direct insurer tends to minimize the limit of its responsibility in order to avoid risk of losses. As the loading factor ξ is considered to stay constant the premium retained by the cedent is effected only by the amount of M an, consequently the expected value \overline{R}_T^M decreases as the share of risk transferred to the reinsurer enlarges, implying bigger premium outflow. However, the standard deviation of the selection $s(R_T^M)$ behaves conversely to the first case, namely lower threshold induces lower fluctuations of claim amount, which appears to be quite intuitive as M is the absolute limit of claims payable by the cedent and if arriving claims exceed the threshold the sum payable by the direct insurer will be equal to the threshold itself.

Following from formula (3.12) the probabilities of \overline{R}^M with $M = \{M_1, M_2, \dots, M_7\}$ to go negative are:

Impact of ξ ($\alpha = 0.2$)				Impact of α ($\xi = 0.1$)			
ξ	M	$P\{\overline{R}^M < 0\}, \%$	$\frac{P\{\overline{R}^M < 0\}}{P\{\overline{R} < 0\}}$	α	M	$P\{\overline{R}^M < 0\}, \%$	$\frac{P\{\overline{R}^M < 0\}}{P\{\overline{R} < 0\}}$
0.1	523	0.39	0.0852	0.01	10452	3.15	0.703
0.11	572	1.05	0.2293	0.02	5226	2.25	0.502
0.12	621	2.59	0.5655	0.03	3484	1.70	0.380
0.13	670	4.53	0.9891	0.04	2613	1.40	0.313
0.14	718	6.48	1.4148	0.05	2090	1.17	0.261
0.15	766	9.08	1.9825	0.1	1045	0.64	0.144
0.2	1000	22.00	4.8035	0.2	522	0.41	0.091

Table 7. The probability of reinsured surplus process to go negative in case of excess of loss contract with utility approach applied

From the simulation results it follows that the probability of the surplus process to ruin increase with the growth of the threshold. The distribution of the actual claim amount is of the prime importance in this aspect due to the fact that, when setting a low threshold the limit of insurer's responsibility will be also low and excessive claims will be payable in the amount of M , while under the higher threshold most of incurred claims will be payable by the cedent in the full volume, inducing a large aggregate claim sum. In this case despite the retained premium is an increasing function of M the cedent's premium may be not sufficient to cover incurred losses. Moreover, if to consider the situation when ξ grows, the ruin probability enlarge much severely in this case, since the outgoing reinsurance premium flow will be increased considerably by the growing loading factor (the respective probability of the original process is exceeded four times in case when ξ is twice bigger than the cedent's loading φ), therefore ξ appears to be indeed crucial parameter when evaluating the excess of loss contract.

3.3.2. Net Profit Variance Minimization Approach

The threshold M with net premium minimization approach in case of excess of loss contract can be derived using the following formula:

$$M = \frac{\gamma\xi}{2},$$

taking ξ and γ as determinative parameters. To estimate the amount of underlying gross premium P_t^M , retained with direct insurer after setting the reinsurance contract for all t , the same approach as in case of utility retention level estimation methodology will be considered (3.10).

The formula of the threshold implies the dependence of M on the parameter γ of the Lagrange multiplier method. Since variable γ is unknown, it can be computed based on assumption that initial insurer fixes a desirable level of net profit c , hence $Var[NP(M)]$ is a subject to this suitable constraint.

From formula (2.17) follows that the expected value of cedent's net revenue can be expressed as

$$E[NP(M)] = P - (1 + \xi)E[S] + \xi E[S_I].$$

Since individual claim amounts are log-normally distributed with known mean μ and standard deviation σ , $E[S]$ can be computed directly as $E[S] = (\sum_{t=1}^T \lambda_t)E[X]$ and total premium charged can be defined in the similar way: $P = (1 + \varphi)[(\sum_{t=1}^T \lambda_t)E[X]]$. Hence, the only unknown argument of the formula is $E[S_I]$, depending on the threshold M . Applying the

limited expected value function of log-normal distribution [12, p.229] and considering the target level of $E[NP(M)]$ to be equal to c , the following expression is obtained:

$$P - (1 + \xi)E[S] + \xi \left(\sum_{t=1}^T \lambda_t \right) \cdot \left[e^{\mu + \frac{\sigma^2}{2}} \cdot \Phi \left(\frac{\ln(M) - \mu - \sigma^2}{\sigma} \right) + M \cdot \left(1 - \Phi \left(\frac{\ln(M) - \mu}{\sigma} \right) \right) \right] = c.$$

Assuming that $M = \frac{\gamma \xi}{2}$ and by setting the previously defined expression to zero (reducing it to the problem of finding of one dimensional root) the parameter γ can be derived.

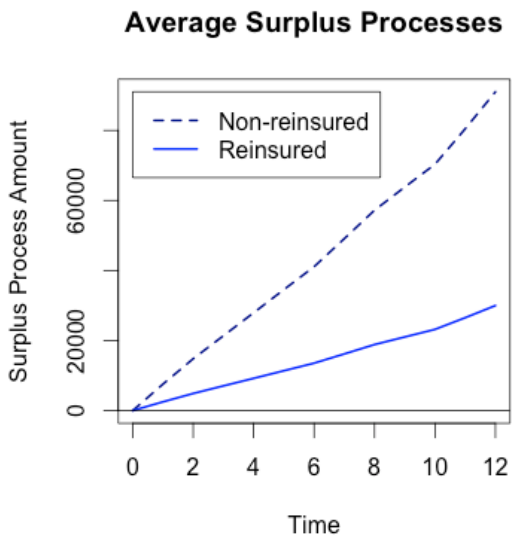


Figure 7: $\{\bar{R}^M, \bar{R}\}$ with net profit variance minimization approach applied
 $(\xi = 0.1, c = 30000, M = 416 \text{ units})$

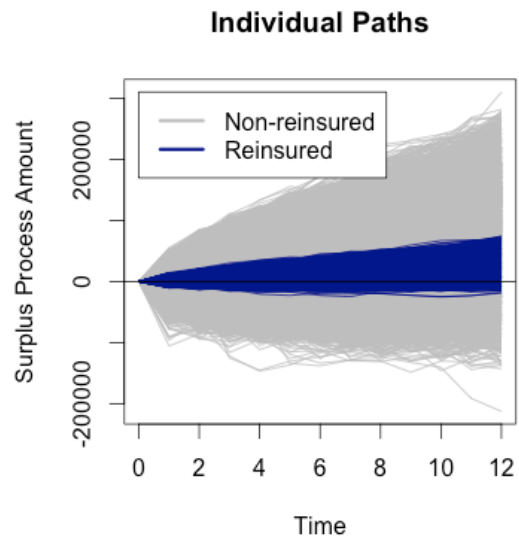


Figure 8: Individual reinsured trajectories R^{Mj} and non-reinsured $R^j, j = 1, 2, \dots, k$

The implementation of net profit minimization approach to the non-proportional treaty generally brings in the same alterations in the cedent's surplus path as in case of utility approach. The principal difference of the net profit variance reduction approach similarly to the case of proportional reinsurance is in the scale of retention parameter, since under the defined parameters the threshold in the current case is 20 % smaller than the respective parameter estimated with utility approach, however the behavior of the average reinsured path comparing to the original one remains the same as with previous optimization methodology.

The range of individual trajectories R^{Mj} and R^j in case of net profit variance minimization approach implies the same conclusions as in the case of utility methodology, despite the upper and lower bounds of the area of possible individual paths are more narrow due to the fact that current methodology stipulates lower retention level than the utility approach. Therefore, the overall impact of the non-proportional treaty on the cedent's surplus process remains the

same, namely the losses are reduced along with the reduction of the profits, i.e. excess of loss treaty is reasonable in terms of securing the direct insurer against excessive losses.

The dependence of threshold on ξ and c , synthesized in Lagrange parameter γ , can be traced by computing parameters (3.3), (3.4) and their further comparison:

ξ	γ	M	Characteristics of the terminal value of surplus process (R_T^M)		Comparison of R_T^M and R_T	
			$\overline{R_T^M}$	$s(R_T^M)$	$\overline{R_T^M} - \overline{R_T}$	$\frac{s(R_T^M)}{s(R_T)}$
0.1	8325	416	29879	10851	-61131	0.2079
0.11	9476	521	29858	13053	-61152	0.2501
0.12	10341	620	29845	14974	-61165	0.2869
0.13	11000	715	29840	16663	-61170	0.3193
0.14	11506	805	29831	18157	-61179	0.3479
0.15	11895	892	29827	19494	-61183	0.3735
0.2	12820	1282	29818	24579	-61192	0.4709

Table 8. The impact of reinsurance premium loading factor ξ ($c = 30000$)

c	γ	M	Characteristics of the terminal value of surplus process (R_T^M)		Comparison of R_T^M and R_T	
			$\overline{R_T^M}$	$s(R_T^M)$	$\overline{R_T^M} - \overline{R_T}$	$\frac{s(R_T^M)}{s(R_T)}$
5000	1206	60	4999	1741	-86267	0.0338
10000	2439	122	10001	3490	-81266	0.0678
15000	3736	187	15005	5263	-76262	0.1023
20000	5129	256	20007	7071	-71260	0.1374
25000	6648	332	24998	8923	-66269	0.1734
30000	8325	416	29989	10824	-61278	0.2104
35000	10200	510	34969	12780	-56298	0.2484

Table 9. The impact of expected amount of net profit c ($\xi = 0.1$)

Based on the simulated results the following conclusions can be formulated:

1) Considering the case of the increasing premium loading factor ξ and respectively growing threshold M (based on the same arguments as in case of utility approach), the expected value of the selection of the terminal values of the surplus process $\overline{R_T^M}$ presents a slight decrease with the raise of loading factor. Since the expected value is heavily dependent on the amount of retained premium, which decreases with the growth of ξ , but increases as M enlarge, it can be concluded that despite the impact of these two factors the retained premium volume and amount of incurred claims remain balanced. The standard deviation of terminal values $s(R_T^M)$ increases with the growth of ξ , since the higher limit allows higher variance of the incurred claims, however still remains significantly smaller than the original level in non-reinsurance case.

2) In case when the cedent fixes higher level of expected net profit c the threshold increases respectively, implying that the higher proportion of the total premium will be kept (since in the given case the alterations of the reinsurance premium depend only on changing c the effect of ξ is eliminated). Consequently, both the expected value \overline{R}_T^M and standard deviation $s(R_T^M)$ are growing.

Following from formula (3.11) the probabilities of reinsured surplus process R^M to obtain negative values (considering all $M = \{M_1, M_2, \dots, M_q\}$ with either ξ or c staying constant) are:

Impact of ξ , $c = 30000$				Impact of c , $\xi = 0.1$			
ξ	M	$P\{\overline{R}^M < 0\}, \%$	$\frac{P\{\overline{R}^M < 0\}}{P\{\overline{R} < 0\}}$	c	M	$P\{\overline{R}^M < 0\}, \%$	$\frac{P\{\overline{R}^M < 0\}}{P\{\overline{R} < 0\}}$
0.1	416	0.24	0.0550	5000	60	0.34	0.0767
0.11	521	1.06	0.2431	10000	122	0.35	0.0790
0.12	620	2.26	0.5183	15000	187	0.35	0.0790
0.13	715	3.64	0.8349	20000	256	0.38	0.0858
0.14	805	5.03	1.1537	25000	332	0.42	0.0948
0.15	892	6.50	1.4908	30000	416	0.44	0.0993
0.2	1282	11.49	2.6353	35000	510	0.51	0.1151

Table 10. The probability of reinsured surplus process to go negative in case of excess of loss contract with net profit variance minimization approach applied

When the threshold is increasing the ruin probability behaves the same way in both cases, however the range of alteration differs depending on the initial parameter inducing the growth of the threshold M . The increase of loading factor, similarly to the utility approach case, implies significant growth of ruin probability, namely for ξ twice bigger than the direct insurer's loading factor φ the ruin probability of the average reinsured process exceeds the respective parameter of the original process for more than two times, proving that the reinsurer's loading factor is the determinative parameter when evaluating the reasonableness of purchasing the reinsurance cover.

In case when the cedent enlarge the target level of the net revenue the increasing threshold stipulates the growth of the ruin probability of reinsured process, however the computed probabilities remain smaller than the one of the non-reinsured process, due to the fact that as the reinsurer's premium loading remains equal to the cedent's the retained premium depends only on M , therefore the probability that the total claim amount will exceed the volume of premium kept by insurer stays relatively small.

Hence, the decision concerning setting the non-proportional reinsurance agreement should consider two main conclusions derived from the results of simulations, namely:

- the excess of loss treaty may potentially induce the ruin of the surplus process in case

the reinsurer's premium loading is higher than the one the cedent is using, since the price of reinsurance cover represented by the premium asked by the reinsuring party is high, consequently the reinsurance agreement appears to be unreasonable from the perspective of the ruin probability;

– nevertheless, if the cedent aims to reduce the variance of the surplus process and ensure the prevention of significant losses, the underlying risks should be reinsured up to some extent, since the lower is the threshold the less variable are possible terminal surplus amount by the end of the insurance period.

Consequently, the implementation of an excess of loss treaty requires thorough comparison of the expected outcomes of purchasing the reinsurance cover under specified parameters with the operational goals fixed by the cedent, since the non-proportional treaty introduce a double effect on the insurer's performance: higher lower limit of direct insurer's responsibility certainly reduce the variance of possible financial result, however the ruin probability may increase due to the higher price paid for the reinsurance coverage.

3.4. The Comparison of Risk Process Paths under the Investigated Methodologies

To make the final conclusion regarding the effectiveness of examined methodologies it terms of the direct insurer's risk process, the comparison between the surplus path in non-reinsurance case and under two investigated types of contracts with two different retention estimation methodologies applied can be performed.

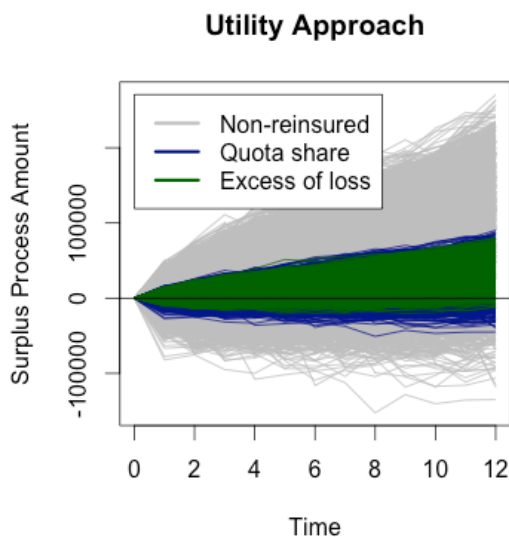


Figure 9: Individual reinsured trajectories $R^{\beta_{1j}}, R^{M_{1j}}$ and $R^j, j = 1, 2, \dots, k$, under utility approach

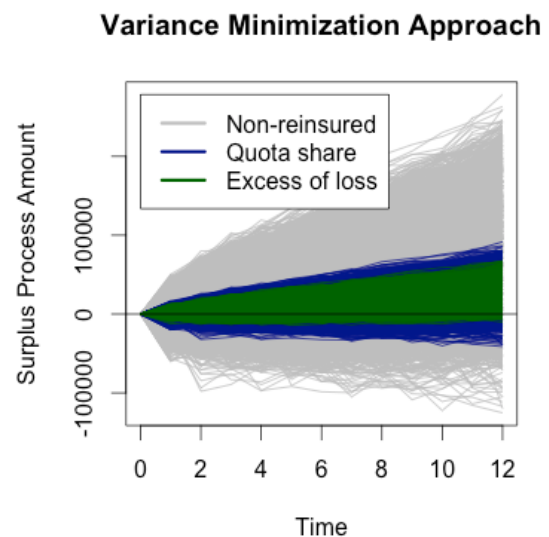


Figure 10: Individual reinsured trajectories $R^{\beta_{2j}}, R^{M_{2j}}$ and $R^j, j = 1, 2, \dots, k$, under variance minimization approach

$$(\xi = \eta = 0.1, \alpha = 0.2, c = 30000)$$

Hence, under the specified conditions application of both optimization methodologies proved that the quota share reinsurance treaty performs better, due to the fact that in average the original surplus process maintains the growing tendency and the proportional reinsurance cover certainly retains the particular proportion of the positive trend equal to the retention rate.

The excess of loss treaty under the specified set of determinative parameters generally ensures the cedent against excessive losses and lower the probability of negative financial outcome, however it should be considered that the non-proportional treaty may induce increase of insurer's ruin probability even in case of the slight growth of the price of reinsurance cover.

Generally, the average expected terminal values of surplus paths under all analyzed types of reinsurance treaties and retention estimation approaches over k iterations have the following order in terms of their absolute value:

$$E[R_T^{\beta_1}] > E[R_T^{\beta_2}] > E[R_T^{M_1}] > E[R_T^{M_2}].$$

Since the utility approach applied to quota share agreement provides the highest average return, it can be concluded that considering the assumed parameters of the cedent's risk process the given methodology is more reasonable, due to the fact that in general risk process is expected to maintain positive slope and therefore higher cession rates implied by the variance minimization approach ensure higher share of revenue retained by the direct insurer, i.e. its higher profitability. However, in practice the application of the utility optimization technique raises the question of the utility function to be chosen for the derivation of the retention parameter, requiring additional thorough investigation.

CONCLUSIONS

The optimal retention level being a principal parameter of the reinsurance treaty requires derivation of specific numerical approaches to its estimation followed by detailed analysis of possible outcomes of the application of the determined cession rates in practice. The given problem appears to be complex as different types of reinsurance agreements may influence direct insurer's financial performance in various ways, not necessarily ensuring cedent's solvency. Therefore, the aim of this thesis was to investigate the reinsurance techniques both in terms of their mathematical formulation and through their impact on the financial performance of the direct insurer, considering that the optimization of reinsurance agreement involves two layers of the decision making procedure: choosing the type of agreement and the selection of the retention estimation methodology to determine the scope of reinsurance cover under the settled contract.

The share of risk retained within the direct insurer's responsibility in case of a particular type of treaty can be estimated using several methodologies, each aiming to fulfill certain objective of insurer's performance and consequently appealing to different mathematical concepts. Two examined approaches, the utility and net profit variance minimization methodologies, stipulate complete algorithms of defining of retention rates. Under both techniques the share of retained risk (or threshold) is mathematically formulated on the basis of performed derivation procedures, each taking particular target from the cedent's perspective as a starting point. Namely maximizing the cedent's wealth by the end of period in case of utility approach and minimizing the variance of expected net revenue under the second methodology are settled as goals, which should be fulfilled in case of each type of reinsurance agreement executed.

The principal difference between the possible proportional and non-proportional treaties, independently of the methodology of retention level estimation is that in case of the first type of contract settled the incurred claims will be reduced proportionally to the retention rate securing the direct insurer from losses in the respective share. However, under the non-proportional agreement the reduction of losses is not guaranteed, since in some specific cases setting an excess of loss agreement followed by the transfer of determined premium amount to reinsurer increase the risk of disparity of retained premium quantity and total amount of incurred claims retained under the cedent's responsibility.

The obtained simulation results have proved that the scope of the reinsurance cover may influence the cedent's performance differently depending on the type of settled contract and

the primary conditions of the reinsurance operations. In this aspect the pattern of influence of the initial parameters determining the retention level of the proportional reinsurance agreement under both approaches is straightforward, inducing the reasonable alterations of the cession rate and consequently proportional shift of the original surplus path within the quota share contract.

However, in case of non-proportional agreement the optimization of the threshold as a key parameter of the reinsurance contract appears to be heavily dependent on the initial conditions under which the contract is settled. The retention limit in this case may have a dual impact on the cedent's financial results, since the modeled risk process has a more complex relation with both the initial factors implying the particular level of reinsurance cover and the parameter of reinsurance agreement as a final result of optimization procedure. Thus the expected outcomes of direct insurer's activity with the reinsurance treaty applied may vary considerably due to the changing underlying parameters.

Consequently, from the simulation of the cedent's surplus process it follows that the change of premium loading factor of the reinsuring party representing the price of the purchased cover, followed by the respective shift of the threshold, stipulates the expected alterations in the cedent's final financial outcomes different from the changes induced by parameters other than the reinsurance loading rate. Therefore, the estimation of the reasonable retention parameter of the non-proportional reinsurance treaty involves the indeed complex analysis of the outstanding factors of insurer's activity, their impact on the reinsurance parameter and the expected outcomes resulting from the specified conditions of the agreement. Since the correct evaluation of the retention limit directly impacts the riskiness of the company's operations, the choice of the scope of the reinsurance cover should follow from the objectives settled by the direct insurer with respect to estimated expected outcomes induced by the selected parameter of the reinsurance agreement.

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