DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
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INDREK ZOLK

The commuting bounded approximation property of Banach spaces

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Supervisor:

Full Prof. Eve Oja, Cand. Sc.
University of Tartu
Tartu, Estonia

Opponents:

Assoc. Prof. Terje Hõim, Ph.D.
Florida Atlantic University
Florida, USA

Full Prof. Olav Nygaard, Dr. Scient.
University of Agder
Kristiansand, Norway

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Chapter 1

Introduction

Firstly, it would be a pity not to retell the evergreen folkloristic story behind the approximation problem. Having done that, we continue with a self-contained summary of the thesis. In the end of this chapter, we shall fix some notation and give references to sources that contain well-known general concepts and results in functional analysis.

1.1 Historical roots of the field

In the 1930s, the mathematical life of Lwów (then in Poland, now Львів, in the Ukraine) was intense. Notable members of the *Lwów School of Mathematics* included Stefan Banach, Władysław Orlicz, Stanisław Saks, Hugo Steinhaus Stanisław Mazur, Stanisław Ulam, Juliusz Schauder, Herman Auerbach, and others.

According to Ulam's memories, it must have been Banach who had suggested keeping track of some of the problems occupying the group of mathematicians there. Apart from the more official meetings, there were frequent informal discussions held in coffee houses "Roma" or "The Scottish Coffee House" located near the University building. The problems were written down in a large notebook which was deposited with the headwaiter of "The Scottish Coffee House". Later, after the war, the problems from the notebook were published as the "Scottish Book" [Scottish].

In the "Scottish Book", the Problem 153 is the following.

Given is a continuous function $f(x, y)$ defined for $0 \leqslant x, y \leqslant 1$

and a number $\varepsilon > 0$; do there exist numbers $a_1, \dots, a_n$, $b_1, \dots, b_n$, $c_1, \dots, c_n$ with the property that

$$\left| f(x,y) - \sum_{k=1}^{n} c_k f(a_k, y) f(x, b_k) \right| \leqslant \varepsilon$$

in the interval $0 \leqslant x, y \leqslant 1$?

This problem had been proposed by Mazur in November 6, 1936, and the prize for solving it was exceptional: a live goose. (Other problems had smaller prizes, such as free dinner or bottle of whiskey, etc.)

Grothendieck proved in his famous Memoir [Gro, "Proposition" 37] that Problem 153 is equivalent to the *approximation problem*: *do all Banach spaces have the approximation property?* In other words, is it true for every Banach space X that, given any compact set $K \subset X$ and a number $\varepsilon > 0$, one can find a bounded linear finite-rank operator T on X such that $\sup_{x \in K} \|Tx - x\| < \varepsilon$?

In fact, the "Scottish Book" was not the first source where the approximation problem had been touched. Banach's book "Théorie des opérations linéaires" from 1932 [B] that created functional analysis as an independent discipline of mathematics, contains at least two hints on the approximation problem.

On page 237, Banach considers a result originating from Mazur's remark: let (U_n) be a sequence of compact linear operators on a Banach space E such that $U_n x \to x$ for every $x \in E$, then the relative compactness of a subset $G \subset E$ is equivalent to the fact that the convergence $U_n \to I_E$ is uniform on G. Here Banach essentially considers the bounded compact approximation property.

On page 111, Banach emphasized: *On ne sait pas si tout espace du type (B) séparable admet une base*[1]. This question, the *basis problem*, is related to the approximation problem in the way that every space with a basis also has the approximation property. Hence, the negative solution for the approximation problem implies the negative solution for the basis problem.

Both problems were in the focus of analysts for a long time, until the negative solution was given by Per Enflo in 1972 [E]. The live goose was then indeed given to Enflo by Mazur (see, e.g., [Kałuża], for a photo of this remarkable event in 1972).

Nowadays the field of approximation properties attracts many researchers, since it contains a number of problems that have not been solved for a long time. For instance, two of such famous open problems are: *is the bounded*

[1] It is not known whether every separable Banach space has a Schauder basis.

approximation property always commuting (for separable spaces) and *is the approximation property of a dual space always metric?*

1.2 Summary of the thesis

The main aim of the thesis has been to investigate the commuting bounded approximation property (and also its compact version). On the one hand, the property is in general weaker than the commuting metric approximation property or the finite-dimensional decomposition property. On the other hand, it is stronger than the bounded approximation property. (At least formally) in between the bounded approximation property and the commuting bounded approximation property there is a new concept defined in the thesis: the asymptotically commuting bounded approximation property.

The thesis has been organized as follows.

Chapter 1 contains a short historic overview of the approximation problem, a summary of the thesis and some technical remarks on the notation used in the thesis.

In Chapter 2 we make the reader familiar with several versions of approximation properties, including the approximation property and its compact version, the bounded (compact) approximation property (including the 1-bounded, in other words, the metric (compact) approximation property), and the commuting bounded (compact) approximation property. We present concepts and results that are needed in the following chapters or that might be required to obtain a holistic background on the subject.

In Chapter 3 we prove that the metric compact approximation of the identity of the space X_W due to Willis [W] is commuting. This shows that the space X_W has the commuting metric compact approximation property. Since X_W fails the approximation property as shown in [W], we establish now that the commuting bounded compact approximation property and the approximation property are different properties.

Chapter 4 relies heavily on [O4] and [GS]. In 1988, Godefroy and Saphar [GS] demonstrated how the geometric structure (being M-embedded) of a separable Banach space permits to lift the commuting bounded approximation property to its dual space. We extend their result in a number of ways, omitting the assumption on separability as well as making use of a more general structural framework that we call the $M(a,B,c)$-inequality. Note that M-embedded Banach spaces are precisely those that satisfy the $M(1,\{-1\},1)$-

inequality. Among others, we prove that if a Banach space satisfies the $M(a,B,c)$-inequality and has a λ-bounded (compact) approximation property (where λ must not exceed $\max|B|+c$), then both the space and its dual space enjoy the metric (compact) approximation property.

The concept of the $M(a,B,c)$-inequality can be found implicitely in [O4], it was considered there while investigating intensively property $M^*(a,B,c)$, another general structural property that enables simultaneously describe a large class of ideals of Banach spaces.

Chapter 5 focuses on an aspect of the space X_{JS}, a space whose description was published in [JO] but created already in 1996 by Johnson and Schechtman. The space is remarkable for the fact that it fails the metric approximation property but has the bounded approximation property. In 2001, Godefroy [G] proved that X_{JS} has the commuting 8-bounded approximation property. Godefroy also wrote that no effort had been made to tighten the constant 8. In Chapter 5 we show that X_{JS} has the commuting 6-bounded approximation property. It is still open whether the constant 6 is sharp.

Chapter 6 coins a new term: the asymptotically commuting bounded approximation property. For separable spaces it coincides with the commuting bounded approximation property. In the general setting we prove that if a Banach space has the asymptotically commuting bounded approximation property, then it has a strong form of the separable local complementation property. We note that in view of this result it is not clear whether the commuting bounded approximation property implies the separable complementation property (a fact that has been claimed to be true in [C2, Theorem 9.3]).

Chapters 3 and 4 are based on [OZ1], Chapter 5 is based on [Z], and Chapter 6 on [OZ2].

1.3 Notation

Our notation is standard.

In a Banach (or normed linear) space X (over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), we denote the unit sphere by S_X and the closed unit ball by B_X. For a set $A\subset X$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span} A$, its convex hull by $\operatorname{conv} A$, and its absolutely convex hull by $\operatorname{absconv} A$. The norm closures of the three latter sets are denoted by $\overline{\operatorname{span}}\, A$, $\overline{\operatorname{conv}}\, A$, and $\overline{\operatorname{absconv}}\, A$, respectively. For closures with respect to other topologies, we mark the topology or specific space separately, such as $\overline{\operatorname{conv}}^{w^*} A$, etc.

For Banach spaces X and Y, we denote the Banach space of bounded linear operators from X to Y by $\mathscr{L}(X,Y)$ and its closed subspace of compact linear operators by $\mathscr{K}(X,Y)$. The normed space of finite-rank linear operators from X to Y will be denoted by $\mathscr{F}(X,Y)$. If $X=Y$, we shall write $\mathscr{L}(X)$, $\mathscr{K}(X)$, or $\mathscr{F}(X)$, respectively. For an operator $T: X \to Y$, we denote $\operatorname{ran} T = \{Tx : x \in X\}$, the range of T, and $\ker T = \{x \in X : Tx = 0\}$, the kernel of T. The restriction of T on a subset A will be denoted by $T|_A$. The identity operator on X will be denoted by I_X, that is, $I_X x = x$ for every $x \in X$.

We are going to use the canonical embedding $j_X : X \to X^{**}$, being defined by

$$(j_X x)(x^*) = x^*(x), \qquad x^* \in X^*, \qquad x \in X,$$

and the canonical projection $\pi_X : X^{***} \to X^{***}$ onto $\operatorname{ran} j_{X^*}$, being defined by

$$\pi_X = j_{X^*}(j_X)^*.$$

Usually, we shall not write out j_X and regard a Banach space X as a subspace of its bidual X^{**}.

Recall that the *Banach-Mazur distance* between isomorphic Banach spaces X and Y is defined as

$$d_{BM}(X,Y) = \inf\left\{\|\mathscr{J}\|\left\|\mathscr{J}^{-1}\right\| : \mathscr{J} \text{ is an isomorphism from } X \text{ onto } Y\right\},$$

and if X and Y are not isomorphic, then $d_{BM}(X,Y) = \infty$.

Well-known basic notions and theorems of the theory of Banach spaces and topological vector spaces (such as the Hausdorff theorem, the Auerbach lemma, the Minkowski functional, the Hahn-Banach theorem, the Alaoglu theorem, the Goldstine theorem, etc.) are used without referring to their wording. If required, the adequate background information can be found, for instance, in [Day], [D], [FHHMPZ], [LTz I], and [SW].

Chapter 2

Approximation properties in general

In this chapter we introduce the terms needed in the following chapters, namely we define several versions of approximation properties, including the finite-dimensional decomposition, Schauder basis and others, at the same time giving insight to some obvious or well-known results on these concepts. To acquire more background information on these properties, we refer the reader to an excellent survey by Casazza [C2] which describes the state of the field as it was ten years ago.

2.1 The (compact) approximation property and its bounded version

Let X be a Banach space.

Definition 2.1. If the following condition holds:

> for every compact set $K \subset X$ and every $\varepsilon > 0$ there exists an operator $T \in \mathscr{K}(X)$ such that $\|Tx - x\| < \varepsilon$,

then X is said to have the *compact approximation property* (CAP). If we can always choose T from $\mathscr{F}(X)$, then X is said to have the *approximation property* (AP).

As it was mentioned in Chapter 1, the problem whether every Banach space has the AP, is fairly old. The ground-breaking negative solution by Enflo was the following. (For the concept of Schauder basis, see Definition 2.28.)

Theorem 2.2 ([E, Theorem 1]). *There exists a separable reflexive Banach space B with a sequence (M_n) of finite-dimensional subspaces,* $\dim M_n \to \infty$ *when $n \to \infty$, and a constant C such that for every T of finite rank*

$$\| T|_{M_n} - I_{M_n} \| \geqslant 1 - \frac{C\|T\|}{\log \dim M_n}.$$

In particular, B does not have the AP and B does not have a Schauder basis.

For the case of the CAP, the first example was the very same space B in Theorem 2.2. In [JSz], it has been pointed out that Figiel had noted: *the criterion used by Enflo [E] to guarantee that a separable space fails the AP, actually guarantees that it fails the CAP*. Among other examples, it was proven by Figiel [F] and Davie [Davie] that ℓ_p, where $p \in (2,\infty)$, contains a closed subspace failing the AP (due to Figiel's note, also failing the CAP), and by Szankowski [Sz1] (see [LTz II, Theorem 1.g.4, p. 107]) that ℓ_p, where $p \in [1,2)$, contains a closed subspace failing the CAP.

For a space having the CAP and failing the AP, we refer to [W] and Chapter 3. (Hence the CAP and the AP are different properties.)

However, most of the classical spaces, such as c_0, $C[a,b]$, ℓ_p, $L_p(a,b)$, where $1 \leqslant p < \infty$, satisfy the basis property (see Definition 2.28); in fact, they enjoy all the approximation properties described throughout this chapter. Most examples failing different approximation properties are highly artificial; perhaps the easiest example "to write down" failing the AP is $\mathscr{L}(\ell_2)$, again due to Szankowski [Sz2].

Definition 2.3. If there exists a real number λ such that the following condition holds:

> for every compact set $K \subset X$ and every $\varepsilon > 0$ there exists an operator $T \in \mathscr{K}(X)$ such that $\|T\| \leqslant \lambda$ and $\|Tx - x\| < \varepsilon$

then X is said to have the *λ-bounded compact approximation property* (*λ-bounded* CAP). If we can always choose T from $\mathscr{F}(X)$, then X is said to have the *λ-bounded approximation property* (*λ-bounded* AP). (See also Remark 2.4.)

Remark 2.4. In all definitions of approximations, if the value of λ is not important, we omit the string "λ-".

Remark 2.5. The least possible value of λ is 1: indeed, if $\|T\| \leqslant \lambda < 1$, then for any $x \in S_X$ we have

$$\|Tx - x\| \geqslant \|x\| - \|Tx\| \geqslant 1 - \sup_{x \in B_X} \|Tx\| = 1 - \|T\| \geqslant 1 - \lambda,$$

the lower bound being a positive constant, thus disabling the possibility to have $Tx - x$ as small as wanted for suitable T.

The 1-bounded (C)AP is called the *metric* (C)AP, or the *metric* (*compact*) *approximation property*, for long.

For a Banach space failing the metric AP, but having the bounded AP, we refer to Chapter 5 for references and a result. For very recent examples of Banach spaces failing the bounded AP, but still having the AP, we refer to the paper by Figiel, Johnson, and Pełczyński [FJP, Corollary 1.13].

The (C)AP and the bounded (C)AP are inherited to *complemented* subspaces, i.e. to such closed subspaces Y of a Banach space X for which there exists a projection $P \in \mathscr{L}(X)$ onto Y.

Proposition 2.6. *Let Y be a complemented subspace of a Banach space X. Let X have the (C)AP or the bounded (C)AP. Then Y also has it.*

Proof. A projection onto Y is identity on Y, since for every $y \in Y$ there is $x \in X$ such that $Px = y$, hence

$$Py = PPx = Px = y.$$

Now fix an $\varepsilon > 0$ and a compact set K in Y, then K is compact in X and we find a $T \in \mathscr{F}(X)$ (resp., $T \in \mathscr{K}(X)$) such that $\|Ty - y\| < \varepsilon$, $y \in K$. Then

$$\|PTy - Py\| \leqslant \|P\| \, \|Ty - y\| < \|P\| \, \varepsilon \qquad \forall y \in K.$$

Hence $PT|_Y \in \mathscr{F}(Y)$ (resp., $PT|_Y \in \mathscr{K}(Y)$) is the required operator to show that Y has the (C)AP. For the bounded version, if λ is a uniform bound on the norms of T, then $\|P\| \, \lambda$ is a uniform bound on the norms of $PT|_Y$. □

The conditions in Definitions 2.1 and 2.3 can be written down in the language of *convergence to the identity operator uniformly on compact subsets.* Recall that a *net* (x_α) is a function $(x_\alpha) : \alpha \mapsto x_\alpha$ from a directed set of indices α.

Proposition 2.7. *A Banach space X has the (C)AP if and only if there exists a net $(T_\alpha) \subset \mathscr{F}(X)$ (resp., $(T_\alpha) \subset \mathscr{K}(X)$) converging to the identity operator uniformly on compact subsets, i.e. for any compact set $K \subset X$ and a number $\varepsilon > 0$ there exists an index α_0 such that*

$$\alpha \succcurlyeq \alpha_0 \qquad \Rightarrow \qquad \|T_\alpha x - x\| < \varepsilon \quad \forall x \in K.$$

For the case of the λ-bounded (C)AP, the equivalent condition is the same but together with the restriction $\sup\limits_\alpha \|T_\alpha\| \leqslant \lambda$.

Proof. Firstly, if such a net exists, then for any compact set $K \subset X$ and a number $\varepsilon > 0$ we take T_{α_0} that meets the needs of Definitions 2.1 or 2.3, yielding that X has the (λ-bounded) (C)AP.

Let now X have the (C)AP. We order the pairs (K, ε) (where K is compact and $\varepsilon > 0$) to obtain a directed set as follows:

$$(K_1, \varepsilon_1) \preccurlyeq (K_2, \varepsilon_2) \qquad \Leftrightarrow \qquad K_1 \subset K_2 \wedge \varepsilon_1 \geqslant \varepsilon_2.$$

For every such pair $\alpha = (K, \varepsilon)$ we obtain an operator T_α for which $\|T_\alpha\| \leqslant \lambda$ and $\|T_\alpha x - x\| < \varepsilon$ for every $x \in K$.

Now, let us have a compact set $K \subset X$ and a number $\varepsilon > 0$. We denote $\alpha_0 = (K, \varepsilon)$. Having $(K_1, \varepsilon_1) =: \alpha \succcurlyeq \alpha_0$, there holds

$$\|T_\alpha x - x\| < \varepsilon_1 \leqslant \varepsilon$$

for all elements $x \in K_1$, hence for all $x \in K \subset K_1$. Therefore there exists a net (T_α) such that $T_\alpha \to I_X$ uniformly on compact subsets.

For the bounded version, we see that all the elements from the net shall not exceed λ by norm. □

It is quite straightforward to verify that the λ-bounded (C)AP can be defined using a *strongly* (i.e. pointwise) converging net of operators.

Proposition 2.8. *A Banach space X has the λ-bounded (C)AP if and only if there exists a net $(T_\alpha) \subset \mathscr{F}(X)$ (resp., $(T_\alpha) \subset \mathscr{K}(X)$) such that $\|T_\alpha\| \leqslant \lambda$ and $T_\alpha x \to x$ for every $x \in X$.*

Proof. Let X have the λ-bounded (C)AP. By Proposition 2.7, we have a net $(T_\alpha) \subset \mathscr{F}(X)$ (resp., $(T_\alpha) \subset \mathscr{K}(X)$) such that $T_\alpha \to I_X$ uniformly on compact subsets. Since $\{x\}$ is a compact subset, $T_\alpha x \to x$ for every $x \in X$.

Now let there exist a net (T_α) of finite-rank (resp., compact) operators on X such that $\|T_\alpha\| \leqslant \lambda$ and $T_\alpha x \to x$ for every $x \in X$. Fix a compact set K and a real number $\varepsilon > 0$. By the Hausdorff theorem, one can find elements $x_1, \dots, x_n \in K$ such that for every $x \in K$, there exists an index $j = 1, \dots, n$ for which $\left\|x - x_j\right\| < \dfrac{\varepsilon}{3\lambda}$. Since $T_\alpha x \to x$ for every x, one can find an index α_0 such that

$$\left\|T_{\alpha_0} x_j - x_j\right\| < \frac{\varepsilon}{3} \qquad \forall j = 1, \dots, n.$$

Let be given an $x \in K$. We find an $x_j \in \{x_1, \dots, x_n\}$ such that $\left\|x - x_j\right\| < \dfrac{\varepsilon}{3\lambda}$. Therefore

$$\left\|T_{\alpha_0} x - x\right\| \leqslant \left\|T_{\alpha_0}\right\| \left\|x - x_j\right\| + \left\|T_{\alpha_0} x_j - x_j\right\| + \left\|x - x_j\right\| < \varepsilon,$$

as required. □

Remark 2.9. Unlikely to the bounded (C)AP, one cannot define the (C)AP by a net converging strongly to the identity. Namely, every Banach space has such a net.

To see this, we order all finite sets of a Banach space X by the ordering

$$F_1 \preccurlyeq F_2 \qquad \Leftrightarrow \qquad F_1 \subset F_2.$$

Now $\mathscr{F} := \{F : F \text{ is finite}, F \subset X\}$ is a directed set.

For every finite set $F := \{x_1, \dots, x_n\} \subset X$, the Auerbach lemma gives a projection P_F onto $\operatorname{span} F$ such that $\|P\| \leqslant \dim \operatorname{span} F \leqslant n$. In particular, for every $k = 1, \dots, n$ we have $P_F x_k = x_k$. Now (P_F) converges strongly to I_X. Indeed, let us be given an $x \in X$, then for every $F \succcurlyeq \{x\}$ we even have $P_F x = x$, hence by way $P_F x \to x$.

Definition 2.10. A net $(T_\alpha) \subset \mathscr{K}(X)$ is called a *compact approximation of the identity* (CAI) provided $T_\alpha x \to x$ for any $x \in X$. In particular, if $(T_\alpha) \subset \mathscr{F}(X)$, then (T_α) is called an *approximation of the identity* (AI). If there is a λ such that for a (C)AI (T_α) there holds $\sup_\alpha \|T_\alpha\| \leqslant \lambda$, then (T_α) is called a *λ-bounded (compact) approximation of the identity* (λ-bounded (C)AI). (See also Remark 2.4.)

A 1-bounded (C)AI is called a *metric* (C)AI, or a *metric* (*compact*) *approximation of the identity,* for long. Bearing in mind Remark 2.5, any λ-bounded (C)AI must have $\lambda \geqslant 1$.

Recall that if (x_α) and (y_β) are nets with directed sets A and B of their indices, then (y_β) is called a *subnet* of (x_α) if there exists a function $h : B \to A$ satisfying the conditions

(i) $y_\beta = x_{h(\beta)}$ for all $\beta \in B$,
(ii) $\beta_1 \preccurlyeq \beta_2$ implies $h(\beta_1) \preccurlyeq h(\beta_2)$ for all $\beta_1, \beta_2 \in B$,
(iii) for every $\alpha \in A$ there exists a $\beta \in B$ such that $h(\beta) \succcurlyeq \alpha$.

It is obvious that every subnet of a (C)AI (or a λ-bounded (C)AI) is also a (C)AI (resp., a λ-bounded (C)AI).

The next proposition shows that an AI must contain operators of arbitrarily large rank.

Proposition 2.11. *Let X be an infinite-dimensional Banach space and (T_α) be an AI of X. Then for any $m \in \mathbb{N}$ there exists an index α_0 such that $\alpha \succcurlyeq \alpha_0$ implies* $\dim T_\alpha \geqslant m$.

Proof. Consider a subspace E of X such that $\dim E = m$. As B_E is a compact set, we use Proposition 2.7 to find an α_0 such that

$$\alpha \succcurlyeq \alpha_0 \quad \Rightarrow \quad \|T_\alpha|_E - I_E\| = \sup_{x \in B_E} \|T_\alpha x - x\| < \frac{1}{m}.$$

We claim that such operators T_α work.

Let $P \in \mathscr{L}(X, E)$ be a projection onto E for which $\|P\| \leqslant m$ (due to the Auerbach lemma). Then P is the identity on E. Now $PT_\alpha|_E \in \mathscr{L}(E)$ is a bijection on E. Indeed,

$$\begin{aligned} \|PT_\alpha|_E - I_E\| &= \sup_{x \in B_E} \|PT_\alpha x - x\| = \sup_{x \in B_E} \|PT_\alpha x - Px\| \leqslant \\ &\leqslant \|P\| \sup_{x \in B_E} \|T_\alpha x - x\| < 1, \end{aligned}$$

which shows that $PT_\alpha|_E$ is invertible.

We conclude that

$$\dim \operatorname{ran} T_\alpha \geqslant \dim \operatorname{ran} T_\alpha|_E \geqslant \dim \operatorname{ran} PT_\alpha|_E = \dim E = m. \qquad \square$$

For the following condition equivalent to the bounded AP we slightly perturb the operators of the bounded AI in order to obtain that they coincide with the identity on suitable finite-dimensional subspaces.

Proposition 2.12 ([C2, Theorem 3.3])**.** *A Banach space X has the bounded AP if and only if there exists a $\lambda' \geqslant 1$ so that for every finite-dimensional subspace $E \subset X$ there is an operator $T \in \mathscr{F}(X)$ such that $\|T\| \leqslant \lambda'$ and $T|_E = I_E$.*

We may compare this condition with the Auerbach lemma where for every finite-dimensional subspace $E \subset X$ one also obtains an operator $T \in \mathscr{F}(X)$ satisfying $T|_E = I_E$. That operator is a projection onto E, but the bound is not uniform: $\|T\| \leqslant \dim E$. The bound $\dim E$ can be made better: a result from 1971 by Kadets and Snobar [KC] establishes $\sqrt{\dim E}$. Uniform bound, however, could never be possible, since this would mean that all Banach spaces would have the bounded AP.

The proof of Proposition 2.12 relies heavily on the following result.

Proposition 2.13 ([JRZ, Lemma 2.4])**.** *Let X be a Banach space, let F be an n-dimensional subspace of X and let $T : X \to F$ be onto. Let $k \leqslant n$ and let E be a k-dimensional subspace of X such that $\|T|_E - I_E\| < \varepsilon < 1$, where $\dfrac{\varepsilon k}{1-\varepsilon} < 1$.*

Then there is a rank n operator S on X such that $S|_E = I_E$, $\|S - T\| < \dfrac{\varepsilon k \|T\|}{1-\varepsilon}$, and $\operatorname{ran} S^* = \operatorname{ran} T^*$. *Moreover, if T is a projection, then S can be chosen to be a projection.*

We point out that the condition $k \leqslant n$ is not really a restriction, in view of the proof of Proposition 2.11. Indeed, there it has been shown that if $\|T|_E - I_E\| < \varepsilon$ for a number ε small enough, then the condition $\dim E \leqslant \dim \operatorname{ran} T$ follows automatically.

Proof of Proposition 2.12. Let X have the λ-bounded AP. Fix a finite-dimensional subspace E, find an operator $T \in \mathscr{F}(X)$ such that $\|T|_E - I_E\| < \varepsilon \dim E < 1$ (Proposition 2.7) where ε is so small that also $\dfrac{\varepsilon \dim E}{1-\varepsilon} < 1$. Now have $\operatorname{ran} T$ in the role of F in Proposition 2.13 and obtain an operator $S \in \mathscr{F}(X)$ such that $S|_E = I_E$ and

$$\|S\| \leqslant \|S - T\| + \|T\| \leqslant \frac{\varepsilon k \|T\|}{1-\varepsilon} + \|T\| \leqslant \frac{\varepsilon k \lambda}{1-\varepsilon} + \lambda \leqslant 2\lambda.$$

On the other hand, if there is a uniform bound λ' such that any finite-dimensional subspace can have a λ'-bounded $T_E \in \mathscr{F}(X)$ such that $T_E x = x$ for all $x \in E$, then X has clearly the λ'-bounded AP. Indeed, order all the finite-dimensional subspaces by inclusion (denote the order by $\preccurlyeq$), this will give an ordering on the obtained operators T_E. Being given an element $x \in X$, we see that

$$E \succcurlyeq \operatorname{span}\{x\} \quad \Rightarrow \quad T_E x = x$$

meaning by far that $T_E x \to x$. $\square$

Making $\varepsilon > 0$ in the proof of Proposition 2.12 arbitrarily small, we can have the first addend $\dfrac{\varepsilon k \lambda}{1-\varepsilon}$ as small as desired, since $\lim\limits_{\varepsilon \to 0+} \dfrac{\varepsilon k \lambda}{1-\varepsilon} = 0$.

Remark 2.14. If X is a separable Banach space, then we can use a sequence in the net definition (Proposition 2.8) of the λ-bounded (C)AP. Indeed, let us have $X = \overline{\{x_1, x_2, \dots\}}$ and find subsequently the λ-bounded operators $T_1, T_2, \dots \in \mathscr{F}(X)$ (resp., $\mathscr{K}(X)$) such that

$$\begin{aligned}
&\|T_1 x_1 - x_1\| < 1,\\
&\|T_2 x_1 - x_1\| < \frac{1}{2}, \quad \|T_2 x_2 - x_2\| < \frac{1}{2},\\
&\dots\\
&\|T_n x_k - x_k\| < \frac{1}{n} \quad k = 1, \dots, n,\\
&\dots
\end{aligned}$$

then for any $\varepsilon > 0$ we can at first approximate $\|x - x_k\| < \dfrac{\varepsilon}{3\lambda}$ for some k and after that find an N (with $N \geqslant k$) for which

$$n \geqslant N \qquad \Rightarrow \qquad \|T_n x_k - x_k\| < \frac{\varepsilon}{3}.$$

Putting these two estimations together gives us $\|T_n x - x\| \to 0$.

On the other hand, if X has a (C)AI (T_n) (i.e., a (C)AI that is a sequence), then

$$X = \overline{\bigcup_{n=1}^{\infty} \operatorname{ran} T_n}$$

which means that X is separable.

What is more, a (C)AI (T_n) (i.e., a (C)AI that is a sequence) is always bounded, due to the Banach-Steinhaus theorem.

For separable Banach spaces we can use Proposition 2.12 to have even more.

Proposition 2.15 ([C2, Corollary 3.4])**.** *A separable Banach space X has the λ-bounded AP if and only if there is a sequence $(S_n) \subset \mathscr{F}(X)$ such that*

(i) $S_n x \to x$ *for any* $x \in X$,

(ii) $S_m S_n = S_n$ *for all* $m > n$,

(iii) $\limsup\limits_{n} \|S_n\| \leqslant \lambda$.

Proof. Let a separable Banach space X have the λ-bounded AP. We have $X = \overline{\bigcup\limits_{k=1}^{\infty} E_k}$ where $\{0\} \neq E_1 \subset E_2 \subset \ldots$ is a chain of finite-dimensional subspaces of X.

Let us have a λ-bounded AI (T_n) (see Remark 2.14). For the base of induction, find S_1 by use of E_1, ε_1, a T_{n_1}, and Proposition 2.13. Then S_1 is the identity on E_1 whereas

$$\|S_1 - T_1\| < \frac{\varepsilon_1 \dim E_1 \left\| T_{n_1} \right\|}{1 - \varepsilon_1}.$$

The step of induction can be performed as follows. We denote

$$E'_{k+1} := \operatorname{span}\left(E_{k+1} \cup \bigcup_{j=1}^{k} \operatorname{ran} S_j \right),$$

take an ε_{k+1} so small that $\max\left(\varepsilon_{k+1}\dim E'_{k+1}, \dfrac{\varepsilon_{k+1}\dim E'_{k+1}}{1-\varepsilon_{k+1}}\right) < 1$, and find a $T_{n_{k+1}}$ such that

$$\left\| T_{n_{k+1}}|_{E'_{k+1}} - I_{E'_{k+1}} \right\| < \varepsilon_{k+1}.$$

Inputting these ingredients into Proposition 2.13 gives an S_{k+1} that is the identity on E'_{k+1} (hence also on E_{k+1}). Therefore $S_{k+1}S_n = S_n$ for all $n < k+1$ due to $\operatorname{ran} S_j \subset E'_{k+1}$, $j = 1, \ldots, k$, and

$$\left\| S_{k+1} - T_{n_{k+1}} \right\| < \frac{\varepsilon_{k+1}\dim E'_{k+1}\, \|T_{k+1}\|}{1-\varepsilon_{k+1}}.$$

Demanding also $\varepsilon_k \to 0$, the "only if" part is done.

Assume now that there exists a sequence $(S_n) \subset \mathscr{F}(X)$ such that $S_n x \to x$ for any $x \in X$, $S_m S_n = S_n$ for all $m > n$, and $\limsup_n \|S_n\| \leqslant \lambda$. Switching to a subnet (S_{n_k}), we have $\lambda_0 := \lim_k \left\|S_{n_k}\right\| \leqslant \lambda$. Now denote

$$T_k = \frac{\lambda_0}{\left\|S_{n_k}\right\|} S_{n_k}, \qquad k \in \mathbb{N}.$$

Then (T_k) is a λ-bounded AI. Indeed,

$$\|T_k x - x\| = \left\| \frac{\lambda_0}{\left\|S_{n_k}\right\|} S_{n_k} x - x \right\| \leqslant \left\| \frac{\lambda_0}{\left\|S_{n_k}\right\|} - 1 \right\| \left\|S_{n_k}\right\| \|x\| + \left\| S_{n_k} x - x \right\| \to 0$$

and $\|T_k\| = \left\| \dfrac{\lambda_0}{\left\|S_{n_k}\right\|} S_{n_k} \right\| = \lambda_0$ for all k, giving $\sup_k \|T_k\| \leqslant \lambda$. $\square$

Remark 2.16. The "only if" part of the proof of Proposition 2.15 shows the following. For a separable Banach space the λ-bounded AI (T_n) implies the existence of an AI (S_k) such that $S_m S_n = S_n$ for all indices $m > n$, $\limsup_k \|S_k\| \leqslant \lambda$ and for some subsequence of natural number (n_k) there holds $\left\| S_k - T_{n_k} \right\| \to 0$.

In the present days, investigations include the extensions of the classical notions of approximation properties.

One way is to consider more general or different types of compactness for the sets K in Definition 2.1, e.g. [OT] on approximation using weakly compact sets, and [SK1], [SK2], [DOPS] on approximation using p-compact sets.

One can also consider different restrictions on the approximating operators. Following [LO2], X has the *weak λ-bounded AP* if for an arbitrary Banach space Y and an operator $S \in \mathscr{W}(X,Y)$ (*weakly compact operators*, i.e., those that map bounded sets to relatively weakly compact sets) there exists a net $(T_\alpha) \subset \mathscr{F}(X)$ such that $T_\alpha \to I_X$ uniformly on compact subsets of X whereas $\limsup_\alpha \|ST_\alpha\| \leqslant \lambda \|S\|$. The weak bounded AP has been intensively studied by Oja, Å. Lima, and V. Lima (see, e.g., [LO2], [V.L], [O8]).

The *strong AP* of X (meaning that for an arbitrary separable reflexive Banach space Z and an operator $S \in \mathscr{K}(X,Z)$ there exists a bounded net $(T_\alpha) \subset \mathscr{F}(X,Z)$ such that $T_\alpha x \to Sx$ for all $x \in X$) has been studied by Oja in [O7]. This property is (at least formally) weaker than the weak bounded AP, but strictly stronger than the AP.

Another way is to consider different classes of operators instead of $\mathscr{K}(X)$ or $\mathscr{F}(X)$ in Definition 2.1. For references in this direction we suggest to look at [LMO].

2.2 The commuting bounded approximation property

Let X be a Banach space.

Definition 2.17. A (C)AI (T_α) is called *commuting* if $T_\alpha T_\beta = T_\beta T_\alpha$ for all α and β. A commuting (C)AI (T_α) is called a *λ-commuting bounded* (C)AI if $\limsup \|T_\alpha\| \leqslant \lambda$. A 1-commuting bounded (C)AI is called a *commuting metric* (C)AI. (See also Remark 2.4.)

It is obvious that every subnet of a commuting (C)AI (or a λ-commuting bounded (C)AI) is also a commuting (C)AI (resp., a λ-commuting bounded (C)AI).

The concept of λ-commuting bounded (C)AI enables to isolate another approximation property.

Definition 2.18. If there is a λ-commuting bounded (C)AI (respectively, a commuting metric (C)AI), then X is said to have the *λ-commuting bounded* (C)AP (respectively, the *commuting metric* (C)AP). (See also Remark 2.4.)

This type of concept of the commuting bounded approximation property is due to Casazza [C2].

It is interesting to make a historical note here: in [J] one finds a similar concept (for separable spaces) due to Rosenthal, where the commutativity assumption is stronger: $T_m T_n = T_{\min(m,n)}$ whenever $m \neq n$. In [J] it is written that

Rosenthal had observed that, for separable spaces, his concept is equivalent to the (our) λ-commuting bounded approximation property, i.e., that Rosenthal had already proven (twenty years before) the important equivalence (i)$\Leftrightarrow$(ii) of Corollary 2.21, due to Casazza and Kalton [CK].

The AP and the commuting bounded CAP are different properties, as can be seen from Chapter 3. It is an open question whether the bounded AP implies the commuting bounded AP.

Though the condition $\limsup_{\alpha} \|T_\alpha\| \leqslant \lambda$ is seemingly weaker than $\sup_{\alpha} \|T_\alpha\| \leqslant \lambda$, the λ-commuting bounded (C)AP implies trivially the λ-bounded (C)AP, as can be seen from the following proposition.

Proposition 2.19. *A Banach space X has the λ-commuting bounded (C)AP if and only if X has a commuting (C)AI (T_α) together with one of the following conditions:*

(i) $\limsup_{\alpha} \|T_\alpha\| \leqslant \lambda$; (ii) $\liminf_{\alpha} \|T_\alpha\| \leqslant \lambda$;

(iii) $\lim_{\alpha} \|T_\alpha\| \leqslant \lambda$; (iv) $\sup_{\alpha} \|T_\alpha\| \leqslant \lambda$;

Proof. It is clear that (iii) implies (i) and (ii). The other direction can be obtained by switching to a subnet.

It is also clear that (iv) implies (i) since $\limsup_{\alpha} \|T_\alpha\| \leqslant \sup_{\alpha} \|T_\alpha\|$.

The argument why (iii) implies (iv) is similar to the proof of the "if" part of Proposition 2.15. Let (T_α) be such a commuting (C)AI for which $\lambda_0 := \lim_{\alpha} \|T_\alpha\| \leqslant \lambda$. Now denote

$$S_\alpha = \lambda_0 \frac{T_\alpha}{\|T_\alpha\|}.$$

Then S_α is a commuting (C)AI: indeed,

$$\begin{aligned}
S_{\alpha_1} S_{\alpha_2} &= \frac{\lambda_0^2}{\|T_{\alpha_1}\| \, \|T_{\alpha_2}\|} T_{\alpha_1} T_{\alpha_2} = \frac{\lambda_0^2}{\|T_{\alpha_1}\| \, \|T_{\alpha_2}\|} T_{\alpha_2} T_{\alpha_1} = S_{\alpha_2} S_{\alpha_1}, \\
\|S_\alpha x - x\| &= \left\| \frac{\lambda_0}{\|T_\alpha\|} T_\alpha x - x \right\| \leqslant \left| \frac{\lambda_0}{\|T_\alpha\|} - 1 \right| \|T_\alpha\| \, \|x\| + \|T_\alpha x - x\| \to 0.
\end{aligned}$$

What is more,

$$\|S_\alpha\| = \lambda_0 \qquad \forall \alpha,$$

hence $\sup_{\alpha} \|S_\alpha\| \leqslant \lambda$. $\square$

We are going to need all the arguments of the proof of the next result in a slightly different context in Chapter 6 while proving Theorem 6.9 (see pp. 74–84).

Theorem 2.20 ([CK, Proposition 2.1]). *Let X be a separable Banach space with an AI (T_n) such that $T_nT_m = T_m$ if $n > m$, and $\limsup_n \|T_n\| \leqslant \lambda$. If $\sum_n \|T_nT_{n+1} - T_{n+1}T_n\| < \infty$, then X has an AI (S_n) satisfying $S_nS_m = S_{\min(n,m)}$ for all $n \neq m$, and $\limsup_n \|S_n\| \leqslant \lambda$.*

For the sake of completeness, we shall prove the next result originating from [CK], which relies on Theorem 2.20, in detail.

Corollary 2.21 ([CK, Propositions 2.2 and 2.3]). *For a separable Banach space X, the following statements are equivalent:*

(i) *X has the λ-commuting bounded AP.*

(ii) *There is an AI (T_n) on X with $T_nT_m = T_{\min(n,m)}$ for all $n \neq m$, and $\limsup_n \|T_n\| \leqslant \lambda$.*

(iii) *There is an AI (T_n) on X with $\lim_n \|T_nT_m - T_mT_n\| = 0$ for all m, and $\limsup_n \|T_n\| \leqslant \lambda$.*

(iv) *There is an AI (T_n) on X with $\lim_{m,n} \|T_nT_m - T_mT_n\| = 0$, and $\limsup_n \|T_n\| \leqslant \lambda$.*

Proof. The condition (ii) obviously implies all the other conditions. Also, (i) easily implies (ii), by using Theorem 2.20.

Let us show that (iii) or (iv) is sufficient to fulfill the assumptions of Theorem 2.20.

In both cases, we shall first find an AI (S_k) such that $S_mS_n = S_n$ if $m > n$, and for a subsequence (T_{n_k}) of (T_k) we have $\lim_{k\to\infty} \left\|S_k - T_{n_k}\right\| = 0$ (see Remark 2.16).

(iii). Assume that the original AI (T_n) on X satisfies $\lim_n \|T_nT_m - T_mT_n\| = 0$ for all m, and $\limsup_n \|T_n\| \leqslant \lambda$. We find an AI (S_n) such that $S_mS_n = S_n$ if $m > n$, and for a subsequence (T_{k_n}) of (T_n) we have $\lim_{n\to\infty} \left\|S_n - T_{k_n}\right\| = 0$. This implies that the limes superior of the right hand side of the inequality

$$\|S_n\| \leqslant \left\|S_n - T_{k_n}\right\| + \left\|T_{k_n}\right\|$$

is not greater than λ, giving $\limsup_n \|S_n\| \leqslant \lambda$.

Let us denote by λ' a number strictly greater than λ. If needed, we omit a finite number of members from the beginning, so obtaining

$$\sup_n \|S_n\| \leqslant \lambda', \qquad \sup_n \|T_n\| \leqslant \lambda'.$$

We also have $\lim_n \limsup_m \|S_m S_n - S_n S_m\| = 0$. Indeed, fix an index m and a number ε, assuming $0 < \varepsilon \leqslant 1$. Clearly

$$\|S_m S_n - S_n S_m\| \leqslant \left\|S_m S_n - T_{k_m} T_{k_n}\right\| + \left\|T_{k_m} T_{k_n} - T_{k_n} T_{k_m}\right\| + \left\|T_{k_n} T_{k_m} - S_n S_m\right\|.$$

Let N be an index for which

$$n \geqslant N \quad \Rightarrow \quad \begin{cases} \left\|S_n - T_{k_n}\right\| < \dfrac{\varepsilon}{\lambda'}, \\ \left\|T_{k_m} T_{k_n} - T_{k_n} T_{k_m}\right\| < \varepsilon \end{cases}$$

Hence for the case $n \geqslant N$ we have

$$\begin{array}{rcl} \left\|S_m S_n - T_{k_m} T_{k_n}\right\| & \leqslant & \|S_m\| \left\|S_n - T_{k_n}\right\| + \left\|T_{k_n}\right\| \left\|S_m - T_{k_m}\right\| < \\ & < & \varepsilon + \lambda' \left\|S_m - T_{k_m}\right\|, \\ \left\|T_{k_m} T_{k_n} - T_{k_n} T_{k_m}\right\| & < & \varepsilon, \\ \left\|T_{k_n} T_{k_m} - S_n S_m\right\| & \leqslant & \left\|T_{k_n}\right\| \left\|T_{k_m} - S_m\right\| + \|S_m\| \left\|T_{k_n} - S_n\right\| < \\ & < & \varepsilon + \lambda' \left\|S_m - T_{k_m}\right\|, \end{array}$$

or altogether

$$\|S_m S_n - S_n S_m\| < 2\lambda' \left\|S_m - T_{k_m}\right\| + \varepsilon.$$

Thus for all indices m we have

$$\limsup_{n\to\infty} \|S_m S_n - S_n S_m\| \leqslant 2\lambda' \left\|S_m - T_{k_m}\right\|.$$

Therefore

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|S_m S_n - S_n S_m\| = 0.$$

Next, we choose a subsequence of (S_n) inductively such that $\left\|S_{l_n} S_{l_{n+1}} - S_{l_{n+1}} S_{l_n}\right\| \leqslant \dfrac{1}{2^n}$. For this task let l_1 be an index such that

$$\limsup_n \left\|S_{l_1} S_n - S_n S_{l_1}\right\| < \frac{1}{2}.$$

Pass to a subsequence (call it again (S_n)) so that

$$\lim_n \left\|S_{l_1} S_n - S_n S_{l_1}\right\| < \frac{1}{2}.$$

Let $l_2 > l_1$ be such that

$$\left\| S_{l_1} S_{l_2} - S_{l_2} S_{l_1} \right\| < \frac{1}{2}, \qquad \limsup_n \left\| S_{l_2} S_n - S_n S_{l_2} \right\| < \frac{1}{4}.$$

Pass to a subsequence (call it again (S_n)) so that

$$\lim_n \left\| S_{l_2} S_n - S_n S_{l_2} \right\| < \frac{1}{4}.$$

In general, having chosen l_n, let $l_{n+1} > l_n$ be such that

$$\left\| S_{l_n} S_{l_{n+1}} - S_{l_{n+1}} S_{l_n} \right\| < \frac{1}{2^n}, \qquad \limsup_n \left\| S_{l_{n+1}} S_n - S_n S_{l_{n+1}} \right\| < \frac{1}{2^{n+1}}.$$

Passing to a subsequence and calling it again (S_n), we have the conditions

$$\left\| S_{l_n} S_{l_{n+1}} - S_{l_{n+1}} S_{l_n} \right\| < \frac{1}{2^n}, \qquad \lim_n \left\| S_{l_{n+1}} S_n - S_n S_{l_{n+1}} \right\| < \frac{1}{2^{n+1}}.$$

Having denoted $U_n = S_{l_n}$, we are done: $\sum_n \|U_n U_{n+1} - U_{n+1} U_n\| \leqslant 1$.

(iv). Assume now we have an AI (T_n) on X with $\lim_{m,n} \|T_n T_m - T_m T_n\| = 0$, and $\limsup_n \|T_n\| \leqslant \lambda$. As above, we again find an AI (S_n) such that $S_m S_n = S_n$ if $m > n$, and for a subsequence (T_{k_n}) of (T_n) we have $\lim_{n\to\infty} \left\| S_n - T_{k_n} \right\| = 0$. In a similar manner we have $\limsup_n \|S_n\| \leqslant \lambda$.

This time we have also $\lim_{m,n} \|S_m S_n - S_n S_m\| = 0$. Indeed, fix an ε such that $0 < \varepsilon \leqslant 1$, and find an index N for which $m, n \geqslant N$ implies the following conditions:

$$\begin{aligned} \left\| S_n - V_{k_n} \right\| &< \frac{\varepsilon}{\lambda + 1} \\ \|S_n\| &< \lambda + \varepsilon \\ \left\| V_{k_m} V_{k_n} - V_{k_n} V_{k_m} \right\| &< \varepsilon. \end{aligned}$$

Now the inequality

$$\|S_m S_n - S_n S_m\| \leqslant \left\| S_m S_n - T_{k_m} T_{k_n} \right\| + \left\| T_{k_m} T_{k_n} - T_{k_n} T_{k_m} \right\| + \left\| T_{k_n} T_{k_m} - S_n S_m \right\|.$$

enables to estimate all the addends separately, giving altogether

$$\|S_m S_n - S_n S_m\| < 5\varepsilon.$$

This means that

$$\lim_{m,n\to\infty} \|S_m S_n - S_n S_m\| = 0.$$

Now we again go for $\|S_{l_n}S_{l_{n+1}} - S_{l_{n+1}}S_{l_n}\| \leqslant \frac{1}{2^n}$. We choose l_1 such that

$$\forall m > l_1 \qquad \|S_m S_{l_1} - S_{l_1} S_m\| \leqslant \frac{1}{2^1}.$$

In general, having chosen l_n, we choose $l_{n+1} > l_n$ such that

$$\forall m > l_{n+1} \qquad \|S_m S_{l_{n+1}} - S_{l_{n+1}} S_m\| \leqslant \frac{1}{2^{n+1}}$$

Denoting $U_n = S_{l_n}$, we are done: $\sum\limits_n \|U_n U_{n+1} - U_{n+1} U_n\| \leqslant 1$. □

One of the most important positive results concerning approximation properties obtained after the fundamental works of Grothendieck in 1950s is the following theorem due to Casazza and Kalton [CK] from 1990 (see also, e.g., [C2, Theorem 4.6]; for a different proof, see [GK]).

Theorem 2.22 (Casazza, Kalton)**.** *If X is a separable Banach space having the metric AP, then X has the commuting metric AP.*

It is not known whether Theorem 2.22 holds in the case of the metric CAP. As we mentioned, the separable space X_W of Willis not only has the metric CAP, but it also has the commuting metric CAP.

It is not known (even for separable spaces) whether the λ-bounded AP (for $\lambda > 1$) implies the commuting bounded AP (see [C2, Problem 4.2]).

However, in Theorem 4.33 we prove that for certain (large) class of separable Banach spaces and for certain real numbers λ (the upper bound on λ depending on the class), the λ-commuting bounded AP implies the commuting metric AP.

It is not clear whether Theorem 2.22 holds in the non-separable case. An easy testbed for proving the negative could be ℓ_∞ that has the metric AP, but we do not know whether ℓ_∞ fails the commuting metric AP.

Casazza, Kalton, and Wojtaszczyk (see [C2, Theorem 9.3]) have given the following result: *if X is a non-separable Banach space having the commuting bounded AP, then X has the separable complementation property* (meaning that every separable closed subspace of X is contained in a separable subspace which is complemented in X). It appears that their proof might be in error (see Theorem 6.9 and Remark 6.11 in Chapter 6).

It is known that every infinite-dimensional complemented subspace of ℓ_∞ is isomorphic to ℓ_∞ (see [Lin] or, e.g., [LTz I, p. 57]). If [C2, Theorem 9.3] holds, then ℓ_∞ cannot have the commuting metric AP since otherwise ℓ_∞ would have the separable complementation property which is not possible (for example, a separable space c_0 is a closed subspace of ℓ_∞). In this case the non-separable version of Theorem 2.22 would fail as well.

2.3 Stronger properties

Recall that a Banach space X has the bounded AP if and only if there exists a net of uniformly bounded finite-rank operators converging strongly to the identity. Requiring finite-rank projections here gives us a stronger property.

Definition 2.23. If there is a net of finite-rank projections (P_α) on X such that $\limsup_\alpha \|P_\alpha\| \leqslant \lambda$ and $P_\alpha x \to x$ for every $x \in X$, then X is said to have the π_λ-*property*. The π_1-property is called the *metric* π-*property*. A space with the π_λ-property for some λ is said to have the π-*property*.

Obviously every space with the π_λ-property also has the λ-bounded AP. The converse case has been studied by Read and found to be not true (see [C2, p. 295]).

For a separable Banach space, applying Proposition 2.13, we obtain a result similar to Proposition 2.15.

Proposition 2.24 ([C2, Proposition 5.5])**.** *A separable Banach space X has the π_λ-property if and only if there is a sequence $(P_n) \subset \mathscr{F}(X)$ of projections such that*

(i) $P_n x \to x$ *for any* $x \in X$,

(ii) $P_m P_n = P_n$ *for all* $m > n$,

(iii) $\limsup_n \|P_n\| \leqslant \lambda$.

Amalgamating together the requirements of the commuting bounded AP and the π-property, we obtain the finite-dimensional decomposition property.

Definition 2.25. A sequence of bounded linear finite-rank operators (P_m) on X such that $P_m P_n = P_{\min(m,n)}$, $m, n \in \mathbb{N}$, and $\lim_n P_n x = x$ for every $x \in X$, is called a *finite-dimensional decomposition* of X. The number $\sup_n \|P_n\|$ is called the *decomposition constant* of (P_n).

In view of Remark 2.14, every Banach space with a finite-dimensional decomposition is separable. It is clear from the definitions that a separable Banach space having a finite-dimensional decomposition with the decomposition constant λ also has the π_λ-property as well as the λ-commuting bounded AP. The following result shows that the converse holds as well.

Theorem 2.26 ([C1])**.** *A separable Banach space has a finite-dimensional decomposition if and only if it has both the commuting bounded AP and the π-property.*

The finite-dimensional decomposition has also a form closer to basis representation.

Proposition 2.27. *Let X be a Banach space. The following conditions are equivalent.*

(i) *X has a finite-dimensional decomposition.*

(ii) *There exists a sequence of finite-rank projections (p_k) on X such that $p_k p_l = 0$ for every $k \neq l$ and for every $x \in X$ there holds*

$$x = \sum_{k=1}^{\infty} p_k x.$$

(iii) *There exists a sequence of finite-dimensional subspaces (X_k) of X such that every $x \in X$ has a unique representation*

$$x = \sum_{k=1}^{\infty} x_k, \qquad x_k \in X_k, \quad k \in \mathbb{N}.$$

Proof. (i) $\Rightarrow$ (ii) We let $P_0 = 0$ and define

$$p_n = P_n - P_{n-1}, \qquad n \in \mathbb{N}.$$

We easily have $p_n p_n = p_n$ for every n. If $k \neq l$, then

$$p_k p_l = P_{\min(k,l)} - P_{\min(k,l-1)} - P_{\min(k-1,l)} + P_{\min(k-1,l-1)},$$

where without loss of generality we can assume $k < l$. Then $k = \min(k,l) = \min(k, l-1)$ and $k-1 = \min(k-1,l) = \min(k-1,l-1)$, yielding $p_k p_l = 0$.

We see that $\sum_{k=1}^{n} p_k x = P_n x$, hence $x = \sum_{k=1}^{\infty} p_k x$ has been justified by $x = \lim_n P_n x$.

(ii) ⇒ (i) We define

$$P_n = \sum_{k=1}^{n} p_k.$$

Now the equality $x = \lim_n P_n x$ clearly holds. We also have

$$P_m P_n = \left(\sum_{k=1}^{m} p_k\right)\left(\sum_{k=1}^{n} p_k\right) = \sum_{k=1}^{\min(m,n)} p_k^2 = \sum_{k=1}^{\min(m,n)} p_k = P_{\min(m,n)}$$

as required.

(ii) ⇒ (iii) This is almost obvious if we denote $X_n = \operatorname{ran} P_n$, $n \in \mathbb{N}$. The only matter is the uniqueness: if $x = \sum_{k=1}^{\infty} p_k x$ and also $x = \sum_{k=1}^{\infty} p_k x_k$ then applying p_n, $n \in \mathbb{N}$, on the latter equality, we deduce that $p_n x = p_n x_n$. Hence the elements from X_n in the representation of x are indeed unique.

(iii) ⇒ (ii) For every $x \in X$, $x = \sum_{k=1}^{\infty} x_k$ we define $p_n x = x_n$, $n \in \mathbb{N}$. The operators p_n are linear, since for $x = \sum_{k=1}^{\infty} x_k$ and $y = \sum_{k=1}^{\infty} y_k$ the representation of $x + y$ is $\sum_{k=1}^{\infty}(x_k + y_k)$. The operators p_n are projections that satisfy $p_k p_l = 0$ if $k \neq l$ since the representation of x_n is $\underbrace{0+0+\ldots+0}_{n-1 \text{ addends}} + x_n + 0 + \ldots$.

It remains to verify that the operators p_n are bounded. To see this, we point out that the space

$$\mathscr{A} = \left\{ (x_k) : x_k \in X_k, \sum_{k=1}^{\infty} x_k \in X \right\}$$

is a Banach space with respect to the norm $\|(x_k)\| = \sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} x_k \right\|$. Moreover, considering the operator $T : \mathscr{A} \to X$ where

$$T(x_k) = \sum_{k=1}^{\infty} x_k$$

we have that T is a bounded linear bijection between Banach spaces, hence an isomorphism. Therefore there exists an $\alpha > 0$ such that $\alpha \|(x_k)\| \leqslant \|T(x_k)\|$ or, in other words,

$$\alpha \sup_n \left\| \sum_{k=1}^{n} x_k \right\| \leqslant \|x\|.$$

Now

$$\begin{aligned}\|p_n x\| &= \left\|\sum_{k=1}^{n} x_k - \sum_{k=1}^{n-1} x_k\right\| \leqslant \left\|\sum_{k=1}^{n} x_k\right\| + \left\|\sum_{k=1}^{n-1} x_k\right\| \leqslant \\ &\leqslant 2\sup_n \left\|\sum_{k=1}^{n} x_k\right\| \leqslant \frac{2}{\alpha}\cdot\|x\|.\end{aligned}$$

We conclude that the operators p_n, $n \in \mathbb{N}$, are bounded. □

In the more general case when X_k need not be finite-dimensional, the constraint that every $x \in X$ has a unique representation

$$x = \sum_{k=1}^{\infty} x_k, \qquad x_k \in X_k, \quad k \in \mathbb{N},$$

turns the sequence (X_k) into a *Schauder decomposition* X. Therefore a finite-dimensional decomposition is a special case of Schauder decomposition.

If $\dim X_k = 1$ for every k, we reach to the concept of Schauder basis.

Definition 2.28. A Banach space X has a *Schauder basis* (or: X satisfies the *basis property*) if there exists a sequence $(e_k) \subset X$ (*basis elements*) such that for every element $x \in X$ one can find a unique sequence (a_k) of numbers (*coordinates of* x) satisfying

$$x = \sum_{k=1}^{\infty} a_k e_k.$$

A Banach space having the basis property enjoys all the approximation properties (except perhaps the metric approximation property) described throughout this chapter. This is so because the basis projections $P_n \in \mathscr{F}(X)$, $P_n x = \sum_{k=1}^{n} a_k e_k$, $x \in X$, are suitable for most definitions and defining results that include nets (sequences) of operators.

The basis property on X implies that $X = \overline{\operatorname{span}}(e_k)$, yielding that X is separable. For separable Banach spaces, a Schauder basis is much more useful than an algebraic basis (Hamel basis) since an algebraic basis of an infinite-dimensional Banach space is always uncountable.

What is more, the *coordinate functionals* of the Schauder basis $f_n(e_k) = \delta_{kn}$ (yielding $f_n(x) = a_n$ for all $x = \sum_{k=1}^{\infty} a_k e_k$) are always continuous, i.e., elements of X^*. On the contrary, one can define the coordinate functionals associated with the algebraic basis E of X: for $e \in E$, one defines $f_e(e) = 1$ and $f_e(\varepsilon) = 0$ where $\varepsilon \in E \setminus \{e\}$, and extends f_e linearly to all elements of X. There are always non-continuous functionals among the coordinate functionals that have been generated by the algebraic basis E of an infinite-dimensional Banach space.

However, in the following, we shall work in terms general enough such that we shall never see the basis property again. Roughly speaking, the "best" approximation properties that we are going to use or touch are the commuting metric CAP in Chapter 3, the commuting metric AP in Chapters 4 and 6, and the finite-dimensional decomposition (with the constant $\lambda \leqslant 6$) in Chapter 5.

Chapter 3

The metric compact approximation of the identity of Willis is commuting

In this chapter we prove that the approximation property is different from the commuting metric compact approximation property. More precisely, in 1992, Willis [W] constructed a separable Banach space X_W failing the AP but having the metric CAP. We shall show that the metric compact approximation of the identity constructed by Willis in the space X_W is commuting. Hence, the commuting metric compact approximation property does not imply the approximation property. The chapter is based on [OZ1].

3.1 Bochner integral

For the sake of completeness, we shall take a short excurse to integral theory in order to use the concept of Bochner integral. This integral works on functions $f : \Omega \to X$, where $\Omega = (\Omega, \mu)$ is a measure space and X is a Banach space. We need the case where $\mu(\Omega) < \infty$. The path to follow is similar to that of the Lebesgue integral, only absolute values have been replaced by norms.

We define simple μ-measurable functions, define the integral at first on these functions and after that we define when an arbitrary function is Bochner integrable.

Definition 3.1. A function $f : \Omega \to X$ is called *simple* if it has a finite number of different values.

In other words, f is a simple function if and only if we can have

$$\Omega = \bigcup_{k=1}^{n} E_k, \qquad k \neq l \Rightarrow E_k \cap E_l = \emptyset,$$

and $f(\omega) = x_k$ if $\omega \in E_k$, $k = 1, \ldots, n$. Hence we may write down

$$f = \sum_{k=1}^{n} \chi_{E_k} x_k.$$

Definition 3.2. Let $f : \Omega \to X$ be a simple function. It is said that f is μ-*measurable* if in the expression of $f = \sum_{k=1}^{n} \chi_{E_k} x_k$ all the sets E_k, $k = 1, \ldots, n$, are μ-measurable.

For a function $f : \Omega \to X$ we denote by $\|f\|$ the norm function of f, i.e.

$$\|f\|(\omega) = \|f(\omega)\|, \qquad \omega \in \Omega.$$

Definition 3.3. Let $f : \Omega \to X$ be a function. It is said that f is μ-*measurable* if there exists a sequence (f_n) such that every f_n is a μ-measurable simple function and

$$\lim_n \|f_n - f\| = 0 \qquad \mu\text{-almost everywhere.}$$

It is straightforward to verify that if a function $f : \Omega \to X$ is μ-measurable then $\|f\| : \Omega \to \mathbb{R}$ is also μ-measurable.

Definition 3.4. Let $f = \sum_{k=1}^{n} \chi_{E_k} x_k$ be a μ-measurable simple function. The *Bochner integral* of f over a μ-measurable subset $E \subset \Omega$ is

$$\int_E f \, \mathrm{d}\mu := \sum_{k=1}^{n} \mu(E_k \cap E) x_k.$$

For a μ-measurable simple function $f : \Omega \to X$ one needs only the triangle inequality to justify

$$\left\| \int_E f \, \mathrm{d}\mu \right\| \leqslant \int_E \|f\| \, \mathrm{d}\mu.$$

Definition 3.5. It is said that a μ-measurable function $f : \Omega \to X$ is *Bochner integrable* if there exists a sequence (f_n) of μ-measurable simple functions such that

$$\lim_n \int_\Omega \|f_n - f\| \,\mathrm{d}\mu = 0.$$

If that is the case, then for every μ-measurable subset $E \subset \Omega$ one can define the *Bochner integral* of f over E by

$$\int_E f \,\mathrm{d}\mu := \lim_n \int_E f_n \,\mathrm{d}\mu.$$

The Definition 3.5 is correct: if f is Bochner integrable, then the limit $\lim_n \int_E f_n \,\mathrm{d}\mu$ exists and is independent on the choice of the sequence (f_n).

Proposition 3.6. *Let $\lambda \in \mathbb{K}$ be a scalar. If functions $f, g : \Omega \to X$ are Bochner integrable, then $f + \lambda g$ is Bochner integrable as well and*

$$\int_E (f + \lambda g) \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu + \lambda \int_E g \,\mathrm{d}\mu.$$

In general an easily accessible criterion to check the Bochner integrability of a function is the following.

Proposition 3.7 (Bochner's Criterion)**.** *If a function $f : \Omega \to X$ is μ-measurable, then f is Bochner integrable if and only if $\|f\|$ is (Lebesgue) integrable.*

The justification of the following two facts is straightforward.

Proposition 3.8. *If a function $f : \Omega \to X$ is Bochner integrable, then*

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\| \leqslant \int_\Omega \|f\| \,\mathrm{d}\mu.$$

Proposition 3.9. *Let a function $f : \Omega \to X$ be Bochner integrable. If $T \in \mathscr{L}(X, Y)$, then the function $Tf : \Omega \to Y$ is Bochner integrable and*

$$\int_\Omega Tf \,\mathrm{d}\mu = T\left(\int_\Omega f \,\mathrm{d}\mu\right).$$

3.2 Willis space

First we shortly describe the construction of X_W due to Willis [W]. Let X be a Banach space which does not have the AP. Then there exists a compact set

$K \subset X$ such that the identity operator cannot be approximated on K by finite-rank operators. It may be supposed, in view of a theorem by Grothendieck (see, e.g., [LTz I, Proposition 1.e.2]), that $K = \overline{\operatorname{conv}}\{x_k : k \in \mathbb{N}\}$ where $\|x_k\| \leqslant 1$ for all $k \in \mathbb{N}$ and $\|x_k\| \downarrow 0$.

Put

$$U_t = \overline{\operatorname{absconv}}\left\{\frac{x_k}{\|x_k\|^t} : k \in \mathbb{N}\right\}$$

for every arbitrarily fixed $t \in (0,1)$. Define $Y_t = \operatorname{span} U_t$.

Lemma 3.10. *Having defined U_t and Y_t as above, the following holds.*

1) *U_t is a closed compact absolutely convex subset of X.*

2) *Y_t is a Banach space with respect to the norm*

$$|||x|||_t = \inf\{\lambda > 0 : x \in \lambda U_t\}, \qquad x \in Y_t,$$

and with the unit ball U_t.

3) *If $s < t$, then $Y_s \subset Y_t$ and $\|y\| \leqslant |||y|||_t \leqslant |||y|||_s$, $y \in Y_s$.*

Proof. The claim 1) is obvious: U_t is a closure, hence closed; it is an absolutely convex hull, hence absolutely convex. Since the set $\left\{\frac{x_k}{\|x_k\|^t} : k \in \mathbb{N}\right\} \cup \{0\}$ is compact, the set $\left\{\frac{x_k}{\|x_k\|^t} : k \in \mathbb{N}\right\}$ is relatively compact and having applied Mazur's Theorem (absolutely convex hull preserves relative compactness) we see that U_t is compact.

The core of the claim 2) lies in the fact that $|||\cdot|||_t$ is actually the Minkowski functional p_{U_t}. It is well known that if U is closed bounded absolutely convex non-empty subset of a normed space X, then $\operatorname{span} U$ is a normed space with respect to the Minkowski functional p_U, whereas U is its unit ball. If, moreover, U is compact, then $(\operatorname{span} U, p_U)$ is complete.

If $s < t$, then

$$\frac{x_k}{\|x_k\|^s} = \|x_k\|^{t-s}\frac{x_k}{\|x_k\|^t} \in \operatorname{absconv}\left\{\frac{x_k}{\|x_k\|^t} : k \in \mathbb{N}\right\}$$

for every $k \in \mathbb{N}$. This proves that $U_s \subset U_t$. The inclusion of the spaces itself follows, since

$$\frac{y}{|||y|||_s} \in U_s \subset U_t \subset B_X, \qquad y \in Y_s,\ y \neq 0,$$

giving also the inequalities of the norms. □

Consider the following space of functions on $(0,1)$ with values in X:

$$W = \operatorname{span}\{\chi_{(s,t)}\, y : 0 < s < t < 1;\ y \in Y_s\},$$

where $\chi_{(s,t)}$ is the characteristic function of (s,t). If $f \in W$, then $f(r) \in Y_r$ for all $r \in (0,1)$. Define a norm on W by

$$\|f\|_W = \int_0^1 |||f(r)|||_r \, dr, \qquad f \in W.$$

The *Willis space* X_W is the completion of W with respect to $\|\cdot\|_W$.

Theorem 3.11 ([W])**.** *The space X_W has the metric CAP but fails the AP.*

3.3 The result

We emphasize that the only thing that is new here is the "commuting" part. Everything else has been done by Willis [W].

Theorem 3.12. *The space X_W has the commuting metric CAP.*

Proof. We start with the description of the metric CAI $(T_n) = (T_n)_{n=1}^{\infty}$ by Willis and after that we shall show that the operators of this CAI commute.

For each $r \in (0,1)$, define $S_r(\chi_{(s,t)}\, y) = \chi_{(s+r,t+r)}\, y$ for every s, t, $0 < s < t < 1$, and $y \in Y_s$. Then extend S_r to W by linearity and after that, since S_r is a contraction mapping, extend S_r to X_W by continuity. Now define, for each n, operators $T_n : X_W \to X_W$, by

$$T_n f = n \int_0^{1/n} S_r f \, dr, \qquad f \in X_W.$$

It is proven in [W] that (T_n) is a CAI of X_W satisfying $\|T_n\| \leqslant 1$ for each n.

In order to obtain that $T_n T_m = T_m T_n$ for any m and n, by the definitions of X_W and W, it suffices to show the commutativity of (T_n) on the elements $\chi_{(s,t)}\, y$, where $0 < s < t < 1$ and $y \in U_s$. In fact, having done that, the commutativity of (T_n) extends to W and then to X_W due to the linearity and continuity of T_n.

Let us first prove the commutativity of (T_n) on the elements $\chi_{(s,t)}\, x_k$, where $0 < s < t < 1$ and $k \in \mathbb{N}$. By the definition of T_n, it is straightforward to verify that, for all $\varphi \in L_1(0,1)$,

$$T_n(\varphi x_k) = \left(\varphi * \left(n \chi_{(0,1/n)}\right)\right) x_k,$$

where $*$ denotes the usual convolution product over $(0,1)$ (this equality was used in [W] to show the compactness of T_n). Hence,

$$\begin{aligned} T_m\left(T_n\left(\chi_{(s,t)}x_k\right)\right) &= T_m\left(\left(\chi_{(s,t)} * n\chi_{(0,1/n)}\right)x_k\right) \\ &= \left(\left(\chi_{(s,t)} * n\chi_{(0,1/n)}\right) * m\chi_{(0,1/m)}\right)x_k \\ &= mn\left(\left(\chi_{(s,t)} * \chi_{(0,1/n)}\right) * \chi_{(0,1/m)}\right)x_k. \end{aligned}$$

Since, by the Fubini-Tonelli theorem,

$$\left(\left(\chi_{(s,t)} * \chi_{(0,1/n)}\right) * \chi_{(0,1/m)}\right)(r) = \int_0^{1/m}\int_0^{1/n} \chi_{(s,t)}(r-(u+v))\,du\,dv,$$

we clearly have

$$T_m\left(T_n\left(\chi_{(s,t)}x_k\right)\right) = T_n\left(T_m\left(\chi_{(s,t)}x_k\right)\right)$$

as required.

Denote

$$K = \operatorname{absconv}\left\{\frac{x_k}{\|x_k\|^s} : k \in \mathbb{N}\right\}.$$

By the above, we have the commutativity of (T_n) on the elements $\chi_{(s,t)}x$ with $x \in K$. To prove the commutativity on the elements $\chi_{(s,t)}y$ with $y \in U_s$, it clearly suffices to show that for any $\varepsilon > 0$, $s+\varepsilon < t$, there exists $x \in K$ such that

$$\|\chi_{(s,t)}y - \chi_{(s,t)}x\|_W < 3\varepsilon.$$

Recall that $U_s = \overline{K}^X$, closure of K in X. We shall prove that $\overline{K}^X$ equals $\overline{K}^{Y_{s+\varepsilon}}$, closure of K in $Y_{s+\varepsilon}$.

First, let us notice that $\overline{K}^{Y_{s+\varepsilon}} \subset \overline{K}^X$ (by Lemma 3.10, 3))). Since

$$\left|\left|\left|\frac{x_k}{\|x_k\|^s}\right|\right|\right|_{s+\varepsilon} = \|x_k\|^\varepsilon \left|\left|\left|\frac{x_k}{\|x_k\|^{s+\varepsilon}}\right|\right|\right|_{s+\varepsilon} \to 0$$

(we used that $\|x_k\| \to 0$ and $x_k/\|x_k\|^{s+\varepsilon} \in U_{s+\varepsilon}$), we have the convergence $\dfrac{x_k}{\|x_k\|^s} \to 0$ in $Y_{s+\varepsilon}$. Having applied Mazur's theorem, we obtain that $\overline{K}^{Y_{s+\varepsilon}}$ is compact in $Y_{s+\varepsilon}$. Since the identity operator from $Y_{s+\varepsilon}$ to X is continuous (see Lemma 3.10, 3))), $\overline{K}^{Y_{s+\varepsilon}}$ is compact in X as well. Now $\overline{K}^X \subset \overline{\overline{K}^{Y_{s+\varepsilon}}}^X = \overline{K}^{Y_{s+\varepsilon}}$. Therefore $\overline{K}^{Y_{s+\varepsilon}} = \overline{K}^X$.

Since $y \in U_s = \overline{K}^X = \overline{K}^{Y_{s+\varepsilon}}$, there exists $x \in K$ such that $|||y-x|||_{s+\varepsilon} < \varepsilon$. Hence, using Lemma 3.10, we have

$$
\begin{aligned}
\|\chi_{(s,t)}y - \chi_{(s,t)}x\|_W &= \int_0^1 |||\chi_{(s,t)}(r)y - \chi_{(s,t)}(r)x|||_r \, dr \\
&= \int_s^t |||y-x|||_r \, dr = \\
&= \int_s^{s+\varepsilon} |||y-x|||_r \, dr + \int_{s+\varepsilon}^t |||y-x|||_r \, dr \\
&\leqslant \varepsilon |||y-x|||_s + |||y-x|||_{s+\varepsilon} \\
&< \varepsilon(|||y|||_s + |||x|||_s) + \varepsilon \\
&\leqslant 2\varepsilon + \varepsilon = 3\varepsilon
\end{aligned}
$$

as desired. □

Note that relying on the reflexive version of Willis space [W], Oja [O6] has constructed a Banach space X with Schauder basis such that its all duals are separable, its odd duals X^*, X^{***}, $\ldots$, have the *metric CAP with conjugate operators* (i.e., have a metric CAI whose operators are conjugate operators), and its even duals X^{**}, X^{****}, $\ldots$, have the metric CAP, but fail the metric CAP with conjugate operators.

Chapter 4

The $M(a,B,c)$-inequality

In this chapter we introduce a geometric property of a Banach space, the $M(a,B,c)$-inequality.

The main result is the following. If the property is fulfilled for a Banach space X, then for every λ-commuting bounded compact approximation of the identity $(T_\alpha) \subset \mathcal{K}(X)$ (having $1 \leqslant \lambda < \max|B| + c$), also the net of conjugates $(T_\alpha^*) \subset \mathcal{K}(X^*)$ is a (λ-commuting bounded) compact approximation of the identity.

We shall derive many corollaries from the main result, using several concepts like the Radon-Nikodým property, weakly* strongly exposed points, etc. Among other results, we shall prove that if a Banach space X satisfying the $M(a,B,c)$-inequality has a λ-commuting bounded (compact) approximation of the identity (with $1 \leqslant \lambda < \max|B| + c$), then both X and X^* enjoy the metric (compact) approximation property.

A number of corollaries will make use of special cases of the $M(a,B,c)$-inequality and other structural properties.

The chapter is based on [OZ1].

4.1 The concept

Throughout this chapter, $B \subset \mathbb{K}$ will be a compact set and $a,c \geqslant 0$. We write $\max|B|$, meaning $\max\limits_{b\in B}|b|$. Since B is compact, we can always find a $b_0 \in B$ for which $|b_0| = \max\limits_{b\in B}|b|$.

Recall that we denote by π_X the canonical projection of X^{***} onto $\operatorname{ran} j_{X^*}$ where $j_{X^*} : X^* \to X^{***}$ is the canonical embedding.

Definition 4.1. It is said that a Banach space X is *M-embedded* if

$$\|x^{***} - \pi_X x^{***}\| + \|\pi_X x^{***}\| = \|x^{***}\| \quad \forall x^{***} \in X^{***}.$$

It is clear that the left hand side of the inequality is never less than the right hand side.

Definition 4.2. We shall say that a Banach space X *satisfies the $M(a,B,c)$-inequality* if

$$\|ax^{***} + b\pi_X x^{***}\| + c\|\pi_X x^{***}\| \leqslant \|x^{***}\| \quad \forall b \in B, \quad \forall x^{***} \in X^{***}.$$

Having compared the two definitions, it is clear that being M-embedded means precisely satisfying the $M(1,\{-1\},1)$-inequality.

The $M(a,B,c)$-inequality was occasionally used in [O4] to characterize a large class of ideals of compact operators, providing, in particular, an alternative unified and easier approach to the theories of M-, u-, and h-ideals of compact operators (see [O4, Section 4] for results and references).

The $M(a,B,c)$-inequality follows from property $M^*(a,B,c)$ (see Proposition 4.46). The latter structural property was introduced in [O4] (see also [O3]) to characterize intrinsically a large class of shrinking metric (C)AI, including, e.g., those related to M-, u-, and h-ideals.

Note that for every Banach space satisfying the $M(a,B,c)$-inequality there must hold $|a+b|+c \leqslant 1$. Indeed, if in such a space one uses an element $x^* \in X^*$ such that $\|x^*\| = 1$ and denotes $x^{***} = j_{X^*} x^*$ (giving $x^{***} = \pi_X x^{***}$), then

$$\begin{aligned}
|a+b|+c &= \|ax^* + bx^*\| + c\|x^*\| = \\
&= \|ax^{***} + b\pi_X x^{***}\| + c\|\pi_X x^{***}\| \leqslant \\
&\leqslant \|x^{***}\| = \|x^*\| = 1.
\end{aligned}$$

Since for a reflexive Banach space X any element $x^{***} \in X^{***}$ satifies the condition $x^{***} = \pi_X x^{***}$, every reflexive Banach space satisfies the $M(a,B,c)$-inequality.

The $M(a,B,c)$-inequality inherits to closed subspaces and quotient spaces, as can be seen from the following result.

Proposition 4.3. *If a Banach space X satisfies the $M(a,B,c)$-inequality, then any closed subspace Y of a quotient space of X satisfies the $M(a,B,c)$-inequality.*

Proof. Consider a closed subspace Y of a Banach space X satisfying the $M(a,B,c)$-inequality. Denote by $i : Y \to X$ the natural embedding. It is well known and straightforward to check that $\pi_Y i^{***} = i^{***}\pi_X$ and that $i^{**} : Y^{**} \to X^{**}$ is isometric.

Fix $y^{***} \in Y^{***}$ arbitrarily and define a functional $z^{***} \in \left(\operatorname{ran} i^{**}\right)^*$ by $z^{***}\left(i^{**}y^{**}\right) = y^{***}\left(y^{**}\right)$. Then $\|z^{***}\| \leqslant \|y^{***}\|$. Thus for a norm-preserving extension x^{***} of z^{***}, one has $\|x^{***}\| \leqslant \|y^{***}\|$ and $i^{***}x^{***} = y^{***}$. Hence,

$$
\begin{aligned}
\left\|ay^{***} + b\pi_Y y^{***}\right\| + c\left\|\pi_Y y^{***}\right\| &= \left\|ai^{***}x^{***} + b\pi_Y i^{***}x^{***}\right\| \\
&\quad + c\left\|\pi_Y i^{***}x^{***}\right\| \\
&= \left\|i^{***}\left(ax^{***}\right) + i^{***}\left(b\pi_X x^{***}\right)\right\| \\
&\quad + c\left\|i^{***}\pi_X x^{***}\right\| \\
&\leqslant \left\|ax^{***} + b\pi_X x^{***}\right\| + c\left\|\pi_X x^{***}\right\| \\
&\leqslant \left\|x^{***}\right\| \leqslant \left\|y^{***}\right\|,
\end{aligned}
$$

giving that Y satisfies the $M(a,B,c)$-inequality.

Inheritance by quotient spaces follows similarly, using that q^{***} is isometric and $q^{***}\pi_{X/Y} = \pi_X q^{***}$ for the quotient mapping $q : X \to X/Y$. □

4.2 The main result

We start from the result dating back to 1988, when Godefroy and Saphar [GS] demonstrated how the geometric structure of a separable Banach space permits to lift the commuting bounded AP from the space to its dual space.

Definition 4.4. A (C)AI (T_α) of a Banach space X is called *shrinking* if $T_\alpha^* x^* \to x^*$ for every $x^* \in X^*$.

Theorem 4.5 ([GS, Proposition 4.3])**.** *Let X be an M-embedded separable Banach space. If $1 \leqslant \lambda < 2$, then every commuting AI $(T_n)_{n=1}^\infty$ of X such that $\sup\|T_n\| \leqslant \lambda$ is shrinking.*

The following theorem is the main result of this chapter. In particular, it extends Theorem 4.5 to *non-separable* Banach spaces and *compact* approximations of the identity.

Theorem 4.6. *Let X be a Banach space that satisfies the $M(a,B,c)$-inequality. If $1 \leqslant \lambda < \max|B| + c$, then every λ-commuting bounded CAI of X is shrinking.*

Note that since

$$|b| - a + c \leqslant |a+b| + c \leqslant 1,$$

the maximal value of $\max|B| + c$ can never exceed $1 + a$.

Theorem 4.6 applies, for instance, to the subspaces of non-separable M-embedded spaces like $c_0(\Gamma)$ or $\mathcal{K}(\ell_p(\Gamma))$, $1 < p < \infty$, the Banach space of compact operators on $\ell_p(\Gamma)$, where Γ is an uncountable set. It also applies, e.g., to $\mathcal{K}(d(w,p))$, $1 < p < \infty$, the Banach space of compact operators on the Lorentz sequence space $d(w,p)$, which is not M-embedded (see [Hen], or, e.g., [HWW, p. 305]), but satisfies the inequality in Theorem 4.6 with $a = 1$, $B = \{-2\}$, and $c = 0$ (see Section 4.6 below).

The applications to Theorem 4.6 will be given in Sections 4.5–4.7. Among other things, we prove (see Theorem 4.33) that if a Banach space X satisfying the $M(a,B,c)$-inequality has a λ-commuting bounded (C)AI with $1 \leqslant \lambda < \max|B| + c$, then both X and X^* enjoy the metric (C)AP.

The proof of Theorem 4.6 below will develop the idea of the original proof of Theorem 4.5 due to Godefroy and Saphar (see [GS, Proposition 4.3]) (notice that an alternative proof was recently given by Godefroy in [G, Theorem VI.I]), and it will apply, among others, techniques from the paper [O4] by Oja.

Let us recall that the *characteristic* $r(V)$ of a subspace V of X^* is defined by

$$r(V) = \max\left\{r \geqslant 0 : rB_{X^*} \subset \overline{B_V}^{w^*}\right\}.$$

Obviously, $r(V) \leqslant 1$. We shall need the following auxiliary result which is implicitly contained in [O4, proof of Theorem 4.1]. We include a proof for completeness.

Lemma 4.7. *If a Banach space X satisfies the $M(a,B,c)$-inequality with $\max|B| + c > 1$, then*

$$r(V) \leqslant \frac{1}{\max|B| + c} < 1$$

for any proper closed subspace V of X^*.

Proof. Since $r(V) \leqslant r(W)$ if $V \subset W$, it is enough to consider the case when $V = \ker x^{**}$, where $x^{**} \in S_{X^{**}}$. Let $\beta = \max|B| = b \operatorname{sgn} b$ for some $b \in B$. We have

$$\begin{aligned}\|((a \operatorname{sgn} b) I_{X^{***}} + \beta\pi_X + c\pi_X)x^{***}\| &\leqslant |\operatorname{sgn} b| \|(aI_{X^{***}} + b\pi_X)x^{***}\| \\ &\quad + c\|\pi_X x^{***}\| \leqslant \|x^{***}\|,\end{aligned}$$

$x^{***} \in X^{***}$, and therefore

$$\left\| I_{X^{***}} + \frac{\beta + c}{a \operatorname{sgn} b} \pi_X \right\| \leqslant \frac{1}{a}.$$

(Notice that $a > 0$, since $a = 0$ would easily imply that $\beta + c \leqslant 1$.) Applying a characterization due to Godefroy, Kalton, and Saphar [GKS, Proposition 2.3], this condition implies the existence of a net (x_ν) in B_X converging weakly* to x^{**} in X^{**} such that

$$\limsup_{\nu} \left\| x^{**} + \frac{\beta + c}{a \operatorname{sgn} b} x_\nu \right\| \leqslant \frac{1}{a}.$$

By a well-known theorem due to Dixmier [Dixmier, Theorem 7],

$$r(V) = \inf_{x \in S_X} \sup_{x^* \in B_V} |x^*(x)|.$$

Hence,

$$\begin{aligned} (\beta + c) r(V) &\leqslant (\beta + c) \inf_{\nu} \sup_{x^* \in B_V} |x^*(x_\nu)| = \\ &= a \inf_{\nu} \sup_{x^* \in B_V} \left| x^* \left(\frac{\beta + c}{a \operatorname{sgn} b} x_\nu \right) \right| = \\ &= a \inf_{\nu} \sup_{x^* \in B_V} \left| x^{**}(x^*) + x^* \left(\frac{\beta + c}{a \operatorname{sgn} b} x_\nu \right) \right| \leqslant 1 \end{aligned}$$

by the above inequality. This completes the proof. □

Proof of Theorem 4.6. Let X be a Banach space. We assume that X satisfies the $M(a, B, c)$-inequality. Let $1 \leqslant \lambda < \max|B| + c$ and let $(T_\alpha) \subset \mathcal{K}(X)$ be a λ-commuting bounded CAI of X. We need to show that

$$T_\alpha^* x^* \to x^*, \qquad x^* \in X^*.$$

Since $T_\alpha x \to x$ for all $x \in X$, we clearly have that $T_\alpha^* x^* \to x^*$ weakly* in X^* for all $x^* \in X^*$.

Denoting

$$V = \overline{\operatorname{span}} \bigcup_{\alpha} \operatorname{ran} T_\alpha^*,$$

the closed subspace of X^* generated by the subspaces $\operatorname{ran} T_\alpha^*$, let us first prove that

$$T_\alpha^* v \to v, \qquad v \in V.$$

Consider $v \in V$. Fixing $\varepsilon > 0$ arbitrarily, we find $x^* \in \operatorname{span}\bigcup_\alpha \operatorname{ran} T_\alpha^*$ so that $\|x^* - v\| < \varepsilon$. Hence $x^* = T_{\alpha_0}^* y^*$ for some index α_0 and some $y^* \in X^*$. Due to the compactness of $T_{\alpha_0}^*$ and the weak* convergence of the bounded net $(T_\alpha^* y^*)$ to y^*, we have $T_{\alpha_0}^* T_\alpha^* y^* \to T_{\alpha_0}^* y^*$. As the net (T_α^*) is commuting, we obtain

$$T_\alpha^* x^* = T_\alpha^* T_{\alpha_0}^* y^* = T_{\alpha_0}^* T_\alpha^* y^* \to T_{\alpha_0}^* y^* = x^*.$$

Now

$$\begin{aligned}\limsup_\alpha \|T_\alpha^* v - v\| \;&\leqslant\; \Big(\limsup_\alpha \|T_\alpha^*\|\Big)\|x^* - v\| + \limsup_\alpha \|T_\alpha^* x^* - x^*\| \\ &\quad + \|x^* - v\| < (\lambda+1)\varepsilon,\end{aligned}$$

yielding that $\lim_\alpha \|T_\alpha^* v - v\| = 0$.

To conclude the proof, it suffices to show that $V = X^*$. If we had $V \neq X^*$, then by Lemma 4.7, we would have $r(V) \leqslant 1/(\max|B| + c)$. This is not the case, however. Indeed, by assumption,

$$0 < \limsup_\alpha \|T_\alpha\| \leqslant \lambda < \max|B| + c.$$

By passing to a subnet, we may assume that

$$0 < \tau := \lim_\alpha \|T_\alpha\| < \max|B| + c.$$

But then, since $T_\alpha^* x^* \to x^*$ weakly* in X^* for all $x^* \in X^*$, also

$$\frac{T_\alpha^* x^*}{\|T_\alpha\|} \to \frac{x^*}{\tau}$$

weakly* in X^*. This immediately implies that

$$\frac{x^*}{\tau} \in \overline{B_V}^{w^*}, \qquad x^* \in B_{X^*}.$$

Hence, $r(V) \geqslant 1/\tau$, and therefore $r(V) > 1/(\max|B| + c)$, a contradiction. □

Proposition 4.3 allows us to point out the following immediate extension of Theorem 4.6.

Theorem 4.8. *Let a Banach space satisfy the $M(a,B,c)$-inequality and let X be a closed subspace of its quotient space. If $1 \leqslant \lambda < \max|B| + c$, then every λ-commuting bounded CAI of X is shrinking.*

Corollary 4.9. *Let a Banach space satisfy the $M(a,B,c)$-inequality with* $\max|B|+c>1$ *and let X be a closed subspace of its quotient space. Let (e_n) be a basic sequence in X. If the basis constant of (e_n) is strictly less than* $\max|B|+c$, *then (e_n) is shrinking.*

Proof. Apply Theorem 4.8 to $\overline{\operatorname{span}}\{e_1,e_2,\dots\}\subset X$ (which is a closed subspace of a quotient space of a Banach space satisfying the $M(a,B,c)$-inequality) and to the sequence of partial sum projections associated with (e_n). ☐

Corollary 4.9 extends [Γ, Corollaries 1 and 2], [GS, Corollary 4.4], and [O4, Corollary 1.8].

Below, we shall formulate several results for Banach spaces X satisfying the $M(a,B,c)$-inequality or having property $M^*(a,B,c)$. Notice that they actually hold for any closed subspace of a quotient space of X because both the $M(a,B,c)$-inequality and property $M^*(a,B,c)$ are inherited by subspaces and quotient spaces (see Proposition 4.3 and [O4, Section 1]).

4.3 The $M(r,s)$-inequality

The next definition follows [CN] and [HO].

Definition 4.10. Let $r,s\geqslant 0$. We say that a Banach space X *satisfies the $M(r,s)$-inequality* if

$$r\|\pi_X x^{***}\|+s\|x^{***}-\pi_X x^{***}\|\leqslant\|x^{***}\|\qquad\forall x^{***}\in X^{***}.$$

It is clear that satisfying the $M(r,s)$-inequality precisely means satisfying the $M(s,\{-s\},r)$-inequality. Also, X is M-embedded if and only if X satisfies the $M(1,1)$-inequality.

In [CN], the Godefroy-Saphar theorem (Theorem 4.5) was extended from M-embedded spaces to spaces satisfying the $M(r,s)$-inequality with $r+s>1$. Let us point out its extension to non-separable Banach spaces and compact approximations of the identity. This is an evident special case of Theorem 4.6.

Corollary 4.11. *Let X be a Banach space satisfying the $M(r,s)$-inequality. If $1\leqslant\lambda<r+s$, then every λ-commuting bounded CAI of X is shrinking.*

4.4 The Radon-Nikodým property and exposed points

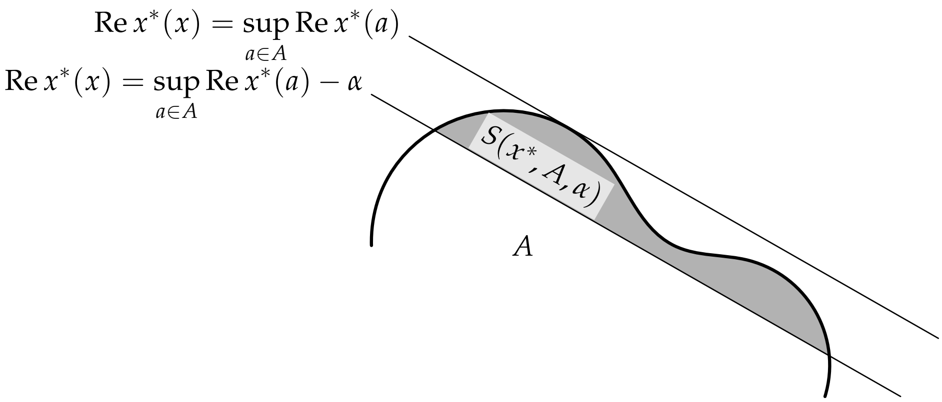
This section is based on Phelps's monograph [Ph] and the book on integral theory of vector-valued functions by Diestel and Uhl [DU]. We introduce the notions of the Radon-Nikodým property and (weakly*) (strongly) exposed points that will be needed in the following sections.

At first we define the terms of slice and dentable subset in a Banach space X and their weak* counterparts for a subset in the dual X^*.

Definition 4.12. Let A be a non-empty subset in X. Choose an arbitrary number $\alpha > 0$ and a functional $x^* \in X^*$. The subset $S(x^*, A, \alpha) \subset X$ where

$$S(x^*, A, \alpha) = \left\{ x \in A : \operatorname{Re} x^*(x) > \sup_{a \in A} \operatorname{Re} x^*(a) - \alpha \right\}$$

is called a *slice* of A.



The gray area is the slice of $A \subset X$ corresponding to a number $\alpha > 0$ and a functional $x^* \in X^*$ (see Definition 4.12).

Definition 4.13. Let A^* be a non-empty subset in X^*. Choose an arbitrary number $\alpha > 0$ and an element $x \in X$. The subset $S(x, A^*, \alpha) \subset X^*$ where

$$S(x, A^*, \alpha) = \left\{ x^* \in A^* : \operatorname{Re} x^*(x) > \sup_{a^* \in A^*} \operatorname{Re} a^*(x) - \alpha \right\}$$

is called a *weak* slice* of A^*.

Definition 4.14. It is said that a non-empty subset A of X is *dentable* if for arbitrarily small $\varepsilon > 0$ there exists a functional $x^* \in X^*$ and a number $\alpha > 0$ such that

$$\operatorname{diam} S(x^*, A, \alpha) < \varepsilon.$$

Definition 4.15. It is said that a non-empty subset A^* of X^* is *weakly* dentable* if for arbitrarily small $\varepsilon > 0$ there exists an element $x \in X$ and a number $\alpha > 0$ such that

$$\operatorname{diam} S(x, A^*, \alpha) < \varepsilon.$$

Definition 4.16. It is said that a Banach space has the *Radon-Nikodým property* if its every non-empty bounded subset is dentable.

For the Radon-Nikodým property, we refer to [DU, Chapter VII] whose end contains 29 equivalent formulations of the property as well as a representable list of spaces that do and that do not have the Radon-Nikodým property. For instance, reflexive spaces, separable dual spaces, $\ell_1(\Gamma)$ for any Γ, $\mathscr{L}(\ell_p, \ell_q)$ for $1 \leqslant q < p < \infty$ have the Radon-Nikodým property, whereas c_0, c, ℓ_∞, $\mathscr{K}(X)$ and $\mathscr{L}(X)$ where $X = \ell_p$, do not.

Definition 4.17. It is said that X is an *Asplund space* if for every separable subspace $E \subset X$ the dual space E^* is separable.

Theorem 4.18 ([Ph, Theorem 5.7])**.** *A Banach space X is an Asplund space if and only if X^* has the Radon-Nikodým property.*

In the following we shall give the definitions of strongly exposed and exposed points, together with their weak* counterparts.

Definition 4.19. Let C be a closed convex set in X. A point $x \in C$ is called *strongly exposed* if for some non-null functional $x^* \in X^*$ we have $x \in S(x^*, C, \alpha)$ for every $\alpha > 0$ and $\lim_{\alpha \to 0} \operatorname{diam} S(x^*, C, \alpha) = 0$. The suitable functional x^* is called *strongly exposing* and it is said that it *strongly exposes* x. The set of all strongly exposed points of a closed convex set $C \subset X$ is denoted $\operatorname{sexp} C$.

Proposition 4.20. *Let C be a closed convex set in X. Let us have $x^* \in X^*$ and $x \in C$. Then x^* strongly exposes x if and only if for every sequence $(x_n) \subset C$ there holds*

$$\operatorname{Re} x^*(x_n) \to \sup_{c \in C} \operatorname{Re} x^*(c) \quad \Rightarrow \quad \|x_n - x\| \to 0.$$

Proof. Let x^* strongly expose x. This means that a functional x^* and a point $x \in C$ are such that

$$\operatorname{Re} x^*(x) > \sup_{c \in C} \operatorname{Re} x^*(c) - \alpha \quad \forall\, \alpha > 0, \qquad \lim_{\alpha \to 0} \operatorname{diam} S(x^*, C, \alpha) = 0.$$

The definition of supremum enables us to find a sequence $(x_n) \subset C$ for which $\operatorname{Re} x^*(x_n) \to \sup_{c \in C} \operatorname{Re} x^*(c)$. Letting $\alpha \to 0$ shows that $\operatorname{Re} x^*(x) = \sup_{c \in C} \operatorname{Re} x^*(c)$. Hence we have $\operatorname{Re} x^*(x_n) \to \operatorname{Re} x^*(x)$.

This also shows that for any fixed $\alpha > 0$, starting from some index n_0, the elements x_n belong to the slice $S(x^*, C, \alpha)$. Assuming now that $\|x_n - x\| > \varepsilon$ for some $\varepsilon > 0$ and all indices n, due to $x \in S(x^*, C, \alpha)$ we have $\operatorname{diam} S(x^*, C, \alpha) \geqslant \|x_n - x\| > \varepsilon$ which contradicts the fact that x^* strongly exposes x. The "only if" part is done.

Now let there hold

$$\operatorname{Re} x^*(x_n) \to \sup_{c\in C} \operatorname{Re} x^*(c) \quad \Rightarrow \quad \|x_n - x\| \to 0$$

for every sequence $(x_n) \subset C$. Using the definition of a supremum, we find a sequence $(x_n) \subset C$ such that $\operatorname{Re} x^*(x_n) \to \sup_{c\in C} \operatorname{Re} x^*(c)$. Hence $\|x_n - x\| \to 0$, which easily yields that $x^*(x_n) \to x^*(x)$. Since $\operatorname{Re} x^*(x_n)$ must converge to a single point, we conclude that $\operatorname{Re} x^*(x) = \sup_{c\in C} \operatorname{Re} x^*(c)$. Now $\operatorname{Re} x^*(x) > \operatorname{Re} x^*(x) - \alpha$ is trivially true for any $\alpha > 0$, hence $x \in S(x^*, C, \alpha)$ for every $\alpha > 0$.

It remains to show that $\lim_{\alpha\to 0} \operatorname{diam} S(x^*, C, \alpha) = 0$. Assume the contrary. Then there exists a number $\varepsilon > 0$ and a decaying sequence (α_n) such that

$$\operatorname{diam} S(x^*, C, \alpha_n) > 2\varepsilon$$

for every n. Thus there exist sequences $(\tilde{x}_n), (\bar{x}_n) \subset C$ such that

$$2\varepsilon < \|\tilde{x}_n - \bar{x}_n\| \leqslant \|\tilde{x}_n - x\| + \|\bar{x}_n - x\|.$$

We see that at least one of the addends in the right hand side must exceed ε. Hence for every index n, we select $x'_n \in S(x^*, C, \alpha_n)$ such that $\left\|x'_n - x\right\| > \varepsilon$.

The condition $x'_n \in S(x^*, C, \alpha_n)$ means that $\operatorname{Re} x^*(x'_n) > \sup_{c\in C} \operatorname{Re} x^*(c) - \alpha_n$, hence $\operatorname{Re} x^*(x'_n) \to \sup_{c\in C} \operatorname{Re} x^*(c)$, which gives $\left\|x'_n - x\right\| \to 0$, contradicting $\left\|x'_n - x\right\| > \varepsilon$. Therefore also $\lim_{\alpha\to 0} \operatorname{diam} S(x^*, C, \alpha) = 0$.

We have shown that x^* strongly exposes x. □

Definition 4.21. Let C^* be a closed convex set in X^*. A functional $x^* \in C^*$ is called *weakly* strongly exposed* if for some non-null element $x \in X$ we have $x^* \in S(x, C^*, \alpha)$ for every $\alpha > 0$ and $\lim_{\alpha\to 0} \operatorname{diam} S(x, C^*, \alpha) = 0$. It is said that the element x *weakly* strongly exposes* x^*. The set of all weakly* strongly exposed points of a closed convex set $C \subset X$ is denoted w^*-sexp C.

Similar proposition to Proposition 4.20 can we proven in the weak* setting.

Proposition 4.22. *Let C^* be a closed convex set in X^*. Let us have $x^* \in X^*$ and $x \in X$. Then x weakly* strongly exposes x^* if and only if for every sequence $(x_n^*) \subset C^*$ there holds*

$$\operatorname{Re} x_n^*(x) \to \sup_{c^* \in C^*} \operatorname{Re} c^*(x) \quad \Rightarrow \quad \|x_n^* - x^*\| \to 0.$$

Definition 4.23. Let C be a closed convex set in X. A point $x \in C$ is called *exposed* if for some non-null functional $x^* \in X^*$ we have $\operatorname{Re} x^*(x) = \sup_{c \in C} \operatorname{Re} x^*(c)$ and $\operatorname{Re} x^*(y) < \sup_{c \in C} \operatorname{Re} x^*(c)$ for every $y \in C$ different from x. The functional x^* is called *exposing* and it is said that x^* *exposes* x. The set of all exposed points of a closed convex set $C \subset X$ is denoted $\exp C$.

Proposition 4.24. *Let x be a point in a closed convex set $C \subset X$. If x is strongly exposed, then x is exposed.*

Proof. The fact that $x \in S(x^*, C, \alpha)$ for every $\alpha > 0$ gives that $\operatorname{Re} x^*(x) = \sup_{c \in C} \operatorname{Re} x^*(c)$. If for some $y \in C$, $y \neq x$, we also had $\operatorname{Re} x^*(y) = \sup_{c \in C} \operatorname{Re} x^*(c)$, then having chosen $x_n = y$ in Proposition 4.20, we would have $x_n \to x$, a contradiction. $\square$

Definition 4.25. Let C^* be a closed convex set in X^*. A functional $x^* \in C^*$ is called *weakly* exposed* if for some non-null element $x \in X$ we have $\operatorname{Re} x^*(x) = \sup_{c^* \in C^*} \operatorname{Re} c^*(x)$ and $\operatorname{Re} y^*(x) < \sup_{c^* \in C^*} \operatorname{Re} c^*(x)$ for every $y^* \in C^*$ different from x^*. It is said that the element x *weakly* exposes* x^*. The set of all weakly* exposed points of a closed convex set $C \subset X$ is denoted w^*-$\exp C$.

The following obvious proposition can be proved similarly to Proposition 4.24.

Proposition 4.26. *Let x^* be a functional in a closed convex set $C^* \subset X^*$. If x^* is weakly* strongly exposed, then it is weakly* exposed.*

We refer an example from [Ph, p. 83] showing that exposed points need not be strongly exposed. Namely, let C be the closed convex hull of the orthogonal basis vectors (e_n) in ℓ_2. Then 0 is an exposed point of C, but not strongly exposed, since every slice C containing 0 has diameter at least $\sqrt{2}$.

Theorem 4.27 ([Ph, Theorem 5.12])**.** *The dual space X^* has Radon-Nikodým property if and only if for every weakly* compact convex subset $C^* \subset X^*$ there holds*

$$C = \overline{\operatorname{conv}}^{w^*} w^*\text{-}\operatorname{sexp} C.$$

In Lemma 4.30 we shall see that the $M(a,B,c)$-inequality (in the case when $\max|B|+c>1$) implies the Radon-Nikodým property.

Definition 4.28. A closed subspace Y of a dual space X^* is called *norming* if $\|x\| = \sup_{x^*\in B_V} \left|x^*(x)\right|$ for any $x \in X$.

Due to the Hahn-Banach theorem, it is clear that the whole X^* can recover the norm of X, hence X^* itself is norming. The following lemma shows that the norm of a separable Banach space is always separably determined.

Lemma 4.29. *For any separable Banach space X there exists a separable closed norming subspace of X^*.*

Proof. Let $X = \overline{\{x_k : k \in \mathbb{N}\}}$. For every x_k we find a functional $x_k^* \in S_{X^*}$ such that $x_k^*(x_k) = \|x_k\|$. Now the required norming subspace is

$$V = \overline{\operatorname{span}}\left\{x_k^* : k \in \mathbb{N}\right\}.$$

□

4.5 The $M(a,B,c)$-inequality and the metric (compact) approximation property

Theorem 4.32 below shows that the isomorphic assumptions in Theorem 4.6 imply an isometric conclusion. Its proof will use the following auxiliary result which is implicitly contained in [O4, proof of Theorem 4.1]. We include a proof for completeness.

Lemma 4.30. *If a Banach space X satisfies the $M(a,B,c)$-inequality with $\max|B| + c > 1$, then X^* has the Radon-Nikodým property and $X^* = \overline{\operatorname{span}}(w^*\text{-sexp}\, B_{X^*})$.*

Proof. If a closed subspace V of X^* is norming, then $r(V) = 1$. Therefore it is clear from Lemma 4.7 that X^* contains no proper norming closed subspace. As closed subspaces of X inherit the $M(a,B,c)$-inequality, their duals contain no proper norming closed subspace either. Since the dual space of any separable subspace of X contains a separable norming subspace (Lemma 4.29), it must be separable. Hence, X^* has the Radon-Nikodým property (Theorem 4.18). By Theorem 4.27, we have $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(w^*\text{-sexp}\, B_{X^*})$. This clearly implies that $\overline{\operatorname{span}}(w^*\text{-sexp}\, B_{X^*})$ is a norming subspace of X^*. Therefore $X^* = \overline{\operatorname{span}}(w^*\text{-sexp}\, B_{X^*})$, because X^* contains no proper norming closed subspace. □

Corollary 4.31. *Let X be a Banach space satisfying the $M(a,B,c)$-inequality with $\max|B|+c>1$. If X has a metric (C)AI, then it is shrinking.*

Proof. Let $T_\nu x \to x$ for all $x \in X$ and $\sup\|T_\nu\| \leqslant 1$. By Lemma 4.30, $X^* = \overline{\operatorname{span}}(w^*\text{-sexp}\, B_{X^*})$, and therefore it suffices to show that $T_\nu^* x^* \to x^*$ whenever $x^* \in B_{X^*}$ is weakly* strongly exposed by some $x \neq 0$.

Let $x^* \in B_{X^*}$ be such a functional that there exists a non-null element $x \in X$ satisfying $\sup\limits_{f\in B_{X^*}} \operatorname{Re} f(x) - \operatorname{Re} x^*(x) < \alpha$ for all numbers $\alpha > 0$ and $\lim\limits_{\alpha\to 0} \operatorname{diam} S(x, B_{X^*}, \alpha) = 0$. Fix a number $\varepsilon > 0$, then the last convergence implies the existence of an $\alpha_\varepsilon > 0$ such that

$$y^* \in S(x, B_{X^*}, \alpha_\varepsilon) \quad \Rightarrow \quad \left\|y^* - x^*\right\| < \varepsilon \qquad \forall\, y^* \in X^*.$$

The convergence $(T_\nu^* x^*)(x) \to x^*(x)$ gives an index ν_0 such that

$$\nu \succcurlyeq \nu_0 \quad \Rightarrow \quad \left|\left(T_\nu^* x^*\right)(x) - x^*(x)\right| < \frac{\alpha_\varepsilon}{2}.$$

Hence if $\nu \succcurlyeq \nu_0$, we have

$$\begin{aligned}
\sup_{f\in B_{X^*}} \operatorname{Re} f(x) - \operatorname{Re}(T_\nu^* x^*)(x) &= \left(\sup_{f\in B_{X^*}} \operatorname{Re} f(x) - \operatorname{Re} x^*(x)\right) + \\
&\qquad + \left(\operatorname{Re} x^*(x) - \operatorname{Re}(T_\nu^* x^*)(x)\right) \leqslant \\
&\leqslant \frac{\alpha_\varepsilon}{2} + \left|x^*(x) - (T_\nu^* x^*)(x)\right| < \\
&< \alpha_\varepsilon,
\end{aligned}$$

yielding that $T_\nu^* x^* \in S(x, B_{X^*}, \alpha_\varepsilon)$. This gives $\left\|T_\nu^* x^* - x^*\right\| < \varepsilon$, as required. □

Corollary 4.31 may be applied in the following context.

Theorem 4.32. *Let X be a Banach space satisfying the $M(a,B,c)$-inequality and let $1 \leqslant \lambda < \max|B|+c$. If X has a λ-commuting bounded (C)AI, then X has a shrinking metric (C)AI. In particular, both X and X^* have the metric (C)AP.*

Proof. By Theorem 4.6, X has a shrinking (C)AI and, by Lemma 4.30, X^* has the Radon-Nikodým property. But it is well known (this is an extension of classical results of Grothendieck) that whenever a Banach space, the dual of which has the Radon-Nikodým property, has a shrinking (C)AI, it also has a shrinking metric (C)AI (see [GS, Corollary 1.6 and its proof] and [DU, p. 246] (for the case of AI)). □

If X is separable in Theorem 4.32, then X^* also is (because, by Lemma 4.30, X^* has the Radon-Nikodým property). But then both X and X^*, being separable, by the Casazza-Kalton theorem (Theorem 2.22), have the commuting metric AP whenever they have the metric AP. This proves the following result, which contains [G, Theorem VI.1] (as a special case when X is M-embedded), slightly improving it.

Theorem 4.33. *Let X be a separable Banach space satisfying the $M(a,B,c)$-inequality and let $1 \leqslant \lambda < \max|B| + c$. If X has a λ-commuting bounded AI, then both X and X^* have the commuting metric AP.*

The following result (an extension of [G, Corollary VI.2]) concerns the case when the metric AP passes from one space to the other satisfying the $M(a,B,c)$-inequality, if only the spaces do not lie too apart.

Corollary 4.34. *Let X be a separable Banach space satisfying the $M(a,B,c)$-inequality with $\max|B|+c > 1$. If there exists a Banach space Y with the metric AP such that $d_{BM}(X,Y) < \max|B|+c$, then both X and X^* have the commuting metric AP.*

Proof. The assumption $d_{BM}(X,Y) < \max|B| + c$ ensures the existence of λ, $1 \leqslant \lambda < \max|B| + c$, together with an isomorphism $\mathscr{J} : X \to Y$ for which $\|\mathscr{J}\| \left\|\mathscr{J}^{-1}\right\| < \lambda$. In particular, Y is separable. Therefore the Casazza-Kalton theorem (Theorem 2.22) gives that if Y has the metric AP, then Y has the commuting metric AP as well. Now suppose there exists a sequence (S_n) of finite-rank operators on Y such that $\limsup \|S_n\| \leqslant 1$, $S_n y \to y$ for all $y \in Y$, and $S_m S_n = S_n S_m$ for all m, n. Since the sequence of operators $T_n = \mathscr{J}^{-1} S_n \mathscr{J}$ is a λ-commuting bounded AI for X, Theorem 4.33 applies. □

4.6 Ideals and the $M(a,B,c)$-inequality for compact operators

We follow the definitions from [GKS].

Definition 4.35. A closed subspace $\mathscr{K}$ of a Banach space $\mathscr{L}$ is said to be an *ideal* in $\mathscr{L}$ if there exists a norm one projection P on $\mathscr{L}^*$ with $\ker P = \mathscr{K}^\perp = \{f \in \mathscr{L}^* : f|_{\mathscr{K}} = 0\}$. In this case, we shall say that P is an *ideal projection.*

Definition 4.36. If $\|Pf\| + \|f - Pf\| = \|f\|$ for all $f \in \mathscr{L}^*$, then $\mathscr{K}$ is called an *M-ideal* in $\mathscr{L}$.

The class of M-ideals has been extensively studied by many authors (see, e.g., the monograph [HWW] for results and references).

Definition 4.37. If there are $r,s \in (0,1]$ such that $r\|Pf\| + s\|f - Pf\| \leqslant \|f\|$ for all $f \in \mathscr{L}^*$, then we say that $\mathscr{K}$ is an $M(r,s)$-*ideal* in $\mathscr{L}$.

The $M(r,s)$-ideals of compact operators have been studied by several authors (see, e.g., [HOP] for references).

Definition 4.38. If $\|I - 2P\| = 1$, then $\mathscr{K}$ is called a u-*ideal* in $\mathscr{L}$. If $\|I - (1+\lambda)P\| = 1$ whenever $|\lambda| = 1$, then $\mathscr{K}$ is called an h-*ideal* in $\mathscr{L}$.

A deep study of u- and h-ideals was made in [GKS].

Finally, let us note that every Banach space $\mathscr{K}$ is an ideal in $\mathscr{K}^{**}$ with respect to the canonical projection $\pi_{\mathscr{K}}$ of $\mathscr{K}^{***}$ onto $\mathscr{K}^*$.

In the next section, we shall apply Theorem 4.32 to infer some new sufficient conditions for $\mathscr{K}(X)$ to be an M-, u-, or h-ideal in $\mathscr{L}(X)$. But now, we are going to use implicitly ideals of compact operators to show that in many natural cases, $\mathscr{K}(X)$ satisfies the $M(a,B,c)$-inequality with $\max|B| + c > 1$.

Proposition 4.39. *Let* $\max|B| > 1$. *If a Banach space* X *is reflexive and has a metric AI* (T_α) *such that*

$$\limsup_{\alpha} \|I_X + bT_\alpha\| \leqslant 1 \qquad \forall b \in B,$$

then $\mathscr{K}(X)$ *satisfies the* $M(1,B,0)$-*inequality.*

Proof. Since every AI of a reflexive space is shrinking, we get immediately from [O4, Corollary 4.5, 3°⇒1°] that $\mathscr{K}(X)$ is an ideal in $\mathscr{L}(X)$ with an ideal projection P such that $\|I_{\mathscr{L}(X)^*} + bP\| \leqslant 1$ for all $b \in B$. Since $\mathscr{K}(X)$ is an ideal in $\mathscr{L}(X)$ and X is reflexive, the ideal projection is unique (see [CNO, Proposition 3.2]). But by the well-known Grothendieck's classics (see, e.g., [DU, p. 247]), $\mathscr{L}(X) = \mathscr{K}(X)^{**}$ (because X is reflexive and has the AP). Therefore $P = \pi_{\mathscr{K}(X)}$ and $\mathscr{K}(X)$ satisfies the $M(1,B,0)$-inequality. ☐

Corollary 4.40. *If a Banach space* X *is reflexive and has a* 1-*unconditional basis, then* $\mathscr{K}(X)$ *satisfies the* $M(1,\{-2\},0)$-*inequality.*

Proof. Denoting the partial sum projections by (P_n), we have $\|P_n\| = 1$ and $\|I_X - P_n\| = 1$ for all indices n. Hence

$$\|I_X - 2P_n\| \leqslant \frac{1}{2}(\|I_X\| + \|I_X - P_n\|) \leqslant 1,$$

and it remains to apply Proposition 4.39. ☐

Reflexive Banach spaces having a 1-unconditional basis are, for instance, ℓ_p $(1<p<\infty)$, the Orlicz sequence space ℓ_M whenever both the Orlicz function M and its complementary function M^* satisfy the Δ_2-condition at zero (see, e.g., [LTz I, p. 148]), the Lorentz sequence space $d(w,p)$ $(1<p<\infty)$ (see, e.g., [LTz I, p. 178]).

4.7 Property $M^*(a,B,c)$ and ideals of compact operators

In this section we are going to use property $M^*(a,B,c)$, a structural property that was introduced in [O3] in order to give a uniform approach to several properties, including property (M^*), (wM^*) and others.

Let X be a Banach space.

Definition 4.41. It is said that X has *property* (M^*) if

$$\limsup_\nu \|x^* + x_\nu^*\| = \limsup_\nu \|y^* + x_\nu^*\|$$

for any functionals $x^*, y^* \in X^*$ where $\|x^*\| = \|y^*\|$ and for any bounded net $(x_\nu^*) \subset X^*$ converging weakly* to null.

The following Kalton-Werner-Lima-Oja theorem is a basic result of the theory of M-ideals of compact operators.

Theorem 4.42. *For a Banach space X, $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ if and only if X has property (M^*) and the metric CAP.*

The sequential version of property (M^*) was introduced by Kalton [K2]. For separable X, Theorem 4.42 was proven by Kalton and Werner [KW], a simpler proof was given in [L2]. For arbitrary (non-separable) X, the proof is due to Oja [O2]. Known shortest proof to Theorem 4.42 has been given in [O4], and a direct "non-separable" proof in [NP].

Definition 4.43. It is said that X has *property* (wM^*) if

$$\limsup_\nu \|2x^* - x_\nu^*\| = \limsup_\nu \|x_\nu^*\|$$

for any bounded net $(x_\nu^*) \subset X^*$ converging weakly* to $x^* \in X^*$.

Property (wM^*), a weak version of property (M^*), was introduced by Lima in [L2]. For a reflexive space X, Lima obtained a similar condition: $\mathscr{K}(X)$ is an u-ideal in $\mathscr{L}(X)$ if and only if X has property (wM^*) and the metric CAP.

Definition 4.44. It is said that a complex Banach space X has *complex property* (wM^*) if

$$\limsup_{\nu} \|x_\nu^* + bx^*\| \leq \limsup_{\nu} \|x_\nu^*\| \qquad \forall b \in \{b \in \mathbb{C} : |b+1| = 1\}$$

for any bounded net $(x_\nu^*) \subset X^*$ converging weakly* to $x^* \in X^*$.

Complex property (wM^*) was introduced by Oja in [O3].

In [O3] properties (M^*) and (wM^*) have been generalized as follows.

Definition 4.45. It is said that X has *property* $M^*(a,B,c)$ if

$$\limsup_{\nu} \|ax_\nu^* + bx^* + cy^*\| \leq \limsup_{\nu} \|x_\nu^*\| \qquad \forall b \in \{b \in \mathbb{C} : |b+1| = 1\}$$

for any functionals $x^*, y^* \in X^*$, $\|y^*\| \leq \|x^*\|$, and for any bounded net $(x_\nu^*) \subset X^*$ converging weakly* to x^*.

We can easily see that:

- property (M^*) is precisely property $M^*(1,\{-1\},1)$;
- property (wM^*) is precisely property $M^*(1,\{-2\},0)$;
- complex property (wM^*) is precisely property $M^*(1,\{b \in \mathbb{C} : |b+1| = 1\},0)$;
- property (M^*) implies property (wM^*);
- property (M^*) implies property $M^*(1,\{b \in \mathbb{C} : |b+1| \leq 1-c\},0)$ for every $c \in [0,1]$;
- for a complex Banach space X, property (M^*) implies complex property (wM^*);

The spaces $c_0(\Gamma)$ and $\ell_p(\Gamma)$, $1 < p < \infty$, satisfy property (M^*), but the Lorentz sequence spaces $d(w,p)$ do not. The Lorentz sequence spaces $d(w,p)$, $1 < p < \infty$, and, more generally, Banach spaces with a shrinking 1-unconditional basis have property (wM^*) and, in the case of complex scalars, the complex property (wM^*) (see [L2, Theorem 4.2] and [O4, Lemma 1.1]).

Property $M^*(a,B,c)$, hence also all its special cases, inherit to subspaces and quotient spaces (see [O4, Section 1]).

Proposition 4.46 ([O4, Proposition 1.3])**.** *If X has property $M^*(a,B,c)$, then X satisfies the $M(a,B,c)$-inequality.*

The proof of the proposition is essentially the same as the proof in [HWW, p. 298] of the fact that property (M^*) of X implies X being M-embedded.

Proof. Let X have property $M^*(a,B,c)$. Fix a $x^{***} \in X^{***}$ and $b \in B$. Let a positive number $\alpha < 1$ be chosen arbitrarily. Since

$$\left\| ax^{***} + b\pi_X x^{***} \right\| = \sup_{\|x^{**}\|=1} \operatorname{Re}\left(ax^{***} + b\pi_X x^{***} \right)(x^{**}),$$

we can find an element x^{**} from the unit sphere of X^{**} such that

$$\operatorname{Re}\left(ax^{***} + b\pi_X x^{***} \right)(x^{**}) > \alpha \left\| ax^{***} + b\pi_X x^{***} \right\|.$$

As $\left\| x^{**} \right\| = 1$, there exists a $x_0^* \in S_{X^*}$ such that $x^{**}\left(x_0^*\right) > \alpha$. Letting now $x^* = \left\| \pi_X x^{***} \right\| x_0^*$, we have $\left\| x^* \right\| = \left\| \pi_X x^{***} \right\|$ and

$$x^{**}(x^*) > \alpha \left\| \pi_X x^{***} \right\|. \tag{4.1}$$

Due to weakly* closedness of $\left\| x^{***} \right\| B_{X^{***}}$ (the Goldstine theorem) we find a net $\left(x_\nu^*\right) \subset X^*$ such that $j_{X^*} x_\nu^* \to w^{***}$ weakly* in X^{***} and $\left\| x_\nu^* \right\| \leqslant \left\| x^{***} \right\|$.

Since

$$x^{**}\left(ax_\nu^* + b(j_X)^* x^{***} \right) \to \left(ax^{***} + b\pi_X x^{***} \right)(x^{**}),$$

we may assume that the members of the net $\left(x_\nu^*\right)$ have been chosen such that

$$\operatorname{Re} x^{**}\left(ax_\nu^* + b(j_X)^* x^{***} \right) > \alpha \left\| ax^{***} + b\pi_X x^{***} \right\|. \tag{4.2}$$

What is more, we see that $x_\nu^* \to (j_X)^* x^{***}$ weakly* in X^*.

Putting together property $M^*(a,B,c)$ and the inequalities (4.1) and (4.2), we have

$$\begin{aligned}
\left\| x^{***} \right\| &\geqslant \limsup_\nu \left\| x_\nu^* \right\| \geqslant \\
&\geqslant \limsup_\nu \left\| ax_\nu^* + b\left(j_X\right)^* x^{***} + c\left(j_X\right)^* x^{***} \right\| \geqslant \\
&\geqslant \limsup_\nu \operatorname{Re} x^{**}\left(ax_\nu^* + b\left(j_X\right)^* x^{***} + c\left(j_X\right)^* x^{***} \right) \geqslant \\
&\geqslant \alpha\left(\left\| ax^{***} + b\pi_X x^{***} \right\| + \left\| c\pi_X x^{***} \right\| \right),
\end{aligned} \tag{4.3}$$

Going $\alpha \uparrow 1$ in (4.3), we obtain

$$\left\| ax^{***} + b\pi_X x^{***} \right\| + c\left\| \pi_X x^{***} \right\| \leqslant \left\| x^{***} \right\|$$

as required. $\square$

For an example showing that the $M(a,B,c)$-inequality need not imply property $M^*(a,B,c)$, we refer to [OZ1, Example 4.15].

Relying on Theorem 4.32 and the main result of [O4], we shall see in the next Theorem 4.47 that this stronger property ensures the existence of a shrinking metric (C)AI having certain important features.

Theorem 4.47. *Let X be a Banach space having property $M^*(a,B,c)$ with $\max|B|+c>1$ and let $1\leqslant\lambda<\max|B|+c$. If X has a λ-commuting bounded (C)AI, then for any $S\in B_{\mathcal{K}(X)}$, there exists a shrinking metric (C)AI (T_α) of X such that*

$$\limsup_{\alpha}\|aI_X+bT_\alpha+cS\|\leqslant 1\qquad\forall b\in B.$$

Proof. Property $M^*(a,B,c)$ of X implies that X satisfies the $M(a,B,c)$-inequality (see Proposition 4.46). Hence, by Theorem 4.32, X has the metric (C)AP. But then, since X has property $M^*(a,B,c)$ with $\max|B|+c>1$, by [O4, Theorem 3.5, $2^\circ\Rightarrow1^\circ$], we have (T_α) as desired. □

In contrast to the previous results of this chapter which are mostly interesting for non-reflexive spaces, Theorem 4.47 and the corollaries below are also interesting for reflexive spaces. All these results are new even in the separable case. The conclusion of Theorem 4.47 was known before to hold under the assumptions of property $M^*(a,B,c)$ and the metric (C)AP (see [O4, Theorem 3.5]). We do not know whether Theorem 4.47 holds without the commutativity assumption.

We shall now apply Theorem 4.47 to particular cases of property $M^*(a,B,c)$. Our first application shows that the metric CAI in the criterion when $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ can be replaced by a λ-commuting bounded CAI with $\lambda<2$.

Corollary 4.48. *Let X be a Banach space having property (M^*) and let $1\leqslant\lambda<2$. If X has a λ-commuting bounded CAI, then $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$.*

Proof. By Theorem 4.47, for any $S\in\mathcal{K}(X)$, $\|S\|\leqslant1$, there exists a shrinking metric CAI (T_α) of X such that

$$\limsup_{\alpha}\|I_X-T_\alpha+S\|\leqslant 1.$$

It remains to apply [O4, Corollary 4.3, $7^\circ\Rightarrow1^\circ$]. □

In the next definition, we follow [LO1].

Definition 4.49. A closed subspace $\mathscr{K}$ of a Banach space $\mathscr{L}$ is said to have the *unique ideal property* if there is at most one ideal projection, that is, at most one norm one projection P on $\mathscr{L}^*$ with $\ker P = \mathscr{K}^{\perp}$.

It is well known that M-ideals have the unique ideal property. By [CNO, Proposition 3.2], $\mathscr{K}(X)$ has the unique ideal property in any subspace $\mathscr{L}$ of $\mathscr{L}(X)$ containing $\mathscr{K}(X)$ and I_X, provided X^* has the Radon-Nikodým property and $X^* = \overline{\operatorname{span}}(w^*\text{-sexp}\, B_{X^*})$. Due to [O4, Corollary 1.6], this is the case in the corollaries below.

Corollary 4.50. *Let X be a Banach space having property $M^*(s,\{-s\},r)$ with $r+s>1$, $r,s\in(0,1]$, and let $1\leqslant\lambda<r+s$. If X has a λ-commuting bounded CAI, then $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{I}(X) := \operatorname{span}(\mathscr{K}(X)\cup\{I_X\})$ having the unique ideal property.*

Proof. By Theorem 4.32, X has the metric CAP. Moreover, in [O4, Remark 2 on p. 2818] it is proven that property $M^*(s,\{-s\},r)$ implies the following stronger property. For all $T\in B_{\mathscr{I}(X)}$, one has

$$\limsup_{\nu}\|sT^*(x_\nu^*-x^*)+ry^*\|\leqslant\limsup_{\nu}\|x_\nu^*\|$$

whenever $x^*,y^*\in X^*$ satisfy $\|y^*\|\leqslant\|x^*\|$, and (x_ν^*) is a bounded net converging weakly* to x^* in X^*. Therefore it only remains to apply [O4, Theorem 4.1, $2^\circ\Rightarrow1^\circ$]. ☐

Remark 4.51. In the special case of property $(M^*) = M(1,\{-1\},1)$, Corollary 4.50 yields that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{I}(X)$. By a theorem due to Kalton [K2] (established in [K2] for separable X and extended to arbitrary (non-separable) X in [O1]) (see, e.g. [HWW, p. 299]), $\mathscr{K}(X)$ is an M-ideal in $\mathscr{I}(X)$ if and only if $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$. Thus we regain Corollary 4.48. We do not know whether $\mathscr{I}(X)$ in Corollary 4.50 can be replaced by $\mathscr{L}(X)$. It is not known whether $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{L}(X)$ whenever $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{I}(X)$. Notice that in [HJO], it is proven that if $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{I}(X)$, then it is an $M(r^2,s^2)$-ideal in $\mathscr{L}(X)$.

Corollary 4.52. *Let X be a Banach space having property $M^*(a,B,0)$ with $\max|B|>1$, and let $1\leqslant\lambda<\max|B|$. If X has a λ-commuting bounded CAI, then $\mathscr{K}(X)$ is an ideal in $\mathscr{L}(X)$ having the unique ideal property and the ideal projection P satisfies $\|aI_{\mathscr{L}(X)^*}+bP\|\leqslant1$ for all $b\in B$.*

Proof. By Theorem 4.47, (for $S=0$) there exists a shrinking metric CAI (T_α) of X such that

$$\limsup_{\alpha}\|aI_X+bT_\alpha\|\leqslant1\qquad\forall b\in B.$$

It remains to apply [O4, Corollary 4.5, 3°⇒1°] □

The most important particular cases of Corollary 4.52 are those concerning u-ideals and h-ideals of compact operators.

Corollary 4.53. *Let X be a Banach space having property (wM^*) and let $1 \leqslant \lambda < 2$. If X has a λ-commuting bounded CAI, then $\mathcal{K}(X)$ is an u-ideal in $\mathcal{L}(X)$ having the unique ideal property.*

Corollary 4.54. *Let X be a complex Banach space having complex property (wM^*) and let $1 \leqslant \lambda < 2$. If X has a λ-commuting bounded CAI, then $\mathcal{K}(X)$ is an h-ideal in $\mathcal{L}(X)$ having the unique ideal property.*

Chapter 5

The Johnson-Schechtman space has the commuting 6-bounded approximation property

In this chapter we show that the closed subspace X_{JS} of c_0 constructed by Johnson and Schechtman in 1996 has the commuting λ-bounded AP with $\lambda \leqslant 6$. This slightly improves the proof by Godefroy [G, Theorem VI.3] in 2001 where it was established that $\lambda \leqslant 8$. The chapter is based on [Z].

5.1 The setting

It is a well-known result of Grothendieck [Gro, Chapter I, "Proposition" 37] that if there exists a Banach space which fails the AP, then there also exists a closed subspace of c_0 that fails the AP (see, e.g., [LTz I, p. 37]). Hence, relying on Enflo's theorem [E], let $Y = \overline{\cup_n Y_n}$ be a closed subspace of c_0 failing the AP, where (Y_n) is an increasing sequence of finite-dimensional subspaces of Y. We denote by $c(Y_n)$ and $c_0(Y_n)$ the Banach spaces of norm-convergent sequences and norm-decaying sequences $(y_n) \subset Y$, respectively, where $y_n \in Y_n$, $n \in \mathbb{N}$, with respect to the supremum norm. It is clear that $c_0(Y_n)$ is a closed subspace of $c(Y_n)$.

The *Johnson-Schechtman space* X_{JS} (constructed by Johnson and Schechtman in 1996 and published in [JO]) is an isomorphic copy of $c(Y_n)$ in c_0. The key points of the construction are the Sobczyk theorem (Theorem 5.1) and [JZ, p.

51, observation of Lindenstrauss]: *if a Banach space X has a closed subspace Y so that both the subspace Y and the quotient space X/Y embed isomorphically into c_0, then so does X itself.*

We denote the coordinate functionals on c_0 by e_k^*, $k \in \mathbb{N}$.

Theorem 5.1 ([S]; see also [G, Theorem II.1])**.** *Let X be a separable Banach space, and Y a closed subspace of X. Let $T \in \mathscr{L}(Y, c_0)$. Then there exists an operator $\tilde{T} \in \mathscr{L}(X, c_0)$ such that*

1) $\tilde{T}x = \left(\left(x_k^* - t_k^*\right)(x)\right)_k$, *where x_k^* are Hahn-Banach extensions of functionals $y_k^* := T^* e_k^* \in Y^*$;*
2) $x_k^*, t_k^* \in \|T\| B_{X^*}$, $k \in \mathbb{N}$*;*
3) t_k^* *is null on Y for all $k \in \mathbb{N}$;*
4) $\tilde{T}|_Y = T$*;*
5) $\|\tilde{T}\| \leqslant 2\|T\|$.

Godefroy has proven in [G, Theorem VI.3] that X_{JS} has a finite-dimensional decomposition with the decomposition constant not exceeding 8. He wrote in [G, Ch. VII, §VI] that no effort had been made in the proof to tighten the constant and it is unlikely that 8 were the critical value. The main aim of this chapter is to tighten the constant to 6.

5.2 The main result

The following – the main result of this chapter – is a slight improvement of [G, Theorem VI.3].

Theorem 5.2. *The Johnson-Schechtman space X_{JS} has a finite-dimensional decomposition with the decomposition constant not greater than 6, but X_{JS} fails the metric AP.*

The proof in [G] goes in two parts: first the construction of X_{JS} and the finite-dimensional decomposition, and second, showing that X_{JS} fails the metric AP. We need to go through only the first part. For the second part, we refer the reader to [G, p. 21].

Proof. Let $Y = \overline{\cup_n Y_n}$ be a closed subspace of c_0 failing the AP, $\dim Y_n < \infty$, $n \in \mathbb{N}$, and $Y_1 \subset Y_2 \subset \ldots$. We define a quotient map $L: c(Y_n) \to Y$ by $L(y_n) = \lim_n y_n$, thus $\ker L = c_0(Y_n)$, yielding that $Y = c(Y_n)/c_0(Y_n)$.

We take an isometric embedding $T : c_0(Y_n) \to c_0$. For instance, having $(y_n) \in c_0(Y_n)$, where $y_n = \left(\xi_n^k\right)_k \in c_0$, we can define

$$T(y_n) = \left(\xi_1^1,\ \xi_2^1, \xi_1^2,\ \xi_3^1, \xi_2^2, \xi_1^3,\ \xi_4^1, \xi_3^2, \xi_2^3, \xi_1^4,\ \dots\right).$$

It is straightforward to verify that $T \in \mathscr{L}(c_0(Y_n), c_0)$ and T is isometric.

We shall use Theorem 5.1 in the situation when $X = c(Y_n)$ and $Y = c_0(Y_n)$. Theorem 5.1 allows to extend the operator T to an operator $\tilde{T} \in \mathscr{L}(c(Y_n), c_0)$ such that $\tilde{T}|_{c_0(Y_n)} = T$ and $\|\tilde{T}\| \leqslant 2\|T\| = 2$. We also need the expression of $\tilde{T}(y_n) = \left(\left(x_k^* - t_k^*\right)(y_n)\right)_k$, where x_k^* are Hahn-Banach extensions of functionals $y_k^* = T^* e_k^* \in c_0(Y_n)^*$, also $x_k^*, t_k^* \in \|T\| B_{c(Y_n)^*}$ and t_k^* is null on $c_0(Y_n)$ for all k.

Now the subspace $c_0(Y_n)$ and the quotient space Y of $c(Y_n)$ isomorphically embed into c_0. It can be easily verified that the operator $V : c(Y_n) \to c_0 \oplus_\infty c_0 \cong c_0$, defined by $V(y_n) = \left(\tilde{T}(y_n), L(y_n)\right)$, is an isomorphism into c_0 with $\|V\| \leqslant 2$.

The next step in [G, proof of Theorem VI.3] yields $\left\|V^{-1}|_{\operatorname{ran} V}\right\| \leqslant 4$; we shall present an argument that gives $\left\|V^{-1}|_{\operatorname{ran} V}\right\| \leqslant 3$.

Assume that $\left\|V^{-1}|_{\operatorname{ran} V}\right\| > 3$. As

$$\left\|V^{-1}|_{\operatorname{ran} V}\right\| = \sup_{\substack{\|\tilde{T}(y_n)\|<1 \\ \lim\limits_n \|y_n\|<1}} \left\|(y_n)\right\|,$$

there exists a sequence $(y_n) \in c(Y_n)$ such that $\left\|(y_n)\right\| > 3$, $\lim\limits_n \left\|y_n\right\| < 1$ and $\left\|\tilde{T}(y_n)\right\| < 1$. Let $N \in \mathbb{N}$ be an index such that $\sup\limits_{n \geqslant N} \left\|y_n\right\| < 1$. Split (y_n) into two parts: $\left(y_n^0\right) = \left(y_1, \dots, y_{N-1}, 0, 0, \dots\right)$ and $\left(y_n^1\right) = (y_n) - \left(y_n^0\right)$. Of course $\left\|\left(y_n^0\right)\right\| > 3$, $\left(y_n^0\right) \in c_0(Y_n)$ and $\left\|\left(y_n^1\right)\right\| < 1$.

Due to the inequality $\sup\limits_k \left|y_k^*\left(y_n^0\right)\right| = \left\|T\left(y_n^0\right)\right\| = \left\|\left(y_n^0\right)\right\| > 3$, we find an index $m \in \mathbb{N}$ for which $\left|y_m^*\left(y_n^0\right)\right| > 3$. As $\sup\limits_k \left|x_k^*((y_n)) - t_k^*((y_n))\right| = \left\|\tilde{T}(y_n)\right\| < 1$, we also have the inequality $\left|x_m^*((y_n)) - t_m^*((y_n))\right| < 1$. Bearing in mind that $t_m^*\left((y_n)\right) = t_m^*\left(\left(y_n^1\right)\right)$, we have

$$\begin{aligned}
\left|x_m^*((y_n))\right| + 1 &\leqslant \left|x_m^*((y_n)) - t_m^*((y_n))\right| + \left|t_m^*((y_n^1))\right| + 1 < \\
&< 2 + \left\|t_m^*\right\| \left\|\left(y_n^1\right)\right\| < 3 < \left|y_m^*((y_n^0))\right| = \\
&= \left|x_m^*((y_n^0))\right| \leqslant \left|x_m^*((y_n^0)) + x_m^*((y_n^1))\right| + \left|x_m^*((y_n^1))\right| \leqslant \\
&\leqslant \left|x_m^*((y_n))\right| + \left\|x_m^*\right\| \left\|\left(y_n^1\right)\right\| < \left|x_m^*((y_n))\right| + 1,
\end{aligned}$$

a contradiction. Therefore $\left\|V^{-1}|_{\operatorname{ran} V}\right\| \leqslant 3$.

We denote $X_{JS} = \operatorname{ran} V$ and $P_m = VQ_mV^{-1}$, where $Q_m(y_n) = (y_1,\dots,y_{m-1},y_m,y_m,\dots)$, $m \in \mathbb{N}$. It is straightforward to verify that (P_m) is a finite-dimensional decomposition of X_{JS} and $\sup_m \|P_m\| \leqslant 6$. □

Before making a remark, we need to go through some definitions.

Definition 5.3. A class of operators $\mathscr{A}$, $\mathscr{F} \subset \mathscr{A} \subset \mathscr{L}$, where $\mathscr{F}$ and $\mathscr{L}$ consist of all finite-rank and bounded linear operators, respectively, is said to be a *Banach operator ideal* if for all Banach spaces X and Y the following conditions hold:

1) the *component* $\mathscr{A}(X,Y) = \mathscr{A} \cap \mathscr{L}(X,Y)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathscr{A}}$,
2) for every $x^* \in X$ and $y \in Y$ the one-dimensional operator $T \in \mathscr{F}(X,Y)$, $Tx = x^*(x)y$, $x \in X$, satisfies $\|T\|_{\mathscr{A}} = \|x^*\| \cdot \|y\|$,
3) for every $A \in \mathscr{L}$, $T \in \mathscr{A}$, $B \in \mathscr{L}$ we have $BTA \in \mathscr{A}$ and $\|BTA\|_{\mathscr{A}} \leqslant \|B\| \cdot \|T\|_{\mathscr{A}} \cdot \|A\|$.

Definition 5.4. Let $T \in \mathscr{L}(X,Y)$. The operator T is said to be *weakly compact* if $T(B_X)$ is relatively weakly compact.

The operator T is said to be *strictly singular* if for any infinite-dimensional subspace $Z \subset X$ and every $\varepsilon > 0$ there exists an element $z \in Z$ such that $\|Tz\| < \varepsilon\|z\|$.

The operator T is said to be *completely continuous* if it maps every weakly convergent sequence to a norm-convergent sequence.

The operator ideal norms with respect to the Banach operator ideals of weakly compact operators, strictly singular operators, and completely continuous operators coincide with the usual operator norm.

Definition 5.5. A Banach space X is said to have the *metric $\mathscr{A}$-approximation property* if for every compact set $K \subset X$ and every $\varepsilon > 0$ there exists an operator $T \in \mathscr{A}(X)$ such that $\|Tx - x\| < \varepsilon$ for all $x \in K$.

Remark 5.6. By [O6, Corollary 2.5 and Remark 2.4], X_{JS} fails the metric $\mathscr{A}$-approximation property for any operator ideal $\mathscr{A}$ which is contained in the union of weakly compact, strictly singular, and completely continuous operators.

Remark 5.7. By [O7, Corollary 3.8], there exist a separable reflexive Banach space Z and a compact linear operator $T : X_{JS} \to Z$ such that for every net (T_α) of finite-rank operators from X_{JS} to Z converging strongly to T, there holds $\sup_\alpha \|T_\alpha\| > \|T\|$; in particular, X_{JS} fails the weak metric approximation property (see [LO2]).

Since a finite-dimensional decomposition with constant λ implies the commuting λ-bounded AP, the following corollary is immediate.

Corollary 5.8. *The Johnson-Schechtman space X_{JS} has the commuting λ-bounded AP with $\lambda \leqslant 6$.*

Note that the proof of Theorem 5.2 is useful for any closed subspace of c_0 as a starting point, yielding a finite-dimensional decomposition with the decomposition constant not greater than 6 on the constructed space.

Since a finite-dimensional decomposition with constant λ implies the λ-commuting bounded AP (see also Theorem 2.26), every Banach space constructed in this manner has the commuting bounded AP, hence this construction cannot provide any information on a well-known open problem whether every Banach space with the bounded AP has the commuting bounded AP (see remarks after Theorem 2.22).

5.3 Applications

It was already defined in Chapter 4 (see Definition 4.1) that X is called *M-embedded* if the canonical projection π_X from X^{***} onto X^* satisfies the inequality

$$\|x^{***} - \pi_X x^{***}\| + \|\pi_X x^{***}\| \leqslant \|x^{***}\|, \qquad x^{***} \in X^{***}.$$

M-embeddedness inherits to closed subspaces and quotient spaces (see Proposition 4.3). A well-known example of an M-embedded Banach space is c_0. Therefore also X_{JS} is M-embedded.

The following result is a special case of Corollary 4.34.

Theorem 5.9. *Let X be a separable M-embedded space. If there exists a Banach space Y with the metric AP such that $d_{BM}(X,Y) < 2$, then X has the metric AP.*

Merging the last result (note that it also applies to X_{JS}) with the facts that $V : c(Y_n) \to X_{JS}$ is an isomorphism, $\|V\| \left\|V^{-1}\right\| \leqslant 6$, and $c(Y_n)$ has the metric AP, we have

Corollary 5.10. *For every Banach space Y with the metric AP, there holds $d_{BM}(X_{JS}, Y) \geqslant 2$. On the other hand, there exists a Banach space Y with the metric AP for which $d_{BM}(X_{JS}, Y) \leqslant 6$.*

The question which is the greatest value of λ that would guarantee the metric AP to pass over from a Banach space Y to any separable M-embedded space X with $d_{BM}(X, Y) < \lambda$, is yet open.

Chapter 6

Asymptotically commuting bounded approximation property

This chapter coins a new term: the asymptotically commuting bounded approximation property. The main result is the following: if a Banach space has the asymptotically λ-commuting bounded approximation property, then it has a strong form of the separable local λ-complementation property. The chapter is based on [OZ2].

6.1 The concept

Definition 6.1. We say that a Banach space X has the *asymptotically λ-commuting bounded approximation property* if there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that

1) $S_\alpha x \to x$ for every element $x \in X$;
2) $\limsup_\alpha \|S_\alpha\| \leqslant \lambda$;
3) $\lim_\alpha \left\| S_\alpha S_\beta - S_\beta S_\alpha \right\| = 0$ for all indices β.

A net of operators (S_α) satisfying these conditions is called an *asymptotically λ-commuting bounded approximation of the identity*. (See also Remark 2.4.)

Recall that the definition of the λ-commuting bounded (C)AP is alike, only the third condition is stronger: $S_\alpha S_\beta = S_\beta S_\alpha$ for all indices α, β. Hence, the λ-

commuting bounded (C)AP trivially implies the asymptotically λ-commuting bounded (C)AP.

Proposition 6.2. *Let X be a Banach space. If the dual space X^* has the λ-bounded AP, then both X and X^* enjoy the asymptotically λ-commuting bounded AP.*

Proof. It is a known fact (see, e.g., [C2, Proposition 3.5]) that if a dual space X^* has the λ-bounded AP, then X^* has the *λ-duality bounded AP*, i.e. we can find a net $(S_\alpha) \subset \mathscr{F}(X)$ such that

1) $S_\alpha x \to x$ for every element $x \in X$;
2) $S_\alpha^* x^* \to x^*$ for every element $x^* \in X^*$;
3) $\|S_\alpha\| = \left\|S_\alpha^*\right\| \leqslant \lambda$ for all indices α.

Using this fact, we find a net $(S_\alpha) \subset \mathscr{F}(X)$ (then also $(S_\alpha^*) \subset \mathscr{F}(X^*)$) such that $S_\alpha x \to x$, $S_\alpha^* x^* \to x^*$ where the convergences are uniform on compact sets, and $\|S_\alpha\| = \left\|S_\alpha^*\right\| \leqslant \lambda$. The justification for the uniformness of convergences on compact sets is similar to that of in Chapter 2, proof of Proposition 2.7, namely due to the Hausdorff theorem we choose a finite ε-net on the compact set and approximate all the elements of the net well enough.

Therefore

$$\left\|S_\alpha S_\beta - S_\beta\right\| = \sup_{x \in B_X} \left\|(S_\alpha - I)\left(S_\beta x\right)\right\| = \sup_{y \in S_\beta(B_X)} \left\|(S_\alpha - I)\, y\right\| \to 0$$

since a finite-rank, hence a compact operator S_β maps the unit ball B_X to a relatively compact set. Similarly

$$\left\|S_\alpha^* S_\beta^* - S_\beta^*\right\| = \sup_{x^* \in B_{X^*}} \left\|\left(S_\alpha^* - I\right)\left(S_\beta^* x\right)\right\| = \sup_{y^* \in S_\beta^*(B_{X^*})} \left\|\left(S_\alpha^* - I\right) y\right\| \to 0.$$

Now for all indices β we have

$$\begin{aligned} \left\|S_\alpha S_\beta - S_\beta S_\alpha\right\| &\leqslant \left\|S_\alpha S_\beta - S_\beta\right\| + \left\|S_\beta - S_\beta S_\alpha\right\| = \\ &= \left\|S_\alpha S_\beta - S_\beta\right\| + \left\|S_\alpha^* S_\beta^* - S_\beta^*\right\| \xrightarrow[\alpha]{} 0. \quad \square \end{aligned}$$

Proposition 6.2 applies, among others, to ℓ_∞. Namely, ℓ_∞^* has the metric AP, hence by Proposition 6.2, ℓ_∞ has the asymptotically commuting bounded AP. In [C2, Corollary 9.4], it has been asserted that ℓ_∞ does not have the commuting bounded AP. This, however, seems to be an open problem whether ℓ_∞ has the commuting bounded AP or not (see Remark 6.11).

The aim of the following result is to say that in the case of separable Banach spaces, the asymptotically commuting bounded AP gives nothing new, it coincides with the commuting bounded AP. Therefore, the difference from the commuting bounded AP may arise only in the case of non-separable Banach spaces.

Proposition 6.3. *For a separable Banach space X the λ-commuting bounded AP and the asymptotically λ-commuting bounded AP are equivalent properties. If a separable Banach space X is a dual space, then X has the asymptotically λ-commuting bounded AP if and only if X has the metric AP if and only if X has the commuting metric AP.*

Proof. It has been proven in [CK, Corollary 2.3] that a separable Banach space X has the λ-commuting bounded AP if and only if X has a λ-bounded AI $(T_n) \subset \mathscr{F}(X)$ such that $\lim_n \|T_n T_m - T_m T_n\| = 0$ for every m. The proof in detail for this result has also been written out in Chapter 2, Corollary 2.21.

It is a known fact that for a separable dual space Y^* the AP and the metric AP coincide (see, e.g., [C2, Theorem 3.6]). Hence, if a separable Banach space $X = Y^*$ has the asymptotically λ-commuting bounded AP, it has the bounded AP, therefore the AP, hence the metric AP. By a famous result by Casazza and Kalton (see Theorem 2.22), for a separable Banach space the metric AP and the commuting metric AP coincide. Finally, if X has the commuting metric AP, it also has the asymptotically λ-commuting bounded AP for any λ. $\square$

6.2 The separable (local) complementation property

Recall that a closed subspace Y of a Banach space X is *complemented* in X if there exists a projection $P \in \mathscr{L}(X)$ onto Y, i.e. $P^2 = P$ and $\operatorname{ran} P = Y$. For the main result of this chapter, we shall need the following definition.

Definition 6.4. It is said that a closed subspace Y of a Banach space X is *locally complemented* in X if there exists a constant $\lambda \geqslant 1$ such that whenever E is a finite-dimensional subspace of X and a number $\varepsilon > 0$, there exists a linear operator $T : E \to Y$ with $Tx = x$ for all $x \in E \cap Y$ and $\|T\| \leqslant \lambda + \varepsilon$. If λ works, then it is said that Y is *locally λ-complemented* in X.

It is clear that a complemented subspace is locally complemented. Indeed, let Y be a complemented subspace of X, with the projection P. Fix a finite-dimensional subspace E of X and a number $\varepsilon > 0$. We define $T = P|_E$. Then

$T \in \mathscr{L}(E, Y)$. If $x \in E \cap Y$, we have $x = Px$ since P is a projection and $Y = \operatorname{ran} P$. It is also clear that $\|P|_E\| \leqslant \|P\|$. Thus the working λ is $\|P\|$ and we conclude that Y is locally $\|P\|$-complemented.

Definition 6.5. Let X be a Banach space and let Y be a closed subspace of X. It is said that an operator $\Phi \in \mathscr{L}(Y^*, X^*)$ is an *extension operator* if $\left(\Phi y^*\right)|_Y = y^*$ for every $y^* \in Y^*$, i.e.

$$\left(\Phi y^*\right)(y) = y^*(y), \qquad y \in Y, \qquad y^* \in Y^*.$$

It has been proven independently by Fakhoury [Fak] and Kalton [K1] that the existence of an extension operator ensures the local complementation of a closed subspace. We quote an improvement from [OP].

Proposition 6.6 ([OP, Corollary 3.3])**.** *Let X be a Banach space and let Y be a closed subspace of X such that there is an extension operator $\Phi \in \mathscr{L}(Y^*, X^*)$. Let $E \subset X$ and $F \subset Y^*$ be finite-dimensional subspaces, and let $\varepsilon > 0$. Then there is a linear operator $T : E \to Y$ such that $Ty = y$ for all $y \in E \cap Y$, $y^*(Tx) = \Phi y^*(x)$ for all $x \in E$ and $y^* \in F$, and $\|T\| \leqslant \|\Phi\| + \varepsilon$. In particular, Y is locally $\|\Phi\|$-complemented in X.*

Definition 6.7. A non-separable Banach space X is said to have the *separable complementation property* if for every separable closed subspace Y in X, there is a separable closed subspace Z with $Y \subset Z \subset X$ and Z is complemented in X.

"Definition" 6.8. We say that a non-separable Banach space X has the *separable local λ-complementation property* if for every separable closed subspace Y in X, there is a separable closed subspace Z with $Y \subset Z \subset X$ and Z is locally λ-complemented in X. If the value of λ is not important, we say that X has the *separable local complementation property.*

We are using the quotation marks in the latter definition, since, by [HM] or [SY], every non-separable Banach space X has the separable local 1-complementation property.

6.3 The main result

The following is the main result of this chapter. For short, it states that if a Banach space has the asymptotically λ-commuting bounded AP, it has a strong version of the separable local λ-complementation property. Among other spaces, the result applies to ℓ_∞.

It is worth emphasizing (see Remark 6.11) that it is not clear whether the commuting bounded AP for a non-separable Banach space implies the separable complementation property (as claimed in [C2, Theorem 9.3]).

Theorem 6.9. *Let X be a Banach space with the asymptotically λ-commuting bounded AP. Let Y be a separable closed subspace of X. Then there exists a separable closed subspace Z of X such that $Y \subset Z$, and there exists a sequence $(R_n) \subset \mathscr{F}(X,Z)$ such that the following conditions hold:*

(i) $R_n R_m = R_m R_n = R_m$ *whenever* $n > m$;

(ii) $\operatorname{ran} R_n = \operatorname{ran} R_n^2$ *for every* n;

(iii) $\limsup\limits_n \|R_n\| \leqslant \lambda$;

(iv) *the sequence* $(R_n|_Z) \subset \mathscr{F}(Z)$ *is a* λ*-commuting bounded AI on* Z *(hence* Z *has the* λ*-commuting bounded AP);*

(v) Z *is locally* λ*-complemented in* X.

For the proof, we shall construct sequences of operators, making their properties subsequently better and better. The last sequence to be constructed will be (R_n) that meets all the claims of the theorem.

Construction of (T_n) and Z

Assume that $Y = \overline{\{y_1, y_2, \ldots\}}$. Choose a decaying sequence of positive reals (ε_n) such that $\varepsilon_n \leqslant 1$ for every n and $\sum\limits_n \varepsilon_n < \infty$.

Denote $E_1 = \operatorname{span}\{y_1\}$ and $d_1 = \dim E_1$. Let $P_1 \in \mathscr{L}(X)$ be a projection onto E_1; such a projection exists due to Auerbach Lemma where it is also established that $\|P_1\| \leqslant d_1$.

The unit ball B_{E_1} is compact (since E_1 is finite-dimensional), hence the assumption of the theorem ensures the existence of an operator $S_{\alpha_1} \in \mathscr{F}(X)$ such that $\left\|S_{\alpha_1} x - x\right\| \leqslant \dfrac{\varepsilon_1}{2d_1}$ for every $x \in B_{E_1}$ and $\left\|S_{\alpha_1}\right\| \leqslant \lambda + \dfrac{\varepsilon_1}{2}$.

Denote

$$T_1 = S_{\alpha_1} + P_1 - S_{\alpha_1} P_1.$$

The operator T_1 is the identity operator on E_1. Indeed, if $x \in E_1$, i.e. $x = P_1 y$ for some y, then $T_1 x = S_{\alpha_1}(P_1 y) + P_1(P_1 y) - S_{\alpha_1} P_1(P_1 y) = P_1 y = x$. What is more,

$$\begin{aligned} \left\| T_1 - S_{\alpha_1} \right\| &= \left\| S_{\alpha_1} P_1 - P_1 \right\| = \sup_{y \in P_1(B_X)} \left\| S_{\alpha_1} y - y \right\| \leqslant \\ &\leqslant \sup_{y \in \|P_1\| B_{E_1}} \left\| S_{\alpha_1} y - y \right\| \leqslant \|P_1\| \cdot \frac{\varepsilon_1}{2d_1} = \frac{\varepsilon_1}{2}. \end{aligned}$$

Now assume we have operators $S_{\alpha_1}, \ldots, S_{\alpha_n} \in \mathscr{F}(X)$, and $T_1, \ldots, T_n \in \mathscr{F}(X)$ such that

(a) $T_n y_k = y_k$ for every $k = 1, \ldots, n$,

(b) $T_n T_m = T_m$ for every $m = 1, \ldots, n-1$,

(c) $\left\| T_m - S_{\alpha_m} \right\| \leqslant \dfrac{\varepsilon_m}{2}$, $m = 1, \ldots, n$,

(d) $\left\| S_{\alpha_m} \right\| \leqslant \lambda + \dfrac{\varepsilon_m}{2}$, $m = 1, \ldots, n$,

(e) $\|T_n T_m - T_m T_n\| \leqslant 2(\lambda + 1)(\varepsilon_n + \varepsilon_m)$ for every $m = 1, \ldots, n-1$.

It is clear from the constraint on (ε_n) that the last condition also implies

$$\sum_n \|T_n T_{n+1} - T_{n+1} T_n\| < \infty.$$

We construct an operator T_{n+1} such that the similar conditions hold.

Denote

$$E_{n+1} = \operatorname{span}\left(\{y_1, \ldots, y_{n+1}\} \cup \bigcup_{m=1}^{n} \operatorname{ran} T_m \right).$$

Then E_{n+1} is finite-dimensional, let $d_{n+1} = \dim E_{n+1}$. Denote by $P_{n+1} : X \to E_{n+1}$ a projection onto E_{n+1} such that $\|P_{n+1}\| \leqslant d_{n+1}$.

The unit ball $B_{E_{n+1}}$ is compact. The asymptotically λ-commuting bounded AI (S_α) gives an index α' such that

$$\alpha \succcurlyeq \alpha' \Rightarrow \begin{cases} \left\| S_{\alpha_m} S_\alpha - S_\alpha S_{\alpha_m} \right\| \leqslant \varepsilon_{n+1} & \forall m = 1, \ldots, n, \\ \|S_\alpha x - x\| \leqslant \dfrac{\varepsilon_{n+1}}{2d_{n+1}} & \forall x \in B_{E_{n+1}}. \end{cases}$$

We choose α_{n+1} such that these two conditions as well as (d), i.e.

$$\left\| S_{\alpha_{n+1}} \right\| \leqslant \lambda + \frac{\varepsilon_{n+1}}{2}$$

hold.

Denote

$$T_{n+1} = S_{\alpha_{n+1}} + P_{n+1} - S_{\alpha_{n+1}} P_{n+1}.$$

We shall verify that the conditions (a), (b), (c), (e) hold for T_{n+1}.

(a), (b): For every $x \in E_{n+1}$ we have $x = P_{n+1} y$ where $y \in X$ and hence

$$T_{n+1} x = S_{\alpha_{n+1}}(P_{n+1} y) + P_{n+1}(P_{n+1} y) - S_{\alpha_1} P_{n+1}(P_{n+1} y) = P_{n+1} y = x.$$

Therefore $T_{n+1} y_k = y_k$ for every $k = 1, \ldots, n+1$ and $T_{n+1} T_m = T_m$ for every $m = 1, \ldots, n$.

(c): We obtain that

$$\begin{aligned} \left\| T_{n+1} - S_{\alpha_{n+1}} \right\| &= \left\| S_{\alpha_{n+1}} P_{n+1} - P_{n+1} \right\| = \sup_{y \in P_{n+1}(B_X)} \left\| S_{\alpha_{n+1}} y - y \right\| \leqslant \\ &\leqslant \sup_{y \in \|P_{n+1}\| B_{E_{n+1}}} \left\| S_{\alpha_{n+1}} y - y \right\| \leqslant \|P_{n+1}\| \cdot \frac{\varepsilon_{n+1}}{2 d_{n+1}} = \frac{\varepsilon_{n+1}}{2}. \end{aligned}$$

(e): Fix an index $m \in \{1, \ldots, n\}$. Then

$$\begin{aligned} \|T_{n+1} T_m - T_m T_{n+1}\| &\leqslant \left\| T_{n+1} T_m - S_{\alpha_{n+1}} T_m \right\| + \left\| S_{\alpha_{n+1}} T_m - S_{\alpha_{n+1}} S_{\alpha_m} \right\| + \\ &\quad + \left\| S_{\alpha_{n+1}} S_{\alpha_m} - S_{\alpha_m} S_{\alpha_{n+1}} \right\| + \\ &\quad + \left\| S_{\alpha_m} S_{\alpha_{n+1}} - T_m S_{\alpha_{n+1}} \right\| + \\ &\quad + \left\| T_m S_{\alpha_{n+1}} - T_m T_{n+1} \right\| \leqslant \\ &\leqslant 2 \|T_m\| \left\| T_{n+1} - S_{\alpha_{n+1}} \right\| + 2 \left\| S_{\alpha_{n+1}} \right\| \left\| T_m - S_{\alpha_m} \right\| + \\ &\quad + \left\| S_{\alpha_{n+1}} S_{\alpha_m} - S_{\alpha_m} S_{\alpha_{n+1}} \right\|. \end{aligned}$$

Since

$$\|T_m\| \leqslant \left\| T_m - S_{\alpha_m} \right\| + \left\| S_{\alpha_m} \right\| \leqslant \frac{\varepsilon_m}{2} + \left(\lambda + \frac{\varepsilon_m}{2} \right) \leqslant \lambda + \varepsilon_m,$$

we have

$$\begin{aligned} \|T_{n+1} T_m - T_m T_{n+1}\| &\leqslant 2(\lambda + \varepsilon_m)\varepsilon_{n+1} + 2\left(\lambda + \frac{\varepsilon_{n+1}}{2} \right) \varepsilon_m + \varepsilon_{n+1} = \\ &= 2\lambda \varepsilon_{n+1} + 2\lambda \varepsilon_m + 2\varepsilon_m \varepsilon_{n+1} + \varepsilon_{n+1}(\varepsilon_m + 1) \leqslant \\ &\leqslant 2(\lambda + 1)(\varepsilon_{n+1} + \varepsilon_m). \end{aligned}$$

The inductive step, hence also the construction of the sequence (T_n), has been completed.

Denote

$$Z = \{z \in X : z = \lim T_n z\}.$$

Then $Y \subset Z$ and $Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} T_n$.

Indeed, fix an element $y \in Y$ and a number $\varepsilon > 0$, then there exists an index N such that $\|y - y_N\| < \varepsilon$. If $n \geqslant N$, then

$$\begin{aligned} \|T_n y - y\| &\leqslant \|T_n y - T_n y_N\| + \|T_n y_N - y_N\| + \|y_N - y\| < \\ &< (\lambda+1)\varepsilon + 0 + \varepsilon. \end{aligned}$$

Hence $\|T_n y - y\| \to 0$, meaning that $y \in Z$.

It is evident that $Z \subset \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} T_n$.

To prove that $\overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} T_n \subset Z$, it suffices to show the inclusions $\operatorname{ran} T_n \subset Z$ and the closedness of Z.

Fix an element $T_n x \in \operatorname{ran} T_n$, then $T_m T_n x = T_n x$ whenever $m > n$, hence $T_m T_n x \to T_n x$, yielding that $\operatorname{ran} T_n \subset Z$ for every n.

Now let $x \in X$ be an element for which $x_m \to x$ while $(x_m) \subset Z$. We verify that $T_n x \to x$. Fix an $\varepsilon > 0$, find an M such that $\|x_M - x\| < \varepsilon$, and an N such that $n \geqslant N$ implies $\|T_n x_M - x_M\| < \varepsilon$. Then

$$\|x - T_n x\| \leqslant \|x - x_M\| + \|x_M - T_n x_M\| + \|T_n\| \, \|x_M - x\| < \varepsilon + \varepsilon + (\lambda+1)\varepsilon,$$

showing that Z is closed.

Up to now, we have constructed a sequence $(T_n) \subset \mathscr{F}(X)$ with the following properties:

(Ta) $T_n y_m = y_m$ for all indices $n \geqslant m$;

(Tb) $T_n T_m = T_m$ for all indices $n > m$;

(Tc) $\|T_n\| \leqslant \lambda + \varepsilon_n$ for all indices n;

(Td) $\|T_n T_m - T_m T_n\| \leqslant 2(\lambda+1)(\varepsilon_n + \varepsilon_m)$ for all indices $m = 1, \dots, n-1$;

(Te) $Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} T_n$ is a separable space, $Y \subset Z$, and $T_n z \to z$ for every element $z \in Z$.

Z is locally λ-complemented in X

Next we shall verify that Z is locally λ-complemented in X. For this it suffices to show (see Proposition 6.6) that there exists an extension operator $\Phi \in \mathscr{L}(Z^*, X^*)$ such that $\|\Phi\| \leqslant \lambda$.

We construct the extension operator Φ as follows. Consider the operators $T_n \in \mathscr{L}(X,Z)$, $n \in \mathbb{N}$. Then $\left\| T_n^* z^* \right\| \leqslant (\lambda + \varepsilon_n) \left\| z^* \right\|$ for every functional $z^* \in Z^*$. Hence

$$\left(T_n^* z^*\right)_{z^* \in Z^*} \in \prod_{z^* \in Z^*} (\lambda + \varepsilon_n) \left\| z^* \right\| B_{X^*}$$

while every factor $(\lambda + \varepsilon_n) \left\| z^* \right\| B_{X^*}$ is a w^*-compact set in X^* due to the Alaoglu theorem. Due to the Tychonoff theorem the direct product itself is compact in the product topology, therefore the net (sequence) $\left(\left(T_n^* z^*\right)_{z^* \in Z^*}\right)_{n \in \mathbb{N}}$ contains a pointwise converging subnet $\left(\left(T_{n(\nu)}^* z^*\right)_{z^* \in Z^*}\right)_{\nu}$. We denote

$$\Phi z^* = \lim_{\nu} T_{n(\nu)}^* z^*.$$

The definition of Φ is correct in the sense that Φz^* is an element of X^*. Indeed, Φz^* is linear since all the ingredients are linear; Φz^* is bounded as

$$\begin{aligned} \left|(\Phi z^*)(z)\right| &= \left|\left(\lim_{\nu} T_{n(\nu)}^* z^*\right)(z)\right| = \left|\lim_{\nu} \left(T_{n(\nu)}^* z^*(z)\right)\right| = \lim_{\nu} \left| T_{n(\nu)}^* z^*(z)\right| \leqslant \\ &\leqslant \limsup_{\nu} \left\| T_{n(\nu)}^* \right\| \left\| z^* \right\| \|z\| \leqslant \lambda \left\| z^* \right\| \|z\| . \end{aligned}$$

From the inequality we also see that Φ itself is bounded and $\|\Phi\| \leqslant \lambda$. The operator Φ is obviously linear, hence $\Phi \in \mathscr{L}(Z^*, X^*)$.

The operator Φ is an extension operator since for every $z^* \in Z^*$ we have

$$\begin{aligned} z^*(z) &= z^*\left(\lim_{\nu} T_{n(\nu)} z\right) = \lim_{\nu} z^*\left(T_{n(\nu)} z\right) = \\ &= \lim_{\nu} \left(\left(T_{n(\nu)}^* z^*\right)(z)\right) = \left(\lim_{\nu} T_{n(\nu)}^* z^*\right)(z) = \\ &= \left(\Phi z^*\right)(z). \end{aligned}$$

At this point, all the claims of our theorem have been proved, except (i) and (ii). The argumentation for this final step has been essentially done in [CK, proof of Proposition 2.1]. For the sake of completeness, we present here the proof in detail.

Construction of (U_n)

We construct the operators $U_n \in \mathscr{F}(X)$, $n \in \mathbb{N}$, that possess the same properties as the operators T_n, i.e.

(Ua) $U_n y_m = y_m$ for every $n \geqslant m$;

(Ub) $U_n U_m = U_m$ for every $n > m$;

(Uc) $\limsup_n \|U_n\| \leqslant \lambda$;

(Ud) $\|U_n U_m - U_m U_n\| \leqslant 4(\lambda+1)(\varepsilon_n + \varepsilon_m)$ for every $m = 1, \ldots, n-1$;

(Ue) $Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} U_n$ is a separable space, $Y \subset Z$, and $U_n z \to z$ for every $z \in Z$,

but in addition,

(Uf) $\operatorname{ran} U_n = \operatorname{ran} U_n^2 = \operatorname{ran} T_n$.

For every $n \in \mathbb{N}$ we take a projection $Q_n \in \mathscr{L}(X)$ onto $\operatorname{ran} T_n$ and an operator

$$U_n = (1-\alpha_n) T_n + \alpha_n Q_n,$$

where the numbers $\alpha_n \in (0,1)$ have been chosen such that $-\dfrac{\alpha_n}{1-\alpha_n}$ is not an eigenvalue of the operator $T_n|_{\operatorname{ran} T_n} \in \mathscr{F}(\operatorname{ran} T)$, $n \in \mathbb{N}$, $\delta_n \leqslant \varepsilon_n$, $n \in \mathbb{N}$, and $\sum_n \delta_n \leqslant 1$, where $\delta_n := \alpha_n (\|Q_n\| + \|T_n\|)$.

(Ua): Since for all indices $n \geqslant m$, we have $T_n y_m = y_m$, there also holds $Q_n y_m = y_m$, since a projection is the identity on its range. Hence

$$U_n y_m = (1-\alpha_n) y_m + \alpha_n y_m = y_m, \qquad n \geqslant m.$$

(Ub): Let us have $n > m$. By a simple calculation we obtain $U_n U_m - U_m = 0$ due to the equalities $T_n T_m = T_m$, $T_n Q_m = Q_m$, $Q_n T_m = T_m$, $Q_n Q_m = Q_m$.

(Uc): Since $\|U_n - T_n\| = \alpha_n \|Q_n - T_n\| \leqslant \delta_n$, we have

$$\limsup_n \|U_n\| \leqslant \limsup_n \|T_n\| + \lim_n \|U_n - T_n\| \leqslant \lambda.$$

(Ud): Due to $\|U_n - T_n\| \leqslant \delta_n$, we can also well estimate $\|U_n U_m - U_m U_n\|$. Namely, assuming that for all n there holds $\|T_n\| \leqslant \lambda + 1$, $\|U_n\| \leqslant \lambda + 1$ (for every T_n this is true since $\varepsilon_n \leqslant 1$; if not true for some U_n, we can omit some members from the beginning while not harming other properties), we have

$$\begin{aligned}
\|U_n U_m - T_n U_m\| &\leqslant (\lambda+1)\delta_n, \\
\|T_n U_m - T_n T_m\| &\leqslant (\lambda+1)\delta_m, \\
\|T_n T_m - T_m T_n\| &\leqslant 2(\lambda+1)(\varepsilon_m + \varepsilon_n), \\
\|T_m T_n - T_m U_n\| &\leqslant (\lambda+1)\delta_n, \\
\|T_m U_n - U_m U_n\| &\leqslant (\lambda+1)\delta_m.
\end{aligned}$$

Summing up all these inequalities, we obtain that

$$\begin{aligned}\|U_nU_m - U_mU_n\| &\leqslant 2(\lambda+1)(\varepsilon_m+\varepsilon_n+\delta_m+\delta_n)\\ &\leqslant 4(\lambda+1)(\varepsilon_m+\varepsilon_n), \qquad m=1,\dots,n-1.\end{aligned}$$

This again implies

$$\sum_n \|U_nU_{n+1} - U_{n+1}U_n\| < \infty$$

due to the choice of (ε_n).

$(U\mathrm{f})$: It is clear that $\operatorname{ran} U_n^2 \subset \operatorname{ran} U_n \subset \operatorname{ran} T_n$. We are going to verify that $\operatorname{ran} T_n \subset \operatorname{ran} U_n^2$ which would show that $\operatorname{ran} U_n = \operatorname{ran} U_n^2 = \operatorname{ran} T_n$.

Consider the operator $U_n|_{\operatorname{ran} T_n} \in \mathscr{L}(\operatorname{ran} T_n)$, we see that it is injective. Indeed, let us have $x \in \operatorname{ran} T_n$, $x \neq 0$, then

$$U_nx = (1-\alpha_n)T_nx + \alpha_nQ_nx = (1-\alpha_n)T_nx + \alpha_nx \neq 0$$

due to the fact that $-\dfrac{\alpha_n}{1-\alpha_n}$ is not an eigenvalue of $T_n|_{\operatorname{ran} T_n}$.

Now the operator $U_n^2|_{\operatorname{ran} T_n} \in \mathscr{L}(\operatorname{ran} T_n)$ is injective as well: if $x \in \operatorname{ran} T_n$ is such that $U_n^2x = U_n(U_nx) = 0$, we must have $U_nx = 0$, hence $x = 0$. Therefore $U_n^2|_{\operatorname{ran} T_n}$ is surjective. Hence $\operatorname{ran} T_n \subset \operatorname{ran} U_n^2$.

$(U\mathrm{e})$: Since $\operatorname{ran} T_n = \operatorname{ran} U_n$, we have

$$Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} U_n.$$

The sequence (U_n) is an approximation of the identity on Z. Indeed, fix an element $z \in Z$, then

$$\|U_nz - z\| \leqslant \|U_n - T_n\|\,\|z\| + \|T_nz - z\| \leqslant \delta_n\,\|z\| + \|T_nz - z\| \to 0.$$

Remark. The above proof developed the first five lines of the proof of [CK, Proposition 2.1].

Construction of (V_n)

Next, we shall construct the operators $V_n \in \mathscr{F}(X)$, $n \in \mathbb{N}$, that possess quite the same properties as the operators U_n, i.e.

$(V\mathrm{a})$ $V_ny_m = y_m$ for all indices $n \geqslant m$;

(Vb) $V_n V_m = V_m$ for all indices $n > m$;

(Vc) $\limsup\limits_n \|V_n\| \leqslant \lambda$;

(Vd) $\|V_n V_m - V_m V_n\| \leqslant K \|U_{m-1} U_m - U_m U_{m-1}\|$ for some constant $K > 0$ and all indices $m > n$;

(Ve) $Z = \overline{\operatorname{span}} \bigcup\limits_{n=1}^{\infty} \operatorname{ran} V_n$ is a separable space, $Y \subset Z$, and $V_n z \to z$ for every element $z \in Z$.

(Vf) $\operatorname{ran} V_n = \operatorname{ran} V_n^2 = \operatorname{ran} U_n$.

Note that the previous condition (Ud)

$$\|U_n U_m - U_m U_n\| \leqslant 4(\lambda + 1)(\varepsilon_m + \varepsilon_n).$$

or its consequence

$$\sum_n \|U_n U_{n+1} - U_{n+1} U_n\| < \infty$$

has been replaced by a stronger condition (Vd) where the bound depends only on the greater index $m > n$.

Denote

$$\gamma_n = \|U_n U_{n+1} - U_{n+1} U_n\|, \quad n \in \mathbb{N}.$$

We define

$$A(n,k) = \prod_{j=n}^{n+k-1} U_j, \qquad k, n \in \mathbb{N}, \qquad A(n,0) = I.$$

The desired operators V_n will be the limits (by k) of $A(n,k)$.

For every $k \in \mathbb{N}$ it is easy to check the equality

$$A(n,k+1) = A(n,k) + A(n,k-1)\,(U_{n+k-1} U_{n+k} - U_{n+k} U_{n+k-1})$$

due to the property $U_n U_m = U_m$ if $n > m$. Thus having denoted

$$M_n(k) = \max_{1 \leqslant l \leqslant k} \|A(n,l)\|,$$

we have $M_n(1) = \|A(n,1)\| = \|U_n\|$. As

$$\|A(n,k+1)\| \leqslant \|A(n,k)\| + \|A(n,k-1)\|\, \gamma_{n+k-1},$$

we also have

$$\begin{aligned} M_n(k+1) &= \max(M_n(k), \|A(n,k+1)\|) \leqslant \\ &\leqslant \max\big(M_n(k), M_n(k) + M_n(k)\gamma_{n+k-1}\big) \leqslant \\ &\leqslant M_n(k)(1 + \gamma_{n+k-1}). \end{aligned}$$

Therefore

$$\|A(n,k)\| \leqslant \|U_n\| \prod_{j=n}^{\infty}(1+\gamma_j)$$

because

$$\frac{M_n(k+1)}{M_n(1)} = \prod_{j=1}^{k} \frac{M_n(j+1)}{M_n(j)} \leqslant \prod_{j=1}^{k}(1+\gamma_{n+j-1}) \leqslant \prod_{j=n}^{\infty}(1+\gamma_j).$$

Since $\sum_{j=n}^{\infty} \gamma_j < \infty$, the product $\prod_{j=n}^{\infty}(1+\gamma_j)$ converges, indicating that there is a uniform bound L on all sequences $(\|A(n,k)\|)_k$.

The sequence $(\|A(n,k)\|)_k$ converges since it is a Cauchy sequence. Indeed, for every $k \in \mathbb{N}$ we have

$$\|A(n,k+1) - A(n,k)\| \leqslant L\gamma_{n+k-1}.$$

Assuming $l > k$, we now have

$$\|A(n,l) - A(n,k)\| \leqslant L \sum_{j=n+k-1}^{n+l-1} \gamma_j \xrightarrow[k,l]{} 0.$$

For every n we define

$$V_n = \lim_{k\to\infty} A(n,k) = \prod_{j=n}^{\infty} U_j.$$

$(V\mathrm{a})$: We have $V_n y_m = y_m$ for every $n \geqslant m$, since $V_n U_m = U_m$ for the case $n > m$, and $V_n U_n = U_n^2$ (indeed, $U_n^2 = (U_n U_{n+1} \ldots U_{n+k-1})\, U_n$ from where going to the limit as $k \to \infty$ yields $U_n^2 = V_n U_n$).

$(V\mathrm{b})$: If $n > m$, we also have

$$V_n V_m = \lim_k V_n U_m U_{m+1} \ldots U_{m+k-1} = V_m$$

since $V_n U_m = U_m$.

$(V\mathrm{f})$: We have $\operatorname{ran} V_n = \operatorname{ran} V_n^2 = \operatorname{ran} U_n$, hence the operators V_n are of finite rank. Indeed, we have

$$\operatorname{ran} U_n = \operatorname{ran} U_n^2 \subset \operatorname{ran} V_n^2 \subset \operatorname{ran} V_n \subset \operatorname{ran} U_n.$$

More precisely, the inclusion $\operatorname{ran} U_n^2 \subset \operatorname{ran} V_n^2$ holds because $U_n^2 = V_n U_n$, as shown before, and for every $x \in X$ there is an element y such that $U_n x = U_n^2 y$; hence,

$$U_n^2 x = V_n U_n x = V_n U_n^2 y = V_n V_n U_n y = V_n^2 y.$$

The inclusion $\operatorname{ran} V_n \subset \operatorname{ran} U_n$ holds since for every $x \in \operatorname{ran} V_n$ there exists an element y such that

$$x = V_n y = \lim_{k\to\infty} \big(A(n,k)y\big).$$

For all indices k, we have $A(n,k)y \in \operatorname{ran} U_n$; as $\operatorname{ran} U_n$ is closed (it is finite-dimensional), we also have $x \in \operatorname{ran} U_n$.

(Vd): We have the estimate

$$\|V_n\| = \lim_{k\to\infty} \|A(n,k)\| \leqslant \prod_{j=n}^{\infty} (1+\gamma_j)\,\|U_n\|.$$

As the remainder of a converging infinite product vanishes (i.e. converges to 1), we find that

$$\limsup_n \|V_n\| \leqslant \limsup_n \|U_n\| \leqslant \lambda.$$

(Ve): Since $\operatorname{ran} U_n = \operatorname{ran} V_n$, we have

$$Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} V_n.$$

The sequence (V_n) is an approximation of the identity on Z. Indeed, fix an element $z \in Z$, then

$$\|V_{n+1}z - z\| \leqslant \|V_{n+1}z - V_{n+1}U_n z\| + \|U_n z - z\| \leqslant (\|V_{n+1}\| + 1)\,\|U_n z - z\| \to 0,$$

since $V_{n+1}U_n = U_n$ for every n.

(Vd): We are going to obtain a vanishing bound on $\|V_n V_m - V_m V_n\|$ depending only on m, where $m > n$. For any k where $n + k - 1 \geqslant m$, we have

$$\begin{aligned}
A(n,k)V_m &= \lim_{j\to\infty} (U_n U_{n+1} \dots U_{n+k-1})\, U_m A(m+1, j-1) = \\
&= \lim_{j\to\infty} U_n U_{n+1} \dots U_{m-1} U_m U_m A(m+1, j-1) = \\
&= \lim_{j\to\infty} A(n, m-n+1) A(m, j) = \\
&= A(n, m-n+1)\, V_m.
\end{aligned}$$

Therefore

$$V_n V_m = \lim_{k\to\infty} A(n,k) V_m = A(n, m-n+1) V_m = A(n, m-n-1) U_{m-1} U_m V_m$$

and

$$V_m V_n = V_n = A(n, m-n-1) U_m U_{m-1} V_m.$$

Denote $K = L \sup_n \|V_n\|$, then

$$\|V_n V_m - V_m V_n\| \leqslant \|A(n, m-n-1)\| \, \|U_{m-1} U_m - U_m U_{m-1}\| \, \|V_m\| \leqslant K \gamma_{m-1}$$

whenever $m > n$.

Remark. The above proof developed the first half-page of the proof of [CK, Proposition 2.1].

Construction of (R_n)

As the final step of the proof, we construct a sequence $(R_n) \in \mathscr{F}(X)$ such that

(Ra) for $l < k$ we have $R_k R_l = R_l R_k = R_l$;

(Rb) $\operatorname{ran} R_k = \operatorname{ran} R_k^2 = \operatorname{ran} V_{n_k}$ for every k;

(Rc) $\limsup_n \|R_n\| \leqslant \lambda$;

(Rd) $Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} R_n$;

(Re) $R_n z \to z$ for every element $z \in Z$.

For this end we shall choose an increasing sequence of positive integers (n_k) and operators $R_k \in \mathscr{L}(X, Z)$ such that the following conditions hold for every k:

(1) for $l < k$ we have $R_k R_l = R_l R_k = R_l$;

(2) $\operatorname{ran} R_k^2 = \operatorname{ran} R_k = \operatorname{ran} V_{n_k}$;

(3) R_k is a polynomial in $V_{n_1}, \dots, V_{n_k}$;

(4) $\left\| R_k - V_{n_k} \right\| < 2^{-k}$.

At first we choose $n_1 = 1$ and $R_1 = V_1$, then conditions (1)–(4) hold trivially.

Now suppose we have already chosen $n_1, \dots, n_k$ and $R_1, \dots, R_k$ that satisfy (1)–(4).

Denote $r_k = R_k|_{\operatorname{ran} R_k}$. The operator $r_k \in \mathscr{L}(\operatorname{ran} R_k)$ is invertible, since it is onto $\operatorname{ran} R_k^2 = \operatorname{ran} R_k$. Denote $w_k = (r_k)^{-1} \in \mathscr{L}(\operatorname{ran} R_k)$, then $w_k R_k x = x$ for every $x \in \operatorname{ran} R_k$. The operator w_k is a polynomial in r_k. Indeed, if $\dim R_k = d$, then $\dim \mathscr{L}(\operatorname{ran} R_k) = d^2$ and there exist coefficients a_i, $\sum_{j=0}^{d^2} |a_i| > 0$, such that

$$a_0 I + a_1 r_k + \ldots + a_{d^2} r_k^{d^2} = 0.$$

From this equation we can express w_k by carrying the first non-null member $a_i r_k^i$ to the right and multiplying both sides by $a_i^{-1} w_k^i$. We obtain $w_k = p(r_k)$ where p is a polynomial.

Denote $W_k = p(R_k) \in \mathscr{F}(X)$. Since R_k is a polynomial in $V_{n_1}, \dots, V_{n_k}$, for $m > n$ there holds $V_n(I - V_m) = V_n - V_n V_m = V_m V_n - V_n V_m$, and $\|V_n V_m - V_m V_n\|$ vanishes if $m \to \infty$, we have

$$\lim_{m\to\infty} \|R_k(I - V_m)\| = 0.$$

Using this, we find an index n_{k+1} such that $n_{k+1} > n_k$ and

$$\left\|R_k(I - V_{n_{k+1}})\right\| < \frac{1}{2^{k+1}\,\|W_k\|}.$$

We define

$$R_{k+1} = V_{n_{k+1}} + W_k R_k (I - V_{n_{k+1}}).$$

It remains to verify that conditions (1)–(4) hold for R_{k+1}.

Condition (3) holds trivially since W_k is a polynomial in R_k, hence $W_k R_k$ is a polynomial in $V_{n_1}, \dots, V_{n_k}$, thus altogether R_{k+1} is a polynomial in $V_{n_1}, \dots, V_{n_k}, V_{n_{k+1}}$.

Condition (4) is also easily verifiable:

$$\left\|R_{k+1} - V_{n_{k+1}}\right\| \leqslant \|W_k\| \left\|R_k(I - V_{n_{k+1}})\right\| < \frac{1}{2^{k+1}}.$$

Let us verify (1). We have

$$I - R_{k+1} = I - V_{n_{k+1}} - W_k R_k (I - V_{n_{k+1}}) = (I - W_k R_k)(I - V_{n_{k+1}}).$$

Due to the choice of W_k there holds $(I - W_k R_k) R_k = 0$. Since W_k is a polynomial in R_k, we obtain

$$R_k(I - W_k R_k) = R_k - R_k W_k R_k = R_k - W_k R_k^2 = (I - W_k R_k) R_k = 0.$$

We also have $V_{n_{k+1}} R_k = R_k$ as R_k is a polynomial in $V_{n_1}, \dots, V_{n_k}$ and $m > n$ implies $V_m V_n = V_n$. Now putting this together, we get that

$$\begin{aligned} R_k(I - R_{k+1}) &= R_k(I - W_k R_k)(I - V_{n_{k+1}}) = 0, \\ (I - R_{k+1}) R_k &= (I - W_k R_k)(I - V_{n_{k+1}}) R_k = \\ &= (I - W_k R_k)\left(R_k - V_{n_{k+1}}\right) = 0. \end{aligned}$$

We have verified that

$$R_k R_{k+1} = R_{k+1} R_k = R_k.$$

Let now an index l be such that $l < k$. Then

$$\begin{aligned} R_l R_{k+1} &= R_l R_k R_{k+1} = R_l R_k = R_l, \\ R_{k+1} R_l &= R_{k+1} R_k R_l = R_k R_l = R_l. \end{aligned}$$

Finally we shall verify that (2) holds. We have

$$\operatorname{ran} R_{k+1}^2 \subset \operatorname{ran} R_{k+1} \subset \operatorname{ran} V_{n_{k+1}}.$$

The inclusion $\operatorname{ran} R_{k+1} \subset \operatorname{ran} V_{n_{k+1}}$ can be justified by the fact that $W_k R_k (I - V_{n_{k+1}}) x = V_{n_{k+1}} W_k R_k (I - V_{n_{k+1}}) x$ for every $x \in X$.

Consider the operator $R_{k+1}|_{\operatorname{ran} V_{n_{k+1}}}$. We verify that it is injective. Let us have an element $x \in \operatorname{ran} V_{n_{k+1}}$ such that $R_{k+1} x = 0$. As W_k is a polynomial in R_k, we have

$$V_{n_{k+1}} x = W_k R_k \left(I - V_{n_{k+1}} \right) (-x) \in \operatorname{ran} R_k = \operatorname{ran} V_{n_k},$$

hence $V_{n_{k+1}} x = V_{n_k} y$ for some $y \in X$. Thus

$$V_{n_{k+1}} x = V_{n_k} y = V_{n_{k+1}} V_{n_k} y = V_{n_{k+1}} V_{n_{k+1}} x.$$

Since $V_{n_{k+1}}|_{\operatorname{ran} V_{n_{k+1}}}$ is injective, we now have $x = V_{n_{k+1}} x$ and therefore

$$0 = R_{k+1} x = V_{n_{k+1}} x + W_k R_k (I - V_{n_{k+1}} x) = x + W_k R_k x - W_k R_k x = x,$$

indicating that $R_{k+1}|_{\operatorname{ran} V_{n_{k+1}}}$ is injective.

Like we have already done on page 77, we obtain that $R_{k+1}^2|_{\operatorname{ran} V_{n_{k+1}}}$ is injective as well. Therefore it is surjective. Hence $\operatorname{ran} V_{n_{k+1}} \subset \operatorname{ran} R_{k+1}^2$.

We have $R_n y_m = y_m$, if $n \geqslant m$, due to the similar property for the operators V_n.

As all the previous sequences, (R_n) is an approximation of the identity as well. Namely, for every $z \in Z$ we have

$$\|R_k z - z\| \leqslant \left\| R_k - V_{n_k} \right\| \|z\| + \left\| V_{n_k} z - z \right\| \to 0.$$

Hence

$$Z \subset \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} R_n \subset \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} V_n = Z,$$

yielding that

$$Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} R_n.$$

What has done up to now is sufficient since the other conditions (Rd) and (Re) follow from the fact that every R_k is a polynomial in first k members of a subsequence of (V_n).

Remark. The construction of (R_n) and verifying its properties developed the last $\frac{2}{3}$ pages of the proof of [CK, Proposition 2.1].

In conclusion we see that all the claims of the theorem have been proven. □

6.4 Corollaries and remarks

The following corollary is immediate.

Corollary 6.10. *Let X be a non-separable Banach space. If X has the asymptotically λ-commuting bounded AP, then every separable closed subspace of X is contained in a locally λ-complemented separable closed subspace with the λ-commuting bounded AP.*

For the case of the λ-bounded AP, an analogous claim holds. Namely due to [HM] or [SY], every separable closed subspace is contained in a separable locally 1-complemented subspace. Now, if X has the λ-bounded AP, then its every locally 1-complemented subspace has also the λ-bounded AP (for the proof of this fact, see [K1, Theorem 5.1], [L1, Corollary 2], or [O5, Proposition 2.1]). Hence every separable closed subspace of X is contained in a separable locally 1-complemented subspace having the λ-bounded AP.

Remark 6.11. It can be seen from the proof that if $Y = \overline{\{y_1, y_2, \ldots\}}$, $y_n \in Y$, $Y_n = \operatorname{span}\{y_1, \ldots, y_n\}$ (hence, (Y_n) is an increasing sequence of finite-dimensional subspaces of Y whose union is dense in Y), then, moreover, the constructed sequence (R_n) is such that $R_n|_{Y_n} = I_{Y_n}$ for every n and $Z = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} \operatorname{ran} R_n$.

Under these assumptions in the proof of the theorem due to Casazza, Kalton and Wojtaszczyk (see [C2, the proof of Theorem 9.3]) it has been asserted that "by switching to a pointwise convergent subnet of (R_n)" a projection $P \in \mathcal{L}(X)$ with $\operatorname{ran} P = Z$ can be obtained, meaning that Z is complemented in X. However, the assumptions of Theorem 6.9 are met for $X = \ell_\infty$, and it is known that there does not exist any separable complemented subspace in ℓ_∞. This indicates that the proof of the Casazza-Kalton-Wojtaszczyk theorem [C2, the proof of Theorem 9.3] is in error, and it remains an open problem whether a non-separable Banach space with the commuting bounded AP has the separable complementation property or not. In particular, we do not know whether ℓ_∞ has the commuting bounded AP.

Our conjecture is that the Casazza-Kalton-Wojtaszczyk theorem does not hold: the commuting bounded AP of a non-separable Banach space does not imply the separable complementation property.

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Kommuteeruvad tõkestatud aproksimatsiooni-omadused Banachi ruumides

Kokkuvõte

Käesoleva väitekirja põhieesmärk on uurida kommuteeruvat tõkestatud aproksimatsiooniomadust (ja selle kompaktset versiooni). Ühelt poolt on see omadus üldjuhul nõrgem kui kommuteeruv meetriline aproksimatsiooniomadus või lõplikumõõtmelise lahutuse omadus, teiselt poolt aga tugevam kui tõkestatud aproksimatsiooniomadus. Tõkestatud ja kommuteeruva tõkestatud aproksimatsiooniomaduse vahele (vähemalt formaalselt) jääb väitekirjas defineeritud uus mõiste asümptootiliselt kommuteeruv tõkestatud aproksimatsiooniomadus.

Väitekirja esimene peatükk kujutab endast sissejuhatust, mille koosseisu on paigutatud lühike ajalooline ülevaade aproksimatsiooniprobleemist, väitekirja kokkuvõte ning mõningad tehnilised märkused väitekirjas kasutatud tähistuste kohta.

Väitekirja teises peatükis tutvustatakse lugejale aproksimatsiooniomaduste erinevaid versioone, sealhulgas klassikalist aproksimatsiooniomadust ja kompaktset aproksimatsiooniomadust, tõkestatud (kompaktset) aproksimatsiooniomadust (hõlmates muuhulgas 1-tõkestatud, see tähendab, meetrilist (kompaktset) aproksimatsiooniomadust) ning kommuteeruvat tõkestatud (kompaktset) aproksimatsiooniomadust. Välja on toodud mõisted ja tulemused, mis on järgnevate osade mõistmiseks või teemast tervikliku pildi saamiseks vajalikud.

Kolmandas peatükis tõestatakse, et artiklis [W] G. Willise poolt konstrueeritud meetriline kompaktne aproksimeeriv pere Willise ruumil X_W on kommuteeruv. See tähendab, et ruumil X_W on kommuteeruv meetriline kompaktne aproksimatsiooniomadus. Kuna ruumil X_W ei ole aproksimatsiooniomadust, on seega näidatud, et kommuteeruv meetriline kompaktne aproksimatsiooniomadus ja aproksimatsiooniomadus on erinevad omadused.

Neljas peatükk tugineb artiklitele [O4] ning [GS]. Aastal 1988 näitasid G. Godefroy ja P. D. Saphar artiklis [GS], kuidas separaabli Banachi ruumi X geomeetriline struktuur (täpsemalt, M-ideaaliks olemine oma teises kaasruumis) võimaldab tõsta kommuteeruva tõkestatud aproksimatsiooniomaduse ruumilt X kaasruumi X^*. Väitekirjas parendatakse seda tulemust mitmel moel. Muuhulgas näidatakse, et separaabluse eeldusest saab loobuda ning ruumi geomeetria osas kasutatakse eeldusena üldisemat tingimust, mida väitekirjas nimetatakse $M(a,B,c)$-võrratuseks. Märgime, et Banachi ruum on M-

ideaal oma teises kaasruumis parajasti juhul, kui ta rahuldab $M(1,\{-1\},1)$-võrratust.

Rakendusena näidatakse, et kui Banachi ruum rahuldab $M(a,B,c)$-võrratust ning temal on λ-tõkestatud (kompaktne) aproksimatsiooniomadus (kus λ ei ületa arvu $\max|B|+c$), siis nii ruumil endal kui ka tema kaasruumil on meetriline (kompaktne) aproksimatsiooniomadus.

$M(a,B,c)$-võrratus esineb defineerimata kujul juba artiklis [O4], kus seda kasutati omaduse $M^*(a,B,c)$ intensiivsel uurimisel. Omadus $M^*(a,B,c)$ on samuti Banachi ruumide struktuurne omadus, mis võimaldab ühekorraga kirjeldada teatud (suurt) ideaalide klassi Banachi ruumides.

Kolmanda ja neljanda peatüki põhitulemused on ilmunud artiklis [OZ1].

Viiendas peatükis keskendutakse ruumile X_{JS}, mille kirjeldus on avaldatud artiklis [JO], ent mille konstrueerisid juba aastal 1996 W. B. Johnson ja G. Schechtman. Ruum X_{JS} on selle poolest märkimisväärne, et tal ei ole meetrilist aproksimatsiooniomadust, kuid on tõkestatud aproksimatsiooniomadus. Aastal 2001 tõestas G. Godefroy [G], et ruumil X_{JS} on kommuteeruv 8-tõkestatud aproksimatsiooniomadus, ning märkis, et pole tehtud mingeid pingutusi leidmaks väiksemat konstanti 8 asemel. Viiendas peatükis tõestataksegi, et ruumil X_{JS} on kommuteeruv 6-tõkestatud aproksimatsiooniomadus. Siiski jääb lahtiseks, kas konstant 6 on vähim võimalik.

Viienda peatüki põhitulemused on ilmunud artiklis [Z].

Kuuendas peatükis võetakse kasutusele uus mõiste – asümptootiliselt kommuteeruv tõkestatud aproksimatsiooniomadus. Separaablite ruumide korral langeb see mõiste kokku kommuteeruva tõkestatud aproksimatsiooniomadusega. Üldjuhul näidatakse, et kui Banachi ruumil on asümptootiliselt kommuteeruv tõkestatud aproksimatsiooniomadus, siis tal on separaabel lokaalse täiendatavuse omadus tugeval kujul. Märgime, et selle tulemuse valguses pole selge, kas kommuteeruvast tõkestatud aproksimatsiooniomadusest järeldub separaabel täiendatavuse omadus (nagu on väidetud teoreemis [C2, Theorem 9.3]).

Kuuenda peatüki põhitulemusi sisaldava artikli eelvariant on [OZ2].

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Curriculum vitae

NAME: Indrek Zolk
DATE AND PLACE OF BIRTH: 29.01.1982, Tallinn, Estonia
NATIONALITY: Estonian
ADDRESS: Institute of Mathematics, J. Liivi 2, 50409 Tartu, Estonia
PHONE: +372 737 5863
E-MAIL: indrek.zolk@ut.ee

EDUCATION:

1998–2000	Tallinn English College
2000–2005	University of Tartu, bachelor studies in mathematics, B.Sc. in mathematics 2005
2005–2006	University of Tartu, master studies in mathematics, M.Sc. in mathematics 2006
2006–	University of Tartu, doctoral studies in mathematics
2007	University of Tartu, master degree in mathematics teaching

FOREIGN LANGUAGES: English, French, Russian

EMPLOYMENT:

09/2001–08/2002	Estonian Literary Museum, network administrator
10/2002–05/2003	conscription in Estonian Defence Forces, radio operator
08/2003–06/2007	Tartu Art Gymnasium, teacher of information technology
06/2005–05/2006	Urania Com Ltd, software developer
08/2006–06/2008	Tartu Vocational Education Centre, teacher of information technology
01/2008–12/2009	University of Tartu, Faculty of Mathematics and Computer Science, extraordinary research fellow
01/2010–08/2010	University of Tartu, Faculty of Mathematics and Computer Science, extraordinary assistant

FIELDS OF SCIENTIFIC INTEREST:
Functional analysis – geometry of Banach spaces and spaces of operators

Curriculum vitae

NIMI: Indrek Zolk
SÜNNIAEG JA -KOHT: 29.01.1982, Tallinn, Eesti
RAHVUS: eestlane
AADRESS: TÜ matemaatika instituut, J. Liivi 2, 50409 Tartu, Eesti
TELEFON: +372 737 5863
E-POST: indrek.zolk@ut.ee

HARIDUS:

1998–2000	Tallinna Inglise Kolledž
2000–2005	Tartu Ülikool, matemaatika bakalaureuseõpe, BSc matemaatika erialal 2005
2005–2006	Tartu Ülikool, matemaatika magistriõpe, MSc matemaatika erialal 2006
2006–	Tartu Ülikool, matemaatika doktoriõpe
2007	Tartu Ülikool, matemaatikaõpetaja magistrikraad

VÕÕRKEELTE OSKUS: inglise, prantsuse, vene

TEENISTUSKÄIK:

09/2001–08/2002	Eesti Kirjandusmuuseum, arvutivõrgu administraator
10/2002–05/2003	ajateenistus Eesti Kaitseväes, sidemees-radist
08/2003–06/2007	Tartu Kunstigümnaasium, arvutiõpetaja
06/2005–05/2006	OÜ Urania Com, programmeerija
08/2006–06/2008	Tartu Kutsehariduskeskus, infotehnoloogia kutseõpetaja
01/2008–12/2009	Tartu Ülikool, matemaatika-informaatikateaduskond, erakorraline teadur
01/2010–08/2010	Tartu Ülikool, matemaatika-informaatikateaduskond, erakorraline assistent

TEADUSLIKUD HUVID:
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