Monomorphisms in the Category of firm Modules

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Statistical Inference and Algebraic Structures
1. July 2019

Preliminaries

Definition

A set R is called a **ring**, if it is equipped with two binary operations, addition $+\colon\ R\times R\to R$ and multiplicaton $\cdot\colon\ R\times R\to R$, such that the following properties hold:

1. $\forall r,s,t\in R\colon\quad (r+s)+t=r+(s+t)$,
2. $\exists 0\in R\,\forall r\in R\colon\quad 0+r=r+0=r$,
3. $\forall r\in R\,\exists -r\in R\colon\quad r+(-r)=-r+r=0$,
4. $\forall r,s\in R\colon\quad r+s=s+r$,
5. $\forall r,s,t\in R\colon\quad (rs)t=r(st)$,
6. $\forall r,s,t\in R\colon\quad (r+s)t=rt+st \quad \& \quad r(s+t)=rs+rt$.

Definition

A set M is called a **right R-module**, if it is equipped with a binary operation $+\colon M \times M \to M$ and a mapping $\cdot\colon M \times R \to M$, such that the following properties hold:

1. pair $(M, +)$ is an Abelian group,
2. $\forall m, n \in M \, \forall r \in R\colon \quad (m+n)r = mr + nr,$
3. $\forall m \in M \, \forall r, s \in R\colon \quad m(r+s) = mr + ms,$
4. $\forall m \in M \, \forall r, s \in R\colon \quad m(rs) = (mr)s.$

Example (module):

Let S be a ring. Consider the direct power $S^n = \{(s_1, s_2, \ldots, s_n) \mid s_1, s_2, \ldots, s_n \in S\}$. The set S^n is a right module over the ring $\mathrm{Mat}_{n,n}(S)$, where addition is defined componentwise and mapping $S^n \times \mathrm{Mat}_{n,n}(S) \to S^n$ is defined as

$$(s_1, s_2, \ldots, s_n) \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} = \\ = \left(\sum_{k=1}^{n} s_k q_{k1}, \sum_{k=1}^{n} s_k q_{k2}, \ldots, \sum_{k=1}^{n} s_k q_{kn} \right).$$

Definition

A **category** is something that consists of two kinds of things – *objects* and *morphisms*. If $\mathcal{A}$ is a category then its objects form a class, which is denoted as $\mathrm{Ob}(\mathcal{A})$. Every ordered pair of objects (A, B) has a set $\mathrm{Mor}(A, B)$ associated with it, which is called the *set of morphisms from object A to object B*, such that:

1. if $(A, B) \neq (A', B')$, then $\mathrm{Mor}(A, B) \cap \mathrm{Mor}(A', B') = \varnothing$;
2. if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then exists their composition $g \circ f \in \mathrm{Mor}(A, C)$;
3. if the compositions of morphisms $(h \circ g) \circ f$ and $h \circ (g \circ f)$ exist, then they are equal;
4. for every object A, there exists a morphism $1_A \in \mathrm{Mor}(A, A)$, such that $f \circ 1_A = f$ and $1_A \circ g = g$ for every $f \in \mathrm{Mor}(A, B)$ and every $g \in \mathrm{Mor}(C, A)$.

Definition

A morphism $f \in \mathrm{Mor}(A, B)$ in category $\mathcal{A}$ is called a **monomorphism**, if for every $g, h \in \mathrm{Mor}(C, A)$ the following holds

$$f \circ g = f \circ h \implies g = h.$$

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A morphism $f \in \mathrm{Mor}(A, B)$ in category $\mathcal{A}$ is called a **epimorphism**, if for every $g, h \in \mathrm{Mor}(B, C)$ the following holds

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Definition

A monomorphism f is called **extremal**, if $f = g \circ e$, where e is an epimorphism implies that g is an isomorphism.

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A monomorphism $f\colon\ A \to B$ is called **regular**, if it (with the object A) is an equalizer of some morphisms $g, h\colon\ B \to C$.

Description in Mod_R

Mod_R – category of all right R-modules

Theorem

Let R be a ring and f some morphism on the category Mod_R. The following assertions are equivalent:

1. *f is a monomorphism;*

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3. *f is a regular monomorphism;*
4. *f is injective.*

Description in UMod$_R$

Definition

A ring R is called **idempotent**, if $R = RR$, i.e.

$$\forall r \in R\ \exists k \in \mathbb{N}\ \exists r_1, r_1', \ldots, r_k, r_k' \in R: \quad r = r_1 \cdot r_1' + \ldots + r_k \cdot r_k.$$

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Let X be a set. Denote $\mathcal{P}_{\text{fin}}(X)$ as the set of all finite subsets of X.

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Example (idempotent ring 1):

Let X be a set. Denote $\mathcal{P}_{\text{fin}}(X)$ as the set of all finite subsets of X. $(\mathcal{P}_{\text{fin}}(X); \Delta, \cap)$ is an idempotent ring, which does not have an identity element, because

$$\forall A \in \mathcal{P}_{\text{fin}}(X) : \quad A \cap A = A.$$

(Δ – symmetric difference)

Example (idempotent ring 2):

Let $\mathbb{Z}_n$ be the ring integers *modulo n*. Consider the direct sum

$$R = \bigoplus_{k=1}^{\infty} \mathbb{Z}_n$$

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i.e. R is the set of sequences of $\mathbb{Z}_n$ where almost all components are zeros. R is a ring where operations are defined componentwise. There is no unit-element in R, but it is idempotent.

$$\begin{aligned}(\overline{0},\overline{3},\overline{2},\overline{0},\ldots) &= (\overline{0},\overline{3},\overline{0},\overline{0},\ldots) + (\overline{0},\overline{0},\overline{2},\overline{0},\ldots) = \\ &= (\overline{0},\overline{1},\overline{0},\overline{0},\ldots) + (\overline{0},\overline{1},\overline{0},\overline{0},\ldots) + (\overline{0},\overline{1},\overline{0},\overline{0},\ldots) + \\ &\quad + (\overline{0},\overline{0},\overline{1},\overline{0},\ldots) + (\overline{0},\overline{0},\overline{1},\overline{0},\ldots)\end{aligned}$$

Definition

A right R-module is called **unitary**, if $MR = M$, i.e.

$$\forall m \in M\, \exists m_1, \ldots, m_k \in M\, \exists r_1, \ldots, r_k \in R: \quad m = m_1 \cdot r_1 + \ldots + m_k \cdot r_k.$$

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Example (unitary module):

Let R be a ring with an identity element. A right R-module M is unitary if and only if

$$\forall m \in M: \quad m \cdot 1 = m.$$

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$$(\mathrm{Ker}\, f)R = 0.$$

Definition

A right R-module M_R is called **firm**, if the mapping

$$\mu_M\colon\ M\otimes R\to M,\quad \mu_M\left(\sum_{h=1}^{k} m_h\otimes r_h\right)=\sum_{h=1}^{k} m_h\cdot r_h$$

is bijective.

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Corollary

Let R be a ring and $M_R \in \mathrm{UMod}_R$. Then the canonical homomorphism $\mu_M \colon M \otimes R \to M$ is a monomorphism in UMod_R.

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Corollary

Let R be a ring and $M_R \in \mathrm{UMod}_R$. Then the canonical homomorphism $\mu_M \colon M \otimes R \to M$ is a monomorphism in UMod_R.

If M_R in unitary, but not firm, then μ_M is a monomorphism, but is not injective.
Also, for such a module, μ_M is surjective, therefore an epimorphism and a bimorphism.

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4. $(\mathrm{Ker}\, f)R = 0$;
5. *$f = \mu_N \circ (a \otimes 1_R) \circ g$ for some unitary module A_R, an injective homomorphism $a\colon\ A_R \to N_R$ and an isomorphism $g\colon\ M_R \to A \otimes R$;*

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6. $f = h \circ (a \otimes 1_R) \circ g$ *for unitary R-modules A_R and B_R, an injective homomorphism $a\colon\ A_R \to B_R$ and isomorphisms $g\colon\ M_R \to A \otimes R$ and $h\colon\ B \otimes R \to N_R$.*

Lattice of subobjects

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Set $\mathcal{U}(M)$ is a complete modular lattice, where supremum and infimum are defined as

$$B \vee C = B + C,$$
$$B \wedge C = (B \cap C)R,$$

where $B, C \in \mathcal{U}(M)$.

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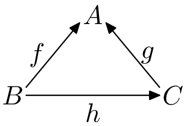

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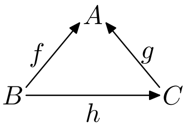

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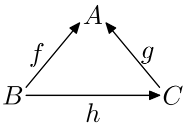

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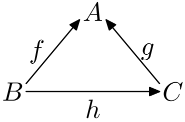

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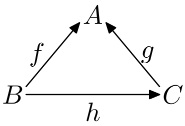

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For $M_R \in \mathrm{FMod}_R$ denote

$$\mathcal{S}(M) = \mathrm{Sub}_{\mathrm{FMod}_R}(M).$$

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This isomorfism is of the form

$$\Psi\colon\ \mathcal{U}(M) \to \mathcal{S}(M), \qquad \Psi(N_R) := \overline{\mu_M \circ (\iota_N \otimes 1_R)},$$

where $\iota_N\colon\ N \to M$ is the inclusion.

$$N \otimes R \xrightarrow{\iota_N \otimes 1_R} M \otimes R \xrightarrow{\mu_M} M$$

Thank you for listening!