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Globalizations of strong partial acts over monoids

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ÜLE MONOIDIDE VAADELDAVATE TUGEVATE OSALISTE POLÜGOONIDE GLOBALISATSIOONID

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Lühikokkuvõte

Olgu S monoid. Tugevaks osaliseks S-polügooniks nimetame sellist osalist S-polügooni, mis tekib kui jätame mingist globaalsest S-polügoonist mingid elemendid ära. Kui A on osaline polügoon, B mingi A globalisatsioon ja B on moodustatud S-polügoonina A elementide poolt, siis ütleme, et B on A-genereeritud. Kellendonk ja Lawson on näidanud, et kui S on rühm, siis igal tugeval osalisel S-polügoonil leidub ühene A-genereeritud globalisatsioon. See aga ei kehti monoidide juhul. Laan ja Kudryavtseva on andnud kaks konstruktsiooni osaliste polügoonide globaliseerimiseks üle poolrühmade, mis ei pruugi olla isomorfsed: tensor-globalisatsioon $A \otimes S$ ja hom-hulk-globalisatisoon A^S. Selles magistritöös defineerime hom-hulk-globalisatsiooni osaliste S-polügoonide morfismidel nii, et saame kovariantse täpse funktori. See funktor aga ei ole reflektor ega koreflektor. Siis näitame, et A-genereeritud globalisatsioonide isomorfismiklassid moodustavad täieliku võre, mis on duaalselt isomorfne võre $\mathrm{Con}\, A \otimes S$ alamvõrega. Lõpuks näitame, et ainsad monoidid, mille puhul kõik tugevad osalised polügoonid on üheselt globaliseeritavad on rühmad.

CERCS teaduseriala: P120 Arvuteooria, väljateooria, algebraline geomeetria, algebra, rühmateooria

Märksõnad: algebra, toime, monoidid

GLOBALIZATIONS OF STRONG PARTIAL ACTS OVER MONOIDS

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Abstract

Let S be a monoid. Then a strong partial S-act is a partial S-act that arises by omitting some elements from a global S-act. If A is a partial act and B a globalization of A that is generated by the elements of A, then we say that B is an A-generated globalization of A. Kellendonk and Lawson have shown that if S is a group, then any strong partial S-act has unique A-generated globalization. This however is not the case for monoids. Laan and Kudryavtseva gave two constructions for globalizing partial semigroup acts: the tensor product globalization $A \otimes S$ and the hom-set globalization A^S. They then showed that these constructions need not be isomorphic. In this thesis we give a definition of the hom-set globalization on morphisms of partial acts which gives a faithful functor from the category of strong partial S-acts to global S-acts which is neither a reflector nor a coreflector. We show that isomorphism classes of A-generated globalizations form a complete lattice that is dual to a sublattice of $\mathrm{Con}\, A \otimes S$. Lastly, we prove that groups are the only monoids for which all strong partial acts are uniquely globalizable.

CERCS research specialisation: P120 Number theory, field theory, algebraic geometry, algebra, group theory

Key Words: algebra, action, monoids

Contents

Introduction

In this thesis we study strong partial actions of monoids. Partial actions of groups, monoids and other structures is currently an active research area. First, the group case was considered by R. Exel in [2]. Partial actions of groups have seen extensive use in the study of C^*-algebras. A natural generalization of a partial group action is a partial monoid action. These were first defined by M. G. Megrelishvili and L. Schröder in [11] (this is what we call a strong partial action). In the monoid (and semigroup) case there are multiple ways to naturally define partial actions. For example in [4] one of the axioms in the definition in [11] is omitted and the notion of incomplete actions in [3] is stronger. For more information on partial actions and their applications to C^*-algebras see [1].

We say that a globalization $(B, \iota, *)$ of a partial S-act A is A-generated, if the image of A in B generates B as an S-act. It is known that a strong partial group action can be uniquely globalized in the sense that there is up to isomorphism only one globalization [6]. Partial acts over monoids however, can have many such globalizations [9, Example 5.5]. In particular, in [9], two constructions of A-generated globalizations of semigroups were considered. The tensor product globalization $A \otimes S$ and the hom-set globalization A^S. The former was first described for the group case in [6] and in the monoid case in [11]. The latter is original to [9]. In [9] it was shown, that if we consider the category of A-generated globalizations, $A \otimes S$ is an initial object and A^S is a terminal object.

In this thesis, we first present some preliminary material on categories, order theory, partial and global acts and congruences of acts. This material is in Section 1. In Section 2, we prove that the hom-set globalization is a faithful functor but not a reflector nor a coreflector. In Section 3, we study the structure of A-generated globalizations of a strong partial act. It turns out that the isomorphism classes of objects of that category form a complete lattice that is anti-isomorphic to a sublattice of $\mathrm{Con}(A \otimes S)$, the lattice of congruences of the S-act $A \otimes S$. Finally, in Section 4, we show that for a monoid S, all strong partial S-acts being uniquely globalizable is equivalent to S being a group. The fact that all strong partial acts over a group are uniquely globalizable was first proven in [6]. The other direction is original to this thesis.

1 Preliminaries

1.1 Categories

For our reference for category theory we use [7] and [10].

Definition 1. A *category* $\mathcal{C}$ consists of a class $\mathrm{Ob}\mathcal{C}$ (the objects of $\mathcal{C}$) and for every pair (A, B) of objects in $\mathrm{Ob}\mathcal{C}$ a set $\mathrm{Hom}(A, B)$ (the set of morphisms from A to B) such that the following axioms apply.

1. If $(A, B) \neq (A', B')$, then $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B') = \emptyset$.

2. If $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, then there exists a morphism $g \circ f \in \mathrm{Hom}(A, C)$ called the *composition* of f and g.

3. If the compositions $(h \circ g) \circ f$ and $h \circ (g \circ f)$ exist, then they are equal.

4. For every object A in $\mathrm{Ob}\mathcal{C}$, there exists a morphism id_A such that for all $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(C, A)$, $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$.

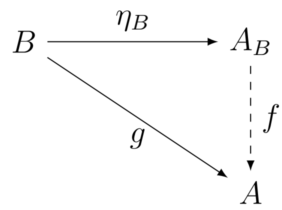
When considering multiple categories, we write $\mathrm{Hom}_{\mathcal{C}}(A, B)$ instead of $\mathrm{Hom}(A, B)$ to avoid confusion.

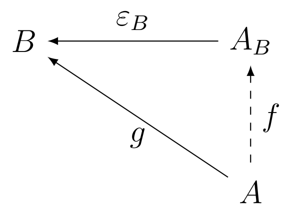
Definition 2 ([10])**.** Let $\mathcal{C}$ be a category. Let $\mathcal{D}$ be a collection of objects and morphisms of $\mathcal{C}$. Then $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if for every morphism in $\mathcal{D}$, its domain and codomain belong to $\mathcal{D}$, for each object A in $\mathcal{D}$, id_A belongs to $\mathcal{D}$ and for every two moprhisms in $\mathcal{D}$ their composite lies in $\mathcal{D}$.

Definition 3 ([10])**.** A subcategory $\mathcal{A}$ of $\mathcal{B}$ is called a *reflective subcategory* in $\mathcal{B}$ if for each $B \in \mathcal{B}$, there is an object A_B in $\mathcal{A}$ and a morphism $\eta_B : B \to A_B$ in $\mathcal{B}$ such that for every morphism $g : B \to A \in \mathcal{A}$ there exists a unique morphism $f : A_B \to A$ in $\mathcal{A}$ such that $g = f \circ \eta_B$.

In this case we call the functor $F\colon \mathcal{B} \to \mathcal{A}$, defined by $F(B) = A_B, F(f) = f$, a *reflector*. *Coreflective subcategories* and *coreflectors* are defined dually.

The following diagrams illustrate the reflective and coreflective subcategories, respectively.





Remark 4. The category $\mathcal{A}$ being a reflective subcategory of $\mathcal{B}$ is equivalent to the inclusion functor $I : \mathcal{A} \to \mathcal{B}$ having a left adjoint $F : \mathcal{B} \to \mathcal{A}$.

Similarly $\mathcal{A}$ being a coreflective subcategory of $\mathcal{B}$ is equivalent to the inclusion functor having a right adjoint.

1.2 Acts and partial acts

In this subsection we present the basics of the theory of partial acts over monoids and their globalizations. We use the theory in [9] that is written for semigroups and specialize it to monoids.

Definition 5. Let S be a set and $\cdot$ a binary operation on S. Then the pair $(S, \cdot)$ is a *semigroup* if the operation $\cdot$ is associative.

We usually denote the binary operation $\cdot$ of a semigroup by juxtaposition. We also usually write S instead of $(S, \cdot)$.

Definition 6. A triple $(S, \cdot, 1)$, where S is a set, $\cdot$ a binary operation and $1 \in S$, is a *monoid* if $(S, \cdot)$ is a semigroup and 1 is an identity element of $(S, \cdot)$.

Definition 7. Let $(M, *, e)$ and $(S, \cdot, 1)$ be monoids. Then a mapping $f\colon M \to S$ is a *morphism of monoids*, if for all $m, n \in M$, $f(m * n) = f(m) \cdot f(n)$ and $f(e) = 1$.

Definition 8. Let S be a semigroup. Let $0 \notin S$. Then $(S \cup \{0\}, \cdot)$ with the operation $\cdot$ defined as

$$a \cdot b = \begin{cases} 0, & \text{if } a = 0 \text{ or } b = 0, \\ ab, & \text{if } a \neq 0 \text{ and } b \neq 0, \end{cases}$$

is a semigroup and we call it the *semigroup S with an adjoined zero element* and 0 the *adjoined zero element.*

Definition 9. Let S be a monoid, A a set and $*\colon A \times S \to A, (a, s) \mapsto a*s$ a mapping. Then $*$ is a *right action* of S on A if

$$\forall x, y \in S \; \forall a \in A\colon (a * x) * y = a * (xy)$$

and

$$\forall x \in A\colon x * 1 = x.$$

In this case A is called a *right S-act* or just an *S-act.*

We can analogously define a left S-act. We will also refer to S-acts as *global S-acts.*

Definition 10. Let X and Y be sets and $Z \subseteq X$. Then a *partial mapping* $f\colon X \to Y$ with domain $\operatorname{dom}(f) = Z$ is a mapping $g\colon Z \to Y$. If $z \in Z$, we say that f is *defined or exists* at z and we set $f(z) = g(z)$. We also write $\exists f(z)$ in this situation.

In this paper we use the definition of partial acts given in [4].

Definition 11. Let S be a monoid and A a set. A partial mapping $\cdot\colon A \times S \to A, (a, s) \mapsto a \cdot s$, is is called a *right partial S-action* on A if the following hold.

1. For all $a \in A$ and $x, y \in S$, if $\exists a \cdot x$ and $\exists (a \cdot x) \cdot y$ then also $\exists a \cdot xy$ and $(a \cdot x) \cdot y = a \cdot xy$.

2. For every $a \in A$, $\exists a \cdot 1$ and $a \cdot 1 = a$.

We call the pair $(A, \cdot)$ a *partial S-act*.

An important notion is that of a strong partial act which arises by deleting elements from global acts. We first provide an equivalent definition in terms of elements and the partial action itself.

Definition 12. Let A be a partial act over a monoid S. Then A is a *strong partial S-act*, if for all $a \in A$ and $x, y \in S$, if $\exists a \cdot x$ and $\exists a \cdot xy$, then $\exists (a \cdot x) \cdot y$ and $a \cdot xy = (a \cdot x) \cdot y$.

Proposition 13. *Let $(A, *)$ be a global act over a monoid S and $B \subseteq A$. Define a partial mapping $\cdot\colon B \times S \to B$ so that*

$$\exists b \cdot s \iff b * s \in B$$

and if $\exists b \cdot s$, then $b \cdot s = b * s$. Then $(B, \cdot)$ is a strong partial act over S.

Proof. We first prove that $(B, \cdot)$ is a partial act. Suppose that $\exists b \cdot s$ and $\exists (b \cdot s) \cdot t$. This is equivalent to $b * s \in B$ and $(b \cdot s) * t \in B$. From the first condition we also get $b * s = b \cdot s$. In combination with the second one it gives $(b * s) * t = (b \cdot s) \cdot t$. Since A is a global act, $(b * s) * t = b * st$. This gives us that $b * st \in B$ or $\exists b \cdot st$

and $b \cdot st = b * st = (b \cdot s) \cdot t$. If $b \in B$, then $b * 1 = b \in B$. Therefore also $\exists b \cdot 1$ and $b \cdot 1 = b * 1 = b$.

For strongness, suppose that $\exists b \cdot s$ and $\exists b \cdot st$, where $b \in B$ and $s, t \in S$. This means that $b * s \in B$ and $b * st \in B$. From the latter we get $(b * s) * t \in B$. Now with the first condition we see that $\exists (b \cdot s) \cdot t$ and $(b \cdot s) \cdot t = (b * s) * t = b \cdot st$. □

The above proposition now lets us easily construct strong partial acts from global acts.

Definition 14. Let S be a monoid and let A and B be partial S-acts. Let $g \colon A \to B$ be a mapping. Then g is a *morphism of partial S-acts* if for all $a \in A$ and $s \in S$,

$$\exists a \cdot s \implies \exists g(a) \cdot s$$

and $g(a) \cdot s = g(a \cdot s)$.

Proposition 15. *The composition of two morphisms of partial S-acts is a morphism of partial S-acts.*

Proof. Let $g \colon A \to B$ and $h \colon B \to C$ be morphisms of partial S-acts. Let $a \in A$ and $s \in S$. Suppose that $\exists a \cdot s$. Then since g is a morphism, $\exists g(a) \cdot s$ and also $g(a \cdot s) = g(a) \cdot s$. Now since h is a morphism, $\exists h(g(a)) \cdot s$ and

$$h(g(a)) \cdot s = h(g(a) \cdot s) = h(g(a \cdot s)).$$

In conclusion, if $\exists a \cdot s$, then $\exists h(g(a)) \cdot s$ and $h(g(a)) \cdot s = h(g(a \cdot s))$ □

Definition 16. Let $(A, \cdot)$ be a partial S-act and $(B, *)$ a global S-act. Let $\iota \colon A \to B$ be an injective mapping. Then the triple $(B, \iota, *)$ is a *globalization* of A if for all $a \in A$ and $s \in S$ the following conditions hold:

G1. $\exists a \cdot s \iff \iota(a) * s \in \iota(A)$;

G2. $\exists a \cdot s \implies \iota(a \cdot s) = \iota(a) * s$.

Definition 17. Let $(A, \cdot)$ be a partial S-act and $(B, \iota, *)$ a globalization of A. Then B is an A-*generated globalization* if for every element $b \in B$, there exist elements $a \in A$ and $s \in S$ such that $b = \iota(a) * s$.

Definition 18. Let A be a partial S-act and let $(B, \iota_B, *)$, $(C, \iota_C, *)$ be globalizations of A. Then a morphism of global S-acts $g\colon B \to C$ is a *morphism of globalizations* of A if for all $a \in A$

$$g(\iota_B(a)) = \iota_C(a).$$

It is easy to see that A-generated globalizations of A with their morphisms form a category. We denote this category by $\mathcal{G}_A(A, S, \cdot)$.

We will use the following result extensively in Section 3.

Proposition 19. [9, Proposition 2.12] *Let S be a monoid and A a partial S-act. In the category $\mathcal{G}_A(A, S, \cdot)$, there is at most one morphism between any two objects and this morphism is always surjective.*

Proof. Suppose that $(B, \iota_B, *)$ and $(C, \iota_C, *)$ are globalizations of A and $f\colon B \to C$ is a morphism of S-acts such that $\iota_C = f \circ \iota_B$ (that is, f is a morphism of globalizations of A).

Suppose that g is another such mapping. Since B is A-generated, we can express any element of B as $\iota_B(a) * s$, where $a \in A$ and $s \in S$. Now we simply use the fact that f and g are morphisms of globalizations:

$$g(\iota_B(a) * s) = g(\iota_B(a)) * s = \iota_C(a) * b = f(\iota_B(a)) * s = f(\iota_B(a) * s).$$

For surjectivity of f, we use that C is A-generated. That is, any element of C can be expressed as $\iota_C(a) * s$, where $a \in A$ and $s \in S$. Now the element $\iota_B(a) * s$ belongs to B and we can apply f:

$$f(\iota_B(a) * s) = f(\iota_B(a)) * s = \iota_C(a) * s.$$

□

Lemma 20. *A bijective morphism of globalizations is an isomorphism of globalizations.*

Proof. Let $(B, \iota_B, *)$ and $(C, \iota_C, *)$ be globalizations of a partial S-act A. Let $g\colon (B, \iota_B, *) \to (C, \iota_C, *)$ be a bijective morphism of globalizations. Then g is a bijective S-act morphism $B \to C$ such that $g \circ \iota_B = \iota_C$.

Since g is a bijection, it has an inverse mapping $g^{-1}\colon C \to B$.

Firstly g^{-1} is a morphism of S-acts since for all $c \in C$ and $s \in S$,

$$g^{-1}(c) * s = g^{-1}(g(g^{-1}(c) * s)) = g^{-1}(g(g^{-1}(c)) * s) = g^{-1}(c * s).$$

Lastly we note that

$$g \circ \iota_B = \iota_C \implies \iota_B = g^{-1} \circ \iota_C.$$

□

1.3 The tensor product globalization

The material in this section based on [9] and originally from [11].

Let S be a monoid and $(A, \cdot)$ a partial S-act. Let $(a, s), (b, r) \in A \times S$. Define the relation

$$(a,s) \to (b,r) \iff \exists u \in S\colon (s = ur, \exists a \cdot u, a \cdot u = b).$$

Now let ρ be the equivalence relation on $A \times S$ generated by $\to$. We denote $(A\times S)/\rho = A \otimes S$ and the ρ-class of (a,s) by $a \otimes s$.

By the definition of ρ,

$$a \cdot s \otimes r = a \otimes sr,$$

whenever $\exists a \cdot s$ in A.

Since ρ is the smallest equivalence, containing $\to$, we have $a \otimes s = b \otimes r$ if and only if there exists a finite sequence

$$(a,s) = (a_0, s_0), (a_1, s_1), \ldots, (a_n, s_n) = (b,r)$$

such that for all $i \in \{0, \ldots, n-1\}$ either $(a_i, s_i) \to (a_{i+1}, s_{i+1})$ or $(a_{i+1}, s_{i+1}) \to (a_i, s_i)$ (see [5], Theorem 1.9.1). Whenever we say "sequence" in this section, we mean a sequence of this type.

We define the S-action on $A \otimes S$ by

$$(a \otimes s) * t = a \otimes st$$

for all $a \otimes s \in A \otimes S$ and $t \in S$.

We call the set $A \otimes S$ together with the action defined above the *tensor product of* A *and* S.

Proposition 21. *The pair* $(A \otimes S, *)$ *is a global* S*-act.*

Proof. We check first that the action is well-defined. That is, for all $t \in S$, if $a \otimes s = b \otimes r$, then $a \otimes st = b \otimes rt$.

We fix $t \in S$ and prove by induction that for all $n \in \mathbb{N}$, if there exists a sequence $(a_0, s_0), \ldots, (a_n, s_n)$, then there exists a sequence $(a_0, s_0 t), \ldots, (a_n, s_n t)$.

If $n = 0$, then $(a, s) = (b, r)$ and clearly the claim holds.

Next suppose that we have a sequence $(a_0, s_0), \ldots, (a_n, s_n), (a_{n+1}, s_{n+1})$. By the induction hypothesis there exists a sequence $(a_0, s_0 t), \ldots, (a_n, s_n t)$.

If $(a_n, s_n) \to (a_{n+1}, s_{n+1})$, then there exists $u \in S$ such that $s_n = u s_{n+1}$, $\exists a_n \cdot u$ and $a_n \cdot u = a_{n+1}$. This gives us that $s_n t = u s_{n+1} t$ from which $(a_n, s_n t) \to (a_{n+1}, s_{n+1} t)$. Hence we have a sequence $(a_0, s_0 t), \ldots, (a_{n+1}, s_{n+1} t)$.

The case $(a_{n+1}, s_{n+1}) \to (a_n, s_n)$ follows symmetrically.

Lastly, we prove that $A \otimes S$ is an S-act. Let $a \otimes s \in A \otimes S$ and $r, t \in S$. Then

$$((a \otimes s) * r) * t = (a \otimes sr) * t = a \otimes (sr)t = a \otimes s(rt) = (a \otimes s) * (rt).$$

□

Define a mapping $\delta \colon A \to A \otimes S$ by $\delta(a) = a \otimes 1$.

Proposition 22. *If $(A, \cdot)$ is a strong partial S-act, then the triple $(A \otimes S, \delta, *)$ is an A-generated globalization of A.*

Proof. Firstly, we prove that δ is an injection. Suppose that $a \otimes 1 = b \otimes 1$. Then there exists a sequence

$$(a, 1) = (a_0, s_0), (a_1, s_1), \ldots, (a_{n-1}, s_{n-1}), (a_n, s_n) = (b, 1)$$

such that for all $i \in \{0, \ldots, n-1\}$ either $(a_i, s_i) \to (a_{i+1}, b_{i+1})$ or $(a_{i+1}, s_{i+1}) \to (a_i, b_i)$.

We now prove by induction on i that for all $i \in \{0, \ldots, n\}$, $\exists a_i \cdot s_i$ and $a_i \cdot s_i = a$. If this holds, then $b = b \cdot 1 = a$ which is what we want.

The base case $i = 0$ is trivial.

Now assume that the claim holds for $i \in \{0, \dots, n-1\}$ and we have a sequence of length $i+1$. Then $\exists a_i \cdot s_i$ and $a_i \cdot s_i = a$ and either $(a_i, s_i) \to (a_{i+1}, s_{i+1})$ or $(a_{i+1}, s_{i+1}) \to (a_i, s_i)$.

In the first case there exists $u \in S$ such that $s_i = us_{i+1}$, $\exists a_i \cdot u$ and $a_i \cdot u = a_{i+1}$. Now

$$a = a_i \cdot s_i = a_i \cdot us_{i+1} = (a_i \cdot u) \cdot s_{i+1} = a_{i+1} \cdot s_{i+1},$$

where the second equality comes from strongness.

Assume now that $(a_{i+1}, s_{i+1}) \to (a_i, s_i)$. In this case there exists $u \in S$ such that $s_{i+1} = us_i$, $\exists a_{i+1} \cdot u$ and $a_{i+1} \cdot u = a_i$. From the induction hypothesis and the fact that A is a partial act we have

$$a_i \cdot s_i = (a_{i+1} \cdot u) \cdot s_i = a_{i+1} \cdot us_i = a_{i+1} \cdot s_{i+1}.$$

Next suppose that $a \cdot s$ is defined in A. Then

$$\delta(a) * s = (a \otimes 1) * s = a \otimes s = a \cdot s \otimes 1 = \delta(a \cdot s).$$

Conversely, if $\delta(a) * s \in \delta(A)$, then there exists $b \in A$ such that $\delta(a) * s = \delta(b) = b \otimes 1$. On the other hand, $\delta(a) * s = a \otimes s$ so $a \otimes s = b \otimes 1$. We can use the same argument with sequences we used above to show that $\exists a \cdot s$ and $a \cdot s = b$.

Lastly, to show that $A \otimes S$ is A-generated, let $a \otimes s \in A \otimes S$. By the definition of the S-action on $A \otimes S$ and by the definiton of δ,

$$a \otimes s = (a \otimes 1) * s = \delta(a) * s.$$

□

It turns out that $A \otimes S$ is an initial object in the category $\mathcal{G}_A(A, S)$.

Proposition 23 ([4]). *Let $(B, \iota, *)$ be a globalization of A. Then there exists a unique morphism of globalizations $g\colon A \otimes S \to B$.*

Proof. Define g by $g(a \otimes s) = \iota(a) * s$. Then g is a morphism of S-acts since for all $a \in A$ and $s, t \in S$

$$g(a \otimes s) * t = (\iota(a) * s) * t = \iota(a) * st = g(a \otimes st)$$

and it is a morphism of globalizations because for all $a \in A$

$$g(\delta(a)) = g(a \otimes 1) = \iota(a) * 1 = \iota(a).$$

For uniqueness, suppose that h is a morphism of globalizations $A \otimes S \to B$. Then for all $a \otimes s \in A \otimes S$

$$\begin{aligned} h(a \otimes s) &= h((a \otimes 1) * s) = h(a \otimes 1) * s = h(\delta(a)) * s = \iota(a) * s \\ &= g(\delta(a)) * s = g(a \otimes 1) * s = g((a \otimes 1) * s) = g(a \otimes s). \end{aligned}$$

□

1.4 The Hom-set globalization

We now describe another way to globalize partial acts. This method is original to [9]. Let S be a monoid and let $(A, \cdot)$ be a strong partial S-act.

Let $f_{a,s}$ denote the partial function $S \to A$ with domain $\{t \in S\colon \exists a \cdot st\}$ such that $f_{a,s}(t) = a \cdot st$ for every $t \in \operatorname{dom} f_{a,s}$. Denote $A^S = \{f_{a,s}\colon a \in A, s \in S\}$. Define an S-action on A^S by $f_{a,s} * t = f_{a,st}$ for all $t \in S$.

Proposition 24. *The pair $(A^S, *)$ is a global S-act.*

Proof. First we prove that the action is well-defined. Suppose that $f_{a,s} = f_{b,r}$. Consider the partial functions $f_{a,s} * t = f_{a,st}$ and $f_{b,r} * t = f_{b,rt}$. Suppose that $p \in S$ and $\exists f_{a,st}(p)$. Then $f_{a,st}(p) = a \cdot stp = a \cdot s(tp)$. From the existence of the last we get $\exists f_{a,s}(tp)$ and from our assumption $\exists f_{b,r}(tp)$ and $f_{a,s}(tp) = f_{b,r}(tp) = b \cdot rtp = f_{b,rt}(p)$. We see that $f_{b,rt}(p)$ is defined whenever $f_{a,st}(p)$ is and in that case they are equal. Symmetrically, we get the converse. In conclusion, we have that $f_{a,st} = f_{a,s} * t$ and $f_{b,rt} = f_{b,r} * t$ have the same domain and they are equal on all arguments of the domain. That is, they are equal as partial functions.

Let $f_{a,s} \in A$ and $t, r \in S$. Then

$$(f_{a,s} * t) * r = f_{a,st} * r = f_{a,str} = f_{a,s} * tr$$

and also

$$f_{a,s} * 1 = f_{a,s1} = f_{a,s}.$$

□

Define the mapping $\lambda\colon A \to A^S$ by

$$\lambda(a) = f_{a,1}.$$

Proposition 25. *If $(A, \cdot)$ is a strong partial S-act, then the triple $(A^S, \lambda, *)$ is an A-generated globalization of A.*

Proof. We first prove that λ is an injection. Let $a, b \in A$ such that $\lambda(a) = \lambda(b)$. That is, $f_{a,1} = f_{b,1}$. Clearly $1 \in \operatorname{dom} f_{a,1}$. Then

$$a = a \cdot 1 = f_{a,1}(1) = f_{b,1}(1) = b \cdot 1 = b.$$

Suppose now that $\exists a \cdot s$ in A. Then $\lambda(a) * s = f_{a,1} * s = f_{a,s}$. We show that $f_{a,s} = f_{a \cdot s,1}$ ($f_{a \cdot s,1} = \lambda(a \cdot s)$). Let $t \in S$. If $\exists f_{a,s}(t)$, then $\exists a \cdot st$ and from strongness we have that $\exists (a \cdot s) \cdot t$ and

$$f_{a,s}(t) = a \cdot st = (a \cdot s) \cdot 1t = f_{a \cdot s,1}(t).$$

Conversely if $\exists f_{a \cdot s,1}(t)$, then $\exists (a \cdot s) \cdot t$ and from the partial act property $\exists a \cdot st$ and

$$f_{a \cdot s,1}(t) = (a \cdot s) \cdot 1t = a \cdot st = f_{a,s}(t).$$

Now suppose that $\lambda(a) * s \in \iota(A)$ for some $a \in A$ and $s \in S$. That is, $\lambda(a) * s = \lambda(b)$ for some $b \in A$. We can write this as

$$f_{a,s} = f_{b,1}.$$

Now $1 \in \operatorname{dom} f_{b,1}$ and $f_{b,1}(1) = b$. Therefore, also $1 \in \operatorname{dom} f_{a,s}$, so $\exists a \cdot s1 = a \cdot s$ and $a \cdot s = f_{a,s}(1) = f_{b,1}(1) = b$.

Finally, we show that A^S is A-generated. Let $f_{a,s} \in A^S$. Then

$$f_{a,s} = f_{a,1} * s = \lambda(a) * s.$$

□

The globalization A^S is the initial object of the category $\mathcal{G}_A(A, S)$.

Proposition 26. *Let $(B, \iota, *)$ be an A-generated globalization of a strong partial S-act A. Then there exists a unique morphism of globalizations $h\colon B \to A^S$.*

Proof. Since B is A-generated, we can express any element of B as $\iota(a) * s$, where $a \in A$ and $s \in S$. Define

$$h(\iota(a) * s) = \lambda(a) * s = f_{a,s}.$$

We show that h is well-defined. Suppose that, in B, $\iota(a) * s = \iota(b) * t$. If $\exists f_{a,s}(r)$, then $\exists a \cdot sr$ and $f_{a,s}(r) = a \cdot sr$. Therefore, since B is a globalization, $\iota(a) * sr = \iota(a \cdot sr)$. Now

$$\iota(a \cdot sr) = \iota(a) * sr = (\iota(a) * s) * r = (\iota(b) * t) * r = \iota(b) * tr \in \iota(A).$$

Therefore, by the definition of globalization, $\exists b \cdot tr$ and $\iota(b) * tr = \iota(b \cdot tr)$. Since ι is injective, $\iota(a \cdot sr) = \iota(b \cdot tr)$ implies $a \cdot sr = b \cdot tr$, so

$$f_{a,s}(r) = a \cdot sr = b \cdot tr = f_{b,t}(r).$$

We see that if $f_{a,s}$ is defined on an element r, so is $f_{b,t}$ and they are equal. The proof of the converse is symmetric.

To show that h is a morphism of globalizations, we first prove that it is a morphism of S-acts. Let $\iota(a) * s$ be any element of B, where $a \in A$ and $s \in S$, and let $t \in S$. Then

$$h(\iota(a) * s) * t = f_{a,s} * t = f_{a,st} = h(\iota(a) * st) = h((\iota(a) * s) * t).$$

Now let $a \in A$. Then

$$h(\iota(a)) = h(\iota(a) * 1) = f_{a,1} = \lambda(a).$$

For uniqueness, let g be another morphism of globalizations $B \to A^S$. Then, for all $\iota(a) * s \in B$,

$$g(\iota(a) * s) = g(\iota(a)) * s = \lambda(a) * s = h(\iota(a)) * s = h(\iota(a) * s),$$

so $g = h$. □

1.5 Complete lattices

For our reference for order theory, we use [5]. The proofs of the propositions in this section can be found there.

Definition 27. Two posets X and Y are *isomorphic*, if there exists a bijective mapping $\phi\colon X \to Y$ such that for all $x, z \in X$,

$$x \leq z \text{ in } X \iff \phi(x) \leq \phi(z) \text{ in } Y.$$

Definition 28. Two posets X and Y are *anti-isomorphic*, if there exists a bijective mapping $\psi\colon X \to Y$ such that for all $x, z \in X$,

$$x \geq z \text{ in } X \iff \psi(x) \leq \psi(z) \text{ in } Y.$$

The mappings ϕ and ψ in the above definitions are called an *isomorphism* and an *anti-isomorphism* respectively.

Definition 29. Let $(X, \leq)$ be a poset. Let $Y \subseteq X$. We call an element $z \in X$ the *join* of Y (denoted $z = \bigvee Y$) if

1. $\forall y \in Y : y \leq z$,

2. $\forall x \in X : (\forall y \in Y : y \leq x) \implies z \leq x$.

We define the *meet* of Y dually and denote it by $\bigwedge Y$.

Definition 30. A poset $(X, \leq)$ is a *complete lattice* if it has all meets and joins for all subsets of X.

Remark 31. A poset is a complete lattice if and only if it has all joins and a least element.

Proposition 32. *If $(X, \leq)$ is a complete lattice and $x \in X$, then the subset $x\downarrow = \{y \in X : y \leq x\} \subseteq X$, sometimes called a downset of x, is also a complete lattice.*

1.6 Congruences of acts

In this subsection, we use the general theory of universal algebras in [5] in the case of acts over semigroups.

An act M over a monoid S can be defined as an algebra for which there exists a unary operation for each element of S that corresponds to the action with that element.

The proofs of the following results are standard and can be found in for example [5].

Let A be a set. Denote the set of binary relations of A by $\mathrm{Rel}A$ and the set of equivalences of A by $\mathrm{Equiv}A$.

If we define the relation $\leq$ on $\mathrm{Rel}A$ by

$$\rho \leq \sigma \iff (\forall a, b \in A : a\rho b \implies a\sigma b),$$

we get a partial order. We call it the *inclusion of relations.*

The set Equiv$A$ is a compete lattice under the inclusion of relations.

Definition 33. Let S be a monoid and $(A, *)$ an S-act. Then a relation ρ is a *congruence* on A if for all $a, b \in A$ and $s \in S$

$$a \,\rho\, b \implies (a * s) \,\rho\, (b * s).$$

For $a \in A$, denote the ρ-class of a by $[a]_\rho$. Denote the set of all congruences on A by $\operatorname{Con} A$.

Proposition 34. *Let A be an S-act and ρ be a congruence of A. Then the following statements are true.*

1. *The set $\operatorname{Con} A$ with the inclusion of relations is a complete lattice.*
2. *For every $\rho \in \operatorname{Con} A$, the set $A/\rho = \{[a]_\rho \colon a \in A\}$ is an S-act with the action defined by*
$$[a]_\rho * s = [a * s]_\rho.$$

The S-act constructed above is called the quotient act *of A by the congruence ρ.*

Definition 35. Let $f \colon A \to B$ be a morphism of S-acts. Let $\ker f$ denote the binary relation on A given by

$$a \ker f\ b \iff f(a) = f(b).$$

The relation just defined is sometimes called the *set theoretic kernel of f.*

Proposition 36. *Let A and B be S-acts. Let ρ be a congruence of A and let $f \colon A \to B$ be a morphism of S-acts. Then the following statements hold.*

1. *The mapping $\pi_\rho \colon A \to A/\rho, a \mapsto [a]_\rho$, is a surjective S-act morphism.*

2. *The relation* $\ker f$ *is a congruence of* A.

3. *The relation* $\rho \leq \ker f$ *holds if and only if* f *factors through* π_ρ *i.e. there exists a morphism of* S*-acts* $\kappa\colon A/\rho \to B$ *such that* $\kappa \circ \pi_\rho = f$.

4. *(Homomorphism theorem for acts) Let* $f\colon A \to B$ *be a surjective morphism of* S*-acts. Then there exists an isomorphism of* S*-acts* $h\colon A/\ker f \to B$ *such that* $h \circ \pi_{\ker f} = f$.

2 Properties of the hom globalization functor

In this section we provide a way to define $(-)^S$ on morphisms of partial acts so that it turns into a faithful functor. We also give examples that show that $(-)^S$ is neither a reflector nor a coreflector.

By SPAct$S$, we denote the full subcategory of the category of partial acts whose objects are strong partial acts.

Definition 37. Let S be a monoid and $g\colon A \to B$ be a morphism of partial S-acts. Then we define a mapping $g^S\colon A^S \to B^S$ by

$$g^S(f_{a,s}) = f_{g(a),s}.$$

Theorem 38. *Let S be a monoid. The mapping $(-)^S\colon \mathrm{SPAct}S \to \mathrm{Act}\,S$ is a faithful covariant functor from the category of strong partial S-acts to the category of S-acts.*

Proof. The mapping $g^S\colon A^S \to B^S$ is a morphism of global S-acts because

$$g^S(f_{a,s}) * t = f_{g(a),s} * t = f_{g(a),st} = g^S(f_{a,st}) = g^S(f_{a,s} * t).$$

We now prove that $(-)^S$ takes identity morphisms to identity morphisms. Let $A \in$ SPAct$S$. If $f_{a,s} \in A^S$, then

$$(1_A)^S(f_{a,s}) = f_{1_A(a),s} = f_{a,s},$$

so 1_A^S is the identity mapping of A^S.

Next we show that $(-)^S$ behaves well with composition of morphisms. Suppose that we have $A, B, C \in \mathrm{SPAct}S$, $g\colon A \to B$, $h\colon B \to C$ and $f_{a,s} \in A^S$. Then

$$(h \circ g)^S(f_{a,s}) = f_{(h\circ g)(a),s} = f_{h(g(a)),s} = h^S(f_{g(a),s}) = (h^S \circ g^S)(f_{a,s}).$$

Lastly, we prove that $(-)^S$ is faithful. Let $A, B \in \mathrm{SPAct}S$. Suppose that $h, g\colon A \to B$ are morphism of partial acts such that $h^S = g^S\colon A^S \to B^S$. Then for all $a \in A$, if $h(a)$ exists, then

$$\exists h(a) = h(a) \cdot 1 = h(a) \cdot 11 \implies 1 \in \mathrm{dom}(f_{h(a),1})$$

and

$$\begin{aligned} h(a) = h(a) \cdot 11 = f_{h(a),1}(1) &= (h^S(f_{a,1}))(1) \\ &= (g^S(f_{a,1}))(1) = f_{g(a),1}(1) = g(a) \cdot 11 = g(a). \end{aligned}$$

□

Note that by 11 we mean the multiplication of the identity element of S by itself in S.

A natural property that one would expect from a globalization functor is that it is a reflector. The functor $(-)^S$ sadly is neither a reflector nor a coreflector. The following two propositions give examples when these properties fail. But first we introduce trivial partial actions of monoids.

Lemma 39. *Let S be a monoid. Then any set X together with a partial action that is defined only for the identity element 1 and $x \cdot 1 = x$ for every $x \in X$ is a strong partial act. Denote it by* $\mathrm{Triv}(X, S)$.

Proof. First we show that it is a partial act. Take $x \in X$ and $s, t \in S$. Suppose that

$x \cdot s$ and $(x \cdot s) \cdot t$ are defined. Then from our definition we see that $s = 1$ and $t = 1$. Hence,

$$(x \cdot s) \cdot t = (x \cdot 1) \cdot 1 = x \cdot 1 = x \cdot 11 = x \cdot st.$$

For strongness assume that $x \cdot s$ and $x \cdot st$ are defined. Then $s = 1$ and $st = 1$. Substituting s into the latter we get $t = 1$. In conclusion

$$x \cdot st = x \cdot 1 = (x \cdot 1) \cdot 1 = (x \cdot s) \cdot t.$$

□

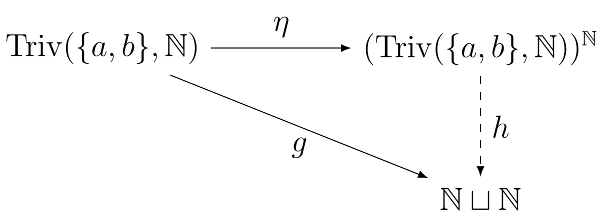
Definition 40. Let $\mathbb{N} \sqcup \mathbb{N}$ be the $\mathbb{N}$-act (where $\mathbb{N} = \{0, 1, 2, \dots\}$) defined as follows. We write

$$\mathbb{N} \sqcup \mathbb{N} = \{n_a : n \in \mathbb{N}\} \cup \{n_b : n \in \mathbb{N}\}.$$

The $\mathbb{N}$-action on $\mathbb{N} \sqcup \mathbb{N}$ is defined by $n_a + m = (n + m)_a$ and $n_b + m = (n + m)_b$.

It is easy to prove using the associativity of the addition of natural numbers, that $\mathbb{N} \sqcup \mathbb{N}$ is an $\mathbb{N}$-act.

Proposition 41. *Let* $(\mathbb{N}, +)$ *be the monoid of natural numbers under usual addition. Let* $\mathrm{Triv}(\{a, b\}, \mathbb{N})$ *and* $\mathbb{N} \sqcup \mathbb{N}$ *be the partial acts as defined above. Let* $g\colon \mathrm{Triv}(\{a, b\}, \mathbb{N}) \to \mathbb{N} \sqcup \mathbb{N}$ *be the partial act morphism that takes* a *to* 0_a *and* b *to* 0_b*. Then there is no morphism of partial acts* $\eta\colon \mathrm{Triv}(\{a, b\}, \mathbb{N}) \to (\mathrm{Triv}(\{a, b\}, \mathbb{N}))^{\mathbb{N}}$ *and morphism of global acts* $h\colon \mathrm{Triv}(\{a, b\}, \mathbb{N}) \to \mathbb{N} \sqcup \mathbb{N}$ *such that* $h \circ \eta = g$.



Proof. Suppose that such η and h do exist. Then $h(\eta(a)) = g(a) = 0_a$ and $h(\eta(b)) = g(b) = 0_b$.

The equality $h(\eta(a) + 1) = h(\eta(b) + 1)$ holds because $f_{x,n} = f_{y,m}$ for all $x, y \in \{a, b\}$ and $m, n \geq 1$. This is true because $x \cdot n$ and $y \cdot m$ do not exist for $m, n \geq 1$ so both functions have empty domain. Now

$$1_a = 0_a + 1 = h(\eta(a)) + 1 = h(\eta(a) + 1) = h(\eta(b) + 1) = h(\eta(b)) + 1 = 0_b + 1 = 1_b,$$

but 1_a and 1_b are in different components of the disjoint union $\mathbb{N} \sqcup \mathbb{N}$, a contradiction. □

Proposition 42. *Let* $\mathrm{Triv}(\{a\}, \mathbb{N})$ *be as in Proposition 41. Then there is no morphism of partial* $\mathbb{N}$*-acts* $\varepsilon\colon \mathrm{Triv}(\{a\}, \mathbb{N})^{\mathbb{N}} \to \mathrm{Triv}(\{a\}, \mathbb{N})$.

Proof. The $\mathbb{N}$-act $\mathrm{Triv}(\{a\}, \mathbb{N})^{\mathbb{N}}$ has two elements $f_{a,0}$ and $f_{a,1}$. Indeed any partial function $f_{a,n}$, where $n \geq 1$ is the empty partial function.

Now the only mapping $\mathrm{Triv}(\{a\}, \mathbb{N})^{\mathbb{N}} \to \mathrm{Triv}(\{a\}, \mathbb{N})$ is the constant function

$$\varepsilon(f_{a,n}) = a.$$

If ε was a morphism of partial $\mathbb{N}$-acts, then since $\exists f_{a,0} \cdot 1$, $\exists \varepsilon(f_{a,0}) \cdot 1 = a \cdot 1$, where $\cdot$ is the partial action of $\mathrm{Triv}(\{a\}, \mathbb{N})$. This is a contradiction with the definition of $\mathrm{Triv}(\{a\}, \mathbb{N})$. □

3 The lattice of A-generated globalizations

From Proposition 19 we know that for a partial act A over a semigroup S, any morphism of globalizations in $\mathcal{G}_A(A, S, \cdot)$ is surjective. We also know that between two objects there is at most one morphism.

If S is a monoid and A is strong, then $A \otimes S$ is an initial object and A^S is a terminal object of the category $\mathcal{G}_A(A, S)$ (see Proposition 23 and Proposition 26).

Define on the class of objects of $\mathcal{G}_A(A, S, \cdot)$ a relation $\leq$ by

$$(B, \iota_B, *) \leq (C, \iota_C, \star) \iff \text{there exists a morphism } (C, \iota_C, \star) \to (B, \iota_B, *) \text{ in } \mathcal{G}_A(A, S, \cdot).$$

This relation is well-defined on the isomorphism classes of objects of $\mathcal{G}_A(A, S, \cdot)$. Indeed, take A-generated globalizations $B \cong B'$ and $C \cong C'$ and suppose that there exists a morphism $C \to B$. Then after composing with the relevant isomorphisms, we get a morphism $C' \to B'$. We denote the collection of isomorphism classes of objects of $\mathcal{G}_A(A, S, \cdot)$ by $\mathcal{L}_A$. Let $[B, \iota_B, *]$ denote the isomorphism class represented by an object $(B, \iota_B, *)$.

The relation $\leq$ on $\mathcal{L}_A$ is reflexive and transitive because $\mathcal{G}_A(A, S, \cdot)$ is a category.

We prove that the relation $\leq$ is antisymmetric. Suppose that $[B, \iota_B, *] \leq [C, \iota_C, \star]$ and $[C, \iota_C, \star] \leq [B, \iota_B, *]$ in $\mathcal{L}_A$. Then there exists $f\colon (C, \iota_C, \star) \to (B, \iota_B, *)$ and $g\colon (B, \iota_B, *) \to (C, \iota_C, \star)$ in $\mathcal{G}_A(A, S)$. Now $f \circ g\colon (B, \iota_B, *) \to (B, \iota_B, *)$. Since morphisms in $\mathcal{G}_A(A, S, \cdot)$ between objects are unique, we must have $f \circ g = 1_{(B, \iota_B, *)}$ and similarly $g \circ f = 1_{(C, \iota_C, \star)}$. Therefore $(B, \iota_B, *)$ and $(C, \iota_C, \star)$ are isomorphic and $[B, \iota_B, *] = [C, \iota_C, \star]$.

Remark 43. Let A be a strong partial act over a monoid S. Consider the set of quotient acts of the global S-act $A \otimes S$. For each quotient $(A \otimes S)/\rho$, we can consider the set of all functions $A \to (A \otimes S)/\rho$. Take the disjoint union of all these sets of functions over all quotients. This is clearly a set. Denote it by H. Now consider the poset $\mathcal{L}_A$. An element of $\mathcal{L}_A$ has a representative $(B, \iota_B, *)$. Since there is a surjective morphism of globalizations $g\colon A \otimes S \to B$, we can write $(B, \iota_B, *) \cong ((A \otimes S)/\rho, r, *)$ for some $\rho \in \operatorname{Con} A \otimes S$ and injective mapping $r\colon A \to (A \otimes S)/\rho$. We see that every element of $\mathcal{L}_A$ can be represented by a quotient act of $A \otimes S$ and a mapping r from A to this quotient. Therefore each element of $\mathcal{L}_A$ can be represented by an element of H. Now if two elements of $\mathcal{L}_A$ are represented by the same element in H, then they must be isomorphic as globalizations. We see that $\mathcal{L}_A$ can be injected into H and is therefore a set.

By Proposition 23, there exists a unique morphism of globalizations $\phi\colon A \otimes S \to A^S$.

Lemma 44. *The morphism $\phi\colon A \otimes S \to A^S$ has the following properties.*

1. *The morphism ϕ is given by $a \otimes s \mapsto f_{a,s}$.*
2. *If $x \in A \otimes S \setminus \delta(A)$, then $\phi(x) \in A^S \setminus \lambda(A)$.*

Proof. 1. From the proof of Proposition 23

$$\phi(a \otimes s) = \lambda(a) * s = f_{a,1} * s = f_{a,s}.$$

2. Let $a \otimes s \in A \otimes S \setminus \delta(A)$. Assume to the contrary, that $\phi(a \otimes s) \in \lambda(A)$. Then for some $b \in A$,

$$f_{a,s} = \phi(a \otimes s) = \lambda(b) = f_{b,1}.$$

Now $b \cdot 11$ is defined, so $1 \in \mathrm{dom}(f_{b,1}) = \mathrm{dom}(f_{a,s})$. Hence $\exists a \cdot s$ and

$$a \cdot s = f_{a,s}(1) = f_{b,1}(1) = b \cdot 1 = b.$$

Lastly, since

$$a \otimes s = a \cdot s \otimes 1 = \delta(a \cdot s) \in \delta(A),$$

we get a contradiction.

□

Lemma 45. *Suppose that* $\rho \in \mathrm{Con}(A \otimes S)$ *and* $\rho \leq \ker \phi$. *Then*

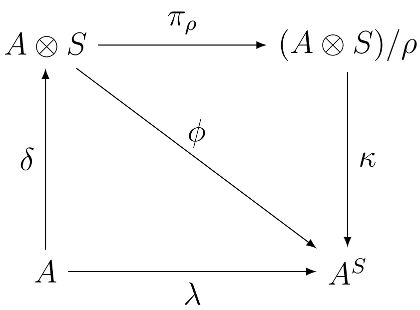
$$((A \otimes S)/\rho, \pi_\rho \circ \delta, *) \in \mathcal{G}_A(A, S, \cdot).$$

Proof. Firstly, we show that $\pi_\rho \circ \delta \colon A \to (A \otimes S)/\rho$ is injective. Suppose that $\pi_\rho(\delta(a)) = \pi_\rho(\delta(b))$, where $a, b \in A$. Then $\delta(a) \ \rho \ \delta(b)$. Now since $\rho \leq \ker \phi$, $\phi(\delta(a)) = \phi(\delta(a))$. Since ϕ is a morphism of globalizations from $(A \otimes S, \delta, *)$ to $(A^S, \lambda, *)$, $\phi \circ \delta = \lambda$. Since λ is injective, ϕ must be injective on $\delta(A)$. Therefore $\delta(a) = \delta(b)$. Since δ is injective, $a = b$.

Suppose that $a \cdot s$ is defined for some $a \in A$ and $s \in S$. Then $\delta(a \cdot s) = \delta(a) * s$. Now we can apply π_ρ to see that $\pi_\rho(\delta(a)) * s = \pi_\rho(\delta(a) * s) = \pi_\rho(\delta(a \cdot s)) \in \pi_\rho(\delta(A))$, where the first equality comes from the fact that π_ρ is an S-act morphism.

Conversely let $\pi_\rho(\delta(a)) * s \in \pi_\rho(\delta(A))$. Firstly, note that $\pi_\rho(\delta(a)) * s = \pi_\rho(\delta(a) * s)$. Since $\rho \leq \ker \phi$, ϕ factors through π_ρ due to Lemma 3. That is $\phi = \kappa \circ \pi_\rho$ for an S-act morphism $\kappa \colon (A \otimes S)/\rho \to A^S$. We get that

$$\kappa(\pi_\rho(\delta(a) * s)) \in \kappa(\pi_\rho(\delta(A))).$$



This implies that $\phi(\delta(a) * s) \in \phi(\delta(A))$. Since ϕ is a morphism of S-acts, $\phi(\delta(a)) * s \in \phi(\delta(A))$. Since ϕ is a morphism of globalizations, $\phi \circ \delta = \lambda$. Therefore

$$\lambda(a) * s \in \lambda(A).$$

Finally since $(A^S, \lambda, *)$ is a globalization of A, $\exists a \cdot s$ in A due to G1.

The fact that the act $(A \otimes S)/\rho$ is A-generated follows from the fact that $A \otimes S$ is A-generated and π_ρ is an S-act morphism. □

The following theorem is the main result of this section. From [9] we know that every A-generated globalization of A is, up to isomorphism, a quotient of the S-act $A \otimes S$ as there is a surjective morphism from $A \otimes S$ to any A-generated globalization. Here we describe which quotients of $A \otimes S$ are A-generated globalizations of A.

Theorem 46. *Let S be a monoid and A a strong partial S-act. Denote by $\ker \phi\downarrow$ the complete sublattice of $\mathrm{Con}\,(A \otimes S)$ that contains all elements below $\ker \phi$ (see Proposition 32). Then $\ker \phi\downarrow$ is as a poset anti-isomorphic to $\mathcal{L}_A$ via the mapping*

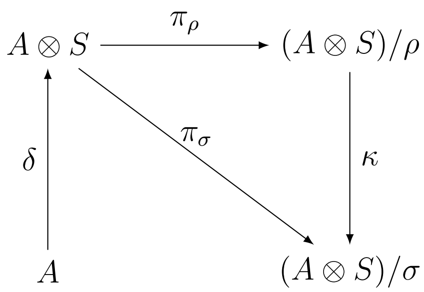
$$\Phi\colon\ \ker \phi\downarrow \to \mathcal{L}_A,\ \ \Phi(\rho) = [(A \otimes S/)\rho, \pi_\rho \circ \delta, *].$$

Proof. Lemma 45 gives us that Φ is well-defined.

We prove that

$$\Phi(\sigma) \leq \Phi(\rho) \iff \rho \leq \sigma.$$

Take $\rho \leq \sigma$ in $\ker \phi \downarrow$. By Lemma 3, π_σ must factor through π_ρ, i.e., there exists a morphism of S-acts $\kappa\colon (A \otimes S)/\rho \to (A \otimes S)/\sigma$ such that $\pi_\sigma = \kappa \circ \pi_\rho$. We are done if we show that κ is a morphism of globalizations $\kappa : ((A \otimes S)/\rho, \pi_\rho \circ \delta, *) \to ((A \otimes S)/\sigma, \pi_\sigma \circ \delta, *)$. We already know that κ is a morphism of S-acts so the only thing left is to note that $\pi_\sigma \circ \delta = \kappa \circ \pi_\rho \circ \delta$.



Conversely let $\Phi(\sigma) \leq \Phi(\rho)$. This means that $[(A\otimes S)/\sigma, \pi_\sigma \circ \delta, *] \leq [(A\otimes S)/\rho, \pi_\rho \circ \delta, *]$ which implies that there exists a unique surjective morphism of S-acts $g\colon (A\otimes S)/\rho \to (A \otimes S)/\sigma$ such that $\pi_\sigma \circ \delta = g \circ \pi_\rho \circ \delta$. Let $\delta(a) * s, \delta(b) * r \in A \otimes S$ be any elements of $A \otimes S$. Since $A \otimes S$ is A-generated, we can choose them in this form without losing generality. Now assume that

$$(\delta(a) * s) \ \rho \ (\delta(b) * r).$$

This is equivalent to

$$\pi_\rho(\delta(a) * s) = \pi_\rho(\delta(b) * r).$$

Since π_ρ is a morphism of S-acts,

$$\pi_\rho(\delta(a)) * s = \pi_\rho(\delta(b)) * s.$$

We now apply g to both sides and use the fact that it is a morphism of S-acts:

$$g(\pi_\rho(\delta(a))) * s = g(\pi_\rho(\delta(b))) * s.$$

Now we use the fact that g is a morphism of globalizations:

$$\pi_\sigma(\delta(a)) * s = \pi_\sigma(\delta(b)) * s.$$

We see that $\pi_\sigma(\delta(a) * s) = \pi_\sigma(\delta(b) * s)$ and finally $(\delta(a) * s)\ \sigma\ (\delta(b) * s)$. We conclude that $\rho \leq \sigma$.

For injectivity suppose that $\Phi(\rho) = \Phi(\sigma)$. Then $\Phi(\rho) \leq \Phi(\sigma)$ and $\Phi(\sigma) \leq \Phi(\rho)$ which imply $\sigma \leq \rho$ and $\rho \leq \sigma$, respectively. From antisymmetry now $\rho = \sigma$.

Lastly, we prove surjectivity of Φ. Let $(B, \iota, *)$ be an A-generated globalization of A. Then, by Proposition 23 and Proposition 3, there exists a surjective morphism of globalizations $g\colon A \otimes S \to B$. Then $\ker g$ is a congruence of $A \otimes S$. We must also check that $\ker g \leq \ker \phi$. Since A^S is a terminal object, there exists unique morphism of globalizations $\kappa\colon B \to A^S$. Now since a composition of morphisms is a morphism and morphisms are unique, $\phi = \kappa \circ g$. Since ϕ factors through g, $\ker g \leq \ker \phi$.

We show that $(B, \iota, *) \cong ((A \otimes S)/\ker g, \pi_{\ker g} \circ \delta, *)$. From the homomorphism theorem, there exists an isomorphism of S-acts $h\colon (A \otimes S)/\ker g \to B$ such that $h \circ \pi_{\ker g} = g$. We also have from the fact that g is a morphism of globalizations that $g \circ \delta = \iota$. Substitute in g to get $h \circ (\pi_{\ker g} \circ \delta) = \iota$. Now h is a bijective morphism of globalizations and by Lemma 20 an isomorphism. □

Corollary 47. *The poset $\mathcal{L}_A$ is a complete lattice.*

4 Monoids for which all strong partial acts are uniquely globalizable

4.1 Globalizations of partial group actions

In this subsection we show that partial acts over groups are uniquely globalizable. The original proof can be found in [6, Proposition 3.3] and is also given in [9, Proposition 5.4].

Definition 48. Let S be a monoid. Then we call a strong partial S-act A *uniquely globalizable* if the lattice $\mathcal{L}_A$ is trivial.

Proposition 49. *Let G be a group and $(A, \cdot)$ a strong partial G-act. Let $(B, \iota, *)$ be an A-generated globalization of $(A, \cdot)$. Then the unique surjective morphim $g\colon A \otimes G \to B, a \otimes s \to \iota(a) * s$ from the proof of Proposition 3 is injective and hence an isomorphism.*

Proof. Suppose that $g(a \otimes x) = g(b \otimes y)$ for some $a, b \in A$ and $x, y \in G$. This is equivalent to $\iota(a) * x = \iota(b) * y$. Apply y^{-1} to both sides: $(\iota(a) * x) * y^{-1} = (\iota(b) * y) * y^{-1}$ or $\iota(a) * xy^{-1} = \iota(b)$.

We see that $\iota(a) * xy^{-1} \in \iota(A)$. Therefore $a \cdot xy^{-1}$ is defined and $\iota(a \cdot xy^{-1}) = \iota(a) * xy^{-1} = \iota(b)$. Form the injectivity of ι we get $a \cdot xy^{-1} = b$.

Finally $a \otimes x = a \otimes xy^{-1}y = (a \cdot xy^{-1}) \otimes y = b \otimes y$. □

Corollary 50. *If G is a group and $(A, \cdot)$ is a strong partial G-act, then $|\mathcal{L}_A| = 1$.*

4.2 Groups are the only monoids for which all strong partial acts are uniquely globalizable

In this subsection, S will denote a monoid.

Lemma 51. *A strong partial S-act A is uniquely globalizable if and only if the surjective morphism $\phi\colon A \otimes S \to A^S, a \otimes s \mapsto f_{a,s}$, is injective.*

Proof. Suppose that the lattice $\mathcal{L}_A$ is trivial. Then there exists up to isomorphism only one A-generated globalization. Therefore ϕ must be an isomorphism and hence injective.

Conversely if the lattice is nontrivial, then $A \otimes S$ and A^S cannot be isomorphic because then all A-generated globalizations would be. Indeed, let $\psi\colon (A^S, \lambda, *) \to (A \otimes S, \delta, *)$ be an isomorphism. Now if $(B, \iota_B, *)$ is an A-generated globalization, then there exist surjective morphisms $g\colon (A \otimes S, \delta, *) \to (B, \iota_B, *)$ and $h\colon (B, \iota_B, *) \to (A^S, \lambda, *)$. Then $\phi = h \circ g$, because there is only one morphism $A \otimes S \to A^S$. Therefore g is injective, hence an isomorphism. In conclusion, all A-generated gobalizations would be isomorphic to $A \otimes S$. $\square$

We now show that the only monoids for which all strong partial acts are uniquely globalizable are groups.

Lemma 52. *The only monoids over which* $\operatorname{Triv}(\{c\}, S)$ *is uniquely globalizable are groups and groups with an externally adjoined zero-element (see Definition 8).*

Proof. Assume that $\operatorname{Triv}(\{c\}, S)$ is uniquely globalizable.

We firstly prove that $\operatorname{Triv}(\{c\}, S) \otimes S \cong S$. If we send $c \otimes s$ to s, then this is clearly a surjective morphism of S-acts.

We show that the mapping $c \otimes s \mapsto s$ is well-defined. Suppose that $(c, s) \to (c, t)$. Then there exists $u \in S$ such that $s = ut$ and $\exists c \cdot u = c$. The last equality gives $u = 1$ and the previous gives $s = t$. Therefore, the equivalence relation in introduced before Proposition 21 is the diagonal relation and two tensors $c \otimes s$ and $c \otimes t$ are equal if and only if $s = t$.

Let us calculate $(\mathrm{Triv}(\{c\}, S))^S$. The partial function $f_{c,s} \colon S \to \mathrm{Triv}(\{c\}, S)$ is defined at $t \in S$ if and only if $st = 1$. Therefore $f_{c,s}$ is not the empty partial function if and only if s has a right inverse. If there were two distinct elements $s, t \in S$ that do not have right inverses, we would have $f_{c,s} = f_{c,t} = \emptyset$ for two distinct elements and in that case the mapping $\phi \colon c \otimes s \mapsto f_{c,s}$ would be non-injective, contradicting Lemma 51.

If every element has a right inverse, S is a group (see [8, Exercise 2.13]).

Suppose that exactly one element $x \in S$ lacks a right inverse. We show that all elements not equal to x form a submonoid. Suppose that $a \in S$ and $b \in S$ are neither equal to x and $ab = x$. Then b has a right inverse β and a has a right inverse α. Multiply both sides by $\beta\alpha$. We get $1 = x\beta\alpha$ but this means that $\beta\alpha$ is a right inverse to x, a contradiction.

Now $S\backslash\{x\}$ is a monoid where every element has a right inverse. Therefore it is a group.

Lastly we prove that x is a zero element. For any $s \in S$, $xs = x$ since if $xs = a \neq x$ then a has a right inverse b and we have $x(sb) = (xs)b = ab = 1$, in other words: x would have a right inverse. We can conduct the same argument to show that $sx = x$.

$\square$

Lemma 53. *The strong partial act* $\mathrm{Triv}(\{c_1, c_2\}, S)$ *is not uniquely globalizable if* S *is a group with an adjoined zero.*

Proof. In the globalization $\text{Triv}(\{c_1, c_2\}, S) \otimes S$ we show that $c_1 \otimes 0 \neq c_2 \otimes 0$.

Suppose that $(c_1, 0) \to (b, t)$. Then there exists $u \in S$ such that $0 = ut$, $\exists c_1 \cdot u$ and $c_1 \cdot u = b$. Since only action by 1 is defined, $u = 1$, $t = 0$ and $b = c_1$.

Now suppose that $(a, s) \to (c_1, 0)$. Then there exists $u \in S$ such that $s = u0 = 0$, $\exists a \cdot u$ and $a \cdot u = c_1$. Again the partial action is only defined for $u = 1$. We see that $a = c_1$ and $s = 0$.

From the above we can conclude that $c_1 \otimes 0 = a \otimes s$ if and only if $a = c_1$ and $s = 0$. In particular $c_1 \otimes 0 \neq c_2 \otimes 0$.

In the globalization $(\text{Triv}(\{c_1, c_2\}, S))^S$, $f_{c_1,0} = f_{c_2,0} = \emptyset$, so $\phi(c_1 \otimes 0) = \phi(c_2 \otimes 0)$. Since ϕ is not injective, $\text{Triv}(\{c_1, c_2\}, S)$ is not uniquely globalizable by Lemma 51. □

We have proved the following theorem.

Theorem 54. *Let S be a monoid. Then all strong partial S-acts are uniquely globalizable if and only if S is a group.*

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